ON THE MEROMORPHIC NON-INTEGRABILITY OF SOME PROBLEMS IN CELESTIAL MECHANICS

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Certifiquem que la present memòria ha estat
realitzada per Sergi Simon i Estrada
i codirigida per nosaltres.

Barcelona, 14 de maig de 2007,

Juan J. Morales-Ruiz

Carles Simó i Torres

A la meva mare,
a la meva germana
i a l'Ainhoa

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Contents

Acknowledgements **i**

1 Introduction **1**
1.1 Integrability of differential systems 1
1.2 Historical note 2
1.3 Original results 4
1.3.1 Homogeneous potentials and N-Body Problems 5
1.3.2 Non-integrability of Hill's Problem 6
1.4 General structure, notation and conventions 6

2 Theoretical background **9**
2.1 Useful results from Algebraic Geometry 9
2.1.1 Preliminaries 9
2.1.2 Linear algebraic groups and Lie algebras 10
2.1.3 Rational invariants 12
2.2 Notions of integrability 15
2.2.1 Integrability of Hamiltonian systems 15
2.2.2 Integrability of linear differential systems 18
2.3 Morales-Ramis theory 21
2.3.1 The general theory 22
2.3.2 Special Morales-Ramis theory: homogeneous potentials . . 23
2.4 Basics in Celestial Mechanics 27
2.4.1 The N-Body Problem 27
2.4.2 Hill's Lunar Problem 35

3 The meromorphic non-integrability of some N-Body Problems **37**
3.1 Introduction 37
3.2 Preliminaries 38
3.2.1 Statement of the main results 38
3.2.2 Setup for the proof 40
3.3 Proofs of Theorems 3.2.2 and 3.2.3 44
3.3.1 Proof of Theorem 3.2.2 44
3.3.2 Proof of Theorem 3.2.3 46
3.3.3 Proof isolate: $N = 2^m$ equal masses 51

4 The meromorphical non-integrability of Hill's Lunar problem **55**
4.1 Introduction 55
4.1.1 Statement of the main results 56
4.2 Proof of Lemma 4.1.1 57
4.2.1 Change of variables 57
4.2.2 Solution of the new equation 57
4.2.3 Singularities of $\phi^2(t)$ 58
4.3 Proof of Lemma 4.1.2 58
4.3.1 Layout of the system 58
4.3.2 Fundamental matrix of the variational equations 59
4.3.3 Relevant facts concerning $\Psi(t)$ 60
4.4 Proof of Theorem 4.1.3 62
4.5 Concluding statements 69

5 Conclusions and work in progress **71**
5.1 Overview 71
5.2 Perspectives on Conjectures 5.1.1 and 5.1.2 72
5.2.1 The N-body problem with arbitrary masses 72
5.2.2 Candidates for a partial result 76
5.3 Hamiltonians with a homogeneous potential 77
5.3.1 Higher variational equations 77

Appendices **79**

A Computations for Theorem 3.2.2 **79**

B Resum **83**
B.1 Introducció 83
B.1.1 Dues nocions d'integrabilitat en sistemes dinàmics 84
B.1.2 Alguns problemes de la Mecànica Celeste 87
B.2 Resultats originals 89
B.2.1 Existència d'una integral primera addicional 90
B.2.2 Problemes de N Cossos 91
B.2.3 La no-integrabilitat del Problema de Hill 93
B.3 Agraïments 94

Chapter 1

Introduction

1.1 Integrability of differential systems

The idea underlying any apprehension of an **integrable** dynamical system is the ability to make global assertions on the system's evolution with respect to time. Although the outcome of such assertions, usually called a **solution**, is fairly easy to characterize, giving the assertions themselves a strict definition has always proven a troublesome task, since each field of study has a specialized notion of "solvability" of its own, seldom equivalent to the others'. The very concept of understanding a dynamical system is already difficult to define since the near-totality of cases will end in a non-trespassed threshold: to wit, the knowledge of a solution in closed form. Plainly speaking: there would be no controversy whatsoever on what integrability means (and hardly any need, by the way, to use such a word as *chaos* in ordinary differential systems) were the general solution of any dynamical system possible to find semi-algorithmically in a finite number of steps – a task nowadays unfeasible. There are attempts at partially circumventing the latter obstacle, most notably the geometrical, also called *qualitative*, theory of differential equations (see [104], [109], [110], [130]) and perhaps most importantly the numerical simulation of solutions of differential equations based on qualitative theoretical results (see [123], [125]), and the computer-assisted proofs these simulations provide for (see [60], [74], [92], [124], [151], [161]), as well as the so-called *algorithmic modeling paradigm* relying on producing models from experimental data ([1]). However, what remains in all cases is an absolute dependence on disciplines (numerics, statistics, even algorithmic geometry) whose domain of application is peripheral to the theoretical groundwork, except when applied by researchers who conceive Mathematics as a science in and of its own rather than as a mere tool.

Thus naturally appears the phenomenon of specialization, so clearly visible in Section 2.2 and not as much a subterfuge as it may be an asset; it is our contention that most of the definitions of and conditions for integrability and non-integrability, including the ones explained in this text, are all their own part of more ambitious endeavors aimed precisely at *integrating* systems, at least those of a certain kind. Such an aim shows up most blatantly in the unrelenting effort at classifying all obstructions to "integrability" of dynamical systems, for instance the presence of certain special functions in their general solution. It may be argued, from a more general perspective, that this is not the shortest path to attain our final goal, but such a perspective is currently not available and there is one leitmotif underlying this outlook which already justifies the whole process wherever it may lead: the act of characterizing first integrals, and more generally systems for which these are easy to find, as anomalies in a wider, much more intricate context. In other words: foreshadowing the computation of exact solutions as a predictable accident in the hope of being able to predict it. Even if this is nothing but an act of self-delusion (as is any model, for that matter), *the inference that it may work if done in a number of proper ways* is currently enough for us.

1.2 Historical note

Arguably the cornerstone of Celestial Mechanics since it originated in Newton's *Principia*, the N-**Body Problem** has long been seen in Astrophysics and Applied Mathematics as an epitome of chaotic behavior; such behavior is retained in a significant amount in every model arising from it, especially by means of simplification. As a matter of fact, as we will recall below, most of the advances made in Applied Mathematics are precisely due to the presence of chaos in mechanical systems directly or indirectly related to many-problems. However, there was a time during which both naïveté and maximalism led philosophers to think otherwise. In keeping with this spirit, P.-S. de Laplace wrote, in 1814, the most famous paradigm in causal determinism:

> "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes." ([71], [72])

In this respect, the basis of modern science is firmly rooted on denial: if such an "intellect", popularly referred to as *Laplace's demon*, were to exist, it would most probably be beyond or outside the Universe, as well as independent and non-binding with respect to it (all of which contradicts Materialism), all present and past states would be knowledgeable (as opposed to Quantum Mechanics and Relativity), the concepts of irreversibility and entropy would be superfluous (hence, virtually obliterating the Second Law of Thermodynamics) and, above all, it would be possible to know all of the laws governing the Universe – a premise against which every single model in the History of Science, from "Panta Rhei" to modern String Theory, may be directed at will if deemed pertinent. Even during Laplace's lifetime, efforts as consistently intelligent as those made by C. F. Gauss were bent strictly on questions of a pragmatic, specific and conceptually subservient sort, such as numerically solving Kepler's equation derived from the two-body problem in [42], rather than engaging in further exercises in futility such as the above quote.

The state of affairs by the mid-twentieth century was thus fairly predictable from the outset, especially for those accepting Science as an endless sediment of partial results assembled in an asymptotical quest for further open questions: Classical Mechanics, as foreseen by Laplace and Newton, were seen by modern physicists as a barren land, a sterile discipline relegated to scholasticism. And yet, advances in Computer Science from the sixties onward, coupled with the insight of a number of researchers, brought about a series of theoretical results in turn showing new light on nonlinear dynamics; these results produced a slight abatement of the ostracism played on Classical Mechanics despite having their roots firmly planted on the perturbation theory by Poincaré and others *precisely undermining classical determinism*; this would come across as an interesting paradox were it not for the fact that the outcome of these results was simply a redistribution of the existing models' underbelly aimed at defining "chaos".

In order to explain such paradox in this and the next paragraphs, we must first mention the fact that the Solar System, in all its complexity, shows a somehow "regular" pattern due to the weakness of gravity and the total predictability of Kepler's two-body problem. In view of this, Euler, Lagrange and Laplace studied increases in the amount of bodies in terms of changes in global stability due to small perturbations of a two-body problem, i.e., saw movement as an addenda to geometry, whereas Hamilton and Jacobi again added geometry as a factor to movement by describing dynamics as phase spaces whose volume was to be kept constant by the flow.

This cumulative intermingling of geometrical and dynamical outlooks was useful for a number of reasons, most importantly the reduction of the dimension via purposefully chosen symmetries, and a serious attempt was already being made late in the nineteenth century at finding corrections to Kepler's problem by a third mass. There was a pending obstacle, though: proving the convergence of the resulting perturbed series. This problem was first glanced upon in the 1880s by K. T. W. Weierstrass who, with the aid of G. Mittag-Leffler and under the auspices of King Oscar of Sweden, favored the announcement of a prize in Acta Mathematica (volume 7, 1885/86) for finding the solution as a uniformly convergent series. The difficulty of finding this global solution as a series, let alone as a convergent one, is inferred from the revised draft of H. Poincaré's attempt which, although thwarted, won the prize and is nowadays considered landmark in the theory of Dynamical Systems. The problem as stated in the terms of the prize was finally solved, except for special cases, by K. F. Sundman in [136] for the Three-body Problem (Theorem 2.4.2) and by Q. D. Wang for the general N-Body Problem (Theorem 2.4.3). See [35] for details on the subject's evolution from Weierstrass and Poincaré's "brilliant failure" onward.

The main success of Poincaré's work was his prediction of divergence as due to the presence of the so-called "small divisors", ever since seen as a marker for the impossibility of predictions on the long-term evolution of systems as complex as the N-Body Problem. Since only a rarity of systems are significantly less complex than the latter Problem or happen to satisfy any of a wide list of requirements for integrability (some of which we will explain further on), the study of small divisors began to focus, during the second half of the past century, on a very specific endeavor: the possible persistence, for a given perturbation of an integrable system, of certain traits or symptoms of integrability. Simply put: how much of the geometry underlying dynamics prevails when perturbing an integrable system into chaos. It was A. N. Kolmogorov who finally found, in 1954 [63], an answer to this question: to wit, that a somehow quantifiable majority of the trajectories of such non-integrable systems are quasi-periodical and may be computed through convergent expressions. V. I. Arnol'd and J. Moser, in the sixties, established further rigorous proofs of this fact, thenceforth known as the *K.A.M. (Kolmogorov-Arnol'd-Moser) Theorem*: besides [63], see [12], [14], [13] and [102].

A long way has been travelled, and is still unconcluded, in order to detect and define chaos as an addendum to an ideal geometric groundwork – in other words, to obtain an axiomatic reassessment of our ignorance, couched on deterministic terminology. Although ancestry as defined by Ph.D. pupilship is not always determinant, it is indeed significant in this case that a direct line of such ancestry may be established from Gauss to Weierstrass and from Weierstrass to Kolmogorov and Moser, as well as from Kolmogorov directly to Arnol'd. See [33].

As for present and future sceneries, the main theme in the current study of chaos is the attempt at transversality between disciplines. In particular, the study of chaos from the *algebraic* point of view is a new, relatively recent trend establishing direct continuity with the preceding and nowadays centered on two stages with more than a trait in common: the line of study initiated by S. L. Ziglin ([162], [163], see also [15]) and the one begun by J.J. Morales-Ruiz and J.-P. Ramis: see [93] and [95]. Ziglin's theory relies strictly on the monodromy generators of the variational equations around a given particular solution, whereas Morales' and Ramis' theory uses linear algebraic groups containing the aforementioned monodromies and is naturally immersed in the *Galois theory of linear differential equations*, which we assume the reader is already familiar with – otherwise, see Section 2.2.2 of this thesis or [93] and [144] for the minimum necessary concepts.

1.3 Original results

Understandably, none of what has been said in the Section 1.1 seems susceptible of conclusive statements at this point, and what is explained in Section 1.2 is highly unlikely to be unified into a single theory in the short term. What is presented in this thesis, instead, is a compendium of algebraic non–integrability proofs for a short array of problems arising from Celestial Mechanics, the original Three-Body Problem among them, as well as a new necessary condition, stronger than mere integrability, which is applied to generalize some of the aforesaid proofs and may in turn be used for a wider class of Hamiltonian systems.

This is done in Chapters 3 and 4, after summarizing in Chapter 2 what is understood as *(meromorphic) integrability* in the Hamiltonian setting where these problems belong. This summary may also be seen as an introduction to some of the topics explained in Section 1.2.

1.3.1 Homogeneous potentials and N-Body Problems

Having Section 1.2 in mind, the N-Body Problem's history of parallel attempts both at looking for new first integrals for it and proving it analytically or meromorphically non-integrable should not come up as a surprise. Even less surprising is the partial success of the latter, especially in recent times thanks to the two parallel lines of study introduced in the last paragraph of Section 1.2. Using a consequence of the new theory by Morales-Ruiz and Ramis as applied to the factorization of linear operators, D. Boucher and J.-A. Weil ([23], [21]) proved the meromorphic non-integrability of the Three-Body Problem. Since the obstruction to integrability arising from the Boucher-Weil approach was precisely the presence of logarithms in the resulting decomposition, this may be seen as an instance of what was said in the last paragraph of Section 1.1. On the other hand, using the Ziglin approach, A. V. Tsygvintsev ([139], [140], [141], [142], [143]) proved the meromorphic non-integrability of the Three-Body Problem and ultimately settled the non-existence of a single additional meromorphic first integral except for three special cases (see Remark 3.3.1). It is finally worth noting that Ziglin ([164, Sections 3.1 and 3.2]) managed to settle strong conditions on the integrability of the Three-Body Problem and the equal-mass N-Body Problem.

Chapter 3 reobtains in simpler ways, strengthens and generalizes the results mentioned in the previous paragraph using the aforementioned theory started in [95] as applied to Hamiltonians of a specific kind: to wit, those which are classical with an integer degree homogeneous potential. Although conjectures and open problems will still prevail (see Chapter 5), the proofs given here are significantly shorter thanks to a significant step forward made in [95, Theorem 3]. Furthermore, using this same Theorem, a new necessary condition is established in Section 2.3.2 on the existence of a single additional integral for any classical conservative system – a condition in turn allowing us to discard the existence of an additional integral for the Three-Body Problem with arbitrary dimension and positive masses (a generalization of Bruns' Theorem 2.4.5, that is) and for the planar N-Body Problem with equal masses if $N = 4, 5, 6$. It must be said that, in the equal-mass case, the only apparent obstacle keeping us from extending Bruns' to an arbitrary amount of bodies was a technical one, namely the structure of a certain algebraic extension of the N^{th} cyclotomic field for general $N \geq 7$.

Specifically, the new results in Chapters 2 and 3 are Theorem 2.1.10 and Corollary 2.3.5, as well as Theorems 3.2.2, 3.2.3 and 3.3.10 and Corollary 3.3.11, as well as Lemmae 3.3.5 and 3.3.6. The Lemmae used in their proofs are mostly a reformulation of known previous results and would hardly qualify as new, although special mention may be made of Lemmae 2.1.7 and 2.1.8. All of the open problems in Chapter 5 find numerical evidence in their favor, gathered for a widespread family of values of N. This is true both for the equal-mass Problem and for a fairly large variety of masses. A word may be said about the impending publication of part of these new results in [99].

1.3.2 Non-integrability of Hill's Problem

Hill's Lunar Problem appears in Celestial Mechanics as a limit case of the Restricted Three-Body Problem, itself a special instance of the problem in the previous paragraph for $N = 3$. Moreover, and aside from the fact that it appears to be the simplest illustration of gravitational dynamics with more than two bodies, Hill's problem provides with information in turn casting light on several other problems in Celestial Mechanics. It contains no parameters and is globally far from any simple well–known problem. Strong numerical evidence of its lack of integrability has been given in the past, although no rigorous proof in this respect had been done in general terms up to this thesis.

In Chapter 4, an algebraic proof of meromorphic non–integrability is presented for Hill's Problem which, rather than exploiting the tools used and found in Chapter 3, avails itself of the deep-set theoretical basis of those tools – not only out of willful diversification, but also because those previous tools were not enough for our purpose. Beyond the novelty of the result itself, thus, Chapter 4 stands as an example of the adequacy of the most general instance of Morales' and Ramis' theory to many significant problems – an instance with whose aid we identified the concrete contributions, embodied in special functions, which probably made this proof so hard to find in the past. Hence, in all its surgical detection of obstructions to integrability, this is one of the places where the thesis is closest to echoing the second paragraph in Section 1.1 without fully conveying it.

All of the Lemmae and Theorems in Chapter 4, that is, those stated in Subsection 4.1.1 (Lemmae 4.1.1 and 4.1.2 and Theorem 4.1.3, and the immediate consequence given in Corollary 4.1.4) are new results. As opposed to the previous Chapter, all that is said in Chapter 4 has already been published, in [98], in a joint work with the advisors of this present thesis.

1.4 General structure, notation and conventions

This thesis consists of four chapters. There will be only one figure derived from numerical simulation (see Section 4.4), since we intended to lean as little as possible on numerical results and only used them for illustration purposes. The first and last chapters will be mainly a compendium of known information except for a new result in Section 2.3.2 and an ensemble of conjectures in Chapter 5. There will be a subject index at the end, in which page numbers will be marked in boldface if the word is defined in the given page, and in regular face if said word is simply mentioned.

Given a field K and a K-vector space V of finite dimension n, $\mathrm{End}_K(V)$ will denote the space of endomorphisms $f : V \to V$ (as opposed to other notations such as $\mathcal{L}_K(V; V)$ or $\mathrm{Hom}_K(K; K)$) and, given $n \in \mathbb{N}$, $\mathcal{M}_n(K)$ will be the alternative way of writing the ring $\mathrm{End}_K(K^n)$ of all square $n \times n$ matrices with their entries in K. Similarly, $\mathrm{GL}(V) \subset \mathrm{End}_K(V)$ will be the group of *invertible* linear transformations and, fixing bases in K^n, the group will be immediately identified with that of *invertible* $n\times n$ matrices and written $\mathrm{GL}_n(K)$; the subgroup of $\mathrm{GL}_n(K)$ comprised of linear transformations whose determinant is equal to the unit element of the group $(K^*, *)$ is denoted as $\mathrm{SL}_n(K)$. $\mathrm{O}_n(\mathbb{R})$ will in turn stand for the set of orthogonal matrices with their entries in $\mathbb{R}$, and $\mathrm{Sp}_{2n}(K)$ will stand for the symplectic group of degree $2n$ over K. Although the underlying set will be a cartesian product in both cases, direct sums will be written differently for algebraic groups $G_1, \ldots, G_n$ (see Section 2.1) and K-vector spaces $V_1, \ldots, V_n$: $G_1 \times \cdots \times G_n$ and $V_1 \oplus \cdots \oplus V_n$, respectively.

There will be a number of cases in which the above field K will be $\mathbb{C}$ by default. This will be the case for vector and matrix functions, for instance, unless stated otherwise. All vectors will be denoted in boldface and their norms will be written in ordinary face. All norms will be assumed *Euclidean* by default, for it is through these that the N-Body Problem finds its simplest known formulation. For every vector whose entries are likely to be broken down in separate vectors of lesser size, at most two different boldface types will be used, albeit with the same letter: for any $n, m \in \mathbb{N}$, a vector in $\mathbb{C}^{nm}$ will be written with italic boldface, $\boldsymbol{q}$ (its norm being q) if the n consecutive m-vectors making up for its entries are also being considered; in such case, these latter will be written in regular boldface, $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbb{C}^m$, their norms written as $q_1, \ldots, q_n$, respectively. If further hierarchy is needed, we will maintain either italic or regular boldface. Vectors will be freely written in concatenation, e.g. $\boldsymbol{z}^T = \left(\boldsymbol{q}^T, \boldsymbol{p}^T\right) = \left(\mathbf{q}_1^T, \ldots, \mathbf{q}_n^T, \mathbf{p}_1^T, \ldots, \mathbf{p}_n^T\right)^T$, but we will avoid the ^{T} superindex unless we have to make specific reference to scalar products, e.g. in Rayleigh quotients. Boldface as described in all of the above considerations will be applied exclusively to constant vectors and vector functions of *one* variable, e.g. $\boldsymbol{q} = \boldsymbol{q}(t)$, whereas vector functions with more than one argument, e.g. $f = f(t, \boldsymbol{q})$, will be written in regular face.

Matrices will be written in capital letters, whether Latin or Greek. Be it for matrices or for vectors, notation will be sometimes implicit by means of subindexes, e.g. $(b_{i,j})_{i,j=1,\ldots,n}$ may stand for $B \in \mathcal{M}_n(K)$ and $(a_i)_{i=1,\ldots,n}$ may stand for a vector $\boldsymbol{a} \in K^n$; the terms inside the parentheses will occasionally stand for whole vectors or matrix blocks instead of single entries. Square roots for diagonal matrices will be defined as usual whenever the original diagonal entries are real and non-negative: $M^{1/2} = \mathrm{diag}\left\{\sqrt{m_{i,i}} : i = 1, \ldots, n\right\}$ if $M = \mathrm{diag}\{m_{i,i} : i = 1, \ldots, n\} \in \mathcal{M}_n$. As for vector functions of one variable, $\boldsymbol{x} : X \subset K \to K^n$, we will occasionally write them as Cartesian products, e.g. $\boldsymbol{x} = x_1 \times \cdots \times x_n$, whenever further reference to their coordinate functions is pertinent.

Since there will only be one independent variable t properly regarded as time, an overdot will stand for $\frac{d}{dt}$ all through the text and $^{(k)}$ will stand for $\frac{d^k}{dt^k}$, $k \geq 4$, whereas $'$ will usually imply derivation with respect to phase variables of Hamiltonian systems. It is worth noting this time variable t will be complex by default all through the text. Γ will often stand for Riemann surfaces, and $\mathbb{P}^1$ will always stand for the (complex) projective line.

Defining the Kronecker delta $\delta_{i,j}$ as usual, $\left\{\boldsymbol{e}_{n,k} = (\delta_{i,k})_{i=1,\ldots,n}^T\right\}$ will be the canonical basis for $\mathbb{R}^n$. Zero vectors and zero and identity matrices will be written with their dimension as a subindex whenever deemed necessary, e.g. $\mathbf{0}_n \in K^n$ or $0_{n\times n}, \mathrm{Id}_n \in \mathcal{M}_n(K)$. $|\cdot|$ will denote absolute value or modulus indistinctively. $\sqrt{-1} = \mathrm{i}$ will always be denoted in Roman, non-italic font. The consideration of points in the plane as either complex numbers or real 2-vectors will also be tacit depending on the context. The determination for complex square roots will be that given by the analytic continuation of the *positive* real square root, i.e. $\sqrt{z} := \sqrt{r}e^{\frac{\mathrm{i}\theta}{2}}$ whenever $z = re^{\mathrm{i}\theta}$ and $\theta \in [0, 2\pi]$.

Chapter 2

Theoretical background

This chapter is devoted to a concise introduction to the theoretical tools used for our main results. Despite its mainly expository nature it contains a new result, proven in Subsection 2.3.2. Basic knowledge will be assumed from the reader concerning complex functions, differential systems, calculus on manifolds, differential forms, group actions, representation theory and invariant theory; readers not acquainted with these themes may first read [2], [3], [13], [55], [68], [93], [131], and [145]. All through the rest of the text, we shall make no significant forays into the topics of special functions, representation theory, Algebraic Geometry and Celestial Mechanics other than the ones made in this chapter.

2.1 Useful results from Algebraic Geometry

See [19], [55], [68], [93], [127] or [131] for technical details and further information.

2.1.1 Preliminaries

From now on, each group G will have its unit element written as e_G, subindex $_G$ being dropped for the most part. We recall calling a subgroup $H \subset G$ **normal** if, for every $x \in G$, $xHx^{-1} = H$. It is straightforward to establish that the kernel of any group homomorphism, as well as the image of a normal subgroup under an *epimorphism* is always a normal subgroup of the source group. A sequence of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_m, \tag{2.1}$$

for any given $m \in \mathbb{N}$, is called a **tower** of subgroups. Tower (2.1) is called **normal** if G_{i+1} is a normal subgroup of G_i for each $i = 0, \ldots, m-1$. A group G is called **solvable** if there is at least one $m \in \mathbb{N}$ such that G has a normal tower (2.1) in which $G_m = \{e_G\}$. It is a known fact that given a normal subgroup $H \subset G$ then G is solvable if and only if H and G/H are solvable; in particular, $f : H \to H' = f(H)$ given, $\ker f$ is a solvable normal subgroup and thus $H/\ker f \simeq H'$ is solvable as well, meaning: *solvability is preserved under group epimorphisms.*

Given a finite-dimensional vector space V over an algebraically closed field K, let S be a finitely-generated K-algebra of K-valued functions on V. Two such algebras are:

1. the K-algebra $K[V]$ of polynomial functions on V, i.e. functions of the form $f = P \circ \varphi : V \to K$, $P : K^n \to K$ being a polynomial, $P \in K[x_1, \ldots, x_n]$, and φ being an isomorphism between V and K^n;

2. and the quotient field of $K[V]$, i.e. the K-algebra $K(V)$ of **rational** functions defined on V, i.e. functions of the form $f = F \circ \varphi : V \to K$, $F : K^n \to K$ being a quotient of polynomials, $P(x_1, \ldots, x_n) / Q(x_1, \ldots, x_n)$ with $P, Q \in K[x_1, \ldots, x_n]$, and again φ being an isomorphism between V and K^n.

If $S = K[V]$ it may be easily proven (e.g. [68, Proposition 5.2 (Chapter 10)]) that the sets $\mathcal{Z}(I)$ of zeros of ideals $I \in S$ are *affine varieties* over K ([55, §1.1]) and thus closed sets of a certain topology called the *Zariski topology* ([55, §1.2]). For the remainder of this Section, any reference to topology will be henceforth set exclusively in either the Zariski topology or the one therefrom induced on subsets or cartesian products.

We recall a topological space X is **irreducible** if two non-empty open subsets of X have a non-empty intersection. In the next results, as said in the previous paragraph, subsets $X \subset V$ will be systematically endowed with the subspace topology induced by the Zariski topology of V. It is easy to establish that V is irreducible ([131, Corollary 1.3.8]) and thus:

Lemma 2.1.1. *Any non-empty open set $A \subset V$ is dense in V.* $\square$

2.1.2 Linear algebraic groups and Lie algebras

Linear algebraic groups

Recall an **algebraic group** over K as being an affine algebraic variety over K endowed with a group structure such, that the two maps $\mu : G \times G \to G$, $\iota : G \to G$ defined by $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$ are morphisms of varieties. In particular, a special type of algebraic group is a **linear algebraic group** which is defined as a Zariski closed subgroup of some $\mathrm{GL}(V)$, V being finite-dimensional K-vector space as above. We also recall ([55, §7.4]) a **morphism of algebraic groups** as being a group homomorphism $\phi : G \to G'$ which is also a morphism of varieties; whenever $G' = \mathrm{GL}_n(K)$ we say morphism ϕ is a **(rational) representation**; in light of this, it is usually advisable to view $\mathrm{GL}(V)$ as an algebraic group all its own, specifying its Zariski topology in an unambiguous way by any arbitrary choice of basis for $V \simeq K^n$ since any such choice in K^n corresponds to an inner automorphism $x \mapsto yxy^{-1}$ in $\mathrm{GL}_n(K)$. Since the product topology in $G_1 \times \cdots \times G_n$ is precisely the initial topology with respect to projection maps $\pi_i : G \to G_i$ defined by $\pi_i(g_1, \ldots, g_n) := g_i$, each of these projections will be continuous with respect to the Zariski topology in G. In particular, if $G_1, \ldots, G_n$ are algebraic groups, then for any connected subgroup $H \subset G_1 \times \cdots \times G_n$ each image $\pi_i(H)$, $i = 1, \ldots, n$, is a connected subgroup of G_i with respect to the Zariski topology in G_i.

A representation is called **faithful** if it is injective. Given any representation $\phi : G \to \mathrm{GL}(V)$ of an algebraic group G, the operation

$$G \times V, \qquad (x, \boldsymbol{v}) \mapsto x \cdot \boldsymbol{v} := \phi(x)\, \boldsymbol{v},$$

is clearly a group action of G on V. In this case V is usually called a (rational) G-**module**. For any algebraic group G acting over V, we call $G\boldsymbol{v} = O(\boldsymbol{v}) = \{g \cdot \boldsymbol{v} : g \in G\}$ the G-**orbit** of $\boldsymbol{v} \in E$. G-module V is called **faithful** if $(x, \boldsymbol{v}) \mapsto x \cdot \boldsymbol{v}$ is faithful as a group action, i.e. if ϕ is a faithful representation. Module V is called **irreducible** if it has exactly two submodules: $\{\boldsymbol{0}\}$ and V itself. More generally, a finite-dimensional G-module V is **completely reducible** if for every submodule $V_1 \subset V$ there is another submodule $V_2 \subset V$ such that $V = V_1 \oplus V_2$ or, equivalently, if V is the direct sum of some of its irreducible submodules.

Given an algebraic group G, the **identity component** G^0 of G is the unique (topologically) irreducible component containing e_G. Any algebraic group has a unique largest normal solvable subgroup, which is automatically closed ([55, Corollary 7.4 and Lemma 17.3(c)]). Its identity component is thus the largest connected normal solvable subgroup of G; it is called the **radical** of G and denoted $R(G)$. The subgroup of $R(G)$ consisting of all its **unipotent** elements (i.e., those elements expressible as the sum of the identity and a nilpotent element) is normal in G; it is called the **unipotent radical** ([55, §19.5]) of G, denoted as $R_u(G)$, and may be characterized as the largest closed, connected, normal subgroup formed by unipotent elements of G. If $R(G)$ is trivial and $G \neq \{e\}$ is connected, G is called **semisimple**; this is the case, for instance, for $\mathrm{SL}_n(K)$ ([55, §19.5]). If G is semisimple, then every G-module V is completely reducible. G is furthermore called **simple** if it has no closed connected normal subgroups other than itself and $\{e\}$; $\mathrm{SL}_n(K)$ is again a valid example ([55, §27.5]).

Lie algebras

Everything defined and asserted in this Subsection is found and verified in detail in [19, Chapter 1, from §3 onward], [55, Chapters 9 and 10], [93, Chapters 2, 3 and 4] or [106, Chapters 1 and 3].

A **Lie algebra** over K is a particular kind of algebra over a field; it is defined as a K-vector space $\mathfrak{a}$ together with a bilinear binary operation $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, called the **Lie bracket**, such that $[\boldsymbol{x}, \boldsymbol{x}] = \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathfrak{a}$ and the **Jacobi identity** holds:

$$[\boldsymbol{x}, [\boldsymbol{y}, \boldsymbol{z}]] + [\boldsymbol{y}, [\boldsymbol{z}, \boldsymbol{x}]] + [\boldsymbol{z}, [\boldsymbol{x}, \boldsymbol{y}]], \qquad \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathfrak{a}.$$

Lie subalgebras will be accordingly defined as subspaces of a Lie algebra which are closed under the Lie bracket. An **ideal** of the Lie algebra $\mathfrak{a}$ is a subspace $\mathfrak{h}$ of $\mathfrak{a}$ such that $[\boldsymbol{a}, \boldsymbol{x}] \in \mathfrak{h}$ for all $\boldsymbol{a} \in \mathfrak{g}$ and $\boldsymbol{x} \in \mathfrak{h}$. All ideals are trivially subalgebras, although the converse is not always true.

The **commutator series** of a Lie algebra $\mathfrak{a}$, sometimes also called the **derived series**, is the sequence of subalgebras recursively defined by $\mathfrak{a}^{k+1} := \left[\mathfrak{a}^k, \mathfrak{a}^k\right]$, $k \geq 0$, with $\mathfrak{a}^0 := \mathfrak{a}$. A Lie algebra $\mathfrak{a}$ is **solvable** if its Lie algebra commutator series $\left\{\mathfrak{a}^k\right\}_k$ vanishes for some k. $\mathfrak{a}$ is **simple** if it is not abelian and has no nonzero proper ideals; it is straightforward to prove that *solvable* implies *not simple* for any Lie algebra. A Lie algebra is **semisimple** if it is a direct sum of simple Lie algebras.

Let G be an algebraic group over $\mathbb{C}$; since, being an affine variety, it may be endowed with the usual complex topology as well as with the Zariski topology, it is actually a **Lie group** ([106, §1 (Chapter 1)]), i.e. a group which is also a differential manifold, such that the group operations are compatible with the differential structure. To every Lie group G we can associate a Lie algebra (whose indication in black letters, $\mathfrak{g}$, is usually the only change in notation), in a way completely summarizing the *local* structure of the group; the underlying vector space of $\mathfrak{g}$ is the tangent space of G at the e_G, and we can heuristically characterize all elements of the Lie algebra as elements of G which are "infinitesimally close" to e_G. We will usually call $\mathfrak{g}$ the **Lie algebra** of G, writing it alternatively as $\mathrm{Lie}\,(G)$. See [55, Chapter 1] for concise definitions and properties. It is also reasonably immediate to prove that the Lie algebra of a semisimple algebraic group is semisimple itself.

We have the following result (see also [93, Proposition 2.2]):

Lemma 2.1.2. $\mathfrak{sl}_2\,(\mathbb{C})$*, i.e. the Lie algebra of* $\mathrm{SL}_2\,(\mathbb{C})$*, has no simple subalgebras other than itself.*

Proof. Indeed, the dimension of $\mathfrak{sl}_2\,(\mathbb{C})$ is three, and thus any proper subalgebra of $\mathfrak{sl}_2\,(\mathbb{C})$ should be of dimension smaller than or equal to two; all such subalgebras are solvable ([93, §2.1]), thus not simple. □

2.1.3 Rational invariants

Let $G \subset \mathrm{GL}\,(V)$ be a linear algebraic group. We may define, as is done in [93, §4.2], the action of G on $\mathbb{C}\,[V]$ or $\mathbb{C}\,(V)$:

$$g \cdot f := f \circ g^{-1}, \qquad g \in G, f \in \mathbb{C}\,(V)\,.$$

We define by $\mathbb{C}\,[V]^G$ (resp. $\mathbb{C}\,(V)^G$) the $\mathbb{C}$-algebra of G-invariant elements of $\mathbb{C}\,[V]$ (resp. $\mathbb{C}\,(V)$); hence the denomination **rational invariant** for any $f \in \mathbb{C}\,(V)^G$. We may furthermore assume G is connected, since G has an invariant if, and only if, G^0 has an invariant; this fact, which is a consequence of the finite index of G^0 in G, may be found proven in the first Lemma of [165, Chapter 1]; see also [15].

For any subgroup G of $\mathrm{GL}\,(V)$, e.g. a linear algebraic group acting over V, the set of G-orbits of G is clearly a partition in V. Moreover, given an algebra of $\mathbb{C}$-valued functions S and a function α which is invariant by G, e.g. $S = \mathbb{C}\,(V)$ and $\alpha \in \mathbb{C}\,(V)^G$, the restriction of α to each of the orbits of G is constant. Furthermore, if G has a non-empty open orbit O, then any invariant of G is constant on O and by extension and the density of the latter (due to Lemma 2.1.1) renders α constant on the whole space V. Thus, *algebraic groups with an open non-empty orbit do not have non-trivial rational invariants*, i.e., their only rational invariants are constants. Conversely, we have the following:

Lemma 2.1.3. *Let G be an algebraic subgroup of* $\mathrm{SL}_2\,(\mathbb{C})$ *with no non-trivial rational invariants with the natural representation of G on $\mathbb{C}^2$. Then, G has an open orbit.*

Proof. As is well-known, the only algebraic subgroups of $\mathrm{SL}_2\,(\mathbb{C})$ with no non-trivial rational invariants are

$$H := \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\},$$

and $\mathrm{SL}_2(\mathbb{C})$ itself. H has the open orbit $(\mathbb{C}\setminus\{0\})\times\mathbb{C}$ and $\mathrm{SL}_2(\mathbb{C})$ has the open orbit $\mathbb{C}^2\setminus\{\mathbf{0}\}$. $\square$

In the following three results, m will be assumed to be an arbitrary natural number.

Proposition 2.1.4. *Let $G = G_1 \times G_2 \times \cdots \times G_m$, G_i being an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$ for each $i = 1, ..., m$. If G has a (non-trivial) rational invariant for the natural representation of G on $(\mathbb{C}^2)^m$, then G_i must have a non-trivial rational invariant for at least one i.*

Proof. Assume each G_i has no non-trivial invariants; then, it has an open orbit $\mathcal{O}_i$. Thus G has an open orbit $\mathcal{O}_1 \times \cdots \times \mathcal{O}_m$ and *reductio ad absurdum* yields the result. $\square$

Corollary 2.1.5. *Let $G = G_1 \times G_2 \times \cdots \times G_m$, G_i being an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$ for each $i = 1, ..., m$. If G has a non-trivial rational invariant, then G_i has a commutative identity component G_i^0 for at least one i.*

Proof. In virtue of the classification of the linear algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$ ([93, Proposition 2.2]) we know that an algebraic subgroup H of $\mathrm{SL}_2(\mathbb{C})$ has non-trivial rational invariants if and only if the identity component H^0 is commutative and the result follows from Proposition 2.1.4. $\square$

Corollary 2.1.6. $\mathrm{SL}_2(\mathbb{C})^m$ *has no non-trivial rational invariants.* $\square$

Lemma 2.1.7. *Let $\mathfrak{g}$ be a simple Lie subalgebra of $\bigoplus_{i=1}^n \mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C})^n)$. Then $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$.*

Proof. For each $i = 1, \ldots, n$ let

$$\pi_i|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{sl}_2(\mathbb{C}), \qquad (\boldsymbol{x}_1, ..., \boldsymbol{x}_n) \mapsto \boldsymbol{x}_i,$$

be the restriction of the canonical projection $\pi_i : \bigoplus_{i=1}^n \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{g}$. There is at least one i such that $\pi_i|_{\mathfrak{g}}(\mathfrak{g}) \neq \{\mathbf{0}\}$, since each element $\boldsymbol{x} = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n) \in \mathfrak{g}$ is precisely equal to $(\pi_1(\boldsymbol{x}), ..., \pi_n(\boldsymbol{x}))$, and were $\pi_i|_{\mathfrak{g}} \equiv \{\mathbf{0}\}$, $i = 1, ..., n$, we would then have $\mathfrak{g} = \{\mathbf{0}\}$. Thus, there is at least one i for which $\pi_i|_{\mathfrak{g}}$ has a non-trivial image $\pi_i(\mathfrak{g}) \neq \{\mathbf{0}\}$, itself a subalgebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ which admits no simple subalgebras other than itself, as said in Lemma 2.1.2; this latter fact implies $\pi_i(\mathfrak{g}) = \mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{g}/\ker \pi_i|_{\mathfrak{g}}$. But $\mathfrak{g}$ is simple as well, and thus the ideal $\ker \pi_i|_{\mathfrak{g}}$ must be either $\{\mathbf{0}\}$ or $\mathfrak{g}$. It is clear that $\ker \pi_i|_{\mathfrak{g}} = \{\mathbf{0}\}$, since $\ker \pi_i|_{\mathfrak{g}} = \mathfrak{g}$ would imply $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{g}/\ker \pi_i|_{\mathfrak{g}} = \{\mathbf{0}\}$ which is obviously absurd. $\square$

Lemma 2.1.8. *Let G be an algebraic group and V a G-module such that G is faithfully represented as a subgroup of $\mathrm{SL}_2(\mathbb{C})^n$,*

$$\rho : G \to \mathrm{SL}_2(\mathbb{C})^n.$$

Assume $\pi_i(G) = \mathrm{SL}_2(\mathbb{C})$ for $i = 1, \ldots, n$,

$$\pi_i : \mathrm{SL}_2(\mathbb{C})^n \to \mathrm{SL}_2(\mathbb{C}), \qquad (A_1, \ldots, A_n) \mapsto A_i,$$

being the i-th projection for each $i = 1, \ldots, n$. Then, the Lie algebra $\mathfrak{g}$ of G satisfies $\mathfrak{g} \simeq \bigoplus_{i=1}^m \mathfrak{sl}_2(\mathbb{C})$ for some $m \leq n$.

Proof. The hypotheses imply V is a completely reducible G-module. In order to further prove G semisimple, let us assume the contrary, i.e. that $R(G) \neq \{e\}$; then not every $\pi_i(R(G))$ would be nontrivial since ρ is injective and thus so is $\rho|_{R(G)}$, i.e. $R(G)$ is represented faithfully as a subgroup of $\mathrm{SL}_2(\mathbb{C})^n$: $R(G) \hookrightarrow \pi_1(R(G)) \times \cdots \times \pi_n(R(G)) \subset \mathrm{SL}_2(\mathbb{C})^n$. But this is absurd since $\pi_i(R(G))$ is trivial, $i = 1, \ldots, n$; indeed, each $\pi_i(R(G)) \subset \mathrm{SL}_2(\mathbb{C})$ is a normal, connected, solvable subgroup of a simple algebraic group since π_i is a group epimorphism and $\mathrm{SL}_2(\mathbb{C})$ is simple. Thus, $\pi_i(R(G)) = \{\mathrm{Id}_2\}$ for each $i = 1, \ldots, n$ implying $R(G) = \{e\}$, i.e. G is a *semisimple* algebraic group. Let $\mathfrak{g} := \mathrm{Lie}(G)$ the corresponding semisimple Lie algebra and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ a decomposition in simple algebras. From Lemma 2.1.7, we know

$$\mathfrak{g}_i \simeq \mathfrak{sl}_2(\mathbb{C}), \qquad i = 1, \ldots, m,$$

and thus $\mathfrak{g} \simeq \bigoplus_{i=1}^m \mathfrak{sl}_2(\mathbb{C})$. □

If G is a semisimple algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})^n$, it is in particular a subset of the symplectic group of a symplectic $\mathbb{C}$-vector space $E \simeq \mathbb{C}^{2n}$, since $\mathrm{SL}_2(\mathbb{C})^n \subset \mathrm{Sp}_n(\mathbb{C})$; Lemma 2.1.8 assures $\mathfrak{g} = \mathrm{Lie}(G) \simeq \bigoplus_{i=1}^m \mathfrak{sl}_2(\mathbb{C})$ for some $m \leq n$ and, in virtue of this, we have

$$\mathfrak{g} \simeq \bigoplus_{i=1}^{m} \mathfrak{sl}_2(\mathbb{C}) \subset \bigoplus_{i=1}^{n} \mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{sp}_n(\mathbb{C}) \simeq \left(S^2 E^*, \{\cdot,\cdot\}\right),$$

the latter isomorphism of Lie algebras being proven in [93, Lemma 3,2], E^* being the dual $\mathbb{C}$-space of E, $S^k E^*$ being the **symmetric algebra** on E^* (that is, the ring of homogeneous quadratic Hamiltonian functions defined over E giving rise to linear, constant-coefficient Hamiltonian fields) and $\{\cdot,\cdot\}$ being the *Poisson bracket* introduced, for instance, in Section 2.2.1 below; see [93, §3.1, 3.4] for more details.

We say that a subalgebra $\mathfrak{g} \subset \mathfrak{sp}_n(\mathbb{C}) \simeq (S^2 E,^* \{\cdot,\cdot\})$ has a **rational invariant** $\alpha \in \mathbb{C}(E)$ if $\{\mathfrak{g}, \alpha\} \equiv 0$. The following is straightforward to verify; see for instance [93, §4.2]:

Lemma 2.1.9. *An algebraic group G has a non-trivial rational invariant if, and only if,* $\mathrm{Lie}(G)$ *has a non-trivial rational invariant.* □

So far we have proven the following train of implications:

1. (Lemma 2.1.6) $\mathrm{SL}_2(\mathbb{C})^m$ has no non-trivial rational invariants;

2. therefore, in virtue of Lemma 2.1.9, $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{C})^m) = \bigoplus_{i=1}^m \mathfrak{sl}_2(\mathbb{C})$ has no non-trivial rational invariants;

3. thus, for any linear algebraic group G satisfying the hypotheses of Lemma 2.1.8, and in virtue of the latter, $\mathrm{Lie}(G) = \mathfrak{g} \simeq \bigoplus_{i=1}^m \mathfrak{sl}_2(\mathbb{C})$ has no non-trivial rational invariants;

4. hence, again in virtue of Lemma 2.1.9, G has no non-trivial rational invariants.

In other words: we have just proven the following:

Theorem 2.1.10. *Let $G \subset \mathrm{SL}_2(\mathbb{C})^n$ be an algebraic group such that the projections $\pi_i(G) = \mathrm{SL}_2(\mathbb{C})$, $i = 1, ..., n$. Then, G has no non-trivial rational invariants.* $\square$

Theorem 2.1.10 will be of key importance for the new result (Corollary 2.3.5) proven in Section 2.3.2.

2.2 Notions of integrability

As said in Section 1.1, specialization is the most immediate symptom in the study of integrability of any given system

$$\dot{\boldsymbol{y}} = f(t, \boldsymbol{y}), \qquad \boldsymbol{y} = y_1 \times y_2 \times \cdots \times y_n : \mathbb{C} \to \mathbb{C}^n. \tag{2.2}$$

The two distinct notions described in this section, adapted to two precise types of dynamical systems, do have a common trait, though: the ability to perform **integration by quadratures**, that is, to express the general solution as an "elementary" function of a finite nested sequence of integrals of "elementary" functions, constants of integration being the parameters of the solution manifold. See [113] for a wider outlook on the subject.

2.2.1 Integrability of Hamiltonian systems

Let us restrict our attention to a very special example of such a system as (2.2). Everything explained here can be found in more detail in [13], [15], [75], [89], [132], [145], and especially [18], [65] and [93].

All assertions and definitions in this Section, save for the hypotheses of Theorem 2.2.2, are made in the *complex* setting as done in [93, Chapter 3] and throughout [95]. Similar assertions and definitions adjusted to *real* bundles and fields may be found in [14], [13], [18] and especially [75].

A **symplectic manifold** is a complex manifold of even dimension $2n$ along with a nondegenerate closed 2-form Ω, called the **symplectic form**, whose nondegeneracy allows the definition of a musical isomorphism of vector bundles,

$$\flat : TM \to T^*M, \quad \flat X = \Omega(X).$$

These manifolds arise naturally as phase spaces of the class of differential systems we are now introducing.

A **Hamiltonian vector field** is a field X_H defined on the symplectic manifold M, such that $X_H = \flat^{-1} \cdot dH$ for some function H, usually called the **Hamiltonian**. The differential equation satisfied by the integral curves of a Hamiltonian vector field is called a **Hamiltonian system**; in virtue of *Darboux's theorem* ([18, Theorem 1.1], [93, Theorem 3.1]), it may be written, in canonical local coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ (referred to as **positions** and **momenta**, respectively), in the following form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n; \tag{2.3}$$

one usually calls these (which will stand for (2.2)) **Hamilton's equations** associated to Hamiltonian H. They may also be written as $\dot{\boldsymbol{z}} = X_H(\boldsymbol{z})$, noting $\boldsymbol{z} := (q_1, \ldots, q_n, p_1, \ldots, p_n)$. This context also allows the definition of **canonical transformations**, i.e. changes of the variables $\boldsymbol{z}$ under which the symplectic form remains invariant; *in other words, under which the Hamiltonian form of the equations is maintained for arbitrary Hamiltonians.* Furthermore, the musical isomorphism $\flat$ allows the adjunction of a Poisson algebra structure on M, bearing Poisson brackets $\{f, g\} = \Omega\left(X_f, X_g\right)$ which in canonical coordinates may be expressed as $\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)$. The following holds:

Proposition 2.2.1. *f is a* ***first integral*** *(that is, a function constant over integral curves) of X_H if, and only if, $\{H, f\} = 0$ (i.e. H and f are* ***in involution****, or* ***commute****). In particular, H is always a first integral of X_H.*

Whenever the idiom **additional first integral** appears, it will be referring to one which is independent and in involution with a certain *known* set of $m < n$ first integrals, be it a singleton $F = \{H\}$ as is the case of Hill's Problem (see Section 2.4.2 and Chapter 4), or the set F of $\frac{1}{2}(d+2)(d+1)$ "classical" integrals for the d-dimensional N-Body Problem (see Section 2.4.1 and Chapter 3).

The following result does not merely provide some Hamiltonians a description of their phase spaces; in most cases, it also confers the whole area a precise notion of integrability; for further details and a proof, see [13, Chapter 10: §49 and §50] or [18, Theorem 1.2 and the remainder of §1.4]. Let X_H be an n-degree-of-freedom real Hamiltonian.

Theorem 2.2.2 (Liouville-Arnol'd)**.** *Assume X_H has n functionally independent first integrals $f_1 = H, f_2, \ldots, f_n$ in pairwise involution. Let $\boldsymbol{a} \in \mathbb{R}^n$ and*

$$M(\boldsymbol{a}) = \{\boldsymbol{z} : f_i(\boldsymbol{z}) = a_i,\ i = 1, \ldots, n\}$$

be a non-critical level manifold of $f_1, \ldots, f_n$. Then,

1. *$M(\boldsymbol{a})$ is an invariant manifold of X_H;*

2. *if compact and connected, $M(\boldsymbol{a})$ is diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and in a neighborhood of the former there exists a coordinate system $(\boldsymbol{I}, \boldsymbol{\phi}) \in \mathbb{R}^n \times \mathbb{T}^n$ in which* (2.3) *read*

$$\dot{I}_i = 0, \quad \dot{\phi}_i = \omega_i, \qquad i = 1, \ldots, n,$$

with $\omega_i = \omega_i(\boldsymbol{I}),\ i = 1, \ldots, n$. In particular, X_H ***can be integrated by quadratures****.* $\square$

Directly after the sufficient condition provided by Theorem 2.2.2,

Definition 2.2.3. *We call system* (2.3) ***integrable in the sense of Liouville-Arnol'd****,* ***completely integrable*** *or simply* ***integrable****, and extend this definition to X_H and H, if* (2.3) *has n functionally independent integrals $f_1 = H, f_2, \ldots, f_n$ in pairwise involution. $\{f_1, \ldots, f_n\}$ is usually called a* ***complete*** *set of independent first integrals.*

We can generalize this definition by allowing a lower cardinality for the set of additional integrals:

Definition 2.2.4. *We call the Hamiltonian* ***partially integrable*** *if there is a set of* $0 < l < n$ *additional first integrals in pairwise involution.*

Obviously, denying Definition 2.2.4 for a given value $0 < l < n$ (as will be the case in part of Chapter 3 for $l = 1$) implies denying Definition 2.2.3. This fact also plays a pivotal role in Subsection 2.3.2 below.

Given a Hamiltonian X_H, there is a number of ways of searching for additional first integrals, although none of them works for all cases – see [51] for more details. One of these ways is using Theorem 2.2.2 directly, i.e. looking for solutions f to the partial differential equation $\{H, f\} = 0$. For exceptional examples in which this method works, see for instance [100] and [101], both owing to the basic work [116] about generalized Noether symmetries. One may also pursue the so-called *integrability in the sense of Hamilton-Jacobi*, i.e. the possibility of finding some explicit canonical variables s_j, r_j, $j = 1, \ldots, n$ separating the Hamilton-Jacobi equation for the action S (see [13, Chapter 10] or [112, Chapter 3]). See also [5] for extensive information on algebraic integrability, in turn related to embeddings of abelian varieties in an affine space through a reedition of ideas by Kowalevskaia and Painlevé.

Remarks 2.2.5.

1. It is explicitly assumed that the first integrals sought after, both in Theorem 2.2.2 and in Definitions 2.2.3 and 2.2.4, are defined *globally*; that is, in no way are we referring to the local integrals existing trivially in virtue of Cauchy's Theorem.

2. Although we restricted everything to $\mathbb{R}$, Hamiltonian formulation may also be defined in the complex setting by allowing t and $\boldsymbol{z}$ to be complex-valued and functions and vector fields to be analytical or meromorphic. The only nuisance to some purposes, though, is the absence of a complex analogue to Theorem 2.2.2 except for special cases (see [93]). The usual procedure is to work with complex meromorphic Hamiltonians which restrict to real for real dependent and independent variables, observing Definitions 2.2.3 and 2.2.4 on the real system and *then* complexifying all variables.

3. From now on, and in tune with what has been said in item 2, whenever we refer to Hamiltonian integrability we will refer to **meromorphic** integrability: additional first integrals, whether in Theorem 2.2.3 or 2.2.4, will be assumed to be *meromorphic* along a subset of a complex manifold. Given a domain Ω in $\mathbb{C}^n$ or any n-dimensional complex manifold, and complex-analytic subset of dimension $n-1$ (or empty) $P \subset \Omega$, we recall a function f defined on $\Omega \backslash P$ is **meromorphic** if for every $\boldsymbol{p} \in P$ there is a neighborhood $U \subset \Omega$ of $\boldsymbol{p}$ and functions ϕ, ψ holomorphic on U without common non-invertible factors in the ring $\mathcal{O}(U)$ of holomorphic functions on U, such that $f \equiv \phi/\psi$ on $U \setminus P$. See also [48, Chapter 8, p. 246] for a precise definition in the context of sheaf theory.

2.2.2 Integrability of linear differential systems

The concept of *integrability* for *linear homogeneous* differential equations is conventionally limited to the possibility of finding their general solution in terms of algebraic functions, integrals and exponentials of known functions or any finite combination of all three. This second notion is naturally inscribed in *differential Galois theory* as will be seen in Definition 2.2.14 and Theorem 2.2.15. Every single fact stated here is described in detail in references [93, Chapter 2], [95, Section 3] and [144, Chapter 1] and, to a lesser degree, Sections 2 and 3 in [15]; Chapters 1 through 6 in [78] may also be useful.

Definition 2.2.6. *Let K be a field. A* ***derivation*** *on K is an additive map $\partial : K \to K$ satisfying the* Leibnitz rule $\partial(ab) = \partial(a)b + a\partial(b)$, $a, b \in K$. *A* ***differential field*** *is a pair (K, ∂_K) consisting of a field and a derivation on it.*

Definition 2.2.7. *An* ***extension of differential fields****, usually noted $L \mid K$, is an inclusion $L \supset K$ such that $\partial_L|_K \equiv \partial_K$.*

(K, ∂_K) given, we henceforth note $\partial = \partial_K$ unless necessary, and use this notation for elements of K^n extending the derivation entrywise. However, we will avoid the notation so frequent in most texts on Galois differential theory $a' = \partial(a)$ so as to be consistent with what was said in Section 1.4.

Definition 2.2.8. *The* ***constants*** *of a differential field (K, ∂) are the elements of the subfield* $\mathrm{Const}(K) := \ker \partial$ *of K.*

All fields and extensions will be assumed to be differential from this point on. We assume characteristic zero for every field considered. The set of all **K-automorphisms** of any differential extension $L \mid K$, (i.e., field isomorphisms $\sigma : L \to L$ such that $\sigma|_K \equiv \mathrm{Id}_K$ and $\partial \circ \sigma \equiv \sigma \circ \partial$) is a group under map composition and will be denoted by $\mathrm{Aut}_K(L)$. Given any $m \in \mathbb{N}$, and using the propagation of morphism axioms of any $\sigma \in \mathrm{Aut}_K(L)$ to elements of $\mathcal{M}_m(K)$,

$$(\sigma a_{i,j})_{1\le i,j\le m}(\sigma b_{i,j})_{1\le i,j\le m} = \left(\sum_{r=1}^{m} \sigma(a_{i,r})\sigma(b_{r,j})\right)_{1\le i,j\le m} = \left(\sigma \sum_{r=1}^{m} a_{i,r}b_{r,j}\right)_{1\le i,j\le m},$$

we will indulge in as many abuses of notation as necessary when extending σ entrywise to any $m \times m$ matrix.

Given a linear homogeneous differential system

$$\partial \boldsymbol{y} = A\boldsymbol{y}, \qquad A \in \mathcal{M}_n(K), \tag{2.4}$$

and an extension $E \mid K$ containing a set V of solutions of (2.4), there is always a minimal differential subfield $L \subset E$ containing both K and the entries of the elements of V; we write $L = K(V)$ and say L is **generated over K as a differential field by the entries of elements of V and using** (2.4). Since (2.4) is linear and homogeneous, V is a $\mathrm{Const}(L)$-vector space of dimension at most n. $\mathrm{Aut}_K(L)$ preserves V and acts on it as a group of linear transformations over $\mathrm{Const}(K)$, and if $\mathrm{Const}(L) = \mathrm{Const}(K)$ the restriction of $\mathrm{Aut}_K(L)$ to V gives a faithful representation $\mathrm{Aut}_K(L) \to \mathrm{GL}(V)$. V owes its relevance to those situations in which it is precisely defined as the maximal set of linearly independent solutions of (2.4), thus establishing the differential analogue of a Galois extension; such an analogue corresponds to the case $\dim_{\mathrm{Const}(L)}(V) = n$ and actually matches the situation in which no new constants are added to K:

Definition 2.2.9. *$L \mid K$ is a **Picard-Vessiot (P–V) extension** for* (2.4) *if*

1. $\mathrm{Const}(L) = \mathrm{Const}(K)$;
2. *there exists a fundamental matrix $\Phi \in \mathrm{GL}_n(L)$ for the equation; and*
3. *L is generated over K as a differential field by the entries of Φ and using* (2.4).

Given a P–V extension $L \mid K$ for (2.4) and an intermediate extension $L \supset L_1 \supset K$ then $L \mid L_1$ is also a P–V extension for some linear ordinary differential system over L_1. We are calling $L \mid K$ a **Picard-Vessiot extension** if it is P–V for some linear ordinary differential system over K; an intrinsic definition may indeed be made, regardless of the equation. For the sake of simplicity and concretion *we are henceforth assuming all fields considered have $\mathbb{C}$ as field of constants.* This assumption also assures existence and uniqueness of P–V extensions.

An essential property of P–V extensions is **normality**:

Lemma 2.2.10. *For any $a \in L \setminus K$, there is a differential K-automorphism σ of L such that $\sigma(a) \neq a$.*

Definition 2.2.11. *If $L \mid K$ is a* P-V *extension for* (2.4)*, then $\mathrm{Aut}_K(L)$ will be denoted $\mathrm{Gal}(L \mid K)$ and called the **Galois differential group of** $L \mid K$ (or of* (2.4)*).*

The Galois differential group of an equation (2.4) is a linear algebraic group; indeed, given a fundamental matrix $\Phi \in \mathrm{GL}_n(L)$, $\sigma(\Phi)$ is also a fundamental matrix and hence $\sigma(\Phi) = \Phi R(\sigma)$ with $R(\sigma) \in \mathrm{GL}_n(\mathbb{C})$, which yields an n-dimensional faithful representation

$$\rho : \mathrm{Gal}(L \mid K) \to \mathrm{GL}_n(\mathbb{C}), \qquad \sigma \mapsto R(\sigma); \tag{2.5}$$

this renders $\mathrm{Gal}(L \mid K)$ a linear group. For a proof of its being also Zariski closed, i.e. a linear algebraic group, see [144, Theorem 1.27]. Furthermore, the *monodromy group* of an equation (2.4), attained through analytical continuation of solutions, is a (generally not Zariski closed) subgroup of the differential Galois group of the corresponding P–V extension. Whenever G is the differential Galois group of some P–V extension, we are identifying elements σ of G with the corresponding matrices $R(\sigma)$ defining representation ρ in (2.5). In other words, we will be dealing indistinctively with the *linear algebraic* group G and the *matrix* group $\rho(G)$.

Remark 2.2.12. Let G be the Galois differential group of the juxtaposition of uncoupled linear differential systems,

$$\partial \boldsymbol{y} = \mathrm{diag}(A_1, A_2, \ldots, A_m)\, \boldsymbol{y}, \qquad A_i \in \mathcal{M}_{n_i}(K), \quad i = 1, \ldots, m \tag{2.6}$$

each subsystem $\partial \mathbf{y}_i = A_i \mathbf{y}_i$ having Galois differential group G_i for $i = 1, \ldots, m$. Then, G is a linear algebraic subgroup of the direct product $G_1 \times \cdots \times G_m$, as may be easily established from the propagation of morphism axioms to matrix blocks (hinted at right after the above definition of K-automorphisms) and the fact that block-diagonal differential system (2.6) admits identically block-diagonal fundamental matrices $\Phi = \operatorname{diag}(\Phi_1, \ldots, \Phi_m)$ and thus just as identically block-diagonal matrix representations (2.5) of the elements of G; it is also straightforward to further prove that if $\pi_1, \ldots, \pi_m$ are the usual projections of $G_1 \times \cdots \times G_m$, then $\pi_i(G) \simeq G_i$ for each $i = 1, \ldots, m$; see [93, Chapter 2] for details as written in a synthetic, coordinate-free formulation.

We now state the so-called *Fundamental Theorem of differential Galois theory*.

Theorem 2.2.13. *Let $L \mid K$ be a Picard-Vessiot extension with common field of constants $\mathbb{C}$, and let $G = \operatorname{Gal}(L \mid K)$, $\mathcal{S}$ the set of closed subgroups of G and $\mathcal{L}$ the set of differential subfields of L. Define*

$$\alpha : \mathcal{S} \to \mathcal{L}, \qquad \alpha(H) = L^H,$$

(L^H being the subfield of L formed by H-invariant elements); and

$$\beta : \mathcal{L} \to \mathcal{S}, \qquad \beta(L_1) = \operatorname{Gal}(L \mid L_1),$$

$\operatorname{Gal}(L \mid L_1)$ *being the subgroup of G of L_1-linear differential automorphisms of L. Then,*

1. *α i β are mutual inverses;*
2. *the following are equivalent,*
   (a) *$H \in \mathcal{S}$ is a* normal *subgroup of G;*
   (b) *$L_1 := \alpha(H) = L^H$ is a* P–V *extension of K;*

   and in such case $\operatorname{Gal}(L \mid L_1) = H$ *and* $\operatorname{Gal}(L_1 \mid K) \simeq G/H$.

As foretold at the start of this subsection, we are now introducing the strict definition of what is to be called an integrable linear differential equation; it is precisely one whose P–V extension falls into the following category:

Definition 2.2.14. *Let K be a differential field. $L \mid K$ is called a* ***Liouville extension*** *if no new constants are added and there exists a tower of extensions*

$$K = L_0 \subset L_1 \subset \cdots \subset L_n = L \tag{2.7}$$

such that for $i = 1, \ldots, n$, $L_i = L_{i-1}(t_i)$ and one of the following holds: either

1. *$\partial t_i \in L_{i-1}$; we say t_i is an* ***integral*** *(of an element of L_{i-1}); or*
2. *$t_i \neq 0$ and $(\partial t_i)/t_i \in L_{i-1}$; in such case, t_i is an* ***exponential*** *(of an integral of an element of L_{i-1}); or*
3. *t_i is algebraic over L_{i-1}.*

If L is a Liouville extension of K and all t_i are integrals (resp. exponentials), we say L is an ***extension by integrals*** *(resp.* ***exponentials****) of K.*

What comes next, finally, is the fundamental characterization of Liouville extensions.

Theorem 2.2.15. *Let L be a Picard-Vessiot extension of K with Galois differential group G. Then, the following are equivalent*

1. *L is a Liouville extension of K;*

2. *the identity component G^0 of G is solvable.*

Moreover, in either case, tower (2.7) *may be chosen so as to render the first extension $K = L_0 \subset L_1$ algebraic.*

Remarks 2.2.16. Regarding P–V extensions defined by integrals or algebraic elements:

1. Any quadrature $\int f$ of an element $f \in K$ is either again in K or **transcendental** (i.e. solution to no polynomial equation with its coefficients in K). Thus, $K\left(\int f\right)$ is either trivial or transcendental.

2. If a Picard-Vessiot extension is defined only by quadrature adjunction,

$$L = K\left(\int f_1, \int f_2, \ldots, \int f_k\right),$$

where $f_1, f_2, \ldots, f_k \in K$, its Galois group is equal to $(\mathbb{C}_+)^s$, $s \leq k$. Here $\mathbb{C}_+$ denotes the additive group of $\mathbb{C}$. Indeed, $\mathrm{Gal}(L \mid K)$ acts on quadratures in an additive manner and the only algebraic subgroups of $\mathbb{C}_+$ are itself and the trivial group. See, for instance, [144, Exercise 1.35(1)].

3. From Theorem 2.2.15 we observe that the denomination "integrability" includes possibility of resolution with algebraic functions. Moreover, generalizing the last sentence in Theorem 2.2.15, all algebraic elements may be inserted in a single extension: the first one, $K = L_0 \subset L_1$. In such case, in virtue of Theorem 2.2.13, $\mathrm{Gal}(L_1 \mid L_0) \simeq G/G^0$, a finite group, where $G = \mathrm{Gal}(L \mid K)$ and G^0 is the identity component of G.

2.3 Morales-Ramis theory

At this point, we need to rely on Sections 2.2.1 and 2.2.2 despite having an initially *real* Hamiltonian; such a reliance is not a problem at all, both because of what was said in Remark 2.2.5(2) and because of the degree of generality 2.2.2 was set upon.

2.3.1 The general theory

Let Γ be an integral curve of a complex Hamiltonian X_H; Γ is a Riemann surface and may be locally parametrized by $\widehat{\boldsymbol{z}}(t)$, $t \in I$ where I is a disc in the complex plane. We may now complete Γ to a new Riemann surface $\overline{\Gamma}$, as detailed in [95, §2.1] (see also [93, §2.3]), by adding equilibrium points, singularities of the vector field and possible points at infinity.

The main Theorem in this Subsection connects the two notions of solvability listed in 2.2.1 and 2.2.2, namely as applied to a *Hamiltonian* X_H and the *linear* variational equations, $\dot{\boldsymbol{\xi}} = X'_H(\widehat{\boldsymbol{z}}(t))\,\boldsymbol{\xi}$ along $\overline{\Gamma}$, respectively. Actually, the Theorem is the ad-hoc implementation of the following heuristic idea: *if a Hamiltonian is integrable, then its variational equations must also be integrable.*

The base field for the P–V extension (i.e. the one containing the coefficients of the variational equations) is the field $\mathcal{M}\left(\overline{\Gamma}\right)$ of *meromorphic* functions defined on the integral curve of X_H.

Theorem 2.3.1 (Morales-Ramis)**.** *Assume there exist n independent meromorphic first integrals in involution for X_H in a neighborhood of an integral curve $\overline{\Gamma}$. Then, the identity component G^0 of the Galois group of the variational equations along $\overline{\Gamma}$ is commutative.*

Proof. See [95, Corollary 8] (or [93, Theorem 4.1]). □

Remarks 2.3.2.

1. Theorem 2.3.1 pivots on a very crucial result ([95, Lemma 9], see also [93, Lemma 4.6]) which is nothing but the ad-hoc implementation of the following premise: every meromorphic first integral of a given dynamical system (2.2), *whether or not Hamiltonian*, yields a non-trivial rational invariant of the Galois group of the variational equations along any integral curve of (2.2). It is the combination of this result with *Ziglin's Lemma* ([93, Lemma 4.3], [95, Lemma 6], [162, p. 184 of the English edition]) as applied to the *junior parts* ([93, §4.2]) of the n first integrals, that builds up the proof of Theorem 2.3.1 by obtaining a Poisson algebra which is invariant by the action of the Galois group G, and thus annihilated by $\mathfrak{g} = \mathrm{Lie}\,(G)$ (recall Lemma 2.1.9).

2. The framework leading to the Morales-Ramis Theorem is a successful step towards generalizing the Ziglin outlook mentioned in Section 1.2. *Ziglin's Theorem* (not to be confused with Ziglin's Lemma) is based on the key idea that m independent meromorphic integrals of X_H must induce m independent rational invariants for the monodromy group: see [162, Theorem 2], [163] and [15, §1]. Although Ziglin did not assume complete integrability in his result (the hypothesis of involutiveness being missing), such an assumption is naturally fulfilled in his theorem if $n = 2$: in this case, Ziglin's Theorem may be obtained as a corollary to Theorem 2.3.1 and part of the results ([95, §4] i.e. [93, §4.1]) leading to it, as done in [95, Corollary 9] or [93, Corollary 4.6]. As a matter of fact, Ziglin's *general* Theorem may also be obtained as a consequence of the Morales-Ramis framework (specifically

from [95, §4 and Lemma 9] i.e. [93, §4.1 and Lemma 4.6]) as stated in [95, Theorem 10] and proven in [31].

2.3.2 Special Morales-Ramis theory: homogeneous potentials

Prior results

This Subsection is nothing but a reenaction of [93, §5.1.2], [95, §7] and [96, §1–3]. Assume X_H is given by a **classical** n-degree-of-freedom Hamiltonian,

$$H(\boldsymbol{q},\boldsymbol{p}) = T + V = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + V(\boldsymbol{q}), \tag{2.8}$$

$V(\boldsymbol{q})$ **being homogeneous of degree** $k \in \mathbb{Z}$. Hamiltonians such as these are by no means generical. The fact V is homogeneous implies the observance of the principle of mechanical similarity ([67]): the orbits on any integral manifold can be rescaled to one of a finite set of such manifolds (typically corresponding to energy values $-1, 0, 1$), i.e. freedom of choice of the energy constant is only countered by discrete gaps in the dynamics generated by V; indeed, transformation $\boldsymbol{q} \mapsto \alpha^2\boldsymbol{q}$, $\boldsymbol{p} \mapsto \alpha^k\boldsymbol{p}$, with possible change in time $t \mapsto it$, yields the new energy $\tilde{H} = (\pm)\,\alpha^{2k}H$ for any given α. In order to see further uses of this fact, as well as generalizations to not necessarily finite values of the energy, see [57], [126] and [158].

X_H defined as above, every vector function $\widehat{\boldsymbol{z}}(t) = \left(\phi(t)\,\boldsymbol{c}, \dot{\phi}(t)\,\boldsymbol{c}\right)$ such that $\ddot{\phi} + \phi^{k-1} = 0$ and $\boldsymbol{c} \in \mathbb{C}^n$ is a solution of $\boldsymbol{c} = V'(\boldsymbol{c})$, is a solution of Hamilton's equations for H, as may be easily proven using the fact that the n entries in vector $V'(\boldsymbol{q})$ are homogeneous polynomials of degree $k-1$. Such a vector $\boldsymbol{c}$ is usually called a **Darboux point** of potential V ([77]).

Writing infinitesimal variations on the canonical variables as $\delta\boldsymbol{q} = \tilde{\boldsymbol{\xi}}$ and $\delta\boldsymbol{p} = \tilde{\boldsymbol{\eta}}$, the equations satisfied by these are

$$\frac{d}{dt}\tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\eta}}, \quad \frac{d}{dt}\tilde{\boldsymbol{\eta}} = -\phi(t)^{k-2}\,V''(\boldsymbol{c})\,\tilde{\boldsymbol{\xi}},$$

or equivalently $\frac{d^2}{dt^2}\tilde{\boldsymbol{\xi}} = -\phi(t)^{k-2}\,V''(\boldsymbol{c})\,\tilde{\boldsymbol{\xi}}$. Assume $V''(\boldsymbol{c})$ is diagonalizable; this is the case, for instance, if $\boldsymbol{c} \in \mathbb{R}^n$. Then, any transformation $\tilde{\boldsymbol{\xi}} = U\boldsymbol{\xi}$, $\tilde{\boldsymbol{\eta}} = U\boldsymbol{\eta}$ with an adequate $U \in \mathrm{GL}_n(\mathbb{C})$ transforms the system, written as

$$\frac{d}{dt}\boldsymbol{\xi} = \boldsymbol{\eta}, \quad \frac{d}{dt}\boldsymbol{\eta} = -\phi(t)^{k-2}\left[U^{-1}V''(\boldsymbol{c})\,U\right]\boldsymbol{\xi},$$

into

$$\frac{d^2}{dt^2}\boldsymbol{\xi} = -\phi(t)^{k-2}\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}\boldsymbol{\xi},$$

where $\{\lambda_1, \ldots, \lambda_n\} = \mathrm{Spec}\,V''(\boldsymbol{c})$.

In other words, along $\widehat{\boldsymbol{z}}$, variational equations may be split into a direct sum $\bigoplus_{i=1}^{n} \mathrm{VE}_i$ of n uncoupled equations, each of the form

$$\frac{d^2\xi_i}{dt^2} + \lambda_i \left[\phi\left(t\right)\right]^{k-2} \xi_i = 0, \qquad i = 1, \ldots, n, \tag{2.9}$$

Furthermore,

$$V''\left(\boldsymbol{c}\right)\boldsymbol{c} = \left(k-1\right)\boldsymbol{c}, \tag{2.10}$$

is easily established as a special case of Euler's Theorem; thus, we may set $\lambda_1 = k - 1$; the corresponding variational equation, VE_1, is trivially integrable. The remaining $n-1$ eigenvalues $\lambda_2, \ldots, \lambda_n$ may be enough to determine the non-integrability de X_H in this special case of [95, Corollary 8]; indeed, (2.9) following [156], the finite branched covering map $\overline{\Gamma} \to \mathbb{P}^1$ is considered, given by $t \mapsto x := \phi\left(t\right)^k$, where $\overline{\Gamma}$ is the compact hyperelliptic Riemann surface of the hyperelliptic curve $w^2 = \frac{2}{k}\left(1 - \phi^k\right)$ (see [93, §4.1.1)], [95, §4.1]). With this covering in consideration, (2.9) are finally written as a system of *hypergeometric differential equations* ([58], [150]) in the new independent variable x, each of them of the form:

$$x\left(1-x\right)\frac{d^2\xi_i}{dx^2} + \left(\frac{k-1}{k} - \frac{3k-2}{2k}x\right)\frac{d\xi_i}{dx} + \frac{\lambda_i}{2k}\xi_i = 0. \tag{2.11}$$

Kimura's table ([62]), in turn owing to Schwarz's ([117]), provides a concise list of those cases in which hypergeometric equations are integrable by quadratures, *i.e. in which the Galois group of* (2.11) *has a solvable identity component.* Both tables were based on properties of the monodromy group ([58]). Adapting both tables to the new hypothesis, namely that the Galois group of each of the variational equations must have a *commutative* identity component, yields the following fundamental result:

Theorem 2.3.3. [95, Theorem 3] (see also [93, Theorem 5.1]) *Assume X_H, given by* (2.8)*, is completely integrable with meromorphic first integrals; let $\boldsymbol{c} \in \mathbb{C}^n$ a solution to $V'\left(\boldsymbol{c}\right) = \boldsymbol{c}$ and assume $V''\left(\boldsymbol{c}\right)$ is diagonalizable; then, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $V''\left(\boldsymbol{c}\right)$ and we define $\lambda_1 = k-1$, each pair $\left(k, \lambda_i\right),\ i = 2, \ldots, n$*

matches one of the following items (p being an arbitrary integer):

TABLE 1

	k	λ		k	λ
1	k	$p+p\left(p-1\right)\frac{k}{2}$	**10**	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
2	2	*arbitrary* $z\in\mathbb{C}$	**11**	3	$-\frac{1}{24}+\frac{1}{24}\left(2+6p\right)^2$
3	-2	*arbitrary* $z\in\mathbb{C}$	**12**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$
4	-5	$\frac{49}{40}-\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$	**13**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$
5	-5	$\frac{49}{40}-\frac{1}{40}\left(4+10p\right)^2$	**14**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
6	-4	$\frac{9}{8}-\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$	**15**	4	$-\frac{1}{8}+\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$
7	-3	$\frac{25}{24}-\frac{1}{24}\left(2+6p\right)^2$	**16**	5	$-\frac{9}{40}+\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$
8	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$	**17**	5	$-\frac{9}{40}+\frac{1}{40}\left(4+10p\right)^2$
9	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$	**18**	k	$\frac{1}{2}\left(\frac{k-1}{k}+p\left(p+1\right)k\right)$

(2.12)

Remarks 2.3.4.

1. Theorem 2.3.3 strengthens what was done by H. Yoshida for $n=2$ from reference [156] onward; indeed, his result, which is not generalizable to $n>2$ in a simple, straightforward manner, pivoted on the use of Ziglin's Theorem in which, as said in Remark 2.3.2(2), complete integrability may only be assumed if $n=2$. Hence, Yoshida's line of study only allowed one non-trivial integer λ_2; besides, it ended up in a wider set of non-integrability regions for λ_2, each with a non-zero Lebesgue measure. Since Yoshida's result is a corollary to Theorem 2.3.3 for $n=2$ ([96, p. 6], see also [93, p. 105]), and since the latter works for arbitrary $n\geq 2$ and restricts the non-integrability regions much further (namely, to discrete sets rather than infinite unions of intervals), Table 1 appears, in expectation for advances concerning the higher variational equations (see Subsection 5.3.1), as the strongest current tool for testing the non-integrability of Hamiltonians of the form (2.8) from the Galoisian viewpoint.

2. It is not difficult to see that, for any given $i=2,\ldots,n$, if λ_i does not appear in Table (2.12), then the Galois group G_i of equation (2.9) is precisely $\mathrm{SL}_2\left(\mathbb{C}\right)$; indeed, the fact λ_i falls out of the Table guarantees the non-solvability of the identity component $\widehat{G}_i^0$ of the Galois group $\widehat{G}_i$ of the hypergeometric equation (2.11). It now only takes recalling the result [95, Theorem 5] (see also [93, Theorem 2.5]), according to which the identity component of the Galois group remains invariant under finite branched coverings. Since $t\mapsto\phi\left(t\right)^k$ is precisely one such covering, G_i^0 is non-commutative. The fact $G_i\subset\mathrm{SL}_2\left(\mathbb{C}\right)$ (due to the absence of $\frac{d\xi_i}{dt}$ in (2.9), see e.g. [93, §2.2]) obviously implies $G_i^0\subset\mathrm{SL}_2\left(\mathbb{C}\right)$ and the fact G_i^0 is not solvable renders $G_i^0=G_i=\mathrm{SL}_2\left(\mathbb{C}\right)$ in virtue of the classification of subgroups of $\mathrm{SL}_2\left(\mathbb{C}\right)$ given in [93, Proposition 2.2] and the analysis done thereof in the last paragraph of [93, §2.1].

Existence of a single additional integral

If X_H has m first integrals $f_1 = H, \dots, f_m$ in pairwise involution and independent over a neighborhood of the integral curve $\overline{\Gamma}$ defined by $\phi(t)\,\boldsymbol{c}$, the *normal variational equations* ([95, §4.3], see also [93, §4.1.3]) are equal to $n-m$ of the initial variational equations; reordering indexes if needed, let us write them as $\mathrm{VE}_{m+1}, \dots, \mathrm{VE}_n$ with corresponding differential Galois groups $G_{m+1}, \dots, G_n$ and let us write the eigenvalues corresponding to $\mathrm{VE}_{m+1}, \dots, \mathrm{VE}_n$ (each of them of the form (2.9)) as $\lambda_1 = k-1, \dots, \lambda_m$ and assume they are all in Table (2.12). In virtue of what was stated in Remark 2.6, the differential Galois group $G_{\mathrm{NVE}} = \mathrm{Gal}\left(\bigoplus_{i=m+1}^{n} \mathrm{VE}_i\right)$ of the normal variational equations satisfies $G_{\mathrm{NVE}} \subset G_{m+1} \times \cdots \times G_n$ and, defining $\pi_{m+1}, \dots, \pi_n$ as the usual projections of $G_{m+1} \times \cdots \times G_n$, $\pi_i\left(G_{\mathrm{NVE}}\right) \simeq G_i$ for $i = m+1, \dots, n$.

Assume none of $\lambda_{m+1}, \dots, \lambda_n$ belongs to Table (2.12); then, in virtue of Remark 2.3.4(2), we have $G_i \simeq \mathrm{SL}_2(\mathbb{C})$ for all $i = m+1, \dots, n$. If there is an additional first integral f which is independent with the set $\{f_1, \dots, f_m\}$, then by Ziglin's Lemma ([93, Lemma 4.3], [95, Lemma 6], Remark 2.3.2(2)) the normal variational equations must have a non-trivial rational first integral $\tilde{f}$ with coefficients in $\mathcal{M}\left(\overline{\Gamma}\right)$ and thus, in virtue of the fundamental lemma referred to in Remark 2.3.2(1) ([95, Lemma 9], see also [93, Lemma 4.6]), G_{NVE} must have a non-trivial rational invariant. However, inclusion $G_{\mathrm{NVE}} \subset G_{m+1} \times \cdots \times G_n$, isomorphisms $G_i \simeq \mathrm{SL}_2(\mathbb{C}), i \geq m+1$ and Remark 2.6 yield a faithful representation of G_{NVE} in $\mathrm{SL}_2(\mathbb{C})^{n-m}$ such that $\pi_i\left(G_{\mathrm{NVE}}\right) \simeq \mathrm{SL}_2(\mathbb{C})$ for each $i = m+1, \dots, n$; thus, Theorem 2.1.10 asserts G_{NVE} has no non-trivial invariant and we arrive at a contradiction.

We may therefore proceed by induction on m; for $m=1$ we have $\{f_1\} = \{H\}$ and eigenvalue $\lambda_1 = k-1$ (linked to f_1 through (2.10)) belongs to item **1** in Table (2.12). For higher m, what has been said in the previous two paragraphs ends the proof for the following:

Corollary 2.3.5. *Let X_H be a Hamiltonian field given by* (2.8). *Let $f_1, \dots, f_m$ be first integrals of X_H in pairwise involution and independent over $\overline{\Gamma}$. Then,*

1. *m of the eigenvalues, say $\lambda_1, \dots, \lambda_m$, belong to Table* 1 *in* (2.12).
2. *If there is a single first integral f independent with $\{f_1, \dots, f_m\}$ on a neighborhood of $\overline{\Gamma}$, then at least one of the eigenvalues $\lambda_{m+1}, \dots, \lambda_n$ belongs to Table* 1. □

See [77] for a parallel attempt at the same goal as that of Corollary 2.3.5. Chapter 3 shows a set of applications of both Theorems 2.12 and 2.3.5. See a further application of both results in [94] for a specific two-degree-of-freedom Hamiltonian.

2.4 Basics in Celestial Mechanics

Even though attempts at explaining the motion of planets have been made since the very dawn of mankind, the origin of Celestial Mechanics as presently known is set in 1687 with the publication of I. Newton's *Principia* ([105]), the coinage of the actual term *mécanique céleste* corresponding to P.-S. Laplace ([69], [70]) and first applied to the specific branch of astronomy studying the motion of celestial bodies under the influence of gravity.

Celestial Mechanics has been, is and will be, arguably for a long time, a palaestra for both astronomers and mathematicians, the tools used ranging from numerical analysis to dynamical systems theory and including stochastic calculus, perturbation theory, topology and, as will be the case here, differential algebra and algebraic geometry. Most of the questions raised nowadays in the study of celestial bodies are essentially related to the Solar System, e.g. orbits of comets and asteroids (especially *NEO*, i.e. Near-Earth Objects), the motion of Jovian moons, Saturn's rings, artificial satellites, accurate ephemeris calculations, exoplanetary systems, etc.: see for instance [44], [45], [46], [49], [137].

2.4.1 The N-Body Problem

Definitions

Let $d, N \geq 2$ be two positive integers. The **(General d-dimensional) N-Body Problem** is the model describing the motion of N mutually interacting point-masses in an Euclidean d-space led solely by their mutual gravitational attraction. It is determined by the initial-value problem given by the $2N$ initial conditions $\mathbf{x}_1(t_0), \dots, \mathbf{x}_N(t_0) \in \mathbb{R}^d$ and $\dot{\mathbf{x}}_1(t_0) \dots, \dot{\mathbf{x}}_N(t_0) \in \mathbb{R}^d$, such that $\mathbf{x}_j(t_0) \neq \mathbf{x}_k(t_0)$ if $j \neq k$, and the system of Nd scalar second-order differential equations

$$m_i \ddot{\mathbf{x}}_i = -G \sum_{k \neq i}^{N} \frac{m_i m_k}{\left\| \mathbf{x}_i - \mathbf{x}_k \right\|^3} \left(\mathbf{x}_i - \mathbf{x}_k \right), \qquad i = 1, \dots, N, \tag{2.13}$$

where, for each $i = 1, \dots, N$, $\mathbf{x}_i \in \mathbb{R}^d$ is a d-dimensional vector function of the time variable t describing the position of a body and m_i is the mass of the body with position $\boldsymbol{q}_i$. G, the gravitational constant, may and will be set equal to one from now on by an appropriate choice of units.

Hamiltonian formulation ensues in a most natural way; defining

$$M = \operatorname{diag}\left(m_1, \dots, m_1, \cdots, m_N, \dots, m_N\right) \in \mathcal{M}_{Nd}\left(\mathbb{R}\right),$$

and assembling the coordinates of our phase space among the Nd-dimensional vectors

$$\boldsymbol{x}(t) = \left(\mathbf{x}_i(t)\right)_{i=1,\dots,N}, \qquad \boldsymbol{y}(t) = \left(\mathbf{y}_i(t)\right)_{i=1,\dots,N} := \left(m_i \dot{\mathbf{x}}_i(t)\right)_{i=1,\dots,N}$$

of **positions** and **momenta**, respectively, the equations of motion may now be expressed as

$$\dot{\boldsymbol{x}} = M^{-1} \boldsymbol{y}, \qquad \dot{\boldsymbol{y}} = -\nabla U_{N,d}(\boldsymbol{x}), \tag{2.14}$$

where $U_{N,d}(\boldsymbol{x}) := -\sum_{1\leq i<k\leq N} \frac{m_i m_k}{\|\mathbf{x}_i - \mathbf{x}_k\|}$ is the **potential function** of the gravitational system. System (2.14) is the set of Hamilton's equations (2.3) linked to the Hamiltonian

$$H_{N,d}(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}\boldsymbol{y}^T M^{-1}\boldsymbol{y} + U_{N,d}(\boldsymbol{x}) . \tag{2.15}$$

Most of the bibliography on the subject deals with either the **planar** ($d = 2$) or **spatial** ($d = 3$) N-Body Problem since raising the dimension of the ambient space deprives the problem of most of its physical significance; it must be said, nevertheless, that further research has been attempted assuming d is an arbitrary integer – needless to say, the reader can already infer that such an assumption is by no means a symptom of confidence in our knowledge of the planar and spatial problems, as may be ascertained in the following chapter.

General solution for the spatial Problem

Defining

$$\Delta_{i,j} := \left\{\boldsymbol{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{Nd} : \mathbf{x}_i = \mathbf{x}_j\right\}, \qquad i \neq j,$$

and $\Delta := \bigcup_{1\leq i<j\leq N} \Delta_{i,j}$, Hamiltonian (2.15) and equations (2.13) (that is, (2.14)) are analytically defined on $\mathbb{R}^{Nd} \setminus \Delta$. The global solution sought after is defined in its maximal interval $(t_-(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0), t_+(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0))$; if a global solution is defined in its maximal interval $(t_-, t_+) = (t_-(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0), t_+(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0))$ and $t_+ < +\infty$ (resp. $t_- > -\infty$), then $\lim_{t\to t_\pm} U_{N,d}(\boldsymbol{x}(t)) = \infty$; this is the case, for instance, if $\boldsymbol{x}(t) \to \Delta$ as $t \to t_\pm$, i.e. in the presence of **collisions**, or values of the time $t^* \in \mathbb{R}$ for which there is a subset $I \subset \{1, \ldots, N\}$, of cardinality greater than one, such that $\lim_{t\to t^*} \mathbf{x}_i(t) = \lim_{t\to t^*} \mathbf{x}_j(t)$ for all $i, j \in I$.

For $N = 2$, the Problem was completely solved by J. Bernoulli in 1710 (see [16], [152]). As was said in Section 1.2, for $N, d = 3$ the open question posed by Mittag-Leffler and Weierstrass was finally solved, except for some exceptional, albeit relevant, cases, by K. F. Sundman. We are now detailing both this and Wang's result for $N \geq 4$ further. Following any introductory text on the subject (e.g. [147, pp. 74–75]), there are three steps implicit in both Sundman's and Wang's aims:

Step 1. *determining if* $(t_-, t_+) = \mathbb{R}$;

Step 2. in either case there exists an open neighborhood $U \subset \mathbb{C}$ of (t_-, t_+) (which may be chosen to be an infinite strip $\{|\mathrm{Im}z| < \omega\}$) such that $(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is analytical in U; *the second step is finding* U.

Step 3. *finding a conformal mapping* $t \mapsto \sigma$ *(which is easily proven to exist) which maps* U *onto the unit disk*; expanding $\phi = (\boldsymbol{x}(t), \boldsymbol{y}(t))$ in the resulting new complex variable, ϕ will converge on the unit disk. This is the series expansion sought after both by Sundman and Wang.

Sundman, as well as others before him, was acquainted with the following:

Lemma 2.4.1. *Facts concerning the N-Body Problem:*

1. *all solutions stopping at total collision have angular momentum (see the definition below)* $\mathbf{I}_A = \mathbf{0}_3$*;*

2. *binary collision is always an algebraic branch point;*

3. *for* $N = 3$*, solutions such that* $(t_-, t_+) \subsetneq \mathbb{R}$ *only stop at collision.* $\square$

Given two solutions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2$ having intervals of definition $(t_1, t_2), (t_2, t_3) \subsetneq \mathbb{R}$, respectively, the classical process of analytical regularization consists, when possible, in finding new phase variables $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \Phi(\boldsymbol{x}, \boldsymbol{y})$ and a new time variable $t = T(\tau)$ such that the (2.14) as expressed in those new variables has a solution $\boldsymbol{\xi} = \boldsymbol{\xi}(\tau), \boldsymbol{\eta} = \boldsymbol{\eta}(\tau)$ which exists in (τ_1, τ_3), where $\tau_1 < \tau_2 < \tau_3$ are such that $T((\tau_1, \tau_2)) = (t_1, t_2)$, $T((\tau_2, \tau_3)) = (t_2, t_3)$, $(\boldsymbol{\xi}, \boldsymbol{\eta})|_{(\tau_1, \tau_2)} \equiv \Phi \circ \boldsymbol{\phi}_1 \circ T$ and $(\boldsymbol{\xi}, \boldsymbol{\eta})|_{(\tau_2, \tau_3)} \equiv \Phi \circ \boldsymbol{\phi}_2 \circ T$. For different types of regularization with more geometrical and physical content, see survey [82].

For $N = 3$ and $\mathbf{I}_A \neq \mathbf{0}_3$, a consequence of Lemma 2.4.1(1 and 3) is that the only possible singularities are caused by binary collisions; in that case, furthermore, Lemma 2.4.1(2) makes it possible to extend the solution through binary collision by regularization. Moreover, with respect to a regularized coordinate system, every solution is defined on all of $\mathbb{R}$. Thus, $\mathbf{I}_A \neq \mathbf{0}_3$ assures Step 1 is fulfilled. Step 2 depends on estimating how far the regularized solution is from the singular set Δ (that is, from *triple* collision); this was precisely the second part of Sundman's approach: finding ω (depending on $\mathbf{I}_A$) such that there are no complex singularities in a strip $U = U(\mathbf{I}_A) = \{|\mathrm{Im}\tau| < \omega\}$ centered around the real axis. Step 3 is then obtained directly:

$$t \mapsto \tau := \int_{t_0}^{t} (U_{N,3}(\boldsymbol{x}) + 1) \mapsto \sigma := \frac{e^{\frac{\pi\tau}{2\omega}} - 1}{e^{\frac{\pi\tau}{2\omega}} + 1}.$$

Hence,

Theorem 2.4.2 (Sundman's Theorem)**.** *For any initial condition* $(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ *such that* $\mathbf{I}_A \neq \mathbf{0}_3$*, there is a new variable* τ *explicitly defined, and a constant* $\omega > 0$ *explicitly given with respect to* $\boldsymbol{x}_0$*,* $\boldsymbol{y}_0$ *and the masses, such that the time* $t = T(\tau)$ *and the positions* $\boldsymbol{x}$ *of the three bodies, as functions of* τ*, are analytical on* $|\mathrm{Im}\, s| < \omega$*. Besides,* $T(-\infty, \infty) = (-\infty, \infty)$ *and there is an explicitly given conformal mapping* $\tau \mapsto \sigma$ *rendering the transformed series a convergent one in the variable* $\{|\sigma| < 1\}$*.*

See [119, §11]; see also the works by Sundman: the original development of the result, i.e. [134] and [135], and the compilation thereof in [136].

As seen above, the key idea in Sundman's work was to regularize the singularities of collisions of two bodies. Such regularization is unfeasible for collisions of larger amounts of bodies, save for special cases; indeed, C. L. Siegel proved that most of the solutions may not be extended analytically beyond collision due to the presence of irrational powers in their series expansion ([119] and [118], see also [86]). Furthermore, the problem for higher N is further aggravated by the more complicated structure of singularities, since not all of them are due to collisions if $N \geq 4$. This was hinted at by P. Painlevé in his now famous conjecture ([108]): namely, that for each $N \geq 4$ the N-Body Problem admits non-collision singularities; von Zeipel proved ([87]) that such singularities require the motion to be unbounded. With this necessary condition in mind, Xia ([153]) and Gerver ([43]) found a proof of Painlevé's conjecture for the spatial Five-Body and a $3N$-Body Problem for a large N, respectively. For a special case of the collinear four-body problem, i.e. four point masses on a straight line under certain restrictions on the masses, [83] proved the existence of unbounded solution in finite time, even if said proof required an infinite number of regularized binary collisions and thus did not prove Painlevé's Conjecture.

Hence, when Q. D. Wang obtained a result analogue to Sundman's for the general N-Body Problem the detection of solutions leading to singularities (including collisions), and thus any attempt at performing Step 1, was completely left off. Instead, he performed Step 2 directly by "blowing up" the time interval to $\mathbb{R}$; this he did with a coordinate transform which is nothing but a modification of McGehee's transform introduced in [86]: defining h as the energy level (i.e. the value of $H_{N,3}$) for the solution $(\boldsymbol{x}(t), \boldsymbol{y}(t))$, and introducing variable u as defined by $(2U_{N,3}(\boldsymbol{x}) + h)^{-1}$ if $h > 0$ and $(2U_{N,3}(\boldsymbol{x}))^{-1}$ if $h \leq 0$, equations (2.13) or (2.14) were then written in terms of

$$\boldsymbol{F} = u^{-1}\boldsymbol{x}, \qquad \boldsymbol{G} = u^{1/2}\boldsymbol{y}.$$

Introducing the new time variable τ such that $\frac{d\tau}{dt} = u^{-3/2}$, Wang proved in [147, Theorems 1 and 2, Lemmae 1] that $\tau\left(\left(t_-\left(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0\right), t_+\left(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0\right)\right)\right) = (-\infty, \infty)$ and that the following holds:

Theorem 2.4.3. *For any given initial condition $(t_0, \boldsymbol{x}_0, \boldsymbol{y}_0)$ of the spatial N-Body Problem, there are constants $A, B > 0$ explicitly given with respect to $\boldsymbol{x}_0, \boldsymbol{y}_0, m_1, \ldots, m_N$ such that $\boldsymbol{F}, \boldsymbol{G}, u, t$ are analytic functions of τ on*

$$U := \left\{|\operatorname{Im}\tau| < Ae^{-B|\operatorname{Re}\tau|}\right\}.$$

As in Sundman's case, Step 3 is immediate to perform from this point on, thus allowing for a corresponding convergent series defined on the unit disk. For more details on Theorem 2.4.3 see also [148, Theorems 1 and 2, Proposition 1], or the first formulation done in [146].

Remarks 2.4.4.

1. This result not only extended Sundman's Theorem 2.4.2 by covering the case of zero angular momentum; it was also more useful in that constants A and B are far easier to estimate than the constant ω in Sundman's Theorem.

2. Although Theorems 2.4.2 and 2.4.3 yield methods for obtaining the terms of a convergent series expression of the global solution, they are in both cases far too slowly convergent and thus of no practical use – not even for numerical computations. This was already said by Wang himself in [147, p. 87] and will be recalled at the beginning of Chapter 3.

Known first integrals

Transformations of the form $\mathbf{x} \mapsto T_{Q,\mathbf{v},\mathbf{w},t}(\mathbf{x}) := Q\mathbf{x} + \mathbf{v} + t\mathbf{w}$, formed by a rotation $Q \in \mathrm{O}_{dN}(\mathbb{R})$ and a translation linear with respect to time, are easily proven to be symmetries of (2.13). $\mathbf{v}$ represents constant translation, and $t\mathbf{w}$ represents the change to a moving frame which moves with a constant velocity $\mathbf{w}$. Since symmetries come paired with first integrals (see [116]), the first step is looking for conserved quantities linked to symmetries as basic as $T_{Q,\mathbf{v},\mathbf{w},t}$. The vector $\mathbf{c}_G(t) := \frac{1}{m}\sum_{i=1}^N m_i \mathbf{x}_i(t)$, where $m = \sum_{i=1}^N m_i$, is the **center of mass** of the configuration $\boldsymbol{x}(t)$. It corresponds to a configuration whose movement is rectilinear and uniform:

$$\ddot{\mathbf{c}}_G = \frac{1}{m}\sum_{i=1}^N m_i \ddot{\mathbf{x}}_i = \frac{1}{m}\sum_{i=1}^N \sum_{j\neq i} \frac{m_i m_j}{\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|^3}(\mathbf{x}_j(t) - \mathbf{x}_i(t)) = 0,$$

due to the symmetry of the expression in the second addition. Thus,

$$\mathbf{c}_G(t) = \mathbf{c}_1 t + \mathbf{c}_2, \qquad \mathbf{c}_i \in \mathbb{R}^d. \tag{2.16}$$

In particular $\mathbf{I}_L := m\mathbf{c}_1 = \sum_{i=1}^N m_i \dot{\mathbf{x}}_i$, usually called the **linear momentum**, is a vector of conserved quantities of the system; the ones associated to translation, that is. The conserved quantities linked to rotation all lie in the **angular momentum** $\mathbf{I}_A = (I_{A,k,l})_{1\le k<l\le d} \in \mathbb{R}^{d(d-1)/2}$,

$$I_{A,k,l} = \sum_{i=1}^N x_{d(i-1)+k}\dot{x}_{d(i-1)+l} - x_{d(i-1)+l}\dot{x}_{d(i-1)+k}, \qquad 1 \le k < l \le d,$$

obviously summing up to a single scalar quantity if $d = 2$: $I_A := \sum_{i=1}^N m_i \mathbf{x}_i \wedge \dot{\mathbf{x}}_i$. In view of (2.16), $\mathbf{c}_G$ can always be assumed fixed at the origin since $T_{\mathrm{Id},-\mathbf{c}_1 t,-\mathbf{c}_2,t}$ is a symmetry for (2.13); except for Definition 2.4.7, we will assume $\mathbf{c}_G = \mathbf{0}$ from now on.

Let us define the scalar product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle := (M\boldsymbol{x})^T \boldsymbol{y}$ in $\mathbb{R}^{Nd}$. The **moment of inertia** for a given solution $\boldsymbol{x}(t)$ of (2.13) is defined as $I(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{x}\rangle$. This is *not* a first integral of the problem but will be useful in the next Subsection.

All in all, the N-body problem has $\frac{1}{2}(d+2)(d+1)$ (so-called *classical*) first integrals (see [149]):

1. $2d$ for the invariance of the linear momentum $\mathbf{I}_L$, i.e. for the uniform linear motion of the center of mass;

2. $d(d-1)/2$ for the invariance of the angular momentum $\mathbf{I}_A$;

3. one for the invariance of the Hamiltonian $H_{N,d}$.

That makes 6 for the planar problem and 10 for the spatial problem. Bruns' theorem, given in 1887, asserts these are the only first integrals algebraic with respect to phase variables for the Three-Body Problem:

Theorem 2.4.5 (Bruns' Theorem, [27])**.** *Every first integral of the spatial Three-Body Problem which is algebraic with respect to positions, momenta and time is an algebraic function of the classical ten first integrals.*

An attempt at extending this result was done by P. Painlevé, namely at proving that any integral depending algebraically on the moments $\mathbf{p}_1, \ldots, \mathbf{p}_N$, regardless of how it depends on the positions $\mathbf{q}_1, \ldots, \mathbf{q}_N$, is a function of the classical integrals. The proof of this assertion, written in [108], is wrong, though; see also [49]. The best generalization of Theorem 2.4.5 known to date is the following:

Theorem 2.4.6 (Julliard's Theorem, [59])**.** *In the d-dimensional N-body problem with $1 \leq d \leq N$, every first integral which is algebraic with respect to positions, momenta and time is an algebraic function of the classical $\frac{1}{2}(d+2)(d+1)$ integrals.*

Our obvious aim, both in Chapter 3 and in the future, is to take the thesis in Theorem 2.4.6 to its most extreme generalization.

Central configurations of the N-body problem

Definition and examples Despite the general lack of faith in finding *simple* closed-form solutions for the N-body problem ([35]), there are special solutions whose orbits allow for a complete qualitative study without having to resort only to the infinite series given in [136], [146] and [147]. Such solutions, called **homographic**, are those preserving the initial figure formed by the bodies, except for homothecies and rotations:

Definition 2.4.7. *A solution $\boldsymbol{x}(t)$ of the N-body problem is called* ***homographic*** *if there are functions $r : J \subset \mathbb{R} \to \mathbb{R}$ and $\Phi : J \subset \mathbb{R} \to \mathrm{SO}_d(\mathbb{R})$ defined on an open interval $J \subset \mathbb{R}$, such that*

$$\mathbf{x}_i(t) - \mathbf{c}_G(t) = r(t)\,\Phi(t)\,(\mathbf{x}_i(t_0) - \mathbf{c}_G(t_0)),$$

Using the homogeneity of $U_{N,d}(\boldsymbol{x})$ and $I(\boldsymbol{x})$ of degree -1 and 2, respectively, the Euler relation for homogeneous functions and the method of Lagrange multipliers, it may be easily proven that initial conditions $\boldsymbol{x}$ of homographic solutions satisfy system

$$U'_{N_d}(\boldsymbol{x}) = \lambda M \boldsymbol{x}, \tag{2.17}$$

where $\lambda > 0$; actually $\lambda = U_{N,d}(\boldsymbol{x})/I(\boldsymbol{x})$. If the bodies are released with zero initial velocity, these initial conditions give rise to simple, explicit **homothetical** solutions of the N-Body Problem (i.e. solutions showing homothetical collapse to the origin).

Definition 2.4.8. *An initial configuration $\boldsymbol{x}(t_0)$ of a homographic solution (i.e. a solution to* (2.17)*) will be called a* ***central configuration****.*

Remark 2.4.9. λ may be set equal to one; indeed, the -2 -homogeneity of $U'_{N,d}$ assures us $U'_{N,d}(\lambda^{\alpha}\boldsymbol{x}) = \lambda^{-2\alpha}U'_{N,d}(\boldsymbol{x})$; thus, assuming $U'_{N,d}(\boldsymbol{x}) = \lambda M\boldsymbol{x}$, defining $\tilde{\boldsymbol{x}} = \lambda\boldsymbol{x}$ and asking for $U'_{N,d}(\tilde{\boldsymbol{x}}) = M\tilde{\boldsymbol{x}}$ to hold, we obtain $\alpha = -1$.

The above remark implies that the set of solutions to (2.17) is independent of the value of λ and thus has the same cardinal as the set of solutions to

$U'(\boldsymbol{x}) = -\lambda^* M\boldsymbol{x}$ for any other $\lambda^* > 0$. Measuring such a cardinal is a fundamental problem in Celestial Mechanics; in order for this problem to make sense, the usual procedure is studying the quotient modulo symmetries of rotation $\mathrm{O}_d(\mathbb{R})$, translation ($\mathbb{R}^d$) and homothecy ($\mathbb{R} \setminus 0$), i.e. counting classes of central configurations modulo these symmetries. For planar central configurations, the set of mutual distances between the bodies may occasionally prove an adequate coordinate system for this quotient space, albeit a rather redundant one since its cardinality is equal to $\binom{N}{2}$ and a set of merely $2N-4$ coordinates suffices in the planar case. See [10].

Examples 2.4.10.

1. Regardless of m_1, m_2, m_3, there exists a central configuration of the Three-Body Problem, called a **Lagrange (triangular) configuration**, consisting of an equilateral triangle whose vertexes are the point-masses (see [66] or Remark 3.2.1 and Section 3.3.1 below).

2. Generalizing Example 1 above, the regular d-simplex is a central configuration of the d-dimensional Problem for any $d \geq 2$ and $N = d+1$ (see [114]): for instance, Lagrange's triangular configuration if $d = 2$ or a regular tetrahedron if $d = 3$ ([73]).

3. Again regardless of m_1, m_2, m_3, each ordering of three bodies arranged on a straight line forms a central configuration, called an **Euler (collinear) configuration** (see [39]).

4. Yet again we may generalize Example 3: for each $N \geq 3$ and each set of positive values $m_1, \ldots, m_N$, N bodies with masses $m_1, \ldots, m_N$ arranged in a straight line lead to $N!/2$ central configurations – one for each ordering of the point-masses; we call these the **Moulton (or Euler-Moulton) configurations** (see [103]).

5. Whenever the masses are equal, regular N-polygons with the point-masses at the vertexes are central configurations, see [30], [107], [111], [154] or Remark 3.2.1 and Lemma 3.3.2. Conversely, for $N > 3$, regular polygons are central configurations if and only if the masses are equal (again [30], [107], [111] or [154]).

6. Whenever N of the masses are equal and an additional mass is allowed into the system, regular N-polygons with the bodies of equal masses at the vertexes and the body corresponding to the isolate mass m_{N+1} placed at the center of the polygon (i.e. the center of mass) are central configurations, see Remark 3.2.1 and Lemma 5.2.7.

7. Depending on N and on the specific masses, other special configurations may be proven to exist. See for instance [40] and [107] for the so-called *pyramidal configurations*, and [50] and [121] for some insight and new results on the case $N = 4$.

Remark 2.4.11. Inasmuch as in Examples 1, 2, 5 and 6, the exact coordinates of the solution in Example 3 may be found explicitly, albeit in a less straightforward way: indeed, for an adequate mutual-distance quotient parameter ρ, the so-called **Euler quintic** holds along any collinear three-body solution:

$$\begin{aligned}(m_2+m_3)+(2m_2+3m_3)\,\rho+(3m_3+m_2)\,\rho^2-(3m_1+m_2)\,\rho^3\\ -(3m_1+2m_2)\,\rho^4-(m_1+m_2)\,\rho^5=0\end{aligned} \tag{2.18}$$

Equation (2.18) may be solved explicitly by transforming P to Bring reduced form $P_B(\rho)=\rho^5-\rho-\beta$ by means of three Tschirnhaus transformations and expressing the roots of $P_B(\rho)$ in terms of generalized hypergeometric functions ${}_4F_3$, although such calculus is not necessary for our study and will be skipped; see [138].

For more information on central configurations, see [91].

Importance of central configurations in Celestial Mechanics There are some facts proving the importance of research in central configurations for the N-body problem:

1. Besides the orbits of the two-body problem, the only known explicit solutions for the N-body problem are homographic orbits, i.e. those having as an initial condition a central configuration.

2. Thanks to Sundman ([136]), we know all orbits beginning or ending at a total collision are asymptotic to a homothetic movement, i.e. the configuration formed by the bodies tends to a central configuration.

3. All changes in the topology of the integral varieties $V_{H,\mathbf{I}_A}$ corresponding to the energy H and the angular momentum $\mathbf{I}_A$ are due to central configurations ([6], [29], [85], [128]). However, the concise description of these varieties with prescribed values of $H,\mathbf{I}_A$ is not even concluded for $N=3$ ([120, §2], [85]).

4. The sixth problem proposed by S. Smale in [129] is whether or not, given $m_1,\ldots,m_N$, the number of classes of central configurations is finite. His program pivoted precisely on the topology of the $V_{H,\mathbf{I}_A}$ so as to pursue topological stability; namely pivoting on the impossibility of transition between connected components. This is useful if $N=3$, since there exist ranges for which $V_{H,\mathbf{I}_A}$ has some connected component projecting on a bounded set of the $\boldsymbol{x}$-space. For $N\geq 4$, however, there is always only one connected component, and it has unbounded $\boldsymbol{x}$-projection: see [120, §2] and, especially, [122].

2.4.2 Hill's Lunar Problem

Hill's Problem (*HP*), usually dubbed *Lunar* as an homage to its earliest motivation, or *planar* in order to distinguish it from its own extension to $\mathbb{R}^3$, is a model originally based on the Moon's motion under the joint influence of Earth and Sun ([52], [53], [54]). A first simplification of the General Three-Body Problem consists in assuming the Moon's mass is negligible and the *primaries* (Earth and Sun) move in circular orbits around their common barycenter; we then have a Hamiltonian system called the *(Planar Circular) Restricted Three-Body Problem* (*RTBP*, see [137]) which is nowadays a fairly approximate model for celestial couples other than Sun-Earth, such as Earth-Moon, Sun-Jupiter, etc. with the negligible mass being, for instance, an Apollo spacecraft, thus making it a dynamical system of paramount importance in some space missions. Let P_1 and P_2 be the primaries, assume $M_{P_1} < M_{P_2}$, let $\mu = M_{P_1}/\left(M_{P_1} + M_{P_2}\right)$ be the (adequately non-binding) choice for a normalized mass unit, and in comes a (so-called **synodical**) rotating coordinate frame whose first axis is spanned by the primaries. Following the previous mass unit choice by a suitable choice of length and time units leads to the best-known equations for the *RTBP*:

$$\left.\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega_\xi, \\ \ddot{\eta} + 2\dot{\xi} &= \Omega_\eta, \end{aligned}\right\} \tag{2.19}$$

where $\Omega(\xi,\eta) := \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}(\xi^2+\eta^2)$ is the *gravitational* plus the *centrifugal* potential and $r_1^2 := (\xi-\mu)^2 + \eta^2$ and $r_2^2 := (\xi - \mu + 1)^2 + \eta^2$ are the respective squared distances between each of the primaries and the massless particle. Setting the lesser primary P_1 as the origin of coordinates and scaling length by $\mu^{1/3}$, *HP* is now defined by taking $\mu \to 0$ in the resulting equations. Thus, the *RTBP* can be written as an $O\left(\mu^{1/3}\right)$ perturbation of *HP* in a neighborhood of the Earth of size $O\left(\mu^{1/3}\right)$ in the initial variables of the *RTBP*. In other words, heuristically speaking, *HP* is the outcome of placing the more massive primary at an infinite distance of the barycenter, yet at the same time endowing it with a "suitably" infinite mass in order to assure both a parallel force field and a finite though considerable influence on the lesser primary. It must be said, though, that this ad-hoc description does not open the door to perturbation theory nor make our problem amenable to the results of K.A.M. theory introduced in Section 1.2: *HP does not* depend on any parameter other than the energy and is therefore far enough, globally, from any known integrable system.

The simplest expression known to date amounts to the polynomial of degree six (2.22) shown below. Everything said from this point owes to [120], [137], and especially, [126]. After following the steps listed above (including the limit-taking in μ), we obtain the best-known equations of *HP*,

$$\left.\begin{aligned} \ddot{\bar{q}}_1 &= -\frac{\bar{q}_1}{\left(\bar{q}_1^2+\bar{q}_2^2\right)^{3/2}} + 2\dot{\bar{q}}_2 + 3\bar{q}_1, \\ \ddot{\bar{q}}_2 &= -\frac{\bar{q}_2}{\left(\bar{q}_1^2+\bar{q}_2^2\right)^{3/2}} - 2\dot{\bar{q}}_1, \end{aligned}\right\} \tag{2.20}$$

The *HP Hamiltonian* for the above equations (2.20) is

$$H_{HP}\left(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}\right) := \bar{p}^2 - \bar{q}^{-1} + \left(\bar{p}_1\bar{q}_2 - \bar{p}_2\bar{q}_1\right) + \frac{1}{2}\left(\bar{q}_2^2 - 2\bar{q}_1^2\right). \tag{2.21}$$

The steps performed in [126] from this point on are a Levi-Civita regularization, a formulation of the problem in the extended phase space, a generalized canonical transformation and a scaling (all four explained in detail in [132]). The final expression is

$$\mathcal{H}_H(\boldsymbol{Q}, \boldsymbol{P}) = H_2 + H_4 + H_6, \tag{2.22}$$

a sum of homogeneous polynomials of degrees 2, 4 and 6, respectively:

$$H_2 = P^2/2 + Q^2/2, \quad H_4 = -2Q^2(P_2 Q_1 - P_1 Q_2), \quad H_6 = -4Q^2(Q_1^4 - 4Q_1^2 Q_2^2 + Q_2^4).$$

Our main statements and proofs in Chapter 4 (that is, sections 4.2 through 4.4) will rely on Hamiltonian (2.22).

Chapter 3

The meromorphic non-integrability of some N-Body Problems

3.1 Introduction

In view of the results by Sundman and Wang mentioned in Subsections 1.3.1 and 2.4.1, i.e. Theorems 2.4.2 and 2.4.3, a case could be made in favor of the Problem's "solvability". But solutions in the form of slowly-converging series not only have low-to-nil numerical utility: neither do they predict the existence of periodic, quasi-periodic, unbounded or colliding orbits, in turn opening further problems whose settlement requires more information than is currently available: stability, central configurations, variational problems, properties of the eight solution, existence of choreographies, Saari's conjecture, etc. An adequate set of conserved quantities could provide such information, but finding such set stands as an obstacle all its own since only the comparatively few *classical* first integrals are known (Section 2.4.1), and any other algebraic first integral would necessarily be an algebraic function of those classical in virtue of Theorems 2.4.5 and 2.4.6. Furthermore, the non-existence of algebraic additional first integrals is no obstacle to the existence to those of a more general class, e.g. analytical or meromorphic.

Section 3.2 exposes the actual goals and paves the way towards them; specifically, Subsection 3.2.1 adapts the contents of Section 2.4.1, states the main results and assesses their degree of novelty separately; whereas Subsection 3.2.2 provides with additional information on the N-Body Problem (and more specifically on its potential, and on consequences of what was presented in Subsection 2.4.1) which, while unnecessary for the requisites of Section 2.4.1, will be extremely useful for the proofs we introduce in the present Chapter. These proofs are finally written in Section 3.3.

3.2 Preliminaries

3.2.1 Statement of the main results

Symplectic change $\boldsymbol{x} = M^{-1/2}\boldsymbol{q}$, $\boldsymbol{y} = M^{1/2}\boldsymbol{p}$ renders $H_{N,d}$ a classical Hamiltonian $\mathcal{H}_{N,d} = \frac{1}{2}p^2 + V_{N,d}(\boldsymbol{q})$ with a potential which is homogeneous of degree -1:

$$V_{N,d}(\boldsymbol{q}) := -\sum_{1\le i<j\le N} \frac{(m_i m_j)^{3/2}}{\left\|\sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j\right\|}. \tag{3.1}$$

In virtue of Theorem 2.3.3, performing the following two steps would prove $\mathcal{H}_{N,d}$ not meromorphically integrable:

Step I either explicitly finding or proving the existence of an adequate constant vector $\boldsymbol{c} \in \mathbb{C}^{2N}$ such that

$$V'_{N,d}(\boldsymbol{c}) = \boldsymbol{c}; \tag{3.2}$$

Assume $V''_{N,d}(\boldsymbol{c})$ is diagonalizable.

Step II proving that at least one of the eigenvalues of $V''_{N,d}(\boldsymbol{c})$ does not belong to the set given by items **1** and **18** in Table (2.12), which happens to be a set of integers:

$$S := \left\{-\frac{p(p-3)}{2} : p \in \mathbb{Z}\right\} = \left\{-\frac{(p+2)(p-1)}{2} : p \in \mathbb{Z}\right\} \subset \mathbb{Z}, \tag{3.3}$$

whose symmetry allows for the assumption $p > 1$; the size of the consecutive gaps in this discrete set is strictly increasing, as is seen in its first elements: $\{1, 0, -2, -5, -9, -14, -20, -27, -35, \ldots\}$.

In virtue of Corollary 2.3.5, isolating an adequate set of eigenvalues and performing the following third step will be enough to discard the existence of even a *single* additional meromorphic integral; in other words, we would prove a generalized version of Theorems 2.4.5 and 2.4.6:

Step III proving that, except for a set $\tilde{S}$ of notable eigenvalues, there is no other eigenvalue of $V''_{N,d}(\mathbf{c})$ in S.

As asserted in Theorem 3.2.2, *this last step has been attained for* $N = 3$; see Subsection 3.3.1 for a proof. See also Chapter 5 for an extended comment regarding higher values of N.

Remark 3.2.1. *Solving* (3.2) *for the general case appears as anything but trivial.* In virtue of Remark 2.4.9, real vector solutions to $V'_{N,d}(\boldsymbol{c}) = \boldsymbol{c}$ correspond exactly to homothetical central configurations, since $M^{1/2}V'_{N,d}(\boldsymbol{q}) = U'_{N,d}\left(M^{-1/2}\boldsymbol{q}\right)$ and thus $U'_{N,d}(\boldsymbol{x}) = M\boldsymbol{x}$ (for $\boldsymbol{x} = M^{-1/2}\boldsymbol{q}$) is equivalent to

$$V'_{N,d}(\boldsymbol{q}) = M^{-1/2}MM^{-1/2}\boldsymbol{q} = \boldsymbol{q}.$$

Were solving (3.2) a straightforward task, so would be computing central configurations; in view of the egregious amount of research involving or needed for the latter, even in special cases, e.g. the lines of study hinted at in [7], [8], [9], [10], [11], [37], [40], [41], [64], [84], [91], [114], or [152], such a premise is arguable at best.

We are proving the following two main results:

Theorem 3.2.2. *For every $d \geq 2$, there is no additional meromorphic first integral for $X_{\mathcal{H}_{3,d}}$ with arbitrary positive masses which is independent with the classical first integrals.*

Theorem 3.2.3. *Let $X_{\tilde{\mathcal{H}}_{N,d}}$ stand for any d-dimensional* equal-mass *N-Body Problem:*

1. *There is no meromorphic additional first integral for the* planar *Problem $X_{\tilde{\mathcal{H}}_{N,2}}$ if $N = 3, 4, 5, 6$.*
2. *For $N \geq 3$ and $d \geq 2$, $X_{\tilde{\mathcal{H}}_{N,d}}$ is not meromorphically integrable in the sense of Liouville.*

Consider any triangular homographic solution (Example 2.4.10(1)) corresponding to energy level zero; such a solution is usually called the *parabolic* Lagrangian solution since the orbit of each of the point-masses is precisely a parabola. By means of Ziglin's Theorem, A. V. Tsygvintsev not only proved there is no complete set of meromorphic first integrals for the *planar* Three-Body Problem in a neighborhood of a parabolic Lagrangian solution; he further transited from this non-integrability proof to one of the absence of a *single* additional integral, except for the three special cases shown in (3.16) below. See [139, Theorems 2 and 4], [140, Theorem 1.1 and Corollary 1.2], [141, Theorems 6.1 and 6.3], [142, Theorem 1.1], [143, Theorem 4.1]. In [164, Section 3.1], S. L. Ziglin himself established a non-integrability proof provided (m_1, m_2, m_3) belongs to the intersection of some neighborhood of $\{m_1 = m_2\} \cup \{m_1 = m_3\} \cup \{m_2 = m_3\}$ in $\mathbb{R}^3_+$ with the set of deleted lines $\bigcup_{k \neq i} \{m_k/m_i \neq 11/12, 1/4, 1/24\}$; this he did exploiting the proximity of the particular solutions with respect to a certain collinear configuration. Although by no means proven valid for a wide set of values of the masses, Ziglin's result had the advantage of considering general dimension d for the point masses. D. Boucher and J.-A. Weil also proved the planar Three-Body Problem non-integrable in [22, Theorem 9] (see also [23, Theorem 2] and [21, Theorem 3]) by using a criterion of their own (e.g. [21, Theorem 2], [22, Theorem 8], [23, Criterion 1]) devised from the Morales-Ramis Theorem 2.3.1 and consisting on the detection of logarithms in the factorization of a certain reduced variational system; the particular solution along which variational equations were reduced and factorized was a Lagrange zero-energy solution, just as in the results by Tsygvintsev. As for the equal-mass N-Body Problem, in [164, Section 3.2] Ziglin allowed one of the masses, say m_N, to be different from the others and made attempts at the very same thesis we use here: to wit, that the trace of the Hessian matrix for $V''_{N,d}(\boldsymbol{c})$ is not contained in $\mathbb{Z}$ for some solution $\boldsymbol{c}$ of (3.2). The main result in [164, Section 3.2] was the existence of at most finitely many values m_N for which the Problem is integrable, although none of these values was actually given.

Theorem 3.2.2 completes the aforementioned results by Tsygvintsev by discarding the three special cases remaining therein. Furthermore, the proof given here is shorter thanks to Theorems 2.3.3 and 2.3.5. Theorem 3.2.2 also completes what was done by S. L. Ziglin in [164, Section 3.1] and complements the non-integrability result by D. Boucher and J.-A. Weil by extending it to arbitrary dimension, besides being a consistent generalization of Bruns' Theorem 2.4.5 and the case $N=3$ of Julliard's Theorem 2.4.6. Theorem 3.2.3, on the other hand, completes the results in [164, Section 3.2], though the tools used here hardly qualify as a theoretical step forward since, as said above, the author of the latter reference shared our aim. A comment will be made in Section 5.2.2 concerning the hypotheses in [164, Section 3.2].

Remark 3.2.4. We must observe that Hamiltonian $\mathcal{H}_{N,d}$ is *not meromorphic*. However, any first integral of $X_{\mathcal{H}_{N,d}}$ (e.g. $\mathcal{H}_{N,d}$ itself), when restricted to a domain of each determination of $\mathcal{H}_{N,d}$, is meromorphic and thus amenable to the whole theory explained so far; see, for instance, [76, pp. 156-157] for more details as applied to a different homogeneous potential.

3.2.2 Setup for the proof

Known eigenvalues

Let us find the exceptional set $\tilde{S}$ hinted at in Step III: it consists of $d+n+1$ eigenvalues, say $\{\lambda_1,\ldots,\lambda_{d+n+1}\}$, all belonging to $\{-2,0,1\}$. d of them, for instance $\lambda_2,\ldots,\lambda_{d+1}$, appear for any solution of Hamilton's equations, and the remaining ones appear specifically for solutions of the form $\phi\boldsymbol{c}$ with $\ddot{\phi}+\phi^{-2}=0$ and $V'_{N,d}(\boldsymbol{c})=\boldsymbol{c}$.

Lemma 3.2.5. *Let $\boldsymbol{q}(t)=(\mathbf{q}_1(t),\ldots,\mathbf{q}_N(t))$ be a solution of the N-Body Problem. Then, d of the eigenvalues of $V''_{N,d}(\boldsymbol{q})$ are identically zero.*

Proof. This results from the invariance of the linear momentum $\mathbf{I}_L$ (Subsection 2.4.1), which after symplectic change $\mathbf{x}_i=\frac{1}{\sqrt{m_i}}\mathbf{q}_i$ and $\mathbf{y}_i=\sqrt{m_i}\mathbf{p}_i$ becomes $\sum_{i=1}^N\sqrt{m_i}\ddot{\mathbf{q}}_i=\mathbf{0}$. Since $\ddot{\mathbf{q}}_i=\dot{\mathbf{p}}_i=-\frac{\partial V_{N,d}}{\partial \mathbf{q}_i}$ for $i=1,\ldots,N$, we obtain

$$\sum_{i=1}^N\sqrt{m_i}\frac{\partial V_{N,d}}{\partial q_{d(i-1)+k}}=0,\qquad k=1,\ldots,d,$$

and derivating these equations with respect to $\boldsymbol{q}$ we obtain d distinct relations of linear dependence between the columns of the Hessian,

$$\sum_{i=1}^N\sqrt{m_i}\frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+k}\partial q_j}=0,\qquad j=1,\ldots,2N,\quad k=1,\ldots,d,$$

rendering $\left\{\sum_{i=1}^N\sqrt{m_i}\boldsymbol{e}_{dN,d(i-1)+j}:j=1,\ldots,d\right\}$ an independent eigensystem for the eigenvalue 0; that alone allows us to write $\lambda_2=\lambda_3=\cdots=\lambda_{d+1}=0$. □

Let $\boldsymbol{q}=\phi(t)\,\boldsymbol{c}$ as above in the next two Lemmae. The first of them takes no other effort in proving than referring the reader back to the consequence (2.10) of Euler's Theorem while setting $k=-1$:

Lemma 3.2.6. *We may write $\lambda_1=-2$.* □

Lemma 3.2.7. $1 \leq n \leq \binom{d}{2}$ *of the eigenvalues, say* $\lambda_{d+2}, \ldots, \lambda_{d+n+1}$*, are equal to* 1.

Proof. This is a consequence of the invariance of the angular momentum; derivating $\mathbf{I}_A$ once after expressing it in coordinates $\boldsymbol{q}$, $\boldsymbol{p}$, we obtain

$$0 = \sum_{i=1}^{N} q_{d(i-1)+k}\ddot{q}_{d(i-1)+l} - q_{d(i-1)+l}\ddot{q}_{d(i-1)+k}, \qquad 1 \leq k < l \leq d,$$

and thus

$$0 = \sum_{i=1}^{N} q_{d(i-1)+k}\frac{\partial V_N}{\partial q_{d(i-1)+l}} - q_{d(i-1)+l}\frac{\partial V_N}{\partial q_{d(i-1)+k}}, \qquad 1 \leq k < l \leq d,$$

which derivated with respect to $\boldsymbol{q}$ yields

$$\begin{aligned} 0 &= \sum_{i=1}^{N}\left(\delta_{d(i-1)+k,j}\frac{\partial V_{N,d}}{\partial q_{d(i-1)+l}} - \delta_{d(i-1)+l,j}\frac{\partial V_{N,d}}{\partial q_{d(i-1)+k}}\right) \\ &\quad + \sum_{i=1}^{N}\left(q_{d(i-1)+k}\frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+l}\partial q_j} - q_{d(i-1)+l}\frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+k}\partial q_j}\right), \\ 1 &\leq k < l \leq d, \qquad j = 1, \ldots, dN; \end{aligned}$$

thus, assuming $\boldsymbol{q} = \phi(t)\,\boldsymbol{c}$ as above we have

$$\begin{aligned} 0 &= \sum_{i=1}^{N}\phi^{-2}\left(\delta_{d(i-1)+k,j}c_{d(i-1)+l} - \delta_{d(i-1)+l,j}c_{d(i-1)+k}\right) \\ &\quad + \sum_{i=1}^{N}\phi^{-2}\left(c_{d(i-1)+k}\frac{\partial^2 V_N}{\partial q_{d(i-1)+l}\partial q_j}(\boldsymbol{c}) - c_{d(i-1)+l}\frac{\partial^2 V_N}{\partial q_{d(i-1)+k}\partial q_j}(\boldsymbol{c})\right), \\ j &= 1, \ldots, dN, \qquad 1 \leq k < l \leq d, \end{aligned}$$

which means $\sum_{i=1}^{N}\boldsymbol{k}_{i,k,l}$ is an eigenvector of $V''_{N,d}(\boldsymbol{c})$ of eigenvalue 1, where $\boldsymbol{k}_{i,k,l} = -c_{d(i-1)+l}\boldsymbol{e}_{dN,d(i-1)+k} + c_{d(i-1)+k}\boldsymbol{e}_{dN,d(i-1)+l}$, for each $1 \leq k < l \leq d$. $\binom{d}{2}$ is clearly an upper bound for the dimension of vector space $\left\langle \sum_{i=1}^{N}\boldsymbol{k}_{i,k,l} : 1 \leq k < l \leq d\right\rangle$. $\square$

Corollary 3.2.8. *Assume* $\boldsymbol{q} = \phi(t)\left(\mathbf{c}_1^T, \ldots, \mathbf{c}_N^T\right)^T$*, where*

$$\mathbf{c}_i = \left(c_{d(i-1)+1}, c_{d(i-1)+2}, 0, \ldots, 0\right)^T, \qquad i = 1, \ldots, N,$$

and there are at least two $\mathbf{c}_{i_1}, \mathbf{c}_{i_2}$ *such that* $c_{d(i_j-1)+1}c_{d(i_j-1)+2} \neq 0$, $j = 1, 2$ *and*

$$\frac{c_{d(i_1-1)+1}}{c_{d(i_1-1)+2}} \neq \frac{c_{d(i_2-1)+1}}{c_{d(i_2-1)+2}}.$$

Then, there are at least $n = 2d - 3$ *eigenvalues equal to one.*

Proof. Let $\tilde{\boldsymbol{c}} = \left(\tilde{\mathbf{c}}_1^T, \ldots, \tilde{\mathbf{c}}_N^T\right)^T$ be the vector formed by shifting the first two entries in each $\mathbf{c}_i$ and multiplying the first of them by -1:

$$\tilde{\mathbf{c}}_i = \left(-c_{d(i-1)+2}, c_{d(i-1)+1}, 0, \ldots, 0\right)^T, \qquad i = 1, \ldots, N.$$

According to the previous Lemma, $\tilde{\boldsymbol{c}} \in \ker\left(V''_{N,d}(\boldsymbol{c}) - \mathrm{Id}_{dN}\right)$. The same Lemma asserts that the set $W \cup \tilde{W} := \{\boldsymbol{v}_k : 3 \leq k \leq d\} \cup \{\tilde{\boldsymbol{v}}_k : 3 \leq k \leq d\}$, where each of its elements is defined as

$$\boldsymbol{v}_k := \left(c_{d(i-1)+1}\mathbf{e}_{d,k}\right)_{i=1,\ldots,N}, \quad \tilde{\boldsymbol{v}}_k := \left(c_{d(i-1)+2}\mathbf{e}_{d,k}\right)_{i=1,\ldots,N}, \qquad k = 3, \ldots, d,$$

is also set of eigenvectors of $V''_{N,d}(\boldsymbol{c})$ for eigenvalue 1, all of them independent with $\tilde{\boldsymbol{c}}$ by hypothesis $c_{d(i_1-1)+1}c_{d(i_1-1)+2} \neq 0$. The dimension of the space spanned by W (resp. $\tilde{W}$) is $d-2$, and any relation of linear independence of a vector of $\boldsymbol{v}_k \in W$ with one vector in $\tilde{\boldsymbol{v}}_l \in \tilde{W}$ would necessarily imply $k = l$; in particular, we would have

$$\frac{c_{d(i_1-1)+1}}{c_{d(i_1-1)+2}} = \frac{c_{d(i_2-1)+1}}{c_{d(i_2-1)+2}},$$

which contradicts our hypothesis. Hence, $\dim W \oplus \tilde{W} = 2d-4$ and adjoining $\tilde{\boldsymbol{c}}$ to $W \cup \tilde{W}$ yields $2d-3$ independent eigenvectors for $V''_{N,d}(\boldsymbol{c})$. $\square$

Notation for the planar case

Defining $\boldsymbol{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ $(\mathbf{q}_i = (q_{2i-1}, q_{2i}), \quad i = 1, \ldots, N$), we have

$$\frac{\partial V_{N,2}}{\partial \mathbf{q}_i} = \sum_{k=1, k\neq i}^{n} \sqrt{m_k}\,(m_i m_k)^{3/2}\, D_{i,k}^{-3}\mathbf{D}_{i,k}, \quad i = 1, \ldots, N, \tag{3.4}$$

where $\mathbf{D}_{i,j} = (d_{2i-1,2j-1}, d_{2i,2j})^T := \sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j$ for each $i, j = 1, \ldots, N$, and we obtain the block expression for the Hessian matrix: $V''_{N,2}(\boldsymbol{q}) = \left(\tilde{U}_{i,j}\right)_{i,j=1,\ldots,N}$, defining

$$\tilde{U}_{i,j} := \begin{cases} -\sqrt{m_i m_j}U_{i,j}, & i \neq j, \\ \sum_{k\neq i} m_k U_{i,k}, & i = j \end{cases} \tag{3.5}$$

where

$$U_{i,j} = U_{j,i} = \begin{cases} 0_{2\times 2}, & i = j, \\ (m_i m_j)^{3/2}\left(d_{2i-1,2j-1}^2 + d_{2i,2j}^2\right)^{-5/2} S_{i,j}, & i < j, \end{cases} \tag{3.6}$$

and

$$S_{i,j} = S_{j,i} := \begin{pmatrix} d_{2i,2j}^2 - 2d_{2i-1,2j-1}^2 & -3d_{2i-1,2j-1}d_{2i,2j} \\ -3d_{2i-1,2j-1}d_{2i,2j} & d_{2i-1,2j-1}^2 - 2d_{2i,2j}^2 \end{pmatrix}, \qquad i \neq j. \tag{3.7}$$

Reduction to the planar case

We are now justifying our future trend to restrict ourselves to $d = 2$. All there is to prove is that, assuming $\boldsymbol{c}$ is embedded in a particular way into a wider ambient space, the only changes in Spec $V''_{N,d}$ are possibly the multiplicity of its existing elements, and possibly the addition of new ones:

Lemma 3.2.9. *For any given $d \geq 2$, let*

$$\boldsymbol{c} : (\mathbf{c}_1, \ldots, \mathbf{c}_N) \in \mathbb{C}^{2d}, \qquad \mathbf{c}_i : (u_{i,1}, u_{i,2}), \qquad i = 1, \ldots, N,$$

be a solution to $V'_{N,2}(\boldsymbol{c}) = \boldsymbol{c}$, and

$$\widetilde{\boldsymbol{c}} : (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_N) \in \mathbb{C}^{Nd}, \qquad \tilde{\mathbf{c}}_i : (u_{i,1}, u_{i,2}, 0, \ldots, 0), \qquad i = 1, \ldots, N.$$

Then, $V'_{N,d}(\widetilde{\boldsymbol{c}}) = \widetilde{\boldsymbol{c}}$ and $\operatorname{Spec} V''_{N,2}(\boldsymbol{c}) \subset \operatorname{Spec} V''_{N,d}(\boldsymbol{c})$.

Proof. $V'_{N,d}(\widetilde{\boldsymbol{c}}) = \widetilde{\boldsymbol{c}}$ is immediate since

$$\left.\frac{\partial V_{N,d}}{\partial \mathbf{q}_i}\right|_{\mathbf{q}_i = \tilde{\mathbf{c}}_i} = \left.\begin{pmatrix} \sum_{k=1, k\neq i}^{n} \sqrt{m_k}\,(m_i m_k)^{3/2} D_{i,k}^{-3} \mathbf{D}_{i,k} \\ \mathbf{0}_{d-2} \end{pmatrix}\right|_{\mathbf{q}_i = \mathbf{c}_i} = \left.\begin{pmatrix} \frac{\partial V_{N,2}}{\partial \mathbf{q}_i} \\ \mathbf{0}_{d-2} \end{pmatrix}\right|_{\mathbf{q}_i = \mathbf{c}_i}.$$

$V''_{N,d}(\widetilde{\boldsymbol{c}})$ takes the following form: $V''_{N,d}(\widetilde{\boldsymbol{c}}) = \left(\tilde{U}_{d,i,j}\right)_{i,j=1,\ldots,N}$, where

$$\tilde{U}_{d,i,j} := \begin{cases} -\sqrt{m_i m_j} U_{d,i,j}, & i \neq j, \\ \sum_{k \neq i} m_k U_{d,i,k}, & i = j \end{cases} \tag{3.8}$$

and the block structure of these matrices will be

$$U_{d,i,j} = \begin{pmatrix} U_{i,j} & \mathbf{0}_{d-2}^T \\ \mathbf{0}_{d-2} & \alpha_{i,j} \mathrm{Id}_{d-2} \end{pmatrix}, \qquad i, j = 1, \ldots, N,$$

where $U_{i,j}$ is defined as in (3.6) and

$$\alpha_{i,j} = \alpha_{j,i} = \begin{cases} 0, & i = j, \\ (m_i m_j)^{3/2} D_{i,j}^{-3}, & i \neq j = 1, \ldots, N. \end{cases}$$

Thus, if

$$\boldsymbol{v} : (\mathbf{v}_1, \ldots, \mathbf{v}_N) \in \mathbb{C}^{2d}, \qquad \mathbf{v}_i : (v_{i,1}, v_{i,2}), \qquad i = 1, \ldots, N,$$

is an eigenvector of $V''_{N,2}(\boldsymbol{c})$, then

$$\widetilde{\boldsymbol{v}} : (\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_N) \in \mathbb{C}^{Nd}, \qquad \tilde{\mathbf{v}}_i : (v_{i,1}, v_{i,2}, 0, \ldots, 0), \qquad i = 1, \ldots, N,$$

is an eigenvector of $V''_{N,d}(\tilde{\boldsymbol{c}})$ for the same eigenvalue. □

We will define $V_N := V_{N,2}$ from now on, and save for indication of the contrary (e.g. for Section 3.3.1), we will assume we are dealing exclusively with the *planar* case.

3.3 Proofs of Theorems 3.2.2 and 3.2.3

3.3.1 Proof of Theorem 3.2.2

Step I in Section 3.2.1 is computing a solution $\boldsymbol{c}$ of (3.2) for $N = 3$. Let us define $m = m_1 + m_2 + m_3$ (which may be always set to 1 by the reader if even simpler calculations are sought all through this section) and $D = m_1m_2 + m_2m_3 + m_3m_1$, and consider vectors of the form $\boldsymbol{c} = m^{-2/3}M^{1/2}\hat{\boldsymbol{c}}$, where $M = (m_i \mathrm{Id}_d)_{i=1,\dots,N}$ as in Subsection 2.4.1 and

$$\hat{\boldsymbol{c}} = \begin{pmatrix} a_2m_2 + a_3m_3 \\ b_2m_2 + b_3m_3 \\ a_3m_3 - a_2(m_1 + m_3) \\ b_3m_3 - b_2(m_1 + m_3) \\ a_2m_2 - a_3(m_1 + m_2) \\ b_2m_2 - b_3(m_1 + m_2) \end{pmatrix} \tag{3.9}$$

and a_2, a_3, b_2, b_3 are solutions to

$$\left(a_2^2 + b_2^2\right)^{3/2} = \left(a_3^2 + b_3^2\right)^{3/2} = \left[(a_2 - a_3)^2 + (b_2 - b_3)^2\right]^{3/2} = 1.$$

See Subsection 5.2.1 for an explanation of such an assumption. An example of such a vector $\hat{\boldsymbol{c}}$ is

$$\hat{\boldsymbol{c}} = \begin{pmatrix} (m_2 + 2m_3)\,\alpha \\ m_2\beta \\ -(m_1 - m_3)\,\alpha \\ -(m_1 + m_3)\,\beta \\ -(2m_1 + m_2)\,\alpha \\ m_2\beta \end{pmatrix}, \tag{3.10}$$

where $\alpha^2 + \beta^2 = 1$ and $\alpha^3 = 1/8$. The possible choices of α and β add up to two such vectors as (3.10), and thus two solutions $\boldsymbol{c} = m^{-2/3}M^{1/2}\hat{\boldsymbol{c}}$ and $\boldsymbol{c}^* = m^{-2/3}M^{1/2}\hat{\boldsymbol{c}}^*$ for (3.2): those corresponding to $\alpha = 1/2$ and $\alpha^* = \frac{-1+\mathrm{i}\sqrt{3}}{4}$, respectively; keeping with what was said in Section 1.4, square roots are taken in their principal determination. A simple, if tedious computation proves $\boldsymbol{c}$ and $\boldsymbol{c}^*$ solutions to (3.2), indeed. *$\boldsymbol{c}$ yields an explicit parametrization for the (homothetical) Lagrange triangular solution (Example 2.4.10(1)).*

The rest of the proof is based on performing both Steps II and III in Section 3.2.1 at a time. The eigenvalues of $V_3''(\boldsymbol{c})$ are $\{-2, 0, 0, 1, \lambda_+, \lambda_-\}$, where

$$\lambda_\pm := -\frac{1}{2} \pm \frac{3\sqrt{m_1^2 + m_2^2 + m_3^2 - m_1m_2 - m_1m_3 - m_2m_3}}{2(m_1 + m_2 + m_3)}.$$

As said in Theorem 2.3.5, the existence of a single additional meromorphic integral for $X_{\mathcal{H}_3}$ implies either $\lambda_+^* \in S$ or $\lambda_-^* \in S$, where $S = \left\{-\frac{1}{2}p(p-3) : p > 1\right\}$, which means (defining $R := \sqrt{m^2 - 3D}$) that $\pm 3R \in \{(p^2 - 3p - 1)\,m : p > 1\}$ and therefore

$$-27(m_1m_2 + m_1m_3 + m_2m_3) \in \left\{m^2(p-1)(p-2)(p-4)(p+1) : p > 1\right\}, \tag{3.11}$$

impossible if $p \in \{2,4\}$ or $p > 4$ since it would have a strictly negative number equaling a non-negative one. For $p = 3$ (3.11) becomes $8m^2 = 27D$, that is,

$$\frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} = \frac{8}{27}. \tag{3.12}$$

Thus, we could at this point assure the absence of an additional meromorphic integral *except when* (3.12) *holds.*

The eigenvalues of $V_3''(\boldsymbol{c}^*)$ are $\left\{-2, 0, 0, 1, \lambda_+^*, \lambda_-^*\right\}$, where $\lambda_\pm^* = -\frac{1}{2} \pm \frac{3\sqrt{A}}{2\sqrt{2}m}$, and

$$A = 2m_1^2 + 2m_2^2 + 2m_3^2 - 5m_1m_2 - 5m_2m_3 + 7m_1m_3 - \mathrm{i}\sqrt{3}(m_1m_2 + m_2m_3 - 5m_1m_3).$$

See Appendix A for details. Again, the thesis in Corollary 2.3.5 amounts to either $\lambda_+^* \in S$ or $\lambda_-^* \in S$, which here becomes $\pm 3\sqrt{A} = (p^2 - 3p - 1)\sqrt{2}m$, and thus

$$A - 2m^2 \in \left\{\frac{2}{9}(p-1)(p-2)(p-4)(p+1)\, m^2 : p > 1\right\};$$

a necessary condition for this to hold with real masses is the vanishing of the imaginary term in A

$$-\mathrm{i}\sqrt{3}\,(m_1m_2 + m_2m_3 - 5m_1m_3) = 0, \tag{3.13}$$

implying $m_1m_2 + m_2m_3 = 5m_1m_3$. Thus,

$$-378 m_1 m_3 = 2(p-1)(p-2)(p-4)(p+1)\, m^2, \tag{3.14}$$

for some $p > 1$. We discard $p = 2, 4$ in (3.14) assuming the strict positiveness of m_1 and m_3. The only integer $p > 1$ for which the right side can be negative is: 3, implying $-378 m_1 m_3 = -16\,(m_1 + m_2 + m_3)^2$. These two constraints arising from (3.13) and (3.14),

$$5m_1m_3 = m_1m_2 + m_2m_3, \qquad \frac{189}{8} m_1 m_3 = (m_1 + m_2 + m_3)^2, \tag{3.15}$$

cannot hold at the same time as condition (3.12). Indeed, the former two substituted into the latter would yield $\frac{(5m_1m_3 + m_1m_3)}{\frac{189}{8} m_1 m_3} = \frac{8}{27}$, i.e. $\frac{16}{63} = \frac{8}{27}$ which is obviously absurd. Thus, either (3.12) holds or both equations in (3.15) hold. In particular, term A in $\lambda_\pm^* = -\frac{1}{2} \pm \frac{3\sqrt{A}}{2\sqrt{2}m}$ does not vanish if (3.12) holds, which implies $\lambda_-^* \neq \lambda_+^*$ and thus $V''(\boldsymbol{c}^*)$ has a *diagonal* Jordan canonical form; indeed, the Jordan blocks for eigenvalues $0, -2, 1$ are already diagonal since the eigenvectors provided by the proofs Lemmae 3.2.5 and 3.2.6 and Corollary 3.2.8 are eigenvectors here as well. In other words, in spite of being complex, the second vector $\boldsymbol{c}^*$ does not prevent the symmetrical matrix from being diagonalizable, and thus amenable to the application of Corollary 2.3.5. The lack of an additional meromorphic first integral for arbitrary $m_1, m_2, m_3 > 0$ is thus proven in the planar case.

Furthermore, for the general case $d \geq 3$, we may embed $\boldsymbol{c}$ and $\boldsymbol{c}^*$ into vectors $\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{c}}^* \in \mathbb{C}^{3d}$ as in Lemma 3.2.9. In virtue of Lemmae 3.2.5 and 3.2.6 and Corollary 3.2.8, we have $d + 1 + 2d - 3 = 3d - 2$ eigenvalues (that is, all of them but two) belonging to $\{-2, 0, 1\}$ and due to the classical first integrals; the remaining two eigenvalues of $V_{3,d}''(\boldsymbol{c})$ (resp. $V_{3,d}''(\boldsymbol{c}^*)$) are $\lambda_\pm$ (resp. $\lambda_\pm^*$) due to Lemma 3.2.9. $\square$

Remarks 3.3.1.

1. It is worth noting that the only case forcing us to resort to a second solution to (3.2) is precisely one of the three cases exceptional to A. V. Tsygvintsev's proof ([139]):
$$\frac{D}{m^2} \in \left\{ \frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{3^2} \right\}. \tag{3.16}$$

2. Yet another valid (and even shorter) proof would be feasible were more knowledge available concerning the collinear solution; see Section 5.2, and especially (5.3), for details.

3. A proof could be attempted at by using Bring forms as in Remark 2.4.11, although the amount of calculations involving generalized hypergeometric functions ${}_4F_3$ appears to be rather cumbersome. We are therefore avoiding this for the sake of simplicity.

3.3.2 Proof of Theorem 3.2.3

In this specific case, since every choice of mass units amounts to a symplectic change in the extended phase space, we may set $m_1 = \cdots = m_N = 1$. Expressions (3.4) and (3.5) may be found explicitly in terms of trigonometric functions if we choose the polygonal configuration (Example 2.4.10 (5)) as a solution to (3.2). Define
$$s_k := \sin \frac{\pi k}{N}, \quad c_k := \cos \frac{\pi k}{N}, \qquad k \in \mathbb{N},$$
and $\zeta = e^{\frac{2\pi \mathrm{i}}{N}} = c_2 + \mathrm{i} s_2$.

Lemma 3.3.2. *Vector* $\boldsymbol{c}_P = (\mathbf{c}_1, \ldots, \mathbf{c}_N)$ *defined by* $\mathbf{c}_j = \beta_N^{1/3} (c_{2j}, s_{2j})$*, where* $\beta_N = \frac{1}{4} \sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$*, is a solution for* $V_N'(\boldsymbol{q}) = \boldsymbol{q}$*.*

Proof. Indeed, assume $\mathbf{c}_j = A\left(\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}\right)$ for some $A > 0$. We have
$$\frac{\partial V_N}{\partial \mathbf{q}_j}(\boldsymbol{c}_P) = \frac{1}{4A^2} \begin{pmatrix} \sum_{k=1}^{N-1} \frac{\cos\frac{2\pi j}{N}}{\sin\frac{\pi}{N}k} \\ \sum_{k=1}^{N-1} \frac{\sin\frac{2\pi j}{N}}{\sin\frac{\pi}{N}k} \end{pmatrix}$$
due to the fact that
$$\sum_{k=1, k\neq j}^{N} \frac{\zeta^j - \zeta^k}{|\zeta^j - \zeta^k|^3} = \zeta^j \sum_{k=1}^{N-1} \frac{1 - (c_{2k} + \mathrm{i} s_{2k})}{|1 - \zeta^k|^3},$$
and, since the imaginary part of this sum satisfies:
$$\sum_{k=1}^{N-1} \frac{s_{2k}}{|1-\zeta^k|^3} = \sum_{k=1}^{N-1} \frac{2 s_k c_k}{8 c_k^3} = \frac{1}{4} \sum_{k=1}^{N-1} \frac{c_k}{s_k^2} = 0,$$
we finally obtain $\zeta^j \sum_{k=1}^{N-1} \frac{1-(c_{2k}+\mathrm{i}s_{2k})}{\left|1-\zeta^k\right|^3} = \frac{1}{4}\zeta^j \sum_{k=1}^{N-1} s_k^{-1}$. Now $V'(\boldsymbol{c}_P) = \boldsymbol{c}_P$ if and only if $\sum_{k=1}^{N-1} \frac{1}{4A^2 s_k} = A$. The latter holds for $A = \beta_N^{1/3}$. $\square$

Let us see how this specific vector simplifies V_N''. Keeping expression (3.5) in consideration we have $d_{2i-1,2j-1} + \mathrm{i} d_{2i,2j} = \beta_N^{1/3}(\zeta^i - \zeta^j)$ which implies

$$S_{i,j} = 2\left(\beta_N^{1/3} s_{i-j}\right)^2 \begin{pmatrix} 3c_{2(i+j)} - 1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)} - 1 \end{pmatrix},$$

for each $1 \le i, j \le N$, and thus

$$\begin{aligned} U_{i,i} &= 0_{2\times 2}, \qquad i = 1, \dots, N, \\ U_{i,j} &= U_{j,i} = \left(2\beta_N^{1/3} s_{i-j}\right)^{-5} S_{i,j} \\ &= \frac{|s_{i-j}|^{-3}}{16\beta_N} \begin{pmatrix} 3c_{2(i+j)} - 1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)} - 1 \end{pmatrix}, \qquad i \ne j, \end{aligned}$$

from which defining

$$\begin{aligned} \tilde{U}_{i,i} &= \sum_{j\ne i} \frac{|s_{i-j}|^{-3}}{16\beta_N} \begin{pmatrix} 3c_{2(i+j)} - 1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)} - 1 \end{pmatrix}, \\ \tilde{U}_{i,j} &= \frac{|s_{i-j}|^{-3}}{16\beta_N} \begin{pmatrix} 1 - 3c_{2(i+j)} & -3s_{2(i+j)} \\ -3s_{2(i+j)} & 3c_{2(i+j)} + 1 \end{pmatrix}, \qquad i \ne j, \end{aligned}$$

we have $V_N''(\boldsymbol{c}_P) = \left(\tilde{U}_{i,j}\right)_{i,j=1,\dots,N}$.

Lemma 3.3.3. *The trace for $V_N''(\boldsymbol{c}_P)$ is equal to $-(N/8)(\alpha_N/\beta_N)$, where $\alpha_N = \sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right)$ and β_N is defined as in Lemma 3.3.2.*

Proof. In virtue of the above simplifications for (3.5), $\mathrm{tr}\,(V_N''(\boldsymbol{c}_P))$ is equal to

$$\mu_N := -\frac{2}{\beta_N} \sum_{1\le k_1<k_2\le N} \left|\zeta^{2k_1} - \zeta^{2k_2}\right|^{-3}.$$

We have $-\frac{\mu_N}{4}\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) = \sum_{1\le k_1<k_2\le N} 2\left|\zeta^{2k_1} - \zeta^{2k_2}\right|^{-3}$; on the other hand, the symmetry of a regular polygon assures

$$\sum_{1\le k_1<k_2\le N} 2\left|2s_{k_2-k_1}\right|^{-3} = N\sum_{k=1}^{N-1} (2s_k)^{-3};$$

thus, $2\mu_N \sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) = -N\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right)$. ◻

Case 1: $N = 3, 4, 5, 6$

We can afford a stronger result than just non-integrability for these values without using Lemma 3.3.3, in view of Corollary 2.3.5. We just have to prove the following

Lemma 3.3.4. *$V_N''(\boldsymbol{c}_P)$, $N = 3, 4, 5, 6$, has only four eigenvalues in S: $\lambda_1 = -2, \lambda_2 = \lambda_3 = 0, \lambda_4 = 1$.*

Proof. The eigenvalues of $V_3''(\boldsymbol{c}_P)$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_{5,6} = -1/2$. Those of $V_4''(\boldsymbol{c}_P)$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5 = \frac{2(5-3\sqrt{2})}{7}, \lambda_{6,7} = \frac{2(\sqrt{2}-4)}{7}, \lambda_8 = \frac{6\sqrt{2}-17}{7}$. $V_5''(\boldsymbol{c}_P)$ has three different non-trivial double eigenvalues:

$$\lambda_{5,6,7,8} = \frac{\sqrt{5}-5 \pm \sqrt{518-222\sqrt{5}}}{4}, \ \lambda_{9,10} = \frac{\sqrt{5}-4}{2}.$$

The eight non-trivial eigenvalues for $V_6''(\boldsymbol{c}_P)$ are

$$\begin{aligned} \lambda_5 &= \frac{4\left(29\sqrt{3}-94\right)}{59}, & \lambda_{6,7} &= \frac{34\sqrt{3}-\sqrt{133465-59584\sqrt{3}}-157}{118}, \\ \lambda_{8,9} &= \frac{2\left(7\sqrt{3}-41\right)}{59}, & \lambda_{10,11} &= \frac{34\sqrt{3}+\sqrt{133465-59584\sqrt{3}}-157}{118}, \\ \lambda_{12} &= \frac{4\left(53-22\sqrt{3}\right)}{59}. \end{aligned}$$

Hence follows item 1 in Theorem 3.2.3. □

Case 2: $N = 7, 8, 9$

Proceeding from Lemma 3.3.3, it is straightforward to see the traces for $V_N''(\boldsymbol{c})$ for these three values of N are non-integers since

$$\begin{aligned} \mu_7 &= -\frac{\sqrt{413+56\sqrt{7}\cos\left(\frac{1}{3}\arctan 3\sqrt{3}\right)}}{2\cos\left(\frac{1}{6}\arctan\frac{3\sqrt{3}}{13}\right)} \in (-12,-11), \\ \mu_8 &= \frac{4\left(-2633+766\sqrt{2}+4\sqrt{118010-68287\sqrt{2}}\right)}{241} \in (-17,-16), \\ \mu_9 &= -\frac{9}{2}\frac{\frac{8\sqrt{3}}{9}+\csc^3\frac{\pi}{9}+\csc^3\frac{2\pi}{9}+\csc^3\frac{4\pi}{9}}{\frac{2\sqrt{3}}{3}+\csc\frac{\pi}{9}+\csc\frac{2\pi}{9}+\csc\frac{4\pi}{9}} \in (-22,-21). \end{aligned}$$

Case 3: $N \geq 10$

We will prove $V_N''(\boldsymbol{c}_P)$ has at least an eigenvalue greater than 1. We know the following holds ([4]),

$$\csc x = \frac{1}{x} + f(x) := \frac{1}{x} + \sum_{k\geq 1} \frac{(-1)^{k-1}2\left(2^{2k-1}-1\right)B_{2k}x^{2k-1}}{(2k)!}, \tag{3.17}$$

f being analytical for $|x| < \pi$ (which obviously holds if $x = \frac{\pi j}{N}$, $j = 1, \ldots, N-1$) and $B_k, k \geq 1$, being the Bernoulli numbers ([4, Chapter 23], [133, §3.3]).

Lemma 3.3.5. *For each $N \geq 10$, $S_N := 2\sum_{j=1}^{N-1}\left(\csc^2\frac{j\pi}{N}-5\right)\csc\frac{j\pi}{N} > 0$.*

Proof. Recall the *Euler-MacLaurin summation formula* ([133, §3.3]): for any $f \in \mathcal{C}^{2s+2}([a,b])$ and $n \in \mathbb{N}$, and defining $h = \frac{b-a}{n}$, the following holds,

$$\sum_{j=0}^{n} f(a+jh) = \frac{\int_a^b f}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{s} h^{2r-1}B_{2r}\frac{f^{(2r-1)}(b)-f^{(2r-1)}(a)}{(2r)!} + R_s,$$

where $R_s = nh^{2s+2}\frac{B_{2s+2}}{(2s+2)!}f^{(2s+2)}(\alpha)$ for some $\alpha \in (a, a+nh)$. Substituting in $a = h = \pi/N$, $n = N-2$, $b = a + hn = \frac{\pi(N-1)}{N}$, $f(x) = 2(\csc^2 x - 5)\csc x$ and $s = 2$, we obtain

$$\begin{aligned}
\frac{\int_a^b f(x)\,dx}{h} &= \frac{2N}{\pi}\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right), \\
\frac{f(a)+f(b)}{2} &= 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N}, \\
hB_2\frac{f'(b)-f'(a)}{2} &= \frac{\pi\cot\frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right)}{3N}, \\
h^3B_4\frac{f'''(b)-f'''(a)}{4!} &= -\frac{\pi^3\csc^6\frac{\pi}{N}\left(742\cos\frac{\pi}{N} + 213\cos\frac{3\pi}{N} + 5\cos\frac{5\pi}{N}\right)}{2880N^3} \\
&> -\frac{\pi^3(742+213+5)\csc^6\frac{\pi}{N}}{2880N^3} = -\frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3},
\end{aligned}$$

and

$$R_2(\alpha) = \frac{\csc^9(\alpha)(N-2)\pi^6 P(\alpha)}{1935360N^6},$$

where $P(x) := 1110231 + 1256972\cos 2x + 206756\cos 4x + 6516\cos 6x + 5\cos 8x$;. In previous formulae, we have used $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ and several trigonometric identities in order to express the different terms in a suitable way for what follows.

Introducing variable $w = \cos 2x$, we may write the function defined by the first three terms in $P(x)$ as

$$\widehat{P}(w) := 903475 + 1256972w + 413512w^2.$$

Then, for each $w \in [-1, 1]$, one has $\widehat{P}'(w) > 0$; hence, for $x \in (0, \pi)$ we obtain $P(x) \geq \widehat{P}(-1) - 6516 - 5 > 0$ and therefore $R_2(\alpha) > 0$, which leads to the following:

$$\begin{aligned}
S_N &= \frac{\int_a^b f}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{2} h^{2r-1}B_{2r}\frac{f^{(2r-1)}(b) - f^{(2r-1)}(a)}{(2r)!} + R_2(\alpha) \\
&> \frac{\int_a^b f(x)\,dx}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{2} h^{2r-1}B_{2r}\frac{f^{(2r-1)}(b) - f^{(2r-1)}(a)}{(2r)!} \\
&> \frac{2N\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right)}{\pi} + 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N} \\
&\quad + \frac{\pi\cot\frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right)}{3N} - \frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3}.
\end{aligned}$$

There is a number of possible ways of proving this latter lower bound strictly positive. For instance, since, for $N \geq 10$, $\cot\frac{\pi}{N} > 3$, we have

$$\begin{aligned}
S_N &> \frac{2N}{\pi}\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right) + 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N} \\
&+ \frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right) - \frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3} \\
&=: \sigma_N.
\end{aligned}$$

The first term in that sum is exactly $\frac{2N}{\pi}F\left(\tan\frac{\pi}{2N}\right)$, where

$$F:(0,\infty)\to\mathbb{R},\qquad F(z):=\frac{z^{-2}-z^2}{4}+9\ln z,$$

is strictly decreasing in $\left(0,\sqrt{5}-2\right)$. Since $\tan\frac{\pi}{2N}<\sqrt{5}-2$ for all $N\geq 10$, we have

$$F\left(\tan\frac{\pi}{2N}\right)\geq F\left(\tan\frac{\pi}{20}\right)>-\frac{20}{3},$$

and thus,

$$\begin{aligned}\sigma_N &> \frac{2N}{\pi}\left(-\frac{20}{3}\right)+2\left(\csc^2\frac{\pi}{N}-5\right)\csc\frac{\pi}{N}+\frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N}-5\right)-\frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3}\\ &> \frac{\csc\frac{\pi}{N}}{3N^3}G_N\left(\csc\frac{\pi}{N}\right),\end{aligned}$$

where

$$G_N(x):=-\pi^3x^5+3N^2(2N+3\pi)x^2-N^2(55N+15\pi),$$

where we have used $\csc(x)>\frac{1}{x}$ for all $x\in(0,\pi)$ (see (3.17)) and thus $-\frac{40N}{3\pi}>-\frac{40}{3}\csc\left(\frac{\pi}{N}\right)$ for all $N\geq 2$. It is immediate that $G'_N(x)>0$ if

$$x\in\left(0,\frac{N}{\pi}\left(\frac{12+18\frac{\pi}{N^2}}{5}\right)^{1/3}\right)\supset\left(0,\frac{N}{\pi}\frac{4}{3}\right).$$

For all $N\geq 3$, the latter interval contains $\left[\frac{N}{\pi},\csc\frac{\pi}{N}\right]$, thus allowing us to lower-bound $G_N\left(\csc\frac{\pi}{N}\right)$ by

$$G_N\left(\frac{N}{\pi}\right)=\frac{N^5}{\pi^2}\left(-1+6+\frac{9\pi}{N}-\frac{55\pi^2}{N^2}-\frac{15\pi^3}{N^4}\right)>0,\qquad N\geq 10.$$

In this way we obtain

$$S_N>\sigma_N>\frac{\csc\left(\frac{\pi}{N}\right)}{3N^3}G\left(\csc\frac{\pi}{N}\right)>0,\qquad N\geq 10.$$

$\square$

Lemma 3.3.6. *For $N\geq 10$, $V''_N(\boldsymbol{c}_P)$ has at least one eigenvalue greater than 1.*

Proof. Indeed, let $A=(a_{i,j})_{i,j=1,\ldots,2N}=V''_N(\boldsymbol{c}_P)$. The Rayleigh quotient for vector $\boldsymbol{v}=\boldsymbol{e}_{2N,2N-1}=(0,0,\cdots,0,1,0)^T$ is

$$\frac{\boldsymbol{v}^TA\boldsymbol{v}}{\boldsymbol{v}^T\boldsymbol{v}}=\frac{\mathbf{v}_N^T\tilde{U}_{N,N}\mathbf{v}_N}{\mathbf{v}_N^T\mathbf{v}_N}=a_{2N-1,2N-1}=\frac{\sum_{j=1}^{N-1}\left(\csc^3 j\frac{\pi}{N}\right)\left(3\cos 2j\frac{\pi}{N}-1\right)}{4\sum_{j=1}^{N-1}\csc j\frac{\pi}{N}},$$

and it will be strictly greater than 1 if and only if

$$\sum_{j=1}^{N-1}\left(3\cos\frac{2j\pi}{N}-1\right)\csc^3\frac{j\pi}{N}-4\sum_{j=1}^{N-1}\csc\frac{j\pi}{N}=\sum_{j=1}^{N-1}2\left(\csc^2\frac{j\pi}{N}-5\right)\csc\frac{j\pi}{N}>0,$$

which we already know holds for $N\geq 10$ by Lemma 3.3.5. Elementary Linear Algebra then yields the existence of at least one eigenvalue $\tilde{\lambda}>1$ for $V''_N(\boldsymbol{c}_P)$. $\square$

Since $\max S=1<\tilde{\lambda}$, $\tilde{\lambda}\notin S$ and this ends the proof for Theorem 3.2.3, item 2. $\square$

3.3.3 Proof isolate: $N = 2^m$ equal masses

For the sake of a (modest) diversification, and in order to show yet another way of confronting issues of non-integrability with arithmetical tools, we include this alternative proof of a weaker version of Theorem 3.2.3, item 2: namely, the case $N = 2^m$ with $m \geq 2$.

We know we can reorder the eigenvalues so as to obtain $\lambda_1 = k - 1 = -2$, $\lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$. These four eigenvalues belong to S. If all of $\lambda_5, \ldots, \lambda_{2N}$ did too, their sum

$$\operatorname{tr}\left(V_N''\left(\boldsymbol{c}_P\right)\right) = -1 + \lambda_5 + \cdots + \lambda_{2N} = -N\left(\frac{\sum_{k=1}^{N-1}\frac{1}{\sin^3\left(\frac{\pi k}{N}\right)}}{2\sum_{k=1}^{N-1}\frac{1}{\sin\left(\frac{\pi k}{N}\right)}}\right), \tag{3.18}$$

would be an integer number μ_N such that $-\infty < \mu_N \leq 2N - 5$ since the only positive term in S is 1.

Proving the trace of $V_N''(\boldsymbol{c})$, i.e. the sum of its eigenvalues, a non-integer will be enough to settle the rest of Corollary 3.3.11; in view of (3.18), such a condition is immediate if we prove that any relation of the form

$$n_1 \sum_{k=1}^{N-1} \csc \frac{\pi}{N} k + n_2 \sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k = 0, \tag{3.19}$$

where $n_1, n_2 \in \mathbb{Z}$, implies $n_1 = n_2 = 0$.

As in the previous Subsection, let $\zeta = \cos\frac{\pi}{N} + \mathrm{i}\sin\frac{\pi}{N}$ be a primitive $2N^{\text{th}}$ root of unity. Then, $\sin\frac{\pi k}{N} = \frac{1}{2\mathrm{i}}\left(\zeta^k - \zeta^{-k}\right)$ for each k, and thus

$$\sum_{k=1}^{N-1} \csc \frac{\pi}{N} k = 2\mathrm{i} \sum_{k=1}^{N-1} \frac{1}{\zeta^k - \zeta^{-k}}, \qquad \sum_{k=0}^{N-1} \csc^3 \frac{\pi}{n} k = -8\mathrm{i} \sum_{k=1}^{N-1} \left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3.$$

Any relation of the form (3.19) would thus yield

$$\sum_{k=1}^{N-1} \frac{1}{\zeta^k - \zeta^{-k}} - \alpha \sum_{k=1}^{N-1} \left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3 = 0,$$

for some $\alpha \in \mathbb{Q}$. Singling out summands with index $N/2$ yields

$$2\sum_{k=1}^{N-1} \frac{1}{\zeta^k - \zeta^{-k}} + \frac{1}{\zeta^{N/2} - \zeta^{-N/2}} = \alpha\left[2\sum_{k=1}^{N-1} \frac{1}{\left(\zeta^k - \zeta^{-k}\right)^3} + \frac{1}{\left(\zeta^{N/2} - \zeta^{-N/2}\right)^3}\right] = 0,$$

which, since $\zeta^{N/2} = \mathrm{i}$, and thus $\zeta^{-N/2} = -\mathrm{i}$, becomes

$$2\sum_{k=1}^{N/2-1} \frac{1}{\zeta^k - \zeta^{-k}} - \frac{\mathrm{i}}{2} = \alpha\left(2\sum_{k=1}^{N/2-1} \frac{1}{\left(\zeta^k - \zeta^{-k}\right)^3} + \frac{\mathrm{i}}{8}\right) \tag{3.20}$$

for some $\alpha \in \mathbb{Q}$. The next lemmae are aimed at proving that such an equation as (3.20) is unfeasible for the only possible value of α, which will be found to be -4.

Remark 3.3.7. We recall that since $\dim_{\mathbb{Q}} \mathbb{Q}(\zeta) = N = 2^m$, the set of roots of unity $\left\{1, \zeta, \ldots, \zeta^{N-1}\right\}$ is rationally independent. So is, therefore, any set $\left\{\zeta^{kj} : 1 \leq j \leq M\right\}$ of cardinality $M \leq N-1$, where $k > 0$ is an arbitrary integer.

We may find two possible expressions of $\frac{1}{\zeta^k - \zeta^{-k}}$ depending on the parity of k:

Lemma 3.3.8. *Let $k = 1, \ldots, N/2$. Then,*

$$\frac{1}{\zeta^k - \zeta^{-k}} = \begin{cases} -\frac{1}{2} \sum_{j=1}^{N/2} \zeta^{(N-2j+1)k}, & k \text{ odd}, \\ -\frac{1}{2} \sum_{j=1}^{2^{m-n-1}} \zeta^{\left(2^{m-n}-2j+1\right)k}, & k = 2^n q,\ q \text{ odd}. \end{cases}$$

Proof. In general, if $u = \zeta^k$ and $\frac{1}{u-u^{-1}} = -\frac{1}{2}\left(u + u^3 + u^5 + \cdots + u^{r-3} + u^{r-1}\right)$ for some $r \leq N$,

$$\begin{aligned} -2 &= \left(u - u^{-1}\right)\left(u + u^3 + u^5 + \cdots + u^{r-3} + u^{r-1}\right) \\ &= u^2 + u^4 + \cdots + u^{r-2} + u^r - \left(1 + u^2 + u^4 + \cdots + u^{r-2}\right) \\ &= u^r - 1, \end{aligned}$$

meaning $\zeta^{kr} = -1$, i.e. $kr = N(2p+1)$ for some $p \in \mathbb{N}$.

1. If k is odd, the facts $kr = N(2p+1)$ and $N = 2^m$ imply $k \mid 2p+1$ and thus $r = \tilde{q}2^m$ for some odd $\tilde{q}$; the minimum value of r satisfying this is $r = 2^m = N$, and indeed $\frac{1}{\zeta^k - \zeta^{-k}} = -\frac{1}{2}\left(\zeta^k + \zeta^{3k} + \cdots + \zeta^{(N-1)k}\right)$ as may be checked multiplying both sides by $\zeta^k - \zeta^{-k}$.

2. For even k we have $k = 2^n s < N/2 = 2^{m-1}$ for some odd integer s, implying $n < m-1$; furthermore, $kr = N(2p+1)$ implies $sr = (1+2p)\,2^{m-n}$; since s is odd, $s \mid 2p+1$ and thus $r = \tilde{q}2^{m-n}$ for some odd $\tilde{q}$; the minimal such r is $r = 2^{m-n}$, and again a simple check indeed assures $\frac{1}{\zeta^k - \zeta^{-k}} = -\frac{1}{2}\left(\zeta^k + \zeta^{3k} + \cdots + \zeta^{\left(2^{m-n}-1\right)k}\right)$.

$\square$

Let $P(\zeta)$ (resp. $Q(\zeta)$) be the polynomial expression of $\sum_{k=1}^{N/2-1} \frac{1}{\zeta^k - \zeta^{-k}}$ (resp. $\sum_{k=1}^{N/2-1} \left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3$) of degree smaller than or equal to $N-1$, attained by reduction via $\zeta^N = -1$. This means (3.20) may be written as $2P(\zeta) - \frac{\mathrm{i}}{2} = \alpha\left(2Q(\zeta) + \frac{\mathrm{i}}{8}\right)$; let us write $P(\zeta) = \sum_{k=0}^{N-1} a_k \zeta^k$ and $Q(\zeta) = \sum_{k=0}^{N-1} b_k \zeta^k$. We are now going to discard cross-contributions to two particular powers of ζ in these polynomials:

Lemma 3.3.9. *Let $\tilde{k} \in \{1, \ldots, N-1\}$. Then,*

1. *if $\tilde{k} = N/2$, $a_{\tilde{k}} = b_{\tilde{k}} = 0$. In particular, $\alpha = -4$.*

2. *If $\tilde{k} = 2^{m-2}$, the only summand $\frac{1}{\zeta^k - \zeta^{-k}}$ in $P(\zeta)$ (resp. $\left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3$ in $Q(\zeta)$) whose polynomial in powers of ζ contains a non-zero coefficient of $\zeta^{\tilde{k}}$, is precisely $\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}$ (resp. $\left(\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}\right)^3$).*

Proof.

1. Let $j \in \{1, \dots, N-1\}$. We may assume $j < N/2$ due to (3.20) and in view of Lemma 3.3.8 there is an even $r_j \in \{2^{m-n}, N\}$ such that

$$\begin{cases} \left(\zeta^j - \zeta^{-j}\right)^{-1} = -\frac{1}{2}\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{(r_j-1)j}\right), \\ \left(\zeta^j - \zeta^{-j}\right)^{-3} = -\frac{1}{8}\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{(r_j-1)j}\right)^3; \end{cases} \tag{3.21}$$

a) if j is odd, $\zeta^j + \zeta^{3j} + \dots + \zeta^{(r_j-1)j}$ includes exclusively odd powers of ζ, i.e. $a_{N/2} = 0$; this is also the case with $\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{(N-1)j}\right)^3$, since it is a polynomial containing powers of the form $\zeta^{(q_1+q_2+q_3)j}$ where $q_1+q_2+q_3 > 1$ is an odd positive integer. Thus, in particular $b_{N/2} = 0$.

b) If j is even, say $j = 2^n q$ with q odd (which implies $n < m-1$), $\zeta^j + \zeta^{3j} + \dots + \zeta^{\left(2^{m-n}-1\right)j}$ consists of powers of the form $\zeta^{\tilde{q}2^n}$ with $\tilde{q}$ odd. These even exponents $\tilde{q}2^n$ are different (mod $2N$) from $2^{m-1} = N/2$. Indeed, any relation of the form $2^n \cdot \tilde{q} = 2^{m-1} + p2^{m+1}$ for some integer p would imply $\tilde{q} = 2^{m-n-1} + p2^{m-n+1}$, impossible since $2^{m-n-1} + p2^{m-n+1}$ is even. Meanwhile, the exponents in $\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{\left(2^{m-n}-1\right)j}\right)^3$ are again of the form $(q_1 + q_2 + q_3)\, j$ as in a), and thus a particular case of the form $\tilde{q}2^n$ just studied, which implies $[(q_1 + q_2 + q_3)\, j]_{2N} \neq [N/2]_{2N}$ and thus $b_{N/2} = 0$.

Thus, for each j neither of the sum expressions in (3.21) contains $\zeta^{N/2}$, implying $a_{N/2} = b_{N/2} = 0$, and since $\mathrm{i} = \zeta^{N/2}$ and the set $\left\{\zeta^{kj} : 1 \leq j \leq N-1\right\}$ is an independent one (Remark 3.3.7), the only contribution to i in each side of (3.20) is precisely the one we singled out of each sum in that equation, i.e. $-\frac{\mathrm{i}}{2} = \alpha\frac{\mathrm{i}}{8}$, meaning $\alpha = -4$.

2. For the same reasons as in item 1, we may restrict to $j \in \{1, \dots, N/2-1\}$.

a) If j is odd, as seen in 1.a) above both $\zeta^j + \zeta^{3j} + \dots + \zeta^{(2^m-1)j}$ and $\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{(2^m-1)j}\right)^3$ are a sum of odd powers of ζ, none of them congruent to the *even* number $\tilde{k} = 2^{m-2}$ (mod $2N$).

b) If $j < 2^{m-1} = N/2$ is even and $j \neq 2^{m-2}$, writing $j = 2^n \cdot q$ for some n and some odd q, implies $n < m-2$ (since $n = m-2$ would imply $q = 1$ and thus $j = 2^{m-2}$) and $\zeta^j + \zeta^{3j} + \dots + \zeta^{\left(2^{m-n}-1\right)j}$, has exponents different modulo N from 2^{m-2} as is proven by the exact same reasoning as in item 1.b) while since every expression of the form $2^{m-n-2} + Q2^{m-n}$, $Q \in \mathbb{Z}$, is even if $n < m-2$. Same applies thus to $\left(\zeta^j + \zeta^{3j} + \dots + \zeta^{\left(2^{m-n}-1\right)j}\right)^3$, as in item 1 *mutatis mutandis.*

$\square$

We finally obtain the result which is central to this Subsection:

Theorem 3.3.10. *For any $N \in \mathbb{N}$ of the form $N = 2^m$, $m \geq 2$, $\sum_{k=1}^{N-1} \csc \frac{\pi}{N}k$ and $\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N}k$ are $\mathbb{Q}$-independent, i.e., any equation of the form* (3.19)*, where $n_1, n_2 \in \mathbb{Z}$, implies $n_1 = n_2 = 0$.*

Proof. As said before, any relation of the form (3.19) may be written in the form (3.20) for some $\alpha \in \mathbb{Q}$. In virtue of item 1 in Lemma 3.3.9, $\alpha = -4$, and (3.20) thus provides for

$$2 \sum_{k=1}^{N/2-1} \frac{1}{\zeta^k - \zeta^{-k}} - \frac{\mathrm{i}}{2} = -4 \left(2 \sum_{k=1}^{N/2-1} \frac{1}{\left(\zeta^k - \zeta^{-k}\right)^3} + \frac{\mathrm{i}}{8} \right),$$

i.e. for $2\sum_{k=0}^{N-1} a_k \zeta^k - \frac{\mathrm{i}}{2} = -4\left(2\sum_{k=0}^{N-1} b_k \zeta^k + \frac{\mathrm{i}}{8}\right)$ (according to the notation introduced immediately prior to Lemma 3.3.9), which in view of Remark 3.3.7 implies $a_k = -4b_k$ for $k = 1, \ldots N-1$. However, let us express $a_{\tilde{k}} = \alpha b_{\tilde{k}}$ for $\tilde{k} = 2^{m-2} = N/4$; this we can do since, in virtue of Lemma 3.3.9 (item 2), we just have to compare the coefficients in $\zeta^{\tilde{k}}$ of $\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}$ and $\left(\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}\right)^3$. Since $\zeta^{4\tilde{k}} = \zeta^N = -1$, we have $\zeta^{6\tilde{k}} = -\mathrm{i} = \zeta^{-2\tilde{k}}$, meaning

$$\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}} = -\frac{1}{2}\left(\zeta^{\tilde{k}} + \zeta^{3\tilde{k}}\right), \qquad \left(\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}\right)^3 = \frac{1}{4}\left(\zeta^{\tilde{k}} + \zeta^{3\tilde{k}}\right),$$

which would imply $-\zeta^{\tilde{k}}/2 = \alpha\zeta^{\tilde{k}}/4$, i.e. $\alpha = -2$, an absurd since we know $\alpha = -4$. □

Hence, the trace of $V_3''(\boldsymbol{c}_P)$, written in (3.18), is irrational, and thus a non–integer; in virtue of Theorem 2.3.3 and Lemma 3.2.9, we conclude the following:

Corollary 3.3.11. *The d-dimensional N-Body Problem with N equal masses is meromorphically non-integrable for $N = 2^m$ with $m \geq 2$.* □

Chapter 4

The meromorphical non-integrability of Hill's Lunar problem

4.1 Introduction

As said in Subsection 1.3.2, Hill's problem is arguably the foremost step of simplification of the 3-Body Problem which still retains dynamical significance and, yet, it still displays most of the numerical evidence inherent to chaotic dynamical systems. Hence, establishing its non-integrability in a rigorous way has long been a tempting, if elusive, goal; for instance, monodromy groups for the normal variational equations apparently yield invariably resonant matrices, thus discarding the application of Ziglin's Theorem (see [162] for details). And the infeasibility of a form $H(\boldsymbol{Q}, \boldsymbol{P}, \varepsilon) = H_0(\boldsymbol{Q}) + \varepsilon H_1$ makes the application of KAM criteria either impossible or impractical. Thus, our ultimate approach has been the use of the most general instance of *Morales–Ramis Theorem* 2.3.1. Thanks to the latter, a symplectic change and a series of minor operations, we have afforded the proof avoiding burdensome calculations and strict dependence on numerical results.

In Chapter 2 we already introduced the basic theory needed. In this chapter, Section 4.1.1 exposes the actual problem and states the main results. The ensuing three Sections are the main body of the proof. Its first part (corresponding to Section 4.2) is based on the **computation of a particular solution of $\boldsymbol{HP}$**; this solution and the sort of integral curve Γ it determines, are in turn useful for the second part, inscribed in Section 4.3 and consisting on the **layout (and a fundamental matrix Ψ) of the variational equations of $\boldsymbol{HP}$ along** Γ. The information we need about the matrix, included in 4.3.3, is actually less than computing the whole of Ψ explicitly, as we will see in Section 4.4: the **study of the Galois differential group of the Picard-Vessiot extension for the aforementioned variational equations**. This will be the concluding part of our proof, using the relevant facts concerning Ψ to apply Morales–Ramis Theorem.

Concerning the recent papers [88], [115] devoted to the very same goal through different techniques, in Section 4.5 we extend on a comment regarding their authors' hypotheses.

As said in Section 1.3.2, the results proven in the following Sections may be found in reference [98]; the author is indebted to his coauthors, J. J. Morales-Ruiz and C. Simó, for the uncountable discussions and teaching about differential Galois theory, group theory and elliptic functions and about Hill's problem, elimination of Coriolis force and variational equations, respectively.

4.1.1 Statement of the main results

A first lemma restricts our study to a particular solution of *HP* contained in an affine submanifold of the phase space $\mathbb{A}^4_{\mathbb{C}}$; we call it an *invariant plane* solution.

Lemma 4.1.1. *$X_{\mathcal{H}}$ has a particular solution (depending on the energy level h) of the form*

$$(Q_1(t), Q_2(t), P_1(t), P_2(t)) = \frac{1}{\sqrt{2}} \left(\phi(t), \mathrm{i}\phi(t), \dot{\phi}(t), \mathrm{i}\dot{\phi}(t)\right). \tag{4.1}$$

For all $0 < h < 1/\left(6\sqrt{3}\right)$, $\phi^2(t)$ is elliptic with two simple *poles in each period parallelogram.*

Using this and properties of the specific elliptic function involved in $\phi(t)$, we then obtain

Lemma 4.1.2. *The variational equations of $X_{\mathcal{H}}$ along solution (4.1) have a fundamental matrix of the form*

$$\Psi(t) = \begin{pmatrix} \Phi_N(t) & \Phi_N(t) \int_0^t V(\tau) d\tau \\ 0 & \Phi_N(t) \end{pmatrix},$$

where

$$\Phi_N(t) = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ \mathrm{i}\dot{\xi}_1(t) & \mathrm{i}\dot{\xi}_2(t) \end{pmatrix}$$

is a fundamental matrix of the normal *variational equations; furthermore, ξ_2 is a linear combination of elliptic functions and nontrivial elliptic integrals of first and second classes, and $\int_0^t V(\tau) d\tau$ is a 2×2 matrix function containing logarithmic terms in its diagonal.*

This allows a careful study of the P–V extension for the variational system, yielding the following

Theorem 4.1.3. *The identity component G^0 of the Galois differential group of variational equations is non-commutative.*

This proven, Theorem 2.3.1 gives the main result:

Corollary 4.1.4. *Hill's problem does not admit a meromorphic integral of motion independent of its Hamiltonian.* $\square$

4.2 Proof of Lemma 4.1.1

4.2.1 Change of variables

Matrix $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$ provides for a symplectic change of variables,

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A \begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{pmatrix}, \qquad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \left(A^{-1}\right)^T \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \end{pmatrix},$$

which in turn transforms Hamiltonian (2.22) into

$$\bar{\mathcal{H}} = \mathrm{i}(\bar{Q}_1\bar{Q}_2 - \bar{P}_1\bar{P}_2) - 4\mathrm{i}(3\bar{Q}_1^4 - 2\bar{Q}_1^2\bar{Q}_2^2 + 3\bar{Q}_2^4)\bar{Q}_1\bar{Q}_2 - 4\bar{Q}_1\bar{Q}_2(\bar{Q}_1\bar{P}_1 - \bar{Q}_2\bar{P}_2).$$

The corresponding differential system $\dot{\bar{z}} = X_{\bar{\mathcal{H}}}(\bar{\boldsymbol{z}})$ now displays two invariant planes

$$\pi_1 : \left\{\bar{Q}_2 = \bar{P}_1 = 0\right\}, \quad \pi_2 : \left\{\bar{Q}_1 = \bar{P}_2 = 0\right\},$$

in any of which all nontrivial information of that system reduces to a hyperelliptic equation,

$$\ddot{\phi} = -\phi + 12\phi^5, \tag{4.2}$$

which through multiplication by $\dot{\phi}$ and subsequent integration becomes

$$\dot{\phi}^2 = -\phi^2 + 4\phi^6 + 2h. \tag{4.3}$$

Defining $w = \phi^2,\ z = 2\phi\dot{\phi}$, we arrive to the system

$$\dot{w} = z, \quad \dot{z} = 4\left(-w + 8w^3 + h\right), \tag{4.4}$$

whose Hamiltonian (at level zero energy) is $\mathcal{K}(w, z) = \frac{1}{2}z^2 + 2w^2 - 8w^4 - 4hw$.

Remark 4.2.1. The fact that in these invariant planes everything becomes simpler has a clear mechanical meaning. Some difficulties appear in (2.22) due to the presence of H_4, which mixes positions and momenta. It corresponds to the Coriolis term coming from the rotating frame. The present choice of variables singles out (complex) planes in which this term becomes zero.

4.2.2 Solution of the new equation

The solution to system (4.4), or equivalently to equation $\dot{w}^2 = -4w^2 + 16w^4 + 8hw$, is the inverse of an elliptic integral:

$$t = \pm \int_0^{w(t)} (-4y^2 + 16y^4 + 8hy)^{-1/2} dy + K_1, \quad K_1 \in \mathbb{C},$$

translation $t \mapsto t - K_1$ being the next obvious step. It is a known fact (see [150, Chapter XX, §20.6 (Example 2, p. 454)]) that given a polynomial of degree four without repeated factors, $p_4(x) = a_4x^4 + 4a_3x^3 + 6a_2x^2 + 4a_1x + a_0$, and defining constants (called *invariants*)

$$g_2 = a_4a_0 - 4a_3a_1 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_4a_1^2 - a_3^2a_0,$$

then the solution for $t = \int_a^{w(t)} (p_4(x))^{-1/2} dx$ is the following:

$$w(t) = a + \frac{\sqrt{p_4(a)}\dot{\wp}(t; g_2, g_3) + \frac{1}{2}\dot{p}_4(a)\left[\wp(t; g_2, g_3) - \frac{1}{24}\ddot{p}_4(a)\right] + \frac{1}{24}p_4(a)\dddot{p}_4(a)}{2\left[\wp(t; g_2, g_3) - \frac{1}{24}\ddot{p}_4(a)\right]^2 - \frac{1}{48}p_4(a)p_4^{(4)}(a)},$$

where $\wp(t; g_2, g_3)$ is the *Weierstrass elliptic function* ([150, Chapter XX, §20.2]). In our specific case, this becomes

$$w(t) = 6h/F(t), \quad z(t) = -18h\dot{\wp}(t; g_2, g_3)/F^2(t),$$

where $F(t) := 3\wp(t; g_2, g_3) + 1$. In particular,

$$\phi_1(t) = \sqrt{6h/F(t)}, \quad \phi_2(t) = -\phi_1(t),$$

are solutions to original equation (4.2). Furthermore, a simple calculation proves $h^* = 1/(6\sqrt{3})$ to be a separatrix value in which $\phi_1^2(t) = \phi_2^2(t)$ breaks down into combinations of hyperbolic functions. In order to step into the next Subsection, we are therefore assuming $0 < h < h^*$.

4.2.3 Singularities of $\phi^2(t)$

We are now proving that, for the above range of h, $w(t)$ has two simple poles in each period parallelogram, the sides of which will be denoted as $2\omega_1, 2\omega_2$, as usual. In virtue of [36, p. 96], expression $1/(\wp(t) - \wp(t^*))$ (in our case, $\wp(t^*) = -1/3$) has exactly two simple poles in $t^*, -t^*$ (mod $2\omega_1, 2\omega_2$), with respective residues $1/\dot{\wp}(t^*)$ and $-1/\dot{\wp}(t^*)$. Therefore, all *double* poles, if any, of $1/(\wp(t) - \wp(t^*))$, expanding around $t = t^*$, are precisely those t^* such that $\dot{\wp}(t^*) = 0$. We have

$$(\dot{\wp}(t; g_2, g_3))^2 = 4(\wp(t; g_2, g_3))^3 - g_2\wp(t; g_2, g_3) - g_3 = 4\wp^3 - \frac{4}{3}\wp - \frac{8}{27} + 64h^2,$$

and every pole (whether double or not) must satisfy $\wp(t^*) = -1/3$; $X = -1/3$ is obviously not a root of $4X^3 - 4X/3 - 8/27 + 64h^2$ unless $h = 0$. This ends the proof. $\square$

4.3 Proof of Lemma 4.1.2

4.3.1 Layout of the system

Reordering the vector of dependent canonical variables as $(\bar{Q}_1, \bar{P}_2, \bar{Q}_2, \bar{P}_1)^T$ and restricting ourselves to the particular solution found in Section 4.2,

$$\bar{Q}_1 = \phi, \quad \bar{Q}_2 = 0, \quad \bar{P}_1 = 0, \quad \bar{P}_2 = \mathrm{i}\dot{\phi},$$

the *variational equations* along that solution are written as

$$\begin{pmatrix} \dot{\bar{\xi}} \\ \dot{\bar{\eta}} \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} & -4w & 0 \\ \mathrm{i}(60w^2-1) & 0 & -4\mathrm{i}z & 4w \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i}(60w^2-1) & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \\ \xi \\ \eta \end{pmatrix}$$

$$=: \begin{pmatrix} A_1 & B_1 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \\ \xi \\ \eta \end{pmatrix}, \tag{4.5}$$

and their lower right block, the *normal variational equations*

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i}(60w^2-1) & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{4.6}$$

that is,

$$\ddot{\xi}(t) = (60w^2(t)-1)\xi(t). \tag{4.7}$$

Next step is to obtain a fundamental matrix for (4.6). An obvious shortcut is to take w as new independent variable and to define $\Xi(w), H(w)$ such that $\xi = \Xi \circ w$ and $\eta = H \circ w$. We have

$$\frac{d^2\Xi}{dw^2} = 4\left(\frac{w-8w^3-h}{wf(w,h)}\right)\frac{d\Xi}{dw} + \frac{60w^2-1}{wf(w,h)}\Xi, \tag{4.8}$$

also expressible in matrix form

$$\begin{pmatrix} \frac{d}{dw}\Xi \\ \frac{d}{dw}H \end{pmatrix} = \frac{1}{\sqrt{wf(w,h)}} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i}(60w^2-1) & 0 \end{pmatrix} \begin{pmatrix} \Xi \\ H \end{pmatrix}, \tag{4.9}$$

where $f = f(w,h) = 4(4w^3 - w + 2h)$.

4.3.2 Fundamental matrix of the variational equations

We are now interested in the fundamental matrix of (4.5). Let us start from the block notation

$$\Psi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{4.10}$$

P, Q, R, S being 2×2 matrices with their entries in some differential field to be described in Section 4.4. We can assume $\Psi(0) = \mathrm{Id}_4$, which, along with the triangular form of (4.5), assures $R \equiv 0$. In particular, the matrix form of the normal variational system (4.6) can be written as $\dot{S} = A_1 S$. Let us now proceed to integrate these normal equations. More precisely, let us explicit all necessary information about the fundamental matrix $\Phi_N(t)$ of (4.6) with initial condition $\Phi_N(0) = \mathrm{Id}_2$.

Using well-known properties of $\dot{\wp}$ and $\ddot{\wp}$, it is easy to prove that $\Xi_1(w) = \sqrt{f(w,h)}$ is a solution of (4.8), and therefore

$$\xi_1(t) = \Xi_1(w(t)) = \dot{\wp}(t; g_2, g_3)\left(3\wp(t; g_2, g_3) + 1\right)^{-3/2},$$

is a solution of (4.7). A first solution of (4.6) is then

$$\begin{pmatrix} \xi_1(t) \\ \eta_1(t) \end{pmatrix} = C_1 \begin{pmatrix} \sqrt{16w^3(t) - 4w(t) + 8h} \\ -2\mathrm{i}\,(12w^2(t) - 1)\sqrt{w(t)} \end{pmatrix}, \quad C_1 \in \mathbb{C}.$$

We now recall *d'Alembert's method* ([56, p. 122]) in order to obtain a second solution of (4.7) independent of ξ_1. This solution is

$$\xi_2(t) = \xi_1(t) \int_0^t \{\xi_1(\tau)\}^{-2}\, d\tau; \tag{4.11}$$

see Subsection 4.3.3 for further details. After recovering our former independent variable t through composition we have a fundamental matrix for the normal variational equations, that is, the block S in (4.10),

$$\Phi_N(t) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \mathrm{i}\dot{\xi}_1 & \mathrm{i}\dot{\xi}_2 \end{pmatrix}.$$

In particular, $P(t) \equiv S(t)$ since they are both fundamental matrices for the same initial value problem. We now compute the block Q in (4.10); the standing equations (in vector form) are

$$\begin{pmatrix} \dot{\bar{\xi}} \\ \dot{\bar{\eta}} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i}(60w^2 - 1) & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} + \begin{pmatrix} -4w & 0 \\ -4\mathrm{i}z & 4w \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{4.12}$$

where $(\xi, \eta)^T$ are the solutions to the normal variational system. Applying variation of constants to (4.12) we obtain

$$Q(t) = \Phi_N(t) \int_0^t V(\tau) d\tau, \tag{4.13}$$

where

$$C(t) = \begin{pmatrix} -4w(t) & 0 \\ -4\mathrm{i}z(t) & 4w(t) \end{pmatrix}, \qquad V(t) = \Phi_N^{-1}(t) C(t) \Phi_N(t).$$

In other words, the fundamental matrix of (4.5) has the form

$$\Psi(t) = \begin{pmatrix} \Phi_N(t) & \Phi_N(t) \int_0^t V(\tau) d\tau \\ 0 & \Phi_N(t) \end{pmatrix}. \tag{4.14}$$

Remark 4.3.1. In view of (4.13), computing Ψ *explicitly* would now only take the computation of four integrals. *The path we are taking, however, is a different one,* although we are keeping in mind all of this notation and the final expression (4.14).

4.3.3 Relevant facts concerning $\Psi(t)$

As said in Section 4.1 and in the above remark, we are not coping with the calculations needed to obtain (4.13) explicitly. Instead, our next aim is to prove only two specific properties of the fundamental matrix Ψ of (4.5), namely the existence of first and second class elliptic integrals and logarithmic terms in its coefficients. The two consecutive steps of transcendence forced by these two new objects will provide the rest of our proof.

Elliptic integrals in Φ_N

Let K be the field of all elliptic functions of the complex plane. We know a solution of (4.7),

$$\xi_1(t) = \sqrt{4w^3(t) - w(t) + 2h},$$

and can obtain a second one using (4.11) and the chain rule. Let us define $\alpha_1, \alpha_2, \alpha_3$ as the values of w for which $f(w,h) = 0$, the functions

$$\beta(w,h) := \arcsin\left(\sqrt{\frac{w(\alpha_3 - \alpha_1)}{\alpha_3(w - \alpha_1)}}\right), \qquad k(h) := \sqrt{\frac{\alpha_3(\alpha_1 - \alpha_2)}{\alpha_2(\alpha_1 - \alpha_3)}},$$

(both attaining complex, nonzero values if $h \in (0, h^*)$ and therefore $w(t) \neq 0$) and let

$$E(\beta|k) := \int_0^\beta (1 - k^2 \sin^2\theta)^{-\frac{1}{2}} d\theta, \qquad F(\beta|k) := \int_0^\beta (1 - k^2 \sin^2\theta)^{\frac{1}{2}} d\theta.$$

be the *elliptic integrals of first and second class*, respectively (see [36], [150]). We then obtain a fundamental matrix for the normal variational equations (4.9),

$$\begin{aligned}\overline{\Phi}_N(w) &= \begin{pmatrix} \Xi_1(w) & \Xi_2(w) \\ H_1(w) & H_2(w) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{f(w,h)} & g_1\left\{f_1 E(\beta|k) + f_2 F(\beta|k) + g_2\right\} \\ 2\mathrm{i}\sqrt{w}(-1 + 12w^2) & \mathrm{i}\frac{d}{dw}(g_1\left\{f_1 E(\beta|k) + f_2 F(\beta|k) + g_2\right\}) \end{pmatrix},\end{aligned}$$

for some $f_1 = f_1(h), f_2 = f_2(h), g_1 = g_1(w,h), g_2 = g_2(w,h)$, the first three non-vanishing if $h \in (0, h^*)$, and the last two linked to w by *algebraic* equations. In particular, this yields our fundamental matrix $\Phi_N(t) = \overline{\Phi}_N(w(t))$ for (4.6).

Remark 4.3.2. The fundamental trait of $E(\beta|k)$ and $F(\beta|k)$ is that *they are transcendental over* K. Indeed, nontrivial elliptic integrals of the first and second classes are not elliptic functions (see [36, Theorem 6.5 and its proof]) and they stem from quadratures; thus, as said in Remark 2.2.16(1), $E(\beta|k)$ and $F(\beta|k)$ cannot be expressed in terms of elliptic functions under any relation of algebraic dependence.

Logarithms in Ψ

Let us prove the existence of terms with nonzero residue in the diagonal of matrix $V(t)$. As

$$\Phi_N(t) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \mathrm{i}\dot{\xi}_1 & \mathrm{i}\dot{\xi}_2 \end{pmatrix}$$

is the fundamental matrix of a Hamiltonian linear system, it is symplectic. The integrand in (4.13) becomes

$$
\begin{aligned}
V(t) &= 4\,\mathrm{i}\begin{pmatrix} -w(\xi_2\dot{\xi}_1+\xi_1\dot{\xi}_2)+\dot{w}\xi_1\xi_2 & -\xi_2(2\dot{\xi}_2 w-\xi_2\dot{w}) \\ (2w\dot{\xi}_1-\dot{w}\xi_1)\xi_1 w(\xi_1\dot{\xi}_2+\dot{\xi}_1\xi_2)-\dot{w}\xi_1\xi_2 & \end{pmatrix} \\
&=: 4\mathrm{i}\begin{pmatrix} u(t) & v_1(t) \\ v_2(t) & -u(t) \end{pmatrix}.
\end{aligned}
$$

For every $h \in (0, h^*)$, and taking profit of what was proved in 4.2.3, we expand these four entries around a simple pole t^* of $w(t)$; expressing only the first term in each power series, we have

$$
\begin{aligned}
w(t) &= C_0(t-t^*)^{-1}+O(1), \\
\xi_1(t) &= 2C_0^{3/2}(t-t^*)^{-3/2}+O\left((t-t^*)^{-1/2}\right), \\
\xi_2(t) &= \frac{C_0^{-3/2}}{8}(t-t^*)^{5/2}+O\left((t-t^*)^{7/2}\right),
\end{aligned}
$$

for some $C_0 = C_0(h) \in \mathbb{C}$; therefore,

$$
\begin{aligned}
u(t) &= -\frac{C_0}{2}(t-t^*)^{-1}+O(1), \\
v_1(t) &= -\frac{3}{32C_0^2}(t-t^*)^{3}+O\left((t-t^*)^{4}\right), \\
v_2(t) &= -8C_0^4(t-t^*)^{-5}+O\left((t-t^*)^{-4}\right).
\end{aligned}
$$

Hence, and except for the only value of h forcing $C_0 = 0$ (i.e. $h = 0$), we have a nonzero residue in $u\,(t)$, which results in the aforementioned logarithmic terms in the diagonal of

$$
\int_0^t V(\tau)d\tau = \begin{pmatrix} \int_0^t u(\tau)d\tau & \int_0^t v_1(\tau)d\tau \\ \int_0^t v_2(\tau)d\tau & -\int_0^t u(\tau)d\tau \end{pmatrix}. \qquad \square
$$

Remark 4.3.3. Same as before, appears a class of functions that cannot be linked algebraically to the former. Indeed, logarithms are special cases of elliptic integrals of the *third* class, which are neither elliptic functions nor elliptic integrals of first or second class (see [36, Theorem 6.5 and its proof] once more), and in this case the logarithms have been obtained through a quadrature. Remark 2.2.16(1) yields the rest.

We thus have a second transcendental extension of fields of functions; it is the combination of this with the previous extension that will ultimately render G^0 non-commutative.

4.4 Proof of Theorem 4.1.3

Let us interpret our results in terms of field extensions. First of all, we note that using coordinates $(x, y) = (\phi, \dot{\phi})$ all solutions of the equation (4.3) roam in the hyperelliptic curve

$$
\Gamma_h := \left\{(x,y) \in \mathbb{C}^2 : y^2 = -x^2+4x^6+2h\right\}.
$$

Denote by VE_{Γ_h} the expression of the variational equations (4.5) and $\widehat{G} := \mathrm{Gal}(EV_{\Gamma_h})$; let $\widehat{G}^0$ be the identity component of $\widehat{G}$. The previous transformation $w = x^2, z = 2xy$ induces a finite branched covering

$$\Gamma_h \to \Lambda_h,$$

where Λ_h is the elliptic curve defined by

$$\Lambda_h := \{z^2/2 + 2w^2 - 8w^4 - 4hw = 0\}$$

and the group $\widehat{G}^0$ does not change; this is a consequence of the basic result [95, Theorem 5] (see also [93, Theorem 2.5]), already mentioned in Remark 2.3.4(2), according to which the identity component of the Galois group remains invariant under covering maps of this sort. We may thus keep with the abuse in notation of calling $\widehat{G}$ and $\widehat{G}^0$ the Galois group of VE_{Γ_h} and its identity component, respectively, *now in variable* w.

Keeping K $(= \mathcal{M}(\Lambda_h))$ as the field of all elliptic functions, let us explicit the Picard-Vessiot extension over K for VE_{Γ_h}.

1. First of all, let us define the extension

$$K \subset K_1 := K(\xi_1, \dot{\xi}_1),$$

based on the adjunction of the first solution ξ_1 of (4.6) and its derivative, which is an algebraic (in fact, quadratic) one. The identity component of the Galois group of this extension is, therefore, trivial.

2. Second of all, adjoining to this new field the solution ξ_2 from (4.11) we obtain the extension

$$K_1 \subset L_1 := K_1(\xi_2, \dot{\xi}_2) = K(\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2),$$

which is *transcendental*, since it is nontrivial and defined exclusively by an adjunction of quadratures (see Remark 4.3.2).

3. Third of all, adjoining the matrix integral from (4.13) to L_1, we have

$$L_1 \subset L_2 := L_1\left(\int_0^t u, \int_0^t v_1, \int_0^t v_2\right),$$

also given by quadratures, nontrivial, and thus transcendental, in virtue of Remark 4.3.3.

So far, the P–V extension $L_2 \mid K$ of the variational equations has been decomposed as a tower of P–V extensions

$$K \subset K_1 \subset L_1 \subset L_2.$$

Let $\widehat{G} := \mathrm{Gal}(L_2 \mid K)$. The fact that each of above extensions results from adjoining either algebraic elements or quadratures renders $L_2 \mid K$ a *Liouville* extension, and thus $\widehat{G}^0$ a *solvable* group. Our aim is to prove that the (stronger) condition demanded by Theorem 2.3.1 is not fulfilled, i.e. $\widehat{G}^0$ *is not commutative.* The proof of this fact has five steps:

STEP 1. Since $L_2 \mid K_1$ is transcendental and $K_1 \mid K$ is algebraic, we may assume the base field of the tower to be K_1, for $\widehat{G}^0 \cong \mathrm{Gal}(L_2 \mid K_1)$; indeed, all of the contributions derived from transcendental elements stay in $\widehat{G}^0$, and the last part of Theorem 2.2.13 (or item 3 in Remark 2.2.16) asserts

$$\widehat{G}/\widehat{G}^0 \cong \mathrm{Gal}(K_1 \mid K).$$

This restricts our study to $\mathrm{Gal}(L_2 \mid K_1)$, besides proving it connected and thus equal to its identity component; in a further abuse of notation, we may call it $\widehat{G}^0$ again.

STEP 2. Let us prove that the elements $R(\sigma)$ of the Galois group $\widehat{G}^0 = \mathrm{Gal}(L_2 \mid K_1)$ are unipotent matrices of the following kind:

$$\widehat{G}^0 = \left\{ \begin{pmatrix} 1 & \mu & A_1 & A_2 \\ 0 & 1 & A_3 & A_4 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in S_0;\ A_1, A_2, A_3, A_4 \in T_0 \right\} \tag{4.15}$$

for some subsets $S_0, T_0 \subset \mathbb{C}$ such that $S_0 \neq \{0\}$.

Indeed, writing $R(\sigma)$ in block notation, $R(\sigma) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, equation $\sigma(\Psi) = \Psi R(\sigma)$ reads

$$\begin{aligned} \sigma(\Psi) &= \begin{pmatrix} \Phi_N(t) & \Phi_N(t)\int_0^t V(\tau)d\tau \\ 0 & \Phi_N(t) \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \\ &= \begin{pmatrix} \Phi_N M_1 + (\Phi_N \int_0^t V) M_3 & \Phi_N M_2 + (\Phi_N \int_0^t V) M_4 \\ \Phi_N M_3 & \Phi_N M_4 \end{pmatrix} \qquad (4.16) \\ &= \begin{pmatrix} \sigma(\Phi_N) & \sigma(\Phi_N \int_0^t V) \\ \sigma(0) & \sigma(\Phi_N) \end{pmatrix}. \qquad (4.17) \end{aligned}$$

From (4.16) and (4.17) we obtain $M_1 = M_4$ and $M_3 = 0$. We are now working on $\mathrm{Gal}(L_2 \mid K_1)$, and the first column of $\Phi_N(t)$ is $(\xi_1(t), i\dot{\xi}(t))^T \in K_1^2$; thus, σ must leave it fixed. That is, defining

$$M_1 = M_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\sigma \begin{pmatrix} \xi_1 & \xi_2 \\ \mathrm{i}\dot{\xi}_1 & \mathrm{i}\dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_1 b + \xi_2 d \\ \mathrm{i}\dot{\xi}_1 & \mathrm{i}\dot{\xi}_1 b + \mathrm{i}\dot{\xi}_2 d \end{pmatrix},$$

so the first column in M_1 and M_4 must be $(a, c)^T = (1, 0)^T$. Their second column must then be of the form $(\mu, 1)^T$ for some $\mu \in \mathbb{C}$, since $\sigma(\Phi_N(t)) = \Phi_N(t) M_1$ is symplectic. This altogether forces the given expression for the diagonal blocks in (4.15).

The actual domain of definition S_0 for μ will be seen in the next step, but we can already assert μ is not identically zero. If it were, then the action of $\widehat{G}^0$

would leave $\xi_2, \dot{\xi}_2 \in L_2$ fixed. This, the definition of L_2, L_1 and the normality of P–V extensions (Lemma 2.2.10) would in turn imply $\xi_2, \dot{\xi}_2 \in K_1$, i.e. we would have elliptic integrals in an algebraic extension of the field of elliptic functions; as said in Remark 4.3.2, this is absurd. Consequently, $S_0 \neq \{0\}$.

Step 3. Let us prove $S_0 = \mathbb{C}$. Indeed, the action of $\widehat{G}^0$ on $\operatorname{diag}(\Phi_{\mathrm{N}}, \Phi_{\mathrm{N}})$ is of the form

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in S_0 \right\}, \tag{4.18}$$

itself a representation of the additive group $\mathbb{C}_+$, which in turn has only two *algebraic* subgroups, namely itself and $\{0\}$; step 2 already discarded the first case, so we are left with $S_0 = \mathbb{C}$.

Step 4. We are now giving a new provisional form to our group. We already know $\sigma\Phi_N = \Phi_N M_1$; let us first study the action of any $\sigma \in \widehat{G}^0$ over the four entries of $\int V$. Applying the identity $\partial \circ \sigma \equiv \sigma \circ \partial$ on $\int V$ and integrating the resulting equation, we obtain

$$\sigma \int \Phi_N^{-1} C \Phi_N = \int \sigma \left(\Phi_N^{-1} C \Phi_N \right) + M, \tag{4.19}$$

for some $M = \begin{pmatrix} \delta & \gamma \\ \beta & \kappa \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$. Besides, using $\sigma C = C$ we have

$$\sigma(\Phi_N^{-1} C \Phi_N) = (\sigma\Phi_N)^{-1} C (\sigma\Phi_N) = M_1^{-1} (\Phi_N^{-1} C \Phi_N) M_1, \tag{4.20}$$

which translates (4.19) into

$$\sigma \int \Phi_N^{-1} C \Phi_N = M_1^{-1} \left(\int \Phi_N^{-1} C \Phi_N \right) M_1 + M, \tag{4.21}$$

that is, the following separate actions of σ on the entries of $\int (\Phi_N^{-1} C \Phi_N)$:

$$\int \begin{pmatrix} u \\ v_1 \\ v_2 \\ -u \end{pmatrix} \mapsto \int \begin{pmatrix} u - \mu v_2 \\ 2\mu u - \mu^2 v_2 \\ v_2 \\ \mu v_2 - u \end{pmatrix} + \begin{pmatrix} \delta \\ \gamma \\ \beta \\ \kappa \end{pmatrix}, \tag{4.22}$$

the first and fourth components of which readily imply $\delta = -\kappa$.

On the other hand, (4.20) allows us to write (4.16) in the equivalent form

$$\sigma(\Psi) = \begin{pmatrix} \sigma\Phi_N & (\sigma\Phi_N) \left[M_1^{-1} M_2 + \int \sigma(\Phi_N^{-1} C \Phi_N) \right] \\ 0 & \sigma\Phi_N \end{pmatrix}.$$

Morphism axioms (and (4.17)) render the latter's upper right block equal to $(\sigma\Phi_N)(\sigma \int \Phi_N^{-1} C \Phi_N)$, and thus force the following to hold,

$$\sigma \left(\int \Phi_N^{-1} C \Phi_N \right) = M_1^{-1} M_2 + \int \sigma(\Phi_N^{-1} C \Phi_N),$$

which along with (4.21) yields $M_1^{-1}M_2 = M$. This gives us the explicit form for the upper 2×2 block in the generic expression (4.15) for $R(\sigma)$:

$$M_2 = M_1 M = \begin{pmatrix} -\kappa+\mu\beta & \gamma+\mu\kappa \\ \beta & \kappa \end{pmatrix}.$$

In particular,

$$\widehat{G}^0 = \left\{ \begin{pmatrix} 1 & \mu & -\kappa+\mu\beta & \gamma+\mu\kappa \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu\in\mathbb{C}, \kappa\in S_1, \beta\in S_2, \gamma\in S_3 \right\} \tag{4.23}$$

for some subsets $S_1, S_2, S_3 \subset \mathbb{C}$.

Step 5. Further specification of the domains of definition of κ,β,γ will finish our proof. We already know $S_0=\mathbb{C}$ is the domain for μ. Given any $a_{\mu,\kappa,\beta,\gamma}\in\widehat{G}^0$, we have

$$\begin{aligned} a_{\mu,\kappa,\beta,\gamma} &= \begin{pmatrix} 1 & \mu & -\kappa+\mu\beta & \gamma+\mu\kappa \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & \kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \\ &=: U_\mu V_\kappa W_\beta X_\gamma, \end{aligned}$$

Assume, for the moment, $S_1=S_2=S_3=\mathbb{C}$. Defining G and H as the subgroups of $\widehat{G}^0$ generated by U_μ and $V_\kappa W_\beta X_\gamma$, respectively,

$$G = \left\{ \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu\in\mathbb{C} \right\}, \quad H = \left\{ \begin{pmatrix} 1 & 0 & -\kappa & \gamma \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \kappa,\beta,\gamma\in\mathbb{C} \right\},$$

G is a representation of $\mathbb{C}_+$, and H, unlike G, is a normal subgroup of $\widehat{G}^0$. The facts $G\cap H = \mathrm{Id}_4$ and $\widehat{G}^0 = GH$ therefore prove $\widehat{G}^0$ to be the *semidirect product* ([55]) of $G\simeq\mathbb{C}_+$ and H.

Consider, besides, the three subgroups of H formed by matrices of the form $V_\kappa, W_\beta, X_\gamma$, respectively; they are all normal subgroups of H and representations of $\mathbb{C}_+$, and their pairwise intersections are $\{\mathrm{Id}_4\}$. Therefore, writing $\times$ as the direct product and $\ltimes$ as the semidirect product, we have

$$\widehat{G}^0 = G\ltimes H \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+\times\mathbb{C}_+\times\mathbb{C}_+).$$

So far we have assumed $S_1=S_2=S_3=\mathbb{C}$; were that false for any of them, say, S_i, it would still have to be the underlying set of an *algebraic* subgroup of the additive group $\mathbb{C}_+$, since each of κ,β,γ comes from one quadrature; indeed, if we consider L_1 as our base group, we have $\mathrm{Gal}(L_2 \mid L_1) = H$ and $\mu = 0$ in formula (4.22), which in turn yields *additive* actions on $\int V$:

$$\int u \mapsto \int u - \kappa, \quad \int v_1 \mapsto \int v_1 + \gamma, \quad \int v_2 \mapsto \int v_2 + \beta. \tag{4.24}$$

Parameters κ, β, γ thus belong to an algebraic subgroup of $\mathbb{C}_+$ (i.e., $\mathbb{C}$ or $\{0\}$), so

$$S_i \in \{\{0\}, \mathbb{C}\}, \quad i = 1, 2, 3.$$

(recall Remark 2.2.16(2)). However, κ *is not* identically zero. If it were, (4.24) would then prove $\int u$ invariant under any $\sigma \in \widehat{G}^0$; this, the logarithm in $\int u$ and Remark 4.3.3 are obviously in contradiction with the normality of $L_2 \mid K$ established in Lemma 2.2.10. γ is *not* identically zero, either; otherwise, the product in $\widehat{G}^0$ would not be defined. Therefore,

$$S_1 = S_3 = \mathbb{C}, \qquad S_2 \in \{\{0\}, \mathbb{C}\}. \tag{4.25}$$

Resetting K_1 as our base field in order to obtain the remaining parameter μ, and using both the factorisation $a_{\mu,\kappa,\beta,\gamma} = U_\mu V_\kappa W_\beta X_\gamma$ and the isomorphism provided by the second part of Fundamental Theorem 2.2.13,

$$\mathrm{Gal}(L_1 \mid K_1) \simeq \mathrm{Gal}(L_2 \mid K_1)/\mathrm{Gal}(L_2 \mid L_1),$$

we actually have, in this case, a splitting of $\mathrm{Gal}(L_2 \mid K_1)$ as the semidirect product

$$\widehat{G}^0 = G \ltimes H = G \ltimes \left(\widehat{G}^0/G\right) \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+ \times S_2).$$

Both this and condition (4.25) force $\widehat{G}^0$ to be isomorphic to one of the following:

$$\mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+ \times \mathbb{C}_+) \quad \text{or} \quad \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+),$$

non-commutative, in any case. □

Remarks 4.4.1. Regarding the proof of Theorem 4.1.3:

1. In step 3 the form of (4.18) clearly embodies our need for the *whole* fundamental matrix Ψ; in other words, solving the normal variational equations is not enough to prove Theorem 4.1.3. Indeed, the theorems due to Ziglin and Morales-Ramis are of no use up to this step, since $\tilde{G}$ is a *commutative* group of unipotent (and thus *resonant*) matrices.

2. *An alternative approach to step 4.* Recall the *unipotent radical* ([55, §19.5] or Section 2.1) of G as being the (unique) largest closed, connected, normal subgroup formed by unipotent matrices in G. We know, thanks to [32, p. 27], that the unipotent radical of the symplectic group $\mathrm{Sp}(2, \mathbb{C})$ may be expressed, in an suitable basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$, as follows:

$$R_u(G) = \left\{ \begin{pmatrix} 1 & \mu & \kappa + \mu\beta & \gamma \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & -\mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu, \kappa, \beta, \gamma \in \mathbb{C} \right\},$$

the coordinates being still canonical. Using $\tilde{\boldsymbol{v}}_3 = -\boldsymbol{v}_3$ we transform the fundamental matrix $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$ into $(\boldsymbol{v}_1, \boldsymbol{v}_2, \tilde{\boldsymbol{v}}_3, \boldsymbol{v}_4)$. The fact that these are not canonical coordinates will not affect our result: the symplectic manifold and bundle and the Galois group will remain invariant.

Subsequent changes $\beta \mapsto -\beta$, $\gamma \mapsto \gamma + \mu\kappa$, in this order, make the representation turn into

$$R_u(G) \cong \left\{ \begin{pmatrix} A & AB \\ 0 & A \end{pmatrix} : A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -\kappa & \gamma \\ \beta & \kappa \end{pmatrix}, \mu, \kappa, \beta, \gamma \in \mathbb{C} \right\},$$

that is, exactly in the form (4.23) with $S_i = \mathbb{C},\ i = 1, 2, 3$.

Let us return to $\widehat{G}^0$. The fact that this group is a connected, normal and unipotent subgroup of the symplectic group assures $\widehat{G}^0 \subset R_u(G)$. This is just what was proven in step 4.

3. *An alternative ending to Step 5.* In the general expression of $\widehat{G}^0$, as we know, the domain of definition for μ, κ, γ is all of $\mathbb{C}$, and the one for β is either $\mathbb{C}$ once again or $\{0\}$; given any $a_i := a_{\mu_i,\kappa_i,\beta_i,\gamma_i} \in \widehat{G}^0,\ i = 1, 2$, their commutator $a_1 a_2 a_1^{-1} a_2^{-1}$ is

$$\begin{pmatrix} 1 & 0 & \mu_1\beta_2 - \beta_1\mu_2 & 2(\mu_1\kappa_2 - \mu_2\kappa_1) - \mu_1\beta_2(\mu_1 + 2\mu_2) + \beta_1\mu_2(\mu_2 + 2\mu_1) \\ 0 & 1 & 0 & \beta_1\mu_2 - \mu_1\beta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is now a simple exercise to verify this is not identically equal to Id_4, which also proves $\widehat{G}^0$ non-commutative.

4. *Some comments on preliminary methodology and checks.*

Reliance on numerics not only provided significant preliminary information prior to the actual proof; it also shed some light into the ensuing algebraic framework, namely in the relationship between the monodromy group and the presumably larger one $\widehat{G}^0 = \mathrm{Gal}(L_2 \mid K_1)$ containing it. We first considered system (4.2), along with the related variational equations as given in (4.5), from a numerical point of view. Clearly, for $h \in (0, h^*)$ (4.2) has both a real and a purely imaginary period (with $\dot{\phi}$ real in both cases). It is enough to take $(\phi(0), \dot{\phi}(0)) = (0, \sqrt{2h})$ as initial conditions and then real or imaginary times, respectively.

Let M_1 and M_2 be the monodromy matrices along the real and the imaginary periods, respectively. These matrices have the common structure

$$M = \begin{pmatrix} 1 & p & q & 0 \\ 0 & 1 & 0 & -q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which, of course, turns out to be a particular case of (4.23). In the real period case $p = a\mathrm{i}$, $q = b(1 - \mathrm{i})$ has been found, and in the imaginary

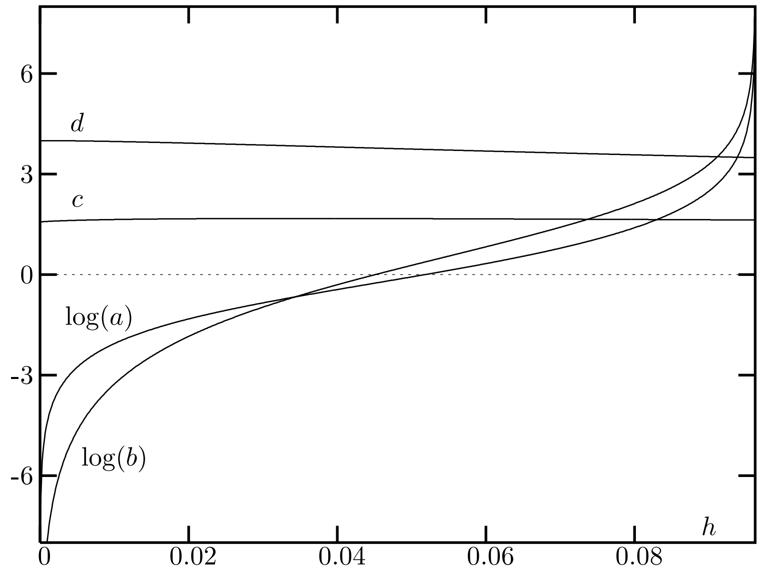

Figure 4.1: Values of a, b, c, d, first two in logarithmic scale

period case $p = c$, $q = d(1 + \mathrm{i})$, with $a, b, c, d \in \mathbb{R}$, all of them positive. The computed values of a, b, c, d are displayed, as a function of h, in Figure 4.4.1. Let p_1, q_1 and p_2, q_2 be the p, q entries in M_1 and M_2. These matrices commute if and only if $\Delta := p_1q_2 - q_1p_2 = (\mathrm{i} - 1)(ad + bc) = 0$. From the positive character of a, b, c, d it follows $\Delta \neq 0$ for all $h \in (0, h^*)$. Furthermore, the coefficient of $(\mathrm{i} - 1)$ in Δ is far away from zero, except for small h, a domain amenable to perturbative computations.

The group generated by M_1 and M_2, which is a subgroup of the monodromy one and, hence, a subgroup of the Galois group, has the same structure as in (4.23) with $\beta = 0$. This is in favor of the second of the options presented for $\widehat{G}^0$, i.e., $\widehat{G}^0 \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+)$.

4.5 Concluding statements

In reference [88], the authors start from (2.21) expressed in polar canonical coordinates and with scalings leading to the Hamiltonian $\tilde{H}_{b,\omega} = H_0 + \omega^2 H_2$, where ω^2 is assumed small enough, H_0 is the Hamiltonian of Kepler's classical problem in a reference frame rotating with angular velocity b, and $\tilde{H}_{\omega,\omega}$ is Hill's Hamiltonian. The strategy followed henceforth is based in proving there is no first integral Φ at a time independent of $\tilde{H}_{b,\omega}$ and analytical with respect to ω in an open neighborhood of $\omega = 0$.

The authors presumably afford their non-integrability proof restricting it to first integrals which are analytical with respect to the conjugate variables *and the parameter* ω; in other words, their proof does not deny, for instance, the existence of additional first integrals meromorphic *with respect to phase variables* and satisfying the Liouville-Arnol'd hypotheses. That denial, which discards any restriction concerning ω, comes precisely from our proof.

As for reference [115], the proof given there is of *algebraic* non-integrability;

using his own generalisation of a method nearly 100 years old, the author establishes there is no second integral of motion for *HP* which is polynomial with respect to phase variables at a given arbitrary level of energy. Spurious parameters such as momentum are not considered here, but the constraint of *algebraic* dependence is still far stronger than our hypothesis of *meromorphic* dependence on canonical variables.

Chapter 5

Conclusions and work in progress

5.1 Overview

We have proven the non-integrability of Hill's problem using the most general instance of the Morales-Ramis Theorem. Furthermore, with the aid of a special case of the aforementioned Theorem we have established a necessary condition on the existence of a single additional first integral for Hamiltonian systems with a homogeneous potential. Using this condition we have generalized Theorems 2.4.5 and 2.4.6 for $N = 3$ with arbitrary masses, and for $N = 3, 4, 5, 6$ with equal masses. Finally, we have proven the non-integrability of the N-Body Problem for $N \geq 7$ equal masses.

Proving non–integrability for the given instances of the N-Body Problem required nothing but the exploration of the eigenvalues of a given matrix, with the advantage of knowing four of them explicitly: $-2, 0, 0, 1$. Thus, whether it be for generalizations of Bruns' Theorem or just for proofs of non-integrability, not all variational equations were needed but those *not* corresponding to these four eigenvalues – this is exactly what transpires from the reduction of variational systems and the introduction of normal variational equations in Section 2.3.2. Hill's Problem, however, required the whole variational system since only thanks to the special functions introduced in the process of variation of constants was it possible to assure the presence of obstructions to integrability.

The main goal of the present thesis was presenting a number of (old and new) possible ways of proving Hamiltonian non-integrability, rather than exhausting all possible open problems that might appear. Both classical and non-classical Hamiltonians have been considered, although everything has been done using the first variational equations along known particular solutions. Our immediate goal at this point is proving one of the following:

Conjecture 5.1.1 (Non-integrability of the N-Body Problem)**.** *Regardless of the masses $m_1, \ldots, m_N > 0$, the d-dimensional N-Body Problem has no set of dN meromorphic first integrals independent and in pairwise involution.*

Conjecture 5.1.2. *Except for an identifiable, zero-measure family $\mathfrak{M} \in \mathbb{R}_+^N$ of mass vectors $(m_1, \ldots, m_N)$, the d-dimensional N-Body Problem has no meromorphic first integral independent and in involution with the classical ones.*

The latter, which in some sense may be seen as a generalization of Bruns' Theorem 2.4.5, obviously implies the former whenever $(m_1, \ldots, m_N) \notin \mathfrak{M}$, although the difference in complexity between both can only be a source of speculation at this point. Besides, proving any of these will definitely call for a further extension of our present knowledge regarding central configurations and Galois differential theory. Indeed, in spite of the apparent simplicity of our intermediate goal (proving the non-integer character of *some* or *all* of the eigenvalues of a matrix except for a known set of masses), the drawbacks and troubles in proving conjectures 5.1.1 and 5.1.2 attest the epistemic frailty present in many a problem in modern Applied Mathematics: a first glance at the steps leading to Chapter 3 shows too many imbrications (Celestial Mechanics, Hamiltonian dynamics, Number Theory, Invariant Theory, special function theory, Algebraic Geometry...) for such an insignificant final obstacle. As a matter of fact, the powerful theoretical background used, especially a framework as profound and seamlessly built as is Picard-Vessiot theory, appears to be nothing but a series of open doors thanks to a number of strong previous leaps forward (e.g. Theorems 2.3.1 and 2.3.3), leaving the "mere" isolation of a matrix spectrum as the only apparently insurmountable obstacle. Being both a source of immediate frustration and a promising source of further discoveries, this seeming incongruity sets a mood at once bleak and optimistic for any researcher for reasons probably needless to clarify to the reader at this point: *we need to formulate all arithmetical and dynamical problems arisen throughout the process in a wider setting* – one in which the solutions to each and all of these problems will be special cases of a more powerful theory with fringe benefits of its own.

Needless to say, *our goal is to find such a setting*, even if our present attempts, whose remnants are shown in Section 5.2, end up having an easy resolution in an immediate future. This wider setting, besides considering generalized hypergeometric functions and higher variational equations (see Section 5.3 for more details), will very probably step on to characterize the Galois groups of these higher variational equations ([97]), and finally exploring the difference between integrating certain Hamiltonians and proving them non-integrable. The Tannakian approach ([34]) will very likely play a part in this endeavour.

5.2 Perspectives on Conjectures 5.1.1 and 5.1.2

5.2.1 The N-body problem with arbitrary masses

Numerical exploration does suggest special values of the masses for which at least one of the eigenvalues of V_N'' may belong to Table (2.12). Refining of these values has been done in order to obtain generalizations of relation (3.16) – to no avail. Thus, most of what follows for arbitrary masses would be more likely applied to Conjecture 5.1.1 than to Conjecture 5.1.2.

Main lines of study

Let $\boldsymbol{c}_L = (\mathbf{c}_1, \ldots, \mathbf{c}_N) \in \mathbb{R}^{Nd}$ be the collinear solution defined in Section 2.4.1. We assume

$$\mathbf{c}_i : (\sqrt{m_1} c_i, 0, \ldots, 0)\,, \qquad i = 1, \ldots, N, \tag{5.1}$$

are, respectively, the coordinates of the bodies of masses $m_1, \ldots, m_N$. Tracing the steps in Moulton's existence and unicity proof it is easy to prove there exists such a solution as (5.1).

Eigenvalues for the collinear solution The very particular form of $\boldsymbol{c}_L$ allows for a more specific version of Lemma 3.2.9. $V''_N(\boldsymbol{c}_L) = (V_{i,j})_{i,j=1,\ldots,N}$, where for each $i, j = 1, \ldots, N$ we have

$$\begin{aligned} V_{i,i} &= \left(\sum_{k \neq i, k=1}^{N} \frac{m_k}{|c_i - c_k|^3} \right) A, \qquad 1 \leq i \leq N, \\ V_{i,j} &= V_{j,i} = -\frac{\sqrt{m_i}\sqrt{m_j}}{|c_i - c_j|^3} A, \qquad 1 \leq i < j \leq N, \end{aligned}$$

where $A = \begin{pmatrix} -2 & \mathbf{0}^T \\ \mathbf{0} & \mathrm{Id}_{d-1} \end{pmatrix}$. The following appears to be a direct consequence of this:

Conjecture 5.2.1. *The following holds:*

$$\mathrm{Spec}\left(V''_{N,d}(\boldsymbol{c}_L)\right) = \{\mu_1, \ldots, \mu_N, -2\mu_1, \ldots, -2\mu_N\}\,,$$

where $\mu_i \geq 0$ and $-2\mu_i$ has multiplicity $d-1$ for every $i = 1, \ldots, N$.

Hence, we will cling to the planar collinear solution

$$\boldsymbol{c}_L : (\sqrt{m_1} c_1, 0, \sqrt{m_2} c_2, 0, \sqrt{m_3} c_3, 0, \ldots, \sqrt{m_N} c_N, 0)\,.$$

The main line of study A property which seems true for all values numerically tested is:

Conjecture 5.2.2. *There is at least an $i = 1, \ldots, N$ such that $\sum_{k \neq i, k=1}^{N} \frac{m_k}{|c_k - c_i|^3} > 1$.*

The known result closest resembling our goal is apparently what was done for $m_1 = \cdots = m_N = m$ in [28], although deviating one, two or more of the masses away from the common value m has consequences still unknown to us. Anyway, proving Conjecture 5.2.2 proves Conjecture 5.1.1. Indeed, we have

$$V''_N(\boldsymbol{c}_L) = \mathrm{diag}\left\{ \begin{pmatrix} -2\sum_{k \neq i, k=1}^{N} \frac{m_k}{|c_k - c_i|^3} & 0 \\ 0 & \sum_{k \neq i, k=1}^{N} \frac{m_k}{|c_k - c_i|^3} \end{pmatrix} : 1 \leq i \leq N \right\} + B_N,$$

B_N being null along its three main diagonals; hence, inasmuch as was done in Subsection 3.3.2, we may now proceed to search for vectors yielding a Rayleigh quotient greater than 1. One such vector is $\boldsymbol{w}_i := \boldsymbol{e}_{2N,2i}$ (i as in Conjecture 5.2.2), since the following holds:

$$\frac{\boldsymbol{w}_i^T A \boldsymbol{w}_i}{\boldsymbol{w}_i^T \boldsymbol{w}_i} = \sum_{k \neq i, k=1}^{N} \frac{m_k}{|c_i - c_k|^3} > 1;$$

this proves the existence of an eigenvalue strictly greater than one, and thus not belonging to $S = \left\{-\frac{1}{2}p\,(p-3) : p > 1\right\}$.

Second line of study Again, let $\boldsymbol{c}_L$ be the Moulton collinear solution to $V_N'(\boldsymbol{c}) = \boldsymbol{c}$. We know $\operatorname{Spec}(V_N''(\boldsymbol{c}_L)) = \{\mu_1, \ldots, \mu_N, -2\mu_1, \ldots, -2\mu_N\}$ from Conjecture 5.2.1. We are now proving the following:

Lemma 5.2.3. *Assume all of the eigenvalues of $V_N''(\boldsymbol{c}_L)$ belong in Table (2.12). Then, they all belong to $\tilde{S} = \{-2, 0, 1\}$.*

Proof. For any $\lambda = -\frac{1}{2}p\,(p-3) \in S$, assume $\lambda = -2\mu$ for some other $\mu \in \tilde{S}$. Then defining $\mu = -\frac{1}{2}q\,(q-3)$, we would have

$$-\frac{1}{2}p\,(p-3) = q\,(q-3)\,, \tag{5.2}$$

implying $p = p_\pm = \frac{3}{2} \pm \frac{1}{2}\sqrt{\Delta}$, where $\Delta = -8q^2 + 24q + 9$. $\Delta \geq 0$ only holds for $q \in \left[3\left(2-\sqrt{6}\right)/4, 3\left(2+\sqrt{6}\right)/4\right] \subset (-1, 4)$, and for $q = 0, 1, 2, 3$ the corresponding values of $p_\pm$ are easily proven to yield either -2 or 0 for both sides of (5.2). ☐

Hence, if we prove the following we are done with Conjecture 5.1.1:

Conjecture 5.2.4. *There is at least one eigenvalue of $V_N''(\boldsymbol{c}_L)$ not belonging to $\{-2, 0, 1\}$.*

Numerical evidence of this is overwhelming.

Other possibilities

Since only four of the eigenvalues are known for sure and little is known about central configurations allowing us to make some disquisitions of a qualitative sort, most of the remaining possible methods of proving Conjectures 5.1.1 and 5.1.2 are likely to be dead-end sidings, at least if we are expecting simple proofs for these conjectures.

1. Matrix deflation is already useless for $N = 3$ in the Euler collinear case $\boldsymbol{c}_L$ and arguably remains so for higher N: if we choose for instance null-vectors

$$v_1 : \left(\sqrt{m_1}, 0, \sqrt{m_2}, 0, \sqrt{m_3}, 0\right), \qquad v_2 : \left(\sqrt{m_1}, 0, \sqrt{m_2}, 0, \sqrt{m_3}\right),$$

for the corresponding 6×6 and 5×5 matrices to be deflated with, respectively, it is easy to see that $\operatorname{Spec} V_N''(\boldsymbol{c}_L) = \{-2, 0, 0, 1, \lambda, -2\lambda\}$, where

$$\lambda = -1 + \frac{m_1 + m_2}{|c_1 - c_2|^3} + \frac{m_1 + m_3}{|c_1 - c_3|^3} + \frac{m_2 + m_3}{|c_2 - c_3|^3}. \tag{5.3}$$

Proving that one or both of λ and -2λ lies outside $\tilde{S}$ is as open a problem as the one posed in Conjecture 5.2.2 and requires more knowledge on the collinear solution than we currently have.

2. Another apparent dead end is the use of a more general family of solutions than the one appearing in Section 3.3.1. It may be shown that a solution for $V_N'(\boldsymbol{c}) = \boldsymbol{c}$ is $\widehat{\boldsymbol{c}} = \left(\sum_{k=1}^N m_k\right)^{-2/3} \boldsymbol{c}$, where

$$\begin{aligned}
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \sqrt{m_1} \sum_{k\neq 1} \begin{pmatrix} a_k m_k \\ b_k m_k \end{pmatrix}, \\
\begin{pmatrix} c_{2i-1} \\ c_{2i} \end{pmatrix} &= \sqrt{m_i} \left[\sum_{k\neq i, k\geq 2} m_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} - \left(\sum_{k\neq i, k\geq 2} m_k \right) \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right], \quad i \geq 2,
\end{aligned}$$

and $a_2, \ldots, a_N, b_2, \ldots, b_N$ are solutions to

$$\begin{aligned}
\left(a_i^2 + b_i^2\right)^{3/2} &= 1, \qquad i = 2, \ldots, N, \\
\left((a_i - a_j)^2 + (b_i - b_j)^2\right)^{3/2} &= 1, \qquad i \neq j = 2, \ldots, N.
\end{aligned}$$

A special case for $N = 3$ is the solution (3.9) used Section 3.3.1. The problem, though, is assuring the existence of such a set $\{a_2, \ldots, a_N, b_2, \ldots, b_N\} \subset \mathbb{C}$ when $N \geq 4$. Another problem is determining how many solutions of (3.2) *do not* match pattern $\widehat{\boldsymbol{c}}$; in particular, determining whether or not (3.9) and collinear solutions are the only possible complex solutions of (3.2) for $N = 3$.

3. A formula of the sorts of

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} (A - z\mathrm{Id}_{2N})^{-1} f(z)\, dz, \tag{5.4}$$

where $f(z) = \sum_{k=0}^\infty a_k z^k$ is any given analytical function with a matrix counterpart $f(A) := \sum_{k=0}^\infty a_k A^k$ and Spec $A \subset \Omega$, is hardly of any use here no matter how simple f is, since everything basically boils down to observing obstructions to an equality such as (5.4) on the complementary of a *discrete* set and this is arguably the opposite of the way a proper proof works, especially considering our scarce knowledge of the Hessian matrix A. This is especially evident when trying to compute, for instance, the *matrix sine* $f(A) = \sin(\pi A) := \frac{1}{2\mathrm{i}} [\exp(\mathrm{i}\pi A) - \exp(-\mathrm{i}\pi A)]$, the matrix exponential $\exp : \mathcal{M}_{2N\times 2N}(\mathbb{C}) \to \mathcal{M}_{2N\times 2N}(\mathbb{C})$ being defined as usual. Proving $\sin(\pi A)$ has not a single zero (resp. at least a non-zero) eigenvalue would establish Conjecture 5.1.2 (resp. 5.1.1), but finding plausible properties (or patterns, for that matter) for the infinite series involved requires a knowledge on A which we currently don't have, not even for the relatively sparse form $A = V_N''(\boldsymbol{c}_L)$ it has in the collinear case.

4. Geršgorin and Bauer-Fike bounds ([133, §6.9]) are probably just as useless here since numerical evidence yields non-void pairwise intersection of nearly all of the disks containing the eigenvalues for a widespread set of values of the masses.

5. Finally, and in spite of some distant similarities, the reduction of $V_N''(\boldsymbol{c})$ to a Toeplitz matrix ([20], [47]) seems difficult to perform, even for solutions such as those given by the polygonal and collinear configurations. Hence, none of the well-known results of detection of extreme eigenvalues for such matrices is likely to hold here, at least not regardless of N and $\boldsymbol{c}$.

5.2.2 Candidates for a partial result

The N-body problem with equal masses

We already generalized Bruns' Theorem for this special case with $N \leq 6$, and proved non-integrability for $N \geq 7$. Let $\boldsymbol{c}_P$ be the polygonal solution (Example 2.4.10(5) and Section 3.3.2). Numerical evidence supports the following fact for all $N \geq 3$: Spec $V_N''(\boldsymbol{c}_P) = \tilde{S} \cup \{\mu_1, \ldots, \mu_n\}$, where $\tilde{S} = \{-2, 0, 1\}$ (-2 and 1 simple, 0 double) and $\mu_1 \leq \cdots \leq \mu_n$, where:

1. if N is even, μ_1 and μ_n are simple, and the remaining $\mu_2, \ldots, \mu_{n-1}$ are double eigenvalues;

2. if N is odd, all of $\mu_1, \ldots, \mu_n$ are double eigenvalues;

and, most importantly:

Conjecture 5.2.5. *There is not a single element in $\{\mu_1, \ldots, \mu_n\}$ belonging to $\tilde{S}$.*

Proving this would obviously prove Conjecture 5.1.2 for equal masses. We may also hint at the following generalization of Theorem 3.3.10, although the result it implies (namely, that the Problem with equal masses is not integrable) has been already obtained by other means in Theorem 3.2.3, item 2:

Conjecture 5.2.6. *For* any $N \in \mathbb{N}$, $N \geq 7$, $\sum_{k=1}^{N-1} \csc \frac{\pi}{N} k$ *and* $\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k$ *are $\mathbb{Q}$-independent.*

The $N+1$-body problem with N equal masses

Assume $m_1 = \cdots = m_N = 1$ and $m_{N+1} > 0$ is the additional mass. The next two Lemmae are as immediate to prove as Lemmae 3.3.2 and 3.3.3:

Lemma 5.2.7. *The vector $\boldsymbol{c}_C = \tilde{\beta}_N^{1/3} (\mathbf{c}_1, \ldots, \mathbf{c}_N, \mathbf{c}_{N+1})$, defined by*

$$\mathbf{c}_j = (c_{2j-1}, c_{2j}) = \begin{cases} \left(\cos \frac{2\pi j}{N}, \sin \frac{2\pi j}{N}\right), & j < N+1, \\ (0,0), & j = N+1, \end{cases} \tag{5.5}$$

where $\tilde{\beta}_N := m_{N+1} + \frac{1}{4} \sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$, is a solution for $V_{N+1}'(\boldsymbol{c}) = \boldsymbol{c}$. $\square$

Lemma 5.2.8. *The trace for $V_{N+1}''(\boldsymbol{c}_C)$ is equal to*

$$\tilde{\mu}_N := -\frac{N}{2} \frac{\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8\left(m_{N+1} + 1\right)}{\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) + 4 m_{N+1}}. \quad \square$$

Observation of Lemma 3.3.5 for $N \geq 10$ and a direct check for $N < 0$ assure the following fact: $\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8 > 2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$ for all N; hence, we have

$$\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8\left(m_{N+1}+1\right) > 2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) + 8m_{N+1},$$

and thus $\frac{\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right)+8(m_{N+1}+1)}{2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)+8m_{N+1}} > 1$; hence, as was already stated in reference [164, Section 3.2]:

Corollary 5.2.9. *Given N, $\operatorname{tr} V''_{N+1}(\boldsymbol{c}_C)$ is a non-integer for all but a finite number of values of $m_{N+1} > 0$. The cardinality of this exceptional set depends on N.* $\square$

Let $\boldsymbol{c}_C$ be as in Lemma 5.2.7. Numerics seem to corroborate the following assertions:

Conjecture 5.2.10. *$V''_{N+1}(\boldsymbol{c}_C)$ has at least an eigenvalue $\lambda > 1$.*

Conjecture 5.2.11. *$V''_{N+1}(\boldsymbol{c}_C)$ has all of its eigenvalues out of S, except for -2 and 1 (simple) and 0 (double).*

Proving these would settle the matter for Conjectures 5.1.1 and 5.1.2, respectively on $\mathcal{H}_{N+1}$ with arbitrary $m_{N+1} > 0$ and $m_1 = \cdots = m_N$.

The Spatial Four-Body Problem

Let $\boldsymbol{c}_T = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) \in \mathbb{R}^{12}$ be a vector such that $V''_{4,3}(\boldsymbol{c}_T) = \boldsymbol{c}_T$ and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ are the vertexes of a regular tetrahedron. Such a vector exists in virtue of Remark 3.2.1 and what was said in Example 2.4.10(2), and in turn yields a homographic solution for the three-dimensional Four-Body Problem. The following appears to hold:

Conjecture 5.2.12. *The eigenvalues of $V''_{4,3}(\boldsymbol{c}_T)$ are*

$$\lambda_1 = -2, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = \lambda_6 = \lambda_7 = 1, \quad \lambda_8, \ldots, \lambda_{12},$$

at least one of $\lambda_8, \ldots, \lambda_{12}$ being a non-integer.

A stretch may be attempted by asking for Conjecture 5.1.2 to hold, at least for a generic family of masses m_1, m_2, m_3, m_4. $\boldsymbol{c}_T$, as is the case for the triangular solution used in Subsection 3.3.1, is fairly easy to compute; the main drawback here is computing the eigenvalues of $V''_{4,3}(\boldsymbol{c}_T)$.

5.3 Hamiltonians with a homogeneous potential

5.3.1 Higher variational equations

All of what follows is the product of a personal communication from J.-P. Ramis during a short-term stay in Toulouse in 2005 as well as a couple of conversations with J.-P. Ramis and J.-A. Weil in Luminy and Barcelona in 2006.

The first variational equations along solutions of the form $\phi(t)\,\boldsymbol{c}$ such that (3.2) holds are expressible in terms of hypergeometric functions, as was seen in Subsection 2.3.2. A first step should be done forward into expressing higher-order variational equations along those solutions in terms of generalized hypergeometric functions; the most general instance of such functions for which a significant amount of study has been done is the *Meijer G-function* ([38, §5.3]),

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{array}{c} a_1\cdots a_p \\ b_1\cdots b_q \end{array}\right) := \frac{1}{2\pi\mathrm{i}}\int \frac{\prod_{j=1}^{m}\Gamma(\beta_j-\tau)\prod_{j=1}^{m}\Gamma(1-\alpha_j+\tau)\,x^{\tau}}{\prod_{j=n+1}^{p}\Gamma(\alpha_j-\tau)\prod_{j=m+1}^{q}\Gamma(1-\beta_j+\tau)}d\tau \tag{5.6}$$

where $m,n,p,q\in\mathbb{N}$. The change $t\mapsto x$ will probably involve a branched covering much in the way explained in Subsection 2.3.2. Hence, the study of monodromy and Galois groups done by Yoshida, Morales-Ruiz and Ramis is here substituted in by the computation of those groups for differential equations with functions of the form (5.6). Since higher variational equations are solvable by quadratures along any known integral curve (using variation of constants), the corresponding linear differential operators given by (5.6) are reducible; this places us in the least studied case, since most of the bibliography concerning a Galoisian approach to generalized hypergeometric functions corresponds to the irreducible case (e.g. [17], [61]). The most reliable sources concerning this are probably [24], [25], [26] and [90], in which relevant information has been collected on the Galois group G of these operators: for instance, that G is the semi-direct product of a reductive group (computable in terms of the first variational equations), and its unipotent radical; furthermore, a thorough study has been made of this unipotent radical in the first three references, for instance concerning its usual commutativity. However, it is still not clear whether or not this information (especially the non-trivial direct product structure, which we already found in Subsection 4.4) is useful for our purposes here. And even if it were, and the aforesaid direct product were to yield families of masses $m_1,\ldots,m_N$ for which the identity component of G is non-commutative, the task would still remain to find such families – a rather involved task ahead of us, considering we have not one but N parameters to work with.

Appendix A

Computations for Theorem 3.2.2

We have, using the notation in Subsection 3.2.2,

$$\begin{aligned}
\mathbf{D}_{1,2} &= \begin{pmatrix} d_{1,3} \\ d_{2,4} \end{pmatrix} := \sqrt{m_2}\mathbf{q}_1 - \sqrt{m_1}\mathbf{q}_2 = \sqrt{m_1 m_2} m^{1/3} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \\
\mathbf{D}_{1,3} &= \begin{pmatrix} d_{1,5} \\ d_{2,6} \end{pmatrix} := \sqrt{m_3}\mathbf{q}_1 - \sqrt{m_1}\mathbf{q}_3 = \sqrt{m_1 m_3} m^{1/3} \begin{pmatrix} 2\alpha \\ 0 \end{pmatrix}, \\
\mathbf{D}_{2,3} &= \begin{pmatrix} d_{3,5} \\ d_{4,6} \end{pmatrix} := \sqrt{m_3}\mathbf{q}_2 - \sqrt{m_2}\mathbf{q}_3 = \sqrt{m_2 m_3} m^{1/3} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix},
\end{aligned}$$

and thus, using the fact that $\alpha^2 + \beta^2 = 1$,

$$\begin{aligned}
\tilde{D}_{1,2} &= \sqrt{d_{1,3}^2 + d_{2,4}^2} = \sqrt{\left(\alpha^2 + \beta^2\right) m_1 m_2 m^{2/3}} = \sqrt{m_1 m_2 m^{2/3}}, \\
\tilde{D}_{1,3} &= \sqrt{d_{1,5}^2 + d_{2,6}^2} = 2\sqrt{\alpha^2 m_1 m_3 m^{2/3}}, \\
\tilde{D}_{2,3} &= \sqrt{d_{3,5}^2 + d_{4,6}^2} = \sqrt{\left(\alpha^2 + \beta^2\right) m_2 m_3 m^{2/3}} = \sqrt{m_2 m_3 m^{2/3}};
\end{aligned}$$

take into consideration $\tilde{D}_{1,2}, \tilde{D}_{1,3}, \tilde{D}_{2,3}$ need not be Euclidean norms (hence the unusual notation, as opposed to the one introduced in Section 1.4), though this will be the case if the terms inside the parentheses are real. Furthermore, we will at this point assume that either $\alpha \in (0, \infty)$ or $\alpha = re^{\theta \mathrm{i}}$, with $\theta \in [0, \pi)$, as is the case in the proof of Theorem 3.2.2: $\alpha = \frac{1}{2}, \frac{-1+\sqrt{3}\mathrm{i}}{4}$. In both cases, we have $\sqrt{\alpha^2} = \alpha$ according to our positive determination of the square root.

We know, using the notation in Subsection 3.2.2, that

$$V_3''(q) = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{1,2} & A_{2,2} & A_{2,3} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{pmatrix},$$

where

$$
\begin{aligned}
A_{1,1} &= m_1^{\frac{3}{2}} \begin{pmatrix} \frac{\left(\tilde{D}_{1,2}^2 - 3d_{1,3}^2\right) m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} + \frac{\left(\tilde{D}_{1,3}^2 - 3d_{1,5}^2\right) m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} & -\frac{3d_{1,3}d_{2,4}m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} - \frac{3d_{1,5}d_{2,6}m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} \\ -\frac{3d_{1,3}d_{2,4}m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} - \frac{3d_{1,5}d_{2,6}m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} & \frac{\left(\tilde{D}_{1,2}^2 - 3d_{2,4}^2\right) m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} + \frac{\left(\tilde{D}_{1,3}^2 - 3d_{2,6}^2\right) m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} \end{pmatrix}, \\
A_{1,2} &= \frac{m_1^2 m_2^2}{\tilde{D}_{1,2}^5} \begin{pmatrix} 3d_{1,3}^2 - \tilde{D}_{1,2}^2 & 3d_{1,3}d_{2,4} \\ 3d_{1,3}d_{2,4} & 3d_{2,4}^2 - \tilde{D}_{1,2}^2 \end{pmatrix}, \\
A_{1,3} &= \frac{m_1^2 m_3^2}{\tilde{D}_{1,3}^5} \begin{pmatrix} 3d_{1,5}^2 - \tilde{D}_{1,3}^2 & 3d_{1,5}d_{2,6} \\ 3d_{1,5}d_{2,6} & 3d_{2,6}^2 - \tilde{D}_{1,3}^2 \end{pmatrix}, \\
A_{2,2} &= m_2^{\frac{3}{2}} \begin{pmatrix} \frac{\left(\tilde{D}_{1,2}^2 - 3d_{1,3}^2\right) m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} + \frac{\left(\tilde{D}_{2,3}^2 - 3d_{3,5}^2\right) m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & -\frac{3d_{1,3}d_{2,4}m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} - \frac{3d_{3,5}d_{4,6}m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \\ -\frac{3d_{1,3}d_{2,4}m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} - \frac{3d_{3,5}d_{4,6}m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & \frac{\left(\tilde{D}_{1,2}^2 - 3d_{2,4}^2\right) m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5} + \frac{\left(\tilde{D}_{2,3}^2 - 3d_{4,6}^2\right) m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \end{pmatrix}, \\
A_{2,3} &= \frac{m_2^2 m_3^2}{\tilde{D}_{2,3}^5} \begin{pmatrix} 3d_{3,5}^2 - \tilde{D}_{2,3}^2 & 3d_{3,5}d_{4,6} \\ 3d_{3,5}d_{4,6} & 3d_{4,6}^2 - \tilde{D}_{2,3}^2 \end{pmatrix}, \\
A_{3,3} &= m_3^{\frac{3}{2}} \begin{pmatrix} \frac{\left(\tilde{D}_{1,3}^2 - 3d_{1,5}^2\right) m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} + \frac{\left(\tilde{D}_{2,3}^2 - 3d_{3,5}^2\right) m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & -\frac{3d_{1,5}d_{2,6}m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} - \frac{3d_{3,5}d_{4,6}m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \\ -\frac{3d_{1,5}d_{2,6}m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} - \frac{3d_{3,5}d_{4,6}m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & \frac{\left(\tilde{D}_{1,3}^2 - 3d_{2,6}^2\right) m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} + \frac{\left(\tilde{D}_{2,3}^2 - 3d_{4,6}^2\right) m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \end{pmatrix}.
\end{aligned}
$$

In this case, thus, we have

$$
\begin{aligned}
A_{1,1} &= \frac{1}{m} \begin{pmatrix} \frac{4\left(1-3\alpha^2\right)m_2 - m_3\alpha^{-3}}{4} & -3\alpha\beta m_2 \\ -3\alpha\beta m_2 & \frac{8\left(1-3\beta^2\right)m_2 + m_3\alpha^{-3}}{8} \end{pmatrix}, \\
A_{1,2} &= \frac{\sqrt{m_1}\sqrt{m_2}}{m} \begin{pmatrix} 3\alpha^2 - 1 & 3\alpha\beta \\ 3\alpha\beta & 3\beta^2 - 1 \end{pmatrix}, \\
A_{1,3} &= \frac{\sqrt{m_1}\sqrt{m_3}}{m} \begin{pmatrix} \alpha^{-3}/4 & 0 \\ 0 & -\alpha^{-3}/8 \end{pmatrix}, \\
A_{2,2} &= \frac{1}{m} \begin{pmatrix} \left(1-3\alpha^2\right)\left(m_1+m_3\right) & 3\alpha\beta\left(m_3-m_1\right) \\ 3\alpha\beta\left(m_3-m_1\right) & \left(1-3\beta^2\right)\left(m_1+m_3\right) \end{pmatrix}, \\
A_{2,3} &= \frac{\sqrt{m_2}\sqrt{m_3}}{m} \begin{pmatrix} -1+3\alpha^2 & -3\alpha\beta \\ -3\alpha\beta & -1+3\beta^2 \end{pmatrix}, \\
A_{3,3} &= \frac{1}{m} \begin{pmatrix} \frac{4\left(1-3\alpha^2\right)m_2 - m_1\alpha^{-3}}{4} & 3\alpha\beta m_2 \\ 3\alpha\beta m_2 & \frac{8\left(1-3\beta^2\right)m_2 + \alpha^{-3}m_1}{8} \end{pmatrix},
\end{aligned}
$$

which under the assumption $\alpha^3 = 1/8$ become

$$
\begin{aligned}
A_{1,1} &= \frac{1}{m}\begin{pmatrix} m_2\left(1-3\alpha^2\right)-2m_3 & -3\alpha\beta m_2 \\ -3\alpha\beta m_2 & m_2\left(1-3\beta^2\right)+m_3 \end{pmatrix}, \\
A_{1,2} &= \frac{\sqrt{m_1 m_2}}{m}\begin{pmatrix} 3\alpha^2-1 & 3\alpha\beta \\ 3\alpha\beta & 3\beta^2-1 \end{pmatrix}, \\
A_{1,3} &= \frac{\sqrt{m_1 m_3}}{m}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \\
A_{2,2} &= \frac{1}{m}\begin{pmatrix} \left(1-3\alpha^2\right)\left(m_1+m_3\right) & 3\alpha\beta\left(m_3-m_1\right) \\ 3\alpha\beta\left(m_3-m_1\right) & \left(1-3\beta^2\right)\left(m_1+m_3\right) \end{pmatrix}, \\
A_{2,3} &= \frac{\sqrt{m_2 m_3}}{m}\begin{pmatrix} 3\alpha^2-1 & -3\alpha\beta \\ -3\alpha\beta & 3\beta^2-1 \end{pmatrix}, \\
A_{3,3} &= \frac{1}{m}\begin{pmatrix} -2m_1+\left(1-3\alpha^2\right)m_2 & 3\alpha\beta m_2 \\ 3\alpha\beta m_2 & m_1+\left(1-3\beta^2\right)m_2 \end{pmatrix}
\end{aligned}
$$

The characteristic polynomial for $V_3''(c)$ is $P(x) = x^2(x-1)\frac{Q(x)}{m^2}$, where

$$Q(x) = p_1p_3m_1^2 + p_1^2(x-1)m_2^2 + p_1p_3m_3^2 + p_1p_2m_1m_2 + 2(2+x)p_4m_1m_3 + p_1p_2m_2m_3,$$

and

$$
\begin{aligned}
p_1(x) &= x + 3\left(\alpha^2+\beta^2\right) - 1, \\
p_2(x) &= 3\alpha^2(x-1) + 3\beta^2(x+2) + (x-1)(2x+1), \\
p_3(x) &= (x+2)(x-1), \\
p_4(x) &= (x-1)\left(x+3\beta^2-1\right) + 3\alpha^2\left(x+6\beta^2-1\right);
\end{aligned}
$$

substituting in $\alpha^2+\beta^2 = 1$ once again, we obtain

$$
\begin{aligned}
p_1(x) &= x+2, \\
p_2(x) &= 2x^2+2x+6\beta^2-3\alpha^2-1, \\
p_4(x) &= x^2+x+18\alpha^2\beta^2-2,
\end{aligned}
$$

and thus

$$P(x) = x^2(x-1)(x+2)\frac{Q(x)}{m^2},$$

$Q(x) := p_3m_1^2 + p_1(x-1)m_2^2 + p_3m_3^2 + p_2m_1m_2 + p_2m_2m_3 + 2p_4m_1m_3$ having six roots: $-2, 0, 0, 1, \lambda_+, \lambda_-$ where $\lambda_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}\sqrt{\rho}}{2m}$ and

$$
\begin{aligned}
\rho = \; & 3m_1^2 + 3m_2^2 + 2\left(1+2\alpha^2-4\beta^2\right)m_2m_3 + 3m_3^2 \\
& + 2m_1\left[m_2\left(1+2\alpha^2-4\beta^2\right) + 2m_3\left(1-8\alpha^2\beta^2\right)\right],
\end{aligned}
$$

which assuming once again that $\beta^2 = 1-\alpha^2$ becomes

$$\rho = 3\left(m_1^2+m_2^2+m_3^2+2\left(2\alpha^2-1\right)\left(m_2m_3+m_1m_2\right)+m_3+8\alpha^2\left(\alpha^2-1\right)m_3\right),$$

and assuming $\alpha^4 = \alpha/8$ we finally obtain

$$\rho = 3\left(m_1^2+m_2^2+m_3^2+2\left(2\alpha^2-1\right)\left(m_2m_3+m_1m_2\right)-2\left(8\alpha^2-\alpha-1\right)m_1m_3\right).$$

For $\alpha = 1/2$, we obtain $\rho = 3\left(m_1^2 + m_2^2 + m_3^2 - m_2 m_1 - m_2 m_3 - m_1 m_3\right)$, as was the case for the *real* eigenvalues $\lambda_\pm$ in Subsection 3.3.1, whereas, defining

$$\begin{aligned} B_1 &= 2m_1^2 + 2m_2^2 + 2m_3^2 - 5m_1 m_2 - 5m_2 m_3 + 7m_1 m_3, \\ B_2 &= \sqrt{3}\left(m_1 m_2 + m_2 m_3 - 5m_1 m_3\right), \end{aligned}$$

for $\alpha = \frac{-1+\sqrt{3}\mathrm{i}}{4}$ we have the discriminant $\rho = \frac{3(B_1 - \mathrm{i}B_2)}{2}$, precisely the one appearing in the *complex* eigenvalues $\lambda_\pm^*$ in Subsection 3.3.1.

Appendix B

Resum

B.1 Introducció

Tota definició d'**integrabilitat** en sistemes dinàmics es pot resumir en la possibilitat de fer afirmacions de caràcter global sobre l'evolució temporal dels dits sistemes. Si bé el resultat de tals afirmacions, habitualment anomenat *solució*, no planteja obstacles seriosos quant a definicions, les afirmacions "per se" acostumen a ser difícils de caracteritzar rigorosament, atesa la varietat de nocions de resolubilitat, cadascuna adaptada a un camp d'estudi concret. Hi ha, a més, un llindar que cap àrea d'estudi traspassa: la possibilitat de calcular la solució de forma semi-algorísmica; és per la presència d'aquest darrer obstacle que la comprensió d'un sistema dinàmic roman escorada a estudis parcials i alternatius en vistes a detectar comportament caòtic o propietats (dinàmica periòdica, acotació de les solucions, etc.) de validesa essencialment compensatòria.

La majoria d'aquests intents parcials se situen fonamentalment dins l'àmbit de l'anomenada teoria *qualitativa* d'equacions diferencials ([104], [109], [110], [130]) i, especialment, en les simulacions numèriques que aquesta teoria genera ([60], [74], [92], [123], [124], [125], [151], [161]). Malgrat tot, el sentiment que en darrera instància se'n desprèn és el d'una total dependència de disciplines (anàlisi numèrica, estadística, àdhuc geometria algorísmica) el domini d'aplicació de les quals és sovint més pràctic que no pas teòric.

És lògic per tant que, a falta d'un model determinista vàlid, i en vista de la profusió de problemes a estudiar, aparegui el fenomen de l'especialització en l'estudi dels sistemes dinàmics. Això no obstant, hom podria argumentar que el dit fenomen, i en especial la majoria de les definicions i condicions d'integrabilitat i no-integrabilitat, incloent-hi les descrites en aquesta tesi, formen part d'una agenda molt més ambiciosa destinada precisament a *integrar* sistemes, si més no els de determinats tipus; és simptomàtic d'aquest fet l'esforç constant i explícit a detectar, un cop definida una noció concreta d'"integrabilitat", totes les possibles obstruccions a la "integrabilitat" d'un determinat tipus de sistema, sovint materialitzades en l'aparició de certes funcions transcendents a llur solució general.

B.1.1 Dues nocions d'integrabilitat en sistemes dinàmics

Presentació

Les dues definicions d'integrabilitat que serveixen de punt de partida d'aquesta tesi són:

1. La integrabilitat de sistemes *hamiltonians*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n; \tag{B.1}$$

també escrits de la forma $\dot{\boldsymbol{z}} = X_H(\boldsymbol{z})$, essent n *el nombre de graus de llibertat* del sistema, $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})$ el vector de *posicions* $\boldsymbol{q}$ i *moments* $\boldsymbol{p}$, i H la *funció hamiltoniana.*

2. La integrabilitat de sistemes *lineals*

$$\dot{\boldsymbol{y}} = A(t)\,\boldsymbol{y}. \tag{B.2}$$

Sistemes hamiltonians

El sistema (B.1), i per extensió el camp vectorial X_H i àdhuc la funció H, s'anomena *integrable en el sentit de Liouville-Arnol'd* o *completament integrable* si existeixen tantes integrals primeres de (B.1) com graus de llibertat té el sistema, $f_1, \ldots, f_n$ (una de les quals sempre es pot suposar igual a H) independents i en involució. $\{f_1, \ldots, f_n\}$ s'acostuma a anomenar un conjunt *complet* d'integrals primeres.

És habitual també fer referència a la més general noció d'integrabilitat *parcial*, que es defineix com a l'existència d'un nombre potser menor que n d'integrals primeres $f_1, \ldots, f_k$ de (B.1), independents i en involució dues a dues. Més en general, l'epítet *addicional* s'aplica a una integral primera independent i en involució amb cert conjunt *conegut* de $m < n$ integrals primeres de (B.1), ja sigui un simplet $F = \{H\}$ en el cas del Problema de Hill (vegeu Secció 2.4.2 i Capítol 4), o el conjunt F de $\frac{1}{2}(d+2)(d+1)$ integrals "clàssiques" que introduirem més avall per al Problema d-dimensional de N Cossos (vegeu Secció 2.4.1 i Capítol 3).

En el nostre cas, i per tal de poder treballar dins el context *complex* al qual s'insereix la teoria de Morales-Ramis, suposarem que les integrals primeres conegudes i addicionals, formin o no un conjunt complet, són *meromorfes*, tot treballant amb hamiltonians complexos per als quals la restricció a temps i variables de fase reals restringeixin el valor del hamiltonià a la recta real.

Sistemes lineals

La segona de les nocions d'integrabilitat presentades a la Secció B.1.1 es concreta tradicionalment en la possibilitat de trobar-ne la solució general com a combinació de funcions algebraiques, quadratures (és a dir, integrals de funcions conegudes) i exponencials de funcions conegudes, i llurs inverses, i s'insereix de forma natural dins la teoria que descriurem a continuació, la qual, mantenint el format de l'equació (B.2), generalitza de forma axiomàtica els conjunts de funcions als quals pertanyen els coeficients de la matriu del sistema i els d'una matriu fonamental qualsevol. Vegeu la Secció 2.2.2 d'aquesta tesi o les referències [93] i [144] per a més detalls. Donat un sistema diferencial lineal a coeficients en un cos diferencial (K, ∂) (per exemple, $\left(\mathbb{C}(t)\,, \frac{d}{dt}\right)$),

$$\partial \boldsymbol{y} = A\boldsymbol{y}, \qquad A \in \mathcal{M}_n\,(K)\,, \tag{B.3}$$

la *teoria de Galois diferencial* assegura l'existència, i estudia les propietats:

- d'un cos diferencial $K \supset \mathbb{C}(t)$ que conté tots els coeficients d'una matriu fonamental $\Psi = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n]$ de (B.3);
- d'un grup algebraic G associat a K, anomenat el *grup de Galois diferencial* de (B.3) o de l'extensió diferencial $\mathbb{C}(t) \mid K$, i tal que
  - G actua sobre el $\mathbb{C}$-espai vectorial $\langle \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n \rangle$ de solucions com a un grup de transformacions lineals sobre $\mathbb{C}$;
  - el *grup de monodromia* del sistema (B.3) és un subgrup de G.

En el context galoisià, la integrabilitat de (B.3) *es defineix equivalent a la resolubilitat de la component de la identitat G^0 del grup de Galois diferencial G de* (B.3).

Val a dir, però, que de vegades és convenient tractar els grups de Galois com a grups de Lie, atès que, malgrat que la classificació de grups algebraics i grups de Lie és relativament anàloga, i ambdós donen lloc a les mateixes àlgebres de Lie, les representacions dels grups algebraics requereixen la substitució de les tècniques infinitesimals usades en grups diferencials per tècniques de la Geometria algebraica, la topologia de la qual, batejada en honor a O. Zariski ([55]), és relativament feble i font, per tant, de multitud de complicacions tècniques.

Teoria general de Morales-Ramis

Sigui Γ una corba integral d'un hamiltonià complex X_H; definint $\overline{\Gamma}$ com a la completació de la superfície de Riemann Γ mitjançant adjunció de singularitats i punts d'equilibri, el principal resultat de la teoria de Morales-Ramis connecta les dues nocions d'integrabilitat introduïdes a la Secció anterior: la integrabilitat hamiltoniana de X_H, i la integrabilitat del sistema *lineal* d'equacions variacionals, $\dot{\boldsymbol{\xi}} = X_H'\,(\widehat{\boldsymbol{z}}\,(t))\,\boldsymbol{\xi}$ al llarg de $\overline{\Gamma}$, respectivament. De fet, el Teorema és la implementació ad-hoc de la idea heurística següent: *si un hamiltonià és integrable, aleshores les seves equacions variacionals han d'ésser també integrables.*

Teorema B.1.1 (Morales-Ramis)**.** *Suposem que existeixen n integrals primeres de X_H, meromorfes independents i en involució en un entorn de la corba integral $\overline{\Gamma}$. Aleshores, la component de la identitat del grup de Galois diferencial G de les equacions variacionals al llarg de $\overline{\Gamma}$ és commutativa.*

Vegeu [95, Corol·lari 8] o [93, Teorema 4.1].

Un resultat essencial per a la demostració d'aquest Teorema ([95, Lema 9], vegeu també [93, Lema 4.6]) afirma el següent: si existeix una integral meromorfa

f d'un sistema $\dot{\boldsymbol{z}} = X(\boldsymbol{z})$, *hamiltonià o no*, aleshores el grup de Galois del sistema variacional $\dot{\boldsymbol{\xi}} = X(\widehat{\boldsymbol{z}}(t))\,\boldsymbol{\xi}$ al llarg de qualsevol corba solució $\widehat{z}(t)$ té un invariant racional no trivial.

La importància del Teorema de Morales-Ramis radica en diversos fets, dos dels quals afecten directament el desenvolupament d'aquesta tesi. En primer lloc, els lemes que serveixen de rerafons teòric i demostració per a aquest Teorema són alhora una generalització consistent del *Teorema de Ziglin* ([95, Teorema 10], [162, Teorema 2]), possiblement el resultat més complet de què hom disposava, abans del Teorema B.1.1, per a detectar obstruccions a la integrabilitat per a sistemes de dos graus de llibertat; el fet que el resultat de Ziglin sols accepta la integrabilitat completa com a hipòtesi per a $n = 2$ converteix, doncs, la manca de restricció sobre el nombre de graus de llibertat en un avantatge importantíssim per al Teorema B.1.1. Un valor afegit del Teorema de Morales-Ramis és el fet que G, en ser algebraic, és sovint més senzill de calcular o d'estudiar que el grup de monodromia.

Cas especial: potencials homogenis Sigui

$$H(\boldsymbol{q},\boldsymbol{p}) = T + V = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + V(\boldsymbol{q}), \tag{B.4}$$

un hamiltonià clàssic, de potencial homogeni $V(\boldsymbol{q})$ amb grau d'homogeneïtat $k \in \mathbb{Z}$. Aleshores, tota funció producte de funció escalar i vector constant $\widehat{\boldsymbol{z}}(t) = \left(\phi(t)\,\boldsymbol{c}, \dot{\phi}(t)\,\boldsymbol{c}\right)$, tal que $\ddot{\phi} + \phi^{k-1} = 0$ i $\boldsymbol{c} \in \mathbb{C}^n$ és solució de

$$\boldsymbol{c} = V'(\boldsymbol{c}), \tag{B.5}$$

és una solució de les equacions de Hamilton $\dot{\boldsymbol{z}} = X_H(\boldsymbol{z})$. La matriu $V''(\boldsymbol{c})$ sempre té $k-1$ entre els seus valors propis, i si a més és diagonalitzable aleshores una conjugació matricial adient, seguida del recobriment ramificat $t \mapsto x := \phi(t)^k$, transforma el sistema variacional $\dot{\boldsymbol{\xi}} = X'_H(\widehat{\boldsymbol{z}}(t))\,\boldsymbol{\xi}$ en un sistema desacoblat d'equacions diferencials *hipergeomètriques* ([58], [150]) en x:

$$x(1-x)\frac{d^2\xi_i}{dx^2} + \left(\frac{k-1}{k} - \frac{3k-2}{2k}x\right)\frac{d\xi_i}{dx} + \frac{\lambda_i}{2k}\xi_i = 0, \qquad i = 1,\ldots,n. \tag{B.6}$$

Adaptant aleshores el resultat previ [62] de Kimura dedicat a equacions de la forma (B.6), fou obtingut el següent resultat fonamental ([95, Teorema 3], vegeu també [93, Teorema 5.1])

Teorema B.1.2. *Sigui* (B.4) *un hamiltonià clàssic completament integrable amb integrals primeres meromorfes; sigui $\boldsymbol{c} \in \mathbb{C}^n$ una solució de $V'(\boldsymbol{c}) = \boldsymbol{c}$ i suposem que $V''(\boldsymbol{c})$ és diagonalizable; aleshores, si $\lambda_1,\ldots,\lambda_n$ són els valors propis de $V''(\boldsymbol{c})$ i definim $\lambda_1 = k-1$, tot parell (k,λ_i), $i = 2,\ldots,n$, pertany a la següent* *taula (essent p un enter arbitrari):*

Taula 1						
	k	λ			k	λ
1	k	$p+p\,(p-1)\,\frac{k}{2}$		**10**	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
2	2	*arbitrary* $z\in\mathbb{C}$		**11**	3	$-\frac{1}{24}+\frac{1}{24}\left(2+6p\right)^2$
3	-2	*arbitrary* $z\in\mathbb{C}$		**12**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$
4	-5	$\frac{49}{40}-\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$		**13**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$
5	-5	$\frac{49}{40}-\frac{1}{40}\left(4+10p\right)^2$		**14**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
6	-4	$\frac{9}{8}-\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$		**15**	4	$-\frac{1}{8}+\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$
7	-3	$\frac{25}{24}-\frac{1}{24}\left(2+6p\right)^2$		**16**	5	$-\frac{9}{40}+\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$
8	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$		**17**	5	$-\frac{9}{40}+\frac{1}{40}\left(4+10p\right)^2$
9	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$		**18**	k	$\frac{1}{2}\left(\frac{k-1}{k}+p\,(p+1)\,k\right)$

(B.7)

Aquest resultat reforça l'obtingut anteriorment per H. Yoshida per a $n=2$ ([156]) per dos motius fonamentals. En primer lloc, el procediment seguit per Yoshida, ancorat en una aplicació directa del Teorema de Ziglin ([162]), no era directament generalitzable a $n>2$. En segon lloc, la condició necessària obtinguda per Yoshida era la pertinença del valor propi addicional a un conjunt de mesura no nul·la, tesi aquesta immediatament superada en caràcter restrictiu pel conjunt *discret* al qual resta limitat cadascun dels valors propis addicionals pel Teorema B.1.2. Per això la Taula (B.7) es presenta, a l'espera d'avenços en l'enfocament galoisià de les variacionals superiors (vegeu Subsecció 5.3.1), com a eina preponderant de detecció de la no-integrabilitat de hamiltonians de la forma (B.4).

B.1.2 Alguns problemes de la Mecànica Celeste

El Problema de N Cossos

Possiblement la pedra angular de la Mecànica Celeste des que va aparèixer esmentat per primer cop als *Principia* de Newton, el *Problema (General d-dimensional) de N Cossos* és el model que descriu el moviment, dins l'espai euclidià de d dimensions, de N cossos conduïts únicament per llur atracció gravitatòria mútua. És determinat pel problema de valors inicials amb format per les $2N$ condicions inicials $\mathbf{x}_1(t_0),\ldots,\mathbf{x}_N(t_0)\in\mathbb{R}^d$ i $\dot{\mathbf{x}}_1(t_0)\ldots,\dot{\mathbf{x}}_N(t_0)\in\mathbb{R}^d$, tals que $\mathbf{x}_j(t_0)\neq\mathbf{x}_k(t_0)$ si $j\neq k$ i el sistema de Nd equacions diferencials escalars de segon ordre

$$m_i\ddot{\mathbf{x}}_i=-G\sum_{k\neq i}^{N}\frac{m_im_k}{\left\|\mathbf{x}_i-\mathbf{x}_k\right\|^3}\left(\mathbf{x}_i-\mathbf{x}_k\right),\qquad i=1,\ldots,N, \tag{B.8}$$

essent $\mathbf{x}_i\in\mathbb{R}^d$, per a cada $i=1,\ldots,N$, la funció vectorial d-dimensional que descriu de la variable temporal t que descriu la posició del cos amb massa m_i.

Podem suposar, prèvia elecció d'unitats adients, que la constant gravitatòria G és igual a 1. Definint

$$M = \operatorname{diag}(m_1, \ldots, m_1, \cdots, m_N, \ldots, m_N) \in \mathcal{M}_{Nd}(\mathbb{R}),$$

i distribuint les coordenades de l'espai de fases entre els vectors Nd-dimensionals

$$\boldsymbol{x}(t) = (\mathbf{x}_i(t))_{i=1,\ldots,N}, \qquad \boldsymbol{y}(t) = (\mathbf{y}_i(t))_{i=1,\ldots,N} := (m_i \dot{\mathbf{x}}_i(t))_{i=1,\ldots,N}$$

de posicions i moments, respectivament, les equacions del moviment es poden expressar de la forma següent:

$$\dot{\boldsymbol{x}} = M^{-1}\boldsymbol{y}, \qquad \dot{\boldsymbol{y}} = -\nabla U_{N,d}(\boldsymbol{x}), \tag{B.9}$$

essent $U_{N,d}(\boldsymbol{x}) := -\sum_{1 \leq i < k \leq N} \frac{m_i m_k}{\|\mathbf{x}_i - \mathbf{x}_k\|}$ la *funció potencial* del sistema gravitatori. (B.9) no és sinó el sistema de Hamilton (B.1) associat al hamiltonià

$$H_{N,d}(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}\boldsymbol{y}^T M^{-1}\boldsymbol{y} + U_{N,d}(\boldsymbol{x}). \tag{B.10}$$

El Problema de N Cossos ha estat considerat d'antuvi un epítom del comportament caòtic en sistemes dinàmics, fins al punt que hom considera que tal comportament es transmet a tots els models derivats del Problema, especialment als obtinguts a través de la simplificació. De fet, la majoria dels avenços assolits en Matemàtica Aplicada es deuen precisament a la presència de caos en sistemes mecànics directament o indirecta relacionats amb problemes gravitatoris de diversos cossos.

Fou justament l'avaluació de les possibilitats reals de resoldre el Problema de N Cossos el que féu que, a finals del segle XIX, diverses línies d'investigació endegades des de França i Alemanya confluïssin al ja famós concurs convocat pel Rei Òscar de Suècia, l'any 1885, a través del volum 7 d'Acta Mathematica. La bases del concurs, proposat per K. T. W. Weierstrass i G. Mittag-Leffler, requerien la demostració de l'existència de la solució com a sèrie uniformement convergent. La prova fefaent de la dificultat de trobar aquesta solució és el seguit de repercussions derivades de la monografia presentada per H. Poincaré a concurs: tot i contenir un error, l'intent de Poincaré guanyà el premi i es considera avui dia un text fundacional en la història dels sistemes dinàmics.

El problema obert tal i com apareixia publicat a les bases del premi fou finalment resolt, llevat de sengles conjunts de condicions inicials, per K. F. Sundman ([136]) per a $N = 3$ i per Q. D. Wang per al cas general $N \geq 3$. I malgrat que les condicions inicials per a les quals el resultat de Sundman no es podia aplicar directament es limitaven a les corresponents a moment angular zero, les condicions inicials refractàries a l'aplicació del resultat de Wang eren indetectables atesa la possible existència, per a $N \geq 3$, de singularitats no provinents de col·lisions (l'anomenada *conjectura de Painlevé*). A més, i malgrat que les contribucions de Sundman i Wang permeten el càlcul successiu d'un desenvolupament asimptòtic de la solució per a determinades condicions inicials, persisteixen problemes oberts la resolució dels quals requereix quelcom més que l'esmentat desenvolupament en sèrie. Bastaria, potser, el coneixement d'un conjunt adequat d'integrals primeres, potser meromorfes respecte de les variables de fase, *justament la condició que veiem en aquesta tesi que no es pot produir per a determinats valors de $N \geq 3$ i de les masses.*

Hom coneix $\frac{1}{2}(d+2)(d+1)$ integrals primeres del Problema de N Cossos, sovint anomenades *clàssiques* (vegeu [149]) i totes elles algebraiques respecte les coordenades de $\boldsymbol{q}$ i $\boldsymbol{p}$ i el temps t : $2d$ degudes a la invariància del moment lineal $\mathbf{I}_L$, $d(d-1)/2$ lligades a la invàriancia del moment angular $\mathbf{I}_A$; i una provinent de la invariància del hamiltonià $H_{N,d}$. Sabem, en virtut del *Teorema de Bruns* ([27]), que per a $N, d = 3$ tota integral primera algebraica respecte de les variables de fase i el temps és una funció algebraica de les integrals clàssiques. Una primera generalització d'aquest resultat, que anomenarem *Teorema de Julliard* ([59]), afirma que tota integral primera del Problema d-dimensional ($1 \leq d \leq N$) de N Cossos algebraica respecte de $\boldsymbol{q}$, $\boldsymbol{p}$ i t és una funció algebraica de les $\frac{1}{2}(d+2)(d+1)$ integrals clàssiques.

El Problema de Hill

El *problema Lunar de Hill* s'esdevé en Mecànica Celeste com a cas límit del Problema Restringit de Tres Cossos, al seu torn un cas especial del Problema de la Secció anterior per a $N = 3$. A més, i a banda del fet de ser en aparença la il·lustració més simple de la dinàmica gravitatòria en més de dos cossos, el problema de Hill permet d'obtenir informació addicional considerablement útil per a d'altres problemes de l'Astrofísica.

El hamiltonià en qüestió es pot expressar, prèvia regularització de Levi-Civita i reformulació a l'espai de fases estès (vegeu [126]), de la forma següent:

$$\mathcal{H}_H(\boldsymbol{Q}, \boldsymbol{P}) = H_2 + H_4 + H_6, \tag{B.11}$$

essent H_2, H_4, H_6 polinomis homogenis de graus 2, 4 i 6, respectivament:

$$H_2 = P^2/2 + Q^2/2, \quad H_4 = -2Q^2(P_2Q_1 - P_1Q_2), \quad H_6 = -4Q^2(Q_1^4 - 4Q_1^2Q_2^2 + Q_2^4).$$

Pel fet de contenir paràmetres i trobar-se globalment llunyà de qualsevol sistema integrable conegut, una primera conclusió o inferència assenyada fóra la no integrabilitat de $X_{\mathcal{H}_H}$; tal suposició es veu reforçada per la ingent quantitat de resultats numèrics (vegeu [126] un cop més, per exemple) en favor del caràcter caòtic del hamiltonià. Això no obstant, no s'havia obtingut fins als resultats d'aquesta tesi una demostració rigorosa de no-integrabilitat meromorfa i els pocs resultats parcials obtinguts abans es limitaven a una integral primera addicional algebraica ([115]) o bé analítica respecte les variables de fase i d'un paràmetre addicional espuri ω ([88]).

B.2 Resultats originals

Aquesta tesi presenta un compendi de demostracions de no-integrabilitat per a tres dels sistemes dinàmics, provinents de la Mecànica Celeste, descrits a la Secció B.1.2 anterior: el Problema de Hill, el Problema de Tres Cossos i el Problema de $N \geq 3$ Cossos amb masses iguals.

A més, presentem també una nova condició necessària per a l'existència d'una sola integral primera addicional en potencials homogenis arbitraris. És justament aquest darrer resultat el que permet generalitzar a integrals addicionals meromorfes, a més, els Teoremes de Bruns i Julliard per a $N = 3$ i $d \geq 2$, i per a $N = 3, 4, 5, 6$ masses iguals al pla.

B.2.1 Existència d'una integral primera addicional

Usant propietats fonamentals de la Geometria Algebraica obtenim a la Secció 2.1 el següent resultat:

Lema B.2.1. *Sigui* $\mathfrak{g}$ *una subàlgebra simple de Lie de*

$$\bigoplus_{i=1}^{n} \mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}\left(\mathrm{SL}_2(\mathbb{C})^n\right).$$

Aleshores, $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$.

Usant el Lema B.2.1 obtenim el següent:

Lema B.2.2. *Sigui* G *un grup algebraic que admet una representació fidel com a subgrup de* $\mathrm{SL}_2(\mathbb{C})^n$,

$$\rho : G \to \mathrm{SL}_2(\mathbb{C})^n,$$

tal que $\pi_i(G) = \mathrm{SL}_2(\mathbb{C})$ *per* $i = 1, \dots, n$*, essent cada*

$$\pi_i : \mathrm{SL}_2(\mathbb{C})^n \to \mathrm{SL}_2(\mathbb{C}), \qquad (A_1, \dots, A_n) \mapsto A_i,$$

la i-èssima projecció. Aleshores, existeix un $m \leq n$ *tal que* $\mathfrak{g} = \mathrm{Lie}(G) \simeq \bigoplus_{i=1}^{m} \mathfrak{sl}_2(\mathbb{C})$.

L'ús del Lema B.2.2, unit al fet que la possessió d'un invariant racional no trivial, cas de verificar-se, ho fa simultàniament per un grup algebraic G i la seva àlgebra $\mathrm{Lie}(G)$, ens permet concloure el resultat fonamental:

Teorema B.2.3. *Sigui* $G \subset \mathrm{SL}_2(\mathbb{C})^n$ *un grup algebraic tal que* $\pi_i(G) = \mathrm{SL}_2(\mathbb{C})$, $i = 1, ..., n$*. Aleshores,* G *no té invariants racionals no trivials.*

Usant aquest darrer resultat podem obtenir la primera contribució genuïnament original d'aquesta tesi, que, en constituir una condició necessària per a l'existència d'una sola integral addicional, complementa el Teorema B.1.2 i proporciona una condició suficient de no-integrabilitat *parcial*:

Corol·lari B.2.4. *Sigui* (B.4) *un hamiltonià clàssic amb* $m \leq n$ *integrals primeres meromorfes* $f_1, \dots, f_m$ *independents i en involució dues a dues, i sigui* $\boldsymbol{c} \in \mathbb{C}^n$ *una solució de* $V'(\boldsymbol{c}) = \boldsymbol{c}$ *tal que* $V''(\boldsymbol{c})$ *sigui diagonalizable. Siguin* $f_1, \dots, f_m$ *integrals primeres meromorfes de* X_H*, independents i en involució dues a dues sobre* $\overline{\Gamma}$*. Aleshores,*

1. *m dels valors propis, que escriurem $\lambda_1, \dots, \lambda_m$, pertanyen a la Taula* 1 *de* (B.7).

2. *Si existeix almenys una integral primera f independent del conjunt $\{f_1, \dots, f_m\}$ en un entorn de $\overline{\Gamma}$, aleshores almenys un dels valors propis $\lambda_{m+1}, \dots, \lambda_n$ pertany a la Taula* 1.

B.2.2 Problemes de N Cossos

Tenim, doncs, dos objectius clars a curt i mig termini, que només hem assolit en part dins d'aquesta tesi:

Conjectura B.2.5 (No-integrabilitat del Problema de N Cossos)**.** *Independentment dels valors de les masses $m_1, \ldots, m_N > 0$, el Problema d-dimensional de N Cossos no té cap conjunt de dN integrals primeres meromorfes independents i en involució dues a dues.*

Conjectura B.2.6. *Llevat d'un conjunt identificable i de mesura zero $\mathfrak{M} \in \mathbb{R}_+^N$ de vectors de masses $(m_1, \ldots, m_N)$, el Problema d-dimensional de N Cossos no té cap integral primera meromorfa independent de les integrals clàssiques.*

Observem que demostrar la segona conjectura implicaria demostrar la primera per a $(m_1, \ldots, m_N) \notin \mathfrak{M}$, i constituïria en un cert sentit la generalització del Teorema de Bruns per a N arbitrari.

Usant una especialització de la teoria de Morales-Ramis aplicada a la factorització d'operadors lineals, més concretament la descripció de la presència de logaritmes com a condició suficient d'obstrucció a la integrabilitat, D. Boucher i J.-A. Weil ([23], [21]) provaren la Conjectura B.2.5 per a $d = 2$ i $N = 3$ masses arbitràries. Per altra banda, i fent ús del Teorema de Ziglin, A. V. Tsygvintsev ([139], [140], [141], [142], [143]) demostrà, també per a $N = 3$ i $d = 2$, la Conjectura B.2.5, així com la Conjectura B.2.5 llevat de tres casos especials:

$$\frac{m_1 m_2 + m_2 m_3 + m_1 m_3}{(m_1 + m_2 + m_3)^2} \in \left\{ \frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{3^2} \right\}.$$

Nous resultats

Gràcies al fet fonamental que podem reduir el hamiltonià del Problema de Tres Cossos a un hamiltonià *clàssic* $\mathcal{H}_{N,d} = \frac{1}{2}p^2 + V_{N,d}(\boldsymbol{q})$ amb potencial homogeni de grau -1,

$$V_{N,d}(\boldsymbol{q}) := -\sum_{1 \le i < j \le N} \frac{(m_i m_j)^{3/2}}{\left\| \sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j \right\|}, \tag{B.12}$$

l'aplicació dels dos resultats introduïts a les Seccions B.1.1 i B.2.1, és a dir el (ja conegut) Teorema B.1.2 i el (nou) Corol·lari B.2.4, ens permet obtenir tres nous resultats de no-integrabilitat, el primer i part del segon dels quals estableixen, a més, l'absència d'una integral primera addicional. Són els descrits a continuació.

El primer d'ells simplifica, reobté i completa els resultats de Boucher, Weil i Tsygvintsev per a $N = 3$ i per a dimensió i masses $d \ge 2$ i $m_1, m_2, m_3 > 0$ arbitràries, demostrant per tant la Conjectura B.2.6 en tota la seva generalitat per a $N = 3$ i $\mathfrak{M} = \varnothing$, la qual cosa constitueix una novetat i generalitza el Teorema de Bruns, el darrer resultat de Tsygvintsev i el Teorema de Julliard per a $N = 3$. En particular, demostra també (per tercer cop en menys de deu anys) la Conjectura B.2.5. El resultat en qüestió és el següent:

Teorema B.2.7. *Per a cada $d \ge 2$, no existeix cap integral primera meromorfa del Problema d-dimensional de Tres Cossos amb masses arbitràries positives i independent de les integrals clàssiques.*

La demostració d'aquest Teorema es basa en l'aplicació del Corol·lari B.2.4 a dues solucions concretes i explícitament calculades $\boldsymbol{c}, \boldsymbol{c}^*$ de (B.5). La primera d'elles és la *configuració triangular de Lagrange* ([66]), i la segona és una solució de coordenades complexes. Obtenim valors propis

$$\operatorname{Spec}\left(V''_{3,2}\left(\boldsymbol{c}\right)\right) = \{-2,0,0,1,\lambda_\pm\}, \qquad \operatorname{Spec}\left(V''_{3,2}\left(\boldsymbol{c}^*\right)\right) = \left\{-2,0,0,1,\lambda^*_\pm\right\},$$

essent els primers quatre valors propis $-2,0,0,1$ lligats a les integrals primeres clàssiques i per tant identificables amb els valors propis $\lambda_1,\ldots,\lambda_m$ de l'apartat 1 del Corol·lari B.2.4. L'existència d'una integral addicional del Problema (pla) de Tres Cossos implicaria la pertinença de $\lambda_\pm$ i $\lambda^*_\pm$ a la Taula (B.7), la qual cosa implicaria en particular tres relacions de dependència algebraica entre les masses m_1, m_2, m_3 que no es poden produir simultàniament. Un darrer argument d'increment de la dimensió estén el resultat a $d \geq 2$ arbitrària.

El segon nou resultat relatiu al Problema de $N \geq 4$ Cossos prové d'un primer intent d'ampliar el resultat anterior a un $N \geq 4$ arbitrari:

Teorema B.2.8. *Sigui $X_{\tilde{\mathcal{H}}_{N,d}}$ qualsevol Problema de N Cossos d-dimensional* amb masses iguals. *Aleshores,*

1. *No existeix una integral meromorfa addicional per al Problema* pla $X_{\tilde{\mathcal{H}}_{N,2}}$ *si* $N = 3,4,5,6$.

2. *Per a $N \geq 3$ i $d \geq 2$, $X_{\tilde{\mathcal{H}}_{N,d}}$ no és integrable amb integrals primeres meromorfes.*

El primer apartat és a més una obstrucció a la existència d'una integral primera addicional, la qual cosa generalitza el cas $d = 2$ del Teorema de Julliard per a $N = 3,4,5,6$ masses iguals. El segon apartat no nega l'existència d'una integral addicional i es limita a demostrar la no-integrabilitat meromorfa en el sentit de Lioville-Arnol'd.

La solució de $V'_N\left(\boldsymbol{c}\right) = \boldsymbol{c}$ emprada en aquest cas és la solució *poligonal* $\boldsymbol{c} = \boldsymbol{c}_P$ del Problema amb masses iguals ([111]), expressable en termes de les arrels N-èssimes de la unitat. Els Lemes previs usats en la demostració són dos. El primer és una aplicació de la fórmula d'Euler-Maclaurin:

Lema B.2.9. *Per a cada $N \geq 10$, $S_N := 2\sum_{j=1}^{N-1}\left(\csc^2\frac{j\pi}{N} - 5\right)\csc\frac{j\pi}{N} > 0$.*

De l'anterior es dedueix el següent, prèvia expressió compacta de la matriu hessiana $V''_3\left(\boldsymbol{c}\right)$

Lema B.2.10. *Per a $N \geq 10$, $V''_N\left(\boldsymbol{c}_P\right)$ té almenys un valor propi més gran que* 1.

El tercer i darrer resultat consisteix en una demostració alternativa del punt 2 del Teorema B.2.8 per al cas particular en què $N = 2^m$, $m \geq 2$. L'eina principal és el següent resultat aritmètic:

Teorema B.2.11. *Per a cada $N \in \mathbb{N}$ de la forma $N = 2^m$, $m \geq 2$, les expressions $\sum_{k=1}^{N-1} \csc \frac{\pi}{N} k$ i $\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k$ són racionalment independents, és a dir, tota equació de la forma*

$$n_1 \sum_{k=1}^{N-1} \csc \frac{\pi}{N} k + n_2 \sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k = 0,$$

essent $n_1, n_2 \in \mathbb{Z}$, implica $n_1 = n_2 = 0$.

Atès que la suma dels valors propis de la matriu hessiana $V_N''(\boldsymbol{c}_P)$ és precisament

$$\operatorname{tr}\left(V_N''(\boldsymbol{c}_P)\right) = -\frac{N}{2} \frac{\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k}{\sum_{k=1}^{N-1} \csc \frac{\pi}{N} k},$$

podem concloure per tant que:

Teorema B.2.12. *El Problema de N Cossos amb N masses iguals no és integrable amb integrals primeres meromorfes si $N = 2^m$ amb $m \geq 2$.*

Igual que en el cas de potencials homogenis, tots els resultats dels quals hem escrit la demostració es poden qualificar de nous, si bé la majoria d'ells són senzills i d'una importància accessòria i no han estat enunciats, per tant, en aquest resum. Vegeu els Capítols 2 i 3 per més detalls, així com el capítol 5 per tal de copsar possibilitats d'estudi futures de les Conjectures B.2.5 i B.2.6.

Val a dir que els Teoremes B.2.4, B.2.7 and B.2.12, així com els Lemes previs descrits més amunt, apareixeran publicats a [99].

B.2.3 La no-integrabilitat del Problema de Hill

Al Capítol 4 presentem una demostració de no-integrabilitat meromorfa – demostració aquesta que, en comptes d'explotar les eines, conegudes i noves, emprades per al Problema de N Cossos (és a dir, el Teorema B.1.2 i el Corol·lari B.2.4), fa ús de la base teòrica d'aquestes eines, és a dir el Teorema general de Morales-Ramis B.1.1. La necessitat de recórrer al fons teòric obeeix no sols a l'ànim de diversificar l'estudi, sinó també a la dificultat de transformar el hamiltonià (B.11) a forma clàssica (B.4) amb potencial homogeni. Més enllà de la novetat del resultat en sí, per tant, el Capítol 4 es presenta com a paradigma de la utilitat del Teorema B.1.1 a hamiltonians significatius d'índole general. A més, el dit Teorema ha permès identificar les contribucions concretes, en forma de funcions especials, que probablement feren tan difícil aquesta demostració en el passat. Justament aquesta detecció d'obstruccions a la integrabilitat és el lloc de la tesi on més propers ens trobem al comentari fet al final de la Secció B.1.

Tots els Lemes i Teoremes del Capítol 4 són nous, i els enunciem a continuació a mode de resum. Un primer resultat proporciona la solució particular, i per tant la corba integral, necessària per a l'aplicació del Teorema B.1.1:

Lema B.2.13. *$X_{\mathcal{H}}$ té una solució particular*

$$\left(Q_1(t), Q_2(t), P_1(t), P_2(t)\right) = \frac{1}{\sqrt{2}} \left(\phi(t), \mathrm{i}\phi(t), \dot{\phi}(t), \mathrm{i}\dot{\phi}(t)\right), \tag{B.13}$$

tal que, per a tot valor $0 < h < 1/\left(6\sqrt{3}\right)$ *del nivell d'energia* h, $\phi^2(t)$ *és el·líptica amb dos pols simples en cada paral·lelogram periòdic.*

Usant el Lema B.2.13 i les propietats de la funció el·líptica $\phi^2(t)$ en qüestió, obtenim:

Lema B.2.14. *Les equacions variacionals de* $X_{\mathcal{H}}$ *al llarg de la solució* (B.13) *tenen una matriu fonamental de la forma*

$$\Psi(t) = \begin{pmatrix} \Phi_N(t) & \Phi_N(t)\int_0^t V(\tau)d\tau \\ 0 & \Phi_N(t) \end{pmatrix},$$

essent

$$\Phi_N(t) = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ \mathrm{i}\dot{\xi}_1(t) & \mathrm{i}\dot{\xi}_2(t) \end{pmatrix}$$

una matriu fonamental de les equacions normals*; a més,* ξ_2 *és una combinació lineal de funcions el·líptiques i integrals el·líptiques no trivials de primera i segona classe, i* $\int_0^t V(\tau)d\tau$ *és una funció matricial* 2×2 *amb logaritmes a la diagonal.*

Això permet un estudi acurat de l'extensió de Picard-Vessiot del sistema variacional gràcies al qual podem afirmar:

Teorema B.2.15. *La component de la identitat* G^0 *del grup de Galois diferencial de les equacions variacionals al llarg de la solució particular és no-commutatiu.*

En virtut, finalment, del Teorema B.1.1, obtenim el resultat principal:

Corol·lari B.2.16. *El problema de Hill no admet una integral del moviment meromorfa independent del seu hamiltonià.*

A diferència del Problema de N Cossos o els estudis de potencials homogenis, aquests nous resultats del Capítol 4 han estat publicats, a [98], en un treball conjunt amb els directors de la present tesi, l'autor de la qual els està agraït per les incomptables discussions i lliçons sobre teoria de Galois diferencial, teoria de grups, funcions el·líptiques i sobre el problema de Hill, l'eliminació de la força de Coriolis i les equacions variacionals, entre d'altres.

B.3 Agraïments

En primer lloc, vull fer palesa la meva gratitud als meus directors, Juan J. Morales-Ruiz i Carles Simó, pel camí recorregut conjuntament fins ara. Vull agrair el Juan el fet d'haver-me introduït dins la teoria de Galois diferencial, possiblement un dels racons més elegants, bells i fascinants de la Matemàtica actual, alhora que de sorprenent potència en les aplicacions pràctiques – valgui com a exemple la teoria que ell mateix creà juntament amb en Jean-Pierre Ramis fa no molt. També li estic agraït per la seva inesgotable paciència, pel seu suport continu, pel seu ànim encoratjador, per la seva vocació instructiva i pels seus esforços a ensenyar-me les eines bàsiques (i les no tan bàsiques) de l'enfoc galoisià dels sistemes diferencials. Li estic agraït al Carles per presentar-me en Juan tant bon punt vaig manifestar-li el meu interès en la integrabilitat, per proporcionar-me els ambiciosos problemes oberts que han estat resolts en part aquí, per guiar-me pel dens territori de les equacions diferencials en general i dels sistemes hamiltonians en particular, i per transmetre'm una afecció i un interès duradors envers la Mecànica Celest i, per què no dir-ho, per fer-me veure que un cert esprit temerari i un gust pels "tours de force" matemàtics no en tenen pas res, de dolent, ans al contrari. Gràcies a tots dos per ensenyar-me les coses que no surten als llibres, i, el que és més important, per ensenyar-me a buscar-les pel meu compte.

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Index

G-module, 11, **11**, 13, 14, 19
  completely reducible, 11, **11**, 14
  faithful, **11**, 13, 19, 26
  irreducible, 11, **11**
G-orbit, **11**, 12, 13
K-automorphism
  differential, **18**, 19, 20, 64, 65, 67
N-Body Problem, 2, 3, 5, 7, **27**, 29, 30, 32, 54, 71
  $N = 2^m$, 51, 54
  $N = 3$, 3–5, 29–31, 33–35, 39, 71
    no additional integral, 5, 39
    planar, 39
    spatial, 31
  $N = 4$, 5, 33, 39, 47, 71
    spatial, 77
  central configuration, 32, **32**, 33, 34, 37, 38
  collision, **28**, 29, 30, 34
  Euler configuration, **33**, 34, 39, 46, 74, 75
  first integrals, 5, 31, 32
    classical, 16, 31, **31**, 32, 37, 39, 45, 71
  Hamiltonian, **28**, 30, 31, 38, 40
  homographic solution, 32, **32**, 34, 39, 77
  homothetical solution, **32**, 38, 44
  Lagrange configuration, 33, **33**, 39, 44
    parabolic, **39**
  Moulton configuration, 30, **33**, 73–76
  with $N - 1$ equal masses, 76
  with equal masses, 5, 39, 51, 54, 71, 76

affine variety, 10, 11
algebra, 9–13
  Lie, *see* Lie algebra
  Poisson, *see* Poisson algebra
  symmetric, **14**
algebraic group, 4, 7, 10, **10**, 11–15, 19–21, 66, 67
  linear, 4, **10**, 12–14, 19, 20
  morphism, **10**
  semisimple, **11**, 12, 14
  simple, **11**
angular momentum, 29–31, **31**, 34, 41

Bernoulli
  Two-Body Problem solution, 28
Boucher-Weil
  proof for the Three-Body Problem, 5, 39
Bruns
  Theorem, 5, **31**, 38, 40, 71, 72, 76

canonical transformation, **16**, 36, 46, 55, 57
Celestial Mechanics, 2, 4, 6, 9, 27, 33, 34, 72
center of mass, 31, **31**, 33
chaos, 1
constants
  field of, **18**, 19, 20

Darboux
  point, **23**
  Theorem, 15
derivation, **18**
differential Galois group, 19, **19**, 20–22, 24–26, 55, 56, 63, 64, 68, 69, 72, 78
differential Galois theory, 4, 18
dynamical systems, 15
  algorithmic modeling, 1
  Hamiltonian, **15**
  integrable, 1
    meromorphically, 4
  qualitative theory, 1

computer-assisted proofs, 1
numerical simulation, 1

Euler
configuration, **33**, 34, 39, 46, 74, 75
quintic, **34**, 46
exponentials, **20**
extension
by exponentials, **21**
by integrals, **21**
Liouville, **20**, 21, 63
of differential fields, **18**
Picard-Vessiot, 19, **19**, 20–22, 55, 56, 63, 65
of a given equation, **19**

field
differential, **18**
first integrals, 2, 5, 16, **16**, 17, 22, 24, 26, 31, 32, 37, 39, 40, 45, 69, 71
additional, 5, **16**, 17, 26, 37–39, 44, 45, 69, 71
in involution, commuting, 16, **16**, 17, 22, 26, 71
Four-Body Problem, *see* $N = 4$ in N-Body Problem

Hamiltonian
function, **15**
system, **15**
vector field, **15**
Hill's Lunar Problem, 6, 35, **35**, 55, 56, 69, 70
equations, **35**
Hamiltonian
original, **35**
polynomial, **36**, 56, 57
meromorphic non-integrability, **56**

identity component, 11, **11**, 13, 21, 22, 24, 25, 56, 63, 64, 78
integrability, 1, 15
for linear systems, 20
Hamilton-Jacobi, **17**
Liouville-Arnol'd, **16**
partial, **17**
integrals, *see* quadratures
invariant
of a linear algebraic group, **12**
rational, 12–15, 22, 26
irreducible topological space, **10**, 11

Jacobi identity, **11**
Julliard
Theorem, 32, **32**, 37, 38, 40, 71

Lagrange
configuration, 33, **33**, 39, 44
parabolic, **39**
Laplace
on determinism, 2
Lie algebra, 11, **11**, 12, 13
commutator series, **11**
ideal, **11**
of a group, **12**
semisimple, **11**, 12, 14
simple, 11, **11**, 12–14
solvable, 11, **11**, 12
subalgebra, **11**
Lie bracket, 11, **11**
Lie group, **12**
linear momentum, 31, **31**, 40

meromorphic
first integral, 5, 17, 22, 24, 37–40, 44, 45, 56, 69–71
function, **17**
integrability, **17**
moment of inertia, **31**
Morales-Ramis
Theorem, 22, **22**, 39, 55, 56, 64, 71, 72
Theory, 4, 22, 55, 56, 64, 67, 71, 72
Morales-Ramis Theorem, 22, 67
Morales-Ramis Theory, 22, 39
Moulton
configuration, 30, **33**, 73–76
musical isomorphism, **15**

Noether symmetry, 17
normal extension, 19, 65, 67
normal subgroup group, **9**

Painlevé
conjecture, **30**
Poincaré
N-Body Problem monograph, 3

Poisson algebra, 16, 22
Poisson bracket, 14, 16

quadratures, 15, 16, **20**, 21, 24, 61, 63, 78

radical, **11**
  unipotent, *see* unipotent radical
rational
  invariant, 12–15, 22, 26
rational function, **10**
rational invariant
  of a Poisson algebra, **14**
representation
  faithful, **10**, 11, 19
  rational, **10**
Restricted Three-Body Problem, 6, 35

small divisors, 3
solution, 1
solvable group, **9**
Sundman
  Three-Body Problem solution, 3, 28, 29, **29**, 30, 37
symplectic
  change of variables, *see* canonical transformation
  form, **15**
  manifold, 15, **15**, 68

Theorem
  Bruns, 5, **31**, 38, 40, 71, 72, 76
  fundamental (differential Galois theory), **20**, 21, 64, 67
  Julliard, 32, **32**, 37, 38, 40, 71
  Kolmogorov-Arnol'd-Moser (K.A.M.), 4, 35
  Liouville-Arnol'd, 15, 16, **16**, 17
  Morales-Ramis, 22, **22**, 39, 55, 56, 64, 67, 71, 72
Three-Body Problem, *see* $N = 3$ in N-Body Problem
tower of subgroups, **9**
  normal, **9**
transcendental element, **21**, 61–64
Tsygvintsev
  proof for the Three-Body Problem, 5, 39, 46
    exceptional cases, 5, 39, **46**

unipotent element, **11**
unipotent radical, **11**, 67, 78

Wang
  N-Body Problem solution, 3, 28, 30, **30**, 32, 37

Zariski topology, 10
Ziglin
  Lemma, 22, 26
  proof for the Three-Body Problem, 5, 39
  proofs for some N-Body Problems, 5, 39
  Theorem, 4, 5, 22, 25, 39, 55, 67