Projective Forcing

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Contents

Chapter 1
INTRODUCTION

In this introduction we give the historical background of the themes developed in this dissertation and state the main results. We also set up the notation and state some facts that we will use further on.

Remarks and notation

We work in Zermelo-Fraenkel Set Theory with the Axiom of Choice, ZFC. Our basic references are: T. Jech, *Set Theory* (Academic Press, 1978) and K. Kunen, *Set Theory. An Introduction to Independence Proofs* (North Holland, 1980). With ZFC^* we refer to a finite fragment of ZFC sufficient for the proof at hand.

α, β, γ, δ, η, ν, ξ, ζ denote ordinal numbers. We reserve κ, λ, μ for cardinal numbers. ω is the set of all finite ordinals, or natural numbers, which we denote by i, j, k, l, m, n.

ω^ω is the Baire space. i.e., the set of all functions from ω into ω viewed as product space. 2^ω is the Cantor space. i.e., the set of all functions from ω into $\{0,1\}$ viewed as product space. Finally, R is the real line with the topology generated by open intervals with rational endpoints.

There is a close relationship between these three spaces: they are "almost" homeomorphic, that is, there exists an homeomorphism from any of them into any other except for a countable subset (see [L2], VII.3). So, most of the topological and regularity questions can be transferred from one of these spaces to another. Since all the questions that we shall consider are of this kind, we will work in the space that is most convenient for each particular problem. In each case we will indicate explicitly what space we are working on.

Strictly speaking, only the members of R are real numbers. But, by our previous remarks, we also call real numbers or, for short, reals the elements of ω^ω and 2^ω. We use a, b, c, r, s, t, x, y, z to denote reals and reserve f, g, h for elements of ω^ω and 2^ω.

These three spaces are polish spaces: separable complete metric spaces. In general, given a polish space X, we say that a subset A of X is a *Borel set* iff* it belongs to smallest σ-algebra of subsets of X containing all open sets. More explicitly:

Definition *Let X be a polish space. For every countable ordinal α, ($\alpha \geq 1$) we define, by recursion on ω_1, the collection $\underset{\sim}{\Sigma}^0_\alpha$ of subsets of X:*

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* "iff" abreviates "if and only if"

- $A \in \underset{\sim}{\Sigma}^0_1$ *iff* A *is an open subset of* X.
- $A \in \underset{\sim}{\Sigma}^0_\alpha$ *iff there exists a sequence* $\langle B_n : n \in \omega\rangle$ *such that for every* $n \in \omega$ *there is* $\beta_n < \alpha$ *with* $B_n \in \underset{\sim}{\Sigma}^0_{\beta_n}$ *and* $A = \bigcup_{n\in\omega} \tilde{B}_n$, *where* $\tilde{B}_n$ *is the complement of* B_n.

We also define the collections $\underset{\sim}{\Pi}^0_\alpha$ *and* $\underset{\sim}{\Delta}^0_\alpha$, $1 \leq \alpha < \omega_1$*:* $\underset{\sim}{\Pi}^0_\alpha = \{\tilde{A} : A \in \underset{\sim}{\Sigma}^0_\alpha\}$ *and* $\underset{\sim}{\Delta}^0_\alpha = \underset{\sim}{\Sigma}^0_\alpha \cap \underset{\sim}{\Pi}^0_\alpha$. *Finally,* $B \subseteq X$ *is a* Borel set *iff there is* $\alpha < \omega_1$ *such that* $B \in \underset{\sim}{\Sigma}^0_\alpha$.

A subset A of polish space X is an *analytic set* iff it is the projection of a Borel subset of $X \times Y$, where Y is some polish space. Also, A is analytic iff it is the projection of a closed subset of $X \times \omega^\omega$. Clearly, every Borel set is analytic.

We define the projective sets of a polish space X as follows:

Definition *For every* $n \geq 1$, *we define the collection of* $\underset{\sim}{\Sigma}^1_n$ *of subsets of* X:

- $A \in \underset{\sim}{\Sigma}^1_1$ *iff* A *is an analytic subset of* X.
- $A \in \underset{\sim}{\Sigma}^1_{n+1}$ *iff* $A = \{x : \exists y \langle x, y\rangle \in B\}$ *for some* $B \subseteq X \times \omega^\omega$ *such that* $\tilde{B} \in \underset{\sim}{\Sigma}^1_n$.

We also define the collections $\underset{\sim}{\Pi}^1_n$ *and* $\underset{\sim}{\Delta}^1_n$ *as follows:* $\underset{\sim}{\Pi}^1_n = \{A : \tilde{A} \in \underset{\sim}{\Sigma}^1_n\}$ *and* $\underset{\sim}{\Delta}^1_n = \underset{\sim}{\Sigma}^1_n \cap \underset{\sim}{\Pi}^1_n$. *A subset* A *of* X *is a* projective set *iff for some* $n \geq 1$, $A \in \underset{\sim}{\Sigma}^1_n$.

M. Suslin showed in 1917 ([Su]) that the $\underset{\sim}{\Delta}^1_1$ sets of every polish space are precisely the Borel sets.

The projective sets of the Baire space are definable in the following way: For every $n \geq 1$ and every $a \in \omega^\omega$,

- If n is odd, then:
  - A is $\Sigma^1_n(a)$ iff for all $x \in \omega^\omega$
    $$x \in A \text{ iff } \exists z_1 \forall z_2 ... \exists z_n \forall m \langle x \upharpoonright m, z_1 \upharpoonright m, z_2 \upharpoonright m, ..., z_n \upharpoonright m\rangle \in R$$
    where R is an arithmetical relation on a. i.e., definable in the model $\langle V_\omega, \in, a\rangle$.
  - A is $\Pi^1_n(a)$ iff for all $x \in \omega^\omega$
    $$x \in A \text{ iff } \forall z_1 \exists z_2 ... \forall z_n \exists m \langle x \upharpoonright m, z_1 \upharpoonright m, z_2 \upharpoonright m, ..., z_n \upharpoonright m\rangle \in R$$
    where R is an arithmetical relation on a.
- If n is even, then:
  - A is $\Sigma^1_n(a)$ iff for all $x \in \omega^\omega$
    $$x \in A \text{ iff } \exists z_1 \forall z_2 ... \forall z_n \exists m \langle x \upharpoonright m, z_1 \upharpoonright m, z_2 \upharpoonright m, ..., z_n \upharpoonright m\rangle \in R$$
    where R is an arithmetical relation on a.

– A is $\Pi^1_n(a)$ iff

$$x \in A \text{ iff } \forall z_1 \exists z_2 ... \exists z_n \forall m \langle x \restriction m, z_1 \restriction m, z_2 \restriction m, ..., z_n \restriction m \rangle \in R$$

where R is an arithmetical relation on a.

Finally, for every $n \geq 1$, $\underset{\sim}{\Sigma}^1_n = \bigcup_{a \in \omega^\omega} \Sigma^1_n(a)$ and $\underset{\sim}{\Pi}^1_n = \bigcup_{a \in \omega^\omega} \Pi^1_n(a)$.

Forcing

A forcing notion $\mathsf{P} = \langle P, \leq_P \rangle$ is a set P with a binary relation $\leq_P$ on P that is reflexive and transitive. We do not require that $\leq_P$ be antisymmetric nor that P have a $\leq_P$-least element, although this is almost always the case. p, q and r (possibly with subindexes) denote elements of P, which we call *conditions (of the forcing)*. $p \leq_P q$ means that "p extends q". i.e., p gives more information than q on the generic object that is added by forcing with P. $p \perp_P q$ means that p and q are incompatible conditions. i.e., there is no $r \in P$ such that $r \leq_P p$ and $r \leq_P q$. P is *separative* if for all $p, q \in P$, if $p \nleq_P q$, then there exists $r \in P$ such that $r \leq_P p$ and $r \perp_P q$. $\Vdash_{\mathsf{P}}$ denotes the forcing relation of P. σ, τ, ρ, $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{P}$, $\dot{Q}$, $\dot{\mathsf{P}}$, $\dot{\mathsf{Q}}$, $\dot{G}$ (possibly with subindexes) denote P-names. $\check{n}$, $\check{a}$, $\check{\mathsf{P}}$ denote canonical P-names for elements in the ground model. If G is a P-generic filter over some model V, then $\tau[G]$, $\dot{a}[G]$, $\check{n}[G]$ denote the evaluations of τ, $\dot{a}$ and $\check{n}$ by G in $V[G]$.

Sometimes we do forcing with complete Boolean algebras. If B is a complete Boolean algebra, then $\langle B \setminus \{0_{\mathsf{B}}\}, \leq_{\mathsf{B}} \rangle$, where $\leq_{\mathsf{B}}$ is the canonical partial order from B, is a forcing notion. It is a well-known fact that every separative poset P can be densely embedded into an unique (up to isomorphism) complete Boolean algebra, the completion of P, r.o. (P). So, all forcing arguments can be carried out with complete Boolean algebras. The main advantage of working with complete Boolean algebras is that for every formula $\varphi(x_1, ..., x_n)$ and all B-names $\tau_1, ..., \tau_n$ there exists a condition $[\![\varphi(\tau_1, ..., \tau_n)]\!]_{\mathsf{B}}$ in B such that for every generic filter G over V,

$$V[G] \models \varphi(\tau_1[G], ..., \tau_n[G]) \text{ iff } [\![\varphi(\tau_1, ..., \tau_n)]\!]_{\mathsf{B}} \in G.$$

Namely, $[\![\varphi(\tau_1, ..., \tau_n)]\!]_{\mathsf{B}} = \sup(\{p \in B \setminus \{0_{\mathsf{B}}\} : p \Vdash_{\mathsf{B}} \varphi(\tau_1, ..., \tau_n)\})$ (see [Ku], VII.7 or [J2], 2.18).

Descriptive Set Theory and forcing

Descriptive Set Theory is the study of definable sets of reals, mainly the projective sets of reals. One of the main objectives in this field is to look into the *regularity properties* of projective sets, like the Lebesgue measurability, the property of Baire or the perfect set property.

We use the following notation: we write $\underset{\sim}{\Sigma}^1_n(L)$ ($\underset{\sim}{\Pi}^1_n(L)$, $\underset{\sim}{\Delta}^1_n(L)$) if every $\underset{\sim}{\Sigma}^1_n$ (respectively, $\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$) set of reals is Lebesgue measurable. We write $\underset{\sim}{\Sigma}^1_n(B)$ ($\underset{\sim}{\Pi}^1_n(B)$, $\underset{\sim}{\Delta}^1_n(B)$) if every $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$) set of reals has the property of Baire. Finally, we write $\underset{\sim}{\Sigma}^1_n(P)$ ($\underset{\sim}{\Pi}^1_n(P)$, $\underset{\sim}{\Delta}^1_n(P)$) if every $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$) has the perfect set property.

The first results on regularity properties of projective sets date from the beginning of the 20th century: in 1917, N. Luzin ([Lu1]) proved $\undertilde{\Sigma}^1_1(L)$, in 1923, N. Luzin and W. Sierpiński ([Lu-Si]) proved $\undertilde{\Sigma}^1_1(B)$, and, in 1917 M. Šuslin proved $\undertilde{\Sigma}^1_1(P)$ (see [Lu2]).

But it was quickly realized the difficulty to extend this results to more complex projective sets. If a set of reals is Lebesgue measurable, then so is its complementary set. The same occurs with the property of Baire. So, the results mentioned above imply $\undertilde{\Pi}^1_1(L)$ and $\undertilde{\Pi}^1_1(B)$. Therefore, the question about the measurability and the property of Baire was for sets in the $\undertilde{\Delta}^1_2$ level. For the perfect set property, the question remained at the $\undertilde{\Pi}^1_1$ level.

Then K. Gödel obtained in 1938 the first negative results: in the model of constructible sets, L, used by him to prove the consistency of the Generalized Continuum Hypothesis and the Axiom of Choice with ZF ([Gö]) there exists a $\undertilde{\Delta}^1_2$ set which is not Lebesgue measurable and does not have the property of Baire. He also noticed that in this model there is a $\undertilde{\Pi}^1_1$ set without the perfect set property.

After the method of forcing was invented by Cohen to prove the independence of the Continuum Hypothesis and the Axiom of Choice from ZF, it has been applied to produce a vast array of consistency results in Descriptive Set Theory. For instance, R. Solovay ([So]) proved that $ZFC+$"Every projective set of reals is Lebesgue measurable and has the property of Baire" is consistent, supposing that $ZFC+$"There exists an inaccessible cardinal" also is. We can find other applications of forcing in Descriptive Set Theory in the proof of S. Shelah ([Sh1]) that the existence of an inaccessible cardinal is a necessary hypothesis in order to obtain a model of ZFC where all projective set of reals (in fact, all $\undertilde{\Sigma}^1_3$) are Lebesgue measurable, and in the proof that this large cardinal hypothesis is not necessary in order to obtain a model of ZFC where all projective sets of reals have the property of Baire.

Cohen Forcing ([C]) which adds a real that does not belong to any meager Borel set with code in the ground model, and Random forcing ([So]) which produces a real not belonging to any null Borel set with code in the ground model, play an essential role in Solovay's proof. Shelah used the Amoeba forcing ([M-So]) which adds a measure one set of Random reals over the ground model, to show that the inaccessible cardinal assumption is necessary to find a model of ZFC where all projective sets of reals are Lebesgue measurable. He also used Amoeba forcing for Category ([M-So]) which adds a comeager set of Cohen reals over the ground model, for the aforementioned result on the property of Baire.

Looking at the partially ordered sets used in these and other forcing arguments in Descriptive Set Theory, we realize that they are, essentially, sets of reals definable in a simple way: they are projective partial orderings. By a projective partial ordering we mean a partially ordered set (a poset, for short) where the ordering and the incompatibility relation are projective subsets of the real plane. Cohen, Random, Amoeba and Amoeba for the Category forcing notions are Borel ($\undertilde{\Delta}^1_1$). Other examples of projective, in fact Borel, and ccc forcing notions can be found in [He], [To2], [Ju-R-Sh], and more complicated projective forcing notions in [B2].

Thus, the study of the projective forcing notions arise in a natural way. It starts in [Sh1], where the absoluteness properties of $\undertilde{\Sigma}^1_2$ sets are exploited. It continues in [Ju-Sh1], [Ju-Sh3] and [Ju-Sh4]. In [Ju-Sh1], H. Judah and S. Shelah develop a general theory for ω-Suslin forcing notions. i.e., $\utilde{\Sigma}^1_1$ posets. Their work was continued in [B1] and in [B-Ju] restricted to $\utilde{\Sigma}^1_1$ ccc posets. Other papers, as for instance [G-Sh1], [G-Sh2], [Sh2], [Ju-R] and [Ju-R-Sh], study some other properties of $\utilde{\Sigma}^1_1$ ccc forcing notions.

In Section 2.1, we generalize these results by developing a general theory of projective and projective ccc posets. We give the basic definitions and facts and we compute the complexity of some sets of reals associated to these forcing notions, such as the set of codes of the posets of reals of a given complexity, the set of codes of maximal antichains of a poset and the set of codes of simple names for reals. We finish this section showing that forcing a projective formula by means of a projective ccc poset is a projective relation and computing its complexity:

Theorem 2.1.23 *Let P be a projective ccc poset and let $\theta(x)$ be a Σ^1_k (Π^1_k) formula with $k \geq 2$. Then the relation*

$$R(p,\tau) \leftrightarrow p \in \mathsf{P} \wedge \tau \text{ is a simple } \mathsf{P}\text{-name for a real} \wedge p \Vdash_{\mathsf{P}} \theta(\tau))$$

is a projective relation. Moreover,

1. *If P is a $\utilde{\Sigma}^1_n$ poset, then R is $\utilde{\Sigma}^1_{n+k-1}$ ($\utilde{\Pi}^1_{n+k-1}$).*
2. *If P is a $\utilde{\Pi}^1_n$ poset, then R is $\utilde{\Sigma}^1_{n+k}$ ($\utilde{\Pi}^1_{n+k}$).*

The remainder of the second chapter is devoted to Martin's Axiom restricted to projective posets. Martin's Axiom (henceforth, MA) was formulated for the first time by A. Martin and R. Solovay in [M-So]: *For every ccc poset P and every family $\{A_\alpha : \alpha < \kappa\}$, $\kappa < 2^{\aleph_0}$, of maximal antichains of P there exists a filter $G \subseteq P$ such that for every $\alpha < \kappa$, $A_\alpha \cap G \neq \emptyset$.* We obtain Martin's Axiom for projective posets by adding the condition that P is a projective poset. R. Solovay and S. Tennenbaum showed the consistency of MA with $ZFC + \neg CH$ ([So-T]). MA has become a powerful tool for consistency results. For instance, A. Martin and R. Solovay show in their paper that MA implies that there are no Suslin trees, and that it implies the additivity of the Lebesgue measure, the additivity of category, $\utilde{\Sigma}^1_2(L)$ and $\utilde{\Sigma}^1_2(B)$.

In [Ju-Sh1], H. Judah and S. Shelah define the Martin's Axiom for $\utilde{\Sigma}^1_1$ posets, $MA(\utilde{\Sigma}^1_1)$, as the restriction of MA to $\utilde{\Sigma}^1_1$ posets and show that although $MA(\utilde{\Sigma}^1_1)$ implies the additivity of the Lebesgue measure, it is weaker than MA. Since all consequences of $MA(\utilde{\Sigma}^1_1)$ in [Ju-Sh1] are consequences of the additivity of measure, they asked if they are equivalent. The negative answer was given by J. Bagaria and H. Judah in [B1] (see also [B-Ju]), where they build a transitive model where the additivity of the Lebesgue measure holds and Martin's Axiom fails for a Borel ccc poset. Moreover, they give a combinatorial characterization of Martin's Axiom restricted to the Amoeba poset and show its equivalence to the additivity of the Lebesgue measure.

In Section 2.2, we define Martin's Axiom for projective posets, $MA(Proj)$, and we show:

Theorem 2.2.14 *(GCH) Let κ be a regular cardinal which is not the successor of a cardinal of countable cofinality. Then there is an iteration of projective and ccc posets such that whenever G is a generic filter for the iteration,*

$$V[G] \models MA(Proj) \wedge 2^{\aleph_0} = \kappa.$$

As Theorem 2.2.14 states, we build a model of $ZFC + MA(Proj) + \neg CH$ by iterating only projective ccc posets. We want to proceed imitating the usual proof of consistency of $MA + \neg CH$ with ZFC. But when we check that $MA(Proj)$ holds in the final generic extension, some difficulties arise. Firstly, A. Levy ([L1]) showed that if $ZFC+$ "There exists an inaccessible cardinal" is consistent, so is $ZFC+$"Every uncountable projective set of reals has the cardinality of the continuum", where the continuum is as large as you want. Hence, we cannot assume that $MA(Proj)$ is equivalent to $MA(Proj)$ restricted to posets with cardinality less than the continuum, a crucial fact in the proof of the consistency of $ZFC+MA+\neg CH$. Secondly, and more important, in general, projective formulas fail to be absolute for transitive models of ZF. Since we only force with posets which are defined by projective formulas at some stage of the iteration, given a projective poset in the final generic extension, there is a priori no reason to ensure that we have forced with this same poset at some stage along of the iteration. However, as we shall see, we can arrange the iteration in such a way so that the projective formulas are absolute for *sufficiently-many* models along of the iteration, where "sufficiently" means for a ω_1-closed and unbounded subset of κ (ω_1-club subset of κ, for short). This notion is a generalization of a club subset of κ: a ω_1-club subset of κ is an unbounded subset of κ closed under supremums of sequences of length γ, $\omega_1 \leq \mathrm{cf}(\gamma) < \kappa$, of its elements. So, we begin this section by showing some properties of λ-club subsets of κ, analogous to the well-known properties of club subsets of κ, which we use in the proof of Theorem 2.2.14.

In Section 2.3, we show that $MA(Proj)$ is weaker than MA. We first collapse a weakly-compact cardinal κ onto ω_1 using the Levy-collapsing poset $Coll(\omega, <\kappa)$. This allows us to apply a version of a well-known theorem of K. Kunen (Theorem 2.3.2) to projective ccc posets and prove that they are indestructible-ccc in every ccc generic extension of the collapse. Then, we use Theorem 2.2.14 to prove the following:

Theorem 2.3.18 *Let κ be a weakly-compact cardinal and let $V_0 = L[C]$, where C is a $Coll(\omega, <\kappa)$-generic filter over L. Suppose that $\varphi(x)$ is a formula of the language of Set Theory such that:*

1. *For every $X \subseteq \omega^\omega$, there are posets $\mathsf{P}_0^X, \ldots, \mathsf{P}_n^X$ such that*

$$ZFC \vdash (\varphi(X) \leftrightarrow \mathsf{P}_0^X, \ldots, \mathsf{P}_n^X \text{ are ccc posets}).$$

2. *For every $X \subseteq \omega^\omega$, $\varphi(X)$ is preserved under direct limits of finite support iterations of ccc forcing notions.*

Moreover, suppose that there exists a ccc generic extension V_1 of V_0 and $A \in V_1$ such that $V_1 \models \varphi(A)$. Then there is a ccc poset $\mathsf{P} \in V_1$ such that whenever G is a

$\mathbb{P}$*-generic filter over* V_1,

$$V_1[G] \models MA(Proj) \wedge \neg CH \wedge \varphi(A).$$

This theorem essentially says that $ZFC + MA(Proj) + \neg CH$ is consistent with the existence of almost any kind of uncountable structure that $MA + \neg CH$ forbids. Then, in the following subsections, we apply Theorem 2.3.18 to show the consistency, modulo a weakly-compact cardinal, of "There exists a Suslin tree", "There exists a non-strong gap in ω^ω" and of "There exists an entangled set of reals" with $ZFC + MA(Proj) + \neg CH$, in spite of there being no such structures under $MA + \neg CH$. Thus we improve a result of H. Judah and S. Shelah in [Ju-Sh1], where they show that $ZFC + MA(\utilde{\Sigma}^1_1) + \neg CH +$ "There exists a Suslin tree" is consistent. Finally, in the last subsection of the Section 2.3, we show that in some cases we do not need to collapse a weakly-compact cardinal in order to obtain the consistency with $MA(Proj) + \neg CH$ of statements like "There exists a cardinal κ less than the continuum such that $2^{\aleph_0} < 2^\kappa$" or "No set of reals is a Q set", which are false under $MA + \neg CH$. Note that, since Martin's axiom for σ-centered posets (for short, $MA(\sigma$-centered$)$) implies that for every $\kappa < 2^{\aleph_0}$, $2^\kappa = 2^{\aleph_0}$, we obtain a model of $MA(Proj) + \neg CH$ where $MA(\sigma$-centered$)$ does not hold. Therefore, $MA(Proj)$ does not imply $MA(\sigma$-centered$)$.

Generic absoluteness for projective ccc posets

In the third chapter, we study the absoluteness properties of projective formulas between a model and its generic extensions.

Definition *Let M, N be transitive models of ZF such that $M \subseteq N$. A* formula *$\varphi(v_0, ..., v_n)$ of the language of Set Theory is* absolute for *M* and *N iff for all $x_0, ..., x_n \in M$,*

$$M \models \varphi(x_0, ..., x_n) \text{ iff } N \models \varphi(x_0, ..., x_n).$$

A model M of ZF is $\utilde{\Sigma}^1_n$-absolute *$(n \geq 1)$ iff every Σ^1_n formula is absolute between M and every model N of ZF such that $M \subseteq N$. M is* projective absolute *iff M is $\utilde{\Sigma}^1_n$-absolute for all $n \geq 1$.*

It is a well-known fact that every transitive model of ZF is $\utilde{\Sigma}^1_1$-absolute. Shoenfield's Absoluteness Theorem, [Sho] (see [J2], Theorem 98) states that every transitive model of $ZF + DC$ containing all countable ordinals is $\utilde{\Sigma}^1_2$-absolute. This is the best result we can prove in ZFC: it is easy to find two transitive models of ZFC with all countable ordinals, $M \subseteq N$, such that $\utilde{\Sigma}^1_3$-absoluteness for M, N fails. For instance, if $M \models$ "$V = L$", then "There is a non-constructible real" is a Σ^1_3 sentence true in all generic extensions of M that add a real but it is false in M.

However, if we only take into consideration generic extensions of a transitive model of ZFC, instead of all extensions, the situation changes significantly.

Definition *Let M be a model of ZF and let Γ be a class of posets. M is* $\utilde{\Sigma}^1_n$-absolute for Γ *$(n \geq 1)$ iff every Σ^1_n formula is absolute between M and every generic extension* *of M by a poset* $\mathsf{P} \in \Gamma$*. M is* projective absolute for Γ *iff M is $\underset{\sim}{\Sigma}^1_n$-absolute for Γ for all $n \geq 1$. M is* $\underset{\sim}{\Sigma}^1_n$-absolute for P *(*projective absolute for P*) iff M is $\underset{\sim}{\Sigma}^1_n$-absolute (projective absolute) for* $\{\mathsf{P}\}$*.*

We are interested in the generic absoluteness and projective generic absoluteness properties because they are a simple way to turn consistent statements into true statements. Roughly speaking, given some statement, we force with an appropriate forcing notion showing that the statement is true in the generic extension, thereby concluding, by absoluteness, that it is true in the ground model. Since most of the regularity properties of projective sets can be expressed by means of projective sentences which may be forced by an appropriate forcing notion, projective generic absoluteness properties provide us with a new way to prove some of these regularity properties. For instance, H. Judah uses this way of reasoning in [Ju] to give a new proof of $\underset{\sim}{\Sigma}^1_1(L)$ and $\underset{\sim}{\Sigma}^1_1(B)$. Moreover, from [Ju-Sh2], [M-So] and [B1] we have that $\underset{\sim}{\Delta}^1_2(L)$, $\underset{\sim}{\Delta}^1_2(B)$, $\underset{\sim}{\Sigma}^1_2(L)$ and $\underset{\sim}{\Sigma}^1_2(B)$ are equivalent, respectively, to $\underset{\sim}{\Sigma}^1_3$-absoluteness for Random, Cohen, Amoeba and Amoeba for Category posets. From [Ju] and [B1], we have that $\underset{\sim}{\Sigma}^1_3$-absoluteness for Amoeba plus $\underset{\sim}{\Sigma}^1_4$-absoluteness for Random implies $\underset{\sim}{\Delta}^1_3(L)$ and that $\underset{\sim}{\Sigma}^1_3$-absoluteness for Amoeba for category plus $\underset{\sim}{\Sigma}^1_4$-absoluteness for Cohen implies $\underset{\sim}{\Delta}^1_3(B)$. Finally, from [Ju] and [Br], we have that $\underset{\sim}{\Sigma}^1_4$-absoluteness for Amoeba implies $\underset{\sim}{\Sigma}^1_3(L)$ and from [Ju] and [Br-Ju-Sh] we have that $\underset{\sim}{\Sigma}^1_4$-absoluteness for Hechler forcing implies $\underset{\sim}{\Sigma}^1_3(B)$. Thus, projective generic absoluteness properties for projective posets imply regularity properties for projective sets of low complexity.

Note that Shoenfield's Absoluteness Theorem implies that every transitive model of ZFC containing all countable ordinals is $\underset{\sim}{\Sigma}^1_2$-absolute for all forcing notions. And, as we remarked above, if $M \models$ "$V = L$", then M is not $\underset{\sim}{\Sigma}^1_3$-absolute for any poset that adds reals, as for instance, the Cohen poset. Therefore, in general, transitive models of ZFC are not $\underset{\sim}{\Sigma}^1_3$-absolute for the Cohen poset.

However, there is a close relationship between Martin's axiom, and in general Forcing axioms, and generic absoluteness of formulas of the language of Set Theory for certain classes of posets. Recall the Levy hierarchy of formulas of language of Set Theory:

Definition *A formula* is Σ_0 *(also Π_0) iff all its quantifiers are bounded; i.e., they all are of the form $\exists x \in y$* or $\forall x \in y$*. A formula* is Σ_{n+1} *iff it is of the form $\exists x_0 ... x_n \psi$ where ψ is Π_n and it* is Π_{n+1} *iff it is of the form $\forall x_0 ... x_n \psi$ where ψ is Σ_n. A class C (a property P, a relation R) is a* Σ_n-class (Σ_n-property, a Σ_n-relation) *iff $x \in C$ ($P(x)$, $R(x,y)$) can be written as a Σ_n-formula. A function F is a* Σ_n-function *iff the relation $y = F(x)$ is a Σ_n-relation. Similarly for Π_n. A property P is a* Δ_n-property *iff it is both Σ_n and Π_n (similarly for relations and functions)*

If A is a set, then we say that φ is a $\Sigma_1(A)$ formula iff it is a Σ_1 formula and all its parameters belongs to A. In [B3], it was proved that MA is equivalent to the absoluteness of $\Sigma_1(\mathcal{P}(\kappa))$ formulas with $\kappa < 2^{\aleph_0}$, for all ccc posets. So, Martin's axiom is a sort of generic absoluteness axiom between a model and its ccc generic extensions for a class of simple formulas. In [B3] it is also shown that all models of $ZFC + MA_{\omega_1}$ are $\underset{\sim}{\Sigma}^1_3$-absolute for the class of ccc posets. But MA_{ω_1} is stronger than

$\underset{\sim}{\Sigma}^1_3$-absoluteness for ccc posets: if there exists a weakly-compact cardinal, then one can force to obtain a transitive model of ZFC which is projective absolute for all ccc posets and where MA_{ω_1} is false.

In order to have $\underset{\sim}{\Sigma}^1_4$-absoluteness we need an inaccessible cardinal. More precisely, $\underset{\sim}{\Sigma}^1_3$-absoluteness for Random forcing plus $\underset{\sim}{\Sigma}^1_4$-absoluteness for Cohen forcing implies that ω_1 is an inaccessible cardinal in L ([B1], see also [B-Ju]). So, in order to obtain a transitive model of ZFC $\underset{\sim}{\Sigma}^1_4$-absolute for ccc posets we need a large cardinal hypothesis. In fact, the existence of an inaccessible cardinal is enough to obtain transitive models of ZFC projective absolute, not only $\underset{\sim}{\Sigma}^1_4$-absolute, for all $\underset{\sim}{\Sigma}^1_1$ ccc posets ([B1], see also [B-Ju]).

In the third chapter of this work, we extend all these results. We begin in Section 3.1 by studying one of the main notions of this chapter, namely, the *Solovay models over a model* V. Essentially, a Solovay model over a transitive model V of ZFC is the class of all sets constructible from real numbers, $L(\mathsf{R})$, of a generic extension of V obtained by Levy-collapsing an inaccessible cardinal in V. Solovay models are, of course, the models discovered by Solovay [So] in which every set of reals is Lebesgue measurable and has the property of Baire. We shall prove:

Lemma 3.1.6 *Suppose that $L(\mathsf{R})^M$ and $L(\mathsf{R})^N$ are Solovay models over V such that $\mathsf{R}^M \subseteq \mathsf{R}^N$ and $\omega_1^M = \omega_1^N$. Then there is an elementary embedding $j : L(\mathsf{R})^M \to L(\mathsf{R})^N$ which is the identity on the reals and the ordinals.*

So, all formulas with ordinals and reals as parameters, and hence all projective formulas, are absolute between two Solovay models over V with the same first uncountable cardinal. Thus, whenever the property of $L(\mathsf{R})$ of being a Solovay model is preserved under forcing notions that do not collapse ω_1, a strong form of generic absoluteness occurs. We call it $L(\mathsf{R})$*-absoluteness.*

Then, also in Section 3.1, we give some consequences of $L(\mathsf{R})$-absoluteness and of projective absoluteness. We first show that absoluteness for Borel ccc (Suslin ccc) implies two-step projective absoluteness, a stronger form of generic projective absoluteness, for Borel ccc (Suslin ccc) posets. Then, we prove a weak version of this fact for more complex projective ccc posets and for ccc posets in $L(\mathsf{R})$. Then, to motivate the interest of generic absoluteness properties for classes of definable and ccc posets, we prove that a form of generic absoluteness for ccc posets in $L(\mathsf{R})$ implies that every projective set of reals is Lebesgue measurable and has the property of Baire. We also show, using an argument of H. Woodin, that projective absoluteness for Borel ccc posets implies that there are no uncountable projective well-orderings of reals and, hence, that ω_1 is an inaccessible cardinal in L.

The remainder of the Chapter 3 is devoted to study the consistency strength of generic absoluteness properties for several classes of forcing notions. We begin, in Section 3.2, by extending the results of [B1] and [B-Ju] to $\underset{\sim}{\Sigma}^1_3$ ccc posets. We prove the following:

Theorem 3.2.1 *Suppose $L(\mathsf{R})^M$ is a Solovay model over V and P is a $\underset{\sim}{\Sigma}^1_3$ and ccc poset in M. Then the $L(\mathsf{R})$ of any P-generic extension of M is also a Solovay model over V.*

Note that (by Lemma 3.1.6) this theorem implies more than $L(\mathsf{R})$-absoluteness for $\underset{\sim}{\Sigma}^1_3$ ccc posets. It implies $L(\mathsf{R})$-two-step absoluteness, namely,

Definition *Let V be a model of ZFC. V is* $L(\mathsf{R})$-two-step absolute for P and $\dot{\mathsf{Q}}$ *iff for every P-generic filter G over V and every $\dot{\mathsf{Q}}[G]$-generic filter H over $V[G]$, there is an elementary embedding*

$$j : L(\mathsf{R})^{V[G]} \to L(\mathsf{R})^{V[G][H]}.$$

that fixes all ordinals and reals. Let Γ be a class of posets, V is $L(\mathsf{R})$-two-step absolute for Γ *iff for all $\mathsf{P} \in \Gamma$ and every P-name for a poset $\dot{\mathsf{Q}}$ such that $\Vdash_{\mathsf{P}}$ "$\dot{\mathsf{Q}} \in \Gamma$", V is $L(\mathsf{R})$-two-step absolute for P and $\dot{\mathsf{Q}}$.*

As a corollary, from [B1] (see also [B-Ju]) and Theorem 3.2.1, we have:

Corollary 3.2.8 *The following are equiconsistent (modulo ZFC):*

1. *There exists an inaccessible cardinal.*
2. *$L(\mathsf{R})$-two-step absoluteness for $\underset{\sim}{\Sigma}^1_3$ ccc forcing notions.*
3. *$\underset{\sim}{\Sigma}^1_4$-absoluteness under Cohen and Random forcing.*

In the next section, Section 3.3, we deal with generic absoluteness for all projective and ccc sets. We prove the following theorem:

Theorem 3.3.1 *The following are equiconsistent (modulo ZFC)*

1. *There exists a Σ_ω-Mahlo cardinal.*
2. *$L(\mathsf{R})$-two-step absoluteness for projective and ccc posets.*
3. *$\underset{\sim}{\Sigma}^1_4$-absoluteness for projective and ccc posets.*

We begin this section by defining the Δ_n-Mahlo, Σ_n-Mahlo, Π_n-Mahlo and Σ_ω-Mahlo cardinals and studying their reflection properties. They are definable versions of Mahlo cardinals: κ is a Σ_n-Mahlo cardinal iff κ is an inaccessible cardinal such that the set of inaccessible cardinals below κ has non empty intersection with all clubs of κ which are definable by means of a Σ_n formula in V_κ (similarly for Δ_n-Mahlo and Π_n-Mahlo). κ is Σ_ω-Mahlo cardinal iff κ is Σ_n-Mahlo for all $n \in \omega$. Note that all these cardinals are below a Mahlo cardinal. The following subsection is devoted to prove (1) implies (2) of Theorem 3.3.1. We do it by showing that every projective and ccc forcing extension of a Σ_ω-Solovay model over V, the $L(\mathsf{R})$ of a generic extension of V obtained by Levy-collapsing a Σ_ω-Mahlo cardinal in V, is also a Σ_ω-Mahlo Solovay model over V. In the next subsection, we prove (3) implies (1) of the theorem. More precisely, we prove that $\underset{\sim}{\Sigma}^1_4$-absoluteness for projective and ccc forcing notions implies that ω_1 is a Σ_ω-Mahlo cardinal in L.

In Section 3.4, we study the projective generic absoluteness by Levy-collapsing a Mahlo cardinal. First we show that every extension of a Mahlo Solovay model over V by means of a ccc subposet of projective poset is a Mahlo Solovay model over V. As consequence, we get that every σ-linked extension of a Mahlo Solovay model over V is a Mahlo Solovay model over V. J. Brendle, H. Judah and S. Shelah ([Br-Ju-Sh]) have shown that $\underset{\sim}{\Sigma}^1_4$-absoluteness for Hechler forcing, a σ-centered forcing notion, implies that ω_1 is an inaccessible cardinal in L. On the other hand, A. R. D. Mathias (see [B-F]) using an argument of R. Jensen showed that if ω_1 is an inaccessible cardinal in L and $\underset{\sim}{\Sigma}^1_4$-absoluteness for σ-centered posets holds, then ω_1 is a Mahlo cardinal in L. So, we conclude:

Theorem 3.4.18 *The following are equiconsistent (modulo ZFC)*

1. *There exists a Mahlo cardinal.*
2. $L(\mathsf{R})$*-two-step absoluteness for ccc subposets of projective posets.*
3. $L(\mathsf{R})$*-two step absoluteness for σ-linked posets.*
4. $L(\mathsf{R})$*-two-step absoluteness for σ-centered posets.*
5. $\underset{\sim}{\Sigma}^1_4$*-absoluteness for σ-centered posets.*
6. $\underset{\sim}{\Sigma}^1_4$*-absoluteness for σ-centered subposets of Borel posets.*

In the last section, Section 3.5, we remark that Theorem 2.3.2 of K. Kunen, can be rephrased as saying that every ccc generic extension of a weakly-compact Solovay model over V is also weakly-compact Solovay model over V, where weakly-compact Solovay model is Solovay model obtained by Levy-collapsing a weakly-compact cardinal. Then, using a version of an argument of L. Harrington and S. Shelah ([H-Sh]), we show

Theorem 3.5.5 *The following are equiconsistent (modulo ZFC)*

1. *There exists a weakly-compact cardinal.*
2. $L(\mathsf{R})$*-two step absoluteness for ccc forcing notions.*
3. $L(\mathsf{R})$*-two step absoluteness for Knaster forcing notions.*
4. *One step $\underset{\sim}{\Sigma}^1_4$-absoluteness for Knaster forcing notions.*

The Levy-collapsing forcing notion

We use the Levy-collapsing forcing notion in almost every section of this work. Here, we state the definitions and the main facts about this forcing notion. We will use them without further comment.

Definition *For every ordinal α, $Coll\,(\omega, \alpha)$ is the following partial order:*

- $p \in Coll\,(\omega, \alpha)$ *iff* p *is a function,* $\mathrm{dom}\,(p) \subseteq \omega$ *is finite and* $\mathrm{rec}\,(p) \subseteq \alpha$.
- $p \leq q$ *iff* $q \subseteq p$

The Levy-collapse for α, $Coll\,(\omega, < \alpha)$, *is the product with finite support of* $Coll\,(\omega, \beta)$, *all* $\beta < \alpha$. *i.e.,*

- $p \in Coll\,(\omega, < \alpha)$ *iff* $p \in \prod_{\beta<\alpha} Coll\,(\omega, \beta)$ *and* $p\,(\beta) = \emptyset$ *for all but finitely many* β.

For $p \in Coll\,(\omega, < \alpha)$, $\mathrm{supp}\,(p) = \{\beta < \alpha : p\,(\beta) \neq \emptyset\}$.

- $p \leq q$ *iff* $q\,(\beta) \subseteq p\,(\beta)$, *all* $\beta < \alpha$.

Therefore, for every $p, q \in Coll\,(\omega, \alpha)$, $p \perp q$ iff p, q are incompatible functions. i.e., there is $\beta \in \mathrm{dom}\,(p) \cap \mathrm{dom}\,(q)$ such that $p\,(\beta) \neq q\,(\beta)$. And for every $p, q \in Coll\,(\omega, < \alpha)$, $p \perp q$ iff there is $\beta \in \mathrm{dom}\,(p) \cap \mathrm{dom}\,(q)$ such that $p\,(\beta), q\,(\beta)$ are incompatible functions.

Note that our notation departs from the usual one. We use $Coll\,(\omega, < \alpha)$ for the Levy-collapsing poset (and not $Lv\,(\alpha)$ as [Ku]) instead of the Levy-collapsing algebra, as in [J2].

Thus forcing with $Coll\,(\omega, \alpha)$ collapses α onto ω, and forcing with $Coll\,(\omega, < \alpha)$ collapses all ordinals less than α onto ω.

Remark *We can regard the conditions* $p \in Coll\,(\omega < \alpha)$ *as functions on subsets of* $\alpha \times \omega$ *such that*

1. $\mathrm{dom}\,(p)$ *is finite.*
2. $p\,(\beta, n) < \beta$, *all* $\langle \beta, n \rangle \in \mathrm{dom}\,(p)$.

Then $p \leq q$ *iff* $q \subseteq p$.

The following basic facts about the Levy-collapse can be found in [K], III.10, [Ku], VII.8, or [J2], IV.25, so we state them here without proof:

Fact *Let* κ *be a regular uncountable cardinal. Then,*

1. $Coll\,(\omega, < \kappa)$ *is a* κ*-cc partially ordered set.*

2. $\Vdash_{Coll(\omega,<\kappa)}$ "$\check{\kappa} = \omega_1$".

Definition *A partially ordered set* P *is* almost-homogeneous *iff for all* $p, q \in \mathsf{P}$ *there exists an automorphism* h *of* P *such that* $h(p)$ *and* q *are compatible.*

Fact

1. *If a partially ordered set* P *is almost-homogeneous, then for every formula* $\varphi(x_1, .., x_n)$ *and* $a_1, ..., a_n \in V$*, either* $\Vdash_{\mathsf{P}} \varphi(\check{a}_1, ..., \check{a}_n)$ *or else* $\Vdash_{\mathsf{P}} \neg\varphi(\check{a}_1, ..., \check{a}_n)$.
2. $Coll(\omega, <\alpha)$ *is an almost-homogeneous partially ordered set.*

Robert Solovay [So] showed the following property of the Levy-collapse forcing notion which plays a crucial role in his proof of the consistency of $ZFC+$"All projective sets of reals are Lebesgue measurable and have the property of Baire":

Lemma (Factor Lemma) *Suppose that* κ *is an uncountable regular cardinal and* G *is a* $Coll(\omega, <\kappa)$*-generic filter over* V*. Then for every countable set of ordinals* x *in* $V[G]$ *there is a* $Coll(\omega, <\kappa)$*-generic filter* H *over* $V[x]$ *such that* $V[x][H] = V[G]$.

We also use that the Levy-collapse is a very simple definable forcing notion:

Fact *The function* $\alpha \mapsto Coll(\omega, <\alpha)$ *is* Δ_1*. So,* $Coll(\omega, <\alpha)$ *is a partially ordered set which is* Δ_1 *definable with* α *as parameter. Moreover, if* κ *is a regular uncountable cardinal, then* $Coll(\omega, <\kappa)$ *is* Δ_1*-definable over* V_κ *without parameters. That is for every* p, q*, "*$p \in Coll(\omega, <\kappa)$*" is a* Δ_1 *property and "*$p \leq q$*" and "*$p \perp q$*" are* Δ_1 *relations.*

Proof. Note that $x = Coll(\omega, <\alpha)$ iff α is an ordinal and there is a function f such that $\mathrm{dom}(f) = \alpha$ and $(\forall \xi \in \alpha)(\xi + 1 \in \alpha \to f(\xi+1) = Coll(\omega, \xi) \times f(\xi))$ and $(\forall \xi \in \alpha)(\xi \text{ is limit} \to f(\xi) = \bigcup \mathrm{rec}(f \restriction \xi))$. Since for every ξ, $x = Coll(\omega, \xi)$ is Δ_1, the above is a Σ_1 definition of the function $\alpha \mapsto Coll(\omega, <\alpha)$. Since its domain is the Δ_0 class of all ordinals, it is a Δ_1 function (see [J2], Lemma 14.2).

Suppose now that κ is a regular uncountable cardinal. Then, since $\kappa = V_\kappa \cap ON$, for every p, $p \in Coll(\omega, <\alpha)$ iff V_κ satisfies that p is a function and $\mathrm{dom}(p)$ is a finite subset of ordered pairs and for every $\langle \beta, n \rangle \in \mathrm{dom}(p)$, β is an ordinal and $n \in \omega$ and $p(\beta, n) \in \beta$. Since "x is finite" is a Δ_1 property of x and all other notions involved in the definition of $Coll(\omega, <\kappa)$ are Δ_0 on p, "$p \in Coll(\omega, <\kappa)$" is a property of p which is Δ_1-definable without parameters over V_κ. Since the ordering of $Coll(\omega, <\kappa)$ is the inverse inclusion, it is clear that "$p \leq q$" is a relation which is Δ_1-definable without parameters over V_κ. Finally, since for all p, q, V_κ satisfies that $p \perp q$ iff

$$V_\kappa \models p, q \in Coll(\omega, <\kappa) \wedge (\exists \langle \beta, n \rangle \in \mathrm{dom}(p) \cap \mathrm{dom}(q))(p(\beta, n) \neq q(\beta, n)),$$

"$p \perp q$" is also a relation which is Δ_1-definable without parameters over V_κ. ■

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Chapter 2
PROJECTIVE FORCING

2.1 Projective posets

2.1.1 Definitions and basic facts

Definition 2.1.1 *A* $\underset{\sim}{\Sigma}^1_n$ poset $(n \geq 1)$ *is a triple* $\langle P, \leq_P, \perp_P \rangle$*, where* $\leq_P$ *is a* $\underset{\sim}{\Sigma}^1_n$ *subset of* $\omega^\omega \times \omega^\omega$*,* $P = \text{field}(\leq_P)$*,* $\langle P, \leq_P \rangle$ *is a partial order, and* $\perp_P$ *is a* $\underset{\sim}{\Sigma}^1_n$ *subset of* $\omega^\omega \times \omega^\omega$ *contained in* $P \times P$ *such that for every* $x, y \in P$*,* $x \perp_P y$ *iff for no* $z \in P$*, both* $z \leq_P x$ *and* $z \leq_P y$*; i.e., iff* x, y *are incompatible. Similarly, we define* $\underset{\sim}{\Pi}^1_n$ *posets by substituting* $\underset{\sim}{\Sigma}^1_n$ *for* $\underset{\sim}{\Pi}^1_n$ *in the above definition. A* $\underset{\sim}{\Delta}^1_n$ *poset is a poset that is both* $\underset{\sim}{\Sigma}^1_n$ *and* $\underset{\sim}{\Pi}^1_n$*. Finally,* $\langle P, \leq_P, \perp_P \rangle$ *is a projective poset iff there exists* $n \geq 1$ *such that* $\langle P, \leq_P, \perp_P \rangle$ *is a* $\underset{\sim}{\Sigma}^1_n$ *poset.*

Fact 2.1.2 *Let* $\langle P, \leq_P, \perp_P \rangle$ *be a* $\underset{\sim}{\Sigma}^1_n$ *poset. Then,*

1. P *is a* $\underset{\sim}{\Sigma}^1_n$ *subset of* ω^ω*.*

2. $\perp_P$ *is a* $\underset{\sim}{\Delta}^1_n$ *subset of* $\omega^\omega \times \omega^\omega$*.*

Proof. (1) $x \in P$ iff $\exists y\, (x \leq_P y \vee y \leq_P x)$.

(2) $x \perp_P y$ iff $\forall z\, (z \leq_P x \rightarrow \neg z \leq_P y)$. So $\perp_P$ is both $\underset{\sim}{\Sigma}^1_n$ and $\underset{\sim}{\Pi}^1_n$. Hence, $\perp_P$ is a $\underset{\sim}{\Delta}^1_n$ subset of $\omega^\omega \times \omega^\omega$. ■

We shall refer to $\underset{\sim}{\Delta}^1_1$ posets as *Borel posets*, to $\underset{\sim}{\Sigma}^1_1$ posets as *Suslin posets* and to $\underset{\sim}{\Pi}^1_1$ posets as *co-Suslin posets*.

Following the standard notation, we will write P for $\langle P, \leq_P, \perp_P \rangle$ and, usually, we also write P instead of $P = \text{field}\,(\leq_P)$. If P is a projective poset, there are projective formulas $\varphi(x, y)$ and $\psi(x, y)$ such that $\leq_P = \{\langle x, y \rangle : \varphi(x, y)\}$ and $\perp_P = \{\langle x, y \rangle : \psi(x, y)\}$ (see [J2], 40.6).

Suppose M is a transitive class, A is a projective set, and M contains the parameters of the projective formula that defines A. Then A^M denotes the set in M defined by the relativization to M of this formula.

Definition 2.1.3 *Let* M *be a transitive model of* ZF*.* M *is a* $\underset{\sim}{\Sigma}^1_n$-correct ($\underset{\sim}{\Pi}^1_n$-correct) model *iff for every* $\underset{\sim}{\Sigma}^1_n$ *(respectively,* $\underset{\sim}{\Pi}^1_n$*) set* A *such that* M *contains the parameter of the formula that defines* A*,* A *is absolute for* M*; i.e.,* $A^M = A \cap M$*.*

So, if P is a projective poset and M is a model of ZF that contains all the parameters of the formulas that define P, then we say that P *is absolute for* M iff $\leq_P^M = \leq_P \cap M$ and $\perp_P^M = \perp_P \cap M$.

Remark 2.1.4 *Suppose that M a transitive model of ZF. Then, since for every $n \geq 1$, the $\undertilde{\Pi}^1_n$ sets are the complements of $\undertilde{\Sigma}^1_n$ sets, M is $\undertilde{\Sigma}^1_n$-correct iff M is $\undertilde{\Pi}^1_n$-correct.*

Fact 2.1.5
1. *Every transitive model of ZF is $\undertilde{\Sigma}^1_1$-correct.*
2. *Every transitive model of $ZF + DC$ containing all countable ordinals is $\undertilde{\Sigma}^1_2$-correct.*

Proof. (1) Since "to be a well-founded relation" is a Δ_1 property and, so, absolute for these models. (2) By Shoenfield's Absoluteness Theorem. ([J2], Theorem 98). ■

Corollary 2.1.6
1. *Let P be a Borel, Suslin or co-Suslin poset. Then P is absolute for every transitive model of ZF containing the parameters of the formulas that define P.*
2. *Let P be a $\undertilde{\Delta}^1_2$, $\undertilde{\Sigma}^1_2$ or $\undertilde{\Pi}^1_2$ poset. Then P is absolute for every transitive model of $ZF + DC$ containing all the countable ordinals and the parameters of the formulas that define P.* ■

Remark 2.1.7 *If P is a projective poset, then we may assume, by merging all the parameters in the projective formulas that define P, that there is only one parameter in its definition.*

If M is a transitive model of ZF, then "$\mathsf{P} \in M$" should be interpreted as "M contains the parameter of the definition of P".

Fact 2.1.8 *Every projective poset can be coded by a real. Moreover,*

1. *The set of all codes of $\undertilde{\Sigma}^1_n$ posets is a $\undertilde{\Pi}^1_{n+1}$ subset of ω^ω.*
2. *The set of all codes of $\undertilde{\Pi}^1_n$ posets is a $\undertilde{\Pi}^1_{n+2}$ subset of ω^ω.*

Proof. Let $\mathcal{U}$ be a universal $\undertilde{\Sigma}^1_n$ set in $\omega^\omega \times \omega^\omega \times \omega^\omega$ as in [J2], 39.4.; i.e., $\mathcal{U}$ is a Σ^1_n and for every $\undertilde{\Sigma}^1_n$ set X in $\omega^\omega \times \omega^\omega$, there exists $a \in \omega^\omega$ such that $X = \{\langle x, y\rangle : \langle x, y, a\rangle \in \mathcal{U}\}$.

Let $h : \omega^\omega \to \omega^\omega \times \omega^\omega$ be the one-to-one and onto function given by $h(a) = \langle a_0, a_1\rangle$ iff for all $n \in \omega$, $a_i(n) = a(2n + i)$, where $i \in \{0, 1\}$.

Let P be a $\undertilde{\Sigma}^1_n$ poset. Then there is a $a \in \omega^\omega$ such that

(i) $\leq_P = \{\langle x, y\rangle : \langle x, y, a_0\rangle \in \mathcal{U}\}$

(ii) $\perp_P = \{\langle x, y\rangle : \langle x, y, a_1\rangle \in \mathcal{U}\}$

We say that *a codes* P. Similarly for $\undertilde{\Pi}^1_n$ posets using a universal $\undertilde{\Pi}^1_n$ set.

On the other hand, to every $a \in \omega^\omega$ we can associate two relations $\leq_a$, $\perp_a \subseteq \omega^\omega \times \omega^\omega$ such that:

(i) $\leq_a = \{\langle x, y\rangle : \langle x, y, a_0\rangle \in \mathcal{U}\}$

(ii) $\perp_a = \{\langle x, y\rangle : \langle x, y, a_1\rangle \in \mathcal{U}\}$

where $h(a) = \langle a_0, a_1\rangle$.

We say that $a \in \omega^\omega$ *codes a* $\underset{\sim}{\Sigma}^1_n$ *poset* if

(i) $\langle \text{field}(\leq_a), \leq_a\rangle$ is a partial ordering

(ii) $(\forall x, y \in \text{field}(\leq_a))(x \perp_a y \leftrightarrow \neg(\exists z \in \text{field}(\leq_a))(z \leq_a x \wedge z \leq_a y))$

Now, since $x \in \text{field}(\leq_a)$ iff $\exists y(x \leq_a y \vee y \leq_a x)$, $\text{field}(\leq_a)$ is a $\Sigma^1_n(a)$ subset of ω^ω. Since $\langle \text{field}(\leq_a), \leq_a\rangle$ is a partial ordering iff

- $(\forall x \in \text{field}(\leq_a))(x \leq_a x)$ $(\Pi^1_{n+1}(a))$
- $(\forall xy \in \text{field}(\leq_a))(x \leq_a y \wedge y \leq_a x \rightarrow x = y)$ $(\Pi^1_n(a))$
- $(\forall xyz \in \text{field}(\leq_a))(x \leq_a y \wedge y \leq_a z \rightarrow x \leq_a z)$ $(\Pi^1_{n+1}(a))$

(i) is $\Pi^1_{n+1}(a)$. Also, since $\text{field}(\leq_a)$ is $\Sigma^1_n(a)$, (ii) is $\Pi^1_{n+1}(a)$. It follows that "a codes a $\underset{\sim}{\Sigma}^1_n$ poset" is a Π^1_{n+1} condition on a.

In the same way, we can define $a \in \omega^\omega$ *codes a* $\underset{\sim}{\Pi}^1_n$ *poset* and show that this is a Π^1_{n+2} condition on a. Indeed, since $\text{field}(\leq_a)$ is $\Sigma^1_{n+1}(a)$, (i) is $\Pi^1_{n+1}(a)$ and (ii) is $\Pi^1_{n+2}(a)$. ∎

Corollary 2.1.9 *Let $N \subseteq M$ be a transitive models of ZF. Let $a \in \omega^\omega \cap N$. Then*

1. *If N, M are $\underset{\sim}{\Pi}^1_{n+1}$-correct, then*

$$N \models \text{“}a \text{ codes a } \underset{\sim}{\Sigma}^1_n \text{ poset”} \text{ iff } M \models \text{“}a \text{ codes a } \underset{\sim}{\Sigma}^1_n \text{ poset”}$$

2. *If N, M are $\underset{\sim}{\Pi}^1_{n+2}$-correct, then*

$$N \models \text{“}a \text{ codes a } \underset{\sim}{\Pi}^1_n \text{ poset”} \text{ iff } M \models \text{“}a \text{ codes a } \underset{\sim}{\Pi}^1_n \text{ poset”}$$

Moreover, if P^N_a and P^M_a denote the posets coded by a in N and M respectively, then $\mathsf{P}^N_a = \mathsf{P}^M_a \cap N$.

Proof. (1) and (2) follow from the fact above and from absoluteness for Π^1_{n+1} and Π^1_{n+2} formulas of, respectively, $\underset{\sim}{\Pi}^1_{n+1}$-correct and $\underset{\sim}{\Pi}^1_{n+2}$-correct transitive models of ZF. The second part is clear since $\mathcal{U}^N = \mathcal{U}^M \cap N$. ∎

Corollary 2.1.10 *Let $N \subseteq M$ be a transitive models of $ZF + DC$ containing all countable ordinals. Let $a \in \omega^\omega \cap N$. Then,*

$$N \models \text{“}a \text{ codes a Suslin poset”} \text{ iff } M \models \text{“}a \text{ codes a Suslin poset”}$$

Moreover, if P^N_a and P^M_a denote the posets coded by a in N and M respectively, then $\mathsf{P}^N_a = \mathsf{P}^M_a \cap N$.

Proof. By Shoenfield's Absoluteness Theorem ([J2], Theorem 98) these models are $\underset{\sim}{\Pi}^1_2$-correct. ■

Definition 2.1.11 *Let* P, Q *be posets.* $i : \mathsf{P} \to \mathsf{Q}$ *is a* dense embedding *iff*

1. $(\forall p, p' \in \mathsf{P})\,(p \leq_P p' \to i(p) \leq_Q i(p'))$
2. $(\forall p, p' \in \mathsf{P})\,(p \perp_P p' \to i(p) \perp_Q i(p'))$
3. i"P *is dense in* Q*. i.e.,* $(\forall q \in \mathsf{Q})\,(\exists p \in \mathsf{P})\,(i(p) \leq_Q q)$

Fact 2.1.12 *Let* P*,* Q *be posets in a transitive model* M*. Suppose* $i : \mathsf{P} \to \mathsf{Q}$ *is a dense embedding lying in* M*. Then,*

1. *If* G *is* P*-generic over* M*, then* $H = \{q \in \mathsf{Q} : (\exists p \in G)\,(i\,(p) \leq_Q q)\}$ *is* Q*-generic over* M *and* $M[H] = M[G]$*.*
2. *If* H *is* Q*-generic over* M*, then* $G = i^{-1}(H)$ *is* P*-generic over* M *and* $M[G] = M[H]$*.*

Proof. See [J2], 17.4, or [Ku], VII.7.11. ■

Remark 2.1.13 *In view of this fact, if* P *and* Q *are equivalent forcing notions and* P *is projective, then we will also say that* Q *is projective (and with the same complexity as* P*). This is, of course, just a convenient abuse of language since* Q *need not be projective in the strict sense of the definition 2.1.1.*

2.1.2 Projective forcing with the countable chain condition

Definition 2.1.14 *A poset* P *satisfies the* countable chain condition *(or, for short, is* ccc*) iff every antichain of* P *(i.e., every set of pairwise incompatible elements) is at most countable.*

Fact 2.1.15 *If* P *is a projective and ccc poset, then every antichain of* P *can be coded by a real. Moreover, we can set up the coding so that,*

1. *If* P *is a* $\underset{\sim}{\Sigma}^1_n$ *poset, then "*x *codes a maximal antichain of* P*" is a* $\underset{\sim}{\Pi}^1_n$ *predicate.*
2. *If* P *is a* $\underset{\sim}{\Pi}^1_n$ *poset, then "*x *codes a maximal antichain of* P*" is a* $\underset{\sim}{\Pi}^1_{n+1}$ *predicate.*

Proof. Since P is a projective and ccc poset, every antichain A is a countable set of reals. So, we can write A as sequence $\langle p_n : n < \omega\rangle$. But any such a sequence of reals can be recursively coded by a real. e.g., let $J : \omega \times \omega \to \omega$ be the standard, one-to-one and onto, pairing function given by $J(n, m) = 2^n(2m + 1) - 1$ and define $a \in \omega^\omega$ by $a(J(n, m)) = p_n(m)$. So, $p_n = \{\langle i, j\rangle : a(J(n, i)) = j\}$ and a codes an antichain of P.

Now, *a codes a maximal antichain of* P iff

(i) $\forall nm\,(n \neq m \to \{\langle i,j\rangle : a(J(n,i)) = j\} \perp_P \{\langle i,j\rangle : a(J(m,i)) = j\})$

(ii) $\neg\exists y \forall n (y \perp_P \{\langle i,j\rangle : a(J(n,i)) = j\})$

Hence, if P is a $\underset{\sim}{\Sigma}^1_n$ poset, then, since $\perp_P$ is $\underset{\sim}{\Delta}^1_n$, (i) is $\underset{\sim}{\Delta}^1_n$ and (ii) is $\underset{\sim}{\Pi}^1_n$. If P is a $\underset{\sim}{\Pi}^1_n$ poset, then, since $\perp_P$ is $\underset{\sim}{\Pi}^1_n$, (i) is $\underset{\sim}{\Pi}^1_n$ and (ii) is $\underset{\sim}{\Pi}^1_{n+1}$. ■

Corollary 2.1.16 *Let $N \subseteq M$ be transitive models of ZF and let P be a projective and ccc poset with $\mathsf{P} \in N$. Suppose $a \in \omega^\omega \cap N$. Then,*

1. *If N, M are $\underset{\sim}{\Pi}^1_n$-correct and P is $\underset{\sim}{\Sigma}^1_n$, then*

$$N \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^N\text{”}$$
$$\text{iff}$$
$$M \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^M\text{”}.$$

2. *If N, M are $\underset{\sim}{\Pi}^1_{n+1}$-correct and P is $\underset{\sim}{\Pi}^1_n$, then*

$$N \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^N\text{”}$$
$$\text{iff}$$
$$M \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^M\text{”}.$$

Moreover, if A_N and A_M denote the maximal antichains of P^N and P^M coded by a in N and M respectively, then $A_N = A_M$.

Proof. (1) and (2) follows from the fact above and from absoluteness of Π^1_n and Π^1_{n+1} formulas for, respectively, $\underset{\sim}{\Pi}^1_n$-correct and $\underset{\sim}{\Pi}^1_{n+1}$-correct transitive models of ZF. The last sentence of the corollary is clear since the coding is recursive. ■

Corollary 2.1.17

1. *Let $N \subseteq M$ be a transitive models of ZF and let P be a Suslin and ccc poset with $\mathsf{P} \in N$. Suppose $a \in \omega^\omega \cap N$. Then,*

$$N \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^N\text{”}$$
$$\text{iff}$$
$$M \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^M\text{”}.$$

2. *Let $N \subseteq M$ be a transitive models of $ZF + DC$ containing all countable ordinals and let P be a $\underset{\sim}{\Sigma}^1_2$ and ccc poset with $\mathsf{P} \in N$. Suppose $a \in \omega^\omega \cap N$. Then,*

$$N \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^N\text{”}$$
$$\text{iff}$$
$$M \models \text{“}a \text{ codes a maximal antichain of } \mathsf{P}^M\text{”}.$$

Moreover, if A_N and A_M denote the maximal antichains of P^N and P^M coded by a in N and M respectively, then $A_N = A_M$.

Proof. These are particular cases of the previous corollary. ■

Definition 2.1.18 *Let* P *be a poset.* τ *is a* P-name for a real *iff* $\Vdash_{\mathsf{P}}$ *"τ is a real ".* τ *is a* simple P-name for a real *if*

1. *The elements of τ are of form $\langle p, \check{n}, \check{m} \rangle$, where $p \in \mathsf{P}$, and $\check{n}, \check{m}$ are the standard P-names for some $n, m \in \omega$.*
2. *For every $n \in \omega$, the set $\{p \in \mathsf{P} : (\exists m \in \omega)(\langle p, \check{n}, \check{m} \rangle \in \tau)\}$ is a maximal antichain of P.*
3. *For every $n, m_0, m_1 \in \omega$, if $\langle p, \check{n}, \check{m}_0 \rangle \in \tau$ and $\langle p, \check{n}, \check{m}_1 \rangle \in \tau$, then $m_0 = m_1$.*

Fact 2.1.19 *For every poset P, and for every P-name τ for a real, there is a simple P-name for a real σ such that $\Vdash_{\mathsf{P}}$ "$\tau = \sigma$".*

Proof. Fix P and τ and suppose $\Vdash_{\mathsf{P}}$ "τ is a real". For every $n \in \omega$, let A_n be a maximal antichain of P such that for every $p \in A_n$, $p \Vdash_{\mathsf{P}}$ "$\tau(\check{n}) = \check{m}$", for some $m \in \omega$. A_n exists since $\Vdash_{\mathsf{P}}$ "τ is a real" and hence for every $p \in \mathsf{P}$ there exists $q \leq_P p$ and $m \in \omega$ such that $q \Vdash_{\mathsf{P}}$ "$\tau(\check{n}) = \check{m}$". Now, let σ be the simple P-name defined by

$$\langle p, \check{n}, \check{m} \rangle \in \sigma \text{ iff } p \in A_n \wedge p \Vdash_{\mathsf{P}} \text{"}\tau(\check{n}) = \check{m}\text{"}. \quad \blacksquare$$

Definition 2.1.20 *Let* P, Q *be partial orderings. A* complete embedding *of* P *into* Q *is a function e from* P *into* Q *such that:*

1. *e is one-to-one.*
2. *e is order-preserving. i.e., for all $p, q \in \mathsf{P}$, if $p \leq_P q$, then $e(p) \leq_Q e(q)$.*
3. *e preserves maximal antichains. i.e., for every maximal antichain A of P, the set $\{e(p) : p \in A\}$ is a maximal antichain of Q.*

We write $\mathsf{P} \lessdot \mathsf{Q}$ when $\mathsf{P} \subseteq \mathsf{Q}$ and the identity function is a complete embedding of P into Q.

Fact 2.1.21 *Let P, Q be posets such that $\mathsf{P} \lessdot \mathsf{Q}$ and let τ be a simple P-name for a real. Then (if we identify the standard P-names for natural numbers with the corresponding standard Q-names) τ is a simple Q-name for a real. Moreover, if $G \subseteq \mathsf{Q}$ is a generic filter, then $G \cap \mathsf{P}$ is a generic filter for P and $\tau[G \cap \mathsf{P}] = \tau[G]$.*

Proof. Since the identity function is an order-preserving map of P into Q which preserves incompatibility, if $G \subseteq \mathsf{Q}$ is a filter, $G \cap \mathsf{P}$ is a filter. If G is a filter on Q, then G is generic over V iff $|G \cap A| = 1$ for every maximal antichain of Q in V. But, since every maximal antichain of P is a maximal antichain of Q, if A is a maximal antichain of P in V, $|(G \cap \mathsf{P}) \cap A| = 1$. It follows that $G \cap \mathsf{P}$ is generic for P.

Let τ be a simple P-name for a real. Since $\mathsf{P} \lessdot \mathsf{Q}$, if we replace the standard P-names for natural numbers for the corresponding standard Q-names, then τ is a simple Q-name for a real and for every $n \in \omega$, $\check{n}[G] = \check{n}[G \cap \mathsf{P}] = n$. So, $\tau[G] = \{\langle n, m \rangle : \langle p, \check{n}, \check{m} \rangle \in \tau \wedge p \in G\} = \tau[G \cap \mathsf{P}]$. $\blacksquare$

Fact 2.1.22 *Let P be a ccc partial ordering with $\mathsf{P} \subseteq \omega^\omega$. Then every simple P-name for a real can be coded by a real. Moreover, the coding can be arranged so that:*

1. *If P is a $\undertilde{\Sigma}^1_n$ poset, then the set of codes of simple P-names for a real is a $\undertilde{\Pi}^1_n$ subset of ω^ω.*

2. *If P is a $\undertilde{\Pi}^1_n$ poset, then the set of codes of simple P-names for a real is a $\undertilde{\Pi}^1_{n+1}$ subset of ω^ω.*

Proof. Let τ be a simple P-name for a real. So,

$$\tau = \bigcup \{\{\langle p, \check{n}, \check{m}_i\rangle : i \in \omega\} : n \in \omega\},$$

where for all $n \in \omega$, $A_n = \{p_i : (\exists m_i \in \omega)(\langle p_i, \check{n}, \check{m}_i\rangle \in \tau)\}$ is a maximal antichain of P. Let $z_n \in \omega^\omega$ be a code for A_n as in the Fact 2.1.15. For every $n \in \omega$, let $B_n = \{m_i : (\exists p_i \in \mathsf{P})(\langle p_i, \check{n}, \check{m}_i\rangle \in \tau)\}$ and let $y_n \in \omega^\omega$ be such that for every $i < \omega$, $y_n(i) = m_i$. So, y_n codes B_n. Let x_n be such that $x_n(m) = J(z_n(m), y_n(m))$. So, x_n codes both z_n and y_n. Finally, let x be a code for $\langle x_n : n < \omega\rangle$ as in the Fact 2.1.15. We say that x codes τ.

Thus, x *codes a simple* P*-name for a real* iff

(i) x codes $\langle x_n : n < \omega\rangle$. i.e., $\forall nm\,(x(J(n, m)) = x_n(m))$

(ii) $\forall n\,(x_n$ codes both z_n and $y_n)$. i.e., $\forall nm\,(x_n(m) = J(z_n(m), y_n(m)))$

(iii) $\forall n\,(z_n$ codes a maximal antichain of $\mathsf{P})$

(iv) $\forall n\,(x_n$ codes both z_n and $y_n \wedge$

$$\wedge \exists ij\,(\{\langle k, l\rangle : z_n(J(i, k)) = l\} = \{\langle k, l\rangle : z_n(J(j, k)) = l\} \rightarrow y_n(i) = y_n(j)))$$

Clearly, (i) ,(ii) and (iv) are arithmetical and (iii) is $\undertilde{\Pi}^1_n$ (if P is $\undertilde{\Sigma}^1_n$) or $\undertilde{\Pi}^1_{n+1}$ (if P is $\undertilde{\Pi}^1_n$). ■

Theorem 2.1.23 *Let P be projective ccc poset and let $\theta(x)$ be a Σ^1_k (Π^1_k) formula with $k \geq 2$. Then the relation*

$$R(p, \tau) \leftrightarrow p \in \mathsf{P} \wedge \tau \textit{ is a simple } \mathsf{P}\textit{-name for a real} \wedge p \Vdash_{\mathsf{P}} \theta(\tau)$$

is a projective relation. Moreover,

1. *If P is a $\undertilde{\Sigma}^1_n$ poset, then R is $\undertilde{\Sigma}^1_{n+k-1}$ $(\undertilde{\Pi}^1_{n+k-1})$.*

2. *If P is a $\undertilde{\Pi}^1_n$ poset, then R is $\undertilde{\Sigma}^1_{n+k}$ $(\undertilde{\Pi}^1_{n+k})$.*

Proof. We prove it only for Σ^1_k formulas, $k \geq 2$. The proof for Π^1_k formulas is analogous. We proceed by induction on k:

$\underline{k+1}$: Let $\theta(x)$ be a Σ^1_{k+1} formula, $k \geq 2$. So, $\theta(x)$ is of the form $\exists y \neg\psi(x,y)$, where ψ is Σ^1_k. Assume the fact holds for Σ^1_k formulas and suppose that $p \in \mathsf{P}$ and τ is a simple P-name for a real such that $p \Vdash_{\mathsf{P}} \theta(\tau)$. By Maximal Principle ([Ku], VII.8.2) and Fact 2.1.19,

$$\begin{aligned} p \Vdash_{\mathsf{P}} \theta(\tau) \ &\text{iff} \ \ p \Vdash_{\mathsf{P}} \exists y \neg\psi(\tau, y) \\ &\text{iff} \ \ \exists\sigma\,(\sigma \text{ is a simple } \mathsf{P}\text{-name for a real} \wedge p \Vdash_{\mathsf{P}} \neg\psi(\tau,\sigma)) \\ &\text{iff} \ \ \exists\sigma\,(\sigma \text{ is a simple } \mathsf{P}\text{-name for a real} \wedge \forall q\,(q \leq_P p \to \neg q \Vdash_{\mathsf{P}} \psi(\tau,\sigma)) \end{aligned}$$

Now, by inductive hypothesis, "$q \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$" is $\underset{\sim}{\Sigma}^1_{n+k-1}$ (if P is $\underset{\sim}{\Sigma}^1_n$) or $\underset{\sim}{\Sigma}^1_{n+k}$ (if P is $\underset{\sim}{\Pi}^1_n$). Therefore, the last sentence is $\underset{\sim}{\Sigma}^1_{n+k}$ ($\underset{\sim}{\Sigma}^1_{n+k+1}$, respectively).

$\underline{k=2}$: Then, $\theta(x) = \exists y\, \psi(x,y)$, where $\psi(x,y)$ is a Π^1_1 formula. So, by the Maximal Principle,

$$p \Vdash_{\mathsf{P}} \theta(\tau) \text{ iff } \exists\sigma(\sigma \text{ is a simple } \mathsf{P}\text{-name for a real } \wedge p \Vdash_{\mathsf{P}} \psi(\tau,\sigma))$$

Suppose that $\varphi_{\leq}(x,y)$ and $\varphi_{\perp}(x,y)$ are the projective formulas that define P with parameter $a \in \omega^\omega$. Let $\mathrm{wf}(x)$ be the predicate "x is a well-founded relation". Then, $\mathrm{wf}(x)$ is a Δ_1 predicate (see [J2], 14.3). Therefore there exists a finite set S of axioms of ZF such that

$$ZF \vdash \forall M\,(M \text{ transitive } \wedge M \models \bigwedge S \to \mathrm{wf}(x) \text{ is absolute for } M),$$

where $\bigwedge S$ denotes the conjunction of the sentences in S (see [Ku], IV, Exercise 17). Let ZFC^* be a finite set of axioms of ZFC containing all axioms which are needed to define the forcing relation in a model and to prove the Forcing Theorem and including the set S.

Claim 2.1.24 *The following are equivalent:*

1. $p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$.
2. *For every transitive model M of ZFC^* containing a, p and (the codes of) τ and σ and such that $\mathsf{P}^M \lessdot \mathsf{P}$, $M \models$ "$p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$".*
3. *There exists a transitive and countable model M of ZFC^* containing a, p and (the codes of) τ and σ and such that $\mathsf{P}^M \lessdot \mathsf{P}$, $M \models$ "$p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$".*

Proof. ($1 \Rightarrow 2$) Let M be as in (2). Assume M does not satisfy $p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$. Then, since $M \models ZFC^*$, there is $q \in M$ such that $q \leq_P p$ and $M \models$ "$q \Vdash_{\mathsf{P}} \neg\psi(\tau,\sigma)$". Let G be a P-generic filter over V with $q \in G$. Then, since G is closed upwards, $p \in G$. Let $G' = G \cap M$. Since $\mathsf{P}^M \lessdot \mathsf{P}$, by the Fact 2.1.21, G' is P-generic over M. Also, since $q \in G'$, $M[G'] \models \neg\psi(\tau[G'],\sigma[G'])$. But, given that $\tau,\sigma \in M$ and $\mathsf{P}^M \lessdot \mathsf{P}$, by Fact 2.1.21, we have that $\tau[G'] = \tau[G]$ and $\sigma[G'] = \sigma[G]$. So, $M[G] \models \neg\psi(\tau[G],\sigma[G])$, which is a Σ^1_1 formula. Since $M \models ZFC^*$, also $M[G] \models ZFC^*$, and therefore the

Σ^1_1 formulas are absolute for $M[G]$ as being a well-founded relation is absolute for models of ZFC^*. This implies $V[G] \models \neg\psi(\tau[G], \sigma[G])$, which contradicts (1).

(2 $\Rightarrow$ 3) Since ZFC^* is a finite fragment of ZFC, by the Reflection Principle, there exists an ordinal $\alpha > \omega + 2$ such that for every formula of $ZFC^* \cup \{\varphi_{\leq}(x,y), \varphi_{\perp}(x,y)\}$ is absolute for V_α. Let $X = \{a, p, \tau, \sigma, TC(\tau), TC(\sigma)\}$ and let M the Skolem hull of X in V_α. So $M \preccurlyeq V_\alpha$, and, since $|X| = \omega$, M is countable. Moreover, M is extensional since V_α satisfies the axiom of extensionality. Without loss of generality, we can suppose that M is transitive (if not, since Mostowski's collapsing preserves sets of reals, we collapse it). Clearly, $M \models ZFC^*$ and $a, p, \tau, \sigma \in M$. Since $M \preccurlyeq V_\alpha$, for every pair of reals $b_1, b_2 \in M$, if $M \models \varphi_{\leq}(b_1, b_2)$ then $V_\alpha \models \varphi_{\leq}(b_1, b_2)$ and hence, by absoluteness of $\varphi_{\leq}(x,y)$ for V_α, $\varphi_{\leq}(b_1, b_2)$. Therefore, $\mathsf{P}^M \subseteq \mathsf{P}$. Suppose that A_M is a maximal antichain of P^M in M. Then, there is a code for it, that is, there is a real $b \in M$ such that $M \models$ "b codes A_M". Now, since $M \preccurlyeq V_\alpha$, $V_\alpha \models$ "b codes A_{V_α}" and since the coding is recursive $A_M = A_{V_\alpha}$. So, A_M is a maximal antichain of P^{V_α}. Therefore, $\mathsf{P}^M \lessdot \mathsf{P}^{V_\alpha}$. But, by the choice of V_α, $\mathsf{P}^{V_\alpha} \lessdot \mathsf{P}$. So, $\mathsf{P}^M \lessdot \mathsf{P}$. Then, by (2), $M \models$ "$p \Vdash_{\mathsf{P}} \psi(\tau, \sigma)$".

(3 $\Rightarrow$ 1) Let M be a countable and transitive model of ZFC^* containing a, p, τ, σ and such that $\mathsf{P}^M \lessdot \mathsf{P}$ and $M \models$ "$p \Vdash_{\mathsf{P}} \psi(\tau, \sigma)$". Let $G \subseteq \mathsf{P}$ a generic filter over V with $p \in G$. Since $\mathsf{P}^M \lessdot \mathsf{P}$, $G' = G \cap \mathsf{P}^M$ is a P^M-generic filter over M and $p \in G'$. So, since $M \models ZFC^*$,

$$M[G'] \models \psi(\tau[G'], \sigma[G']).$$

Moreover, by Fact 2.1.21, $\tau[G'] = \tau[G]$ and $\sigma[G'] = \sigma[G]$. So,

$$M[G] \models \psi(\tau[G], \sigma[G]).$$

But then, since $M \models ZFC^*$, $M[G] \models ZFC^*$, and therefore the Π^1_1 formulas are absolute for $M[G]$. Thus,

$$V[G] \models \psi(\tau[G], \sigma[G]).$$

This proves the claim. ■

Now, (2) of the claim above holds iff for every transitive set M,

$$\langle M, \in \rangle \models ZFC^* \wedge a, p, \tau, \sigma \in M \wedge \mathsf{P}^M \lessdot \mathsf{P} \rightarrow \langle M, \in \rangle \models \text{“}p \Vdash_{\mathsf{P}} \psi(\tau, \sigma)\text{”}$$

iff for every well-founded extensional relation E over ω,

$$\begin{gathered}\langle \omega, E \rangle \models ZFC^* \wedge \exists n_0 n_1 n_2 n_3 (\pi_E(n_0) = a \wedge \pi_E(n_1) = p \wedge \pi_E(n_2) = \tau \wedge \\ \wedge \pi_E(n_3) = \sigma) \wedge \forall nm (\langle \omega, E \rangle \models \varphi_{\leq}(n, m, n_0) \rightarrow \varphi_{\leq}(\pi_E(m), \pi_E(m), a)) \wedge \\ \wedge \forall n (\langle \omega, E \rangle \models n \text{ codes a max. antichain} \rightarrow \pi_E(n) \text{ codes a max.antichain}) \rightarrow \\ \rightarrow \langle \omega, E \rangle \models \text{“}p \Vdash_{\mathsf{P}} \psi(\tau, \sigma)\text{”}\end{gathered}$$

where π_E is a transitive collapse of $\langle \omega, E \rangle$ onto $\langle M, \in \rangle$; iff

$$\begin{gathered}\forall z (z \in WF \wedge \langle \omega, E_z \rangle \models ZFC^* \wedge \exists n_0, n_1, n_2, n_3 (\pi_{E_z}(n_0) = a \wedge \pi_{E_z}(n_1) = p \wedge \\ \wedge \pi_{E_z}(n_2) = \tau \wedge \pi_{E_z}(n_3) = \sigma) \wedge \\ \wedge \forall nm (\langle \omega, E_z \rangle \models \varphi_1(n, m, n_0) \rightarrow \varphi_1(\pi_{E_z}(n), \pi_{E_z}(m), a)) \wedge \\ \wedge \forall n (\langle \omega, E_z \rangle \models n \text{ codes a max. antichain} \rightarrow \pi_{E_z}(n) \text{ codes a max. antichain}) \rightarrow \\ \rightarrow \langle \omega, E_z \rangle \models \text{“}p \Vdash_{\mathsf{P}} \psi(\tau, \sigma)\text{”})\end{gathered}$$

Now, WF is Π^1_1 (see [J2], 40.2). $\langle\omega, E_z\rangle \models ZFC^*$, $\langle\omega, E_z\rangle \models$ "$p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$" and $\pi_{E_z}(n) = x$ are all arithmetical relations in z (see [J2], 41.1). Moreover, the formula

$$\forall nm\,(\langle\omega, E_z\rangle \models \varphi_{\leq}(n,m,n_0) \to \varphi_{\leq}(\pi_{E_z}(n), \pi_{E_z}(m), a))$$

is as complex as the poset. Finally, if P is a $\utilde{\Sigma}^1_n$ ($\utilde{\Pi}^1_n$) poset, the formula

$$\forall n(\langle\omega, E_z\rangle \models n \text{ codes a max. antichain} \to \pi_{E_z}(n) \text{ codes a max. antichain})$$

is $\utilde{\Pi}^1_n$ ($\utilde{\Pi}^1_{n+1}$, respectively). It follows that (2) is $\utilde{\Pi}^1_{n+1}$.

Similarly, (3) iff

$$\begin{gathered}
\exists z\,(z \in WF \wedge \langle\omega, E_z\rangle \models ZFC^* \wedge \exists n_0 n_1 n_2 n_3\,(\pi_{E_z}(n_0) = a \wedge \\
\wedge \pi_{E_z}(n_1) = p \wedge \pi_{E_z}(n_2) = \tau \wedge \pi_{E_z}(n_3) = \sigma) \wedge \\
\wedge \forall nm\,(\langle\omega, E_z\rangle \models \varphi_{\leq}(n,m,n_0) \to \varphi_{\leq}(\pi_{E_z}(n), \pi_{E_z}(m), a)) \wedge \\
\wedge \forall n\,(\langle\omega, E_z\rangle \models n \text{ codes a max. antichain} \to \pi_{E_z}(n) \text{ codes a max. antichain}) \wedge \\
\wedge \langle\omega, E_z\rangle \models \text{“}p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)\text{”})
\end{gathered}$$

But this is $\utilde{\Sigma}^1_{n+1}$ (if P is $\utilde{\Sigma}^1_n$), or $\utilde{\Sigma}^1_{n+2}$ (if P is $\utilde{\Pi}^1_n$).

Hence, if P is $\utilde{\Sigma}^1_n$, then "$p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$" is $\utilde{\Delta}^1_{n+1}$. Therefore, "$p \Vdash_{\mathsf{P}} \theta(\tau)$" is $\utilde{\Sigma}^1_{n+1}$. On the other hand, if P is $\utilde{\Pi}^1_n$ then "$p \Vdash_{\mathsf{P}} \psi(\tau,\sigma)$" is $\utilde{\Delta}^1_{n+2}$. Hence, "$p \Vdash_{\mathsf{P}} \theta(\tau)$", is $\utilde{\Sigma}^1_{n+2}$. ■

2.2 Martin's Axiom for projective posets

Definition 2.2.1 *Let Γ be a class of posets.* Martin's Axiom for Γ*, henceforth denoted by $MA(\Gamma)$, is the following statement:* For every ccc poset $\mathsf{P} \in \Gamma$ and for every family $\{A_\alpha : \alpha < \kappa\}$, $\kappa < 2^{\aleph_0}$, of maximal antichains of P, there exists $G \subseteq \mathsf{P}$ directed such that for every $\alpha < \kappa$, $G \cap A_\alpha \neq \emptyset$.

Martin's Axiom*, in the sequel denoted by MA, is $MA(\Gamma)$, where Γ is the class of all posets.*

Definition 2.2.2 Martin's Axiom for projective posets*, $MA(Proj)$, is Martin's Axiom restricted to the class of projective posets. i.e., $MA(\Gamma)$ where Γ is the class of projective posets. Similarly, for every $n \geq 1$, we define* Martin's Axiom for $\utilde{\Sigma}^1_n$ posets*, $MA(\utilde{\Sigma}^1_n)$,* Martin's Axiom for $\utilde{\Pi}^1_n$ posets*, $MA(\utilde{\Pi}^1_n)$, and* Martin's Axiom for $\utilde{\Delta}^1_n$ posets*, $MA(\utilde{\Delta}^1_n)$.*

In this section we shall construct a model of $MA(Proj) + \neg CH$ by iterating only projective posets. As it was pointed out in the Introduction, the main difficulty of the proof comes from the fact that, in general, projective formulas are not absolute. However, as we shall see, we can arrange the construction so that they are absolute for "sufficiently" many models along of the iteration, where "sufficiently" means for a ω_1-club set.

2.2.1 λ-club and λ-stationary sets

Definition 2.2.3 *Let $\kappa > \lambda$ be regular cardinals. A set $C \subseteq \kappa$ is* λ-closed *iff for every ordinal γ with $\lambda \leq \mathrm{cf}(\gamma) < \kappa$ and for every increasing sequence $\langle \beta_\xi : \xi < \gamma \rangle$ of elements of C, $\sup_{\xi<\gamma}(\beta_\xi) \in C$. A set $C \subseteq \kappa$ is* unbounded in κ *iff for every $\alpha < \kappa$ there is $\beta > \alpha$ such that $\beta \in C$. A set $C \subseteq \kappa$ is a* λ-club in κ *iff C is a λ-closed and unbounded in κ.*

Remark 2.2.4 *Let $\kappa > \omega$ be a regular cardinal. $C \subseteq \kappa$ is a club in κ iff C is an ω-club in κ.*

The following facts are well-known. We give a proof for the convenience of the reader.

Fact 2.2.5 *Let $\kappa > \lambda$ be regular cardinals. The intersection of any family of less than κ λ-club subsets of κ is a λ-club in κ.*

Proof. Let $\{C_\alpha : \alpha < \beta\}$, with $\beta < \kappa$, be a family of λ-club subsets of κ. Let $C = \bigcap_{\alpha<\beta} C_\alpha$. It is easy to see that C is λ-closed. To show that C is unbounded, let $\gamma < \kappa$. We construct a sequence $\langle \xi_\eta : \eta < \lambda \rangle$ by induction:

$\underline{\eta = 0}$: Then $\xi_0 = \sup_{\alpha<\beta}(\zeta^0_\alpha)$ where $\langle \zeta^0_\alpha : \alpha < \beta \rangle$ is defined by an other induction:

$\underline{\alpha = 0}$: Let ζ^0_0 be the least $\nu \in C_0$ greater than γ.

$\underline{\alpha > 0}$: Let ζ^0_α be the least $\nu \in C_\alpha$ greater than $\sup_{\delta<\alpha}(\zeta^0_\delta)$.

Since κ is regular and for every $\alpha < \beta$, C_α is unbounded, this sequence is well-defined.

$\underline{\eta > 0}$: Then $\xi_\eta = \sup_{\alpha<\beta}(\zeta^0_\alpha)$ where $\langle \zeta^\eta_\alpha : \alpha < \beta \rangle$ is defined by an other induction:

$\underline{\alpha = 0}$: Let ζ^η_0 be the least $\nu \in C_0$ greater than $\sup_{\delta<\eta}(\xi_\delta)$.

$\underline{\alpha > 0}$: Let ζ^η_α be the least $\nu \in C_\alpha$ greater than $\sup_{\delta<\alpha}(\zeta^\eta_\delta)$.

As in the previous case, these sequences are well defined.

Let $\delta = \sup_{\eta<\lambda}(\xi_\eta)$. By regularity of κ, $\delta < \kappa$. Clearly, $\gamma < \zeta^0_0 < \delta$. Finally, since for every $\alpha < \beta$, $\delta = \sup_{\eta<\lambda}(\zeta^\eta_\alpha)$, δ is the supremum of a λ-sequence of elements of C_α. So, for every $\alpha < \beta$, $\delta \in C_\alpha$ and, hence, $\delta \in C$. ■

Fact 2.2.6 *Let $\kappa > \lambda$ be regular cardinals. The diagonal intersection of a κ-sequence of λ-club subsets of κ is a λ-club in κ.*

Proof. Let $\langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of λ-club subsets in κ. Let $D = \triangle_{\alpha<\kappa} C_\alpha$. D is easily seen to be λ-closed. To see that D is unbounded in κ, let $\gamma < \kappa$. We construct a λ-sequence $\langle \xi_\eta : \eta < \lambda \rangle$ by induction:

$\underline{\eta = 0}$: Then ξ_0 is the least $\nu \in \bigcap_{\alpha<\gamma} C_\alpha$ greater than γ.

$\underline{\eta > 0}$: Let $\zeta = \sup_{\delta<\eta}(\xi_\delta)$. Then ξ_η is the least $\nu \in \bigcap_{\alpha<\zeta} C_\alpha$ greater than ζ.

Since κ is regular and, by Fact 2.2.5, for every $\beta < \kappa$, $\bigcap_{\alpha<\beta} C_\alpha$ is λ-club set in κ, this sequence is well-defined.

Let $\delta = \sup_{\eta<\lambda}(\xi_\eta)$. By regularity of κ, $\delta < \kappa$. Clearly, $\gamma < \xi_0 < \delta$. Finally, since for every $\beta < \lambda$, $\langle \xi_\eta : \beta \leq \eta < \lambda \rangle$ is a λ-sequence of elements of $\bigcap_{\alpha<\xi_\beta} C_\alpha$ and $\delta = \sup_{\beta\leq\eta<\lambda}(\xi_\eta)$, for every $\beta < \lambda$, $\delta \in \bigcap_{\alpha<\xi_\beta} C_\alpha$. Hence, $\delta \in \bigcap_{\alpha<\delta} C_\alpha$ and therefore, $\delta \in D$. ■

Definition 2.2.7 *Let $\kappa > \lambda$ be regular cardinals. A function $f : \kappa \to \kappa$ is* λ-normal *if it is* increasing *(if $\alpha < \beta$, then $f(\alpha) < f(\beta)$) and* λ-continuous *($f(\alpha) = \sup_{\beta<\alpha}(f(\beta))$, for every limit $\alpha < \kappa$ with $\mathrm{cf}(\alpha) \geq \lambda$).*

Fact 2.2.8 *Let $\kappa > \lambda$ be regular cardinals. $C \subseteq \kappa$ is a λ-club in κ iff there exists a λ-normal function f such that $\mathrm{ran}(f) = C$.*

Proof. ($\Rightarrow$) Let $C \subseteq \kappa$ be a λ-club in κ. Define $f : \kappa \to \kappa$ by

$$f(\alpha) = \text{ the least ordinal of } C \setminus \{f(\beta) : \beta < \alpha\}.$$

Since κ is regular, $|C| = \kappa$ and so f is well-defined. Clearly, f is increasing. If $\langle f(\beta) : \beta < \gamma\rangle$ is a sequence with γ limit and $\mathrm{cf}(\gamma) \geq \lambda$, then $\sup_{\beta<\gamma}(f(\beta)) \in C$. But $\sup_{\beta<\gamma}(f(\beta))$ is the least ordinal in C greater than every $f(\beta)$, $\beta < \gamma$. So, $f(\gamma) = \sup_{\beta<\gamma}(f(\beta))$.

($\Leftarrow$) Let f be a λ-normal function. Since f is increasing, for every $\alpha < \kappa$, $\alpha \leq f(\alpha) < f(\alpha + 1)$. So, $\mathrm{ran}(f)$ is unbounded. Let $\langle \alpha_\xi : \xi < \gamma\rangle$ an increasing sequence of elements of $\mathrm{ran}(f)$ with γ limit and $\mathrm{cf}(\gamma) \geq \lambda$. Let $\langle \beta_\xi : \xi < \gamma\rangle$ be the sequence of elements of κ such that for every $\xi < \gamma$, $f(\beta_\xi) = \alpha_\xi$. Then, since f is λ-continuous

$$f(\sup_{\xi<\gamma}(\beta_\xi)) = \sup_{\xi<\gamma}\left(f\left(\beta_\xi\right)\right) = \sup_{\xi<\gamma}(\alpha_\xi).$$

Hence, $\sup_{\xi<\gamma}(\alpha_\xi) \in \mathrm{ran}(f)$. ∎

Definition 2.2.9 *Let $\kappa > \lambda$ be regular cardinals. A set $S \subseteq \kappa$ is* λ-stationary *iff $S \cap C \neq \emptyset$ for every λ-club in κ.*

Fact 2.2.10 *Let $\kappa > \lambda$ be regular cardinals. Then,*

1. *Every λ-club subset in κ is a λ-stationary set in κ.*

2. *If C is a λ-club in κ and S is a λ-stationary set in κ, then $C \cap S$ is a λ-stationary set in κ.*

3. *Every λ-stationary set in κ is unbounded in κ. So, every λ-stationary set in κ has cardinality κ.*

Proof. (1) By Fact 2.2.5. (2) Since $(S \cap C) \cap C' = S \cap (C \cap C') \neq \emptyset$. (3) Since the set $\{\beta < \kappa : \alpha \leq \beta\}$ is a λ-club for every $\alpha < \kappa$. ∎

Theorem 2.2.11 (S. Ulam) *Let κ be a successor cardinal and let λ be a regular cardinal $\kappa > \lambda$. Then any λ-stationary set in κ is the disjoint union of κ λ-stationary subsets.*

Proof. Let $\kappa = \mu^+$. For every $\alpha < \kappa$, let $f_\alpha : \alpha \to \mu$ be a one-to-one function. Now, for every $\beta < \kappa$ and every $\xi < \mu$, let

$$X_\beta^\xi = \{\alpha < \kappa : f_\alpha(\beta) = \xi\}.$$

Then, since every f_α is a one-to-one function, for any $\xi < \mu$, if $\beta \neq \gamma$, then $X_\beta^\xi \cap X_\gamma^\xi = \emptyset$. Furthermore, for every $\beta < \kappa$, $\bigcup_{\xi<\mu} X_\beta^\xi = \{\alpha < \kappa : \beta < \alpha\}$.

Let S be a λ-stationary subset of κ. Then, for every $\beta < \kappa$, $\bigcup_{\xi<\mu} X_\beta^\xi \cap S$ (by Fact 2.2.10) is a λ-stationary set. Furthermore, for every $\beta < \kappa$, there is $\xi < \mu$ such that $X_\beta^\xi \cap S$ is λ-stationary. Otherwise, for every $\xi < \mu$, let C_ξ be a λ-club such that $(X_\beta^\xi \cap S) \cap C_\xi = \emptyset$. So $\bigcap_{\xi<\mu} C_\xi$ is a λ-club such that,

$$\bigcup_{\xi<\mu} (X_\beta^\xi \cap S) \cap \bigcap_{\xi<\mu} C_\xi = (\bigcup_{\xi<\mu} X_\beta^\xi \cap S) \cap \bigcap_{\xi<\mu} C_\xi = \emptyset.$$

A contradiction.

We define $h : \kappa \to \mu$ by

$$h(\beta) = \text{ the least } \xi < \mu \text{ such that } X_\beta^\xi \cap S \text{ is a } \lambda\text{-stationary set.}$$

Since $\kappa = \mu^+$, there is $\xi < \mu$ such that $|\{\beta < \kappa : h(\beta) = \xi\}| = \kappa$. Let ξ be the least such ordinal. Then, $\{X_\beta^\xi \cap S : h(\beta) = \xi\}$ is the desired set. ■

Corollary 2.2.12 *Let $\kappa > \lambda$ be regular cardinals. Then κ is the disjoint union of κ λ-stationary subsets in κ.*

Proof. If κ is a successor cardinal, then apply Theorem 2.2.11. If κ is a limit cardinal, and hence weakly inaccessible, then there are κ regular cardinals μ, $\lambda \leq \mu < \kappa$, and for each such μ, $\{\gamma < \kappa : \mathrm{cf}(\gamma) = \mu\}$ is a λ-stationary in κ. ■

2.2.2 Forcing iteration of projective posets

Lemma 2.2.13 *Let $\kappa > \aleph_1$ be a regular cardinal and let P be the direct limit of an iteration $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \kappa\rangle$ with finite support of ccc forcing notions such that for each $\alpha < \kappa$, $\Vdash_\alpha$ "$2^{\aleph_0} < \kappa$". Let G be a P-generic filter over V. Assume that $\varphi(x)$ is a projective formula with parameter $a \in V[G]$. Then, for every $b \in V[G]$,*

$$V[G] \models \varphi(b)$$

iff there is an ω_1-club $C \subseteq \kappa$ such that for all $\alpha \in C$,

$$V[G_\alpha] \models \varphi(b).$$

Proof. ($\Rightarrow$) We prove this direction by induction on the complexity of the projective formulas:

<u>$n = 1$</u>: That is, $\varphi(x)$ is a $\Sigma_1^1(a)$ formula or a $\Pi_1^1(a)$ formula. Since $a, b \in \omega^\omega \cap V[G]$, there is an ordinal $\alpha < \kappa$ such that $a, b \in V[G_\alpha]$ (see [Ku], VIII.5.14). By absoluteness of Σ_1^1 (Π_1^1) formulas for transitive models of ZF, for every β, $\alpha \leq \beta < \kappa$,

$$V[G_\beta] \models \varphi(b).$$

So, since P preserves cofinalities, $C = \{\beta < \kappa : \alpha \leq \beta\}$ is the desired set.

$\underline{n+1}$: Let $\varphi(x) = \exists y\, \psi(x,y)$ be a $\Sigma^1_{n+1}(a)$ formula. Take a witness $c \in V[G]$ so that

$$V[G] \models \psi(b,c).$$

Since $\psi(b,c)$ is a $\Pi^1_n(a)$ formula, the lemma follows by inductive hypothesis.

So, suppose $\varphi(x)$ is $\Pi^1_{n+1}(a)$. Then, $\varphi(x) = \forall y\, \psi(x,y)$ where $\psi(x,y)$ is a $\Sigma^1_n(a)$ formula. We define an ω_1-club $C = \{\alpha_\xi : \xi < \kappa\}$ by induction on $\xi < \kappa$.

$\underline{\xi = 0}$: α_0 is the supremum of an increasing sequence $\langle \beta_\gamma : \gamma < \omega_1 \rangle$ of ordinals satisfying

$$\text{For all } \delta < \gamma < \omega_1 \text{ and all reals } r \in V[G_{\beta_\delta}], V[G_{\beta_\gamma}] \models \psi(b,r). \qquad (*)$$

We define this sequence by another induction on $\gamma < \omega_1$:

$\gamma = 0$: Let $\zeta < \kappa$ be the least ordinal such that $a, b \in V[G_\zeta]$. Since $V[G_\zeta] \models 2^{\aleph_0} = \lambda < \kappa$, let $\langle r^0_\eta : \eta < \lambda \rangle$ be an enumeration of all the reals in $V[G_\zeta]$. Since ψ is $\Sigma^1_n(a)$ and for each $\eta < \lambda$,

$$V[G] \models \psi(b, r^0_\eta),$$

by inductive hypothesis for every $\eta < \lambda$ there is an ω_1-club $C^0_\eta \subseteq \kappa$ such that for all $\nu \in C^0_\eta$,

$$V[G_\nu] \models \psi(b, r^0_\eta).$$

Let $D^0_0 = \bigcap_{\eta<\lambda} C^0_\eta$. D^0_0 is an ω_1-club. Let β_0 be the least $\nu \in D^0_0$ greater than ζ.

$\gamma > 0$: Let $\zeta = \sup_{\delta<\gamma}(\beta_\delta)$. Let $\langle r^\gamma_\eta : \eta < \lambda \rangle$, $\lambda < \kappa$, be an enumeration of all reals in $V[G_\zeta]$. As in the previous case, for every $\eta < \lambda$, there is an ω_1-club $C^\gamma_\eta \subseteq \kappa$ such that for all $\nu \in C^\gamma_\eta$,

$$V[G_\nu] \models \psi(b, r^\gamma_\eta).$$

Let $D^0_\gamma = \bigcap_{\eta<\lambda} C^\gamma_\eta$. Then D^0_γ and $\bigcap_{\delta\leq\gamma} D^0_\delta$ are ω_1-clubs. Let β_γ the least ordinal in $\bigcap_{\delta\leq\gamma} D^0_\eta$ greater than ζ.

Thus, $\langle \beta_\gamma : \gamma < \omega_1 \rangle$ is an increasing sequence of ordinals satisfying $(*)$. Let $\alpha_0 = \sup_{\gamma<\omega_1}(\beta_\gamma)$.

To see that

$$V[G_{\alpha_0}] \models \forall y \psi(b,y),$$

suppose that $r \in V[G_{\alpha_0}]$ is a real. Since P preserves cofinalities, $\mathrm{cf}(\alpha_0) = \omega_1$, and so, there exists a least $\gamma < \omega_1$ and $\eta < \lambda < \kappa$ such that $r = r^\gamma_\eta \in V[G_{\beta_\gamma}]$. But, for every $\delta \geq \gamma$, $\beta_\delta \in C^\gamma_\eta$. So, $\langle \beta_\delta : \gamma \leq \delta < \omega_1 \rangle$ is an increasing ω_1-sequence of elements of C^γ_η with supremum α_0. Hence, $\alpha_0 \in C^\gamma_\eta$ and, therefore,

$$V[G_{\alpha_0}] \models \psi(b,r).$$

$\underline{\xi > 0}$: We take two cases into consideration.
Case 1: $\mathrm{cf}(\xi) \leq \omega$. We define α_ξ in the same way as in case $\xi = 0$; i.e., as the supremum of an increasing sequence $\langle \beta_\gamma : \gamma < \omega_1 \rangle$ of ordinals satisfying $(*)$. We obtain this sequence like before except for $\gamma = 0$. In this case, we define β_0 as the least ordinal $\zeta \in \bigcap_{\nu<\xi} D_\nu$ greater than $\sup_{\nu<\xi}(\alpha_\nu)$, where for every $\nu < \xi$, $D_\nu = \bigcap_{\gamma<\omega_1} D^\nu_\gamma$.
Case 2: $\mathrm{cf}(\xi) > \omega$. Let $\alpha_\xi = \sup_{\nu<\xi}(\alpha_\nu)$. We now show that

$$V[G_{\alpha_\xi}] \models \forall y \psi(b, y).$$

Since $\mathrm{cf}(\alpha_\xi) = \mathrm{cf}(\xi) = \mu > \omega$, we can fix a subsequence $\langle \beta_\gamma : \gamma < \mu \rangle$ of $\langle \alpha_\nu : \nu < \xi \rangle$ with supremum α_ξ. Let $r \in V[G_{\alpha_\xi}]$ be a real. Let $\gamma < \mu$ be the least ordinal such that $r \in V[G_{\beta_\gamma}]$. But, if $\gamma \leq \delta < \mu$, then $\beta_\delta \in D_{\beta_\gamma}$. So, $\langle \beta_\delta : \gamma \leq \delta < \mu \rangle$ is a μ-sequence of elements of D_{β_γ} with supremum α_ξ. Hence, $\alpha_\xi \in D_{\beta_\gamma}$ and, therefore,

$$V[G_{\alpha_\xi}] \models \psi(b, r).$$

Since P preserves cofinalities, $C = \{\alpha_\xi : \xi < \kappa\}$ is the range of an ω_1-normal function and therefore C is an ω_1-club contained in κ.

($\Leftarrow$) Assume that there is an ω_1-club $C \subseteq \kappa$ such that for all $\alpha \in C$,

$$V[G_\alpha] \models \varphi(b),$$

and suppose that

$$V[G] \not\models \varphi(b).$$

Then, $V[G] \models \neg\varphi(b)$ and, by ($\Rightarrow$), there exists an ω_1-club D such that for all $\alpha \in D$, $V[G_\alpha] \models \neg\varphi(b)$. By Fact 2.2.5, $C \cap D \neq \emptyset$. But if $\alpha \in C \cap D$, then

$$V[G_\alpha] \models \varphi(b) \wedge \neg\varphi(b).$$

A contradiction. ■

Theorem 2.2.14 *(GCH) Let κ be a regular cardinal which is not the successor of a cardinal of countable cofinality. Then there is an iteration of projective and ccc posets such that whenever G is a generic filter for the iteration,*

$$V[G] \models MA(Proj) \wedge 2^{\aleph_0} = \kappa.$$

Proof. We divide the proof in two parts: first, we construct the poset P and, second, we show that forcing with this poset gives a model of $MA(Proj)$ and $2^{\aleph_0} = \kappa$.

(I) Construction of P: To start with, we fix a function π from κ onto $\kappa \times \kappa$ such that for every $\beta, \gamma < \kappa$,

1. $(\forall \alpha < \kappa)\,(\pi(\alpha) = \langle \beta, \gamma \rangle \rightarrow \beta \leq \alpha)$.
2. $S_{\beta,\gamma} = \{\alpha \in \kappa : \pi(\alpha) = \langle \beta, \gamma \rangle\}$ is an ω_1-stationary set in κ.

There exists a such a function. Indeed, by Corollary 2.2.12, κ is the disjoint union of $\{X_\xi : \xi < \kappa\}$, a family of ω_1-stationary subsets of κ. So, let f be a one-to-one function from $\kappa \times \kappa$ onto κ and define π by:

$$\pi(\alpha) = \begin{cases} \langle \beta, \gamma \rangle, & \text{if } \alpha \in X_{f(\beta,\gamma)} \text{ and } \beta \leq \alpha \\ \langle 0, 0 \rangle, & \text{if } \alpha \in X_{f(\beta,\gamma)} \text{ and } \beta > \alpha \end{cases}$$

Clearly π satisfies (1), and $S_{\beta,\gamma} = X_{f(\beta,\gamma)} \cap \{\alpha \in \kappa : \beta \leq \alpha\}$ is an ω_1-stationary set, for every $\beta, \gamma < \kappa$. So, π also satisfies (2). We shall use π as a bookkeeping function to ensure that we force with all projective ccc posets that appear along the iteration.

We obtain the poset P as the direct limit of an iteration $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \kappa \rangle$ with finite support such that for every $\alpha < \kappa$,

$$\Vdash_\alpha \text{“}\dot{\mathsf{Q}}_\alpha \text{ is a projective ccc poset”}.$$

We proceed by induction on $\alpha < \kappa$: suppose that for every $\beta < \alpha$, P_β and $\dot{\mathsf{Q}}_\beta$ have been already defined and satisfy that:

1. P_β is a ccc poset.
2. $\Vdash_\beta$ “$2^{\aleph_0} < \kappa$”.
3. $\Vdash_\beta$ “$\dot{\mathsf{Q}}_\beta$ is a projective ccc poset”.

Then we define P_α and show that it satisfies (1)-(3):

$\underline{\alpha = 0}$: Let P_0 the trivial poset.

$\underline{\alpha = \beta + 1}$: Let $\mathsf{P}_\alpha = \mathsf{P}_\beta * \dot{\mathsf{Q}}_\beta$. Since P_β is a ccc poset and $\Vdash_\beta$ “$\dot{\mathsf{Q}}_\beta$ is a projective ccc poset”, P_α is a ccc poset. So, P_α satisfies (1).

Moreover, on one hand, it is easy to see that $|\mathsf{P}_\beta| < \kappa$. On the other hand, since $\Vdash_\beta$ “$2^{\aleph_0} < \kappa$” and $\Vdash_\beta$ “$\dot{\mathsf{Q}}_\beta$ is a projective poset”, $\Vdash_\beta$ “$|\dot{\mathsf{Q}}_\beta| < \kappa$”. So, $|\mathsf{P}_\alpha| = |\mathsf{P}_\beta * \dot{\mathsf{Q}}_\beta| < \kappa$. But then, since κ is a regular cardinal and GCH holds, $\kappa^{\aleph_0} = \kappa$. So, since P_α is a ccc poset, $\Vdash_\alpha$ “$2^{\aleph_0} < \kappa$”. Therefore, P_α satisfies (2).

Given P_α, it only remains to find $\dot{\mathsf{Q}}_\alpha$ satisfying (3). Since $\Vdash_\alpha$ “$2^{\aleph_0} < \kappa$”,

$$V[G_\alpha] \models \text{“There are less than } \kappa \text{ projective posets”}.$$

Let $\langle \dot{\mathsf{Q}}_\alpha^\gamma : \gamma < \kappa \rangle$ be a sequence of all P_α-names (where for some $\eta < \kappa$, for every $\gamma \geq \eta$, $\dot{\mathsf{Q}}_\alpha^\gamma$ is the trivial poset) such that:

$$\Vdash_\alpha \text{“}\langle \dot{\mathsf{Q}}_\alpha^\gamma : \gamma < \kappa \rangle \text{ enumerates all projective posets”}.$$

There is a such a sequence by the Maximal Principle.

Suppose that $\pi(\alpha) = \langle \delta, \gamma \rangle$. Since $\delta \leq \alpha$, $\dot{\mathsf{Q}}_\delta^\gamma = \langle \dot{Q}_\delta^\gamma, \dot{\leq}_\delta^\gamma, \dot{\perp}_\delta^\gamma \rangle$ has been already defined; i.e., there are projective formulas $\varphi_\leq(x, y; z)$ and $\varphi_\perp(x, y; z)$ and a simple P_δ-name for a real $\dot{a}$ that define the projective poset $\dot{\mathsf{Q}}_\beta^\gamma$. i.e.,

$$\Vdash_\delta \text{“}\dot{Q}_\delta^\gamma = \text{field}(\dot{\leq}_\delta^\gamma)$$
$$\Vdash_\delta \text{“}(\forall x, y \in \omega^\omega)(x \dot{\leq}_\delta^\gamma y \leftrightarrow \varphi_\leq(x, y; \dot{a}))\text{”}$$
$$\Vdash_\delta \text{“}(\forall x, y \in \omega^\omega)(x \dot{\perp}_\delta^\gamma y \leftrightarrow \varphi_\perp(x, y; \dot{a}))\text{”}$$

Let $\dot{\mathsf{Q}} = \langle \dot{Q}, \dot{\leq}, \dot{\perp} \rangle$ be a P_α-name such that

$$\begin{aligned}&\Vdash_\alpha \text{"}\dot{Q} = \text{field}(\dot{\leq})\\ &\Vdash_\alpha \text{"}(\forall x, y \in \omega^\omega)(x \dot{\leq} y \leftrightarrow \varphi_{\leq}(x, y; \dot{a}))\text{"}\\ &\Vdash_\alpha \text{"}(\forall x, y \in \omega^\omega)(x \dot{\perp} y \leftrightarrow \varphi_{\perp}(x, y; \dot{a}))\text{"}\end{aligned}$$

Then we put $\dot{\mathsf{Q}}_\alpha = \dot{\mathsf{Q}}$, providing that $\Vdash_\alpha$ "$\dot{\mathsf{Q}}$ is a ccc poset", and $\dot{\mathsf{Q}}_\alpha = \{\emptyset\}$, otherwise. So

$$\Vdash_\alpha \text{"}\dot{\mathsf{Q}}_\alpha \text{ is a projective ccc poset"},$$

and P_α satisfies (3).

<u>α limit</u>: Then P_α is the direct limit of $\langle \mathsf{P}_\beta, \dot{\mathsf{Q}}_\beta : \beta < \alpha \rangle$. Since for every $\beta < \alpha$, $\Vdash_\beta$ "$\dot{\mathsf{Q}}_\beta$ is a ccc poset", P_α satisfies (1).

Since we are working with an iteration with finite support,

$$|\mathsf{P}_\alpha| = |\bigcup_{\beta<\alpha} \mathsf{P}_\beta| = \sum_{\beta<\alpha} |\mathsf{P}_\beta|$$

But κ is a regular cardinal, $\alpha < \kappa$ and, by inductive hypothesis, for all $\beta < \alpha$, $|\mathsf{P}_\beta| < \kappa$. So, $|\mathsf{P}_\alpha| < \kappa$. But then, as above, $\Vdash_\alpha$ "$2^{\aleph_0} < \kappa$" and P_α satisfies (2).

We can see that P_α satisfies (3) as in the previous case.

This completes the construction of the P_α's, $\alpha < \kappa$. P is its direct limit.

(II) If G is a P-generic filter over V, then $V[G] \models MA(Proj) \wedge 2^{\aleph_0} = \kappa$: We fix a P-generic filter G over V.

We first show that $V[G] \models 2^{\aleph_0} = \kappa$. Since P is a ccc poset which is the direct limit of a finite support iteration of length κ of posets of cardinality less than κ and κ is a regular cardinal, $|\mathsf{P}| \leq \kappa$. Furthermore, by GCH, $\kappa^{\aleph_0} = \kappa$. Hence,

$$\Vdash_{\mathsf{P}} \text{"}2^{\aleph_0} \leq \kappa\text{"}.$$

On the other hand, if Q is the Cohen poset for adding one generic real, then, since it is a projective ccc poset, Q is the denotation of $\dot{\mathsf{Q}}_\alpha$ for arbitrary large $\alpha < \kappa$. Each time we force with one of these $\dot{\mathsf{Q}}_\alpha$ one more new real is added and therefore we have

$$\Vdash_{\mathsf{P}} \text{"}2^{\aleph_0} \geq \kappa\text{"}.$$

We now show that $V[G] \models MA(Proj)$. Let Q be a projective ccc poset in $V[G]$. Suppose that $\leq_Q = \left\{\langle x, y \rangle : \varphi_{\leq}(x, y)\right\}$ and that $\perp_Q = \{\langle x, y \rangle : \varphi_{\perp}(x, y)\}$, where $\varphi_{\leq}(x, y)$ and $\varphi_{\perp}(x, y)$ are projective formulas with parameter $a \in \omega^\omega \cap V[G]$. Let $\{A_i : i < \mu\}$, $\mu < \kappa$, be a family of maximal antichains of Q in $V[G]$.

We code every maximal antichain A_i, $i < \mu$, by a real a_i like in Fact 2.1.15. Hence, for every $i < \mu$,

$$V[G] \models \text{"}a_i \text{ codes a maximal antichain of } \mathsf{Q}\text{"}.$$

Note that, by Fact 2.1.15, the right-hand side is a projective sentence. So, by Lemma 2.2.13, for every $i < \mu$, there exists an ω_1-club $C_i \subseteq \kappa$ such that for every $\xi \in C_i$,

$$V[G_\xi] \models \text{"}a_i \text{ codes a maximal antichain of } \mathsf{Q}^\xi\text{"},$$

where Q^{ξ} denotes the poset in $V[G_\xi]$ which is defined by the same formulas that define the poset Q in $V[G]$. Let $C = \bigcap_{i<\mu} C_i$. Then, C is an ω_1-club such that for every $i < \mu$ and every $\xi \in C$,

$$V[G_\xi] \Vdash \text{“}a_i \text{ codes a maximal antichain of } \mathsf{Q}^{\xi}\text{”}.$$

Moreover, since the coding is recursive, if for each $\xi \in C$, A_i^{ξ} denotes the maximal antichain of Q^{ξ} coded by a_i, then $A_i^{\xi} = A_i$.

Claim 2.2.15 *There exists an ω_1-club $D \subseteq \kappa$ such that for every $\alpha \in D$ and for all reals $r, r' \in V[G_\alpha]$,*

$$\begin{array}{lcl} V[G_\alpha] \Vdash \varphi_{\leq}(r, r') & \textit{iff} & V[G] \Vdash \varphi_{\leq}(r, r') \\ V[G_\alpha] \Vdash \varphi_{\perp}(r, r') & \textit{iff} & V[G] \Vdash \varphi_{\perp}(r, r') \end{array}$$

Proof. Let $\langle\langle r, r'\rangle_\gamma : \gamma < \kappa\rangle$ be an enumeration of all pairs of reals in $V[G]$. Let C be the following set:

$$C = \{\beta < \kappa : \langle\langle r, r'\rangle_\gamma : \gamma < \beta\rangle \text{ enumerates all pairs of reals of } V[G_\beta]\}.$$

It is easy to see that C is a ω_1-closed set. To see that C is unbounded, given an ordinal $\alpha < \kappa$, we construct by induction a ω_1-sequence $\langle \alpha_\xi : \alpha < \omega_1\rangle$ of ordinals as follows:

$\underline{\xi = 0}$: $\alpha_0 = \alpha$

$\underline{\xi > 0}$: α_ξ is the least ordinal greater than $\sup_{\zeta<\xi}(\alpha_\zeta)$ such that for every $\zeta < \xi$, $\langle\langle r, r'\rangle_\gamma : \gamma < \alpha_\xi\rangle$ enumerates all pairs of reals of $V[G_{\alpha_\zeta}]$.

Clearly $\alpha < \sup_{\xi<\omega_1}(\alpha_\xi) \in C$.

By Lemma 2.2.13, for every pair of reals $\langle r, r'\rangle_\gamma$, $\gamma < \kappa$, there exists an ω_1-club E_γ such that for every $\xi \in E_\gamma$,

$$V[G_\xi] \Vdash \varphi_{\leq}(r, r') \quad \text{iff} \quad V[G] \Vdash \varphi_{\leq}(r, r').$$

For every $\gamma < \kappa$, let $D_\gamma = E_\gamma \cap C$. Let $D_{\varphi_\leq} = \triangle_{\gamma<\kappa} D_\gamma$. So $D_{\varphi_\leq}$ is an ω_1-club (by Fact 2.2.6).

We now show that for every $\xi \in D_{\varphi_\leq}$ and all reals $r, r' \in V[G_\xi]$,

$$V[G_\xi] \Vdash \varphi_{\leq}(r, r') \quad \text{iff} \quad V[G] \Vdash \varphi_{\leq}(r, r').$$

Let $\xi \in D_{\varphi_\leq}$ and $r, r' \in V[G_\xi]$. Since $\xi \in D_{\varphi_\leq}$, for every $\zeta < \xi$, $\xi \in C \cap E_\zeta$. Since $r, r' \in V[G_\xi]$ and $\xi \in C$, there exists $\gamma < \xi$ such that $\langle r, r'\rangle = \langle r, r'\rangle_\gamma$. But $\xi \in E_\gamma$ and so

$$V[G_\xi] \Vdash \varphi_{\leq}(r, r') \quad \text{iff} \quad V[G] \Vdash \varphi_{\leq}(r, r').$$

We obtain $D_{\varphi_\perp}$ in a similar way. Finally, we put $D = D_{\varphi_\leq} \cap D_{\varphi_\perp}$ which, by Fact 2.2.5, is also an ω_1-club. ■

Let D be as in the claim above and we put $E = C \cap D$. Clearly, $E \neq \emptyset$. So, pick $\beta \in E$. Since $\beta \in D$, for all reals $r, r' \in V[G_\beta]$,

$$V[G_\beta] \models \varphi_\perp(r, r') \quad \text{iff} \quad V[G] \models \varphi_\perp(r, r').$$

Thus, every antichain of Q^β in $V[G_\beta]$ is an antichain of Q in $V[G]$. Therefore, since Q is a ccc poset in $V[G]$ and forcing with P preserves cardinals, Q^β is a ccc poset in $V[G_\beta]$. Hence, there exists a P_β-name $\dot{\mathsf{Q}}^\gamma_\beta$ such that $\mathsf{Q}^\beta = \dot{\mathsf{Q}}^\gamma_\beta[G_\beta]$. Since $S_{\gamma,\beta}$ is an ω_1-stationary subset of κ, $E \cap S_{\gamma,\beta} \neq \emptyset$. Let $\eta \in E \cap S_{\gamma,\beta}$. So, Q^η is a projective ccc poset and $\mathsf{Q}^\eta = \dot{\mathsf{Q}}_\eta[G_\eta]$. But then, $V[G_{\eta+1}]$ contains a Q^η-generic filter H over $V[G_\eta]$. Since $\eta \in C$, for every $i < \mu$, $A^\eta_i = A_i$ is a maximal antichain of Q^η in $V[G_\eta]$. Hence, for every $i < \mu$, $H \cap A^\eta_i \neq \emptyset$. Finally, since $\eta \in D$, for all reals $r, r' \in V[G_\eta]$,

$$V[G_\eta] \models \varphi_\leq(r, r') \quad \text{iff} \quad V[G] \models \varphi_\leq(r, r').$$

Thus, H is a directed subset of Q in $V[G]$ such that for every $i < \mu$, $H \cap A_i \neq \emptyset$. Hence, $V[G] \models MA(Proj)$. ■

Theorem 2.2.16 *(GCH) Let κ be a regular cardinal which is not the successor of a cardinal of countable cofinality. Then for every $n \geq 1$, there is an iteration of $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$) ccc posets such that whenever G is a generic filter for the iteration, $V[G]$ satisfies $MA(\underset{\sim}{\Sigma}^1_n)$ ($MA(\underset{\sim}{\Pi}^1_n)$, $MA(\underset{\sim}{\Delta}^1_n)$) and $2^{\aleph_0} = \kappa$.*

Proof. As in Theorem 2.2.14 but using only ccc $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$, $\underset{\sim}{\Delta}^1_n$) posets. ■

We finish this section by remarking that with a similar argument as in Theorem 2.2.14, we can improve a result from [Ju-R]. The following is an alternative axiom to $MA_\kappa(\Gamma)$ introduced in [vD-F]:

Definition 2.2.17 *Let Γ be a class of posets and let κ be any cardinal.* κ-Anti Martin's Axiom for Γ, *henceforth denoted by $AMA_\kappa(\Gamma)$, is the following statement:* For every ccc poset $\mathsf{P} \in \Gamma$ there exists a family $\{G_\alpha : \alpha < \kappa\}$ of filters on P such that for every maximal antichain $A \subseteq \mathsf{P}$ there exists $\alpha < \kappa$ such that for every $\beta \geq \alpha$, $G_\beta \cap A \neq \emptyset$.

Theorem 2.2.18 *(GCH) Let κ be cardinal such that $\omega_1 < \mathrm{cf}(\kappa)$ and κ is not the successor of a cardinal of countable cofinality. Then there is an iteration of projective ccc posets such that whenever G is a generic filter for the iteration,*

$$V[G] \models AMA_{\mathrm{cf}(\kappa)}(Proj) \wedge 2^{\aleph_0} = \kappa.$$

Proof. Note that for all results of 2.2.1 on λ-clubs and λ-stationary subsets of κ only need that $\mathrm{cf}(\lambda) < \mathrm{cf}(\kappa)$. So, the theorem follows with a similar argument to that 2.2.14. ■

Remark 2.2.19 *The case for consistency of $AMA_\lambda(Proj) \wedge 2^{\aleph_0} = \kappa$ with $\omega_1 \leq \lambda \leq \kappa$ for λ regular and $\kappa^{\aleph_0} = \kappa$ has been solved in [R-Sh].*

2.3 $MA(Proj)$ is weaker than MA

2.3.1 Projective forcing after collapsing a weakly-compact cardinal

Fact 2.3.1 *Let V be a transitive model of ZF. Then, V and $L(\mathsf{R})$ are projective absolute. i.e., for every projective formula $\varphi(x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \in \mathsf{R} \cap V$,*

$$V \models \varphi(b_1, \ldots, b_n) \text{ iff } L(\mathsf{R}) \models \varphi(b_1, \ldots, b_n). \blacksquare$$

We need the following form of a theorem of K. Kunen. The proof is taken from [To1]:

Theorem 2.3.2 *Let κ be a weakly-compact cardinal and let $V_0 = V[C_0]$, where C_0 is a $Coll(\omega, < \kappa)$-generic filter over V. Suppose that P is a ccc poset in V_0 and G is a P-generic filter over V_0. Then, $L(\mathsf{R})^{V_0}$ and $L(\mathsf{R})^{V_0[G]}$ satisfy the same sentences of Set Theory with parameters in V.*

Proof. Let κ be a weakly-compact cardinal and let $\dot{\mathsf{P}}$ be a $Coll(\omega, < \kappa)$-name for a ccc poset. Let $\mathsf{S} = Coll(\omega, < \kappa) * \dot{\mathsf{P}}$.

We need the following lemma of K. Kunen (see [H-Sh], Lemma 1). We work with complete Boolean algebras in order to simplify the proof.

Lemma 2.3.3 *Let κ be a weakly-compact cardinal and let B a complete and κ-cc Boolean algebra. Then for every $Z \subseteq \mathsf{B}$, if $|Z| < \kappa$, there exists a complete subalgebra B^* of B such that $Z \subseteq \mathsf{B}^*$ and $|\mathsf{B}^*| < \kappa$.*

Proof. Let κ be a weakly-compact cardinal, B a κ-cc complete Boolean algebra and $Z \subseteq \mathsf{B}$ such that $|Z| < \kappa$. Let C be a complete subalgebra of B containing Z. Since B is κ-cc and $|Z| < \kappa$, $|\mathsf{C}| \leq \kappa$. So, without loss of generality, we may assume that $\mathsf{C} = \langle \kappa, +, \cdot, -, 0, 1 \rangle$.

Let $U_1 = \{a \subseteq \kappa : a \text{ is a maximal antichain of } \mathsf{C}\}$. Since C is κ-cc, for all $a \in U_1$, $|a| < \kappa$. So, $U_1 \subseteq \kappa^{<\kappa}$. Let $U_2 = \{\langle x, \alpha \rangle : x \subseteq \kappa \wedge |x| < \kappa \wedge \alpha \in \kappa \wedge \alpha = \sup_{\mathsf{C}}(x)\}$. And let $U_3 = \{Z\}$.

Let σ be the conjunction of the following sentences of the second order language of type $\{\in, U_1, U_2, U_3, +, \cdot, -, 0, 1\}$:

1. C is a Boolean algebra and $Z \subseteq \mathsf{C}$; i.e., the conjunction of Boolean algebra axioms and $\forall x (U_3 x \to x \subseteq \kappa)$ (first order).

2. $\forall x (x \subseteq \kappa \wedge |x| < \kappa \to \exists z U_2(x, z))$. i.e., C is κ-complete (first order).

3. $\forall X (X \subseteq \kappa \wedge \forall yz (Xy \wedge Xz \wedge y \neq z \to y \cdot z = 0) \wedge$
$$\wedge \forall z (z < \kappa \wedge z \neq 0 \to \exists y (Xy \wedge \neg y \cdot z = 0)) \to \exists x (U_1 x \wedge x = X))$$
i.e., every maximal antichain of C belongs to U_1 (second order and Π^1_1).

Since $\mathsf{C} = \langle \kappa, +, \cdot, -, 0, 1 \rangle$ is a complete Boolean algebra, $Z \subseteq \kappa$ and $U_1 \subseteq \kappa^{<\kappa}$ is the set of all maximal antichains of C,

$$\langle V_\kappa, \in, \kappa, +, \cdot, -, 0, 1, U_1, U_2, U_3 \rangle \models \sigma.$$

Then, since σ is a Π^1_1 sentence, by Π^1_1-indescribability of κ, there is $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, \kappa \cap V_\alpha, + \cap V_\alpha, \cdot \cap V_\alpha, - \cap V_\alpha, 0, 1, U_1 \cap V_\alpha, U_2 \cap V_\alpha, U_3 \cap V_\alpha \rangle \models \sigma.$$

Let $\mathsf{B}^* = \mathsf{C} \cap V_\alpha$. Then, by (1) $\mathsf{B}^* = \langle \alpha, +, \cdot, -, 0, 1 \rangle$ is a subalgebra of C such that $Z \subseteq \mathsf{B}^*$. Clearly, $|\mathsf{B}^*| = \alpha < \kappa$. By (2), B^* is a α-complete Boolean algebra and, by (3), every maximal antichain of B^* belongs to $U_1 \cap V_\alpha = U_1 \cap \alpha^{<\alpha}$. Hence B^* is a α-cc Boolean algebra. Therefore, B^* is a complete Boolean algebra and it is a subalgebra of C.

So, it only remains to see that B^* is a complete subalgebra of C. Let $X \subseteq \alpha$. If $|X| < \alpha$, then $\langle X, \sup_{\mathsf{B}^*}(X) \rangle \in U_2 \cap V_\alpha$, so $\langle X, \sup_{\mathsf{B}^*}(X) \rangle \in U_2$ and therefore, $\sup_{\mathsf{B}^*}(X) = \sup_{\mathsf{C}}(X)$. If $|X| = \alpha$, then, since B^* is α-cc, $\sup_{\mathsf{B}^*}(X) = 1$. Further, let $X' = \{u \in \alpha : (\exists v \in X)\, (u \leq_{\mathsf{B}^*} v)\}$. Then, since X' is open in B^*, there exists a maximal antichain A in B^* such that $A \subseteq X^{'}$. But then, by (3), $A \in U_1$ and, so, A is a maximal antichain of C. Hence $\sup_{\mathsf{C}}(A) = \sup_{\mathsf{C}}(X') = \sup_{\mathsf{C}}(X) = 1$. Therefore, B^* is a complete subalgebra of C of size less than κ and includes Z. ■

Let E be the poset of all complete embeddings from a complete subalgebra of r.o. (S) of size less than κ into r.o. $(Coll\,(\omega, <\kappa))$, ordered by inverse inclusion. i.e.,

- $h \in \mathsf{E}$ iff h is a complete embedding from a complete subalgebra of r.o. (S) of cardinality less than κ into r.o. $(Coll\,(\omega, <\kappa))$.
- $h \leq h'$ iff $h' \subseteq h$.

By Kripke's Theorem (see [J2], Theorem 62) and lemma above, we know that $\mathsf{E} \neq \emptyset$.

Definition 2.3.4 *Let κ be a cardinal. A poset P is* $<\kappa$-closed *iff whenever $\gamma < \kappa$ and $\langle p_\alpha : \alpha < \gamma \rangle$ is a decreasing sequence of elements of P (i.e., for all $\alpha, \beta < \gamma$, if $\alpha < \beta$, then $p_\beta \leq p_\alpha$) there is $p \in \mathsf{P}$ such that for all $\alpha < \gamma$, $p \leq p_\alpha$.*

Claim 2.3.5 E *is a $<\kappa$-closed poset.*

Proof. Suppose that $\langle h_\alpha : \alpha < \gamma \rangle$, with $\gamma < \kappa$, is a decreasing sequence of elements of E. Let $Z = \bigcup_{\alpha<\gamma} \operatorname{dom}(h_\alpha)$. By regularity of κ, $|Z| < \kappa$. So, since r.o. (S) is a complete and κ-cc Boolean algebra, by Lemma 2.3.3, we have that there is a complete subalgebra (Z) of r.o. (S) including Z of size less than κ. Since $\{h_\alpha : \alpha < \gamma\}$ is a family of complete embeddings pairwise compatible, $h = \bigcup_{\alpha<\gamma} h_\alpha$ is an embedding from Z into r.o. $(Coll\,(\omega, <\kappa))$. But then, we may extend h to a complete embedding h^* from (Z) into r.o. $(Coll\,(\omega, <\kappa))$ (see [J2], 25.12). Then $h^* \in \mathsf{E}$ and for every $\alpha < \gamma$, $h \leq h_\alpha$. ■

Lemma 2.3.6 *Let H be a E-generic filter over V. Then $e = \bigcup H$ is a complete embedding from* r.o. (S) *into* r.o. $(Coll\,(\omega, < \kappa))$*.*

Proof. Let H be a E-generic filter over V and $e = \bigcup H$. Clearly e is an embedding from $\mathrm{dom}\,(e)$ into r.o. $(Coll\,(\omega, < \kappa))$. So, we only need to show that $\mathrm{dom}\,(e) =$ r.o. (S) and that for every maximal antichain $A \subseteq$ r.o. (S), $e"A$ is a maximal antichain of r.o. $(Coll\,(\omega, < \kappa))$.

To see this it will suffice to show that for every $X \subseteq$ r.o. (S) with $|X| < \kappa$, $D_X = \{h \in \mathsf{E} : X \subseteq \mathrm{dom}\,(h)\}$ is a dense subset of E. Since then, on one hand, for every $u \in$ r.o. (S), $H \cap D_{\{u\}} \neq \emptyset$, and so $\mathrm{dom}\,(e) =$ r.o (S). On the other hand, since r.o. (S) is κ-cc, for every maximal antichain $A \subseteq$ r.o. (S), D_A is a dense subset of E. But, if $A \subseteq \mathrm{dom}\,(h)$, then, since $\mathrm{dom}\,(h) \lessdot$ r.o. (S), $h"A$ is a maximal antichain of r.o. $(Coll\,(\omega, < \kappa))$. But $e"A = h"A$. So, e is also a complete embedding.

Now, as in Claim 2.3.5, using Lemma 2.3.3, it is clear that for every $X \subseteq$ r.o. (S) of size less than κ, D_X is a dense subset of E. ■

Suppose that G is a S-generic filter over V. Let H be a E-generic filter over $V\,[G]$. Since E is a poset in V, $G \times H$ is a $\mathsf{S} \times \mathsf{E}$-generic filter over V. Since $\mathsf{S} \times \mathsf{E} \cong \mathsf{E} \times \mathsf{S}$, $V\,[G \times H] = V\,[H \times G] = V\,[H]\,[G]$. Now, since E is a $< \kappa$-closed poset, E does not add new reals. i.e., $\mathsf{R} \cap V[G] = \mathsf{R} \cap V[H][G]$ (see [Ku], VII.6.14). Therefore, all new reals in $V\,[H]\,[G]$ have been added by S. So, for every $p \in \mathsf{S}$, every formula $\varphi\,(v)$ and every $x \in V$,

$$V \models \text{“}p \Vdash_{\mathsf{S}} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”} \text{ iff } V[H] \models \text{“}p \Vdash_{\mathsf{S}} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”}.$$

Similarly, for every $p \in Coll\,(\omega, < \kappa)$, every formula $\varphi\,(v)$ and every $x \in V$,

$$V \models \text{“}p \Vdash_{Coll(\omega,<\kappa)} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”} \text{ iff } V[H] \models \text{“}p \Vdash_{Coll(\omega,<\kappa)} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”}.$$

Let $e \in V[H]$ be the generic complete embedding from S into $Coll\,(\omega, < \kappa)$ given by Lemma 2.3.6 and let i be the canonical embedding from $Coll\,(\omega, < \kappa)$ into $\mathsf{S} = Coll\,(\omega, < \kappa) * \dot{\mathsf{P}}$. Then,

Claim 2.3.7 $V[H] \models \text{“} \Vdash_{\mathsf{S}} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”}$ *iff* $V[H] \models \text{“} \Vdash_{Coll(\omega,<\kappa)} \varphi\,(\check{x})^{L(\mathsf{R})}\text{”}$.

Proof. We show it by induction on the complexity of formulas:

$\underline{n = 0}$: By absoluteness of Σ_0 formulas and the fact that for every S-generic filter G over $V[H]$, there exists a $Coll\,(\omega, < \kappa)$-generic filter C over $V[H]$ such that $e^{-1}\,(C) = G$ and, hence, $V[H][G] = V[H][e^{-1}\,(C)] \subseteq V[H][C]$.

$\underline{n + 1}$: Let $\varphi\,(v) = \exists y \neg\psi\,(v, y)$ where $\psi\,(v, y)$ is a Σ_n formula. Suppose that

$$V[H] \models \text{“} \Vdash_{\mathsf{S}} \exists y \neg\psi\,(\check{x}, y)^{L(\mathsf{R})}\text{”}.$$

So, there exists a S-name $\dot{b}$ in $V[H]$ such that $V[H] \models \text{“} \Vdash_{\mathsf{S}} \neg\psi(\check{x}, \dot{b})^{L(\mathsf{R})}\text{”}$. But then, by inductive hypothesis, $V[H] \models \text{“} \Vdash_{Coll(\omega,<\kappa)} \neg\psi(\check{x}, e_*(\dot{b}))^{L(\mathsf{R})}\text{”}$, where for every S-name τ,

$$e_*\,(\tau) = \{\langle e\,(p)\,, e_*\,(\sigma)\rangle : \langle p, \sigma\rangle \in \tau\}$$

(see [Ku], VII.7.12). Therefore,

$$V[H] \models \text{“} \Vdash_{Coll(\omega,<\kappa)} \exists y \neg\psi(\check{x}, y)^{L(\mathsf{R})} \text{”}.$$

We show that if $V[H] \models$ “$\Vdash_{Coll(\omega,<\kappa)} \exists y \neg\psi(\check{x}, y)^{L(\mathsf{R})}$”, then $V[H] \models$ “$\Vdash_{\mathsf{S}} \exists y \neg\psi(\check{x}, y)^{L(\mathsf{R})}$” as above but using the complete embedding i from $Coll(\omega, <\kappa)$ into S. ■

But, by almost homogeneity of $Coll(\omega, <\kappa)$, we get that for every formula $\varphi(v)$ and every $x \in V$, $\Vdash_{Coll(\omega,<\kappa)}$ “$\varphi(\check{x})$” or $\Vdash_{Coll(\omega,<\kappa)}$ “$\neg\varphi(\check{x})$”. So, by Claim 2.3.7, for every formula $\varphi(v)$ with all its parameters in V and every $x \in V$

$$\begin{aligned} V \models \text{“} \Vdash_{Coll(\omega,<\kappa)} \varphi(\check{x})^{L(\mathsf{R})} \text{”} \quad &\text{iff} \quad V[H] \models \text{“} \Vdash_{Coll(\omega,<\kappa)} \varphi(\check{x})^{L(\mathsf{R})} \text{”} \\ &\text{iff} \quad V[H] \models \text{“} \Vdash_{\mathsf{S}} \varphi(\check{x})^{L(\mathsf{R})} \text{”} \\ &\text{iff} \quad V \models \text{“} \Vdash_{\mathsf{S}} \varphi(\check{x})^{L(\mathsf{R})} \text{”} \end{aligned}$$

This ends the proof of Theorem 2.3.2. ■

Note that, in the Theorem 2.3.2, if all the parameters of the formula are reals, then by the Factor Lemma for the Levy-collapse, we can assume that they are in V_0. So, $L(\mathsf{R})^{V_0}$ and $L(\mathsf{R})^{V_0[G]}$ satisfy the same projective sentences with reals in V_0 as parameters.

Definition 2.3.8 *Let P be a forcing notion, let V be a model of ZFC^* and let $n \geq 1$. V is* $\underset{\sim}{\Sigma}^1_n$-absolute for P *if for every $\underset{\sim}{\Sigma}^1_n$ formula $\varphi(x_1, \ldots, x_n)$ with parameters in V and for every $b_1, \ldots, b_n \in \mathsf{R} \cap V$,*

$$V \models \varphi(b_1, \ldots, b_n) \text{ iff } V[G] \models \varphi(b_1, \ldots, b_n),$$

for every P-generic filter G over V. V is projective absolute for P *iff for every $n \geq 1$, V is $\underset{\sim}{\Sigma}^1_n$ absolute for P. Finally, V is* projective absolute for ccc forcing notions *iff for every ccc poset $\mathsf{P} \in V$, V is projective absolute for P.*

Thus, from the remark following the proof of Theorem 2.3.2, we obtain the following corollary:

Corollary 2.3.9 *Let κ be a weakly-compact cardinal and let C be a $Coll(\omega, <\kappa)$-generic filter over V. Then $V[C]$ is projective absolute for ccc forcing notions.* ■

Lemma 2.3.10 *Let κ be a weakly-compact cardinal and let C_0 be a $Coll(\omega, <\kappa)$-generic filter over V. Suppose that $\mathsf{P} \in V[C_0]$ is a ccc poset and G_0 is a P-generic filter over $V[C_0]$. Then $V[C_0][G_0]$ is projective absolute for ccc forcing notions.*

Proof. Let $V_0 = V[C_0][G_0]$ and suppose that $\mathsf{Q} \in V_0$ is a ccc poset, H is a Q-generic filter over V_0 and $V_0[H] \models \varphi$, where φ is a $\Sigma^1_n(a)$ sentence with parameter $a \in V_0$. We need the following fact:

Fact 2.3.11 *There exists a $Coll\,(\omega, <\kappa)$-generic filter C_1 over V such that $a \in V\,[C_1]$ and $V\,[C_1] \subseteq V\,[C_0]\,[G_0]$.*

Proof. By a Skolem argument, we may assume that $V\,[C_0] \models |\mathsf{P}| \leq \aleph_1$. Let $\dot{a}$ be a simple $Coll\,(\omega, <\kappa) * \dot{\mathsf{P}}$-name for a. Since $\dot{a}$ is of size less than κ and κ is weakly-compact, by 2.3.3, we can find $\mathsf{Q}' \lessdot Coll\,(\omega, <\kappa) * \mathsf{P}$ such that $|\mathsf{Q}'| = \lambda < \kappa$, $\dot{a}$ is a simple Q'-name and there is a dense embedding π from Q' into $Coll\,(\omega, <\lambda + 1)$. As $(C_0 * G_0) \cap \mathsf{Q}'$ is a Q'-generic filter, $\pi\text{''}\,(C_0 * G_0 \cap \mathsf{Q}')$ is a $Coll\,(\omega, <\lambda+1)$-generic filter and $a = \dot{a}\,[C_0 * G_0] = \dot{a}\,[(C_0 * G_0) \cap \mathsf{Q}'] = \pi_*\,(\dot{a})\,[\pi\text{''}\,(C_0 * G_0 \cap \mathsf{Q}')]$. ■

We continue with the proof of Lemma 2.3.10. Fix C_1 as in 2.3.11. So, $a \in V\,[C_1] \subseteq V\,[C_0]\,[G_0]$. Then there exists a ccc generic extension $V\,[C_1]\,[G_1]$ of $V\,[C_1]$ such that $V\,[C_1]\,[G_1] = V\,[C_0]\,[G_0]$ (see, [J2], 25.3). Hence, $V\,[C_1]\,[G_1]\,[H] \models \varphi$. Now, $V\,[C_1]\,[G_1]\,[H] = V[C_1][G_1 * H]$ is a single ccc extension of $V\,[C_1]$. So, since, $a \in V\,[C_1]$, by Corollary 2.3.9, $V\,[C_1] \models \varphi$. Again by 2.3.9, $V\,[C_1]\,[G_1] \models \varphi$. But $V\,[C_1]\,[G_1] = V_0$. Hence, $V_0 \models \varphi$. ■

Corollary 2.3.12 *Let κ be a weakly-compact cardinal and let C be a $Coll\,(\omega, <\kappa)$-generic filter over V. Suppose $\mathsf{P} \in V\,[C]$ is a ccc poset and G is a P-generic filter over $V\,[C]$. Let $V_0 = V\,[C]\,[G]$. Suppose that $\mathsf{Q}_0, \mathsf{Q}_1 \in V_0$ are ccc posets and Q_1 is projective. Then for every Q_0-generic filter H over V_0,*

$$\mathsf{Q}_1^{V_0} \lessdot \mathsf{Q}_1^{V_0[H]}.$$

Proof. Since V_0 is projective absolute for ccc forcing notions. ■

Definition 2.3.13 *Let P be a poset. P is* indestructible-ccc *iff for every ccc poset Q, $\Vdash_{\mathsf{Q}}$ "P is a ccc poset".*

The next theorem shows that after Levy-collapsing a weakly-compact cardinal to ω_1, all projective ccc posets are indestructible-ccc, i.e., they remain ccc in all ccc forcing extensions.

Theorem 2.3.14 *Let κ be a weakly-compact cardinal and let C_0 be a $Coll\,(\omega, <\kappa)$-generic filter over V. If $\mathsf{P}, \mathsf{Q} \in V\,[C_0]$ are ccc posets and Q is projective, then for every P-generic filter G over $V\,[C_0]$,*

$$V\,[C_0]\,[G] \models \text{“}\mathsf{Q} \textit{ is a ccc poset”}.$$

Proof. Fix a projective ccc poset Q in $V\,[C_0]$. By the Factor Lemma for the Levy-collapse we may assume the parameters of the definition of Q are all in the ground model. Further, since $Coll\,(\omega, <\kappa)$ is an almost homogenous poset, we may assume that

$$\Vdash_{Coll(\omega,<\kappa)} \text{“}\dot{\mathsf{Q}} \text{ is a ccc poset”}.$$

Let P be a ccc poset in $V\,[C_0]$ and suppose $\dot{A} = \{\tau_i : i < \kappa\}$ is a $Coll\,(\omega, <\kappa) * \dot{\mathsf{P}}$-name for an uncountable antichain of $\dot{\mathsf{Q}}$.

Let $\mathsf{S} = Coll\,(\omega, <\kappa) * \dot{\mathsf{P}}$ and let E be the set of all complete embeddings from complete subalgebras of the algebra r.o. (S) of size less than κ into r.o.$(Coll\,(\omega, <\kappa))$ ordered by inclusion. By Claim 2.3.5, E is a $<\kappa$-closed poset. So, forcing with E does not add new reals. Hence, projective statements are absolute between the ground model and the E-generic extension.

Let $V_0 = V[H][C_0 * G]$ be a $\mathsf{E} \times \mathsf{S}$-generic extension of V. That is, H is E-generic over V, C_0 is $Coll\,(\omega, <\kappa)$-generic over $V[H]$ and G is $\dot{\mathsf{P}}[C_0]$-generic over $V[H][C_0]$. By the Product Lemma (see [Ku], VIII.1.4), $V_0 = V[C_0 * G][H]$. So, since $\dot{A} \in V$ and projective absoluteness holds between $V[C_0 * G]$ and V_0, we have:

$$V_0 \models \text{“}\dot{A}[C_0 * G] \text{ is an uncountable antichain of } \dot{\mathsf{Q}}[C_0 * G]\text{”}.$$

Let $e \in V[H]$ be the generic complete embedding from S into $Coll\,(\omega, <\kappa)$ given by H. Then there is a $Coll\,(\omega, <\kappa)$-generic C_1 over $V[H]$ such that $C_0 * G = \pi^{-1}(C_1)$. Note that $V[H][C_1]$ is a ccc generic extension of V_0 (see [J2], 25.3) and, hence, the projective formulas are absolute between V_0 and $V[H][C_1]$.

Let $e_*(\dot{A}) = \langle e_*(\tau_i) : i < \kappa \rangle \in V[H]$, the e-image of $\dot{A}$. For every $i < \kappa$, $V[H] \models$ “ $\Vdash_{\mathsf{S}} \tau_i \in \dot{\mathsf{Q}}$”. Thus, since “$\tau_i \in \dot{\mathsf{Q}}$” is a projective formula with τ_i as the only possible non-standard term-parameter and since the projective formulas are absolute between V_0 and $V[H][C_1]$,

$$V[H] \models \text{“ } \Vdash_{Coll(\omega,<\kappa)} e_*(\tau_i) \in \dot{\mathsf{Q}}\text{”}.$$

Hence, for every $i < \kappa$,

$$V[H][C_1] \models \text{“}e_*(\tau_i)[C_1] \in \dot{\mathsf{Q}}[C_1]\text{”}.$$

Since $V[C_1] \models$ “$\dot{\mathsf{Q}}[C_1]$ is a ccc poset”, E is a σ-closed poset and no σ-closed poset can kill the ccc-ness of any ccc poset, $V[C_1][H] \models$ “$\dot{\mathsf{Q}}[C_1]$ is a ccc poset”. So, by the Product Lemma,

$$V[H][C_1] \models \text{“}\dot{\mathsf{Q}}[C_1] \text{ is a ccc poset”}.$$

Thus we can find $i, j < \kappa$ such that

$$V[H][C_1] \models \text{“}e_*(\tau_i)[C_1], e_*(\tau_j)[C_1] \text{ are compatible”}.$$

But, since e is a complete embedding, $e_*(\tau_i)[C_1] = \tau_i[C_0 * G]$ and $e_*(\tau_j)[C_1] = \tau_j[C_0 * G]$. So, since Q is a projective poset, the compatibility relation in Q is also projective, and, by projective absoluteness between V_0 and $V[H][C_1]$,

$$V_0 \models \text{“}\tau_i[C_0 * G], \tau_j[C_0 * G] \text{ are compatible”}.$$

A contradiction. ∎

Corollary 2.3.15 *Let κ be a weakly-compact cardinal and let C_0 be a $Coll\,(\omega, <\kappa)$-generic filter over V, $\mathsf{P} \in V[C_0]$ is a ccc poset and G_0 is a P-generic filter over $V[C_0]$. Let $V_0 = V[C_0][G_0]$. Suppose that $\mathsf{Q}_0, \mathsf{Q}_1 \in V_0$ are ccc posets and Q_1 is projective. Then for every Q_0-generic filter H over V_0,*

$$V_0[H] \models \text{“}\mathsf{Q}_1 \text{ is a ccc poset”}.$$

Proof. Fix a projective and ccc poset Q_1 in V_0. Let $a \in \mathsf{R} \cap V_0$ the parameter of the definition of Q_1. By 2.3.11, we can find a $Coll\,(\omega, <\kappa)$-generic filter C_1 over V such that $a \in V\,[C_1]$ and $V\,[C_1] \subseteq V_0$. Now,

$$V\,[C_1] \models \text{“}\mathsf{Q}_1 \text{ is a projective poset”}.$$

Since $V_0 = V\,[C_1]\,[G_1]$ is a ccc extension of $V\,[C_1]$, V_0 is a projective absolute extension of $V\,[C_1]$ and, hence,

$$V\,[C_1] \models \text{“}\mathsf{Q}_1 \text{ is a ccc poset”}.$$

Since $V_0\,[H] = V\,[C_1]\,[G_1 * H]$ is a single ccc forcing extension of $V\,[C_1]$, by Theorem 2.3.14 we know that

$$V_0\,[H] \models \text{“}\mathsf{Q}_1 \text{ is a ccc poset”}. \quad \blacksquare$$

Lemma 2.3.16 *Let* $\mathsf{P}, \mathsf{Q} \in V$ *be ccc posets and suppose that* Q *is projective. Assume that*

1. *For every* P*-generic filter* G *over* V, $\mathsf{Q}^V \lessdot \mathsf{Q}^{V[G]}$ *and*

2. $\Vdash_{\mathsf{P}}$ *“*Q *is ccc”,*

then $\Vdash_{\mathsf{Q}}$ *“*P *is ccc”.*

Proof. Suppose otherwise. So, there exists $q \in \mathsf{Q}$ such that $q \Vdash_{\mathsf{Q}}$ “P is not ccc”. Hence, $\mathsf{Q} * \mathsf{P}$ is not ccc. Since $\mathsf{P}, \mathsf{Q} \in V$, $\mathsf{Q} * \mathsf{P} \cong \mathsf{Q} \times \mathsf{P} \cong \mathsf{P} \times \mathsf{Q}$, $\mathsf{P} \times \mathsf{Q}$ is not ccc. But, since for every P-generic filter G over V, $\mathsf{Q}^V \lessdot \mathsf{Q}^{V[G]}$, we have $\mathsf{P} \times \mathsf{Q} \lessdot \mathsf{P} * \mathsf{Q}$. Hence, $\mathsf{P} * \mathsf{Q}$ is not ccc. A contradiction, since P is ccc and $\Vdash_{\mathsf{P}}$ “Q is ccc”. $\blacksquare$

Definition 2.3.17 *Let* $\varphi\,(x)$ *be a formula of the language of Set Theory. We say that* $\varphi\,(x)$ is preserved under direct limits of finite support iterations of ccc forcing notions *if, whenever* P *is the direct limit of* $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \nu \rangle$ *a finite support iteration of ccc forcing notions such that for every* $\alpha < \nu$, $V[G_\alpha] \models \varphi(A)$, *then* $V\,[G] \models \varphi(A)$, *where* G *is a* P*-generic filter over* V.

Theorem 2.3.18 *Let* κ *be a weakly-compact cardinal and let* $V_0 = L\,[C]$, *where* C *is a* $Coll\,(\omega, <\kappa)$*-generic filter over* L. *Suppose that* $\varphi\,(x)$ *is a formula of the language of Set Theory such that:*

1. *For every* $X \subseteq \omega^\omega$, *there are posets* $\mathsf{P}_0^X, \ldots, \mathsf{P}_n^X$ *such that*

$$ZFC \vdash (\varphi\,(X) \leftrightarrow \mathsf{P}_0^X, \ldots, \mathsf{P}_n^X \textit{ are ccc posets}).$$

2. *For every* $X \subseteq \omega^\omega$, $\varphi\,(X)$ *is preserved under direct limits of finite support iterations of ccc forcing notions.*

Moreover, suppose that there exists a ccc generic extension V_1 of V_0 and $A \in V_1$ such that $V_1 \models \varphi(A)$. Then there is a ccc poset $\mathsf{P} \in V_1$ such that whenever G is a P-generic filter over V_1,

$$V_1\,[G] \models MA(Proj) \wedge \neg CH \wedge \varphi\,(A)\,.$$

Proof. Let λ be a regular uncountable cardinal in V_1 which is not the successor of a cardinal with cofinality ω. Let P the poset to force $MA\,(Proj) + 2^{\aleph_0} = \lambda$ defined in V_1 as in 2.2.14. Let G be a P-generic filter over V_1. Then $V_1\,[G] \models MA\,(Proj) \wedge 2^{\aleph_0} = \lambda$. Thus it only remains to prove that $V_1\,[G] \models \varphi(A)$. Since P is the direct limit of $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \lambda \rangle$ we prove this by showing, by induction on $\alpha \leq \lambda$, that $V_1[G_\alpha] \models \varphi(A)$.

$\underline{\alpha = 0}$: Obvious, since P_0 is the trivial poset.

$\underline{\alpha = \beta + 1}$: By inductive hypothesis, we have that

$$V_1[G_\beta] \models \varphi(A).$$

We also have that

$$V_1[G_\beta] \models \text{“}\dot{\mathsf{Q}}_\beta[G_\beta] \text{ is a projective ccc poset”}.$$

So, by 2.3.12, for every $i \leq n$ and every P_i^A-generic filter H_i over $V_1[G_\beta]$,

$$\dot{\mathsf{Q}}_\beta[G_\beta]^{V_1[G_\beta]} \lessdot \dot{\mathsf{Q}}_\beta[G_\beta]^{V_1[G_\beta][H_i]}.$$

Further, by 2.3.15, for every $i \leq n$, $V_1\,[G_\beta] \models$ “$\Vdash_{\mathsf{P}_i^A} \dot{\mathsf{Q}}_\beta[G_\beta]$ is a ccc poset”. So, by 2.3.16, for every $i \leq n$,

$$V_1[G_\alpha] \models \text{“}\mathsf{P}_i^A \text{ is a ccc poset”}.$$

But then, by condition (1) of this theorem,

$$V_1[G_\alpha] \models \varphi(A).$$

$\underline{\alpha \text{ limit}}$: Since P_α is the direct limit of $\langle \mathsf{P}_\beta, \dot{\mathsf{Q}}_\beta : \beta < \alpha \rangle$, by condition (2) of this theorem,

$$V_1\,[G_\alpha] \models \varphi(A). \quad \blacksquare$$

The next lemma gives a sufficient condition for a formula to be preserved under direct limits of finite support iterations of ccc forcing notions whenever we can relate the satisfaction of the formula to the nonexistence of certain homogenous sets.

Definition 2.3.19 *Let X be a set. A* 2-coloring of X *is a map π from $X \times X$ onto the set $\{0,1\}$. A set $Y \subseteq X$* is homogeneous for (the 2-coloring) π *iff π"$Y \times Y = \{i\}$, where i is either 0 or 1.*

Lemma 2.3.20 *Let $X \in V$ be an uncountable set and let $\pi \in V$ be a 2-coloring of X. Let P be the direct limit of $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \nu \rangle$, a finite support iteration of ccc forcing notions. Suppose that for every P-generic filter G over V and for every $\alpha < \nu$:*

$$V[G_\alpha] \models \neg (\exists Y \subseteq X)(|Y| = \aleph_1 \wedge Y \text{ is homogeneous})$$

Then $V[G] \models \neg (\exists Y \subseteq X)(|Y| = \aleph_1 \wedge Y$ is homogeneous).

Proof. Suppose the lemma is true for $\alpha < \nu$. Towards a contradiction, let p and $\dot{Y}$ be such that $p \Vdash$ "$\dot{Y} \subseteq X \wedge |\dot{Y}| = \check{\aleph}_1 \wedge (\forall xy \in \dot{Y})(\pi(x,y)) = 0$".

For each $x \in X$, choose $q_x \in \mathsf{P}$ so that $q_x \leq p$ and $q_x \Vdash$ "$x \in \dot{Y}$", if there is such. Otherwise, let $q_x = 0$. Without loss of generality, $B = \{q_x : x \in X \wedge q_x \neq 0\}$ is uncountable. By the Δ-system Lemma (see [Ku], II.1.5), there is $B' \subseteq B$ uncountable such that $\{\mathrm{supp}(q_x) : q_x \in B'\}$ forms a Δ-system with root r. Pick $\delta < \nu$ such that $r \subseteq \delta$.

Fix $\{q_{x_\alpha} : \alpha < \lambda\}$ an enumeration of B'

Claim 2.3.21 *There exists $p' \leq p$ such that $p' \Vdash$ "$\{\alpha : q_{x_\alpha} \in \dot{G}\}$ is uncountable", where $\dot{G}$ is the canonical P-name for the generic.*

Proof. Otherwise, let

$$C = \{\eta : (\exists p' \leq p)(p' \Vdash \text{“}\mathrm{sup}(\{\alpha : q_{x_\alpha} \in \dot{G}\}) = \eta\text{”})\}.$$

Since P is ccc, C is countable. Let $\eta = \max(C)$. So $\eta < \omega_1$. Now, $q_{x_{\eta+1}} \Vdash$ "$\eta + 1 \in \{\alpha : q_{x_\alpha} \in \dot{G}\}$". Hence $q_{x_{\eta+1}} \Vdash$ "$\mathrm{sup}(\{\alpha : q_{x_\alpha} \in \dot{G}\}) > \eta$". A contradiction. This proves the claim. ■

Fix $p' \leq p$ as in the claim. Suppose G_β is a P_β-generic filter with $p' \in G_\beta$ for some β, $\delta \leq \beta < \nu$. The set $\{\alpha : q_{x_\alpha} \restriction \beta \in G_\beta\}$ is uncountable. For otherwise we can extend G_β to a generic G for the whole iteration and then $V[G] \models \{\alpha : q_{x_\alpha} \in \dot{G}\}$ is countable.

Since $V[G_\beta] \models \neg(\exists Y \subseteq X)(|Y| = \aleph_1 \wedge Y$ is 0-homogeneous), we can find $x, y \in X$ such that $q_x \restriction \beta, q_y \restriction \beta \in G_\beta$ and either $\pi(x,y) = 1$ or $\pi(y,x) = 1$. But, since $r \subseteq \beta$, q_x and q_y are compatible. So, we can find $q \leq q_x, q_y$ so that $q \Vdash$ "$q_x, q_y \in \dot{G}$ and either $\pi(x,y) = 1$ or $\pi(y,x) = 1$". This proves the Lemma 2.3.20. ■

2.3.2 Suslin trees

Definition 2.3.22 *A* Suslin tree *T is a tree T (i.e., a partial order $T = \langle T, \leq_T \rangle$ where for every $x \in T$, $\{y \in T : y <_T x\}$ is well-ordered by $<_T$) such that $|T| = \aleph_1$ and every chain and every antichain (i.e., every set of incomparable elements) of T are countable.*

Without loss of generality, we can restrict ourselves to normal Suslin trees. A *normal tree* is a tree T such that $\mathrm{ht}(T) = \omega_1$, T has a unique least point (the root), each level of T is at most countable, if $x \in T$ is not maximal in T, then there exist infinite many $y > x$ at each higher level and, if $\beta < \omega_1$ is a limit ordinal, x, y belong to level β of T and $\{z \in T : z < x\} = \{z \in T : z < y\}$, then $x = y$.

It is a well-known fact that Martin's axiom implies that there are no Suslin trees (see [J2], 23.1, or [Ku], II.5.14). However,

Theorem 2.3.23 *Suppose $Con(ZF + \exists\kappa(\kappa$ is weakly-compact$))$. Then $Con(ZFC + MA(Proj) + \neg CH +$ There exists a Suslin tree$)$.*

Proof. Let κ be a weakly-compact cardinal. Using Theorem 2.3.18 we only need to show that: (1) there is a ccc extension V_1 of $V_0 = L[C]$, where C is any $Coll\,(\omega, <\kappa)$-generic filter over L, and a normal Suslin tree $T \in V_1$, (2) there is a poset P_T such that T is a Suslin tree iff P_T is ccc, and (3) there exists a 2-coloring π of T such that T is a Suslin tree iff there are no homogeneous uncountable subsets of T for π.

It is well-known that the Tennenbaum poset to add a generic Suslin tree is a ccc poset. (See [T] or [J2], Exercise 22.9). Moreover, S. Todorčević has showed that forcing with Cohen poset adds a Suslin tree (see [B2]).

Let $\mathsf{P}_T = \langle T, \geq_T \rangle$. Since T is a normal Suslin tree, P_T is a ccc poset. On the other hand, suppose that T is a normal ω_1-tree and P_T is ccc. Then, since being an antichain of P_T is the same as being an antichain of T, every antichain of T is countable. Moreover, if $B \subseteq T$ is a branch, then, since T is a normal tree, for every $x \in B$ there exists $y_x \in T$ such that $x <_T y_x$ and $y_x \notin B$. But, $\{y_x : x \in B\}$ is an antichain of T. Thus, T is a Suslin tree iff P_T is a ccc poset.

Let $\pi : T \times T \longrightarrow \{0, 1\}$ be a 2-coloring of T defined by:

$$\pi\,(x, y) = \begin{cases} 0, & \text{if } x \not\leq y \text{ and } y \not\leq x \\ 1, & \text{otherwise} \end{cases}$$

Then it is easy to see that $A \subseteq T$ is an antichain iff $\pi"A \times A = \{0\}$. ■

H. Woodin has remarked that the existence of a weakly-compact cardinal is not necessary in order to find a model of $ZFC + MA(Proj) + \neg CH +$ "There exists a Suslin tree": For every poset P and every model of ZFC, let V^{P} denote any P-generic extension of V. Let $MA(Proj)^L$ denote the iteration in length ω_2 for getting $MA(Proj) + 2^{\aleph_0} = \aleph_2$ (see 2.2.14) as defined in L. Let P_T be the Jech's poset for adding a generic Suslin tree T with countable conditions ([J1], see also [J2], Theorem 48). We want to show that $L^{\mathsf{P}_T * MA(Proj)^L}$ is a model of $MA(Proj) + 2^{\aleph_0} = \aleph_2 +$ "There exists a Suslin tree". First note that, since P_T, is a σ-closed poset, it does not add new reals. So, in L^{P_T}, $MA(Proj)^L = MA(Proj)^{L^{\mathsf{P}_T}}$, the iteration in length ω_2 for getting $MA\,(Proj) + 2^{\aleph_0} = \aleph_2$ defined in L^{P_T}. Hence,

$$L^{\mathsf{P}_T * MA(Proj)^L} \Vdash MA(Proj) \wedge 2^{\aleph_0} = \aleph_2.$$

Further, since $\mathsf{P}_T * T$ is also an σ-closed poset and a σ-closed poset cannot kill the countable chain condition of any poset, $MA(Proj)^L$ it is still a ccc poset in $L^{\mathsf{P}_T * T}$. So, by 2.3.16, $L^{\mathsf{P}_T * MA(Proj)^L} \Vdash$ "T is a ccc poset" and hence

$$L^{\mathsf{P}_T * MA(Proj)^L} \Vdash \text{"}T \text{ is a Suslin tree"}.$$

2.3.3 Gaps in ω^ω

Definition 2.3.24 *For $f, g \in \omega^\omega$, we let $f <^* g$ if for all but finitely many $n \in \omega$, $f(n) < g(n)$.*

Suppose γ and δ are ordinals. A (γ, δ)-pregap in $\langle \omega^\omega, <^* \rangle$ is a sequence $\langle g_\alpha, f_\beta : \alpha < \gamma, \beta < \delta \rangle$ such that for every $\alpha < \alpha' < \gamma$, $\beta < \beta' < \delta$ we have that $g_\alpha <^* g_{\alpha'} <^* f_{\beta'} <^* f_\beta$.

Further, $\langle g_\alpha, f_\beta : \alpha < \gamma, \beta < \delta \rangle$ is a (γ, δ)-gap in $\langle \omega^\omega, <^* \rangle$ if for no $h \in \omega^\omega$ it is true that for all $\alpha < \gamma$, $\beta < \delta$, $g_\alpha <^* h <^* f_\beta$. We call such an h a split.

We shall mainly interested in (ω_1, ω_1)-gaps. Thus, henceforth, *pregap* and *gap* will mean (ω_1, ω_1)-pregap and (ω_1, ω_1)-gap, respectively. We need the following facts about pregaps and gaps in ω^ω that can be found in [B-W] or in [S].

Definition 2.3.25 *Given a pregap $G = \langle g_\alpha, f_\alpha : \alpha < \omega_1 \rangle$, let $\mathsf{P}_G \subseteq \omega_1^{<\omega} \times \omega^{<\omega}$ the following poset:*

- $\langle \alpha_0, \ldots, \alpha_{n,} s \rangle \in \mathsf{P}_G$ *iff*

$$(\forall k \geq \mathrm{dom}(s)) \left(\max \{ g_{\alpha_i}(k) : i \leq n \} \leq \min \{ f_{\alpha_i}(k) : i \leq n \} \right).$$

- *We let $\langle \alpha_0, \ldots, \alpha_{n,} s \rangle \leq \langle \beta_0, \ldots, \beta_{m,} t \rangle$ iff:*

  1. $\{\beta_0, \ldots, \beta_m\} \subseteq \{\alpha_0, \ldots, \alpha_n\}$
  2. $t \subseteq s$
  3. *For all $k \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$,*

$$\max \left\{ g_{\beta_i}(k) : i \leq n \right\} \leq s(k) \leq \min \left\{ f_{\beta_i}(k) : i \leq n \right\}.$$

Fact 2.3.26 *Forcing with P_G splits G. More precisely, if H is a P_G-generic filter, then $h = \bigcup \{ s \in \omega^{<\omega} : (\exists \alpha_0 \ldots \alpha_n \in \omega_1) (\langle \alpha_0, \ldots, \alpha_n, s \rangle \in H) \}$ splits G. Moreover, H can be recovered from h. Indeed, $\langle \alpha_0, \ldots, \alpha_n, s \rangle \in H$ iff*

$$s \subseteq h \wedge (\forall k \geq \mathrm{dom}(s)) \left(\max \{ g_{\alpha_i}(k) : i \leq n \} \leq h(k) \leq \min \{ f_{\alpha_i}(k) : i \leq n \} \right). \blacksquare$$

Definition 2.3.27 *A gap G is* strong *if it cannot be split in any ccc forcing extension.*

Lemma 2.3.28 (K. Kunen) *Assume that $G = \langle g_\alpha, f_\alpha : \alpha < \omega_1 \rangle$ is a pregap such that for every $\alpha < \omega_1$, $g_\alpha \leq f_\alpha$. i.e., for every $n \in \omega$, $g_\alpha(n) \leq f_\alpha(n)$. Further, suppose that if $\alpha \neq \beta$, then either g_α £ f_β or g_β £ f_α. Then G is a strong gap.*

Proof. First we will show that G is a gap. Suppose that h splits G. For every $\alpha < \omega_1$, let $\pi(\alpha)$ be the least $n \in \omega$ such that for every $k \geq n$, $g_\alpha(k) \leq h(k) \leq f_\alpha(k)$. Since π is a function from ω_1 into ω, without loss of generality, we may assume that there exists $n_0 \in \omega$ such that for all $\alpha < \omega_1$, $\pi(\alpha) = n_0$. Since there are only countable many ordered pairs $\langle g_\alpha$ ¹ n_0, f_α ¹ $n_0 \rangle$, we also may assume, without loss of generality, that for every $\alpha, \beta < \omega_1$, g_α ¹ $n_0 = g_\beta$ ¹ n_0 and f_α ¹ $n_0 = f_\beta$ ¹ n_0. But then, since $\alpha \neq \beta$ implies $g_\alpha \leq f_\beta$ and $g_\beta \leq f_\alpha$, we get a contradiction. Therefore G is a gap.

It is clear, that, as long as G is not countable, this argument works. So, G is an strong gap. $\blacksquare$

Lemma 2.3.29 (H. Woodin) *If G is a pregap, then P_G is ccc iff G is not a strong gap.*

Proof. ($\Rightarrow$) Clearly, if G is an strong gap, then P_G is not a ccc poset.

($\Leftarrow$) Suppose that P_G is not a ccc poset. Let $\{p_\gamma : \gamma < \omega_1\}$ be an uncountable antichain in P_G.

Let $p_\gamma = \langle \alpha_0^\gamma, ..., \alpha_n^\gamma, s\rangle$, where $n \in \omega$ and $s \in \omega^{<\omega}$ has been stabilized. However, since if $\gamma \neq \gamma'$, p_γ and $p_{\gamma'}$ are incompatible, $\langle \alpha_0^\gamma, ..., \alpha_n^\gamma, \alpha_0^{\gamma'}, ..., \alpha_n^{\gamma'}, s\rangle$ is not a condition.

Therefore, given $\gamma \neq \gamma'$, there exists $k \geq m = \mathrm{dom}\,(s)$, else $\max_{i\leq n}(g_{\alpha_i^\gamma}(k))$ £ $\min_{i\leq n}(f_{\alpha_i^{\gamma'}}(k))$ or $\max_{i\leq n}(g_{\alpha_i^{\gamma'}}(k))$ £ $\min_{i\leq n}(f_{\alpha_i^\gamma}(k))$.

We define, for every $\gamma < \omega_1$, $\bar{g}_\gamma, \bar{f}_\gamma \in \omega^\omega$ as follows:

$$\begin{aligned}
\bar{g}_\gamma(k) &= \begin{cases} 0, & \text{if } k < m \\ \max_{i\leq n} g_{\alpha_i^\gamma}(k)\,, & \text{if } k \geq m \end{cases} \\
\bar{f}_\gamma(k) &= \begin{cases} 0, & \text{if } k < m \\ \min_{i\leq n} f_{\alpha_i^\gamma}(k)\,, & \text{if } k \geq m \end{cases}
\end{aligned}$$

For every $\gamma < \omega_1$, let $\alpha_\gamma = \max\{\alpha_0^\gamma, ..., \alpha_n^\gamma\}$. So, modulo a finite set, $\bar{g}_\gamma = g_{\alpha_\gamma}$ and $\bar{f}_\gamma = f_{\alpha_\gamma}$.

$\{\alpha_\gamma : \gamma < \omega_1\}$ is a unbounded subset of ω_1. Otherwise $\{p_\gamma : \gamma < \omega_1\}$ will not be an uncountable antichain. Therefore, picking a subsequence of G, if necessary, we may assume that $\bar{G} = \langle \bar{g}_\gamma, \bar{f}_\gamma : \gamma < \omega_1\rangle$ is a pregap that determines the same gap that G; i.e., h splits G iff h splits $\bar{G}$.

But, since $\gamma \neq \gamma'$ implies either $\bar{g}_\gamma$ £ $\bar{f}_{\gamma'}$ or $\bar{g}_{\gamma'}$ £ $\bar{f}_\gamma$ and, for every γ, $\bar{g}_\gamma \leq \bar{f}_\gamma$, by Lema 2.3.28, $\bar{G}$ is an strong gap. But then G is also a strong gap. ∎

As an immediate corollary of 2.3.29, we have that Martin's Axiom (plus $\aleph_1 < 2^{\aleph_0}$) implies that every gap is a strong gap.

Definition 2.3.30 *Given a pregap $G = \langle g_\alpha, f_\alpha : \alpha < \omega_1\rangle$, let Q_G be the following poset:*

- *The conditions of Q_G are finite sequences $\left\langle \left\langle \alpha_0, g^*_{\alpha_0}, f^*_{\alpha_0}\right\rangle, \ldots, \left\langle \alpha_n, g^*_{\alpha_n}, f^*_{\alpha_n}\right\rangle\right\rangle$ such that:*
  1. *The α_i are ordinals $< \omega_1$.*
  2. *For each α_i, $g^*_{\alpha_i}$, $f^*_{\alpha_i} \in \omega^{<\omega}$ are perturbations of g_{α_i} and of f_{α_i}, respectively, with the property that if one modifies g_{α_i} and f_{α_i} by $g^*_{\alpha_i}$ and $f^*_{\alpha_i}$ to get g'_{α_i} and f'_{α_i}, then for all $i, j \leq n$, if $i \neq j$, g'_{α_i} £ f'_{α_j} or g'_{α_j} £ f'_{α_i} and for all $i \leq n$, $g'_{\alpha_i} \leq f'_{\alpha_i}$.*
- *The ordering of Q_G is the reversed inclusion.*

Fact 2.3.31 *Forcing with Q_G makes G into a strong gap, provided that ω_1 is not collapsed.*

Proof. Follows from Lemma 2.3.28. ■

Notice that if Q_G is ccc, then G is a gap. For otherwise, a split for G would exist in every generic extension of V, even in the Q_G-generic extension where G should be a strong gap. The converse is also true:

Lemma 2.3.32 *If G is a pregap, then Q_G is ccc iff G is a gap.*

Proof. $(\Rightarrow)$ By the remark before the lemma.

$(\Leftarrow)$ Suppose otherwise. Without loss of generality, we may fix an antichain $\{q_\gamma : \gamma < \omega_1\}$ such that for every $\gamma \neq \gamma'$ the ordinals of q_γ do not appear in $q_{\gamma'}$.

For every $\gamma < \omega_1$, let $G_\gamma, F_\gamma \in \omega^\omega$ be such that:

$$\begin{array}{rcl} G_\gamma(k) & = & \min_i(g'_{\alpha_i^\gamma}(k)) \\ F_\gamma(k) & = & \max_i(f'_{\alpha_i^\gamma}(k)) \end{array}$$

where, for $G_\gamma(k)$, the minimum is taken over the $g'_{\alpha_i^\gamma}$'s obtained from the $g^*_{\alpha_i^\gamma}$'s appearing in q_γ. We compute the $F_\gamma(k)$'s in a similar way.

Since, for $\gamma \neq \gamma'$, the ordinals in q_γ do not belong to $q_{\gamma'}$, the set $\{G_\gamma : \gamma < \omega_1\}$ is unbounded with $\{g_\alpha : \alpha < \omega_1\}$. The same is true for $\{F_\gamma : \gamma < \omega_1\}$ and $\{f_\alpha : \alpha < \omega_1\}$. Further, $G_\gamma \leq F_{\gamma'}$ for $\gamma, \gamma' < \omega_1$. This follows from incompatibility of q_γ and $q_{\gamma'}$ for different γ, γ'. i.e., $q_\gamma \cup q_{\gamma'}$ is not a condition.

Define $H \in \omega^\omega$ as $H(k) = \min\{F_\gamma(k) : \gamma < \omega_1\}$. Therefore, for every $\gamma < \omega_1$, $G_\gamma \leq H \leq F_\gamma$. But this implies that H splits G. A contradiction with the fact that G is a gap. ■

Theorem 2.3.33 *Suppose $Con(ZF + \exists\kappa(\kappa$ is weakly-compact$))$. Then $Con(ZFC + MA(Proj) + \neg CH +$ There exists a non-strong gap$)$.*

Proof. Let κ be a weakly-compact cardinal. By Theorem 2.3.18 we only need to show that: (1) there is a ccc extension V_1 of $V_0 = L[C]$, where C is any $Coll(\omega, <\kappa)$-generic filter over L, and a non-strong gap $G \in V_1$, (2) there are posets $\mathsf{P}_1^G, \ldots, \mathsf{P}_n^G$ such that G is a non-strong gap iff $\mathsf{P}_1^G, \ldots, \mathsf{P}_n^G$ are ccc and, finally, (3) there are 2-colorings of an uncountable set such that G is a non-strong gap iff there do not exist 0-homogeneous sets for the 2-colorings.

S. Todorčević has showed that forcing with the Cohen poset adds a generic non-strong gap (see [S], 49, see also [S], 35 and 48).

Clearly, by 2.3.29 and 2.3.32, a gap G is a non-strong gap iff both P_G and Q_G are ccc posets.

Finally, let $\pi_{\mathsf{P}_G} : \omega_1 \times \omega_1 \longrightarrow \{0,1\}$ and $\pi_{\mathsf{Q}_G} : \omega_1^{<\omega} \times \omega_1^{<\omega} \longrightarrow \{0,1\}$ be 2-colorings defined by:

$$\pi_{\mathsf{P}_G}(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha = \beta \text{ or } g_\alpha \leq f_\beta \text{ or } g_\beta \leq f_\alpha \\ 1, & \text{otherwise} \end{cases}$$

and

$$\pi_{Q_G}(\bar{\alpha}, \bar{\beta}) = \begin{cases} 0, & \text{if } (\exists i \leq n)(\exists j \leq m)(\alpha_i \neq \beta_j \wedge g'_{\alpha_i} \leq f'_{\beta_j} \wedge g'_{\beta_j} \leq f'_{\alpha_i}) \\ 1, & \text{otherwise} \end{cases}$$

where $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_n \rangle$ and $\bar{\beta} = \langle \beta_0, \ldots, \beta_m \rangle$. It is easy to see, that if X is an uncountable subset of ω_1 such that $\pi_{\mathsf{P}_G}{}''X \times X = \{0\}$, then $\langle g_\alpha, f_\alpha : \alpha \in X \rangle$ is a subgap of G (i.e., a subsequence of G that is also a gap) that is a strong gap. Thus G is also a strong gap. On the other hand, if Y is an uncountable subset of $\omega_1^{<\omega}$ such that $\pi_{Q_G}{}''Y \times Y = \{0\}$, then Y is an uncountable antichain of Q_G. So, by 2.3.32, the pregap G is not a gap. ■

2.3.4 Entangled sets of reals

In [A-Sh], S. Shelah defined the notion of entangled set of reals:

Definition 2.3.34 *Let κ be an uncountable cardinal. A set of reals E is κ-*entangled *if $|E| = \kappa$ and for every $n \in \omega$ and every $s \in 2^n$, in every uncountable family $F \subseteq E^n$ of increasing (under the usual ordering of the reals) and pairwise disjoint n-tuples we can find two $x, y \in F$ such that $(\forall i < n)\,(x_i < y_i \rightarrow s\,(i) = 0)$. Let $x\,(s)\,y$ abbreviate the preceding formula.*

We are mainly interested in $\aleph_1$-entangled sets. So, henceforth, *entangled* will mean $\aleph_1$-entangled.

U. Abraham and S. Shelah have showed that if E is a set of $\aleph_1$ Cohen reals, then E is entangled, that CH implies that there exists an entangled set of reals and that MA implies that there are no entangled sets ([A-Sh]). And S. Todorčević has showed that in all Cohen generic extensions there exists an entangled set of reals (see [Be]).

Theorem 2.3.35 *Suppose $Con\,(ZF + \exists\kappa(\kappa$ is weakly-compact$))$. Then $Con(ZFC + MA(Proj) + \neg CH +$ There exists an entangled set of reals$)$*

Proof. We have already remarked that adding a Cohen real to $V_0 = L[C]$, where C is a $Coll\,(\omega, <\kappa)$-generic filter over L, produces an entangled set of reals. Thus, by Theorem 2.3.18, we only need to show that: (1) for every entangled set E there is a poset P_E such that E is an entangled set iff P_E is ccc, and (2) there are 2-colorings such that E is an entangled set iff there are no 0-homogeneous sets for these 2-colorings.

Definition 2.3.36 *Let E be a set of reals of cardinality $\aleph_1$, let $n \in \omega$, $s \in 2^n$ and $F \subseteq E^n$ a set of increasing and pairwise disjoint n-tuples. Then Q_F^s is the following poset:*

- *$p \in \mathsf{Q}_F^s$ iff p is a finite subset of F such that for all distinct $x, y \in p$, either $x\,(s)\,y$ or $y\,(s)\,x$*
- *$p \leq q$ iff $q \subseteq p$.*

Let P_E be the product with finite support of all Q_F^s, $s \in 2^n$, $n \in \omega$, ordered coordinate-wise.

For (1) we show:

Lemma 2.3.37 P_E *is ccc iff* E *is entangled.*

Proof. ($\Rightarrow$) Suppose that E is not entangled. We need the following claim from [B2]:

Claim 2.3.38 *E is entangled iff for every $n \in \omega$, every $s \in 2^n$ and every uncountable set $F \subseteq E^n$ of increasing and pairwise disjoint n-tuples, Q_F^s is ccc.*

Proof. ($\Rightarrow$) Let $n \in \omega$, $F \subseteq E^n$ uncountable and $s \in 2^n$. Let $\{p_\alpha : \alpha < \omega_1\}$ be a uncountable subset of Q_F^s. We may assume that all p_α's have the same size m. For every $\alpha < \omega_1$, we fix an ordering $\langle p_\alpha(i) : i < m \rangle$ of p_α. Let D be a countable dense subset of E. Every p_α can be separated by a sequence of $n \cdot m$ of pairwise disjoint open intervals with endpoints in D. Therefore, we may assume that the sequence that separates the p_α is the same for every $\alpha < \omega_1$. Suppose now that $\alpha, \beta < \omega_1$, $i, j < m$ and $i \neq j$. If $p_\alpha(i)\,(s)\,p_\alpha(j)$, then $p_\alpha(i)\,(s)\,p_\beta(j)$. In a similar way, if $p_\alpha(j)\,(s)\,p_\alpha(i)$, then $p_\beta(j)\,(s)\,p_\beta(i)$, and so $p_\beta(j)\,(s)\,p_\alpha(i)$. Hence, we may assume, without loss of generality, that for every $\alpha, \beta < \omega_1$ and every $i, j < m$, if $i \neq j$, then else $p_\alpha(i)\,(s)\,p_\beta(j)$ or $p_\beta(j)\,(s)\,p_\alpha(i)$. Now, consider every p_α as element of $E^{n \cdot m}$ and let s' the concatenation of n copies of s. Since E is an entangled set, we may find $\alpha, \beta < \omega_1$ such that $\alpha \neq \beta$ and $p_\alpha\,(s')\,p_\beta$. i.e., for every $i < m$, $p_\alpha(i)\,(s)\,p_\beta(i)$.

($\Leftarrow$) Suppose that E is not entangled. Let $n \in \omega$, $s \in 2^n$ and $F \subseteq E^n$ be a counterexample to the entangledness of E. Let $A = \{\{x\} : x \in F\}$. Clearly, $A \subseteq \mathsf{Q}_F^s$ is a set of pairwise incompatible conditions. ■

We continue with the proof of Lemma 2.3.37. From 2.3.38 follows that at least one of the factors of P_E is not ccc. Hence, P_E is not ccc.

($\Leftarrow$) Suppose that E is an entangled set of reals. Then, by 2.3.38, we know that for every $n \in \omega$, every $s \in 2^n$ and every uncountable subset F of increasing and pairwise disjoint n-tuples in E^n, Q_F^s is ccc. Since we are dealing with a product of ccc posets with finite support, we only need to show that the product of a finite, in fact of two, of these posets is ccc. For this we need the following:

Claim 2.3.39 *There is a dense embedding from $\mathsf{Q}_F^s \times \mathsf{Q}_{F'}^{s'}$ into $\mathsf{Q}_{F^\frown F'}^{s^\frown s'}$, where $F^\frown F' = \{x^\frown x' : x \in F \wedge x' \in F'\}$.*

Proof. Define $h : \mathsf{Q}_F^s \times \mathsf{Q}_{F'}^{s'} \longrightarrow \mathsf{Q}_{F^\frown F'}^{s^\frown s'}$ as follows: for every $p \in \mathsf{Q}_F^s$ and $q \in \mathsf{Q}_{F'}^{s'}$, $h(\langle p, q \rangle) = p^\frown q$, where $p^\frown q = \{x^\frown x' : x \in p \wedge x' \in q\}$. h is a dense embedding. ■

So, if $A \subseteq \mathsf{Q}_F^s \times \mathsf{Q}_{F'}^{s'}$, is an antichain, so is $\{h(\langle p, q \rangle) : \langle p, q \rangle \in A\}$ in $\mathsf{Q}_{F^\frown F'}^{s^\frown s'}$. But, since E is an entangled set, by 2.3.38, $\mathsf{Q}_{F^\frown F'}^{s^\frown s'}$ is a ccc poset. Therefore, $\mathsf{Q}_F^s \times \mathsf{Q}_{F'}^{s'}$ is ccc. This ends the proof of Lemma 2.3.37. ■

Continuing with the proof of Theorem 2.3.35, we define for every $n \in \omega$ and every $s \in 2^n$ a 2-coloring $\pi_s : E^n \times E^n \longrightarrow \{0, 1\}$ by:

$$\pi_s(x, y) = \begin{cases} 0, & \text{if } \neg x\,(s)\,y \text{ and } \neg y\,(s)\,x \\ 1, & \text{otherwise} \end{cases}$$

Then, E is an entangled set of reals iff there is no $n \in \omega$, $s \in 2^n$ and an uncountable set $F \subseteq E^n$ of increasing and pairwise disjoint n-tuples such that $\pi" F \times F = \{0\}$. ■

2.3.5 $(\exists \kappa < 2^{\aleph_0})(2^{\aleph_0} < 2^{\kappa})$

Theorem 2.3.40 *Suppose that V satisfies CH plus $2^{\aleph_1} = \aleph_3$. Then there exists a poset $\mathsf{P} \in V$ such that whenever G is a generic filter over V, then*

$$V[G] \Vdash 2^{\aleph_0} = \aleph_2 \wedge MA(Proj) \wedge \aleph_3 \leq 2^{\aleph_1}.$$

Proof. Let P be the direct limit of the iteration $\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha : \alpha < \omega_2 \rangle$ with finite support of projective and ccc posets as in Theorem 2.2.14. Let G be a generic filter over V. Since P is ccc, it preserves cardinalities, and thus,

$$V[G] \Vdash \aleph_3 \leq 2^{\aleph_1}.$$

It only remains to check that in $V[G]$, we have $MA(Proj)$ and $2^{\aleph_0} = \aleph_2$. Since in V, $\aleph_2$ is not a successor of a cardinal with cofinality ω, it only remains to see that for every $\alpha < \omega_2$, $\Vdash_\alpha$ "$2^{\aleph_0} < \aleph_2$". We can show by induction on ω_2 that for every α, $|\mathsf{P}_\alpha| \leq \aleph_1$. Thus, for every $\alpha < \omega_2$, there are at most $\aleph_1^{\aleph_0} = \aleph_1$ many simple P_α-names for reals. Hence, for every $\alpha < \omega_2$, $\Vdash_\alpha$ "$2^{\aleph_0} = \aleph_1$".

$\underline{\alpha = 0}$: Obvious, since P_0 is the trivial poset.

$\underline{\alpha + 1}$: Since, by inductive hypothesis, $|\mathsf{P}_\alpha| \leq \aleph_1$, we have $\Vdash_\alpha$ "$2^{\aleph_0} = \aleph_1$". So, since $\Vdash_\alpha$ "$\dot{\mathsf{Q}}_\alpha$ is a projective ccc poset", $\Vdash_\alpha$ "$|\dot{\mathsf{Q}}_\alpha| \leq \aleph_1$". Hence $|\mathsf{P}_{\alpha+1}| = |\mathsf{P}_\alpha * \dot{\mathsf{Q}}_\alpha| \leq \aleph_1$.

$\underline{\alpha \text{ limit}}$: Since we are working with an iteration with finite support and $\aleph_1$ is a regular cardinal, $|\mathsf{P}_\alpha| = |\bigcup_{\beta<\alpha} \mathsf{P}_\beta| = \sum_{\beta<\alpha} |\mathsf{P}_\beta| \leq \sum_{\beta<\alpha} \aleph_1 = \aleph_1$. ■

Corollary 2.3.41 *Suppose $Con(ZF)$. Then $Con(ZFC + MA(Proj) + \neg CH + (\exists \kappa < 2^{\aleph_0})(2^{\aleph_0} < 2^{\kappa}))$.* ■

Definition 2.3.42 *A poset* P *is* σ-centered *iff there exist a family $\{P_n : n \in \omega\}$ such that $\mathsf{P} = \bigcup_{n\in\omega} P_n$ and for every finite collection $\{p_0, ..., p_k\} \subseteq P_n$, some $n \in \omega$, there exists $p \in \mathsf{P}$ such that $p \leq_P p_0, ..., p_k$.*

Note that every σ-centered poset is ccc.

Definition 2.3.43 Martin's Axiom for σ-centered posets *is $MA(\Gamma)$ for Γ the class of σ-centered posets. We denote it by $MA(\sigma$-centered$)$.*

Since every σ-centered poset is a ccc poset, MA implies $MA(\sigma$-centered$)$. Since $MA(\sigma$-centered$)$ implies $(\forall \kappa < 2^{\aleph_0})(2^{\kappa} = 2^{\aleph_0})$, from above theorem we also obtain the following corollary:

Corollary 2.3.44 *If $Con(ZF)$, then $Con(ZFC + MA(Proj) + \neg CH + \neg MA(\sigma$-centered$))$.* ■

Definition 2.3.45 *A* Q set *is an uncountable set of reals such that every subset of it is a relative F_σ (i.e., $\underset{\sim}{\Sigma}^0_2$).*

Corollary 2.3.46 *If $Con(ZF)$, then $Con(ZFC + MA(Proj) + \neg CH +$ No set of reals is Q set$)$.*

Proof. Since $2^{\aleph_0} < 2^{\aleph_1}$ implies that there are no Q sets (see [Ha]). ■

2.3.6 Final remarks and open questions

1. Let MA (Indestructible-ccc) be Martin's Axiom restricted to indestructible-ccc posets (see Definition 2.3.13). Notice that MA (Indestructible-ccc) does not imply MA (co-Suslin): Let T be a Suslin tree on the reals in L. Let $L[H]$ be a generic extension of L for an iteration of length ω_2 with finite support of posets such that for every $\alpha < \omega_2$,

$$\Vdash_\alpha \text{“}\dot{\mathsf{Q}}_\alpha \text{ is indestructible-ccc”}$$

   Then, $L[H] \models$ “MA(Indestructible-ccc) $\wedge\ 2^{\aleph_0} = \aleph_2$”. Moreover, since we have forced only with indestructible-ccc posets, T remains a Suslin tree in $L[H]$. Since $\omega_1^{L[H]} = \omega_1^L$ and $L[H] \models MA_{\aleph_1}$ (σ-centered), by an argument of A. Martin and R. Solovay ([M-So]), T and every set of $\aleph_1$ reals is a $\utilde{\Pi}^1_1$ set. But, P_T, the ccc poset that adds a unbounded branch to T defined in Theorem 2.3.23, has the same complexity as T. So, $L[H] \not\models MA$ (co-Suslin).

2. Since $\utilde{\Sigma}^1_1$ ccc posets are indestructible-ccc, (see [Ju-Sh1] or [B1] 1.1.1.20), from (1) it follows that MA (Suslin) does not imply MA (co-Suslin). Since $MA(\utilde{\Delta}^1_2)$ implies MA (co-Suslin) and MA (Suslin) implies MA (Borel), MA (Suslin) does not imply $MA(\utilde{\Delta}^1_2)$ and MA (Borel) does not imply MA (co-Suslin). Is it still an open question whether $MA\,(\utilde{\Sigma}^1_n)$ implies $MA\,(\utilde{\Pi}^1_n)$, for $n > 1$. It is also open whether MA (Borel) implies MA (Suslin) and whether MA (co-Suslin) implies $MA(\utilde{\Delta}^1_2)$.

3. Is the assumption of (the consistency of) the existence of a weakly-compact cardinal necessary to obtain the consistency results of 2.3.33 and of 2.3.35?

4. Let $MA\,(L\,(\mathsf{R}))$ be Martin's Axiom restricted to ccc posets, with reals as conditions, that belong to $L\,(\mathsf{R})$. Clearly, MA implies $MA\,(L\,(\mathsf{R}))$ and $MA\,(L\,(\mathsf{R}))$ implies $MA(Proj)$. Can these implications be reversed?

5. Let P be a poset. We say that P is a *proper poset* iff it preserves stationary subsets of $[\lambda]^\omega = \{X \subseteq \lambda : |X| = \omega\}$, for all regular cardinal λ. Clearly, every ccc poset is proper. Let $PFA\,(Proj)$ be the Proper Forcing Axiom restricted to projective posets. Is $PFA\,(Proj)$ weaker than PFA? What is its exact consistency strength?

Chapter 3

GENERIC ABSOLUTENESS FOR PROJECTIVE CCC FORCING

3.1 Solovay models

If κ is an inaccessible cardinal in some model V, then a Solovay model over V is the $L(\mathsf{R})$ of a model M resulting from collapsing κ to ω_1 over V using the Levy-collapse. Thus, if $L(\mathsf{R})^M$ is a Solovay model over V, then M has the following properties:

1. For every $x \in \mathsf{R}$, ω_1 is an inaccessible cardinal in $V[x]$.

2. For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.

Lemma 3.1.1 (H. Woodin) *Suppose that M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is an inaccessible cardinal in $V[x]$ and*

2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset*

Then there exists a forcing notion W such that does not add reals and creates a $Coll(\omega, <\omega_1)$-generic filter C over V such that M and $V[C]$ have the same reals. Thus, W forces that $L(\mathsf{R})^M$ is a Solovay model over V.

Proof. We define the *Woodin pseudo-collapse* W as follows:

- $g \in \mathsf{W}$ iff there exists $\alpha < \omega_1$ such that $g \subseteq Coll(\omega, \leq \alpha)$ is a generic filter over V.

- $g \leq h$ iff $h \subseteq g$.

By (1), for every $g \in \mathsf{W}$, ω_1 is an inaccessible cardinal in $V[g]$ and, hence, for every $\alpha < \omega_1$ there are only countably many antichains of $Coll(\omega, \leq \alpha)$ in $V[g]$. Therefore, for every $\alpha < \omega_1$, $D_\alpha = \{g \in \mathsf{W} : g \cap Coll(\omega, \leq \alpha) \text{ is generic over } V\}$ is a dense subset of W.

Since every $g \in \mathsf{W}$ is a countable set in $L(\mathsf{R})$, given any real x, we can code x and g into a single real y. By (2), $V[y]$ is a generic extension by some countable poset in V. Hence, we can find $\alpha < \omega_1$ and a generic filter $h \subseteq Coll(\omega, \leq \alpha)$ such that $y \in V[h]$. But then, $h \leq g$ and $x \in V[h]$. Therefore, for every real x, $E_x = \{g \in \mathsf{W} : x \in V[g]\}$ is a dense subset of W.

Suppose that H is a W-generic filter over M and let $C = \bigcup H$.

Clearly, $C \subseteq Coll(\omega, <\omega_1)$ is a filter. Since $Coll(\omega, <\omega_1)$ is a ω_1-cc poset, if $A \in V$ is a maximal antichain of $Coll(\omega, <\omega_1)$, then $|A| < \omega_1$ and hence for some $\alpha < \omega_1$, $A \subseteq Coll\,(\omega, \leq \alpha)$. But, then, by density of D_α, $C \cap A \neq \emptyset$. So, C is a $Coll(\omega, <\omega_1)$-generic filter over V.

Notice that if $x \in \mathsf{R} \cap V\,[C]$, then $x \in V[g]$, for some $g \in H$. So $x \in M$. And if $x \in \mathsf{R} \cap M$, then, by density of E_x, $x \in V[g]$ for some $g \in H$. So $x \in V[C]$. This shows that M and $V\,[C]$ have the same reals, hence the same $L\,(\mathsf{R})$.

Finally, we show that W does not add any new ω-sequences, hence no new reals. Suppose $f : \omega \to M$ is such that $f \in M[H]$. We may assume that for some $B \in M$, $f : \omega \to B$. Let $\dot{f}$ be a W-name for f. For every $n \in \omega$, let $g_n \in H$ be such that $g_n \Vdash_{\mathsf{W}}$ "$\dot{f}\,(\check{n}) = (\check{f}\,(n))$. Since $\{g_n : n \in \omega\} \subseteq H$ and H is a filter, $\{g_n : n \in \omega\}$ is a chain. Let α the least ordinal such that $\bigcup_{n\in\omega} g_n \subseteq Coll\,(\omega, \leq \alpha)$. Since ω_1 is regular, $\alpha < \omega_1$. So, $H \cap D_\alpha \neq \emptyset$. Let $g \in H \cap D_\alpha$. Then, for every $n \in \omega$, $g \leq g_n$ and hence

$$f = \{\langle n, x\rangle \in \omega \times B : g \Vdash_{\mathsf{W}} \dot{f}\,(\check{n}) = \check{x}\}.$$

So $f \in M$. ∎

Corollary 3.1.2 *If M is countable, then $L\,(\mathsf{R})^M$ is a Solovay model over V iff M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is an inaccessible cardinal in $V\,[x]$ and*
2. *For every $x \in \mathsf{R}$, $V\,[x]$ is a generic extension of V by some countable poset.* ∎

Remark 3.1.3 *If M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is an inaccessible cardinal in $V\,[x]$ and*
2. *For every $x \in \mathsf{R}$, $V\,[x]$ is a generic extension of V by some countable poset,*

then $L\,(\mathsf{R})^M$ satisfies every sentence with reals x and ordinals as parameters that has Boolean-value 1*, as computed in $V[x]$, in* r.o. $\left(Coll\left(\omega, <\omega_1^M\right)\right)$. *Hence, by [So], every set of reals in $L\,(\mathsf{R})^M$ is measurable, has the Baire property, etc.*

In view of Lemma 3.1.1, we will call a Solovay model any model satisfying (1) and (2) above. So, we re-define:

Definition 3.1.4 *$L\,(\mathsf{R})^M$ is a* Solovay model over *V iff*

1. *For every $x \in \mathsf{R}$, ω_1 is an inaccessible cardinal in $V\,[x]$ and*
2. *For every $x \in \mathsf{R}$, $V\,[x]$ is a generic extension of V by some countable poset.*

We can see that "$L\,(\mathsf{R})$ is a Solovay model over L" is Π^1_4 sentence:

Fact 3.1.5 [B-W] *There exists a Π^1_4 sentence σ such that $L\,(\mathsf{R}) \models \sigma$ iff $L\,(\mathsf{R})$ is a Solovay model over L.*

Proof. We may rewrite (1) and (2) of Definition 3.1.4 as projective sentences. Namely:

1'. $\forall xy(y \in WO \to (\exists z \in WO)(\|y\| < \|z\| \wedge L[x] \models$ "$\|z\|$ is a cardinal")).

2'. $\forall x \exists y(y$ codes a poset $P \wedge x$ is P-generic over $L)$.

where WO is the Π^1_1 set of all $x \in \omega^\omega$ which code a well-ordering of ω and for every $x \in WO$, $\|x\|$ is the order type of the well-ordering coded by x (see [J2] 40.2). So, since "$L[x] \models \|z\|$ is a cardinal" is a $\Pi^1_2(x,z)$ and "y codes a poset $P \wedge x$ is P-generic over L" is $\Pi^1_2(x,y)$, (1') and (2') are Π^1_4. ∎

Lemma 3.1.6 *Suppose that $L(\mathsf{R})^M$ and $L(\mathsf{R})^N$ are Solovay models over V such that $\mathsf{R}^M \subseteq \mathsf{R}^N$ and $\omega_1^M = \omega_1^N$. Then there is an elementary embedding $j : L(\mathsf{R})^M \to L(\mathsf{R})^N$ which is the identity on the reals and the ordinals.*

Proof. Notice that if such an embedding exists, then it is unique and must be defined by:

$$j(A) = \{x \in L(\mathsf{R})^N : L(\mathsf{R})^N \models \varphi(x, \alpha, a)\},$$

where φ is some formula with parameters an ordinal α and a real a, that defines A in $L(\mathsf{R})^M$.

In order to prove that j is well-defined and is an elementary embedding, we only need to show that for every formula $\varphi(x_1, x_2)$, every ordinal α and every real $a \in M$,

$$L(\mathsf{R})^M \models \varphi(\alpha, a) \text{ iff } L(\mathsf{R})^N \models \varphi(\alpha, a).$$

Notice that $Coll(\omega, <\omega_1)^M = Coll(\omega, <\omega_1)^N$. Let $\dot{H}$ and $\dot{H}^*$ be W-terms for $Coll(\omega, <\omega_1)^M$-generic filters over $V[a]$ such that, with W-value $\mathsf{1}$, R^M is the set of reals of $V[a][\dot{H}]$ and R^N is the set of reals of $V[a][\dot{H}^*]$. Then, the following equivalences have all W-value $\mathsf{1}$ in M:

$$L(\mathsf{R}) \models \varphi(\alpha, a) \text{ iff } V[a][\dot{H}] \models \varphi(\alpha, a)^{L(\mathsf{R})} \text{ iff } V[a] \models \text{“} \Vdash_{Coll(\omega, <\omega_1)^M} \varphi(\check{\alpha}, \check{a})^{L(\mathsf{R})} \text{”}$$

(the last one, by almost-homogeneity of the Levy-collapse).

And the following equivalences have all W-value $\mathsf{1}$ in N:

$$L(\mathsf{R}) \models \varphi(\alpha, a) \text{ iff } V[a][\dot{H}^*] \models \varphi(\alpha, a)^{L(\mathsf{R})} \text{ iff } V[a] \models \text{“} \Vdash_{Coll(\omega, <\omega_1)^N} \varphi(\check{\alpha}, \check{a})^{L(\mathsf{R})} \text{”}$$

Hence,

$$L(\mathsf{R})^M \models \varphi(\alpha, a) \text{ iff } L(\mathsf{R})^N \models \varphi(\alpha, a). \quad ∎$$

We are interested in the absoluteness properties of generic extensions of Solovay models under ccc forcing notions.

Definition 3.1.7 *Let V be a model of ZFC. Let $\mathsf{P} \in V$ a forcing notion and let φ be a formula (with parameters in V).* V is φ-absolute for P *iff for every P-generic filter G over V,*

$$V \models \varphi \text{ iff } V[G] \models \varphi.$$

Let Σ be a set of formulas. V is Σ-absolute for P *iff for every $\varphi \in \Sigma$, V is φ-absolute for P. Let Γ be a class of posets.* V is Σ-absolute for Γ *iff for every $\mathsf{P} \in \Gamma$, V is Σ-absolute for P. (Compare with Definition 2.3.8)*

Let P be a forcing notion in V, and $\dot{\mathsf{Q}}$ a P-name for a forcing notion. V is two-step φ-absolute for P and $\dot{\mathsf{Q}}$ *if for every P-generic filter G over V and every $\dot{\mathsf{Q}}[G]$-generic filter H over $V[G]$,*

$$V[G] \models \varphi \text{ iff } V[G][H] \models \varphi.$$

(Note that in two-step absoluteness φ may have parameters in $V[G]$, not just in V). Let Σ be a set of formulas, we define V is two-step Σ-absolute for P and $\dot{\mathsf{Q}}$ *in the obvious way. Let Γ be a class of posets.* V is two-step Σ-absolute for Γ *iff for every $\mathsf{P} \in \Gamma$ and every P-name $\dot{\mathsf{Q}}$ for a forcing notion such that $\Vdash_P$ "$\dot{\mathsf{Q}} \in \dot{\Gamma}$", Σ is absolute for P and $\dot{\mathsf{Q}}$.*

V is $L(\mathsf{R})$-absolute for P *iff for every P-generic filter G over V there exists an elementary embedding*

$$j : L(\mathsf{R}) \to L(\mathsf{R})^{V[G]}$$

that fixes all the ordinals (hence all the reals). For Γ a class of posets, V is $L(\mathsf{R})$-absolute for Γ *iff for every $\mathsf{P} \in \Gamma$, V is $L(\mathsf{R})$-absolute for P.*

V is $L(\mathsf{R})$-two-step absolute for P and $\dot{\mathsf{Q}}$ *iff for every P-generic filter G over V and every $\dot{\mathsf{Q}}[G]$-generic filter H over $V[G]$, there exists an elementary embedding*

$$j : L(\mathsf{R})^{V[G]} \to L(\mathsf{R})^{V[G][H]}$$

that fixes all the ordinals (hence all the reals). Let Γ be a class of posets, we define V is $L(\mathsf{R})$-two-step absolute for Γ *similarly as above.*

Since we are interested mainly in the absoluteness of projective formulas, henceforth, with "*V is absolute*" and with "*V is two-step absolute*" we will mean that V is Σ-absolute or, respectively, that V is two step Σ-absolute where Σ is the set of projective formulas.

We next observe that for some classes of posets, absoluteness implies two-step absoluteness.

Lemma 3.1.8 *Let V be a transitive model of ZFC and let $\varphi(x)$ be a Σ^1_k (Π^1_k) formula. Then there are projective sentences σ_0 and σ_1 such that*

1. *$V \models \sigma_0$ iff V is φ-absolute for Borel ccc posets.*

2. *$V \models \sigma_1$ iff V is φ-absolute for Suslin ccc posets.*

Proof. Suppose that V is a transitive model of ZFC and let $\varphi(x)$ be a Σ^1_k (Π^1_k) formula. Note that for every ccc poset $\mathsf{P} \in V$, identifying every real $a \in \mathsf{R} \cap V$ with its canonical P-name in V, V is φ-absolute for P iff

$$V \models \forall x \left(\left(\Vdash_{\mathsf{P}} \varphi(x) \right) \leftrightarrow \varphi(x) \right).$$

So, V is φ-absolute for Borel ccc posets iff

$$V \models \forall x \forall y (y \text{ is a Borel ccc poset } \rightarrow ((\Vdash_y \varphi(x)) \leftrightarrow \varphi(x))).$$

Since φ is a projective formula, we only need to show that "y is a Borel ccc poset" and "$\Vdash_y \varphi(x)$" are expressible with projective formulas.

Note that, using the Π^1_1 set of codes of Borel subsets of the real plane (see [J2], 42.1), we can code every Borel poset with a real in a such a way that, as in Fact 2.1.8, "y codes a Borel poset" is a Π^1_1 predicate on the reals. Moreover,

Claim 3.1.9 *Let* P *be a Suslin poset. Then, the following are equivalent:*

1. P *is ccc.*
2. *For every transitive well-founded model* M *of* ZF *with* $\mathsf{P} \in M$, $M \models \mathsf{P}$ *is ccc.*
3. *There exists a transitive well-founded model* M *of* ZF *with* $\mathsf{P} \in M$ *such that* $M \models \mathsf{P}$ *is ccc.*

Proof. See [Ju-Sh1]. See also [B1], 1.1.1.17. ■

Then, y is codes a Borel ccc poset iff

(a) y is a code of a Borel poset P and

(b) P is ccc.

But (a) is a Π^1_1 predicate on y and, as in Theorem 2.1.23, we can show that (2) and (3) of the Fact 3.1.9 are $\Pi^1_2(y)$ and $\Sigma^1_2(y)$ respectively. Hence, (b) is $\Delta^1_2(y)$. So, "y codes a Borel ccc poset" is a Δ^1_2 predicate on y.

From Theorem 2.1.23, we have that, if $k \geq 2$, then "$\Vdash_y \varphi(x)$" is a $\underset{\sim}{\Sigma}^1_{k+1}$ relation, if φ is $\underset{\sim}{\Sigma}^1_k$, or a $\underset{\sim}{\Pi}^1_{k+1}$ relation, if φ is $\underset{\sim}{\Pi}^1_k$.

We show (2) of the lemma in a similar way. ■

Corollary 3.1.10 *Suppose that* V *is absolute for Borel (Suslin) ccc posets. Then* V *is two-step absolute for Borel (Suslin) ccc posets.*

Proof. Since for every projective formula φ, "V is φ-absolute for Borel (Suslin) ccc posets" is expressible with a projective sentence, for every Borel (Suslin) ccc extension W of V, W satisfies the same projective formula, and hence, "W is φ-absolute for Borel (Suslin) ccc posets". Therefore, V is two-step absolute for Borel (Suslin) ccc posets. ■

For more complex projective and ccc forcing notions we have a similar result. Recall that for every transitive $M \models ZFC$, if P is a projective poset, then "$\mathsf{P} \in M$" means that the parameters of the definition of P belong to M.

Fact 3.1.11 *Suppose that V is absolute for $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$) and ccc posets. Then, for all $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$) and ccc posets $\mathsf{P}, \mathsf{Q} \in V$, V is two-step absolute for P and Q.*

Proof. We prove it only for $\underset{\sim}{\Sigma}^1_n$ and ccc posets. The proof for $\underset{\sim}{\Pi}^1_n$ and ccc posets is analogous. Suppose V is absolute for $\underset{\sim}{\Sigma}^1_n$ and ccc posets and let P, Q be $\underset{\sim}{\Sigma}^1_n$ and ccc posets in V. Suppose that G is a P-generic filter over V and H is a $\mathsf{Q}^{V[G]}$-generic filter over $V[G]$. We will show that for every projective formula $\varphi(x)$ and every real $a \in V[G]$,

$$V[G][H] \models \varphi(a) \text{ iff } V[G] \models \varphi(a).$$

Suppose that $V[G][H] \models \varphi(a)$. So, $V[G] \models$ "$q \Vdash_{\mathsf{Q}} \varphi(a)$", for some condition $q \in \mathsf{Q}$. Since $\mathsf{Q} \in V$ is a $\underset{\sim}{\Sigma}^1_n$ and ccc poset, by absoluteness for Q,

$$V \models \forall x \forall q((q \Vdash_{\mathsf{Q}} \varphi(x)) \leftrightarrow \varphi(x)).$$

But, by Theorem 2.1.23, the sentence on the right hand is projective and hence, by absoluteness for P,

$$V[G] \models \forall x \forall q((q \Vdash_{\mathsf{Q}} \varphi(x)) \leftrightarrow \varphi(x)).$$

But then, $V[G] \models \varphi(a)$.

The converse follows from the fact that if $V[G][H] \not\models \varphi(a)$, then $V[G][H] \models \neg\varphi(a)$ for all $a \in \mathsf{R} \cap V[G]$. ■

Fact 3.1.12 *$L(\mathsf{R})$-absoluteness for ccc posets in $L(\mathsf{R})$ implies $L(\mathsf{R})$-two-step absoluteness for ccc posets in $L(\mathsf{R})^V$.*

Proof. We show it as in the previous fact, using that every poset P in $L(\mathsf{R})$ is defined by formulas with only ordinals and reals as parameters and the fact that, since P is ccc, we may code every simple P-name for a real with a real. ■

The interest in the generic absoluteness of projective sentences under some class of definable ccc forcing notions can be seen from the following two theorems.

Theorem 3.1.13 *Suppose that V is $L(\mathsf{R})$-two-step absolute for ccc posets in $L(\mathsf{R})$. Then every set of reals in $L(\mathsf{R})$ is Lebesgue measurable and has the Baire property.*

Proof. Let X be a set of reals in $L(\mathsf{R})$. So, there is a formula φ with reals and ordinals as parameters such that

$$\forall x(x \in X \leftrightarrow \varphi(x)).$$

Let $\mathsf{A} = \text{r.o.}(\textit{Amoeba})$ (see [M-So] or [J2], Theorem 106) and let G be a A-generic over V. In $V[G]$ there is a measure-one set of Random reals over V. Let $\dot{r}$ be the canonical term for a Random real and suppose r is one of the Random-generic reals over V added by G.

Claim 3.1.14 $V[G] \models$ "$\varphi(r) \leftrightarrow r \in [\![\varphi(\dot{r})]\!]^{Random}$"

Proof. Work in $V[G]$. Suppose $\varphi(r)$ holds in $V[G]$. Let $U = \{u_n : n \in \omega\}$ where for every $n \in \omega$, $u_n = [\![\check{n} \in \dot{r}]\!]$. So, if B is the complete subalgebra of A generated by U, then $H = G \cap \mathsf{B}$ is a B-generic filter over V and $V[H] = V[r]$ (see [J2], 25.2, Corollary 2). Note that $\mathsf{B} \in L(\mathsf{R})$ since it is definable with parameter $\dot{r}$, the Amoeba term for a Random real which we may assume that is essentially a real.

Let Γ be the canonical B-name for the generic filter. Let $\mathsf{A}/\dot{r}$ be the following poset:

- $\langle \check{q}, p\rangle \in \mathsf{A}/\dot{r}$ iff $q \in \mathsf{A}$ and $p \in \mathsf{B}$ and $(\forall p' \leq p)\,(q \not\perp_{\mathsf{A}} p')$.
- $\langle \check{q}, p\rangle \leq \langle \check{q}', p'\rangle$ iff $q \leq q'$ and $p \leq p'$.

Then, $V[G] = V[r][G]$ (see [Ku], VII, Exercise D.5).

So, $V[G]$ is a ccc forcing extension of $V[r]$ via a definable forcing notion in $V[r]$, with parameter r. So, by $L(\mathsf{R})$-two-step absoluteness, we have $V[r] \models \varphi(r)$.

Since for every Borel-null set B_c with code in V,

$$B_c^{V[r]} = B_c^{V[G]} \cap V[r],$$

$V[r] \models$ "r is a Random real over V". Hence, $V[r]$ is a Random-generic extension of V (see [So] or [J2], 42) and $V[r] \models r \in [\![\varphi(\dot{r})]\!]^{Random}$. So, by Borel absoluteness, $V[G] \models r \in [\![\varphi(\dot{r})]\!]^{Random}$.

We show the converse in a similar way. ■

By the Claim, X is measurable in $V[G]$: Since $V[G] \models \mu(Ra(V)) = 1$, where μ denotes the measure of Lebesgue and $Ra(V)$ denote the set of random reals over V, $V[G] \models \mu(X \triangle [\![\varphi(\dot{r})]\!]^{Random}) = 0$.

But "X is measurable" is a sentence with only reals and ordinals as parameters. Hence, by absoluteness, X is measurable in V.

That all sets of reals in $L(\mathsf{R})$ have the Baire property is proved in a similar way, using Amoeba for category and Cohen forcing notions instead of Amoeba and Random forcing notions, respectively. ■

Also, using absoluteness for Borel ccc posets, we can show, using a result of H. Woodin ([W]), the following:

Theorem 3.1.15 *Suppose that V is absolute for Borel ccc posets. Then there is no uncountable projective well-ordering of reals.*

Proof. Suppose that $X = \langle X, <_X\rangle$ is a projective uncountable well-ordering of reals. Let $\varphi(x, y)$ be a projective formula that defines X with parameter $a \in \omega^\omega$.

Without loss of generality, we may assume that o.t. $(X) = \omega_1$. Otherwise, let $R \subseteq \omega^\omega \times \omega^\omega$ be defined as follows: for every $b, c \in \omega^\omega$,

$$bRc \text{ iff } \{(b)_n : n \in \omega\} = \{d : d <_X c\},$$

where for every $b \in \omega^\omega$ and every $n \in \omega$, $(b)_n = \{\langle i,j\rangle : b(J(i,n)) = j\}$. Note that R is a projective relation. Indeed, for every $b, c \in \omega^\omega$, bRc iff

$$\forall n\, \varphi\left((b)_n, c\right) \wedge \forall x\left(\varphi(x,c) \rightarrow \exists n\ \left(x = (b)_n\right)\right)$$

Therefore, if $\varphi(x,y)$ is $\Sigma^1_n(a)$, R is $\Delta^1_{n+1}(a)$, and if $\varphi(x,y)$ is $\Pi^1_n(a)$, R is $\Pi^1_{n+1}(a)$.

Note that for every real $b \in X$, there exists $x \in \omega^\omega$ such that xRb iff $\rho_{<_X}(b) < \omega_1$, where $\rho_{<_X}$ is the rank function for the well-ordering X. i.e., for every $b \in \omega^\omega$, $\rho_{<_X}(b) = \sup(\{\rho_{<_X}(x) + 1 : x <_X b\})$. Finally, let

$$b <_{X'} c \text{ iff } \exists xy\left(xRb \wedge yRc \wedge b <_X c\right).$$

Then, $X' = \langle X', <_{X'}\rangle$, where $X' = \mathrm{Field}(<_{X'})$, is a projective well-ordering (in fact $\Sigma^1_{n+1}(a)$, if $\varphi(x,y)$ is $\Sigma^1_n(a)$, or $\Sigma^1_{n+2}(a)$, if $\varphi(x,y)$ is $\Pi^1_n(a)$) and o.t.$(X') = \omega_1$.

Claim 3.1.16 *Let* P *be a Borel ccc poset in V. Then for every* P*-generic filter G over V, $\varphi(x,y)$ defines the well-ordering X in $V[G]$.*

Proof. Let $X^{V[G]} = \langle X^{V[G]}, <^{V[G]}_X\rangle$, where $X^{V[G]} = \mathrm{Field}(<^{V[G]}_X)$, and for all $b, c \in \omega^\omega \cap V[G]$, $b <^{V[G]}_X c$ iff $V[G] \models \varphi(b,c)$.

Note that in V the following hold

1. $\forall x\, \neg\varphi(x,x)$,
2. $\forall xy\left(\varphi(x,y) \rightarrow \neg\varphi(y,x)\right)$,
3. $\forall xyz\left(\varphi(x,y) \wedge \varphi(y,z) \rightarrow \varphi(x,z)\right)$,
4. $\forall xy\left(\varphi(x,y) \vee x = y \vee \varphi(y,x)\right)$,
5. $\neg\exists x \forall n\left((x)_n \in X \wedge \varphi\left((x)_{n+1}, (x)_n\right)\right)$,

and (1)-(5) are projective formula with parameters in V. Moreover, $b \in X$ iff $\exists x\left(\varphi(b,x) \vee \varphi(x,b)\right)$. Hence, by absoluteness for Borel ccc posets, $V[G]$ satisfies (1)-(5). Therefore, $X^{V[G]}$ is a projective well-ordering.

Moreover, by absoluteness, for every $b, c \in \omega^\omega \cap V$,

$$V \models \varphi(b,c) \text{ iff } V[G] \models \varphi(b,c).$$

So, $X \subseteq X^{V[G]}$ and o.t.$(X^{V[G]}) \geq \omega_1$. Since

$$V \models \forall x(\exists y\, \varphi(y,x) \rightarrow \exists z\, zRx),$$

by absoluteness,

$$V[G] \models \forall x\left(\exists y\, \varphi(y,x) \rightarrow \exists z\, zRx\right).$$

Therefore, o.t.$(X^{V[G]}) = \omega_1$ and $X = X^{V[G]}$. ■

Claim 3.1.17 *For every $b \in \omega^\omega$, $\omega^\omega \cap L(X,b)$ is a projective set of reals.*

Proof. For every real $x \in X$, let $X_x = \{y \in X : y <_X x\}$. Note that, by reflection in $L(X,b)$, for every $c \in \omega^\omega$, $c \in L(X,b)$ iff $\exists x\,(x \in X \wedge c \in L(X_x, b))$. Define $S \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ as follows: for every $x, y, z \in \omega^\omega$,

$$\langle y, z, x\rangle \in S \text{ iff } z \text{ codes } X_x \text{ and } \langle X_x, <_{X_x}\rangle \cong \langle \omega, E_y\rangle$$

where E_y is the well-ordering on ω coded by y (see [J2] 40.2). S is a projective relation since for all $x, y, z \in \omega^\omega$, $\langle y, z, x\rangle \in S$ iff

$$zRx \wedge y \in WO \wedge \forall nm\left((z)_n <_X (z)_m \leftrightarrow y\,(J\,(n,m)) = 0\right).$$

So, for every real $c \in \omega^\omega$,

$$c \in L(X,b) \text{ iff } \exists xyz\,(x \in X \wedge \langle y,z,x\rangle \in S \wedge r \in L(y,z,b)).$$

But $L(y,z,b) = L[y,z,b]$ and $\omega^\omega \cap L[y,z,b]$ is a $\Sigma^1_2(y,z,b)$ set. So, if X is a $\underset{\sim}{\Sigma}^1_n$ well-ordering, then R is $\underset{\sim}{\Delta}^1_{n+1}$, S is $\underset{\sim}{\Delta}^1_{n+1}$ and, hence, $\omega^\omega \cap L(X,b)$ is $\underset{\sim}{\Sigma}^1_{n+1}$. And if X is a $\underset{\sim}{\Pi}^1_n$ well-ordering, then R is $\underset{\sim}{\Pi}^1_{n+1}$, S also is $\underset{\sim}{\Pi}^1_{n+1}$ and $\omega^\omega \cap L(X,b)$ is $\underset{\sim}{\Sigma}^1_{n+2}$. ■

Continuing with the proof of Theorem 3.1.15, let B be the Random forcing notion and suppose G is a B-generic filter over V. Then for every real $b \in \omega^\omega \cap V$,

$$V[G] \models \text{“There exists a Random real over } L(X,b)\text{”}.$$

Claim 3.1.18 *“There exists a Random real over $L(X,b)$” is a projective sentence.*

Proof. Recall that every Borel set can be coded by a real in such a way that the set of codes of Borel sets is a Π^1_1 set (see [So] or [J2], 42.1). Then, r is a Random real over $L(X,b)$ iff for all real z, if z codes a Borel null set B_z and $z \in L(X,b)$, then $r \notin B_z$.

A set of reals X is null iff

$$\forall n \exists y (y \text{ codes an open set } O_y \wedge X \subseteq O_y \wedge \mu(O_y) \leq \frac{1}{n})).$$

If X is a Borel set, then the expression between parentheses is Π^1_1 (see [J2] 42.4). So, “X is null” is a Σ^1_2 statement about the code of X.

Moreover, for a measurable set X, $\mu(X) > 0$ iff there exists a closed set $F \subseteq X$ such that $\mu(F) > 0$. Therefore, if X is a Borel set, then “X is not null” is a Σ^1_2 statement with parameter the code of X.

Hence, for every Borel set X, “X is null” is a Δ^1_2 statement with parameter the code of X.

By Claim 3.1.17, for every real $b \in \omega^\omega \cap V$, $\omega^\omega \cap L(X,b)$ is a projective set. So, for every real $b \in \omega^\omega \cap V$, “$r$ is a Random real over $L(X,b)$” it is a projective sentence. ■

Now, by absoluteness,

$$V \models \forall x \exists r (r \text{ Random over } L(X,x)).$$

Let C be the Cohen forcing (see [C] or [J2], 42) and suppose c is a Cohen real over V. Since C is a Borel ccc poset and $\forall x \exists r(r$ Random over $L(X,x)))$ is a projective sentence, by absoluteness, and the fact that $X = X^{V[c]}$,

$$V[c] \ {}^{2}\ \forall x \exists r(r \text{ Random over } L(X,x)).$$

So,

$$V[c] \ {}^{2}\ \exists r(r \text{ Random over } L(X,c)).$$

But, since X is a projective uncountable sequence of different reals, this contradicts the following lemma of H. Woodin, [W].

Lemma 3.1.19 (H. Woodin) *Suppose that S is a uncountable sequence of distinct reals and c is a Cohen real over V. Then, in $V[c]$ there is no Random real over $L(S,c)$.*

Proof. Let $S = \langle s_\alpha : \alpha < \lambda \rangle$. Fix in $L(S)$ a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of infinite and almost-disjoint subsets of ω and a enumeration $\langle I_k : \kappa \in \omega \rangle$ of open intervals of R with rational endpoints and such that every interval appears infinitely many times.

Let $c \in \omega^\omega$ a Cohen real over $L(S)$. We work in $L(S,c)$. Define $f, g \in \omega^\omega$ as follows: for every $n \in \omega$,

- $f(n) = c(2n)$.
- $g(n) = c(2n+1)$.

For every $\alpha < \lambda$, let $g_\alpha = g \ {}^{1}\ A_\alpha$.

For every $\alpha < \lambda$ and every $n \in \omega$, $1 \leq n$, we will construct an open set $O_\alpha^n \subseteq \mathsf{R}$ such that $\mu(O_\alpha^n) \leq \frac{1}{n}$.

Fix $\alpha < \lambda$ and $n \in \omega$ with $1 \leq n$. Define a sequence of natural numbers $\langle k_i^{\alpha,n} : i \in \omega \rangle$ recursively:

$\underline{i=0}$: Then, $k_0^{\alpha,n} = \min(\{k \in \omega : g_\alpha(k) = n \wedge \mu(I_{f(k)}) < \frac{1}{n}\})$.

$\underline{i+1}$: Then, $k_{i+1}^{\alpha,n} = \min(\{k \in \omega : g_\alpha(k) = n \wedge \mu(\bigcup_{j \leq i} I_{f(k_j)} \cup I_{f(k)}) < \frac{1}{n}\})$.

We show that $\langle k_i^{\alpha,n} : i \in \omega \rangle$ exists. For every $\varepsilon > 0$ and every $m \in \omega$, we define in $L(S)$,

$$D_{m,\varepsilon} = \{p : \operatorname{dom}(p) > m \wedge (\exists k \in A_\alpha)(p(2k+1) = n \wedge \mu(I_{p(2k)}) < \varepsilon)\}.$$

Claim 3.1.20 *$D_{m,\varepsilon}$ is a dense subset of the Cohen poset.*

Proof. Suppose that $q \in \mathsf{C}$. Let $p \in \mathsf{C}$ be such that:

1. $\operatorname{dom}(p) = 2k+2$, where $k \in A_\alpha$ and $k > \max(\{\operatorname{dom}(q), m\})$.
2. For all $i \in \operatorname{dom}(q)$, $p(i) = q(i)$.
3. $p(2k)$ is the least $i \in \omega$ such that $\mu(I_i) < \varepsilon$.

4. $p(2k+1) = n$.

5. $p(i) = 0$, otherwise.

Then, $p \leq q$ and $p \in D_{m,\varepsilon}$. So, $D_{m,\varepsilon}$ is dense. ■

Since c is a Cohen real over $L(S)$, for every $m \in \omega$ and $\varepsilon > 0$ there exists $p \in D_{m,\varepsilon}$ such that $p \subseteq c$. But then, $\langle k_i^{\alpha,n} : i \in \omega \rangle$ exists.

Let $O_\alpha^n = \bigcup_{i\in\omega} I_{f(k_i^{\alpha,n})}$. So, O_α^n is open and $\mu(O_\alpha^n) \leq \frac{1}{n}$.

Fact 3.1.21 *If $\sigma \subseteq \lambda$, $\sigma \in L(S)$ and σ is infinite, then for all $n \in \omega$, $\bigcup_{\alpha\in\sigma} O_\alpha^n = \mathbb{R}$.*

Proof. Fix $\sigma \subseteq \lambda$ infinite and such that $\sigma \in L(S)$. We will prove that for every $k \geq 1$, $[-k,k] \subseteq \bigcup_{\alpha\in\sigma} O_\alpha^n$.

Suppose otherwise. Then there exists $p \in \mathbb{C}$ such that

$$p \Vdash_{\mathbb{C}} \text{“}(\exists k \geq 1)\left([-k,k] \nsubseteq \textstyle\bigcup_{\alpha\in\sigma} O_\alpha^n\right)\text{”}.$$

Since $\operatorname{dom}(p)$ is finite, there exists $\tau \subseteq \sigma$, τ infinite, such that for every $\alpha, \beta \in \tau$, $\operatorname{dom}(p) \cap A_\alpha = \operatorname{dom}(p) \cap A_\beta$. Then, by definition of the sequences $\langle k_i^{\alpha,n} : i \in \omega \rangle$, there is $m \in \omega$ such that for all $\alpha \in \tau$,

$$\langle k_0, \ldots, k_m \rangle = \langle k_i^{\alpha,n} : i \in \omega \rangle \restriction \operatorname{dom}(p).$$

Let $\varepsilon > \frac{1}{n} - \mu(\bigcup_{i<m} I_{f(k_i)})$. Clearly, $\varepsilon > 0$. Fix $m' \in \omega$ such that $\frac{m' \cdot \varepsilon}{2} > 2k$. Fix $\alpha_0, \ldots, \alpha_{m'-1} \in \tau$ all different. Let $m'' \in \omega$ such that for all $i, j < m'$, $A_{\alpha_i} \cap A_{\alpha_j} \subseteq m''$. Let $q \in \mathbb{C}$ be such that:

1. $\operatorname{dom}(q) = m''$.

2. For all $i \in \operatorname{dom}(p)$, $q(i) = p(i)$.

3. For all $i \in \operatorname{dom}(q) \setminus \operatorname{dom}(p)$, $q(i) = 0$.

Since $n \neq 0$, for all $j < m'$, $\langle k_i^{\alpha_j,n} : i \in \omega \rangle \restriction \operatorname{dom}(p) = \langle k_i^{\alpha_j,n} : i \in \omega \rangle \restriction \operatorname{dom}(q)$.

Since $\frac{m' \cdot \varepsilon}{2} > 2k$, we can find open intervals $J_0, \ldots, J_{m'-1}$ with rational endpoints such that:

a) For every $i < m'$, $\mu(J_i) = \frac{\varepsilon}{2}$.

b) $[-k,k] \subseteq \bigcup_{i<m'} J_i$.

For every $i < m'$, let k_i' be the least $k > \operatorname{dom}(q)$ such that $k \in A_{\alpha_i}$. So, for all $i, j < m'$, if $i \neq j$, then $k_i' \neq k_j'$. Define $q' \in \mathbb{C}$ such that

1. $\operatorname{dom}(q') = 2(\max\{k_i' : i < m'\}) + 2$.

2. For all $i \in \operatorname{dom}(q)$, $q'(i) = q(i)$.

3. For all $i < m'$, $q'(2k_i')$ is the least $j \in \omega$ such that $J_i = I_j$.

4. For all $i < m'$, $q'(2k_i' + 1) = n$.

5. $q'(i) = 0$, otherwise.

But then, $q' \leq p$ and $q' \Vdash_{\mathbb{C}}$ "$[-k, k] \subseteq \bigcup_{i<m'} O^n_{\alpha_i}$". A contradiction. ■

For every $\alpha < \lambda$ and every $n \geq 1$, let $C^n_\alpha = \mathbb{R} \setminus O^n_\alpha$. Then, for all $\alpha < \lambda$, $\mathbb{R} \setminus \bigcup_{1 \leq n < \omega} O^n_\alpha$ is a null set of reals. For every $\alpha < \lambda$ and every $n \geq 1$, let $\dot{C}^n_\alpha$ the canonical $\mathbb{C}$-name in $L(S)$ for C^n_α. Then, by the last claim, the following holds in $L(S)$:

$$\text{For all } \sigma \subseteq \lambda \text{ infinite, all } p \in \mathbb{C} \text{ and all } n \geq 1,\ p \nVdash_{\mathbb{C}} \text{“}\textstyle\bigcap_{\alpha\in\sigma} \dot{C}^n_\alpha \neq \emptyset\text{”}. \qquad (*)$$

Fact 3.1.22 *$(*)$ is true in V*

Proof. For every $p \in \mathbb{C}$, $k \in \omega$ and $n \geq 1$, define in $L(S)$ the following tree $T_{p,k,n}$ of sequences of ordinals less than λ:

- $\langle \alpha_0, ..., \alpha_m \rangle \in T_{p,k,n}$ iff all $\alpha_i < \lambda$ are different ordinals and $p \Vdash_{\mathbb{C}}$ "$[-k, k] \cap \bigcap_{i \leq m} \dot{C}^n_{\alpha_i} \neq \emptyset$"
- $\langle \alpha_0, ..., \alpha_m \rangle \leq \langle \beta_0, ..., \beta_{m'} \rangle$ iff $\langle \alpha_0, ..., \alpha_m \rangle \subseteq \langle \beta_0, ..., \beta_{m'} \rangle$.

Suppose that $(*)$ is false in V. So, there is $\sigma \subseteq \lambda$ infinite, a $q \in \mathbb{C}$ and $n \geq 1$ such that $q \Vdash_{\mathbb{C}}$ "$\bigcap_{\alpha\in\sigma} \dot{C}^n_\alpha \neq \emptyset$". Then there is $p \leq q$ and $k \in \omega$ such that $p \Vdash_{\mathbb{C}}$ "$[-k, k] \cap \bigcap_{\alpha\in\sigma} \dot{C}^n_\alpha \neq \emptyset$". Therefore, $T_{p,k,n}$ has an infinite branch.

If there are $p \in \mathbb{C}$, $n \geq 1$ and $k \in \omega$ such that $T_{p,k,n}$ has an infinite branch, then there exists an infinite sequence $\langle \alpha_i : i < \omega \rangle$ of different ordinals less than λ and $m < \omega$ such that

$$p \Vdash_{\mathbb{C}} \text{“}[-k, k] \cap \textstyle\bigcap_{i \leq m} \dot{C}^n_{\alpha_i} \neq \emptyset\text{”}.$$

But, since for every $m < \omega$, $\bigcap_{i \leq m} C^n_{\alpha_i} \cap [-k, k]$ is a compact set of reals, $\bigcap_{m<\omega} C^n_{\alpha_m} \cap [-k, k] \neq \emptyset$. So, there is $r \leq p$ such that $r \Vdash_{\mathbb{C}}$ "$\bigcap_{m<\omega} \dot{C}^n_{\alpha_m} \neq \emptyset$" and, therefore, $(*)$ is false in V.

So, for all $p \in \mathbb{C}$, all $k \in \omega$ and every $n \geq 1$, $T_{p,k,n}$ is a well-founded tree in $L(S)$. But, by the absoluteness of "being well-founded", $T_{p,k,n}$ is a well-founded tree in V. Therefore, $(*)$ is true in V. ■

We can now finish the proof of Lemma 3.1.19 and of Theorem 3.1.15. Suppose that c is a Cohen real over V and there is a Random real over $L(S, c)$ in $V[c]$. Note that for every $\alpha < \lambda$, $r \in \bigcup_{1 \leq n < \omega} C^n_\alpha$, since $\mathbb{R} \setminus \bigcup_{1 \leq n < \omega} C^n_\alpha$ is null. Pick, in V, a Cohen name $\dot{r}$ for r. For every α, we pick $p_\alpha \in \mathbb{C}$ and $n_\alpha \geq 1$ such that $p_\alpha \Vdash_{\mathbb{C}}$ "$\dot{r} \in \dot{C}^{n_\alpha}_\alpha$". Since $\mathbb{C}$ is countable and λ is uncountable, there is a $\sigma \subseteq \lambda$ infinite, $p \in \mathbb{C}$ and $n \geq 1$ such that for every $\alpha \in \sigma$, $p = p_\alpha$ and $n = n_\alpha$. So, $p \Vdash_{\mathbb{C}}$ "$\bigcap_{\alpha\in\sigma} \dot{C}^n_\alpha \neq \emptyset$", in contradiction with the above Fact. ■

Note that the above theorem really shows that $\underset{\sim}{\Sigma}^1_{n+2}$-absoluteness for Cohen poset and $\underset{\sim}{\Sigma}^1_{n+1}$-absoluteness for Random poset implies that there are no uncountable $\underset{\sim}{\Sigma}^1_n$ well-orderings of the reals. So, as a corollary we get the following result of [B1] (see also [B-Ju])

Corollary 3.1.23 *$\underset{\sim}{\Sigma}^1_4$-absoluteness for Cohen poset and $\underset{\sim}{\Sigma}^1_3$-absoluteness for Random poset implies that ω_1 is an inaccessible cardinal in L.*

Proof. Since for every real $a \in V$, the well-ordering of $\mathsf{R} \cap L[a]$ is $\Sigma^1_2(a)$, there are only countable many reals in $L[a]$. So, $\omega_1^{L[a]} < \omega_1$, for every real a. ■

3.2 Collapsing an inaccessible cardinal

In this section we study the absoluteness properties of Solovay models under ccc forcing extensions. We will show that every $\underset{\sim}{\Sigma}^1_3$ and ccc forcing extension of a Solovay model is also a Solovay model. We will also show in Section 3.3 that this is, in some sense, the optimal result for Solovay models.

Theorem 3.2.1 *Suppose $L(\mathsf{R})^M$ is a Solovay model over V and P is a $\underset{\sim}{\Sigma}^1_3$ and ccc poset in M. Then the $L(\mathsf{R})$ of any P-generic extension of M is also a Solovay model over V.*

Proof. Suppose $L(\mathsf{R})^M$ is a Solovay model over V. Let $\kappa = \omega_1^M$. Force over M with W to obtain a $Coll(\omega, <\kappa)$-generic filter C over V so that $\mathsf{R}^{V[C]} = \mathsf{R}^M$ (Lemma 3.1.1). Let P be a $\underset{\sim}{\Sigma}^1_3$ and ccc poset in M. Notice that for a filter $G \subseteq \mathsf{P}$, G is P-generic over M iff G is P-generic over $V[C]$. Moreover, $\mathsf{R}^{M[G]} = \mathsf{R}^{V[C][G]}$. Thus, to prove the theorem it will be enough to show that every real in $V[C][G]$ is generic over V for a countable poset.

Let $\dot{\mathsf{P}}$ be a $Coll(\omega, <\kappa)$-name for P in V. So, $\Vdash_{Coll(\omega,<\kappa)}$ "$\dot{\mathsf{P}}$ is a ccc poset" and there are a simple $Coll(\omega, <\kappa)$-name for a real $\dot{a}$ and Σ^1_3 formulas $\varphi_{\leq}(x, y; u)$ and $\varphi_{\perp}(x, y; u)$ such that:

1. $\Vdash_{Coll(\omega,<\kappa)}$ "$(\forall xy \in \dot{\omega}^{\omega})(x \dot{\leq}_P y \leftrightarrow \varphi_{\leq}(x, y; \dot{a}))$".
2. $\Vdash_{Coll(\omega,<\kappa)}$ "$(\forall xy \in \dot{\omega}^{\omega})(x \dot{\perp}_P y \leftrightarrow \varphi_{\perp}(x, y; \dot{a}))$".

Let $\mathsf{S} = Coll(\omega, <\kappa) * \dot{\mathsf{P}}$. Note that S is a κ-cc poset.

Notation 3.2.2 *Recall that if A is a projective set and N is a transitive model of (a fragment of) ZFC that contains the parameters of the projective formula that defines A, then A^N denotes the set defined by the relativization to N of this formula.*

- *For every $\alpha < \kappa$ and every $Coll(\omega, <\kappa)$-generic filter C over V let $C_\alpha = C \cap Coll(\omega, <\alpha)$.*
- *For every $Coll(\omega, <\kappa)$-generic filter C over V, let $\mathsf{P} = \dot{\mathsf{P}}[C]$. Moreover, for every $\alpha < \kappa$, if $a = \dot{a}[C]$, the parameter of the definition of P, belongs to $V[C_\alpha]$, $\mathsf{P}_\alpha = \mathsf{P}^{V[C_\alpha]}$. If $a \notin V[C_\alpha]$, then P_α denotes the trivial poset. Let $\dot{\mathsf{P}}_\alpha$ be a $Coll(\omega, <\alpha)$-name for P_α. i.e.,*

  1. $\Vdash_{Coll(\omega,<\alpha)}$ "$(\forall x \in \dot{\omega}^{\omega})(x \in \dot{P}_\alpha \leftrightarrow \varphi_{\leq}(x, x; \dot{a}))$"
  2. $\Vdash_{Coll(\omega,<\alpha)}$ "$(\forall xy \in \dot{\omega}^{\omega})(x \dot{\leq}_{P_\alpha} y \leftrightarrow \varphi_{\leq}(x, y; \dot{a}))$"

3. $\Vdash_{Coll(\omega,<\alpha)}$ "$(\forall xy \in \dot{\omega}^{\omega})(x \dot{\perp}_{P_\alpha} y \leftrightarrow \varphi_\perp(x, y; \dot{a}))$"

We may assume that for every $\alpha < \kappa$, $\Vdash_{Coll(\omega,<\alpha)}$ "$\dot{\mathsf{P}}_\alpha$ is a poset". Finally, for every $\alpha < \kappa$, let $\mathsf{S}_\alpha = Coll\,(\omega, < \alpha) * \dot{\mathsf{P}}_\alpha$.

- *Let $\dot{\mathsf{R}}$ denote the set of all simple $Coll\,(\omega, < \kappa)$-names for reals and for every $\alpha < \kappa$, let $\dot{\mathsf{R}}_\alpha$ be the set of all simple $Coll\,(\omega, < \alpha)$-names for reals.*

Notice that, by Shoenfield's Absoluteness Theorem, if $\xi \leq \xi'$ then $\mathsf{S}_\xi \subseteq \mathsf{S}_{\xi'}$ and, since $Coll\,(\omega, < \kappa)$ is a κ-cc poset, $\mathsf{S} = \bigcup_{\xi<\kappa} \mathsf{S}_\xi$. Thus, for every subposet X of S of cardinality less than κ there exists $\alpha < \kappa$ such that X is a subposet of S_α.

For every $\alpha < \kappa$, let $\xi\,(\alpha)$, if it exists, be the least $\xi < \kappa$ such that for every $\xi' \geq \xi$ the following holds: For every simple $Coll\,(\omega, < \alpha)$-name $\dot{A}$ for a subset of $\dot{\mathsf{R}}_\alpha$, every simple $Coll\,(\omega, < \alpha)$-name for a real $\dot{c}$ and every $q \in Coll(\omega, < \xi')$, if

$$q \Vdash_{Coll(\omega,<\kappa)} \text{“}\dot{A} \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”},$$

then

$$q \Vdash_{Coll(\omega,<\xi')} \text{“}\dot{A} \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”}.$$

Lemma 3.2.3 *For every $\alpha < \kappa$, $\xi\,(\alpha)$ exists.*

Proof. Note that "x codes a maximal antichain of $\dot{\mathsf{P}}$ below $\dot{c}$" iff "x codes a maximal antichain of $\dot{\mathsf{P}}_c$", where P_c is the subposet of P consisting of all conditions of P below c. Since P is a $\undertilde{\Sigma}^1_3$ poset, P_c also is a $\undertilde{\Sigma}^1_3$ poset. So, by Fact 2.1.15, "x codes a maximal antichain of $\dot{\mathsf{P}}$ below $\dot{c}$" is a Π^1_3 predicate on x, and (the codes of) $\dot{c}$ and $\dot{a}$.

In order to simplify the notation, we work with the algebra r.o. $(Coll\,(\omega, < \kappa))$. Fix $\alpha < \kappa$. Note that every subset in $V[C]$ of reals in $V[C_\alpha]$ is countable, and hence can be coded by a real in $V[C]$. So, by Shoenfield's Absoluteness Theorem, for every simple $Coll\,(\omega, < \alpha)$-name $\dot{A}$ for a subset of $\dot{\mathsf{R}}_\alpha$ and every simple $Coll\,(\omega, < \alpha)$-name for a real $\dot{c}$, if

$$[\![\dot{A} \text{ does not code a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}]\!]_{Coll(\omega,<\kappa)} \neq 0,$$

then there exists a final segment $F$of κ such that for every $\xi \in F$,

$$[\![\dot{A} \text{ does not code a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}]\!]_{Coll(\omega,<\kappa)} \Vdash_{Coll(\omega,<\xi)}$$
$$\Vdash_{Coll(\omega,<\xi)} \text{“}\dot{A} \text{ does not code a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”}$$

For all simple $Coll\,(\omega, < \alpha)$-name $\dot{A}$ for a subset of $\dot{\mathsf{R}}_\alpha$ and all simple $Coll\,(\omega, < \alpha)$-name for a real $\dot{c}$, let $D_{\dot{A},\dot{c}}$ be this final segment, if $[\![\dot{A}$ does not code a maximal antichain of $\dot{\mathsf{P}}$ below $\dot{c}]\!]_{Coll(\omega,<\kappa)} \neq 0$. Let $D_{\dot{A},\dot{c}} = \kappa$, otherwise.

Since there are less than κ simple $Coll\,(\omega, < \alpha)$-names $\dot{A}$ for subsets of $\dot{\mathsf{R}}_\alpha$ and less than κ simple $Coll\,(\omega, < \alpha)$-names $\dot{c}$ for a real, the intersection D of all $D_{\dot{A},\dot{c}}$ is a non-empty final segment of κ. Let $\xi(\alpha)$ be the least $\xi \in D$ greater than α. It is easy to check that $\xi(\alpha)$ works. ■

We continue with the proof of Theorem 3.2.1.

For every $\alpha < \kappa$, let $\bar{\mathsf{S}}_\alpha$ be the complete subalgebra of $\mathrm{r.o.}(\mathsf{S})$ generated by S_α. And for every $\alpha \leq \xi < \kappa$, let $\bar{\mathsf{S}}^\xi_\alpha$ be the complete subalgebra of $\bar{\mathsf{S}}_\xi$ generated by S_α.

Claim 3.2.4 *For every $\xi \geq \xi(\alpha)$, $\bar{\mathsf{S}}_\alpha = \bar{\mathsf{S}}^\xi_\alpha$.*

Proof. It is clear that $\mathsf{S}_\alpha \subseteq \bar{\mathsf{S}}^\xi_\alpha$. So, since $\bar{\mathsf{S}}^\xi_\alpha$ is a complete subalgebra of $\mathrm{r.o.}(\mathsf{S})$, $\bar{\mathsf{S}}_\alpha \subseteq \bar{\mathsf{S}}^\xi_\alpha$. To prove that $\bar{\mathsf{S}}^\xi_\alpha \subseteq \bar{\mathsf{S}}_\alpha$, we only need to show that $\bar{\mathsf{S}}_\alpha$ is a complete subalgebra of $\bar{\mathsf{S}}_\xi$. It is clear that $\bar{\mathsf{S}}_\alpha$ is a subalgebra of $\bar{\mathsf{S}}_\xi$. So, let $A \subseteq \mathsf{S}_\alpha$ be a maximal antichain below $\langle p, \dot{c}\rangle \in \mathsf{S}_\alpha$ and suppose that A is not a maximal antichain of S_ξ below $\langle p, \dot{c}\rangle$. Let $\dot{A}_1 = \{\dot{b} \in \dot{\mathsf{R}}_\alpha : (\exists q \in Coll(\omega, <\alpha))(\langle q, \dot{b}\rangle \in A)\}$. Then there exists $q \in Coll(\omega, <\xi)$ such that

$$q \Vdash_{Coll(\omega,<\xi)} \text{“}\dot{A}_1 \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”}$$

Since $\xi \geq \xi(\alpha)$,

$$q \Vdash_{Coll(\omega,<\kappa)} \text{“}\dot{A}_1 \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”}.$$

So, A is no a maximal antichain of S below $\langle p, \dot{c}\rangle$. A contradiction, since $\bar{\mathsf{S}}_\alpha$ is a complete subalgebra of $\mathrm{r.o.}(\mathsf{S})$ and A is a maximal antichain of $\bar{\mathsf{S}}_\alpha$ below $\langle p, \dot{c}\rangle$. ■

Definition 3.2.5 *Let E be the following poset:*

- *$h \in \mathsf{E}$ iff there exists $\alpha < \kappa$ such that h is a complete embedding from $\bar{\mathsf{S}}_\alpha$ into* $\mathrm{r.o.}(Coll(\omega, <\kappa))$.
- *$h \leq h'$ iff $h' \subseteq h$*

Note that the $\bar{\mathsf{S}}_\alpha$, $\alpha < \kappa$, have cardinality less than κ. Hence, by Kripke's Theorem (see [J2], Theorem 62), $\mathsf{E} \neq \emptyset$.

Lemma 3.2.6 *Let $H \subseteq \mathsf{E}$ be a generic filter over V. Then $e = \bigcup_{h \in H} h$ is a complete embedding from* $\mathrm{r.o.}(\mathsf{S})$ *into* $\mathrm{r.o.}(Coll(\omega, <\kappa))$.

Proof. It is clear that $e = \bigcup_{h \in H} h$ is an embedding from $\mathrm{dom}(e)$ into $\mathrm{r.o.}(Coll(\omega, <\kappa))$. Thus to prove the lemma it will be enough to show that for every $h \in \mathsf{E}$ and every $\beta < \kappa$ there is $h' \leq h$ such that $\mathrm{dom}(h') = \bar{\mathsf{S}}_\xi$ and $\xi \geq \beta$.

So, fix h and β. Let α such that h is a complete embedding from $\bar{\mathsf{S}}_\alpha$ into $\mathrm{r.o.}(Coll(\omega, <\kappa))$. Let $\xi \geq \xi(\alpha)$, $\xi \geq \beta$. By claim 3.2.4, $\bar{\mathsf{S}}^\xi_\alpha = \bar{\mathsf{S}}_\alpha$. Hence, (see [J2], 25.12), we can extend h to a complete embedding h' form $\bar{\mathsf{S}}_\xi$ into $\mathrm{r.o.}(Coll(\omega, <\kappa))$.
■

Lemma 3.2.7 *E is a $<\kappa$-closed poset. Hence, forcing with E does not add new bounded subsets of κ. In particular, it does not add reals.*

Proof. Let $\langle h_\alpha : \alpha < \gamma \rangle$, with $\gamma < \kappa$, be a decreasing sequence of elements of E. For each $\alpha < \gamma$, let ξ_α be such that $\mathrm{dom}\,(h_\alpha) = \bar{\mathsf{S}}_{\xi_\alpha}$. Let $\xi_\gamma = \sup_{\alpha<\gamma}(\xi_\alpha)$ and let $\xi \geq \xi_\gamma$. Let $h_\gamma = \bigcup_{\alpha<\gamma} h_\alpha$. Let B the complete subalgebra of $\bar{\mathsf{S}}_\xi$ generated by $\bigcup_{\alpha<\gamma} \bar{\mathsf{S}}_{\xi_\alpha}$. So, h_γ extends uniquely to a complete embedding h from B into r.o. $(Coll\,(\omega, <\kappa))$. Now (see [J2], 25.12) we can extend h to a complete embedding h' from $\bar{\mathsf{S}}_\xi$ into r.o. $(Coll\,(\omega, <\kappa))$. It is clear that h' is below all h_α, $\alpha < \gamma$. ∎

Now we complete the proof of Theorem 3.2.1. We need to show that every real in $V[C * G]$, where $C * G$ is a S-generic filter over V, is generic over V for a countable poset. So, let $\dot{r}$ be a simple S-name for a real, $\dot{r} \in V$. Let $\alpha < \kappa$ be such that $\dot{r}$ is a S_α-name. Suppose H is E^V-generic filter over $V[C * G]$. Notice that by the Product Lemma, $C * G$ is S^V-generic over $V[H]$ and $V[C * G][H] = V[H][C * G]$. We have that $e = \bigcup_{h \in H} h$ completely embeds r.o. (S) onto a complete subalgebra Q of r.o. $(Coll\,(\omega, <\kappa))$. Let $h \in H$ be such that $h : \bar{\mathsf{S}}_\xi \to$ r.o. $(Coll\,(\omega, <\kappa))$ with $\xi \geq \xi\,(\alpha)$ and let $\zeta < \kappa$ such that $h[\bar{\mathsf{S}}_\xi] \subseteq$ r.o. $(Coll\,(\omega, <\zeta))$. Notice that $\mathsf{Q}' = \mathsf{Q} \cap$r.o. $(Coll\,(\omega, <\zeta))$ is a complete subalgebra of Q. Then $g = e[C * G] \cap \mathsf{Q}'$ is a Q'-generic filter over $V[H]$. Further,

$$e_*(\dot{r}) = \{\langle e(p, \dot{b}), \check{n} \rangle : \langle p, \dot{b}, \check{n} \rangle \in \dot{r}\}$$

is a Q'-name and $r = \dot{r}[C * G] = e_*\,(\dot{r})\,[g]$. By Lemma 3.2.7, $\mathsf{Q}' \in V$ and so g is Q'-generic over V. Also, Q' is countable in $V[C * G]$. Also, by Lemma 3.2.7, $g \in V[C * H]$ and $e_*\,(\dot{r}) \in V[C * G]$. Thus, in $V[C * G]$, r belongs to a countable forcing extension of V. ∎

Corollary 3.2.8 *The following are equiconsistent (modulo ZFC):*

1. *There exists an inaccessible cardinal.*
2. $L\,(\mathsf{R})$*-two-step absoluteness for* $\underset{\sim}{\Sigma}^1_3$ *and ccc forcing*
3. $\underset{\sim}{\Sigma}^1_4$*-absoluteness under Cohen and Random forcing notions.*

Proof. (1) implies (2) follows from Theorem 3.2.1 and Lemma 3.1.6. (2) implies (3) is trivial. (3) implies (1) is a result of J. Bagaria and W. H. Woodin (see [B1], 2.1.1.3 or [B-Ju]. See also Corollary 3.1.23). ∎

From Theorem 3.1.15 and the fact that Solovay models are Borel ccc absolute, it readily follows that in a Solovay model there are no uncountable projective well-orderings of reals. Hence, there are no uncountable projective gaps. We will now show that in a Solovay model there are no $\underset{\sim}{\Sigma}^1_3$ Suslin trees.

Definition 3.2.9 *We say that a poset* P *is a* $\underset{\sim}{\Sigma}^1_n$-indestructible-ccc poset *iff for every* $\underset{\sim}{\Sigma}^1_n$ *and ccc poset* Q*,*

$$\Vdash_{\mathsf{Q}} \text{“}\mathsf{P} \textit{ is a ccc poset”}.$$

Similarly, we define the $\underset{\sim}{\Pi}^1_n$-indestructible-ccc posets *and the* $\underset{\sim}{\Delta}^1_n$-indestructible-ccc posets. *Finally,* P *is a* projective-indestructible-ccc poset *iff for all* $n \geq 1$*,* P *is a* $\underset{\sim}{\Sigma}^1_n$*-indestructible-ccc poset*

Theorem 3.2.10 *Let $L(\mathsf{R})^M$ be a Solovay model over V. Then, every ccc poset $\mathsf{P} \in L(\mathsf{R})^M$ is a $\underset{\sim}{\Sigma}^1_3$-indestructible-ccc poset.*

Proof. It is essentially the same proof as that of Theorem 2.3.14.

Suppose $L(\mathsf{R})^M$ is a Solovay model over V. Let $\kappa = \omega_1^M$. Force over M with W to obtain a $Coll(\omega, <\kappa)$-generic filter C_0 over V so that $\mathsf{R}^{V[C_0]} = \mathsf{R}^M$ (Lemma 3.1.1). We work in $V[C_0]$.

Let $\mathsf{P} \in L(\mathsf{R})^{V[C_0]}$ be a ccc poset so that the set, the ordering and the incompatibility relation of P are definable with a real a and an ordinal α as parameters. By the Factor Lemma for the Levy-collapse, we may assume that the parameters of the definition of P are in the ground model V. Further, since $Coll(\omega, <\kappa)$ is an almost-homogeneous poset, we may assume that

$$\Vdash_{Coll(\omega,<\kappa)} \text{“}\dot{\mathsf{P}} \text{ is a ccc poset”}.$$

Let Q be a $\underset{\sim}{\Sigma}^1_3$ and ccc poset in $V[C_0]$. Suppose that there exists a filter $G \subseteq \mathsf{Q}$ generic over $V[C_0]$ such that $V[C_0 * G] \models$ “$\dot{\mathsf{P}}[C_0 * G]$ is not ccc”. Let $\mathsf{S} = Coll(\omega, <\kappa) * \dot{\mathsf{Q}}$ and let $\dot{A} = \{\tau_i : i < \kappa\}$ be a S-name for an uncountable maximal antichain of $\dot{\mathsf{P}}[C_0 * G]$.

By Lemma 3.2.6 and Lemma 3.2.7, we know that there exists a $<\kappa$-closed poset $\mathsf{E} \in V$ such that for every $H \subseteq \mathsf{E}$ generic over V there exists a complete embedding e from S to $Coll(\omega, <\kappa)$ in $V[H]$.

Let $V_0 = V[H][C_0 * G]$ be a $\mathsf{E} \times \mathsf{S}$ forcing extension of V. That is, H is E-generic over V, C_0 is $Coll(\omega, <\kappa)$-generic over $V[H]$ and G is $\dot{\mathsf{Q}}[C_0]$-generic over $V[H][C_0]$. Since E is $<\kappa$-closed, $L(\mathsf{R})^{V[C_0*G]} = L(\mathsf{R})^{V_0}$ and $\omega_1^{V[C_0*G]} = \omega_1^{V_0}$. Hence,

$$V_0 \models \text{“}\dot{A}[C_0 * G] \text{ is an uncountable antichain of } \dot{\mathsf{P}}[C_0 * G]\text{”}.$$

Let $e \in V[H]$ be the generic complete embedding given by H. Then, there is a filter $C_1 \subseteq Coll(\omega, <\kappa)$ generic over $V[H]$ such that $C_0 * G = e^{-1}(C_1)$. Let $e_*(\dot{A}) = \{e_*(\tau_i) : i < \kappa\} \in V[H]$ be the e_*-image of $\dot{A}$. Since, for every $i < \kappa$, $V[H] \models$ “ $\Vdash_{\mathsf{S}} \tau_i \in \dot{\mathsf{P}}$”, “$\tau_i \in \dot{\mathsf{P}}$” is a formula with only a,α and τ_i as parameters and e is a complete embedding,

$$V[H] \models \text{“} \Vdash_{Coll(\omega,<\kappa)} e_*(\tau_i) \in \dot{\mathsf{P}}\text{”}.$$

Hence,

$$V[H][C_1] \models \text{“}e_*(\tau_i)[C_1] \in \dot{\mathsf{P}}[C_1]\text{”},$$

for every $i < \kappa$.

Note that, $V[C_1] \models$ “$\dot{\mathsf{P}}[C_1]$ is a ccc poset” and that, in $V[C_1]$, E is a σ-closed poset. But no σ-closed poset can kill the ccc-ness of a ccc poset. So $V[C_1][H] \models$ “$\dot{\mathsf{P}}[C_1]$ is a ccc poset” and, by the Product Lemma,

$$V[H][C_1] \models \text{“}\dot{\mathsf{P}}[C_1] \text{ is a ccc poset”}.$$

Thus, we can find $i, j < \kappa$, $i \neq j$, such that $V[H][C_1] \models$ “$\tau_i[C_1]$, $\tau_j[C_1]$ are compatible”.

Now, $L(\mathsf{R})^{V[H][C_1]} = L(\mathsf{R})^{V[C_1]}$ and $L(\mathsf{R})^{V[H][C_0]} = L(\mathsf{R})^{V[C_0]}$, since E is a $<\kappa$-closed poset. Hence $L(\mathsf{R})^{V[H][C_1]}$ and $L(\mathsf{R})^{V[H][C_0]}$ are Solovay models over V. Therefore, by Theorem 3.2.1, $L(\mathsf{R})^{V_0}$ is also a Solovay model over V. Moreover, since $\omega_1^{V_0} = \omega_1^{V[H][C_1]}$ and $L(\mathsf{R})^{V_0} \subseteq L(\mathsf{R})^{V[H][C_1]}$, by Lemma 3.1.6, there is an elementary embedding j from $L(\mathsf{R})^{V_0}$ into $L(\mathsf{R})^{V[H][C_1]}$ which is the identity on the reals and ordinals. So, by absoluteness between $L(\mathsf{R})^{V_0}$ and $L(\mathsf{R})^{V[H][C_1]}$ (see [Ku], VII.7.13),

$$V_0 \models \text{“}\tau_i[C_0 * G], \tau_j[C_0 * G] \text{ are compatible”}.$$

But since E is $<\kappa$-closed, $L(\mathsf{R})^{V_0} = L(\mathsf{R})^{V[C_0 * G]}$ and hence,

$$V[C_0 * G] \models \text{“}\tau_i[C_0 * G], \tau_j[C_0 * G] \text{ are compatible”}.$$

A contradiction. ■

Corollary 3.2.11 *If $L(\mathsf{R})^M$ is a Solovay model over V, then in $L(\mathsf{R})^M$ there are no $\underset{\sim}{\Sigma}^1_3$ Suslin trees.*

Proof. If T is a $\underset{\sim}{\Sigma}^1_3$ Suslin tree, then $\mathsf{P}_T = \langle T, \geq_T \rangle$ is $\underset{\sim}{\Sigma}^1_3$ ccc poset. But, in every P_T-generic extension adding an uncountable branch to T, P_T is not ccc. A contradiction with Theorem 3.2.10. ■

3.3 Collapsing a Σ_ω-Mahlo cardinal

This section is devoted to proving the following theorem:

Theorem 3.3.1 *The following are equiconsistent (modulo ZFC)*

1. *There exists a Σ_ω-Mahlo cardinal.*
2. *$L(\mathsf{R})$-two-step absoluteness for projective and ccc posets.*
3. *$\underset{\sim}{\Sigma}^1_4$-absoluteness for projective and ccc posets.*

We first define the Δ_n-Mahlo cardinals, Σ_n-Mahlo cardinals, the Π_n-Mahlo cardinals and the Σ_ω-Mahlo cardinals, we fix their place in the large cardinal hierarchy (below a Mahlo) and study some of their properties that we use in the proof of the theorem. We divide the proof of the theorem in two parts (subsection 3.3.2 and subsection 3.3.3). In the first part we will show that every projective and ccc forcing extension of a Σ_ω-Mahlo Solovay model over V is also a Σ_ω-Mahlo Solovay model in V. In the second part we prove that $\underset{\sim}{\Sigma}^1_4$-absoluteness for all projective and ccc posets implies that ω_1 is a Σ_ω-Mahlo cardinal in L. Finally, we observe that both proofs really give an almost level-by-level proof: the first shows that for all $n \geq 3$, every $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc forcing extension of a Π_n-Mahlo Solovay model over V is a Π_n-Solovay model over V. The second shows that for all $n \geq 2$, $\underset{\sim}{\Sigma}^1_4$-absoluteness for $\underset{\sim}{\Delta}^1_{n+2}$ and ccc forcing notions implies that ω_1 is a Σ_n-Mahlo cardinal in L.

3.3.1 Σ_ω-Mahlo cardinals

Definition 3.3.2 *Let κ be a cardinal. $C \subseteq \kappa$ is a* Σ_n-closed and unbounded *subset of κ, a* Σ_n-club in κ *for short, iff C is a club in κ and there exists a Σ_n formula, $\varphi(x; y)$ and $a \in V_\kappa$ such that for every $\alpha < \kappa$*

$$\alpha \in C \text{ iff } V_\kappa \models \varphi(\alpha; a)$$

i.e., C is definable over V_κ with a Σ_n formula with parameters from V_κ. Similarly, we define Π_n-clubs in κ *by substituting Π_n for Σ_n in the above definition. A* Δ_n-club in κ *is a club in κ that is both Σ_n and Π_n.*

Note that for every cardinal κ and every club $C \subseteq \kappa$, C is a Σ_0-club iff it is a Π_0-club iff it is a Δ_0-club. Moreover, for every $n \in \omega$, if C is a Σ_n-club (Π_n-club, Δ_n-club), then C is a Σ_m-club, Π_m-club and Δ_m-club for every $m > n$.

Note also that for every $n \in \omega$, if C and D are Σ_n-clubs (Π_n-clubs, Δ_n-clubs) then $C \cap D$ is also a Σ_n-club (Π_n-club, Δ_n-club). So for every $n \in \omega$, the collection of all Σ_n-clubs (Π_n-clubs, Δ_n-clubs) has the finite intersection property. So, these collections generate non-principal and proper filters over κ.

Definition 3.3.3 *Let κ be a cardinal. $S \subseteq \kappa$ is a* Σ_n-stationary *subset of κ iff for all Σ_n-club C in κ, $S \cap C \neq \emptyset$. Similarly, we define* Π_n-stationary *and* Δ_n-stationary *subset of κ.*

Note that every stationary subset of κ is a Σ_n-stationary (Π_n-stationary, Δ_n-stationary) set for every $n \in \omega$. Moreover, if C is a Σ_n-club (Π_n-club, Δ_n-club) and S is a Σ_n-stationary (Π_n-stationary, Δ_n-stationary) set, then $C \cap S$ it is also a Σ_n-stationary (Π_n-stationary, Δ_n-stationary) set. Finally, for every $n \in \omega$, every Σ_n-stationary (Π_n-stationary, Δ_n-stationary) subset of κ is unbounded (because the final segments of κ are Σ_0-clubs) and hence, of cardinality κ.

Definition 3.3.4 *Let κ be a cardinal. κ is a* Σ_n-Mahlo cardinal (Π_n-Mahlo cardinal, Δ_n-Mahlo cardinal) *iff κ is an inaccessible cardinal and the set I of all inaccessible cardinals below κ is a Σ_n-stationary (respectively, Π_n-stationary, Δ_n-stationary) subset of κ. Finally, κ is a* Σ_ω-Mahlo cardinal *iff κ is a Σ_n-Mahlo cardinal for all $n \in \omega$.*

Note that a cardinal κ is Σ_0-Mahlo iff it is Π_0-Mahlo iff it is Δ_0-Mahlo. Moreover for every $n \in \omega$, if κ is a Σ_n-Mahlo (Π_n-Mahlo, Δ_n-Mahlo), then κ is Σ_m-Mahlo, Π_m-Mahlo and Δ_m-Mahlo for every $m < n$. In particular, if κ is Σ_ω-Mahlo cardinal, then κ is also Π_n-Mahlo and Δ_n-Mahlo for every $n \in \omega$.

It is obvious that every Mahlo cardinal is a Σ_ω-Mahlo cardinal. Moreover,

Fact 3.3.5 *If κ is a Mahlo cardinal, then the set of Σ_ω-Mahlo cardinals below κ is a stationary subset of κ.*

Proof. Suppose that κ is a Mahlo cardinal. Let C be a club in κ.

Note that if $D \subseteq \lambda$ is a Σ_n-club on λ, $\lambda < \kappa$ inaccessible, for some $n \in \omega$, then $D' = D \cup (\kappa \setminus \lambda)$ is a club in κ and, since $V_x = y$ is a Π_1 function, it is Π_1 definable, with λ as parameter, on V_κ: for all $\xi < \kappa$, $\xi \in D'$ iff

$$V_\kappa \models (\xi < \lambda \wedge Sat\left(V_\lambda, \ulcorner \varphi \urcorner, \xi\right)) \vee \lambda \leq \xi,$$

where φ is the Σ_n formula that defines D in V_λ, $\ulcorner \varphi \urcorner$ is the Gödel number of φ and $Sat\,(v_0, v_1, v_2)$ is the Δ_1 formula that defines the satisfaction relation for sets. i.e., $Sat\,(v_0, v_1, v_2)$ iff the set v_0 satisfies the formula (of Gödel number) v_1 by means of the sequence v_2.

Let $\langle D_\alpha : \alpha < \kappa \rangle$ be an enumeration of all clubs in κ which are first order definable (with parameters) over V_κ and such that for every inaccessible cardinal $\lambda < \kappa$, $\langle D_\alpha : \alpha < \lambda \rangle$ enumerates all clubs in κ such that $D_\alpha \cap \lambda$ is a club in λ which is first order definable on V_λ. Then $D = \Delta_{\alpha<\kappa} D_\alpha$ is a club in κ. Let $E = C \cap D$. Since κ is a Mahlo cardinal,

$$\langle V_\kappa, \in, E \rangle \models \forall \alpha \exists \mu (\alpha < \mu \wedge \mu \text{ is inaccessible} \wedge \mu \in E)$$

and there is an inaccessible cardinal λ such that $\langle V_\lambda, \in, E \cap V_\lambda \rangle \prec \langle V_\kappa, \in, E \rangle$ (see [K], I.6.2). So,

$$\langle V_\lambda, \in, E \cap V_\lambda \rangle \models \forall \alpha \exists \mu (\alpha < \mu \wedge \mu \text{ is inaccessible} \wedge \mu \in E).$$

Notice that λ is a Σ_ω-Mahlo cardinal: Let $D \subseteq \lambda$ be a Σ_n-definable ($n \in \omega$) club in λ. So, for some $\alpha < \lambda$, $D_\alpha \cap \lambda = D$. Let μ be a inaccessible cardinal such that $\alpha < \mu$ and $\mu \in E \cap V_\lambda$. So, $\mu \in D_\alpha$ and hence $\mu \in D$. Moreover, since $E \cap V_\lambda = E \cap \lambda$ is unbounded in λ and E is a club, $\lambda \in E$ and so $\lambda \in C$. Therefore, the set of Σ_ω-Mahlo cardinals below κ is a stationary subset of κ ■

Lemma 3.3.6 *Let $n \geq 1$. Suppose that κ is a cardinal and let $I_n = \{\lambda < \kappa : \lambda$ is inaccessible $\wedge V_\lambda \prec_n V_\kappa\}$. Then,*

1. *κ is a Π_n-Mahlo cardinal iff κ is an inaccessible cardinal and the set I_n is a Π_n-stationary subset of κ.*

2. *If κ is an inaccessible cardinal and I_n is an unbounded subset of κ, then κ is a Σ_n-Mahlo cardinal.*

So, κ is a Σ_ω-Mahlo cardinal iff κ is inaccessible and for every $n \in \omega$, I_n is a Π_n-stationary subset of κ.

Proof. We need the following fact:

Fact 3.3.7 *If κ is an inaccessible cardinal, then for all $n \geq 1$, $C_n = \{\alpha < \kappa : V_\alpha \prec_n V_\kappa\}$ is a Π_n-club on κ.*

Proof. Suppose that κ is inaccessible. It is easy to see that C_n is a closed and unbounded subset of κ (see [K] I.6.1). So, we only need to see that C_n is a Π_n-definable over V_κ. Let $\sigma_n(v_0, v_1)$ the formula that defines the satisfaction relation for Σ_n formulas. i.e., $\sigma_n(v_0, v_1)$ iff v_0 is (the Gödel number of) a Σ_n formula φ, $v_1 \in V_\kappa$ and $V_\kappa \models \varphi(v_1)$. Note that $\sigma_n(v_0, v_1)$ is a Σ_n formula (see [J2] 14.18 and ff.).

We claim that for every $\alpha < \kappa$, $\alpha \in C_n$ iff

$$V_\kappa \models \left(\forall \ulcorner\varphi\urcorner \in \Sigma_n\right)\left(\forall a \in V_\alpha\right)\left(\sigma_n\left(\ulcorner\varphi\urcorner, a\right) \rightarrow Sat\left(V_\alpha, \ulcorner\varphi\urcorner, a\right)\right), \qquad (*)$$

where Σ_n denotes the set of (Gödel numbers of) Σ_n formulas. Since $\sigma_n(v_0, v_1)$ is a Σ_n formula with $n \geq 1$, $Sat(v_0, v_1, v_2)$ is a Δ_1 formula, Σ_n is a Δ_1 definable set and V_α is a Π_1 definable set with parameter α, it is easy to see that the right-hand of $(*)$ is a Π_n formula with α as parameter.

It is clear that if $\alpha < \kappa$ and $\alpha \in C_n$, then $(*)$. Now, we show by induction on $m \leq n$ that if $(*)$ holds for α then for every Σ_m formula $\varphi(x)$ and every $a \in V_\alpha$, $V_\alpha \models \varphi(a)$ iff $V_\kappa \models \varphi(a)$:

$\underline{m = 0}$: Clear since the Σ_0 formulas are absolute between transitive sets containing all the parameters of the formula.

$\underline{m+1}$: By inductive hypothesis the Σ_{m+1} formulas are upward absolute. So suppose that $\varphi(x)$ is a Σ_{m+1} formula, $a \in V_\alpha$ and $V_\kappa \models \varphi(a)$. So, since $m+1 \leq n$, $\varphi(x)$ is a Σ_n formula and so, $V_\kappa \models \sigma_n\left(\ulcorner\varphi\urcorner, a\right)$. Hence, by $(*)$, $V_\kappa \models Sat\left(V_\alpha, \ulcorner\varphi\urcorner, a\right)$. Therefore, $V_\kappa \models \varphi(a)^{V_\alpha}$. i.e., $V_\alpha \models \varphi(a)$.

Hence, if α satisfies $(*)$, then $V_\alpha \preceq_n V_\kappa$. ■

Now, we prove the lemma.

(1) If κ is a Π_n-Mahlo cardinal, then since $C_n = \{\alpha < \kappa : V_\alpha \preceq_n V_\kappa\}$ is a Π_n-club on κ, the set I of all inaccessible cardinals below κ is Π_n-stationary and $I_n = I \cap C_n$, I_n is Π_n-stationary. The other direction is obvious.

(2) Suppose that κ is an inaccessible cardinal and there are unbounded many inaccessible cardinals λ such that $V_\lambda \preceq_n V_\kappa$. Let C be a club on κ which is definable over V_κ by means of a Σ_n formula $\varphi(x)$ with a parameter $a \in V_\kappa$. Then,

$$V_\kappa \models \forall\alpha\exists\beta\left(\alpha < \beta \wedge \varphi(\beta)\right)$$

Since $\varphi(x)$ is a Σ_n formula, the right-hand is a Π_{n+1} sentence with a as parameter. Since there are unbounded many inaccessible cardinals such that $V_\lambda \preceq_n V_\kappa$, there is an inaccessible cardinal λ such that $a \in V_\lambda$ and $V_\lambda \preceq_n V_\kappa$. Therefore,

$$V_\lambda \models \forall\alpha\exists\beta\left(\alpha < \beta \wedge \varphi(\beta)\right)$$

by downward absoluteness of Π_{n+1} formulas for V_λ, V_κ. So, $C \cap \lambda$ is unbounded in λ. But then $\lambda \in C$. ■

Corollary 3.3.8 *For every $n \in \omega$, $n \geq 1$, every Π_n-Mahlo cardinal is a Σ_n-Mahlo cardinal.*

Proof. Follows from (1) and (2) of Lemma 3.3.6. ■

Corollary 3.3.9 *If κ is an inaccessible cardinal limit of inaccessible cardinals, then κ is a Σ_1-Mahlo cardinal.*

Proof. Recall that for every uncountable infinite cardinal $\lambda < \kappa$, $H_\lambda \preccurlyeq_1 H_\kappa$. Since for every inaccessible cardinal μ, $H_\mu = V_\mu$, if κ is an inaccessible cardinal limit of inaccessible cardinals, there are unbounded many inaccessible cardinals λ such that $V_\lambda \preccurlyeq_1 V_\kappa$. So, by (2) of Lemma 3.3.6, κ is a Σ_1-Mahlo cardinal. ■

Note that if κ is a Σ_0-Mahlo cardinal then κ is an inaccessible cardinal limit of inaccessible cardinals. So, Corollary 3.3.9 implies that for every Σ_0-Mahlo cardinal is Σ_1-Mahlo.

Definition 3.3.10 *Let κ be a regular cardinal. κ is* 0-inaccessible *iff κ is an inaccessible cardinal; κ is* α-inaccessible ($\alpha > 0$) *iff for every $\beta < \alpha$, κ is the limit of β-inaccessible cardinals; κ is* hyperinaccessible *iff κ is a κ-inaccessible cardinal.*

It is easy to see that for every α and every κ, "κ is a α-inaccessible cardinal" is a relation between α and κ which is Δ_2 definable over V_κ. So,

Fact 3.3.11 *If κ is a Δ_2-Mahlo cardinal, then κ is an hyperinaccessible cardinal.*

Proof. Suppose that κ is a Δ_2-Mahlo cardinal. We prove that κ is hyperinaccessible by showing by induction on $\alpha < \kappa$ that κ is α-inaccessible: If $\alpha = 0$ or α is limit, it is clear by the definitions of Δ_2-Mahlo cardinal and of α-inaccessible cardinal, respectively. So suppose that $\alpha = \beta + 1$ and assume that κ is a β-inaccessible cardinal. Suppose, towards a contradiction, that κ is not $\beta + 1$-inaccessible. Then there exists $\gamma < \kappa$ such that for all $\mu > \gamma$, if μ is a cardinal limit of δ-inaccessible cardinals with $\delta < \beta$, then μ is a singular cardinal. But then,

$$C = \{\mu < \kappa : \gamma < \mu \wedge (\forall \delta < \beta)\, (\mu \text{ is limit of } \delta\text{-inaccessible cardinals})\}$$

is a Δ_2-club in κ of singular cardinals. A contradiction. ■

Corollary 3.3.12 *The least Σ_1-Mahlo cardinal is not a Δ_2-Mahlo cardinal.* ■

Fact 3.3.13 *The least hyperinaccessible cardinal is not a Π_2-Mahlo cardinal.*

Proof. Let κ be the least hyperinaccessible cardinal and suppose that κ is a Π_2-Mahlo cardinal. Since κ is a hyperinaccessible cardinal,

$$V_\kappa \models \forall \alpha \beta \exists \mu (\beta < \mu \wedge \mu \text{ is } \alpha\text{-inaccessible}).$$

Note that the right hand is a Π_3-sentence. By (1) of Lemma 3.3.6, there exists $\lambda < \kappa$ such that λ is inaccessible and $V_\lambda \preceq_2 V_\kappa$. Since the Π_3-sentences are downward absolute for V_λ, V_κ,

$$V_\lambda \models \forall \alpha \beta \exists \mu (\beta < \mu \wedge \mu \text{ is } \alpha\text{-inaccessible})$$

But, then λ is a hyperinaccessible cardinal. A contradiction. ■

Fact 3.3.14 *Suppose that κ is a Π_{n+1}-Mahlo cardinal with $n \geq 1$. Then the set of Σ_n-Mahlo cardinals below κ is a Π_{n+1}-stationary subset of κ.*

Proof. Let κ be a Π_{n+1}-Mahlo cardinal with $n \geq 1$. Let D be a Π_{n+1}-club on κ. Let $C_{n+1} = \{\alpha < \kappa : V_\alpha \preccurlyeq_{n+1} V_\kappa\}$. By Lemma 3.3.7, C_{n+1} is also a Π_{n+1}-club on κ and so, $E = C_{n+1} \cap D$ is a Π_{n+1}-club on κ. Since κ is a Π_{n+1}-Mahlo cardinal with $n \geq 1$, the set of inaccessible cardinals below κ is Π_{n+1}-stationary.

Let $\lambda \in E$ an inaccessible cardinal. So, $\lambda \in D$.

Suppose that C is a Σ_n-club on λ. Let $\varphi(x)$ be the Σ_n-formula (with possibly parameters in V_λ) such that for every $\alpha < \lambda$,

$$\alpha \in C \text{ iff } V_\lambda \models \varphi(\alpha).$$

Then V_λ satisfies

1. $\forall\alpha\exists\beta(\alpha < \beta \wedge \varphi(\beta))$.
2. $\forall\alpha((\forall\beta < \alpha)(\exists\gamma < \alpha)(\beta < \gamma \wedge \varphi(\gamma)) \rightarrow \varphi(\alpha))$.

Recall that (1) and (2) are Π_{n+1}-formulas. Since $\lambda \in C_{n+1}$, $V_\lambda \preccurlyeq_{n+1} V_\kappa$. Let $C' \subseteq \kappa$ be defined by $\alpha \in C'$ iff $V_\kappa \models \varphi(\alpha)$. Since, $V_\lambda \preccurlyeq_{n+1} V_\kappa$, C' is a Σ_n-club on κ. Hence, since κ is a Π_{n+1}-Mahlo cardinal,

$$V_\kappa \models \exists\mu(\mu \text{ is inaccessible} \wedge \varphi(\mu)).$$

Since φ is a Σ_n formula with $n \geq 1$ and "μ is inaccessible" is Δ_2 predicate on μ, the formula on the right-hand is a Σ_{n+1}-formula and hence it is satisfied by V_λ. But then, λ is a Σ_n-Mahlo cardinal. ■

Corollary 3.3.15 *For all $n \geq 1$, the least Σ_n-Mahlo cardinal is not a Π_{n+1}-Mahlo cardinal.* ■

3.3.2 Absoluteness by collapsing a Σ_ω-Mahlo cardinal

Definition 3.3.16 $L(\mathsf{R})$ *is a* Σ_ω-Mahlo Solovay model over V *iff*

1. *For every $x \in \mathsf{R}$, ω_1 is a Σ_ω-Mahlo cardinal in $V[x]$ and*
2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

Clearly, every Σ_ω-Mahlo Solovay model over V is a Solovay model over V. So, we can give a characterization of Σ_ω-Mahlo Solovay models in the same way as that of Lemma 3.1.1. Namely

Lemma 3.3.17 *Suppose that M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is a Σ_ω-Mahlo cardinal in $V[x]$ and*
2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

Then there exists a forcing notion W *such that does not add reals and creates a* $Coll(\omega, <\omega_1)$*-generic filter* C *over* V *such that* M *and* $V[C]$ *have the same reals. Thus,* W *forces that* $L(\mathsf{R})$ *is a* Σ_ω*-Mahlo Solovay model over* V. ■

Theorem 3.3.18 *Suppose* $L(\mathsf{R})^M$ *is a* Σ_ω*-Mahlo Solovay model over* V *and* P *is a projective and ccc poset in* M*. Then the* $L(\mathsf{R})$ *of any* P*-generic extension of* M *is also a* Σ_ω*-Mahlo Solovay model over* V.

Proof. Suppose $L(\mathsf{R})^M$ is a Σ_ω-Mahlo Solovay model over V. Let $\kappa = \omega_1^M$ and let C be the $Coll(\omega, <\kappa)$-generic filter over V such that, with W-value $\mathsf{1}$, $\mathsf{R}^M = \mathsf{R}^{V[C]}$. Notice that, as in Theorem 3.2.1, to prove the theorem it will be enough to show that for every $n \geq 4$, if P is a $\underset{\sim}{\Sigma}^1_n$ and ccc poset in $V[C]$, then, for every G is P-generic filter over $V[C]$, every real in $V[C][G]$ is generic over V for a countable poset.

Let P be a $\underset{\sim}{\Sigma}^1_n$ and ccc poset in $V[C]$ and let $\dot{\mathsf{P}}$ be a $Coll(\omega, <\kappa)$-name for P in V. So, $\Vdash_{Coll(\omega,<\kappa)}$ "$\dot{\mathsf{P}}$ is a ccc poset" and there are a simple $Coll(\omega, <\kappa)$-name for a real $\dot{a}$ and Σ^1_n formulas $\varphi_\leq(x, y; u)$ and $\varphi_\perp(x, y; u)$ such that:

1. $\Vdash_{Coll(\omega,<\kappa)}$ "$(\forall xy \in \dot{\omega}^\omega)(x \dot{\leq}_P y \leftrightarrow \varphi_\leq(x, y; \dot{a}))$".
2. $\Vdash_{Coll(\omega,<\kappa)}$ "$(\forall xy \in \dot{\omega}^\omega)(x \dot{\perp}_P y \leftrightarrow \varphi_\perp(x, y; \dot{a}))$".

Let $\mathsf{S} = Coll(\omega, <\kappa) * \dot{\mathsf{P}}$. Note that S is a κ-cc poset.

We use the same notational conventions from Theorem 3.2.1.

Definition 3.3.19 *Let* κ *be an inaccessible cardinal. For every* Σ^1_n (Π^1_n) *formula* $\varphi(x)$ *with* $n \geq 2$*, every simple* $Coll(\omega, <\kappa)$*-name* τ *for a real and every* $p \in Coll(\omega, <\kappa)$*, let* $R_\varphi(p, \tau)$ *be defined by induction on the complexity of* $\varphi(x)$ *as follows:*

$\underline{n=2}$: *Then,* $\varphi(x)$ *is* Σ^1_2*. Suppose that* $\varphi(x)$ *is* $\exists y \psi(x, y)$ *where* $\psi(x, y)$ *is* Π^1_1*. Then,* $R_\varphi(p, \tau)$ *iff*

$$\exists\alpha\exists\sigma(\tau, \sigma \text{ are simple } Coll(\omega, <\alpha)\text{-names for reals} \wedge p \Vdash_{Coll(\omega,<\alpha)} \psi(\sigma, \tau))$$

$\underline{n+1}$: *Then,* $\varphi(x)$ *is of the form* $\exists y \neg\psi(x, y)$ *where* $\psi(x, y)$ *is a* Σ^1_n *formula. Then* $R_\varphi(p, \tau)$ *iff*

$$\exists\sigma(\sigma \text{ is a simple } Coll(\omega, <\kappa)\text{-name for a real} \wedge (\forall q \leq p)\neg R_\psi(q, \tau, \sigma))$$

If $\varphi(x)$ *is* Π^1_n *formula with* $n \geq 2$*, then* $R_\varphi(p, \tau)$ *iff* $(\forall q \leq p)\neg R_{\neg\varphi}(p, \tau)$.

Recall (see the Introduction) that for an inaccessible cardinal κ, $Coll(\omega, <\kappa)$ is a poset which is Δ_1-definable with κ as parameter and it is Δ_1-definable without parameters over V_κ. We use this fact to prove the following.

Lemma 3.3.20 *Let* κ *be an inaccessible cardinal. Suppose that* $p \in Coll(\omega, <\kappa)$*,* τ *is a simple* $Coll(\omega, <\kappa)$*-name for a real and* $\varphi(x)$ *is a* Σ^1_n (Π^1_n) *formula with* $n \geq 2$*. Then,* $R_\varphi(p, \tau)$ *is a relation which is* Σ_{n-1} *(respectively,* Π_{n-1}*) definable without parameters over* V_κ.

Proof. We need the following facts:

Fact 3.3.21 *Suppose that κ is a regular uncountable cardinal. Then "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" is a property which is Δ_1-definable with κ as parameter and it is Δ_1-definable without parameters over V_κ.*

Proof. Recall that for every uncountable cardinal, $Coll\,(\omega, <\kappa)$ is a κ-cc poset. Hence, if x is a maximal antichain of $Coll\,(\omega, <\kappa)$, then $|x| < \kappa$. So, by regularity of κ, there is $\alpha < \kappa$ such that x is a maximal antichain of $Coll\,(\omega, <\alpha)$. Hence, "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" iff there is $\alpha < \kappa$ such that:

1. $(\forall p \in x)\,(p \in Coll\,(\omega, <\alpha))$
2. $(\forall pq \in x)\,(p \neq q \rightarrow p \perp q)$
3. $\neg\,(\exists p \in Coll\,(\omega, <\alpha))((\forall q \in x)\,(p \perp q))$

But since $y = Coll\,(\omega, <\alpha)$ is a relation which is Δ_1 definable with α as parameter, "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" is a property which is Δ_1-definable with κ as parameter.

Since for every maximal antichain x of $Coll\,(\omega, <\kappa)$, $|x| < \kappa$ and $x \subseteq V_\kappa$, $x \in V_\kappa$. Note, also, $V_\kappa \models$ "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" iff V_κ satisfies that there is α such that (1)-(3) above holds. So, "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" is a property of x which is Σ_1 definable (without parameters) over V_κ.

Moreover, $V_\kappa \models$ "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" iff V_κ satisfies:

1. $(\forall p \in x)\,(p \in Coll\,(\omega, <\kappa))$
2. $(\forall pq \in x)\,(p \neq q \rightarrow p \perp q)$
3. $\neg\exists p\,(p \in Coll\,(\omega, <\kappa) \wedge (\forall q \in x)\,(p \perp q))$

But, since that $Coll\,(\omega, <\kappa)$ is a Δ_1-definable (without parameters) poset over V_κ, (1) and (2) are Δ_1-predicates on x over V_κ and (3) is a Π_1-predicate on x over V_κ (all without parameters). So, "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" is a property on x which is Π_1-definable without parameters over V_κ.

So, "x is a maximal antichain of $Coll\,(\omega, <\kappa)$" is a property which is Δ_1-definable without parameters over V_κ. ∎

Fact 3.3.22 *Suppose that κ is a regular uncountable cardinal. Then, "τ is a simple $Coll\,(\omega, <\kappa)$-name for a real" is a Δ_1-definable, with κ as parameter, property of τ and Δ_1-definable, without parameters, property of τ over V_κ.*

Proof. Suppose that τ is a simple $Coll\,(\omega, <\kappa)$-name for a real. Identifying the standard $Coll\,(\omega, <\kappa)$-names for natural numbers with elements of ω, τ is a simple $Coll\,(\omega, <\kappa)$-name for a real iff

1. $(\forall x \in \tau)\,(x$ is an ordered triple$\wedge x_0 \in Coll\,(\omega, <\kappa) \wedge x_1 \in \omega \wedge x_2 \in \omega)$.

2. $(\forall n \in \omega)\,(\exists x \in \tau)\,(x_1 = n)$.

3. $(\forall n \in \omega)\,(\{x_0 : x \in \tau \wedge x_1 = n\}$ is a maximal antichain of $Coll\,(\omega, < \kappa))$.

4. $(\forall xy \in \tau)\,(x_0 = y_0 \wedge x_1 = y_1 \rightarrow x_2 = y_2)$

where for an ordered triple x, x_0, x_1 and x_2 denote, respectively, the first, the second and the third coordinates of x. Since the function that sends every ordered triple x to x_i $(0 \leq i \leq 2)$ is a Δ_0 function, (1) and (4) are Δ_0 on τ. Since $Coll\,(\omega, < \kappa)$ is a Δ_1-definable, with κ as parameter, poset (1) is Δ_1 with κ as parameter. Finally, by fact above, (3) is Δ_1 with κ as parameter. So, "τ is a simple $Coll\,(\omega, < \kappa)$-name for a real" is a property which is Δ_1-definable with κ as parameter.

Moreover, if τ is a simple $Coll\,(\omega, < \kappa)$-name for a real, then $\tau \in V_\kappa$. So, as above fact, "τ is a simple $Coll\,(\omega, < \kappa)$-name for a real" is Δ_1-definable without parameters over V_κ. ■

Now, we prove Lemma 3.3.20 for Σ^1_n formulas $(n \geq 2)$ by induction. The proof for Π^1_n formulas is analogous. So, suppose that κ is an inaccessible cardinal. Then $Coll\,(\omega, < \kappa) = Coll\,(\omega, < \kappa)^{V_\kappa}$ and, by above fact, "τ is a simple $Coll\,(\omega, < \kappa)$-name for a real" is a Δ_1 property over V_κ:

$\underline{n+1}$: Let $\varphi(x)$ be a Σ^1_{n+1} formula, $n \geq 2$. So, $\varphi(x)$ is of the form $\exists y \neg \psi(x, y)$, where ψ is Σ^1_n. Assume that the lemma holds for Σ^1_n formulas. Suppose that $p \in Coll\,(\omega, < \kappa)$ and τ is a simple $Coll\,(\omega, < \kappa)$-name for a real. Then, $R_\varphi\,(p, \tau)$ is defined by the formula

$$\exists \sigma (\sigma \text{ is a simple } Coll\,(\omega, < \kappa)\text{-name for a real} \wedge (\forall q \leq p) \neg R_\psi\,(q, \tau, \sigma)).$$

Now, "σ is a simple $Coll\,(\omega, < \kappa)$-name for a real" is a Δ_1 predicate on σ over V_κ and, by inductive hypothesis, $R_\psi\,(q, \tau, \sigma)$ is a Σ_{n-1} relation over V_κ with $n \geq 2$. So, the above is Σ_n a formula with parameters p and τ.

$\underline{n=2}$: Then, $\varphi(x)$ is of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is a Π^1_1 formula. Note that $R_\varphi\,(p, \tau)$ iff

$$\exists \alpha \exists \sigma (\tau, \sigma \text{ are simple } Coll\,(\omega, < \alpha)\text{-names for reals} \wedge p \Vdash_{Coll(\omega, <\alpha)} \psi\,(\sigma, \tau))$$

Since "σ is a simple $Coll\,(\omega, < \alpha)$-name for a real" is a property on σ which is Δ_1-definable with α as parameter over V_κ, we only need to show that "$p \Vdash_{Coll(\omega, <\alpha)} \psi(\tau, \sigma)$" is expressible over V_κ by means of a Σ_1 formula and a Π_1 formula both with α, σ, τ and p as parameters.

As in Theorem 2.1.23, let wf$\,(x)$ be the predicate "x is a well-founded relation". Then, since wf$\,(x)$ is a Δ_1 predicate, there exists a finite set S of axioms of ZF such that

$$ZF \vdash \forall M (M \text{ is transitive} \wedge M \models \bigwedge S \rightarrow \mathrm{wf}\,(x) \text{ is absolute for } M)$$

Let ZFC^* be a finite set of axioms of ZFC containing all axioms which are needed to define the forcing relation in a model, to prove the Forcing Theorem, and including S. Then,

Claim 3.3.23 *The following are equivalent:*

1. $p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$.

2. *For all transitive model M of ZFC^* containing p,τ,σ and α, $M \models$ "$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$".*

3. *There exists a transitive model M of ZFC^* containing p,τ,σ and α and such that $M \models$ "$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$".*

Proof. $(1 \Rightarrow 2)$ Let M be as in (2). Assume M does not satisfy $p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$. Then, since $M \models ZFC^*$, there is $q \in M$ such that $q \leq p$ and $M \models$ "$q \Vdash_{Coll(\omega,<\alpha)} \neg\psi(\tau,\sigma)$". Let C be a $Coll(\omega,<\alpha)$-generic filter over V with $q \in C$. Then, since C is closed upwards, $p \in C$. Since $\alpha \in M$ and M is transitive, $Coll(\omega,<\alpha)^M = Coll(\omega,<\alpha)$ and hence C is $Coll(\omega,<\alpha)$-generic over M. Also, since $q \in C$, $M[C] \models \neg\psi(\tau[C],\sigma[C])$, which is a Σ^1_1 formula. Since $M \models ZFC^*$, also $M[C] \models ZFC^*$. Therefore, the Σ^1_1 formulas are absolute for $M[C]$ as being a well-founded relation is absolute for models of ZFC^*. This implies $V[C] \models \neg\psi(\tau[C],\sigma[C])$, which contradicts (1).

$(2 \Rightarrow 3)$ Since ZFC^* is a finite fragment of ZFC, by the Reflection Principle, there exists an ordinal $\beta > \alpha \geq \omega+2$ such that for every formula of ZFC^* is absolute for V_β. Let $X = \{\alpha, p, \tau, \sigma, TC(\tau), TC(\sigma)\}$ and let M the Skolem hull of X in V_β. So $M \prec V_\beta$. Moreover, M is extensional since V_β satisfies the axiom of extensionality. Without loss of generality, we can suppose that M is transitive (if not, we collapse it with Mostowski's collapsing). Clearly, $M \models ZFC^*$ and $\alpha, p, \tau, \sigma \in M$. Then, by (2), $M \models$ "$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$".

$(3 \Rightarrow 1)$ Let M be a transitive model of ZFC^* containing α, p, τ, σ and $M \models$ "$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$". Let $C \subseteq Coll(\omega,<\alpha)$ a generic filter over V with $p \in C$. Since $\alpha \in M$, $Coll(\omega,<\alpha)^M = Coll(\omega,<\alpha)$. Hence C is a $Coll(\omega,<\alpha)$-generic filter over M and $p \in C$. Since $M \models ZFC^*$, $M[C] \models \psi(\tau[C],\sigma[C])$ and $M[C] \models ZFC^*$. But then, the Π^1_1 formulas are absolute for $M[C]$. Thus,

$$V[C] \models \psi(\tau[C],\sigma[C]).$$

This proves the claim. ■

We finish the proof of Lemma 3.3.20. Note that (2) of the claim above is a Π_1 formula with p,τ,σ and α as parameters and (3) is a Σ_1 formula with p,τ,σ and α as parameters. So, (1) is Δ_1 on p,τ,σ and α. Since κ is inaccessible, $V_\kappa \models ZFC$. So the above claim shows that (1) is Δ_1 on p,τ,σ and α over V_κ. Therefore, $R_\varphi(p,\tau)$ is a Σ_1 relation over V_κ. ■

Lemma 3.3.24 *Let κ be an inaccessible cardinal and let $\varphi(x)$ be a Σ^1_n (Π^1_n) formula with $n \geq 2$. Then, for every $p \in Coll(\omega,<\kappa)$ and every simple $Coll(\omega,<\kappa)$-name for a real τ,*

$$p \Vdash_{Coll(\omega,<\kappa)} \varphi(\tau) \text{ iff } V_\kappa \models R_\varphi(p,\tau)$$

Proof. Recall that $Coll(\omega, <\kappa)^{V_\kappa} = Coll(\omega, <\kappa)$. So, it will be sufficient to prove the lemma for Σ^1_n formulas $(n \geq 2)$ because for every Π^1_n formula $\forall x \varphi(y, x)$, every $p \in Coll(\omega, <\kappa)$ and every simple $Coll(\omega, <\kappa)$-name τ for a real,

$$\begin{aligned} p \Vdash_{Coll(\omega,<\kappa)} \forall x \varphi(\tau, x) \ &\text{iff} \ p \Vdash_{Coll(\omega,<\kappa)} \neg\exists x \neg\varphi(\tau, x) \\ &\text{iff} \ \text{for all } q \leq p,\ \neg q \Vdash_{Coll(\omega,<\kappa)} \exists x \neg\varphi(\tau, x) \\ &\text{iff} \ \text{for all } q \leq p,\ V_\kappa \models \neg R_{\exists x \neg\varphi}(q, \tau) \\ &\text{iff} \ V_\kappa \models (\forall q \leq p)(\neg R_{\exists x \neg\varphi}(q, \tau)) \\ &\text{iff} \ V_\kappa \models R_{\forall x \varphi}(p, \tau) \end{aligned}$$

We prove the lemma by induction on $n \geq 2$:

<u>$n+1$</u>: Let $\varphi(x)$ be a Σ^1_{n+1} formula with $n \geq 2$. So, $\varphi(x)$ is of the form $\exists y \neg\psi(x, y)$, where $\psi(x, y)$ is Σ^1_n. Assume that the lemma holds for Σ^1_n formulas and suppose that $p \in Coll(\omega, <\kappa)$ and τ is a simple $Coll(\omega, <\kappa)$-name for a real. Then, $p \Vdash_{Coll(\omega,<\kappa)} \exists y \neg\psi(\tau, y)$ iff $\exists\sigma$(σ is a simple $Coll(\omega, <\kappa)$-name for a real $\wedge (\forall q \leq p)(\neg q \Vdash_{Coll(\omega,<\kappa)} \psi(\tau, \sigma))$). Fix a witness σ. Then, on one hand,

$$\begin{aligned} &\sigma \text{ is a simple } Coll(\omega, <\kappa)\text{-name for a real iff} \\ &\qquad \text{iff } V_\kappa \models \text{“}\sigma \text{ is a simple } Coll(\omega, <\kappa)\text{-name for a real”} \end{aligned}$$

On the other hand, fix $q \leq p$. Then, by inductive hypothesis

$$\neg q \Vdash_{Coll(\omega,<\kappa)} \psi(\tau, \sigma) \text{ iff } V_\kappa \models \neg R_\psi(q, \tau, \sigma)$$

But, since $Coll(\omega, <\kappa) = Coll(\omega, <\kappa)^{V_\kappa}$,

$$(\forall q \leq p)(\neg q \Vdash_{Coll(\omega,<\kappa)} \varphi(\tau, \sigma)) \text{ iff } V_\kappa \models (\forall q \leq p) \neg R_\psi(q, \tau, \sigma)$$

Therefore, $p \Vdash_{Coll(\omega,<\kappa)} \varphi(\tau)$ iff $V_\kappa \models R_\varphi(p, \tau)$.

<u>$n = 2$</u>: So, $\varphi(x)$ is of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is a Π^1_1 formula. Let $p \in Coll(\omega, <\kappa)$ and τ be a $Coll(\omega, <\kappa)$-name for a real.

Suppose that $p \Vdash_{Coll(\omega,<\kappa)} \varphi(\tau)$. So, since $Coll(\omega, <\kappa)$ is a κ-cc poset, by Shoenfield's Absoluteness Theorem formulas and Maximal Principle,

$$(\exists \alpha < \kappa) \exists\sigma(\sigma, \tau \text{ are simple } Coll(\omega, <\alpha)\text{-name for reals } \wedge p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau, \sigma))$$

Now, since $\alpha < \kappa$ and κ is regular and uncountable, $\alpha, \tau, \sigma \in V_\kappa$ and $V_\kappa \models$ “σ, τ are simple $Coll(\omega, <\alpha)$-names for reals”. So, only remains to show that

$$V_\kappa \models \text{“}p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau, \sigma)\text{”}.$$

But, since κ is inaccessible, $V_\kappa \models ZFC$. Moreover, $p, \tau, \sigma, \alpha \in V_\kappa$ and $\psi(x, y)$ is a Π^1_1 formula. So, by (2) of Claim 3.3.23, $V_\kappa \models$ “$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau, \sigma)$”. So, $V_\kappa \models R_\varphi(p, \tau)$.

Suppose now that $V_\kappa \models R_\varphi(p, \tau)$. i.e.,

$$V_\kappa \models \exists\alpha\exists\sigma(\tau, \sigma \text{ are simple } Coll(\omega, <\alpha)\text{-names for reals } \wedge p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau, \sigma))$$

Let $\alpha, \sigma \in V_\kappa$ such that V_κ satisfies that τ, σ are simple $Coll(\omega, <\alpha)$-names for reals and $p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau, \sigma)$. Then, by absoluteness of Δ_1 predicates with parameters in V_κ, τ and σ are simple $Coll(\omega,<\alpha)$-names for reals. Moreover, since κ is inaccessible, $V_\kappa \models ZFC$. So, since $V_\kappa \models$ "$p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$", by (3) of Claim 3.3.23, $p \Vdash_{Coll(\omega,<\alpha)} \psi(\tau,\sigma)$. But then, since $Coll(\omega,<\alpha) \lessdot Coll(\omega,<\kappa)$ and by Shoenfield's Absoluteness Theorem, $p \Vdash_{Coll(\omega,<\kappa)} \varphi(\tau)$. ■

We continue with the proof of Theorem 3.3.18. Since $\dot{\mathsf{P}}$ is a $Coll(\omega,<\kappa)$-name for a $\underset{\sim}{\Sigma}^1_n$ and ccc poset, by Lemma 3.3.20, for every $p \in Coll(\omega,<\kappa)$ and every simple $Coll(\omega,<\kappa)$-names for reals τ,σ, $R_{\varphi_P}(p,\tau,\dot{a})$, $R_{\varphi_\leq}(p,\tau,\sigma,\dot{a})$ and $R_{\varphi_\perp}(p,\tau,\sigma,\dot{a})$ are Σ_{n-1} relations over V_κ.

Since κ is a Σ_ω-Mahlo cardinal, by Lemma 3.3.6, we can fix an unbounded sequence $\langle \lambda_\xi : \xi<\kappa\rangle$ such that for every $\xi<\kappa$, λ_ξ is an inaccessible cardinal and $V_{\lambda_\xi} \prec_{n-1} V_\kappa$ and $\dot{a}$ is a simple $Coll(\omega,<\lambda_0)$-name for a real.

Let $p \in Coll(\omega,<\kappa)$ and let σ,τ be simple $Coll(\omega,<\kappa)$-names for reals. So there is a least $\xi<\kappa$ such that $p\in Coll(\omega,<\lambda_\xi)$ and σ and τ are simple $Coll(\omega,<\lambda_\xi)$-names for reals. Then, for every $\zeta<\kappa$, $\xi\leq\zeta$, since κ and λ_ζ are inaccessible cardinals, by Lemma 3.3.24,

$$\begin{aligned} p \Vdash_{Coll(\omega,<\kappa)} \varphi_\leq(\sigma,\tau,\dot{a}) \quad &\text{iff} \quad V_\kappa \models R_{\varphi_\leq}(p,\tau,\sigma,\dot{a}) \\ &\text{iff} \quad V_{\lambda_\zeta} \models R_{\varphi_\leq}(p,\tau,\sigma,\dot{a}) \\ &\text{iff} \quad p \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \varphi_\leq(\sigma,\tau,\dot{a}) \end{aligned}$$

And the same holds for the Σ^1_n formula $\varphi_\perp(x,y;z)$.

So, for every $\xi\leq\zeta<\kappa$, every $p\in Coll(\omega,<\lambda_\xi)$ and all simple $Coll(\omega,<\lambda_\xi)$-names for reals τ, σ,

$$\begin{aligned} p \Vdash_{Coll(\omega,<\kappa)} \varphi_P(\tau,\dot{a})\text{''} \quad &\text{iff} \quad p \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \varphi_P(\tau,\dot{a}) \\ p \Vdash_{Coll(\omega,<\kappa)} \varphi_\leq(\tau,\sigma,\dot{a}) \quad &\text{iff} \quad p \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \varphi_\leq(\tau,\sigma,\dot{a}) \\ p \Vdash_{Coll(\omega,<\kappa)} \varphi_\perp(\tau,\sigma,\dot{a}) \quad &\text{iff} \quad p \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \varphi_\perp(\tau,\sigma,\dot{a}) \end{aligned}$$

Hence, if $\xi\leq\zeta$ then $\mathsf{S}_{\lambda_\xi}\subseteq\mathsf{S}_{\lambda_\zeta}$ and, since $Coll(\omega,<\kappa)$ is a κ-cc poset, $\mathsf{S}=\bigcup_{\xi<\kappa}\mathsf{S}_{\lambda_\xi}$. Thus, for every subposet X of S of cardinality less than κ there exists $\xi<\kappa$ such that X is a subposet of S_{λ_ξ}.

Moreover, for every $\xi\leq\zeta<\kappa$, every $p\in Coll(\omega,<\lambda_\xi)$ and all simple $Coll(\omega,<\lambda_\xi)$-names for a real $\tau,\dot{c}$

$$\begin{aligned} &p \Vdash_{Coll(\omega,<\kappa)} \text{“}\tau \text{ codes a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{” iff} \\ &\qquad\qquad \text{iff } p \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \text{“}\tau \text{ codes a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”} \end{aligned}$$

For every $\alpha<\kappa$, let $\xi(\alpha)$, if it exists, be the least $\xi<\kappa$ such that for every $\zeta\geq\xi$ the following holds: For every simple $Coll(\omega,<\lambda_\alpha)$-name $\dot{A}$ for a subset of $\dot{\mathsf{R}}_{\lambda_\alpha}$, every simple $Coll(\omega,<\lambda_\alpha)$-name for a real $\dot{c}$ and every $q\in Coll(\omega,<\lambda_\zeta)$, if

$$q \Vdash_{Coll(\omega,<\kappa)} \text{“}\dot{A} \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”},$$

then

$$q \Vdash_{Coll\left(\omega,<\lambda_\zeta\right)} \text{“}\dot{A} \text{ is not a maximal antichain of } \dot{\mathsf{P}} \text{ below } \dot{c}\text{”}.$$

Lemma 3.3.25 *For every $\alpha < \kappa$, $\xi(\alpha)$ exists.*

Proof. As in Lemma 3.2.3. ■

The rest of the proof of Theorem 3.3.18 is like the proof of Theorem 3.2.1 but using only the λ_ξ $(\xi < \kappa)$ inaccessible cardinals instead of all ordinals below κ. ■

Corollary 3.3.26 *$Con(ZFC + \exists\kappa\,(\kappa$ is a Σ_ω-Mahlo cardinal$))$ implies $Con(ZFC + L(\mathsf{R})$-two-step absoluteness for projective and ccc posets$)$.* ■

Note that the proof of Theorem 3.3.18 really shows that for every $n \geq 3$, every $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc forcing extension of a Π_n-Mahlo Solovay model is a Π_n-Mahlo Solovay model. So,

Corollary 3.3.27 *For every $n \geq 3$, $Con(ZFC + \exists\kappa\,(\kappa$ is a Π_n-Mahlo cardinal$))$ implies $Con(ZFC + L(\mathsf{R})$-two-step absoluteness for $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc posets$)$.* ■

Further, all ccc posets in a Σ_ω-Mahlo Solovay model are projective-indestructible-ccc (see Definition 3.2.9). Hence, in any Σ_ω-Mahlo Solovay model there are no projective Suslin trees. The same holds in every Π_n-Mahlo Solovay model for $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc posets: every ccc poset in a Π_n-Mahlo Solovay model is $\underset{\sim}{\Sigma}^1_{n+1}$-indestructible-ccc and, therefore, there are no $\underset{\sim}{\Sigma}^1_{n+1}$ Suslin trees.

Theorem 3.3.28 *Let $L(\mathsf{R})^M$ be a Σ_ω-Mahlo Solovay model over V. Then, every ccc poset $\mathsf{P} \in L(\mathsf{R})^M$ is a projective-indestructible-ccc poset. More precisely, in any Π_n-Mahlo Solovay model every ccc poset is $\underset{\sim}{\Sigma}^1_{n+1}$-indestructible-ccc.*

Proof. As in Theorem 3.2.10. ■

Corollary 3.3.29 *If $L(\mathsf{R})^M$ is a Σ_ω-Mahlo Solovay model over V, then in $L(\mathsf{R})^M$ there are no projective Suslin trees. More precisely, if $L(\mathsf{R})^M$ is a Π_n-Mahlo Solovay model over V, then in $L(\mathsf{R})^M$ there are no $\underset{\sim}{\Sigma}^1_{n+1}$ Suslin trees.*

3.3.3 The strength of $\underset{\sim}{\Sigma}^1_4$-absoluteness for projective and ccc forcing notions

We will show that $\underset{\sim}{\Sigma}^1_4$-absoluteness for projective and ccc posets implies that ω_1 is Σ_ω-Mahlo in L. In the proof we use an argument of R. Jensen implicit in [Je-So]. We assume that ω_1 is not Σ_n-Mahlo $(n \geq 2)$ in L, so there is a Σ_n-club D on ω_1 of singular cardinals in L. Then, we force with $Coll\,(\omega, <\omega_1)$ to add a function π that, to every real a coding a countable ordinal $\|a\|$ in the generic extension, assigns a real that codes the least singular cardinal in D greater than $\|a\|$. We prove that π has a $Coll\,(\omega, <\omega_1)$-name that can be coded by a $\underset{\sim}{\Delta}^1_{n+2}$ subset of $\omega^\omega \times \omega^\omega$ (Lemma 3.3.31). Then, working in the $Coll\,(\omega, <\omega_1)$-generic extension, we define a version of Solovay's almost-disjoint coding poset, P_π, that adds a generic real b_H that codes π and we show that in every $Coll\,(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$-generic extension $\omega_1^{L[b_H]} = \omega_1$ holds (Lemma 3.3.42). Finally, we prove that, even if P_π was not a projective forcing notion in the $Coll\,(\omega, <\omega_1)$ generic extension, since π has a $Coll\,(\omega, <\omega_1)$-name coded by a

$\underset{\sim}{\Delta}^1_{n+2}$ set of pairs of reals, $Coll\,(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is a $\underset{\sim}{\Delta}^1_{n+2}$ ccc poset (Lemma 3.3.44). So, since "there exists a real x such that $\omega_1^{L[x]} = \omega_1$" is expressible with a Σ^1_4 sentence, this is true in the ground model, in contradiction with the inaccessibility of ω_1 from the reals.

Theorem 3.3.30 *Suppose that V is $\underset{\sim}{\Sigma}^1_4$-absolute for projective and ccc forcing notions. Then ω_1 is a Σ_ω-Mahlo cardinal in L.*

Proof. Suppose, towards a contradiction, that ω_1 is not a Σ_ω-Mahlo cardinal in L. By 3.1.23, ω_1 is an inaccessible cardinal in L. So, there is $n \geq 2$ and a Σ_n-club D on ω_1 of singular cardinals in L. Let $\varphi(x)$ be the Σ_n-formula such that for every $\alpha < \omega_1$, $\alpha \in D$ iff

$$[V_{\omega_1}]^L \models \varphi(\alpha)\,.$$

Since ω_1 is an inaccessible cardinal in L, $[V_{\omega_1}]^L = L_{\omega_1} = [HC]^L$. So, for every $\alpha < \omega_1$, $\alpha \in D$ iff

$$HC \models \varphi(\alpha)^L\,.$$

Since "$x \in L$" is a Σ_1-formula and $\varphi(x)$ is a Σ_n-formula with $n \geq 2$, $\varphi(x)^L$ is also a Σ_n-formula. Therefore, D is a club on ω_1 of singular cardinals in L which is Σ_n-definable over HC.

Let $D^* = \{a \in \omega^\omega : \|a\| \in D\}$. So, D^* is a $\underset{\sim}{\Sigma}^1_{n+1}$ set of reals (see [J2], 41.1)

Note that $Coll\,(\omega, <\omega_1)$ belongs to L and is a ccc poset in V. Moreover,

Lemma 3.3.31 *Suppose C is a $Coll\,(\omega, <\omega_1)$-generic filter over V. Then, there is a function $\pi \in V[C]$ from WO, the set of all reals in $V[C]$ that code a countable ordinal, into D^*, the set of all reals in $V[C]$ that code a countable ordinal in D, such that:*

1. *For every $x \in WO$, $\pi(x)$ is a code for the least ordinal in D greater than $\|x\|$ and for every $x, y \in WO$, if $\|\pi(x)\| = \|\pi(y)\|$, then $\pi(x) = \pi(y)$.*

2. *π has a $Coll\,(\omega, <\omega_1)$-name that can be coded by a $\underset{\sim}{\Delta}^1_{n+2}$ set in $\omega^\omega \times \omega^\omega$.*

Proof. We need the following claims and definitions:

Definition 3.3.32 *We say that τ is a* $Coll\,(\omega, <\omega_1)$-name for a subset of ω *iff $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tau \subseteq \breve{\omega}$ ". We say that τ is a* simple $Coll\,(\omega, <\omega_1)$-name for a subset of ω *iff*

1. *The elements of τ are of form $\langle p, \breve{n}\rangle$, where $p \in Coll\,(\omega, <\omega_1)$ and $\breve{n}$ is a standard $Coll\,(\omega, <\omega_1)$-name for some $n \in \omega$.*

2. *For every $n \in \omega$, the set $A_n = \{p \in Coll\,(\omega, <\omega_1) : \langle p, \breve{n}\rangle \in \tau\}$ is an antichain of $Coll\,(\omega, <\omega_1)$.*

Note that as in Fact 2.1.19, for every $Coll\,(\omega, <\omega_1)$-name for a subset of ω there exists a simple $Coll\,(\omega, <\omega_1)$-name for a subset of ω such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tau = \sigma$". Moreover, for every $\alpha < \omega_1$ there is a simple $Coll\,(\omega, <\omega_1)$-name for a subset of ω, τ, such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \check{\alpha}$". Note also that, since for every $Coll\,(\omega, <\omega_1)$-generic filter C over V, $\omega_1^{L[C]} = \omega_1^{V[C]} = \omega_1$, the same is true for names in L.

Let WO_{ω_1} be the set of all simple $Coll\,(\omega, <\omega_1)$-names σ for a subset of ω such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\sigma\| = \check{\gamma}$" for some $\gamma < \omega_1$. Let $\dot{WO} = Coll\,(\omega, <\omega_1) \times WO_{\omega_1}$.

Definition 3.3.33 *For every $\gamma < \omega_1$ let τ_γ be the $<_L$-least simple $Coll\,(\omega, <\omega_1)$-name τ for a subset of ω such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \check{\gamma}$". Let $B_{\omega_1} = \{\tau_\gamma : \gamma \in D\}$. Let $\dot{B} = Coll\,(\omega, <\omega_1) \times B_{\omega_1}$.*

Define the function π_{ω_1} from the set WO_{ω_1} into B_{ω_1} as follows: for every $\sigma \in WO_{\omega_1}$, $\pi_{\omega_1}(\sigma) = \tau$ iff

1. $\tau \in B_{\omega_1}$
2. $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\sigma\| < \|\tau\|$"
3. $(\forall \rho \in B_{\omega_1})\,(\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\sigma\| < \|\rho\|$" $\rightarrow\ \Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| \leq \|\rho\|$"$)$

Let $\dot{\pi} = Coll\,(\omega, <\omega_1) \times \pi_{\omega_1}$.

Clearly, we have that $\dot{B}$ and π are $Coll\,(\omega, <\omega_1)$-names for, respectively, a subset of $P(\omega)$ and a subset of $P(\omega) \times P(\omega)$. We will show that for every $Coll\,(\omega, <\omega_1)$-generic C over V, $\pi = \dot{\pi}[C]$ satisfies (1) of Lemma 3.3.31.

Claim 3.3.34 *Suppose $C \subseteq Coll\,(\omega, <\omega_1)$ is a generic filter over V and let $B = \dot{B}[C]$. Then, for every $\alpha \in D$ there exists one and only one $a \in B$ such that $\|a\| = \alpha$. i.e., we have:*

1. $V[C] \models \{\|a\| : a \in B\} = D$.
2. $V[C] \models (\forall a, b \in B)\,(\|a\| = \|b\| \rightarrow a = b)$.

Proof. (1) ($\Rightarrow$) Suppose that $V[C] \models \gamma \in D$. Then, since $D \in L$, $L \models \gamma \in D$. But, $\tau_\gamma \in B_{\omega_1}$ and hence $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tau_\gamma \in \dot{B}$". So $V[C] \models$ "$\tau_\gamma[C] \in \dot{B}[C] \wedge \|\tau_\gamma[C]\| = \gamma$". Therefore, $V[C] \models \gamma \in \{\|a\| : a \in B\}$.

($\Leftarrow$) Suppose that $V[C] \models \gamma \in \{\|a\| : a \in B\}$. Then, for some $p \in C$, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$(\exists x \in \dot{B})\,(\|x\| = \check{\gamma})$". Hence, there exists a simple $Coll\,(\omega, <\omega_1)$-name $\dot{a}$ for a subset of ω such that $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{a} \in \dot{B} \wedge \|\dot{a}\| = \check{\gamma}$". But then, there are $q \in C$ and $\tau \in B_{\omega_1}$ such that $q \leq p$ and $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\tau = \dot{a}$".

Since $\tau \in B_{\omega_1}$, for some $\delta \in D$, $\tau = \tau_\delta$ and $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \check{\delta}$". But then, since $q \leq p$, $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \check{\gamma}$", and so $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\check{\gamma} = \check{\delta}$". Therefore, $\gamma = \delta$ and $V[C] \models \gamma \in D$.

(2) Suppose that $a, b \in B$ and $V[C] \models$ "$\|a\| = \|b\|$". Let $\dot{a}$, $\dot{b}$ be simple $Coll\,(\omega, <\omega_1)$-names for subsets of ω such that $a = \dot{a}[C]$ and $b = \dot{b}[C]$. So, there

exists $p \in C$ such that $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{a}, \dot{b} \in \dot{B} \wedge \|\dot{a}\| = \|\dot{b}\|$". Then, as in (1), there are $q \in C$ and $\sigma, \tau \in B_{\omega_1}$ such that $q \leq p$ and

$$q \Vdash_{Coll(\omega,<\omega_1)} \text{“}\sigma = \dot{a} \wedge \tau = \dot{b} \wedge \|\sigma\| = \|\tau\|\text{”}$$

Since $\sigma, \tau \in B_{\omega_1}$, there are countable ordinals $\gamma, \delta \in D$ such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\sigma\| = \breve{\gamma}$" and $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \breve{\delta}$". So, $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{\gamma} = \breve{\delta}$" and hence $\gamma = \delta$. But then, by $<_L$-minimality of τ and σ, $\tau = \sigma$. Therefore,

$$a = \dot{a}\,[C] = \sigma[C] = \tau[C] = \dot{b}\,[C] = b. \quad \blacksquare$$

So, in order to prove (1) of 3.3.31, we only need:

Claim 3.3.35 *Suppose $C \subseteq Coll\,(\omega, <\omega_1)$ is a generic filter over V and $\pi = \dot{\pi}\,[C]$. Then π is a function from WO to B such that for every $a \in WO$, if $\pi\,(a) = b$, then b is the unique code in B of the least ordinal in D greater than $\|a\|$.*

Proof. We work in $V[C]$. Clearly $\pi \subseteq WO \times B$ and π is a function. So, it will suffice to show that:

1. $WO \subseteq \mathrm{dom}\,(\pi)$
2. If $\pi\,(a) = b$, then $b \in B$, $\|a\| < \|b\|$, and for every $c \in B$, if $\|a\| < \|c\|$, then $\|b\| \leq \|c\|$.

(1) Suppose that $a \in WO$. Then, there are a simple $Coll\,(\omega, <\omega_1)$-name $\dot{a}$ and $p \in C$ such that $\dot{a}[C] = a$ and $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{a} \in \dot{WO}$". So, there are $q \in C$ and $\tau \in WO_{\omega_1}$ such that $q \leq p$ and $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\tau = \dot{a}$". Let $\delta < \omega_1$ be such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\tau\| = \breve{\delta}$". Since D is an unbounded subset of ω_1 there is a least $\gamma \in D$ such that $\delta < \gamma$. But then, $\tau_\gamma \in B_{\omega_1}$. So, $\langle \tau, \tau_\gamma \rangle \in \pi_{\omega_1}$ and therefore $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{a} = \tau \in \mathrm{dom}\,(\dot{\pi})$". So, $a \in \mathrm{dom}\,(\pi)$.

(2) Suppose that $\langle a, b \rangle \in \pi$. So there are $p \in C$ and simple $Coll\,(\omega, <\omega_1)$-names $\dot{a}$, $\dot{b}$ such that $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\langle \dot{a}, \dot{b} \rangle \in \dot{\pi}$". Hence, there are $q \in C$ and $\langle \sigma, \tau \rangle \in \pi_{\omega_1}$ such that $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\langle \dot{a}, \dot{b} \rangle = \langle \sigma, \tau \rangle$". Since $\langle \sigma, \tau \rangle \in \pi_{\omega_1} \subseteq WO_{\omega_1} \times B_{\omega_1}$, $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\tau \in \dot{B}$", that is, $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{b} \in \dot{B}$".

Moreover, since $\langle \sigma, \tau \rangle \in \pi_{\omega_1}$, by definition of π_{ω_1}, $\Vdash_{Coll(\omega,<\omega_1)}$ "$\|\sigma\| < \|\tau\|$", and hence, $q \Vdash_{Coll(\omega,<\omega_1)}$ "$\|\dot{a}\| < \|\dot{b}\|$". Therefore, $\|a\| < \|b\|$.

Now Suppose that $c \in B$ and $\|a\| < \|c\|$. Then there are $p' \in C$ and a simple $Coll\,(\omega, <\omega_1)$-name $\dot{c}$ such that $\dot{c}[C] = c$ and $p' \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{c} \in \dot{B} \wedge \|\dot{a}\| < \|\dot{c}\|$". But then, by compatibility in the filter and as in the above case, there are $q' \in C$ and a simple $Coll\,(\omega, <\omega_1)$-name $\rho \in B_{\omega_1}$ such that $q' \leq q, p'$ and $q' \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{c} = \rho$". Therefore,

$$q' \Vdash_{Coll(\omega,<\omega_1)} \text{“}\langle \dot{a}, \dot{b} \rangle = \langle \sigma, \tau \rangle \wedge \langle \sigma, \tau \rangle \in \dot{\pi} \wedge \dot{c} = \rho \wedge \rho \in \dot{B} \wedge \|\sigma\| < \|\rho\|\text{”}.$$

But then, by definition of π_{ω_1},

$$q' \Vdash_{Coll(\omega,<\omega_1)} \text{“}\|\tau\| \leq \|\rho\|\text{”}.$$

Therefore, $q' \Vdash_{Coll(\omega,<\omega_1)}$ "$\|\dot{b}\| \leq \|\dot{c}\|$" and, so, $\|b\| \leq \|c\|$. ■

To prove (2) of Lemma 3.3.31 we need to compute the complexity of the sets involved in the definition of π:

Fact 3.3.36 *$Coll(\omega, <\omega_1)$ is a Δ^1_2 poset.*

Proof. Let $A = \{\langle \alpha, n, \beta \rangle \in \omega_1 \times \omega \times \omega_1 : \beta \in \alpha\}$. We say that $x \in \omega^\omega$ codes $\langle \alpha, n, \beta \rangle \in A$ iff $x(0) = n$, $\|(x')_0\| = \alpha$ and $\|(x')_1\| = \beta$, where for all $n \in \omega$, $x'(n) = x(n+1)$ and for every $i \in \{0,1\}$, $(x)_i = \{\langle k, x(2k+i) \rangle : k \in \omega\}$.

Hence "*x codes a triple in A*" is a Π^1_1 predicate on x: $x \in \omega^\omega$ codes a triple in A iff

$$(x')_0 \in WO \wedge (x')_1 \in WO \wedge \|(x')_1\| < \|(x')_0\|$$

But WO is a Π^1_1 set and $<$ is a Π^1_1 relation.

Let A^* be the Π^1_1 set of codes of elements of A.

Let $p \in Coll(\omega, <\omega_1)$. We may assume that p is lexicographically ordered in the first two coordinates. Suppose that $p = \langle \langle \alpha_i, n_i, \beta_i \rangle : i < n \rangle$ where $n = |\mathrm{dom}(p)|$. Then, we say that $x \in \omega^\omega$ codes p iff $x(0) = n$ and for every $i < n$, $(x')_i$ codes $\langle \alpha_i, n_i, \beta_i \rangle$.

Let $Coll(\omega, <\omega_1)^*$ be the set of all codes of elements of $Coll(\omega, <\omega_1)$. Then, for all $x \in \omega^\omega$, $x \in Coll(\omega, <\omega_1)^*$ iff

1. $(\forall i < x(0))((x')_i \in A^*)$
2. $(\forall ij < x(0))(\|((x')'_i)_0\| = \|((x')'_j)_0\| \wedge (x')_i(0) = (x')_j(0) \rightarrow$
$\rightarrow \|((x')'_i)_1\| = \|((x')'_j)_1\|)$

Now, since (1) is $\Pi^1_1(x)$ and (2) is $\Delta^1_2(x)$, $Coll(\omega, <\omega_1)^*$ is a Δ^1_2 set of reals.

Define $\leq_*$ in $Coll(\omega, <\omega_1)^* \times Coll(\omega, <\omega_1)^*$ by: for all $x, y \in Coll(\omega, <\omega_1)^*$, $x \leq_* y$ iff the sequence coded by x extends the sequence coded by y. Hence, for all $x, y \in \omega^\omega$, $x \leq_* y$ iff

1. $x, y \in Coll(\omega, <\omega_1)^*$
2. $(\forall j < y(0))(\exists i < x(0))(\|((x')'_i)_0\| = \|((y')'_j)_0\| \wedge$
$\wedge (x')_i(0) = (y')_j(0) \wedge \|((x')'_i)_1\| = \|((y')'_j)_1\|)$

So, since (1) is $\Delta^1_2(x, y)$ and (2) is $\Sigma^1_1(x, y)$, $\leq_*$ is a Δ^1_2 relation in $\omega^\omega \times \omega^\omega$. Note that $\leq_*$ is reflexive and transitive relation but not a antisymmetric, and $Coll(\omega, <\omega_1)^* = \mathrm{Field}(\leq_*)$.

Define $\perp_*$ in $Coll(\omega, <\omega_1)^* \times Coll(\omega, <\omega_1)^*$ as follows: for every $x, y \in Coll(\omega, <\omega_1)^*$, $x \perp_* y$ iff x, y code incompatible sequences. So, for all $x, y \in \omega^\omega$, $x \perp_* y$ iff

1. $x, y \in Coll(\omega, <\omega_1)^*$

2. $(\exists i < x(0))(\exists j < y(0))(\|((x')'_i)_0\| = \|((y')'_j)_0\| \wedge$

$$\wedge (x')_i(0) = (y')_j(0) \wedge \|((x')'_i)_1\| \neq \|((y')'_j)_1\|)$$

But then, (1) and (2) are $\Delta^1_2(x, y)$ and hence $\perp_*$ is also a Δ^1_2 relation in $\omega^\omega \times \omega^\omega$.

So, $\langle Coll(\omega, <\omega_1)^*, \leq_*, \perp_* \rangle$ is a Δ^1_2 forcing notion.

Finally, we define $F : Coll(\omega, <\omega_1)^* \to Coll(\omega, <\omega_1)$ by $F(x)$ is the condition coded by x. F is a dense embedding (but not one-to-one) from $Coll(\omega, <\omega_1)^*$ into $Coll(\omega, <\omega_1)$. Therefore $Coll(\omega, <\omega_1)$ is a Δ^1_2 poset. ■

Since $Coll(\omega, <\omega_1)$ is a Δ^1_2 and ccc poset, by Fact 2.1.15, "x codes an antichain of $Coll(\omega, <\omega_1)$" is a Δ^1_2 predicate. Hence, as in Fact 2.1.22, we may code every simple $Coll(\omega, <\omega_1)$-name for a subset of ω with a real in a such a way that "x codes a $Coll(\omega, <\omega_1)$-name for a subset of ω" is a Δ^1_2 predicate and "x is codes a simple $Coll(\omega, <\omega_1)$-name in L for a subset of ω" is the intersection of a Δ^1_2 and a Σ^1_2 predicate. Hence, Σ^1_2.

Fact 3.3.37 *Let $\theta(x)$ be a Σ^1_1 (Π^1_1) formula. Then the relation*

$$R(p, \tau) \leftrightarrow p \in Coll(\omega, <\omega_1) \wedge$$
$$\wedge\, \tau \textit{ is a simple } Coll(\omega, <\omega_1)\textit{-name for a subset of } \omega \wedge$$
$$\wedge\, p \Vdash_{Coll(\omega, <\omega_1)} \theta(\tau)$$

is a Δ^1_2 relation.

Proof. We prove it for Π^1_1 formulas. The case for Σ^1_1 formulas is analogous.

By Claim 2.1.24, for every Σ^1_1 formula $\varphi(x)$, every $p \in Coll(\omega, <\omega_1)$ and every simple $Coll(\omega, <\omega_1)$-name σ for a subset of ω, the following are equivalent:

1. $p \Vdash_{Coll(\omega, <\omega_1)} \varphi(\sigma)$
2. For every transitive model M of ZFC^* containing p and (the code of) σ, and such that $Coll(\omega, <\omega_1)^M \lessdot Coll(\omega, <\omega_1)$, $M \models$ "$p \Vdash_{Coll(\omega, <\omega_1)} \varphi(\sigma)$".
3. There exists a transitive and countable model M of ZFC^* containing p (the code of) σ and such that $Coll(\omega, <\omega_1)^M \lessdot Coll(\omega, <\omega_1)$ $M \models$ "$p \Vdash_{Coll(\omega, <\omega_1)} \varphi(\sigma)$".

For every transitive model M of ZFC^*, $Coll(\omega, <\omega_1)^M = Coll(\omega, <\omega_1^M)$. So, since $\omega_1^M \leq \omega_1$, $Coll(\omega, <\omega_1^M) \lessdot Coll(\omega, <\omega_1)$. Therefore (2) and (3) above are equivalent to, respectively,

2'. For every transitive model M of ZFC^* containing p and (the code of) σ, $M \models$ "$p \Vdash_{Coll(\omega, <\omega_1)} \varphi(\sigma)$".

3'. There exists a transitive and countable model M of ZFC^* containing p (the code of) σ and such that $M \models$ "$p \Vdash_{Coll(\omega, <\omega_1)} \varphi(\sigma)$".

But it is easy to see that (2′) is expressible with a Π^1_2 sentence and (3′) is expressible with a Σ^1_2 sentence, both with p and σ as parameters. ■

Corollary 3.3.38 *Let WO^* be the set of codes of simple $Coll(\omega, <\omega_1)$-names for a subset of ω that belong to WO_{ω_1}. Then WO^* is a Δ^1_2 set of reals.*

Proof. Since $x \in WO$ is a Π^1_1 formula. ■

Claim 3.3.39 *Let B^* be the set of all codes for simple $Coll(\omega, <\omega_1)$-names for a subset of ω belonging to B_{ω_1}. Then B^* is a $\undertilde{\Sigma}^1_{n+1}$ set of reals.*

Proof. We define the relation Dif as follows: for every $x, y \in \omega^\omega$, $xDify$ iff x codes a simple $Coll(\omega, <\omega_1)$-name for a subset of ω different from the simple $Coll(\omega, <\omega_1)$-name coded by y.

So, for every $x, y \in \omega^\omega$, $xDify$ iff x, y code simple $Coll(\omega, <\omega_1)$-names for subsets of ω and

$$\exists nm(J^{-1}(x_n(m))_0 \wedge J^{-1}(y_n(m))_0 \text{ code different elements of } Coll(\omega, <\omega_1) \vee \\ \vee J^{-1}(x_n(m))_1 \neq J^{-1}(y_n(m))_1).$$

(see Fact 3.3.36 and the remark following it). So $xDify$ is a $\Delta^1_2(x, y)$ relation.

Let $<^*_L$ be the following relation: for every $x, y \in \omega^\omega$, $x <^*_L y$ iff x, y code simple $Coll(\omega, <\omega_1)$-names in L for subsets of ω and the simple $Coll(\omega, <\omega_1)$-name coded by x is $<_L$-less than the simple $Coll(\omega, <\omega_1)$-name coded by y. Since "x codes a simple $Coll(\omega, <\omega_1)$-name in L for a subset of ω" is a Σ^1_2 predicate, $<_L$ is a Σ_1-relation over HC and every simple $Coll(\omega, <\omega_1)$-name for a subset of ω is hereditarily countable, $<^*_L$ is a Σ_1 definable relation over HC. Hence $<^*_L$ is a Σ^1_2 relation.

Recall that D^* is the $\undertilde{\Sigma}^1_{n+1}$ set of codes of ordinals in D. Let R be the relation: $R(\sigma, \breve{a})$ iff

$$\sigma \text{ is a simple } Coll(\omega, <\omega_1)\text{-name for a subset of } \omega \wedge \\ \wedge\, a \in WO \wedge \Vdash_{Coll(\omega,<\omega_1)} \text{“} \|\sigma\| = \|\breve{a}\| \text{”}.$$

Note that R is a Δ^1_2 relation (by Fact 3.3.37).

We have that for every $x \in \omega^\omega$, $x \in B^*$ iff

1. x codes a simple $Coll(\omega < \omega_1)$-name in L for a subset of ω.

2. There exists $y \in D^*$ such that $R(x, y)$ and

$$\forall z(z \text{ codes a simple } Coll(\omega, <\omega_1)\text{-name in } L \text{ for a subset of } \omega \wedge \\ \wedge\, xDifz \wedge R(z, y) \rightarrow \neg z <^*_L x)$$

(1) is $\Sigma^1_2(x)$. Since the universally quantified formula in (2) is $\Pi^1_2(x)$ and D^* is $\underset{\sim}{\Sigma}^1_{n+1}$ with $n \geq 2$, (2) is $\underset{\sim}{\Sigma}^1_{n+1}$. So, B^* is a $\underset{\sim}{\Sigma}^1_{n+1}$ set of reals. ■

Let $\pi^* : WO^* \to \omega^\omega$ be such that $\pi^*(x) = y$ iff $\exists\sigma\tau(x$ codes $\sigma \wedge y$ codes $\tau \wedge \pi_{\omega_1}(\sigma) = \tau)$.

We finish the proof of (2) of Lemma 3.3.31 by showing that π^* is a $\underset{\sim}{\Delta}^1_{n+2}$ function in $\omega^\omega \times \omega^\omega$.

Let S and T be the following relations:

$$S(\sigma,\tau) \text{ iff } \sigma,\tau \text{ are simple } Coll(\omega,<\omega_1)\text{-names for subsets of } \omega \wedge \\ \wedge \Vdash_{Coll(\omega,<\omega_1)} \text{“} \|\sigma\| < \|\tau\| \text{”}$$

and

$$T(\sigma,\tau) \text{ iff } \wedge\, \sigma,\tau \text{ are simple } Coll(\omega,<\omega_1)\text{-names for subsets of } \omega \wedge \\ \wedge \Vdash_{Coll(\omega,<\omega_1)} \text{“} \|\sigma\| \leq \|\tau\| \text{”}$$

Since $\|x\| < \|y\|$ is a Π^1_1 formula and $\|x\| \leq \|y\|$ is a Σ^1_1 formula, by Fact 3.3.37, S and T are Δ^1_2 relations between codes for simple $Coll(\omega,<\omega_1)$-names for subset of ω.

So, for every $x, y \in \omega^\omega$, $\langle x,y\rangle \in \pi^*$ iff

1. $x \in WO^*$
2. $y \in B^*$
3. $S(x,y)$
4. $\forall z(z \in B^* \wedge yDifz \wedge S(x,z) \to T(y,z))$

Since (1) is $\Delta^1_2(x)$, (2) is $\underset{\sim}{\Sigma}^1_{n+1}$, (3) is $\Delta^1_2(x,y)$ and, as it can be easily seen, (4) is $\underset{\sim}{\Pi}^1_{n+1}$, π^* is $\underset{\sim}{\Delta}^1_{n+2}$. Actually, it is the intersection of a $\underset{\sim}{\Sigma}^1_{n+1}$ and a $\underset{\sim}{\Pi}^1_{n+1}$ relation. This ends the proof of Lemma 3.3.31. ■

We continue with the proof of Theorem 3.3.30. Working in $V[C]$, we define the Solovay's almost-disjoint forcing for π, P_π, which adds a real that codes π.

Let $\langle s_n : n \in \omega\rangle$ be a recursive enumeration of $2^{<\omega}$, the set of all finite sequences of 0's and 1's such that each sequence is enumerated before any of its proper extensions. Let $\{X_n : n \in \omega\}$ be a recursive partition of ω into infinite pieces. For any $a \subseteq \omega$, let $\tilde{a}$ be the characteristic function of a. For every $a \subseteq \omega$ and every $n \in \omega$, define

$$\begin{aligned} S^a &= \{k \in \omega : s_k \subseteq \tilde{a}\} \\ S^a_n &= \{k \in \omega : s_k \subseteq \tilde{a} \wedge \lg(s_k) \in X_n\} \end{aligned}$$

Note that $\{S^a : a \subseteq \omega\}$ is a perfect maximal almost-disjoint family of infinite subsets of ω. Also, notice that S^a is the disjoint union of the S^a_n, $n \in \omega$.

For any subsets a and b of ω, let

$$b \odot a = \{n \in \omega : b \cap S_n^a \text{ is finite}\}$$

Note that $\odot$ is a Borel operation from $P(\omega) \times P(\omega)$ into $P(\omega)$:

$$b \odot a = c \text{ iff } \forall n \, (n \in c \leftrightarrow \exists m \forall k \, (k \in b \cap S_n^a \rightarrow k < m))$$

and S_n^a is recursive on a and X_n.

Definition 3.3.40 *Suppose C is a $Coll(\omega, <\omega_1)$-generic filter over V. Then, in $V[C]$, let $G(\pi) = \{\langle a, n\rangle : n \in \pi(a)\}$, the* graph of π. *Then, the* Solovay's almost-disjoint coding for π, P_π, *is the following poset:*

- *$\langle s, g\rangle \in \mathsf{P}_\pi$ iff $s \in [\omega]^{<\omega}$ and $g \in [G(\pi)]^{<\omega}$. i.e., $s \subseteq \omega$ is finite and $g \subseteq G(\pi)$ is finite.*
- *$\langle s, g\rangle \leq \langle t, h\rangle$ iff $t \subseteq s$, $h \subseteq g$ and $(\forall \langle a, n\rangle \in h)\,(s \cap S_n^a \subseteq t)$.*

Remark 3.3.41 *Note that in $V[C]$, P_π is a σ-centered poset (see Definition 2.3.42). For $\langle s, g\rangle, \langle s, h\rangle \in \mathsf{P}_\pi$, then $\langle s, g \cup h\rangle \leq \langle s, g\rangle, \langle s, h\rangle$. Hence, any two conditions in P_π with the same first coordinate are compatible.*

We shall see that forcing with P_π over $V[C]$ adds a real that codes π.

Lemma 3.3.42 *Suppose that H is a P_π-generic filter over $V[C]$. Then there is a real $b_H \in P(\omega) \cap V[C][H]$ such that $\omega_1^{L[b_H]} = \omega_1$.*

Proof. Suppose that H is a P_π-generic filter over $V[C]$ and let

$$b_H = \bigcup \{s \in [\omega]^{<\omega} : (\exists g \in [G(\pi)]^{<\omega})\,(\langle s, g\rangle \in H)\}$$

Clearly $b_H \in P(\omega) \cap V[C][H]$.

Claim 3.3.43 *For every $a \in \mathrm{dom}(\pi)$, $b_H \odot a = \pi(a)$.*

Proof. To prove the claim, it will be enough to show that for every $a \in \mathrm{dom}(\pi)$ and $n \in \omega$,

$$S_n^a \cap b_H \text{ is finite iff } n \in \pi(a).$$

Fix $a \in \mathrm{dom}(\pi)$ and $n \in \omega$. Suppose that $n \in \pi(a)$. Then, $D_{a,n} = \{\langle s, g\rangle \in \mathsf{P}_\pi : \langle a, n\rangle \in g\}$ is a dense subset of P_π. For suppose $\langle s, g\rangle \in \mathsf{P}_\pi$. Let $h = g \cup \{\langle a, n\rangle\}$. Then $\langle s, h\rangle \in D_{a,n}$ and $\langle s, h\rangle \leq \langle s, g\rangle$. Let $\langle s, g\rangle \in H \cap D_{a,n}$. Then $\langle a, n\rangle \in g$ and for every $\langle t, h\rangle \in H$, if $\langle t, h\rangle \leq \langle s, g\rangle$, then $S_n^a \cap t \subseteq s$. So, $S_n^a \cap b_H \subseteq s$. Therefore $S_n^a \cap b_H$ is finite.

Suppose now that $n \notin \pi(a)$. Note that for every $m \in \omega$ the set $E_{a,n}^m = \{\langle s, g\rangle \in \mathsf{P}_\pi : (\exists k > m)\,(k \in S_n^a \cap s)\}$ is dense in P_π. Indeed, fix any $m \in \omega$ and $\langle s, g\rangle \in \mathsf{P}_\pi$. Since $\{S^a : a \subseteq \omega\}$ is an almost-disjoint family and for every $a \subseteq \omega$, $S^a = \bigcup_{k \in \omega} S_k^a$,

$S_n^a \setminus \bigcup_{\langle b,j\rangle \in g} S_j^b$ is an infinite set. Let k be the least element of $S_n^a \setminus \bigcup_{\langle b,j\rangle \in g} S_j^b$ greater than m. Put $t = s \cup \{k\}$. Since $m < k \in S_i^a \cap t$, $\langle t, g\rangle \in E_{a,n}^m$. Moreover, $s \subseteq t$ and, since for every $\langle b, j\rangle \in g$, $k \notin S_j^b$, $S_j^b \cap t = S_j^b \cap s \subseteq s$. So, $\langle t, g\rangle \leq \langle s, g\rangle$.

Let $\langle s, g\rangle \in H \cap E_{a,n}^m$. Since $\langle s, g\rangle \in H$, $s \subseteq b_H$. But then, there is $k > m$ such that $k \in S_n^a \cap s \subseteq S_n^a \cap b_H$. We have shown that for every $m \in \omega$ there exists $k > m$ such that $k \in S_n^a \cap b_H$ and so $S_n^a \cap b_H$ is infinite. ■

Let $\{\delta_\xi : \xi < \omega_1\} \in L[b_H]$ be an increasing and continuous enumeration of D. Let $\nu = \omega_1^{L[b_H]}$. We define, by recursion on $\xi < \nu$, a sequence of reals $\langle d_\xi : \xi < \nu\rangle$ such that:

1. For every $\xi < \nu$, $d_\xi \in WO$ and $\|d_\xi\| = \delta_\xi$.

2. For every $\xi < \nu$, $d_\xi \in L[b_H]$.

$\underline{\xi = 0}$: Let $a \in L$ be a code for a well-ordering of ω with $\|a\| = \omega$. Since the S_n^a $(n \in \omega)$ are recursive on a, $S_n^a \in L[b_H]$, for every $n \in \omega$. Since $a \in WO$, $a \in \mathrm{dom}\,(\pi)$. Let $d_0 = \pi\,(a)$. So, $d_0 \in WO$. Moreover, $\|d_0\| = \min\,(D \setminus \omega + 1) = \min\,(D) = \delta_0$ and $d_0 = \pi\,(a) = b_H \odot a \in L[b_H]$.

$\underline{\xi = \eta + 1}$: Suppose that d_η satisfies (1) and (2). Since $d_\eta \in WO$, $d_\eta \in \mathrm{dom}\,(\pi)$. Let $d_{\eta+1} = \pi\,(d_\eta)$. Clearly, by definition of π, $d_{\eta+1} \in B \subseteq WO$ and $\|d_{\eta+1}\| = \delta_{\eta+1} \in D$. Finally, since $d_\eta \in L[b_H]$, for every $n \in \omega$, $S_n^{d_\eta} \in L[b_H]$, and hence $d_{\eta+1} = \pi\,(d_\eta) = b_H \odot d_\eta \in L[b_H]$.

$\underline{\xi \text{ limit}}$: Suppose that for every $\eta < \xi$, d_η has been defined and satisfies (1) and (2). Note that $(d_\eta)_{\eta<\xi} \in L[b_H]$. We work in $L[(d_\eta)_{\eta<\xi}]$. Since for every $\eta < \xi$, $d_\eta \in L[(d_\eta)_{\eta<\xi}]$, $L[(d_\eta)_{\eta<\xi}] \models$ "δ_η is countable" for all $\eta < \xi$. Moreover, since D is a club and for every $\eta < \xi$, $\delta_\eta \in D$, $\delta_\xi = \sup_{\eta<\xi}\,(\delta_\eta) \in D$. But, then $L[(d_\eta)_{\eta<\xi}] \models$ "δ_ξ is singular" and hence, $L[(d_\eta)_{\eta<\xi}] \models$ "δ_ξ is countable". Let d_ξ by the least real in the canonical well-ordering of $L[(d_\eta)_{\eta<\xi}]$ coding a well-ordering of ω with $\|d_\xi\| = \delta_\xi$. So, d_ξ satisfies (1) and, since $L[(d_\eta)_{\eta<\xi}] \subseteq L[b_H]$, $d_\xi \in L[b_H]$; i.e., it satisfies (2).

Suppose now that $\nu < \omega_1$. Then, since $\langle \delta_\xi : \xi < \nu\rangle$ is an increasing sequence of elements of D and $\nu = \sup_{\xi<\nu}\,(\delta_\xi)$, $\nu \in D$. So, for some $\xi < \omega_1$, $\nu = \delta_\xi$. But then, $L[b_H] \models$ "ω_1 is singular". A contradiction. Therefore, $\omega_1^{L[b_H]} = \omega_1$. ■

Note that if V is projective absolute, then P_π is not a projective poset: Since V is projective absolute for projective and ccc posets and $Coll\,(\omega, <\omega_1)$ is a $\underset{\sim}{\Delta}_2^1$ and ccc poset, by the Factor Lemma, $V[C]$ is projective absolute for Borel ccc posets. Indeed, by Fact 3.3.37, every Borel and ccc extension of $V[C]$ is a $\underset{\sim}{\Delta}_2^1$ ccc extension of V. Suppose now, towards a contradiction, that P_π is a projective poset so that B has a projective definition. We define $\leq$ in $B \times B$ as follows: for every $x, y \in B$, $x \leq y$ iff $\|x\| \leq \|y\|$. Then, $\langle B, \leq\rangle$ is an uncountable projective well-ordering. A contradiction with Theorem 3.1.15.

However,

Lemma 3.3.44 *$Coll\,(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is a $\underset{\sim}{\Delta}_{n+2}^1$ poset.*

Proof. We need the following facts and definitions in order to code the coding apparatus of P_π and calculate its complexity.

Definition 3.3.45 *Let $\dot{G}(\pi)$ be the $Coll(\omega,<\omega_1)$-name for the graph of π defined as follows:*

$$\langle p,\sigma,\breve{n}\rangle \in \dot{G}(\pi) \text{ iff } \exists\tau\left(\langle\sigma,\tau\rangle \in \pi_{\omega_1} \wedge \langle p,\breve{n}\rangle \in \tau\right).$$

Let $G(\pi)^*$ be the set of all ordered triples $\langle x,y,n\rangle \in \omega^\omega \times \omega^\omega \times \omega$ such that for some $\langle p,\sigma,\breve{n}\rangle \in \dot{G}(\pi)$, x codes p and y codes σ.

It is clear that for every $p \in Coll(\omega,<\omega_1)$,

$$p \Vdash_{Coll(\omega,<\omega_1)} \text{“}\langle\sigma,\breve{n}\rangle \in \dot{G}(\pi)\text{” iff } p \Vdash_{Coll(\omega,<\omega_1)} \text{“}\breve{n} \in \dot{\pi}(\sigma)\text{”}.$$

Moreover,

Fact 3.3.46 *$G(\pi)^*$ is a $\underset{\sim}{\Delta}^1_{n+2}$ relation in $\omega^\omega \times \omega^\omega \times \omega$.*

Proof. First recall that for every $x,y \in \omega^\omega$ and $n \in \omega$, $\langle x,y,n\rangle \in G(\pi)^*$ iff

1. $x \in Coll(\omega,<\omega_1)^*$
2. $y \in WO^*$
3. $\exists z\left(\langle y,z\rangle \in \pi^* \wedge \exists m\left(x = (\{\langle k,j\rangle : (z)_n(k) = J(j,n)\})_m\right)\right)$

Since (1) is $\Delta^1_2(x)$, (2) is $\Delta^1_2(y)$ and, since π^* is $\underset{\sim}{\Delta}^1_{n+2}$, (3) is $\underset{\sim}{\Sigma}^1_{n+2}$. Therefore, $G(\pi)^*$ is $\underset{\sim}{\Sigma}^1_{n+2}$.

But since π_{ω_1} is a function, if $\langle x,z\rangle, \langle x,z'\rangle \in \pi^*$ and $z \neq z'$, then z and z' are different codes for the same simple $Coll(\omega,<\omega_1)$-name. Therefore, for every $x,y \in \omega^\omega$ and $n \in \omega$, $\langle x,y,n\rangle \in G(\pi)^*$ iff

1. $x \in Coll(\omega,<\omega_1)^*$
2. $y \in WO^*$
3. $\forall z\left(\langle y,z\rangle \in \pi^* \rightarrow \exists m\left(x = (\{\langle k,j\rangle : (z)_n(k) = J(j,n)\})_m\right)\right)$

and since (3) is $\underset{\sim}{\Pi}^1_{n+2}$, $G(\pi)^*$ is also $\underset{\sim}{\Pi}^1_{n+2}$.

So, $G(\pi)^*$ is $\underset{\sim}{\Delta}^1_{n+2}$. ■

Definition 3.3.47 *We say that $\dot{g}$ is a* nice $Coll(\omega,<\omega_1)$-name for a finite subset of $G(\pi)$ *iff for some $n \in \omega$ there exists $\{\langle\sigma_j,\breve{\imath}_j\rangle : j < n\} \subseteq \mathrm{rec}(\dot{G}(\pi))$ such that $\dot{g} \subseteq \dot{G}(\pi)$ is of form*

$$\bigcup\{A_{\langle\sigma_j,i_j\rangle} \times \{\langle\sigma_j,\breve{\imath}_j\rangle\} : j < n\}$$

where for every $j < n$, $A_{\langle\sigma_j,i_j\rangle}$ is an antichain of $Coll(\omega,<\omega_1)$.

Fact 3.3.48 *Suppose that $\dot{h}$ is a $Coll\,(\omega, <\omega_1)$-name such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} \subseteq \dot{G}\,(\pi) \wedge \dot{h}$ is finite" and let $D_{\dot{h}}$ be the set of all $p \in Coll\,(\omega, <\omega_1)$ such that for some nice $Coll\,(\omega, <\omega_1)$-name $\dot{g}$ for a finite subset of $\dot{G}\,(\pi)$, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} = \dot{g}$". Then $D_{\dot{h}}$ is a dense subset of $Coll\,(\omega, <\omega_1)$.*

Proof. Suppose that $\dot{h}$ is a $Coll\,(\omega, <\omega_1)$-name such that $\Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} \subseteq \dot{G}\,(\pi) \wedge \dot{h}$ is finite" and let $p \in Coll\,(\omega, <\omega_1)$. Since $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} \subseteq \dot{G}\,(\pi) \wedge \dot{h}$ is finite", we may find $n \in \omega$ and a condition $p' \leq p$ such that $p' \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} = \{\langle \dot{a}_j, \dot{k}_j\rangle : j < \check{n}\} \subseteq \dot{G}\,(\pi)$". So, for every $j < n$,

$$p' \Vdash_{Coll(\omega,<\omega_1)} \text{“}\langle \dot{a}_j, \dot{k}_j\rangle \in \dot{WO} \times \check{\omega}\text{”}$$

We can find a decreasing sequence $\langle p_j : j < n\rangle$ in $Coll\,(\omega, <\omega_1)$ such that:

1. For every $j < n$, $p_j \leq p'$.
2. For every $j < j' < n$, $p_{j'} \leq p_j$.
3. For every $j < n$, there exists $\sigma_j \in WO_{\omega_1}$ and $i_j \in \omega$ such that $p_j \Vdash_{Coll(\omega,<\omega_1)}$ "$\langle \dot{a}_j, \dot{k}_j\rangle = \langle \sigma_j, \check{\imath}_j\rangle$".

Let $r = p_{n-1}$. Then for every $j < n$,

$$r \Vdash_{Coll(\omega,<\omega_1)} \text{“}\langle \dot{a}_j, \dot{k}_j\rangle = \langle \sigma_j, \check{\imath}_j\rangle\text{”}$$

for some $\sigma_j \in WO_{\omega_1}$ and $i_j \in \omega$. Note that for every $j < n$, $\langle \sigma_j, \check{\imath}_j\rangle \in \operatorname{rec}(\dot{G}\,(\pi))$. Let

$$\dot{g} = \bigcup\{A_{\langle \sigma_j, i_j\rangle} \times \{\langle \sigma_j, \check{\imath}_j\rangle\} : j < n\}$$

where for every $j < n$, $A_{\langle \sigma_j, i_j\rangle} = \{p \in Coll\,(\omega, <\omega_1) : \langle p, \sigma_j, \check{\imath}_j\rangle \in \dot{G}\,(\pi)\}$. It is clear that $\dot{g}$ is a nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$. Since $r \leq p'$,

$$r \Vdash_{Coll(\omega,<\omega_1)} \text{“}\dot{h} = \{\langle \dot{a}_j, \dot{k}_j\rangle : j < \check{n}\} = \{\langle \sigma_j, \check{\imath}_j\rangle : j < \check{n}\} = \dot{g}\text{”}$$

Therefore, $r \in D_{\dot{h}}$ and $D_{\dot{h}}$ is a dense subset of $Coll\,(\omega, <\omega_1)$. ■

Fact 3.3.49 *We may code every nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$ with a real so that the set F^* of all reals that code some nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$ is a $\underset{\sim}{\Delta}^1_{n+2}$ set of reals.*

Proof. Let $\dot{g}$ be a nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$. Suppose that $|\operatorname{rec}(\dot{g})| = n$, where $n \in \omega$. Fix an enumeration $\{\langle \sigma_j, \check{\imath}_j\rangle : j < n\}$ of $\operatorname{rec}(\dot{g})$. For every $j < n$, let $A_j = \{p \in Coll\,(\omega, <\omega_1) : \langle p, \sigma_j, \check{\imath}_j\rangle \in \dot{g}\}$ be the antichain associated to $\langle \sigma_j, \check{\imath}_j\rangle$ in $\dot{g}$. For every $j < n$, let y_j be a code for A_j and let $u_j \in WO^*$ be a code for σ_j. For every $j < n$, let $z_j \in \omega^\omega$ be such that $z_j\,(0) = i_j$ and $z_j' = u_j$. Finally, for all $j < n$, let $x_j \in \omega^\omega$ be such that $(x_j)_0 = y_j$ and $(x_j)_1 = z_j$. Then we say that x codes $\dot{g}$ iff $x\,(0) = n$ and for every $j < n$, $(x')_j = x_j$.

Let F^* be the set of all reals that code some nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$. Then, for all $x \in \omega^\omega$, $x \in F^*$ iff

1. x codes $\langle x_j : j < x(0)\rangle$
2. $(\forall j < x(0))\,(x_j$ codes $\langle y_j, z_j\rangle)$
3. $(\forall j < x(0))\,(y_j$ codes an antichain of $Coll\,(\omega, <\omega_1))$
4. $(\forall j < x(0))\,\forall n(\langle (y_j)_n, z'_j, z_j(0)\rangle \in G(\pi)^*)$

Since (1) and (2) are $\Delta^1_1(x)$, (3) is $\Delta^1_2(x)$ and (4) is $\underset{\sim}{\Delta}^1_{n+2}$, F^* is a $\underset{\sim}{\Delta}^1_{n+2}$ set of reals. ■

Definition 3.3.50 *We define $Caract\,(x,y)$ in $\omega^\omega \times \omega^\omega$ as follows: for all simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and all simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω (i.e., a simple $Coll\,(\omega, <\omega_1)$-name τ for a real such that for every $n \in \omega$, $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tau(\check{n}) \in \{\check{0}, \check{1}\}$"), $Caract\,(\sigma, \tau)$ iff*

$$\Vdash_{Coll(\omega,<\omega_1)} \text{"}\tilde{\sigma} = \tau\text{"}$$

That is, $Caract\,(\sigma,\tau)$ iff $\Vdash_{Coll(\omega,<\omega_1)}$ "τ is the characteristic function of σ".

It is clear that for every simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω there exists a simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω such that $Caract\,(\sigma,\tau)$ and, conversely, for every simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω there is a simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω such that $Caract\,(\sigma,\tau)$. Moreover, for every simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and all simple $Coll\,(\omega, <\omega_1)$-names τ, τ' for reals in 2^ω, if $Caract\,(\sigma,\tau)$ and $Caract\,(\sigma,\tau')$, then $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tau = \tau'$" and, conversely, for every simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω and all simple $Coll\,(\omega, <\omega_1)$-names σ, σ' for subsets of ω, if $Caract\,(\sigma,\tau)$ and $Caract\,(\sigma',\tau)$, then $\Vdash_{Coll(\omega,<\omega_1)}$ "$\sigma = \sigma'$".

Claim 3.3.51 *Let $Caract^* \subseteq \omega^\omega \times \omega^\omega$ be defined by: for all $x, y \in \omega^\omega$, $Caract^*(x,y)$ iff x codes a simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and y codes a simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω and $Caract\,(\sigma,\tau)$. Then $Caract^*$ is a Δ^1_2 relation in $\omega^\omega \times \omega^\omega$.*

Proof. Notice that $Caract^*(x,y)$ iff x codes a simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and y codes a simple $Coll\,(\omega, <\omega_1)$-name for a real in 2^ω and

$$\Vdash_{Coll(\omega,<\omega_1)} \text{"}\tilde{\sigma} = \tau\text{"}.$$

But $\Vdash_{Coll(\omega,<\omega_1)}$ "$\tilde{\sigma} = \tau$" iff $\Vdash_{Coll(\omega,<\omega_1)}$ "$(\forall n \in \check{\omega})\,(n \in \sigma \leftrightarrow \tau(n) = \check{1})$" and $(\forall n \in \omega)\,(n \in x \leftrightarrow y(n) = 1)$ is an arithmetical sentence with x and y as parameters. Thus, by Fact 3.3.37, $Caract^*$ is a Δ^1_2 relation in $\omega^\omega \times \omega^\omega$. ■

Note that $Coll\,(\omega, <\omega_1)$ has the following property: $p_0, ..., p_n \in Coll\,(\omega, <\omega_1)$ are pairwise compatible iff $p_0 \cup ... \cup p_n \in Coll\,(\omega, <\omega_1)$. And, in this case, $p_0 \cup ... \cup p_n \leq p_0, ..., p_n$. So, if $n \in \omega$ and $\{A_i : i \leq n\}$ is a family of antichains of $Coll\,(\omega, <\omega_1)$, then

$$A = \{p_0 \cup ... \cup p_n : (\forall i \leq n)\,(p_i \in A_i) \wedge (\forall i, j \leq n)\,(i \neq j \rightarrow p_i \not\perp p_j)\}$$

is an antichain of $Coll\,(\omega, <\omega_1)$. Moreover, if for every $i \leq n$, A_i is maximal, then A is also maximal.

Let s be a finite sequence of 0's and 1's and $i < \lg(s)$. By $\breve{s}\,(i)$ we denote the canonical $Coll\,(\omega, <\omega_1)$-name for the i-value of s.

Definition 3.3.52 *For every simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω, let $\dot{S}^\tau$ be the following $Coll\,(\omega, <\omega_1)$-name:*

$$\dot{S}^\tau = \{\langle p, \breve{n}\rangle : p \in Coll\,(\omega, <\omega_1) \wedge \\ \wedge (\forall i < \lg(s_n))\, \exists q_i(\langle q_i, \breve{\imath}, \breve{s}_n\,(i)\rangle \in \tau \wedge p = \textstyle\bigcup_{i<\lg(s_n)} q_i))\}.$$

Claim 3.3.53 *For all simple $Coll\,(\omega, <\omega_1)$-name τ for a real in 2^ω, $\dot{S}^\tau$ is a simple $Coll\,(\omega, <\omega_1)$-name for a subset of ω. Moreover, for every $n \in \omega$, $\Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \dot{S}^\tau \leftrightarrow \breve{s}_n \subseteq \tau$".*

Proof. Let τ be a simple $Coll\,(\omega, <\omega_1)$-name for a real in 2^ω. To see that $\dot{S}^\tau$ is a simple $Coll\,(\omega, <\omega_1)$-name for a subset of ω, we only need to show that for every $n \in \omega$, $A_n = \{p \in Coll\,(\omega, <\omega_1) : \langle p, \breve{n}\rangle \in \dot{S}^\tau\}$ is an antichain of $Coll\,(\omega, <\omega_1)$.

For every $i < \lg(s_n)$, let $B_i = \{q \in Coll\,(\omega, <\omega_1) : \langle q, \breve{\imath}, \breve{s}_n\,(i)\rangle \in \tau\}$. Then,

$$A_n = \{\textstyle\bigcup_{i<\lg(s_n)} p_i : p_i \in B_i \wedge (\forall i, j < \lg(s_n))\,(i \neq j \rightarrow p_i \not\perp p_j)\}.$$

But, since for every $i < \lg(s_n)$, $B_i \subseteq \{q \in Coll\,(\omega, <\omega_1) : \langle q, \breve{\imath}, 0\rangle \in \tau \vee \langle q, \breve{\imath}, 1\rangle \in \tau\}$ and τ is a simple $Coll\,(\omega, <\omega_1)$-name for a real in 2^ω, the last set is a maximal antichain and, hence, B_i is an antichain. Therefore, A_n is also an antichain.

Let $C \subseteq Coll\,(\omega, <\omega_1)$ be a generic filter over V such that $V[C] \models$ "$n \in \dot{S}^\tau[C]$". So, for some $p \in C$, $\langle p, \breve{n}\rangle \in \dot{S}^\tau$. Hence, by definition of $\dot{S}^\tau$, for every $i < \lg(s_n)$ there exists q_i such that $\langle q_i, \breve{\imath}, \breve{s}_n\,(i)\rangle \in \tau$ and $p = \bigcup_{i<\lg(s_n)} q_i$. But, since $p \in C$ and for every $i < \lg(s_n)$, $p \leq q_i$, $q_i \in C$ for every $i < \lg(s_n)$. So, for every $i < \lg(s_n)$, $\langle i, s_n\,(i)\rangle \in \tau[C]$. i.e., $V[C] \models$ "$s_n \subseteq \tau[C]$".

Now suppose that $V[C] \models$ "$s_n \subseteq \tau[C]$". For every $i < \lg(s_n)$ there is $q_i \in C$ such that $\langle q_i, \breve{\imath}, \breve{s}_n\,(i)\rangle \in \tau$. But then, by compatibility in the filter, there exists $r \in C$ such that for all $i < \lg(s_n)$, $r \leq q_i$. Therefore, if $p = \bigcup_{i<\lg(s_n)} q_i$, $r \leq p$ and hence $p \in C$. By definition of $\dot{S}^\tau$, $\langle p, \breve{n}\rangle \in \dot{S}^\tau$. Therefore, $V[C] \models$ "$n \in \dot{S}^\tau[C]$". ■

So, for all simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and every simple $Coll\,(\omega, <\omega_1)$-names τ, τ' for reals in 2^ω, if $Caract\,(\sigma, \tau)$ and $Caract\,(\sigma, \tau')$, then $\Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{S}^\tau = \dot{S}^{\tau'}$".

Definition 3.3.54 *Let $Ad^*\,(x, y, k)$ iff x codes a simple $Coll\,(\omega, <\omega_1)$-name σ for a subset of ω and y codes $\dot{S}_k^{\tilde{\sigma}}$, where, recall, $\tilde{\sigma}$ denotes a $Coll\,(\omega, <\omega_1)$-name for the characteristic function of the subset of ω named by σ.*

Fact 3.3.55 *$Ad^*\,(x, y, k)$ is a Σ^1_3 subset of $\omega^\omega \times \omega^\omega \times \omega$.*

Proof. Let $[Coll\,(\omega,<\omega_1)^*]^{<\omega}$ be the set of all finite sequences of reals in $Coll\,(\omega,<\omega_1)^*$ the set of all reals coding a condition in $Coll\,(\omega,<\omega_1)$ (see the proof of Fact 3.3.36). We define $Uni \subseteq [Coll\,(\omega,<\omega_1)^*]^{<\omega} \times Coll\,(\omega,<\omega_1)^*$ by: for all $n \in \omega$, $x_0,...,x_n,x \in Coll\,(\omega,<\omega_1)^*$, $Uni\,(x_0,...,x_n,x)$ iff $x_0,...,x_n$ code compatible conditions and x codes its union. It is easy to see that Uni is a Π^1_2 subset of $(\omega^\omega)^{<\omega}$: for every $x_0,...,x_n,x \in \omega^\omega$, $Uni\,(x_0,...,x_n,x)$ iff

1. $x \leq_* x_0 \wedge ... \wedge x \leq_* x_n$. $(\Delta^1_2\,(x,x_0,...,x_n))$
2. $\forall z\,(z \leq_* x_0 \wedge ... \wedge z \leq_* x_n \rightarrow z \leq_* x)$. $(\Pi^1_2\,(x,x_0,...,x_n))$

Let $R\,(x,y)$ iff x codes a simple $Coll\,(\omega,<\omega_1)$-name τ for a real in 2^ω and y codes $\dot{S}^\tau$. So, for all $x,y \in \omega^\omega$, $R\,(x,y)$ iff:

1. x codes a simple $Coll\,(\omega,<\omega_1)$-name for a real in 2^ω $(\Pi^1_2\,(x))$
2. y codes a simple $Coll\,(\omega,<\omega_1)$-name for a subset of ω $(\Delta^1_2\,(y))$
3. x codes $\langle x_n : n \in \omega\rangle$ and y codes $\langle y_n : n \in \omega\rangle$ $(\Delta^1_1\,(x,y))$
4. $\forall n(x_n$ codes z_n and $u_n \wedge y_n$ codes v_n and $n)$ $(\Delta^1_1\,(x,y))$
5. $\forall n,m\,(\forall i < lg\,(n))\,\exists k(u_i\,(k) = s_n\,(i) \wedge Uni((z_0)_k,...(z_{n-1})_k,(v_n)_m))$
6. $\forall n\,(\forall i,j < lg\,(n))\,\exists k(u_i\,(k) = s_n\,(i) \wedge (z_i)_k \not\perp (z_j)_k \rightarrow$
$$\rightarrow \exists m\, Uni((z_0)_k,...(z_{n-1})_k,(v_n)_m))$$

where, for every $n \in \omega$, $lg\,(n) = \lg\,(s_n)$. So, (5) and (6) are Π^1_2, and hence R also is a Π^1_2 subset of $\omega^\omega \times \omega^\omega$.

Let $S \subseteq \omega^\omega \times \omega^\omega \times \omega$, defined as follows: for every $x,y \in \omega^\omega$and $k \in \omega$, $S\,(x,y,k)$ iff x codes a simple $Coll\,(\omega,<\omega_1)$-name τ for a real in 2^ω and y codes $\dot{S}^\tau_k$. Thus, for all $x,y \in \omega^\omega$ and every $k \in \omega$, $S\,(x,y,k)$ iff

1. x codes a simple $Coll\,(\omega,<\omega_1)$-name for a real in 2^ω $(\Pi^1_2\,(x))$
2. There exists z such that
   (a) $R\,(x,z)$ and
   (b) y codes $\langle y_n : n \in \omega\rangle$ and z codes $\langle z_n : n \in \omega\rangle$ and
   (c) $\forall n(z_n$ codes u_n and $n \wedge y_n$ codes v_n and $J\,(k,n))$ and
   (d) $\forall n,m\,(J\,(k,n) = m \rightarrow v_n = u_m)$

But since $(2a)$ is $\Pi^1_2\,(x,z)$, and $(2b)$-$(2d)$ are $\Delta^1_1\,(y,z)$, (2) is $\Sigma^1_3\,(x,y)$. Hence, S is Σ^1_3.

Finally, since for every $x,y \in \omega^\omega$ and $k \in \omega$, $Ad^*\,(x,y,k)$ iff

$$\exists z\,(Caract^*\,(x,z) \wedge S\,(z,y,k)),$$

Ad^* is also a Σ^1_3 relation in $\omega^\omega \times \omega^\omega \times \omega$. ■

Fact 3.3.56 *For every $p \in Coll(\omega, <\omega_1)$, $s, t \in [\omega]^{<\omega}$ and every $Coll(\omega, <\omega_1)$-name $\dot{S}_i^\tau$, the following are equivalent:*

1. $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{s} \cap \dot{S}_i^\tau \subseteq \breve{t}$"

2. $\forall n(n \in s \wedge (\forall q \leq p)\, \exists r(r \not\perp q \wedge \langle r, \breve{n}\rangle \in \dot{S}_i^\tau) \rightarrow n \in t)$.

Proof. $(1 \Rightarrow 2)$ Suppose that $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{s} \cap \dot{S}_i^\tau \subseteq \breve{t}$". Fix $n \in s$ such that for all $q \leq p$ there exists r such that $q \not\perp r$ and $\langle r, \breve{n}\rangle \in \dot{S}_i^\tau$. Since $n \in s$, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{s}$". Since for every $q \leq p$ there exists $r \in Coll(\omega, <\omega_1)$ such that $q \not\perp r$ and $\langle \breve{n}, r\rangle \in \dot{S}_i^\tau$, the set $\{p' \in Coll(\omega, <\omega_1) : (\exists r \in Coll(\omega, <\omega_1))(p' \leq r \wedge \langle r, \breve{n}\rangle \in \dot{S}_i^\tau)\}$ is dense below p. So, for every $Coll(\omega, <\omega_1)$-generic filter C over V such that $p \in C$, $n \in \dot{S}_i^\tau[C]$. Therefore, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \dot{S}_i^\tau$". So, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{s} \cap \dot{S}_i^\tau$". But then, since $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{s} \cap \dot{S}_i^\tau \subseteq \breve{t}$", $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{t}$". Hence, $n \in t$.

$(2 \Rightarrow 1)$ Suppose (2) and $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{s} \cap \dot{S}_i^\tau$". Since $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{s} \cap \dot{S}_i^\tau$", on one hand, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{s}$", so $n \in s$. On the other hand, since $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \dot{S}_i^\tau$", for every $Coll(\omega, <\omega_1)$-generic filter C over V such that $p \in C$, $n \in \dot{S}_i^\tau[C]$. So, for every $Coll(\omega, <\omega_1)$-generic filter C over V with $p \in C$, there exists $r \in C$ such that $\langle r, \breve{n}\rangle \in \dot{S}_i^\tau$. So, for every $q \leq p$ there exists r such that $q \not\perp r$ and $\langle r, \breve{n}\rangle \in \dot{S}_i^\tau$. But then, by (2), $n \in t$ and hence, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\breve{n} \in \breve{t}$". So, $p \Vdash$ "$\breve{s} \cap \dot{S}_i^\tau \subseteq \breve{t}$". ■

Finally, we will show that $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is a $\underset{\sim}{\Delta}^1_{n+2}$ poset.

By Fact 3.3.48, we may assume that for every $\langle p, \breve{s}, \dot{g}\rangle \in Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$, $\dot{g}$ is a nice $Coll(\omega, <\omega_1)$-name for a finite subset of $\dot{G}(\pi)$.

Let $h : \omega^\omega \rightarrow \omega^\omega \times \omega^\omega \times \omega^\omega$ be the one-to-one and onto function given by $h(x) = \langle x_0, x_1, x_2\rangle$ iff for all $n \in \omega$, $x_i(n) = x(3n + i)$, where $i \in \{0, 1, 2\}$. If $h(x) = \langle x_0, x_1, x_2\rangle$, we say that x *codes* $\langle x_0, x_1, x_2\rangle$.

Recall that F is the dense embedding from $Coll(\omega, <\omega_1)^*$ into $Coll(\omega, <\omega_1)$ of Fact 3.3.36. If $x \in \omega^\omega$, we say that x *codes a condition* $\langle p, \breve{s}, \dot{g}\rangle \in Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ iff x codes $\langle x_0, x_1, x_2\rangle$ and $F(x_0) = p$, x_1 codes s and x_2 codes the nice $Coll(\omega, <\omega_1)$-name $\dot{g}$ for a finite subset of $\dot{G}(\pi)$.

Define $\langle Q^*, \leq^*, \perp^*\rangle$ as follows:

- $Q^* = \{x \in \omega^\omega : x$ codes a condition in $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi\}$
- $x \leq^* y$ iff the condition coded by x extends in $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ the condition coded by y
- $x \perp^* y$ iff x, y code incompatible conditions of $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$

So $x \in Q^*$ iff

1. x codes $\langle x_0, x_1, x_2\rangle$
2. $x_0 \in Coll(\omega, <\omega_1)^*$

3. x_1 codes a finite subset of ω

4. x_2 codes a nice $Coll\,(\omega, <\omega_1)$-name for a finite subset of $\dot{G}\,(\pi)$

where x_1 codes a finite subset of ω iff x_1 is the characteristic function of a finite subset of ω. (1) and (3) are Δ^1_1 predicates on the reals. Note that, by Fact 3.3.36, (2) is $\Delta^1_2\,(x_0)$ and, by Fact 3.3.49, (4) is $\underset{\sim}{\Delta}^1_{n+2}$. Therefore, Q^* is a $\underset{\sim}{\Delta}^1_{n+2}$ subset of ω^ω.

Note that for every $p \in Coll\,(\omega, <\omega_1)$ and all nice $Coll\,(\omega, <\omega_1)$-names $\dot{h}, \dot{g}$ for finite subsets of $\dot{G}\,(\pi)$, $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{h} \subseteq \dot{g}$" iff for all $\langle \sigma, \check{\imath}\rangle \in \mathrm{rec}(\dot{h})$,

$$((\forall q_1 \leq p)\,(\exists q_2 \leq q_1)(\exists \langle q_3, \tau_1, \check{\imath}\rangle \in \dot{h})(q_2 \leq q_3 \wedge q_2 \Vdash_{Coll(\omega,<\omega_1)} \text{“}\sigma = \tau_1\text{”}) \rightarrow$$
$$\rightarrow (\forall r_1 \leq p)(\exists r_2 \leq r_1)(\exists \langle r_3, \tau_2, \check{\imath}\rangle \in \dot{g})(r_2 \leq r_3 \wedge r_2 \Vdash_{Coll(\omega,<\omega_1)} \text{“}\sigma = \tau_2\text{”}))$$

(see [Ku], VII.3.3). Let R be the relation defined by: $R\,(p, \sigma, \tau)$ iff $p \in Coll\,(\omega, <\omega_1)$, σ,τ are simple $Coll\,(\omega, <\omega_1)$-names for reals and $p \Vdash_{Coll(\omega,<\omega_1)}$ "$\sigma = \tau$". Note that R is a Δ^1_2 relation (by Fact 3.3.37). So, using the above equivalence and that given by Fact 3.3.56, for all $x, y \in \omega^\omega$, $x \leq^* y$ iff

1. $x \in Q^*$ and $y \in Q^*$

2. x codes $\langle x_0, x_1, x_2\rangle$ and y codes $\langle y_0, y_1, y_2\rangle$

3. $x_0 \leq_* y_0$, where, recall, $\leq_*$ denotes the ordering relation in $Coll\,(\omega, <\omega_1)^*$

4. $\forall n\,(y_1\,(n) = 1 \rightarrow x_1\,(n) = 1)$.

5. Let $\varphi_{incl}\,(x, y)$ be the conjunction of:

   (a) x_2 codes $\langle a_j : j < x_2\,(0)\rangle$ and y_2 codes $\langle b_j : j < y_2\,(0)\rangle$
   (b) $(\forall j < x_2\,(0))\,(a_j$ codes $\langle a^0_{j,} a^1_j\rangle) \wedge (\forall j < y_2\,(0))\,(b_j$ codes $\langle b^0_{j,} b^1_j\rangle)$
   (c) $(\forall j < x_2\,(0))\,\forall n(\langle (a^0_j)_n, (a^1_j)', a^1_j\,(0)\rangle \in G\,(\pi)^* \wedge$
   $\wedge\,(\forall j < y_2\,(0))\,\forall n(\langle (b^0_j)_n, (b^1_j)', b^1_j\,(0)\rangle \in G\,(\pi)^*$
   (d) $(\forall j < y_2\,(0))\,[\forall z_0(z_0 \leq_* x_0 \rightarrow \exists z_1(z_1 \leq_* z_0 \wedge$
   $\wedge \exists k\,(\exists m < y_2\,(0))\,(z_1 \leq_* (b^0_m)_k \wedge R(z_1, (b^1_j)', (b^1_m)'))))) \rightarrow$
   $\rightarrow \forall z_2(z_2 \leq_* x_0 \rightarrow \exists z_3(z_3 \leq_* z_2 \wedge$
   $\wedge \exists n(\exists l < x_2\,(0))(z_3 \leq_* (a^0_l)_n \wedge R(z_3, (b^1_j)', (a^1_l)'))))))]$

6. Let $\varphi_{presv}\,(x, y)$ be the conjunction of the following:

   (a) x_2 codes $\langle a_j : j < x_2\,(0)\rangle$
   (b) $(\forall j < x_2\,(0))\,(a_j$ codes $\langle a^0_{j,} a^1_j\rangle)$
   (c) $(\forall j < x_2\,(0))\,\forall n(\langle (a^0_j)_n, (a^1_j)', a^1_j\,(0)\rangle \in G\,(\pi)^*$
   (d) $(\forall j < x_2\,(0))\,\forall u(Ad^*((a^0_j)_n, u, a^1_j\,(0)) \wedge$
   $\wedge \forall n(x_1\,(n) = 1 \wedge \forall z_0(z_0 \leq_* x_0 \rightarrow \exists z_1(\neg z_0 \perp_* z_1 \wedge \exists m\,((u)_m = z_1)) \rightarrow$
   $\rightarrow (y)_1\,(n) = 1)))$

We know that (1) is $\underset{\sim}{\Delta}^1_{n+2}$, (2) and (4) are $\Delta^1_1(x,y)$ and (3) is $\Delta^1_2(x,y)$. Note that (5.a) and (5.b) also are $\Delta^1_1(x,y)$ and, by Fact 3.3.46, (5.c) is $\underset{\sim}{\Delta}^1_{n+2}$. Moreover, (5.$d$) is a formula which is equivalent both to a $\Sigma^1_4(x,y)$ formula and to a $\Pi^1_4(x,y)$ formula. Hence, since $n \geq 2$, (5) is $\underset{\sim}{\Delta}^1_{n+2}$. Finally, (6.$a$) and (6.$b$) are $\Delta^1_1(x,y)$, (6.c) is $\underset{\sim}{\Delta}^1_{n+2}$, and, since, by Fact 3.3.55, Ad^* is Σ^1_3, (6.d) is $\Pi^1_3(x,y)$. So, (6) is $\underset{\sim}{\Delta}^1_{n+2}$. Therefore, $\leq^*$ is $\underset{\sim}{\Delta}^1_{n+2}$ relation in $\omega^\omega \times \omega^\omega$ included in $Q^* \times Q^*$.

Finally, note that for every $\langle p, \check{s}, \dot{g}\rangle, \langle q, \check{t}, \dot{h}\rangle \in Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$, $\langle p, \check{s}, \dot{g}\rangle \perp \langle q, \check{t}, \dot{h}\rangle$ iff $p \perp q$ or there exists $\langle \sigma, i\rangle \in \text{rec}(\dot{g} \cup \dot{h})$ such that

$$\exists n(n \in s \wedge (\forall p' \leq p,q)\, \exists r(r \not\perp p' \wedge \langle r, \check{n}\rangle \in \dot{S}^\sigma_i) \wedge n \notin t)$$

or there exists $\langle \sigma, i\rangle \in \text{rec}(\dot{g} \cup \dot{h})$ such that

$$\exists n(n \in t \wedge (\forall p' \leq p,q)\, \exists r(r \not\perp p' \wedge (\langle r, \check{n}\rangle \in \dot{S}^\sigma_i) \wedge n \notin s)$$

Let $\varphi_{incom}(x,y)$ the conjunction of following:

1. x codes $\langle x_0, x_1, x_2\rangle$ and y codes $\langle y_0, y_1, y_2\rangle$
2. x_2 codes $\langle a_j : j < x_2(0)\rangle$ and y_2 codes $\langle b_j : j < y_2(0)\rangle$
3. $(\forall j < x_2(0))\,(a_j \text{ codes } \langle a^0_j, a^1_j\rangle) \wedge (\forall j < y_2(0))\,(b_j \text{ codes } \langle b^0_{j,} b^1_j\rangle)$
4. $(\exists j < \max\{x_2(0), y_2(0)\})\, \exists z(Ad^*((a^1_j)', z, a^1_j(0)) \vee Ad^*((b^1_j)', z, b^1_j(0)))\wedge$
   $\wedge \exists n(x_1(n) = 1 \wedge \forall v(v \leq_* x_0 \wedge v \leq_* y_0 \rightarrow$
   $\rightarrow \exists u(\neg u \perp_* v \wedge \exists m\,((z)_m = u)) \wedge (y)_1(n) = 0)$

(1), (2) and (3) are $\Delta^1_1(x,y)$. Since Ad^* is Σ^1_3, (4) is equivalent both to a Σ^1_3 and a Π^1_3 formula with parameters x, y. Hence, $\varphi_{incom}(x,y)$ is equivalent also to a $\Sigma^1_3(x,y)$ formula and to a $\Pi^1_3(x,y)$ formula. Finally, for all x,$y \in \omega^\omega$, $x \perp^* y$ iff

1. $x \in Q^*$ and $y \in Q^*$
2. x codes $\langle x_0, x_1, x_2\rangle$ and y codes $\langle y_0, y_1, y_2\rangle$
3. $x_0 \perp_* y_0 \vee \varphi_{incom}(x,y) \vee \varphi_{incom}(y,x)$,

where, recall, $\perp_*$ denotes the incompatibility relation in $Coll(\omega, <\omega_1)^*$. So, $\perp^*$ is a $\underset{\sim}{\Delta}^1_{n+2}$ relation in $\omega^\omega \times \omega^\omega$ included in $Q^* \times Q^*$.

Therefore, $\langle Q^*, \leq^*, \perp^*\rangle$ is a $\underset{\sim}{\Delta}^1_{n+2}$ poset.

Let G be the map from Q^* into $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ sending every $x \in Q^*$ to the condition coded by it. It is clear that G is a dense embedding (but not one-to-one) from $\langle Q^*, \leq^*, \perp^*\rangle$ into $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$. Therefore, $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is a $\underset{\sim}{\Delta}^1_{n+2}$ poset. End of proof of Lemma 3.3.44. ■

Now we finish the proof of Theorem 3.3.30: From Lemma 3.3.42, we know that for every $Coll(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$-generic filter $C * H$, $V[C * H]$ ² $\omega_1^{L[b_H]} = \omega_1$. Note that $\omega_1^{L[b_H]} = \omega_1$ is expressible by means of a $\Pi^1_3(b_H)$ sentence:

$$\forall y\,(y \in WO \rightarrow \exists z\,(z \in WO \wedge z \in L[b_H] \wedge \|y\| < \|z\|))\,.$$

So, $\exists x\,(\omega_1^{L[x]} = \omega_1)$ is equivalent to a Σ^1_4 sentence and

$$V[C * H] \models \exists x\,(\omega_1^{L[x]} = \omega_1)$$

Since $Coll\,(\omega, <\omega_1)$ is ccc and $\Vdash_{Coll(\omega,<\omega_1)}$ "$\dot{\mathsf{P}}_\pi$ is σ-centered", $Coll\,(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is ccc. So, by Lemma 3.3.44, $Coll\,(\omega, <\omega_1) * \dot{\mathsf{P}}_\pi$ is a $\utilde{\Delta}^1_{n+2}$ and ccc poset. Therefore, by $\utilde{\Sigma}^1_4$-absoluteness between V and $V[C * H]$,

$$V \models \exists x\,(\omega_1^{L[x]} = \omega_1)$$

in contradiction with the inaccessibility of ω_1 in L. ■

Note that the proof of Theorem 3.3.30 really shows that for every $n \geq 2$, $\utilde{\Sigma}^1_4$-absoluteness for $\utilde{\Delta}^1_{n+2}$ and ccc forcing notions implies that ω_1 is a Σ_n-Mahlo cardinal in L. So,

Corollary 3.3.57 *For every $n \geq 2$, $Con(ZFC + \utilde{\Sigma}^1_4$-absoluteness for $\utilde{\Delta}^1_{n+2}$ and ccc forcing notions) implies $Con(ZFC + \exists\kappa$ (κ is a Σ_n-Mahlo cardinal)).* ■

As consequence of Theorem 3.3.18 and of Theorem 3.3.30, we have proved Theorem 3.3.1:

Theorem 3.3.1 *The following are equiconsistent (modulo ZFC)*

1. *There exists a Σ_ω-Mahlo cardinal.*
2. *$L\,(\mathsf{R})$-two-step absoluteness for projective and ccc posets.*
3. *$\utilde{\Sigma}^1_4$-absoluteness for projective and ccc posets.* ■

3.3.4 Final remarks and open questions

1. For every $n \in \omega$, let σ_n, π_n and δ_n denote, respectively, the least Σ_n-Mahlo cardinal, the least Π_n-Mahlo cardinal and the least Δ_n-Mahlo cardinal. We know that $\delta_0 = \sigma_0 = \pi_0 = \delta_1 = \sigma_1 < \delta_2$ and that for every $n > 1$, $\delta_n \leq \sigma_n \leq \pi_n \leq \delta_{n+1}$. We also know (see Corollary 3.3.15) that $\sigma_n < \pi_{n+1}$. But we do not know if $\delta_n < \sigma_n$ or $\sigma_n < \pi_n$ or $\pi_n < \delta_{n+1}$. What of these inequalities hold?

2. From Theorem 3.3.30 we have that $\utilde{\Sigma}^1_4$-absoluteness for $\utilde{\Delta}^1_4$ and ccc forcing notions implies that ω_1 is a Σ_2-Mahlo cardinal in L. This shows that Theorem 3.2.1 is optimal in the following sense: in order to obtain $\utilde{\Sigma}^1_4$-absoluteness for ccc forcing notions of complexity greater than $\utilde{\Sigma}^1_3$, we need a large cardinal greater than an inaccessible.

3. There is a little gap between Theorem 3.2.1 and Theorem 3.3.30: the case for the $\utilde{\Sigma}^1_4$-absoluteness for $\utilde{\Pi}^1_3$ and ccc posets. We don't know which is its exact consistency strength. Let us call P a *strongly $\utilde{\Pi}^1_3$ poset* iff P is a $\utilde{\Pi}^1_3$ poset and $\perp_P$, the incompatibility relation for P, is a Δ^1_3 subset of the real plane. Then as in Fact 2.1.15, it is easy to see that for every for every strongly $\utilde{\Pi}^1_3$ and ccc poset P and every real $x \in \omega^\omega$, "x codes a maximal antichain of P" it is a $\utilde{\Pi}^1_3$ predicate. So, with a few modifications, the proof of Theorem 3.2.1 also shows that every strongly $\utilde{\Pi}^1_3$ and ccc extension of a Solovay model is a Solovay model.

4. Note that, in spite of Theorem 3.3.1, the proofs of Theorem 3.3.18 and of Theorem 3.3.30 are not optimal for $n \geq 3$, because on one hand, we need a Π_n-Mahlo cardinal in order to obtain $L(\mathsf{R})$-two-step absoluteness for $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc posets (see Corollary 3.3.29). On the other hand, with $\underset{\sim}{\Sigma}^1_4$-absoluteness for $\underset{\sim}{\Delta}^1_{n+2}$ and ccc posets, we only obtain the existence of a Σ_n-Mahlo cardinal in L (see Corollary 3.3.57).

5. Note also that in the case for the $\underset{\sim}{\Sigma}^1_4$-absoluteness for $\underset{\sim}{\Pi}^1_{n+1}$ and ccc posets (with $n \geq 3$) there is a gap between Theorem 3.3.18 and Theorem 3.3.30. As in the case for the $\underset{\sim}{\Pi}^1_3$ and ccc posets, we can define the *strongly* $\underset{\sim}{\Pi}^1_{n+1}$ *posets* and adapt the proof of Theorem 3.3.18 to show that every strongly $\underset{\sim}{\Pi}^1_{n+1}$ and ccc extension of a Π_n-Mahlo Solovay model is a Π_n-Mahlo Solovay model.

6. The following is a natural question: Does $L(\mathsf{R})$-two-step absoluteness for $\underset{\sim}{\Sigma}^1_{n+1}$ and ccc posets imply $\underset{\sim}{\Sigma}^1_4$-absoluteness for $\underset{\sim}{\Delta}^1_{n+2}$ and ccc posets?

3.4 Collapsing a Mahlo cardinal

3.4.1 Absoluteness by collapsing a Mahlo cardinal

Definition 3.4.1 *$L(\mathsf{R})$ is a* Mahlo Solovay model over V *iff*

1. *For every $x \in \mathsf{R}$, ω_1 is a Mahlo cardinal in $V[x]$ and*

2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

As in the case of Solovay models over V and of Σ_ω-Mahlo Solovay models over V, we can give a characterization of Mahlo Solovay models in the same way as that of Lemma 3.1.1. Namely,

Lemma 3.4.2 *Suppose that M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is a Mahlo cardinal in $V[x]$ and*

2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

Then there exists a forcing notion W such that does not add reals and creates a $Coll(\omega, <\omega_1)$-generic filter C over V such that M and $V[C]$ have the same reals. Thus, W forces that $L(\mathsf{R})^M$ is a Mahlo Solovay model over V. ■

Note that P is a subposet of a projective poset Q iff $\mathsf{P} = \langle P, \leq_P, \perp_P \rangle$ where $\leq_P = \leq_Q{}^1\ P$ and $\perp_P = \perp_Q{}^1\ P$ and P is any subset, not necessarily projective, of Q. Hence, P is a subposet of a projective poset iff there are Σ^1_n formulas $\varphi_\leq(x,y)$ and $\varphi_\perp(x,y)$ with $a \in \omega^\omega$ as parameter such that for all reals $x, y \in \omega^\omega$,

$$x \leq_P y \text{ iff } x, y \in P \text{ and } \varphi_\leq(x,y)$$
$$x \perp_P y \text{ iff } x, y \in P \text{ and } \varphi_\perp(x,y)$$

and P is any set of reals.

In order to study the generic absoluteness properties of a Mahlo Solovay model, we need to look into the reflection phenomena of projective sentences along the Levy-collapsing forcing for a Mahlo cardinal.

Lemma 3.4.3 *Suppose that κ is Mahlo cardinal and $C \subseteq Coll\,(\omega, <\kappa)$ is a generic filter over V. Let $\varphi\,(v_0, ..., v_k)$ be a Σ^1_n (Π^1_n) formula. Then for all reals $b_0, ..., b_k \in V\,[C]$,*

$$V\,[C] \models \varphi\,(b_0, ..., b_k)$$

iff there exists a stationary set $S \subseteq \kappa$ of inaccessible cardinals such that for every $\lambda \in S$,

$$V\,[C_\lambda] \models \varphi\,(b_0, ..., b_k)\,.$$

Proof. We write $\bar{v}$ for $v_0, ..., v_k$ and $\bar{b}$ for $b_0, ..., b_k$. Let κ be a Mahlo cardinal and let $C \subseteq Coll\,(\omega, <\kappa)$ be a generic filter over V. For every $\alpha < \kappa$, let $C_\alpha = Coll\,(\omega, <\alpha) \cap C$. Fix an enumeration $\langle r_\alpha : \alpha < \kappa \rangle$ of all reals in $V\,[C]$ such that for every inaccessible cardinal $\lambda < \kappa$, $\langle r_\alpha : \alpha < \lambda \rangle$ enumerates all reals of $V\,[C_\lambda]$. Let I be the stationary set of all inaccessible cardinals below κ. We need the following fact:

Fact 3.4.4 *Suppose that κ is Mahlo cardinal and suppose that $C \subseteq Coll\,(\omega, <\kappa)$ is a generic filter over V. Let $\varphi\,(v_0, ..., v_k)$ be a Σ^1_n (Π^1_n) formula. Then for all reals $b_0, ..., b_k \in V\,[C]$, if*

$$V\,[C] \models \varphi\,(b_0, ..., b_k)$$

then, there exists a club $D \subseteq \kappa$ that for every $\lambda \in D \cap I$,

$$V\,[C_\lambda] \models \varphi\,(b_0, ..., b_k)\,.$$

Proof. We show this fact by induction on the complexity of projective formulas.

$\underline{n = 1}$: Follows from absoluteness of Σ^1_1 and Π^1_1 formulas and the fact that $Coll\,(\omega, <\kappa)$ is a κ-cc poset.

$\underline{n + 1}$: Suppose that for every Σ^1_n and every Π^1_n formula $\psi(\bar{v})$ and every $\bar{b} \in V\,[C]$, if $V\,[C] \models \psi(\bar{b})$, then there exists a club $D \subseteq \kappa$ such that for every $\lambda \in D \cap I$, $V\,[C_\lambda] \models \psi(\bar{b})$.

Σ^1_{n+1} formulas: Follows from inductive hypothesis.

Π^1_{n+1} formulas: Let $\forall x \psi(\bar{v}, x)$ be a Π^1_{n+1} formula with $a \in V\,[C]$ as parameter and let $\bar{b} \in V\,[C]$. Suppose that $V\,[C] \models \forall x \psi(\bar{b}, x)$. Then, for every $\alpha < \kappa$, $V\,[C] \models \psi(\bar{b}, r_\alpha)$. But $\psi(\bar{v}, x)$ is a Σ^1_n formula and therefore, for every $\alpha < \kappa$, there exists a club $D_\alpha \subseteq \kappa$ such that for every $\lambda \in D_\alpha \cap I$, $V\,[C_\lambda] \models \psi(\bar{b}, r_\alpha)$. Let $D = \triangle_{\alpha<\kappa} D_\alpha$. Clearly, for every $\lambda \in D \cap I$, $V\,[C_\lambda] \models \forall x \psi(\bar{b}, x)$. ■

Now we prove the lemma:

$(\Rightarrow)$ Follows from Fact 3.4.4 since $D \cap I$ is a stationary set of inaccessible cardinals.

$(\Leftarrow)$ Let $\varphi\,(\bar{v})$ be a projective formula and let $\bar{b} \in V\,[C]$ be such that there exist a stationary set S of inaccessible cardinals such that for every $\lambda \in S$, $V\,[C_\lambda] \models \varphi(\bar{b})$. Suppose that $V\,[C] \models \neg\varphi(\bar{b})$. Let $D \subseteq \kappa$ be the club from Fact 3.4.4 for $\neg\varphi(\bar{b})$. Let $\lambda \in S \cap D$. Then, since λ is an inaccessible cardinal in D, $V\,[C_\lambda] \models \neg\varphi(\bar{b})$. But, since $\lambda \in S$, $V\,[C_\lambda] \models \varphi(\bar{b})$. A contradiction. ■

Corollary 3.4.5 *There is no projective sentence σ such that $ZFC \vdash$ "$\sigma \leftrightarrow \omega_1$ is a Mahlo cardinal in L".*

Proof. Suppose otherwise. Then, there exists some real a and a Σ^1_n sentence σ with parameter a such that $ZFC \vdash$ "$\sigma \leftrightarrow \omega_1$ is a Mahlo cardinal in L". Let κ be the least Mahlo cardinal in L. Let $C \subseteq Coll\,(\omega, <\kappa)$ be a generic filter over L. Then, $L\,[C] \vDash \sigma$. So, by Lemma 3.4.3, there exists an stationary set $S \subseteq \kappa$ of inaccessible cardinals such that, for every $\lambda \in S$, $L\,[C_\lambda] \vDash \sigma$. So, $L\,[C_\lambda] \vDash$ "ω_1 is a Mahlo cardinal in L" and, by downward absoluteness of Π_1 predicates, $L \vDash$ "$\omega_1^{L[C_\lambda]}$ is a Mahlo cardinal". But $\omega_1^{L[C_\lambda]} < \kappa$. A contradiction with the minimality of κ. ■

Theorem 3.4.6 *Suppose $L\,(\mathsf{R})^M$ is a Mahlo Solovay model over V and P is a ccc subposet of a projective poset in M. Then the $L\,(\mathsf{R})$ of any P-generic extension of M is also a Mahlo Solovay model over V.*

Proof. Suppose that $L\,(\mathsf{R})^M$ is a Mahlo Solovay model over V and let $\kappa = \omega_1^M$ be the Mahlo cardinal in V such that forcing over M with W, we obtain a $Coll\,(\omega, <\kappa)$-generic filter C over V so that $\mathsf{R}^{V[C]} = \mathsf{R}^M$. Let $\mathsf{P} = \langle P, \leq_P, \perp_P \rangle$ be a ccc subposet of a projective poset in $V[C]$. As in Theorem 3.2.1, it will be enough to show that for every P-generic filter G over V, every real in $V[C][G]$ is generic over V for a countable poset.

Let $\dot{\mathsf{P}}$ be a $Coll\,(\omega, <\kappa)$-name for P. Let $a \in \omega^\omega$ be the parameter of the $\underset{\sim}{\Sigma}^1_n$ formulas $\varphi_\leq(x, y; a)$ and $\varphi_\perp(x, y; a)$ that define the ordering $\leq$ and the relation $\perp$ such that $\leq_P = \leq \cap (P \times P)$ and $\perp_P = \perp \cap (P \times P)$ respectively. Suppose that $\dot{a}$ is a simple $Coll\,(\omega, <\kappa)$-name for a.

Let $\mathsf{S} = Coll\,(\omega, <\kappa) * \dot{\mathsf{P}}$. Note that S is a κ-cc poset.

Notation 3.4.7 *We use the same notational conventions from 3.2.1 with only the following two exceptions:*

- *For every $\alpha < \kappa$, $\mathsf{P}_\alpha = \langle P_\alpha, \leq_{P_\alpha}, \perp_{P_\alpha} \rangle$ where $P_\alpha = P \cap V\,[C_\alpha]$ and $\leq_{P_\alpha}$ and $\perp_{P_\alpha}$ are defined in $V\,[C_\alpha]$ by the same formulas $\varphi_\leq(x, y; a)$ and $\varphi_\perp(x, y; a)$ restricted to elements of P_α, whenever $a, P_\alpha \in V\,[C_\alpha]$. Otherwise, P_α is the trivial poset.*

- *We use $\leq$ and $\perp$ only for the relations defined by the formulas $\varphi_\leq(x, y; a)$ and $\varphi_\perp(x, y; a)$.*

Lemma 3.4.8 *Let κ be a Mahlo cardinal, $C \subseteq Coll\,(\omega, <\kappa)$ a generic filter over V and P be ccc subposet of a $\underset{\sim}{\Sigma}^1_n$ poset in $V\,[C]$. Then there exists a stationary set $S \subseteq \kappa$ of inaccessible cardinals such that for every $\lambda \in S$, P_λ is a subposet P.*

Proof. Suppose that $\mathsf{P} \in V\,[C]$ is a ccc subposet of a $\underset{\sim}{\Sigma}^1_n$ poset. Let $\varphi_\leq(x, y)$ and $\varphi_\perp(x, y)$ be Σ^1_n formulas with parameter $a \in V\,[C]$ that define the ordering $\leq$ and the incompatibility relation $\perp$ in $V\,[C]$. Let $\langle \langle r, r' \rangle_\alpha : \alpha < \kappa \rangle$ be an enumeration of all ordered pairs of reals in $V\,[C]$, such that for every inaccessible cardinal $\lambda < \kappa$, $\langle \langle r, r' \rangle_\alpha : \alpha < \lambda \rangle$ enumerates all pairs of reals of $V\,[C_\lambda]$. Since for every real $r, r' \in$

$V[C]$, $\varphi_{\leq}(x,y)$ and $\neg\varphi_{\leq}(x,y)$ are projective formulas, using Fact 3.4.4, for every $\alpha<\kappa$ we can fix a club set $D_\alpha\subseteq\kappa$ such that for every inaccessible cardinal $\lambda\in D_\alpha$,

$$V[C_\lambda]\models\varphi_{\leq}(r,r') \text{ iff } V[C]\models\varphi_{\leq}(r,r'),$$

where $\langle r,r'\rangle_\alpha=\langle r,r'\rangle$. Let $D_{\leq}=\triangle_{\alpha<\kappa}D_\alpha$. Note that for every inaccessible cardinal $\lambda\in D_{\leq}$ and for all reals $r,r'\in V[C_\lambda]$,

$$V[C_\lambda]\models r\leq r' \text{ iff } V[C]\models r\leq r'$$

We define the club $D_\perp$ in the same way but using the formula $\varphi_\perp(x,y)$.

Claim 3.4.9 *There exists a stationary set $S\subseteq\kappa$ of inaccessible cardinals such that for every $\lambda\in S$, $\mathsf{P}_\lambda\in V[C_\lambda]$.*

Proof. For every $\alpha<\kappa$, let

$$D_\alpha=\{\eta<\kappa: P_\alpha\in V[C_\eta]\}.$$

We know that for every $\alpha<\kappa$, $P_\alpha\in V[C]$, $P_\alpha\subseteq\omega^\omega\cap V[C_\alpha]\in V[C_\alpha]$ and $|\omega^\omega\cap V[C_\alpha]|<\kappa$. Hence, there exists $\eta<\kappa$ such that $P_\alpha\in V[C_\eta]$ (see [Ku], VIII.5.14). So, $D_\alpha\neq\emptyset$ and, since for every $\beta<\gamma<\kappa$, $V[C_\beta]\subseteq V[C_\gamma]$, for every $\alpha<\kappa$, D_α is a club in κ.

Let $D=\triangle_{\alpha<\kappa}D_\alpha$. Let I be the stationary of inaccessible cardinals below κ. Let $S=I\cap D$. Let $\lambda\in S$. Then, since $\lambda\in D$, for every $\alpha<\lambda$, $P_\alpha\in V[C_\lambda]$. Since λ is an inaccessible cardinal, for every real b in $V[C_\lambda]$, there exist $\alpha<\lambda$ such that $b\in V[C_\alpha]$. So $P_\lambda=\bigcup_{\alpha<\lambda}P_\alpha$. Thus P_λ belongs to $V[C_\lambda]$. ■

Let $S_{\mathsf{P}}=S\cap D_{\leq}\cap D_\perp$. Clearly, $S_{\mathsf{P}}\subseteq\kappa$ is a stationary set of inaccessible cardinals such that for every $\lambda\in S_{\mathsf{P}}$, P_λ is a subposet of P. ■

Thus for every $\mu,\lambda\in S_{\mathsf{P}}$, if $\mu\leq\lambda$ then $\mathsf{S}_\mu\subseteq\mathsf{S}_\lambda$ and $\mathsf{S}=\bigcup_{\lambda\in S_{\mathsf{P}}}\mathsf{S}_\lambda$. Hence, for every subposet X of S of cardinality $<\kappa$ there exists $\lambda\in S_{\mathsf{P}}$ such that X is a subposet of S_λ.

For every $\lambda\in S_{\mathsf{P}}$, let $\xi(\lambda)$, if it exists, be the least $\xi\in S_{\mathsf{P}}$ such that for all $\mu\in S_{\mathsf{P}}$ such that $\mu\geq\xi$ the following holds: For every simple $Coll(\omega,<\lambda)$-name $\dot{A}$ for a subset of $\dot{\mathsf{R}}_\lambda$, every simple $Coll(\omega,<\lambda)$-name for a real $\dot{c}$ and every $q\in Coll(\omega,<\mu)$, if

$$q\Vdash_{Coll(\omega,<\kappa)}\text{“}\dot{A}\text{ is not a maximal antichain of }\dot{\mathsf{P}}\text{ below }\dot{c}\text{”}$$

then

$$q\Vdash_{Coll(\omega,<\mu)}\text{“}\dot{A}\text{ is not a maximal antichain of }\dot{\mathsf{P}}\text{ below }\dot{c}\text{”}.$$

Lemma 3.4.10 *For every $\lambda\in S_{\mathsf{P}}$, $\xi(\lambda)$ exists.*

Proof. As in Lemma 3.2.3, but using the fact that if A is not a maximal antichain of P then there exists r a real belonging to P which is incompatible with all reals in A. But then, this real belongs to some P_μ for some $\mu \in S_\mathsf{P}$ and the incompatibility relation in P is absolute for $V[C_\mu]$ and $V[C]$, since $\mu \in S_\mathsf{P}$. ■

The rest of the proof of Theorem 3.4.6 is like the proof of Theorem 3.2.1 but using only inaccessible cardinals in S_P instead of all ordinals below κ. ■

As a consequence of Theorem 3.4.6, we will see that being a Mahlo Solovay model is preserved for σ-linked forcing notions.

Definition 3.4.11 *A poset P is* σ-linked *iff there exists a family $\{P_n : n \in \omega\}$ such that $\mathsf{P} = \bigcup_{n\in\omega} P_n$ and for every $n \in \omega$ and every $\{p, q\} \subseteq P_n$, there exists $r \in \mathsf{P}$ such that $r \leq_P p, q$.*

Theorem 3.4.12 *Suppose $L(\mathsf{R})^M$ is a Mahlo Solovay model over V and P is a σ-linked poset in M. Then the $L(\mathsf{R})$ of any P-generic extension of M is also a Mahlo Solovay model over V.*

Proof. In view of Theorem 3.4.6 we only need to prove the following:

Fact 3.4.13 *Every σ-linked poset can be densely embedded into a subposet of a Borel poset.*

Proof. Let P be a σ-linked poset. We find a σ-linked poset X_P which is a subposet of a Borel poset and a dense embedding from P into X_P.

Definition 3.4.14 *Let P be a σ-linked poset. A set $X = \{X_n : n \in \omega\}$ is a* σ-linking family *of P iff $\mathsf{P} = \bigcup_{n\in\omega} X_n$ and for every $n \in \omega$, if $p, p' \in X_n$, then there exists $q \in \mathsf{P}$ such that $q \leq p, p'$. Suppose that X, Y are σ-linking families of P. Y* extends *X iff for every $n \in \omega$, $X_n \subseteq Y_n$. Y is a* maximal σ-linking family *of P iff there is no σ-linking family of P that properly extends Y.*

Claim 3.4.15 *Let P be a σ-linked poset. Then every σ-linking family of P can be extended to a maximal one.*

Proof. By Zorn's Lemma. ■

Definition 3.4.16 *Let P be a σ-linked poset and let Y be some maximal σ-linking family of P. We define X_P as follows: For every $p \in \mathsf{P}$, let $x_P = \{n \in \omega : p \in Y_n\}$. Let*

- *$x_p \in X_\mathsf{P}$ iff $p \in \mathsf{P}$*
- *$x_p \leq_{X_\mathsf{P}} x_q$ iff $x_p \subseteq x_q$.*

So, for every $x_p, x_q \in X_{\mathsf{P}}$,

$$x_p \perp_{X_{\mathsf{P}}} x_q \text{ iff } x_p \cap x_q = \emptyset.$$

Since $x \subseteq y$ and $x \cap y = \emptyset$ are arithmetical, it is clear that for every σ-linked poset P, X_{P} is a subposet of Borel poset.

Claim 3.4.17 *If P is a σ-linked poset, there exists a dense embedding (not necessarily one-to-one) from P onto X_{P}.*

Proof. Let i be the function from P into X_{P} such that $i(p) = x_p$. If $p \leq_P q$ and $p \in Y_n$, then, by maximality of Y, $q \in Y_n$. Hence, $x_p \subseteq x_q$ and so $i(p) = x_p \leq_{X_{\mathsf{P}}} x_q = i(q)$. Moreover, if $p \perp_P q$ then, there is no $n \in \omega$ such that $p, q \in Y_n$. Hence, $x_p \cap x_q = \emptyset$ and, therefore, $i(p) = x_p \perp_{X_{\mathsf{P}}} x_q = i(q)$. Finally, note that $i''\mathsf{P} = X_{\mathsf{P}}$. So i is a dense embedding of P onto X_{P}. This ends the proof of the claim, of Fact 3.4.13 and of Theorem 3.4.12. ■

Note that X_{P} is a σ-linked poset: for every $n \in \omega$, let $X_n = \{x_p : n \in x_p\}$. Then, $\{X_n : n \in \omega\}$ is a σ-linking family of X_{P}.

3.4.2 The strength of $\utilde{\Sigma}^1_4$-absoluteness for σ-centered subposets of Borel posets

Theorem 3.4.18 *The following are equiconsistent (modulo ZFC)*

1. *There exists a Mahlo cardinal.*
2. *$L(\mathsf{R})$-two-step absoluteness for ccc subposets of projective posets.*
3. *$L(\mathsf{R})$-two-step absoluteness for σ-linked posets.*
4. *$L(\mathsf{R})$-two-step absoluteness for σ-centered posets (see Definition 2.3.42).*
5. *$\utilde{\Sigma}^1_4$-absoluteness for σ-centered posets.*
6. *$\utilde{\Sigma}^1_4$-absoluteness for σ-centered subposets of Borel posets.*

Proof. (1) implies (2) follows from Theorem 3.4.6. (2) implies (3) follows from Fact 3.4.13, (3) implies (4), (4) implies (5) and (5) implies (6) are obvious.

(6) implies (1): essentially is a result of A. R. D. Mathias (see [B-F]), which uses a theorem of J. Brendle, H. Judah and S. Shelah (see [Br-Ju-Sh]). For completeness, we give a proof of these results:

Definition 3.4.19 *For all $x, y \in \omega^\omega$, $y \leq^* x$ iff there exists $n \in \omega$ such that for all $m \geq n$, $y(m) \leq x(m)$. In this case, we say that x* dominates *y. Let $F \subseteq \omega^\omega$. $x \in \omega^\omega$* dominates *F iff for all $y \in F$, $y \leq^* x$. In this case, we also say that x is a* dominating real *for F. F is an* unbounded set *iff for every $y \in \omega^\omega$ there exists $x \in F$ such that x £* y.*

Definition 3.4.20 *Let D be the* Hechler forcing *to add a dominating real:*

- $\langle s,x\rangle \in \mathsf{D}$ *iff* $s \in \omega^{<\omega}$, $x \in \omega^{\omega}$, *s and x are increasing and* $s \subseteq x$.
- $\langle s,x\rangle \leq \langle t,y\rangle$ *iff* $t \subseteq s$ *and* $\forall n\,(y\,(n) \leq x\,(n))$.

Remark 3.4.21 *Note that* D *is a Borel poset.* D *also is a σ-centered poset: suppose* $\langle s,x\rangle, \langle s,y\rangle \in \mathsf{D}$. *Let* $z \in \omega^{\omega}$ *such that for every* $n \in \mathrm{dom}\,(s)$, $z\,(n) = s\,(n)$ *and for every* $n \in \omega \setminus \mathrm{dom}\,(s)$, $z\,(n) = x\,(n) + y\,(n)$. *Then* $\langle s,z\rangle \leq \langle s,x\rangle, \langle s,y\rangle$. *So, all conditions in* D *with the same first coordinate are compatible..*

Fact 3.4.22 *If G is a* D*-generic filter over V, then*

$$d = \bigcup \{s \in \omega^{<\omega} : (\exists x \in \omega^{\omega})\,(\langle s,x\rangle \in G)\}$$

is a dominating real for $\omega^{\omega} \cap V$. $\blacksquare$

Theorem 3.4.23 (J. Brendle, H. Judah and S. Shelah) *Assume* $\underset{\sim}{\Sigma}^1_4$*-absoluteness for the Hechler poset holds. Then* ω_1 *is an inaccessible cardinal in* L.

Proof. Suppose V is $\underset{\sim}{\Sigma}^1_4$-absolute for D and ω_1 is not inaccessible in L.

Let $\theta = \exists x(L[x] \cap \omega^{\omega}$ is unbounded). First, note that θ is equivalent to a Σ^1_4 sentence:

$$\exists x \forall y \exists z\,(z \in L[x] \wedge z \not\leq^* y)$$

Since $\leq^*$ is a Borel relation and $\omega^{\omega} \cap L[x]$ is a $\Sigma^1_2\,(x)$ set of reals, the formula between parenthesis is $\Sigma^1_2\,(x,y,z)$ and so θ is Σ^1_4.

Fact 3.4.24 $V \not\models \theta$.

Proof. Fix $x \in \omega^{\omega} \cap V$. Let G be a D-generic filter V and let $d \in \omega^{\omega}$ be the D-generic real over V. Then, for all $z \in \omega^{\omega} \cap V$, $z \leq^* d$ and, since $L[x] \subseteq V$,

$$V[G] \models \forall z\,(z \in L[x] \rightarrow z \leq^* d)\,.$$

So,

$$V[G] \models \exists y \forall z\,(z \in L[x] \rightarrow z \leq^* y)\,.$$

But the right-hand formula is $\Sigma^1_3\,(x)$ and $x \in \omega^{\omega} \cap V$. So, by $\underset{\sim}{\Sigma}^1_4$-absoluteness for D,

$$V \models \exists y \forall z\,(z \in L[x] \rightarrow z \leq^* y)\,.$$

Therefore,

$$V \models \forall x \exists y \forall z\,(z \in L[x] \rightarrow z \leq^* y)\,.$$

i.e., $V \models \neg\theta$. $\blacksquare$

Fact 3.4.25 $V[G] \models \theta$.

Proof. Recall that in the proof of Theorem 3.3.30, we have fixed a recursive enumeration $\langle s_i : i \in \omega\rangle$ of $2^{<\omega}$ such that every finite sequence of 0's and 1's is enumerated before all its proper extensions and recursive partition of ω in infinitely many infinite pieces. Also recall that for every $a \subseteq \omega$, $\tilde{a} : \omega \to \{0,1\}$ denotes its characteristic function. Finally, let $\{S^a : a \subseteq \omega\}$ be the perfect and maximal almost-disjoint family defined in 3.3.30.

Definition 3.4.26 *Let $x \in \omega^\omega$ be an increasing function and let $a \subseteq \omega$ be such that $S^a \cap \operatorname{rec}(x)$ is infinite. We define $h_{a,x} \in \omega^\omega$ as follows: for every $i \in \omega$,*

$$h_{a,x}(i) = k \text{ iff } x(n_i) \in S^a_k,$$

where n_i is the i-th element of $S^a \cap \operatorname{rec}(x)$.

Claim 3.4.27 *For every $a \in P(\omega) \cap V$, $\operatorname{rec}(d) \cap S^a$ is infinite.*

Proof. We only need to show that for every $a \subseteq \omega$ and every $n \in \omega$,

$$D^a_n = \{\langle s,x\rangle : (\exists m \geq n)(s(m) \in S^a)\}$$

is a dense subset of D, for then, by definition of d, for every $n \in \omega$ there is $m \geq n$ such that $d(m) \in S^a$.

Let $\langle s,x\rangle \in \mathsf{D}$. Suppose that $m = \max\{n, \operatorname{dom}(s)\} + 1$ and let $s' \in \omega^{m+1}$ be defined as follows: for every $i \in \operatorname{dom}(s)$, $s'(i) = s(i)$, for every $i \in m \setminus \operatorname{dom}(s)$, $s'(i) = x(i)$ and, finally, let $s'(m)$ be the least $i \in S^a$ such that $i > \max(\{s'(i) : i < m\} \cup \{x(m)\})$.

Let $x' \in \omega^\omega$ be such that for every $i \leq m$, $x'(i) = s'(i)$ and for every $i > m$, $x'(i) = \max\{x(i), x'(i-1)+1\}$.

Since s and x are increasing and $s \subseteq x$, s' is increasing. Moreover, by definition of x', x' is increasing and $s' \subseteq x'$. So, $\langle s',x'\rangle \in \mathsf{D}$. Since $m > n$ and $s'(m) \in S^a$, $\langle s',x'\rangle \in D^a_n$. Finally, by definition of s', $s \subseteq s'$ and, by definition of x', for all $i \in \omega$, $x(i) \leq x'(i)$. Therefore, $\langle s',x'\rangle \leq \langle s,x\rangle$. ■

Definition 3.4.28 *Let $s \in \omega^{<\omega}$ and $D \subseteq \mathsf{D}$. We define the* rank of s in D, $\operatorname{rk}_D(s)$, *by recursion on $\omega^{<\omega}$ as follows:*

1. $\operatorname{rk}_D(s) = 0$ *iff there exists $x \in \omega^\omega$ such that $\langle s,x\rangle \in D$.*
2. $\operatorname{rk}_D(s) = \alpha$ *iff there is no $\beta < \alpha$ such that* $\operatorname{rk}_D(s) = \beta$ *but there are $m \in \omega$ and $\{t_k : k \in \omega\} \subseteq \omega^m$ such that $k \in \omega$, $s \subseteq t_k$, $t_k(\operatorname{dom}(s)) \geq k$ and* $\operatorname{rk}_D(t_k) < \alpha$.
3. $\operatorname{rk}_D(s) = \infty$ *otherwise.*

Note that, since $\omega^{<\omega}$ is countable, for all $s \in \omega^{<\omega}$, either $\operatorname{rk}_D(s) < \omega_1$ or $\operatorname{rk}_D(s) = \infty$. Moreover,

Claim 3.4.29 *Let D be a dense subset of D. Then for every $s \in \omega^{<\omega}$ increasing,* $\operatorname{rk}_D(s) < \omega_1$.

Proof. Suppose that $s \in \omega^m$ is a increasing sequence. First note that if $\mathrm{rk}_D(s) = \infty$, then

$$|\{n \in \omega : \mathrm{rk}_D(s^\frown n) \neq \infty\}| < \omega$$

We define, by recursion on ω, a sequence $\langle k_i : i \in \omega\rangle$ of natural numbers such that for all increasing sequence $t \in \omega^{<\omega}$, if $k_i \leq t(i)$, then $\mathrm{rk}_D\left(s^\frown t\right) = \infty$:

$\underline{i=0}$: Since $\left|\{n \in \omega : \mathrm{rk}_D\left(s^\frown n\right) \neq \infty\}\right| < \omega$, there exists $k \in \omega$ such that for all $k' \geq k$, $k' \in \omega \setminus \{n \in \omega : \mathrm{rk}_D\left(s^\frown n\right) \neq \infty\}$. Let k_0 be the least such k greater than $s(m-1)$.

$\underline{i>0}$: Suppose that $\langle k_j : j < i\rangle$ has been defined. Then, by inductive hypothesis, $\mathrm{rk}_D\left(s^\frown \langle k_j : j < i\rangle\right) = \infty$. So, $\left|\{n \in \omega : \mathrm{rk}_D\left(s^\frown \langle k_j : j < i\rangle\right) \neq \infty\}\right| < \omega$. Therefore, there is $k \in \omega$ such that $k' \geq k$, $k' \in \omega \setminus \{n \in \omega : \mathrm{rk}_D\left(s^\frown \langle k_j : j < i\rangle^\frown n\right) \neq \infty\}$. Let k_i be the least such k greater than k_{i-1}.

Let $x \in \omega^\omega$ defined as follows:

$$x(n) = \begin{cases} s(n), & \text{if } n < m \\ k_{n-m}, & \text{if } n \geq m \end{cases}$$

Then x is increasing and $s \subseteq x$. So, $\langle s, x\rangle \in \mathsf{D}$. But then, by density of D, there is $\langle t, y\rangle \in D$ such that $\langle t, y\rangle \leq \langle s, x\rangle$. Therefore, for all $j \in \mathrm{dom}(t)$, if $m \leq j$, $x(j) \leq y(j) = t(j)$ and, then, by definition of the sequence $\langle k_i : i \in \omega\rangle$, $\mathrm{rk}_D(t) = \infty$. But, since $\langle t, y\rangle \in D$, $\mathrm{rk}_D(t) = 0$. A contradiction. ■

Lemma 3.4.30 *Every real $x \in V[G]$ is eventually different form at most countably many reals $h_{a,d}$ with $a \in P(\omega) \cap V$. i.e., for every $x \in V[G]$,*

$$|\{a \in P(\omega) \cap V : (\exists n \in \omega)(\forall m \geq n)(h_{a,d}(m) \neq x(m))\}| \leq \omega.$$

Proof. Let $\dot{x}$ be a D-name for a real ω^ω. Let, for every $n \in \omega$, $D_n \subseteq \mathsf{D}$ be a dense and open subset such that every $\langle s, y\rangle \in D_n$ decides $\dot{x} \restriction (n+1)$. Let N be a countable elementary submodel of $H(\chi)$ such that $\{S^a : a \in P(\omega) \cap V\}, \dot{x} \in N$. We will prove that if $S^a \notin N$, then,

$$\Vdash_{\mathsf{D}} (\forall n \in \omega)(\exists m \geq n)(\dot{x}(m) = h_{a,d}(m)).$$

Fix $S^a \notin N$ and suppose that $\langle s, y\rangle \in \mathsf{D}$ such that for some $k \in \omega$,

$$\langle s, y\rangle \Vdash_{\mathsf{D}} (\forall m \geq k)(\dot{x}(m) \neq h_{a,d}(m)).$$

Let $l \geq k$ be such that $|\mathrm{rec}(s) \cap S^a| = l$ and let

$$Y = \{t \in \omega^{<\omega} : s \subseteq t \wedge |\mathrm{rec}(t) \cap S^a| = l \wedge (\forall i \in \mathrm{dom}(t) \setminus \mathrm{dom}(s))(y(i) \leq t(i))\}.$$

We fix $t \in Y$ with minimal $\mathrm{rk}_{D_l}(t)$.

Claim 3.4.31 $\mathrm{rk}_{D_l}(t) = 0$

Proof. Suppose otherwise. Since $\mathrm{rk}_{D_l}(t) \neq \infty$, there are $m \in \omega$ and a set $\{t_k : k \in \omega\} \subseteq \omega^m$ of increasing sequences such that for all $k \in \omega$, $\mathrm{rk}_{D_l}(t_k) < \mathrm{rk}_{D_l}(t)$ and $t_k(\mathrm{dom}(t)) > k$. Note that $\{t_k : k \in \omega\} \in N$. For every $j < m - \mathrm{dom}(t)$ we define $Z_j = \{t_k(\mathrm{dom}(t) + j) : k \in \omega\}$. Note that every infinite subsequence of $\{t_k : k \in \omega\}$ also witnesses that $\mathrm{rk}_{D_l}(t) > 0$. So, picking the appropriate subsequence, and since the S^a form a maximal almost-disjoint family, we may assume that $\{t_k : k \in \omega\}$ has the following property:

For every $j < m - \mathrm{dom}(t)$, the exists only one $a_j \subseteq \omega$ such that $Z_j \subseteq S^{a_j}$

Since $N \prec H(\chi)$, we may carry this construction in N.

Since $S^a \notin N$, $Z_j \cap S^a$ is finite for all $j < m - \mathrm{dom}(t)$. Therefore, there is $k \in \omega$ such that $\mathrm{rec}(t_k) \cap S^a = \mathrm{rec}(t) \cap S^a$. So, in particular, $t_k \in Y$ and $\mathrm{rk}_{D_l}(t_k) < \mathrm{rk}_{D_l}(t)$. A contradiction with the minimality of $\mathrm{rk}_{D_l}(t)$ in Y. ■

Since $\mathrm{rk}_{D_l}(t) = 0$, there exists $z \in \omega^\omega$ such that $\langle t, z\rangle \in D_l$. Let $z' \in \omega^\omega$ be such that for all $n \in \omega$, $z'(n) = \max\{y(n), z(n)\}$. Then, $\langle t, z'\rangle \leq \langle s, y\rangle$ and, since $|\mathrm{rec}(t) \cap S^a| < l$, $\langle t, z'\rangle$ decides $\dot{x}(l)$ without deciding $h_{a,d}(l)$.

Suppose that $\langle t, z'\rangle \Vdash_{\mathbb{D}} \dot{x}(l) = j$. Fix $i \geq z'(\mathrm{dom}(t))$ such that $i \in S^a_j$. Then,

$$\langle t^\frown i, z'\rangle \Vdash_{\mathbb{D}} \dot{d}(l) = j = h_{a,d}(l).$$

A contradiction with the election of $\langle s, y\rangle$. ■

We finish the proof of Fact 3.4.25: Since ω_1 is not inaccessible in L, for some real $b \in \omega^\omega \cap V$, $\omega_1^{L[b]} = \omega_1$. Clearly,

$$\{h_{a,d} : a \in P(\omega) \cap L[b]\} \subseteq \omega^\omega \cap L[b][d] \subseteq \omega^\omega \cap V[G]$$

and, by Lemma 3.4.30, $\{h_{a,d} : a \in P(\omega) \cap L[b]\}$ is an unbounded set of reals in $V[G]$. Therefore, $\omega^\omega \cap L[b][d]$ is not countable. Let $x \in \omega^\omega \cap V[G]$ be such that x codes b and d. Then

$$V[G] \models \text{“}\omega^\omega \cap L[x] \text{ is unbounded”}.$$

Therefore, $V[G] \models$ “$\exists x(\omega^\omega \cap L[x]$ is unbounded)” and so $V[G] \models \theta$. ■

Since $V \nvDash \theta$, $V[G] \models \theta$ and θ is a Σ^1_4, V is not $\underset{\sim}{\Sigma}^1_4$-absolute for $\mathbb{D}$. A contradiction. This ends the proof of Theorem 3.4.23. ■

Now we prove (6) implies (1) of Theorem 3.4.18. We will show that $\underset{\sim}{\Sigma}^1_4$-absoluteness for σ-centered subposets of Borel posets implies that ω_1 is a Mahlo cardinal in L. Suppose otherwise. Since $\underset{\sim}{\Sigma}^1_4$-absoluteness for σ-centered subposets of Borel posets implies $\underset{\sim}{\Sigma}^1_4$-absoluteness for Hechler forcing, by Theorem 3.4.23, ω_1 is an inaccessible cardinal in L. So, there exists a club $D \in L$ on ω_1 of singular cardinals in L.

Let $D^* \subseteq P(\omega)$ be the set of all codes of a well ordering of ω of order type ω or an ordinal in D. Let $\pi : D^* \to D^*$ be such that for all $a \in D^*$, $\pi(a)$ is a code for the least ordinal in D greater than $\|a\|$.

Let P_π the Solovay almost-disjoint coding for π (see Definition 3.3.40). Recall that P_π is a σ-centered poset. Hence, by Fact 3.4.13 and the remark following it, P_π can be densely embedded into a σ-centered subposet P_π^* of a Borel poset. But, by Fact 2.1.12 and by Lemma 3.3.42, in every P_π^*-generic extension over V there exists a real x such that $\omega_1^{L[x]} = \omega_1$. But this is expressible with a Σ_4^1 sentence that, by $\underset{\sim}{\Sigma}_4^1$-absoluteness, it is true in V. A contradiction with the inaccessibility of ω_1. ■

3.5 Collapsing a weakly-compact cardinal

Definition 3.5.1 *$L(\mathsf{R})$ is a* weakly-compact Solovay model over V*, henceforth a w-c Solovay model over V, iff*

1. *For every $x \in \mathsf{R}$, ω_1 is a weakly-compact cardinal in $V[x]$ and*
2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

Clearly, every w-c Solovay model over V is a Solovay model over V. So, we can give a characterization of w-c Solovay models in the same way as in Lemma 3.1.1. Namely

Lemma 3.5.2 *Suppose that M satisfies*

1. *For every $x \in \mathsf{R}$, ω_1 is a weakly-compact cardinal in $V[x]$ and*
2. *For every $x \in \mathsf{R}$, $V[x]$ is a generic extension of V by some countable poset.*

Then there exists a forcing notion W such that does not add reals and creates a $Coll(\omega, < \omega_1)$-generic filter C over V such that M and $V[C]$ have the same reals. Thus, W forces that $L(\mathsf{R})^M$ is a w-c Solovay model over V. ■

Note that we can formulate Theorem 2.3.2 in the following way:

Theorem 3.5.3 (K. Kunen) *Suppose $L(\mathsf{R})^M$ is a w-c Solovay model over V and P is a ccc poset in M. Then the $L(\mathsf{R})$ of any P-generic extension of M is also a w-c Solovay model over V.* ■

Definition 3.5.4 *A poset P is* Knaster *iff for all uncountable subset X of P there exists an uncountable $Y \subseteq X$ of pairwise compatible conditions.*

Theorem 3.5.5 *The following are equiconsistent (modulo ZFC):*

1. *There exists a weakly-compact cardinal.*
2. *$L(\mathsf{R})$-two-step absoluteness for ccc forcing notions.*
3. *$L(\mathsf{R})$-two-step absoluteness for Knaster forcing notions.*
4. *$\underset{\sim}{\Sigma}_4^1$-absoluteness for Knaster forcing notions.*

Proof. (1) implies (2) follows from 3.5.3. (2) implies (3) and (3) implies (4) are obvious.

(4) implies (1): Let V be a model of ZFC that is $\underset{\sim}{\Sigma}^1_4$-absolute under Knaster forcing notions. Suppose that ω_1 is not a weakly-compact cardinal in L.

Lemma 3.5.6 (J. Silver) *Suppose that κ is a regular cardinal which is not a weakly compact cardinal in L. Then, in L, there is an Aronszajn tree T on κ such that for every model M of ZFC, if $M \models$ "T has a branch of length κ", then $M \models$ "$\mathrm{cf}\,(\kappa) = \omega$".*

Proof. See [D], 5.1.C. ■

Since ω_1 is not a weakly-compact cardinal in L, we can fix an Aronszajn tree $T \in L$ as given by the lemma. Without loss of generality, we may assume that T has infinitely-many nodes of height 0.

Definition 3.5.7 *For every sequence $(d_\alpha)_{\alpha<\omega_1}$ of reals let $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ be the following poset:*

- *$p \in \mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ iff:*
  1. *p is a function from a finite subset of T into Q, the set of rational numbers.*
  2. $(\forall t, t' \in \mathrm{dom}\,(p))\,(t \leq_T t' \to p\,(t) \leq_Q p\,(t'))$
  3. $(\forall t \in \mathrm{dom}\,(p))\,(\mathrm{ht}_T\,(t) = \omega \cdot \alpha \wedge p\,(t) \in \omega \to p\,(t) \in d_\alpha)$
- *$p \leq q$ iff $q \subseteq p$.*

Lemma 3.5.8 *For every sequence $(d_\alpha)_{\alpha<\omega_1}$, $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ is Knaster poset.*

Proof. Let $(d_\alpha)_{\alpha<\omega_1}$ be any sequence of ω_1 reals. We need the following claim:

Claim 3.5.9 *If T is an Aronszajn tree and for every uncountable subset of $X \subseteq T$, there exists an uncountable subset $Y \subseteq X$ of pairwise incomparable elements, then $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ is a Knaster poset.*

Proof. Let $\{p_\xi : \xi < \omega_1\}$ be an uncountable subset of $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$. By a Δ-system argument we may assume that for every $\xi < \omega_1$, $\mathrm{dom}\,(p_\xi) = x \cup x_\xi$, where $\{x\} \cup \{x_\xi : \xi < \omega_1\}$ is a family of pairwise disjoint finite sets. Moreover, by thinning out the family $\{p_\xi : \xi < \omega_1\}$ we can assume that:

1. For all $\xi, \zeta < \omega_1$, $p_\xi \restriction x = p_\zeta \restriction x$.
2. There exists $n \in \omega$ such that for all $\xi < \omega_1$, $x_\xi = \{t^\xi_i : i \leq n\}$
3. For every $i \leq n$ and every $\xi, \zeta < \omega_1$, $p_\xi(t^\xi_i) = p_\zeta(t^\zeta_i)$
4. For every $\xi < \zeta < \omega_1$, every $t \in x_\xi$ and every $t' \in x_\zeta$, $\mathrm{ht}_T\,(t) < \mathrm{ht}_T\,(t')$.

Since $\bigcup\{x_\xi : \xi < \omega_1\}$ is an uncountable subset of T, there exists $S \subseteq \omega_1$ uncountable such that $\bigcup\{x_\xi : \xi \in S\}$ is a set of pairwise incomparable elements of T.

Note that $p, q \in \mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ are incompatible iff there are $t \in \mathrm{dom}\,(p)$ and $t' \in \mathrm{dom}\,(q)$ such that $t \leq_T t'$ but $p\,(t) \not\leq q\,(t')$. But, since $\bigcup\{x_\xi : \xi \in S\}$ is a set pairwise of incomparable elements of T, $\{p_\xi : \xi \in S\}$ is an uncountable set of pairwise compatible conditions of $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$. So, $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ is Knaster. ■

So, in order to prove Lemma 3.5.8, it only remains to show that for every uncountable set $X \subseteq T$ there exists an uncountable set $Y \subseteq X$ of pairwise incomparable elements of T.

Suppose otherwise. Let $T' \subseteq T$ uncountable such that for every $Y \subseteq T'$ of pairwise incomparable conditions is countable. So, $T' = \langle T', \leq_{T'}\rangle$, where $\leq_{T'} = \leq_T \upharpoonright (T' \times T')$ is a Suslin subtree of T. But then, $\mathsf{P}_{T'} = \langle T', \geq_{T'}\rangle$ is a ccc poset and for every $\mathsf{P}_{T'}$-generic filter G over V, $V[G] \models$ "T has a branch of length ω_1". Hence, by Lemma 3.5.6, $V[G] \models$ "$\mathrm{cf}\,(\omega_1) = \omega$". A contradiction, since $\mathsf{P}_{T'}$ is a ccc poset and hence preserves cofinalities. ■

Claim 3.5.10 *For every $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$-generic filter G over V, there exists a order-preserving function $f \in V\,[G]$, $f : T \longrightarrow \mathsf{Q}$ such that:*

1. *For every $\alpha < \omega_1$, $n \in d_\alpha$ iff there exists $t \in T$ with $ht_T\,(t) = \omega \cdot \alpha$ such that $f\,(t) = n$.*
2. *For every $t \in T$, if $ht_T\,(t)$ is a limit ordinal, then*
$$f\,(t) = \sup\{f\,(t') : t' <_T t\}.$$

Proof. Let G be a $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$-generic filter over V. Let $f = \bigcup G$. Then since G is a filter, f is an order-preserving function and, by genericity, $\mathrm{dom}\,(f) = T$.

By definition of $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$, if there exists $t \in T$ with $ht_T\,(t) = \omega \cdot \alpha$ such that $f\,(t) = n$, then $n \in d_\alpha$. Moreover, since for every $\alpha < \omega_1$ and every $n \in d_\alpha$,

$$D_\alpha^n = \{p \in \mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1}) : (\exists t \in \mathrm{dom}\,(p))\,(ht_T\,(t) = \omega \cdot \alpha \wedge p\,(t) = n)\}$$

is a dense subset of $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$, if $n \in d_\alpha$, then there exists $t \in T$ with $ht_T\,(t) = \omega \cdot \alpha$ such that $f\,(t) = n$. So, (1) holds.

Finally, since for every $t \in T$ with $ht_T\,(t)$ a limit ordinal, every condition $p \in \mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ with $t \in \mathrm{dom}\,(p)$ and every rational number $r < p\,(t)$, the set

$$D_r^p = \{q \in \mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1}) : q \leq p \wedge r \in \mathrm{rec}\,(p)\}$$

is dense below p. Hence, (2) follows. ■

Now we finish the proof of Theorem 3.5.5: Let $(d_\alpha)_{\alpha<\omega_1}$ be a sequence of reals such that for every $\alpha < \omega_1$, $d_\alpha \in WO_\alpha$. Then forcing with $\mathsf{P}(T, (d_\alpha)_{\alpha<\omega_1})$ adds a real $c \in V\,[G]$ such that $(d_\alpha)_{\alpha<\omega_1} \in L\,[T, c]$. Since $T \in L$, $L\,[T, c] = L\,[c]$. Moreover,

$V[G] \models$ "$L[c]$ has uncountably-many reals". But this can be expressed by means of the $\Pi^1_3(c)$ sentence:

$$\neg\exists x \forall y \, (y \in WO \wedge y \in L[c] \rightarrow \| y \| < \| x \|)$$

So, "$\exists x(L[x]$ has uncountably many reals)" is a Σ^1_4 sentence, and by $\underset{\sim}{\Sigma}^1_4$-absoluteness,

$$V \models \exists x(L[x] \text{ has uncontably many reals}).$$

Therefore, there is $a \in V$ such that $\omega_1^{L[a]} = \omega_1$. But, since V is $\underset{\sim}{\Sigma}^1_4$-absolute for Borel ccc posets, by Corollary 3.1.23, $\omega_1^{L[a]} < \omega_1$. A contradiction. Hence, ω_1 is a weakly-compact cardinal in L. ∎

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