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The generalized index of maximum and minimum level and its application in decision making

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Abstract: The index of maximum and minimum level is a very useful technique, especially for decision making, which uses the Hamming distance and the adequacy coefficient in the same problem. In this paper, we suggest a generalization by using generalized and quasi-arithmetic means. As a result, we will get the generalized ordered weighted averaging index of maximum and minimum level (GOWAIMAM) and the Quasi-OWAIMAAM operator. These new aggregation operators generalize a wide range of particular cases such as the generalized index of maximum and minimum level (GIMAM), the OWAIMAM, the ordered weighted quadratic averaging IMAM (OWQAIMAM), and others. We also develop an application of the new approach in a decision making problem about selection of products.
Keywords: Index of maximum and minimum level; OWA operator; Generalized mean; Quasi-arithmetic mean; Decision making.
JEL Classification: C44, C49, D81, D89.

Resumen: El índice del máximo y el mínimo nivel es una técnica muy útil, especialmente para toma de decisiones, que usa la distancia de Hamming y el coeficiente de adecuación en el mismo problema. En este trabajo, se propone una generalización a través de utilizar medias generalizadas y cuasi aritméticas. A estos operadores de agregación, se les denominará el índice del máximo y el mínimo nivel medio ponderado ordenado generalizado (GOWAIMAM) y cuasi aritmético (Quasi-OWAIMAM). Estos nuevos operadores generalizan una amplia gama de casos particulares como el índice del máximo y el mínimo nivel generalizado (GIMAM), el OWAIMAM, y otros. También se desarrolla una aplicación en la toma de decisiones sobre selección de productos.
Palabras clave: Índice del máximo y el mínimo nivel; Operador OWA; Media generalizada; Media cuasi-aritmética; Toma de decisiones.

1. Introduction

The index of maximum and minimum (IMAM) level (J. Gil-Lafuente, 2001; 2002) is a very useful technique that provides similar results than the Hamming distance with some differences that makes it more complete. It includes the Hamming distance and the adequacy coefficient in the same formulation. Since its appearance, it has been used in a wide range of applications such as fuzzy set theory, business decisions, multicriteria decision making, etc. (J. Gil-Lafuente, 2002; Merigó and A.M. Gil-Lafuente, 2007a).

A very common aggregation method is the ordered weighted averaging (OWA) operator (Yager, 1988). It provides a parameterized family of aggregation operators that includes the maximum, the minimum and the average, as special cases. Since its appearance, the OWA operator has been studied by different authors (Beliakov et al., 2007; Calvo et al., 2002; Merigó, 2007; Xu, 2005; Yager, 1992; 1993; 1996a; 2007; Yager and Kacprzyk, 1997). An interesting generalization of the OWA operator is the generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004) that uses generalized means in the aggregation process. Then, we can obtain a wide range of mean operators such as the generalized mean (Dujmovic, 1974; Dyckhoff and Pedrycz, 1984), the weighted generalized mean, the OWA operator, the ordered weighted geometric (OWG) operator (Chiclana et al., 2000; Xu and Da, 2002), etc. The GOWA operator can be further generalized by using quasi-arithmetic means (Beliakov, 2005). Then, the result is the Quasi-OWA operator (Fodor et al., 1995). For further developments on the GOWA and the Quasi-OWA operator, see (Merigó and Casanovas, 2007a; 2007b; Merigó and A.M. Gil-Lafuente, 2007b; Wang and Hao, 2006).

Recently, Merigó and A.M. Gil-Lafuente (2008b) have suggested the use of the OWA operator in the IMAM. They have called it the ordered weighted averaging index of maximum and minimum level (OWAIMAM). Going a step further, in this paper we suggest a generalization of the OWAIMAM by using generalized and quasi-arithmetic means. The result will be the generalized OWAIMAM (GOWAIMAM) and the quasi-arithmetic OWAIMAM (Quasi-OWAIMAM). The main advantage of these operators is that they include a wide range of mean operators such as the normalized IMAM (NIMAM), the weighted IMAM (WIMAM), the OWAIMAM, the generalized IMAM (GIMAM), etc. We will study some of their main properties.

We will also develop an application of the new approach in a decision making problem about the selection of products. We will focus on the selection of apartments because it is one of the main products for the consumers. With the GOWAIMAM, we will be able to evaluate different situations and results depending on the particular case used in the decision process.

In order to do so, this paper is organized as follows. In Section 2, we briefly describe some basic concepts about the IMAM, the OWA and the GOWA operator. In Section 3 we present the GOWAIMAM operator and in Section 4 we study different particular cases. Section 5 introduces the Quasi-OWAIMAM operator and Section 6 develops an application of the OWAIMAM in a decision making problem. Finally, in Section 7 we summarize the main conclusions of the paper.

2. Aggregation operators

In this Section, we briefly review the IMAM, the OWA and the GOWA operator.

2.1. Index of maximum and minimum level

The NIMAM (J. Gil-Lafuente, 2001; 2002) is an index used for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. It is a very useful technique that provides similar results than the Hamming distance but with some differences that makes it more complete. Basically, we could define it as a measure that includes the Hamming distance and the adequacy coefficient (Gil-Aluja, 1998; A.M. Gil-Lafuente, 2005; Kaufmann and Gil-Aluja, 1986; 1987) in the same formulation. For two sets A and B, it can be defined as follows.

Definition 1. A NIMAM of dimension n is a mapping K: $R^n \rightarrow R$ such that:

$$\eta(P,P_j) = \frac{1}{u+v}\left[\sum_u \left|\mu_i(u) - \mu_i^{(j)}(u)\right| + \sum_v \left(0 \vee (\mu_i(v) - \mu_i^{(j)}(v))\right)\right] \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B, and $u + v = n$.

Sometimes, when normalizing the IMAM it is better to give different weights to each individual element. Then, the index is known as the WIMAM. It can be defined as follows.

Definition 2. A WIMAM of dimension n is a mapping K: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_i \in [0,1]$. Then:

$$\eta(P,P_j) = \sum_u w_i(u) \times \left| \mu_i(u) - \mu_i^{(j)}(u) \right| + \sum_v w_i(v) \times \left[0 \vee (\mu_i(v) - \mu_i^{(j)}(v)) \right] \quad (2)$$

where a_i and b_i are the ith arguments of the sets A and B, and $u + v = n$.

2.2. OWA operator

The OWA operator (Yager, 1988) provides a parameterized family of aggregation operators which have been used in many applications (Calvo et al, 2002; Merigó, 2007; Xu, 2005; Yager, 1993; Yager and Kacprzyk, 1997). It can be defined as follows.

Definition 3. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n having the properties:

(1) $w_j \in [0, 1]$

(2) $\sum_{j=1}^{n} w_j = 1$

and such that

$$\text{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (3)$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step we can distinguish between the Descending OWA (DOWA) operator and the Ascending OWA (AOWA) operator (Yager, 1992). Note that the weights of these two operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DOWA and w^*_{n-j+1} the jth weight of the AOWA operator.

The OWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. By choosing a different manifestation of the weighting vector, we are able to obtain different types of aggregation operators such as the maximum, the minimum, the average and the weighted average (Yager, 1988). For example, the maximum is found when $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$. The minimum is obtained when $w_n = 1$ and $w_j = 0$ for all $j \neq n$. The average is found when $w_j = 1/n$ for all j. Other families of OWA operators can be studied in (Merigó, 2007; Xu, 2005; Yager, 1993; 1994; 1996a; 2007; Yager and Filev, 1994; Yager and Kacprzyk, 1997).

2.3. GOWA operator

The generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004) is an aggregation operator that generalizes a wide range of mean operators such as the OWA operator with its particular cases, the ordered weighted geometric (OWG) operator (Chiclana et al., 2000; Herrera et al., 2003; Xu and Da, 2002), the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004) and the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004). It can be defined as follows.

Definition 4. A GOWA operator of dimension n is a mapping GOWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$\text{GOWA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{4}$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, we have to distinguish between the descending generalized OWA (DGOWA) operator and the ascending generalized OWA (AGOWA) operator. Note that it is possible to use them in situations where the highest value is the best result and in situations where the lowest value is the best result. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DGOWA and w^*_{n-j+1} the jth weight of the AGOWA operator.

As it is demonstrated in (Yager, 2004), the GOWA operator is a mean operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and weighted generalized mean. Other families of GOWA operators can be found in (Karayiannis, 2000; Merigó, 2007, Yager, 2004).

If we look to different values of the parameter λ, we can also obtain other special cases as the usual OWA operator, the OWG operator, the OWHA operator and the OWQA operator. When $\lambda = 1$, we obtain the usual OWA operator. When $\lambda = 0$, we obtain the OWG (OWG) operator. When $\lambda = -1$, we obtain the OWHA (OWHA) operator. When λ = 2, we obtain the OWQA (OWQA) operator.

Note that if we replace b^{λ} with a general continuous strictly monotone function $g(b)$, then, the GOWA operator becomes the Quasi-OWA operator. It can be formulated as follows.

Definition 5. A Quasi-OWA operator of dimension n is a mapping QOWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$\text{QOWA}(a_1, a_2,\ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right)\right) \tag{5}$$

where b_j is the jth largest of the a_i.

3. Generalized index of maximum and minimum level

The D-S theory of evidence The IMAM can be generalized by using generalized means. Then, the result is the generalized index of maximum and minimum level (GIMAM). Going a step further, it is also possible to use the weighted generalized mean, obtaining the weighted generalized index of maximum and minimum level (WGIMAM). It can be defined as follows.

Definition 6. A WGIMAM operator of dimension n is a mapping WGIMAM: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

WGIMAM($P_k \rightarrow P$) =

$$= \left(\sum_u w_i(u) \times \left| \mu_i(u) - \mu_i^{(j)}(u) \right|^{\lambda} + \sum_v w_i(v) \times \left[0 \vee (\mu_i(v) - \mu_i^{(j)}(v)) \right]^{\lambda} \right)^{1/\lambda} \quad (6)$$

where μ_i and μ_i^k are the ith arguments of the sets P_k and P, $u + v = n$, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

As we can see, if $w_i = 1/n$, we get the GIMAM operator. Note that if we look to the parameter λ we also find a wide range of mean operators. For example, if λ = 1, we get the weighted IMAM (WIMAM), and if λ = 2, we get the weighted quadratic averaging IMAM (WQAIMAM).

Going a step further, it is possible to present a wider generalization of the WGIMAM operator by using the OWA operator. Then, we get the following.

Definition 7. A GOWAIMAM operator of dimension n is a mapping GOWAIMAM: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} \quad (7)$$

where K_j represents the jth largest of all the $|\mu_i - \mu_i^{(k)}|$ and the $[0 \vee (\mu_i - \mu_i^{(k)})]$; $k = 1,2,\ldots,m$; and λ is a parameter such that $\lambda \in (-\infty, \infty)$. Note that we have given this definition for all R, but we should note that sometimes we may find problems, especially when the arguments are 0. Basically, these problems appear for values in the parameter $\lambda \leq 0$.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending generalized OWAIMAM (DGOWAIMAM) operator and the ascending generalized OWAIMAM (AGOWAIMAM) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DGOWAIMAM and w^*_{n-j+1} the jth weight of the AGOWAIMAM operator.

Analogously to the GOWAIMAM operator, we can suggest an equivalent removal index that it is a dual of the GOWAIMAM because $Q(P_k \rightarrow P) = 1 - K(P_k \rightarrow P)$. We will call it the generalized ordered weighted averaging dual IMAM (GOWADIMAM). It can be defined as follows.

Definition 8. A GOWADIMAM operator of dimension n, is a mapping GOWADIMAM: $R^n \rightarrow R$ that has an associated weighting vector W, with $w_j \in [0,1]$ and the sum of the weights is equal to 1, then:

$$\text{GOWADIMAM}(q_1, q_2, \ldots, q_n) = \left(\sum_{j=1}^{n} w_j Q_j^{\lambda} \right)^{1/\lambda} \tag{8}$$

where Q_j represents the jth largest of all the $[1 - |\mu_i - \mu_i^{(k)}|]$ and the $[1 \wedge (1 - \mu_i + \mu_i^{(k)})]$; and $k = 1,2,\ldots,m$, and λ is a parameter such that $\lambda \in (-\infty, \infty)$. The final result will be a number between [0,1]. Note that in this case, we also find inconsistencies when $\lambda \leq 0$.

In this case, we can also distinguish between the descending GOWADIMAM (DGOWADIMAM) and the ascending GOWADIMAM (AGOWADIMAM) operator. Their weights are related by $w_j = w^*_{n-j+1}$, where w_j

is the jth weight of the DGOWADIMAM and w^*_{n-j+1} the jth weight of the AGOWADIMAM operator.

If K is a vector corresponding to the ordered arguments K_j, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then the GOWAIMAM aggregation can be expressed as:

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = W^T K \tag{9}$$

Also note that the GOWAIMAM operator is commutative, monotonic, bounded and idempotent. These properties can be demonstrated with the following theorems.

Theorem 1 (Monotonicity). Assume f is the GOWAIMAM operator, if $p_i \geq q_i$, for all p_i, then:

$$f(p_1, p_2, \ldots, p_n) \geq f(q_1, q_2, \ldots, q_n) \tag{10}$$

Proof. Let

$$f(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} \tag{11}$$

$$f(q_1, q_2, \ldots, q_n) = \left(\sum_{j=1}^{n} w_j Q_j^{\lambda} \right)^{1/\lambda} \tag{12}$$

Since $p_i \geq q_i$, for all i, it follows that, $p_i \geq q_i$, and then

$$f(p_1, p_2, \ldots, p_n) \geq f(q_1, q_2, \ldots, q_n) \qquad \blacksquare$$

Theorem 2 (Commutativity). Assume f is the GOWAIMAM operator, then:

$$f(p_1, p_2, \ldots, p_n) = f(q_1, q_2, \ldots, q_n) \tag{13}$$

where $(p_1, p_2, \ldots, p_n)$ is any permutation of the arguments $(q_1, q_2, \ldots, q_n)$.

Proof. Let

$$f(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} \tag{14}$$

$$f(q_1, q_2, \ldots, q_n) = \left(\sum_{j=1}^{n} w_j Q_j^{\lambda} \right)^{1/\lambda} \tag{15}$$

Since $(p_1, p_2, \ldots, p_n)$ is a permutation of $(q_1, q_2, \ldots, q_n)$, we have $p_j = q_j$, for all j, and then

$$f(p_1, p_2, \ldots, p_n) = f(q_1, q_2, \ldots, q_n) \qquad \blacksquare$$

Theorem 3 (Idempotency). Assume f is the GOWAIMAM operator, if $p_i = p$, for all p_i, then:

$$f(p_1, p_2, \ldots, p_n) = p \tag{16}$$

Proof. Since $p_i = p$, for all p_i, we have

$$f(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} = \left(\sum_{j=1}^{n} w_j p^{\lambda} \right)^{1/\lambda} = \left(p^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda} \tag{17}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(p_1, p_2, \ldots, p_n) = p \qquad \blacksquare$$

Theorem 4 (Bounded). Assume *f* is the GOWAIMAM operator, then:

$$\text{Min}\{p_i\} \leq f(p_1, p_2, \ldots, p_n) \leq \text{Max}\{p_i\} \tag{18}$$

Proof. Let $\max\{p_i\} = b$, and $\min\{p_i\} = a$, then

$$f(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} w_j b^{\lambda} \right)^{1/\lambda} = \left(b^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda} \tag{19}$$

$$f(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^{\lambda} \right)^{1/\lambda} \geq \left(\sum_{j=1}^{n} w_j a^{\lambda} \right)^{1/\lambda} = \left(a^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda} \tag{20}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(p_1, p_2, \ldots, p_n) \leq b \tag{21}$$

$$f(p_1, p_2, \ldots, p_n) \geq a \tag{22}$$

Therefore,

$$\text{Min}\{p_i\} \leq f\,(p_1, p_2, \ldots, p_n) \leq \text{Max}\{p_i\} \qquad \blacksquare$$

A further interesting problem to consider in the GOWAIMAM operator is the unification point with distance measures. As it was explained in Merigó and A.M. Gil-Lafuente (2007a), the unification point between the IMAM and the Hamming distance appears when $\mu_i \geq \mu_i^{(k)}$ for all i. In the GOWAIMAM operator, we find a similar situation with the difference that now the unification is with the Minkowski distance or with the Minkowski ordered weighted averaging distance (MOWAD) operator (Karayiannis, 2000; Merigó and A.M. Gil-Lafuente, 2008a). Then, we get the following.

Theorem 5. Assume MOWAD(P,P_k) is the MOWAD operator and GOWADIMAM($P_k \rightarrow P$) the GOWADIMAM operator. If $\mu_i \geq \mu_i^{(k)}$ for all i, then:

$$\text{MOWAD}(P,P_k) = \text{GOWADIMAM}(P_k \rightarrow P) \tag{23}$$

Proof. Let

$$\text{MOWAD}(P,P_k) = \sum_{j=1}^{n} w_j \mid \mu_i - \mu_i^{(k)} \mid \tag{24}$$

$$\text{GOWADIMAM}(P_k \rightarrow P) = \left(\sum_{j=1}^{n} w_j Q_j^{\lambda} \right)^{1/\lambda} \tag{25}$$

Since $\mu_i \geq \mu_i^{(k)}$ for all i, $[0 \vee (\mu_i - \mu_i^{(k)})] = (\mu_i - \mu_i^{(k)})$ for all i, then

$$\text{GOWADIMAM}(P_k \rightarrow P) = \sum_{j=1}^{n} w_j (\mu_i - \mu_i^{(k)}) = \text{MOWAD}(P,P_k) \qquad \blacksquare$$

As we can see, the unification appears with the dual IMAM. As it was explained in Merigó and A.M. Gil-Lafuente (2007a), it is possible to distinguish between different types of unifications depending on the situation found such as partial or total unification point. The partial unification point appears if at least one of the alternatives but not all of them enters in a situation of unification point. The total unification point appears if all the alternatives are in a situation of unification point. Note that it is straightforward to prove these unifications by looking to (Merigó and A.M. Gil-Lafuente, 2007a) and following Theorem 5.

Note that this unification has been studied with the general case, but it is also possible to consider different particular cases by giving different values to the parameter λ. For example, if λ = 1, we get the unification found with the IMAM and the Hamming distance. If λ = 2, we get the unification with the quadratic IMAM and the Euclidean distance.

Another interesting issue to analyze is the different measures used for characterizing the weighting vector of the GOWAIMAM operator. Based on the measures developed for the OWA operator by Yager (1988; 1996b; 2002) and for the GOWA (Yager, 2004), they can be defined as follows. The attitudinal character can be formulated as follows.

$$\alpha(W) = \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} \right)^{\lambda} \right)^{1/\lambda} \tag{26}$$

It can be shown that $\alpha \in [0, 1]$. Note that for the optimistic criteria $\alpha(W) = 1$, for the pessimistic criteria $\alpha(W) = 0$, and for the average criteria $\alpha(W) = 0.5$.

The dispersion is a measure that provides the type of information being used (Yager, 1988). It can be defined as follows.

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \tag{27}$$

For example, if $w_j = 1$ for some j, then $H(W) = 0$, and the least amount of information is used. If $w_j = 1/n$ for all j, then, the amount of information used is maximum.

Another interesting measure is the divergence of W (Yager, 2002). It can be defined as follows.

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \tag{28}$$

A further interesting measure that we could study is the balance of W (Yager, 1996b). It can be formulated as follows.

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} \frac{(n+1-2j)}{n-1} w_j \tag{29}$$

Note that these measures can also be used with an ascending order by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DGOWAIMAM and w^*_{n-j+1} the jth weight of the AGOWAIMAM operator.

4. Families of GOWAIMAM operators

In this Section, we analyze different particular cases of the GOWAIMAM operator.

4.1. Analysing the parameter λ

By looking to the parameter λ, we find a wide range of mean operators such as the OWAIMAM, the OWGIMAM, the OWQAIMAM, etc.

When λ = 1, the GOWAIMAM operator becomes the OWAIMAM operator.

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = \sum_{j=1}^{n} w_j K_j \tag{30}$$

Note that it is possible to distinguish between the DOWAIMAM operator and the AOWAIMAM operator. In both cases, the formulation is the same with the difference that the DOWAIMAM operator has a descending order and the AOWAIMAM operator an ascending order. Note that if $w_j = 1/n$, for all i, we get the normalized IMAM (NIMAM) and if the ordered position of j is the same than the position of i, we get the weighted IMAM (WIMAM).

When λ = 0, we get the OWGIMAM operator.

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = \prod_{j=1}^{n} K_j^{\,w_j} \tag{31}$$

In this case, we get the descending OWGIMAM (DOWGIMAM) operator and the ascending OWGIMAM (AOWGIMAM) operator. Note that if $w_j = 1/n$, for all i, we get the normalized geometric IMAM (NGIMAM) and if the ordered position of j is the same than the position of i, we get the weighted geometric IMAM (WGIMAM).

When $\lambda = -1$, we get the OWHAIMAM operator.

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{K_j}} \tag{32}$$

In this case, we obtain the descending OWHAIMAM (DOWHAIMAM) operator and the ascending OWHAIMAM (AOWHAIMAM) operator. In both cases, the formulation is the same although the reordering step is different. Note that if $w_j = 1/n$, for all i, we get the normalized harmonic IMAM (NHIMAM) and if the ordered position of j is the same than the position of i, we get the weighted harmonic IMAM (WHIMAM).

When $\lambda = 2$, we get the OWQAIMAM operator.

$$\text{GOWAIMAM}(p_1, p_2, \ldots, p_n) = \left(\sum_{j=1}^{n} w_j K_j^2 \right)^{1/2} \tag{33}$$

Note that we can distinguish between the descending OWQAIMAM (DOWQAIMAM) operator and the ascending OWQAIMAM (AOWQAIMAM) operator. Note that if $w_j = 1/n$, for all i, we get the normalized quadratic IMAM (NQIMAM) and if the ordered position of j is the same than the position of i, we get the weighted quadratic IMAM (WQIMAM).

4.2. Analysing the weighting vector W

By using a different manifestation in the weighting vector of the GOWAIMAM operator, we are able to obtain different types of aggregation operators. For example, it is possible to obtain the maximum, the minimum, the GIMAM and the WGIMAM operator.

The maximum is found if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. The minimum, if $w_n = 1$ and $w_j = 0$, for all $j \neq n$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any λ, GOWAIMAM$(p_1, p_2,\ldots, p_n) = K_h$, where K_h is the hth largest argument of all the $|\mu_i - \mu_i^{(k)}|$ and the $[0 \vee (\mu_i - \mu_i^{(k)})]$. This case is known as the step-GOWAIMAM operator. The GIMAM is found when $w_j = 1/n$, for all a_i and the WGIMAM obtained when the ordered position of i is the same than j.

Following a similar methodology as it has been developed in (Merigó, 2007; Yager, 1993), we could study other particular cases of the GOWAIMAM operator such as the window-GOWAIMAM, the olympic-GOWAIMAM, the median-GOWAIMAM, the centered-GOWAIMAM operator, the S-GOWAIMAM operator, etc.

For example, when $w_{j*} = 1/m$ for $k \leq j^* \leq k + m - 1$ and $w_{j*} = 0$ for $j^* > k + m$ and $j^* < k$, we are using the window-GOWAIMAM operator. Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, the window-GOWAIMAM becomes the maximum. If $m = 1$, $k = n$, the minimum. And if $m = n$ and $k = 1$, the window-GOWAIMAM is transformed in the GIMAM.

If $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$, we are using the olympic-GOWAIMAM that it is based on the olympic average (Yager, 1996a). Note that if $n = 3$ or $n = 4$, the olympic-GOWAIMAM is transformed in the median-GOWAIMAM and if $m = n - 2$ and $k = 2$, the window-GOWAIMAM is transformed in the olympic-GOWAIMAM.

Note that the median and the weighted median can also be used as GOWAIMAM operators. For the median-GOWAIMAM, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_{j*} = 0$ for all others. If n is even we assign, for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j*} = 0$ for all others. For the weighted median-GOWAIMAM, we select the argument K_h that has the hth largest argument such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Another interesting family is the S-GOWAIMAM operator based on the S-OWA operator (Yager, 1993; Yager and Filev, 1994). It can be divided in three classes: the "orlike", the "andlike" and the generalized S-GOWAIMAM operator. The "orlike" S-GOWAIMAM operator is found when $w_1 = (1/n)(1 - \alpha) + \alpha$, and $w_j = (1/n)(1 - \alpha)$ for $j = 2$ to n with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we get the GIMAM and if $\alpha = 1$, we get the maximum. The "andlike" S-GOWAIMAM operator is found when $w_n = (1/n)(1 - \beta) + \beta$ and $w_j = (1/n)(1 - \beta)$ for $j = 1$ to $n - 1$ with $\beta \in [0, 1]$. Note that in this class, if $\beta = 0$ we get the GIMAM and if $\beta = 1$, the minimum. Finally, the generalized S-GOWAIMAM operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-GOWAIMAM operator becomes the "andlike" S-GOWAIMAM operator and if $\beta = 0$, it becomes the "orlike" S-

GOWAIMAM operator. Also note that if $\alpha + \beta = 1$, we get the generalized Hurwicz criteria.

A further family of aggregation operator that could be used is the centered-GOWAIMAM operator, that it is based on the OWA version (Yager, 2007). We can define a GOWAIMAM operator as a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-GOWAIMAM operator. Note that the GIMAM is an example of this particular case of centered-GOWAIMAM operator. Another particular situation of the centered-GOWAIMAM operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered-GOWAIMAM operator. For this situation, we find the median-GOWAIMAM as a particular case.

5. Quasi-OWAIMAM operator

As it was explained in (Beliakov, 2005), a further generalization of the GOWA operator is possible by using quasi-arithmetic means. Following the same methodology than (Fodor et al., 1995), we can suggest a similar generalization for the GOWAIMAM operator by using quasi-arithmetic means. We will call this generalization, the Quasi-OWAIMAM operator. It can be defined as follows.

Definition 9. A Quasi-OWAIMAM operator of dimension n is a mapping QOWAIMAM: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$\text{QOWAIMAM}(p_1, p_2, \ldots, p_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(K_{(j)}\right)\right) \tag{34}$$

where K_j represents the jth largest of all the $|\mu_i - \mu_i^{(k)}|$ and the $[0 \vee (\mu_i - \mu_i^{(k)})]$; $k = 1,2,\ldots,m$.

As we can see, we replace b^λ with a general continuous strictly monotone function $g(b)$. Note that in the Quasi-OWAIMAM operator we also find problems when the arguments are 0. Basically, these problems appear for values in the parameter $\lambda \leq 0$.

In this case, the weights of the ascending and descending versions are also related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the Quasi-DOWAIMAM and w^*_{n-j+1} the jth weight of the Quasi-AOWAIMAM operator.

Note that it is also possible to suggest an equivalent removal index that it is a dual of the Quasi-OWAIMAM because $Q(P_k \rightarrow P) = 1 - K(P_k \rightarrow P)$. We will call it the Quasi-OWADIMAM.

Also note that all the properties and particular cases commented in the GOWAIMAM operator are also applicable in the Quasi-OWAIMAM operator. For example, if $w_j = 1/n$, for all a_i, then, we get the Quasi-NIMAM operator, and if the ordered position of i is the same than j, then, we get the Quasi-WIMAM.

6. Illustrative example

In the following, we are going to develop an illustrative example where we will see the applicability of the new approach. We are going to consider a decision making problem about selection of products. We will focus on the selection of apartments. We will use different types of GOWAIMAM operators such as the NIMAM, the QIMAM, the WIMAM, the WQIMAM, the OWAIMAM, the AOWAIMAM, the OWQAIMAM, the step-OWAIMAM, the olympic-OWAIMAM, the median-OWAIMAM, etc.

Assume a person wants to buy an apartment and he considers 5 possible alternatives to follow.

- A_1: Apartment A.
- A_2: Apartment B.
- A_3: Apartment C.
- A_4: Apartment D.
- A_5: Apartment E.

In order to evaluate these apartments the decision maker considers different general characteristics about the apartments that can be summarized in 6 characteristics: C_1 = Prize, C_2 = Size, C_3 = Quality, C_4 = Age, C_5 = Zone, C_6 = Connection to other places.

The decision maker evaluates these characteristics that can be summarized in Table 1 depending on the characteristic C_i and the alternative A_k. Note that values near 1 imply that the results are good and values near 0, bad.

Table 1. Expected results

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.7	0.6	0.9	0.9	0.7	0.7
A_2	0.8	0.4	0.7	0.6	0.8	0.9
A_3	0.6	0.7	0.7	0.8	0.9	0.7
A_4	0.5	0.8	0.8	0.8	0.6	0.9
A_5	0.7	0.8	0.9	1	0.8	0.4

The decision maker considers the following weighting vector for all the cases: $W = (0.1, 0.1, 0.1, 0.2, 0.2, 0.3)$. Note that this weighting vector reflects the attitudinal character of the company when using the OWA operator. In order to develop the analysis, the decision maker calculates the results that an ideal apartment should have. The results of the ideal apartment are shown in Table 2.

Table 2. Ideal apartment

	C_1	C_2	C_3	C_4	C_5	C_6
Ideal	0.8	0.9	1	0.8	1	0.8

In this example, we will assume that the decision maker considers the three first characteristics with the Hamming distance and the other three with the adequacy coefficient. The usefulness of the IMAM is that if we believe that the Hamming distance is a good method we can use it, but if we believe that for some characteristics we need a more specific analysis, then, we can use the adequacy coefficient.

With this information, we can aggregate the expected results in order to obtain a representative result for each alternative. First, we are going to consider the NIMAM, the QIMAM, the WIMAM, the WQIMAM and the OWAIMAM operator. The results are shown in Table 3.

Table 3. Aggregated results 1

	NAC	QAC	WAC	WQAC	OWAAC
A_1	0.85	0.857	0.86	0.867	0.81
A_2	0.8	0.818	0.84	0.854	0.73
A_3	0.85	0.855	0.88	0.884	0.81
A_4	0.83	0.846	0.86	0.875	0.77
A_5	0.85	0.859	0.81	0.824	0.8

Now, we are going to consider the results obtained by using other particular cases of the GOWAIMAM operator such as the AOWAIMAM, the OWQAIMAM, the step-OWAIMAM (k=2), the median-OWAIMAM and the olympic-OWAIMAM operator. The results are shown in Table 4.

Table 4. Aggregated results 2

	AOWAAC	OWQAAC	step	median	olympic
A_1	0.89	0.817	0.9	0.9	0.85
A_2	0.86	0.751	1	0.8	0.825
A_3	0.89	0.815	0.9	0.85	0.85
A_4	0.89	0.784	1	0.85	0.85
A_5	0.89	0.812	0.9	0.9	0.875

As we can see, depending on the aggregation operator used the results are different. A_1 is optimal with the NAC, the OWAAC, the AOWAAC, the OWQAAC and the median-OWAAC. A_2 is optimal only with the step-

OWAAC. A_3 with the NAC, the WAC, the WQAC, the OWAAC and the AOWAAC. A_4 with the AOWAAC and the step-OWAAC. Finally, A_5 is optimal with the NAC, the QAC, the AOWAAC, the median-OWAAC and the olympic-OWAAC.

Another interesting issue is to establish an ordering of the alternatives. Note that this is useful when we want to consider more than one alternative. The results are shown in Table 5. Note that $\succ$ means preferred to.

Table 5. Ordering of the strategies

	Ordering		Ordering
NIMAM	$A_1=A_3=A_5 \succ A_2=A_4$	AOWAIMAM	$A_1=A_3=A_4=A_5 \succ A_2$
QIMAM	$A_5 \succ A_1 \succ A_3 \succ A_4 \succ A_2$	OWQAIMAM	$A_1 \succ A_3 \succ A_5 \succ A_4 \succ A_2$
WIMAM	$A_3 \succ A_1=A_4 \succ A_2 \succ A_5$	Step	$A_2=A_4 \succ A_1 \succ A_3 \succ A_5$
WQIMAM	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$	Median	$A_1=A_5 \succ A_3=A_4 \succ A_2$
OWAIMAM	$A_1=A_3 \succ A_5 \succ A_4 \succ A_2$	Olympic	$A_5 \succ A_1=A_3=A_4 \succ A_2$

As we can see, depending on the aggregation operator used, the ordering of the apartments is different. Then, these results may lead to different decisions.

7. Conclusions

We have presented the GOWAIMAM operator. It is a generalization of the OWAIMAM operator by using generalized means. The main advantage of this aggregation operator is that it includes a wide range of mean operators such as the OWAIMAM, the NGIMAM, the WGIMAM, the IMAM, the

OWQAIMAM, etc. Then, with this generalization, we can consider a wide range of results depending on the particular case used. We have further generalized the GOWAIMAM by using quasi-arithmetic means. As a result we have obtained the Quasi-OWAIMAM operator.

We have also developed an application of the new approach in a decision making problem about selection of products, and more specifically, selection of apartments. We have seen that depending on the particular type of GOWAIMAM operator used, the results are different and they may lead to different decisions.

In future research, we expect to develop further extensions of the GOWAIMAM operator by adding new characteristics in the problem such as the use of inducing orders and applying it to other decision making problems.

References

Beliakov, G. (2005) "Learning Weights in the Generalized OWA Operators", Fuzzy Optimization and Decision Making, 4, 119-130.

Beliakov, G., A. Pradera and T. Calvo (2007) "Aggregation Functions: A guide for practitioners", Springer-Verlag, Berlin.

Calvo, T., G. Mayor and R. Mesiar (2002) "Aggregation Operators: New Trends and Applications", Physica-Verlag, New York.

Chiclana, F., F. Herrera and E. Herrera-Viedma (2000) "The ordered weighted geometric operator: Properties and application", Proceedings of the 8th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Madrid, Spain, pp. 985-991.

Dujmovic, J. (1974) “Weighted conjunctive and disjunctive means and their application in system evaluation”, Publikacije Elektrotechnickog Faculteta Beograd, Serija Matematika i Fizika, 483, pp. 147-158.

Dyckhoff, H. and W. Pedrycz (1984) “Generalized means as model of compensative connectives”, Fuzzy Sets and Systems, 14, 143-154.

Figueira, J., S. Greco and M. Ehrgott (2005) “Multiple criteria decision analysis: state of the art surveys”, Springer. Boston.

Fodor, J., J.L. Marichal and M. Roubens (1995) “Characterization of the ordered weighted averaging operators”, IEEE Transactions on Fuzzy Systems, 3, 236-240.

Gil-Aluja, J. (1998) “The interactive management of human resources in uncertainty”, Kluwer Academic Publishers, Dordrecht.

Gil-Aluja, J. (1999) “Elements for a theory of decision in uncertainty”, Kluwer Academic Publishers, Dordrecht.

Gil-Aluja, J. (2001) “Handbook of management under uncertainty”, Kluwer Academic Publishers, Dordrecht.

Gil-Lafuente, A.M. (2005) “Fuzzy logic in financial analysis”, Springer, Berlin.

Gil-Lafuente, J. (2001) “El “índice del máximo y mínimo nivel” en la optimización del fichaje de un deportista”, X Congreso Internacional de la Asociación Europea de Dirección y Economía de la Empresa (AEDEM), Reggio Calabria, Italia, pp. 439-443.

Gil-Lafuente, J. (2002) “Algoritmos para la excelencia: Claves para el éxito en la gestión deportiva” (In Spanish), Ed. Milladoiro, Vigo.

Herrera, F., E. Herrera-Viedma and F. Chiclana (2003) “A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making”, International Journal of Intelligent Systems, 18, 689-707.

Karayiannis, N. (2000) “Soft Learning Vector Quantization and Clustering Algorithms Based on Ordered Weighted Aggregation Operators”, IEEE Transactions on Neural Networks, 11, 1093-1105.

Kaufmann, A. and J. Gil-Aluja (1986) “Introducción de la teoría de los subconjuntos borrosos a la gestión de las empresas” (In Spanish), Ed. Milladoiro, Santiago de Compostela.

Kaufmann, A. and J. Gil-Aluja (1987) “Técnicas operativas de gestión para el tratamiento de la incertidumbre” (In Spanish), Ed. Hispano-europea, Barcelona.

Merigó, J.M. (2007) “Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión empresarial”, Unpublished thesis (In Spanish), Department of Business Administration, University of Barcelona.

Merigó, J.M. and M. Casanovas (2007a) “The fuzzy generalized OWA operator”. Proceedings of the XIVth SIGEF Congress. Poiana-Brasov, Romania, pp. 504-517.

Merigó, J.M. and M. Casanovas (2007b) “The uncertain generalized OWA operator and its application in the selection of financial strategies”. Proceedings of the AEDEM International Conference. Krakow, Poland, pp. 547-556.

Merigó, J.M. and A.M. Gil-Lafuente (2007a): “Unification point in methods for the selection of financial products”, Fuzzy Economic Review, 13, 35-50, 2007.

Merigó, J.M. and A.M. Gil-Lafuente (2007b) “The induced generalized OWA operator”, Proceedings of the EUSFLAT conference, Ostrava, Czech Republic, Vol. 2, pp. 463-470.

Merigó, J.M. and A.M. Gil-Lafuente (2008a) “Using the OWA operator in the Minkowski distance”, International Journal of Computer Science, (submitted for publication).

Merigó, J.M. and A.M. Gil-Lafuente (2008b) "Using the OWA operator in the index of maximum and minimum level", Proceedings of the AMSE International Conference, Mallorca, Spain, (submitted for publication).

Wang, J.H. and J. Hao (2006) "A new version of 2-tuple fuzzy linguistic representation model for computing with words". IEEE Transactions on Fuzzy Systems, 14, 435-445.

Xu, Z.S. (2005) "An Overview of Methods for Determining OWA Weights", International Journal of Intelligent Systems, 20, 843-865.

Xu, Z.S. and Q.L. Da (2002) "The Ordered Weighted Geometric Averaging Operators", International Journal of Intelligent Systems, 17, 709-716.

Yager, R.R. (1988) "On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making", IEEE Transactions on Systems, Man and Cybernetics B, 18, 183-190.

Yager, R.R. (1992) "On generalized measures of realization in uncertain environments", Theory and Decision, 33, 41-69.

Yager, R.R. (1993) "Families of OWA operators", Fuzzy Sets and Systems, 59, 125-148.

Yager, R.R. (1996a) "Quantifier guided aggregation using OWA operators", International Journal of Intelligent Systems, 11, 49-73.

Yager, R.R. (1996b) "Constrained OWA Aggregation", Fuzzy Sets and Systems, 81, 89-101.

Yager, R.R. (2002) "Heavy OWA Operators", Fuzzy Optimization and Decision Making, 1, 379-397.

Yager, R.R. (2004) "Generalized OWA Aggregation Operators", Fuzzy Optimization and Decision Making, 3, 93-107.

Yager, R.R. (2007) "Centered OWA operators", Soft Computing, 11, 631-639.

Yager, R.R. and D.P. Filev (1994) "Parameterized andlike and orlike OWA Operators", International Journal of General Systems, 22, 297-316.

Yager, R.R. and J. Kacprzyk (1997) “The Ordered Weighted Averaging Operators: Theory and Applications”, Kluwer Academic Publishers, Norwell, MA.