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# Forcing Arguments in Infinite Ramsey Theory

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# Forcing Arguments in Infinite Ramsey Theory

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*Con mucho cariño para aquellos*
*cuyo amor incondicional me*
*acompaña a pesar de la distancia.*

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# Introduction

This is a contribution to combinatorial set theory, specifically to infinite Ramsey Theory, which deals with partitions of infinite sets. The basic pigeon hole principle states that for every partition of the set of all natural numbers in finitely-many classes there is an infinite set of natural numbers that is included in some one class. Ramsey's Theorem, which can be seen as a generalization of this simple result, is about partitions of the set $[\mathbb{N}]^k$ of all $k$-element sets of natural numbers. It states that for every $k \geq 1$ and every partition of $[\mathbb{N}]^k$ into finitely-many classes, there is an infinite subset $M$ of $\mathbb{N}$ such that all $k$-element subsets of $M$ belong to some same class. Such a set is said to be *homogeneous* for the partition. In Ramsey's own formulation (Ramsey, [19], p.264), the theorem reads as follows.

**Theorem 0.1** (Ramsey). *Let $\Gamma$ be an infinite class, and $\mu$ and $r$ positive numbers; and let all those sub-classes of $\Gamma$ which have exactly $r$ numbers, or, as we may say, let all $r-$combinations of the members of $\Gamma$ be divided in any manner into $\mu$ mutually exclusive classes $C_i$ $(i = 1, 2, \ldots, \mu)$, so that every $r-$combination is a member of one and only one $C_i$; then assuming the axiom of selections, $\Gamma$ must contain an infinite sub-class $\triangle$ such that all the $r-$combinations of the members of $\triangle$ belong to the same $C_i$.*

In [13], Neil Hindman proved a Ramsey-like result that was conjectured by Graham and Rotschild in [9]. Hindman's Theorem asserts that if the set of all natural numbers is divided into two classes, one of the classes contains an infinite set such that all finite sums of distinct members of the set remain in the same class. Hindman's original proof was greatly simplified, though the same basic ideas were used, by James Baumgartner in [2].

An important reference in the area of Ramsey theory is Stevo Todorčević's book, *Introduction to Ramsey Spaces* [24], presenting a general procedure to transfer any Ramsey theoretic principle to higher, and especially infinite, dimension going beyond Ellentuck's topological Ramsey theory.

We will give proofs of Ramsey's and Hindman's theorems which rely on forcing arguments. After this, we will be concerned with the particular partial orders used in each case, with the aim of studying their basic properties and their relation to other similar forcing notions. The partial order used to prove Ramsey's Theorem is, from the point of view of forcing, equivalent to Mathias forcing. The analysis of the partial order arising in the proof of Hindman's Theorem, which we denote by $\mathbb{P}_{\mathrm{FIN}}$, will be the object of the $4^{\mathrm{th}}$ chapter of the thesis.

A summary of our work follows.

In the first chapter we give some basic definitions and state several known theorems that we will need. We explain the set-theoretic notation used and we describe some forcing notions that will be useful in the sequel. Our notation is generally standard, and when it is not it will be sufficiently explained. Thus, although most of the theorems recorded in this first, preliminary chapter, will be stated without proof, it will be duly indicated where a proof can be found.

Chapter 2 is devoted to a proof of Ramsey's Theorem in which forcing is used to produce a homogeneous set for the relevant partition. The partial order involved is equivalent to Mathias forcing.

In Chapter 3 we modify Baumgartner's proof of Hindman's Theorem to define a partial order, denoted by $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$, from which we get by a forcing argument a suitable homogeneous set. Here $\mathscr{C}$ is an infinite set of finite subsets of $\mathbb{N}$, and $\mathscr{D}'$ is an infinite block sequences such that $\mathscr{C}$ is large for $\mathscr{D}'$. $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$ adds an infinite block sequence of finite subsets of natural numbers with the property that all finite unions of its elements belong to $\mathscr{C}$. Our proof follows closely Baumgartner's, and the fact that $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$ has the required properties is secured by the propositions proved in [2] (see also [10]). The partial order $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$ is similar both to the one due to Pierre Matet in [17] and to Mathias forcing. This prompts the question whether it is equivalent to one of them or to none, which can only be answered by studying $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$, which we do in chapter 4.

In chapter 4 we first show that, for some dense set of conditions $(A, \mathscr{D})$ of $\mathbb{P}_{\mathscr{C},\mathscr{D}}$, the partial order $\mathbb{P}_{\mathscr{C},\mathscr{D}}$ bellow $(A, \mathscr{D})$ is equivalent to a more manageable partial order, which we denote by $\mathbb{P}_{\mathrm{FIN}}$, thus studying $\mathbb{P}_{FIN}$ allow us to know $\mathbb{P}_{\mathscr{C},\mathscr{D}}$, where $\mathscr{C}$ is large for $\mathscr{D}$. For this reason, from now on, we will restrict our attention to $\mathbb{P}_{FIN}$. From a $\mathbb{P}_{\mathrm{FIN}}$-generic filter an infinite block sequence can be defined, from which, in turn, the generic filter can be reconstructed, roughly as a Mathias generic filter can be reconstructed from a Mathias real. In section 4.1 we prove that $\mathbb{P}_{\mathrm{FIN}}$ is not equivalent to Matet forcing. This we do by showing that $\mathbb{P}_{\mathrm{FIN}}$ adds a dominating real, thus also a splitting real (see [11]). But Andreas Blass and Claude Laflamme proved (independently and unpublished) that Matet forcing preserves $p$-point ultrafilters. Their result can be recovered from Theorem 4 in Tod Eisworth's article [8], from which follows that Matet forcing does not add splitting reals.

Still in section 4.1 we prove that $\mathbb{P}_{\mathrm{FIN}}$ adds a Mathias real by using Mathias characterization of a Mathias real in [18] according to which $x \subseteq \omega$ is a Mathias real over $V$ iff $x$ diagonalizes every maximal almost disjoint family in $V$. In fact, we prove that if $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$ is the generic block sequence of finite sets of natural numbers added by forcing with $\mathbb{P}_{\mathrm{FIN}}$, then both $\{\min d_i : i \in \omega\}$ and $\{\max d_i : i \in \omega\}$ are Mathias reals. In section 4.2 we prove that $\mathbb{P}_{\mathrm{FIN}}$ is equivalent to a two-step iteration of a $\sigma$-closed and a $\sigma$-centered forcing notions. In section 4.3 we prove that $\mathbb{P}_{\mathrm{FIN}}$ satisfies Axiom A and in section 4.4 that, as Mathias forcing, it has the pure decision property. In section 4.5 we prove that $\mathbb{P}_{\mathrm{FIN}}$ does not add Cohen reals. So far, all the properties we have found of $\mathbb{P}_{\mathrm{FIN}}$ are also shared by Mathias forcing.

The question remains, then, whether $\mathbb{P}_{\mathrm{FIN}}$ is equivalent to Mathias forcing. This we solve in Chapter 5 by first showing in section 5.1 that $\mathbb{P}_{\mathrm{FIN}}$ adds a

Matet real and then, in section 5.2, that Mathias forcing does not add a Matet real, thus concluding that $\mathbb{P}_{\mathrm{FIN}}$ and Mathias forcing are not equivalent forcing notions.

In the last section we explore another forcing notion, denoted by $\mathbb{M}_2$, which was introduced by Saharon Shelah and Otmar Spinas in [23] and is a kind of product of two copies of Mathias forcing. The reason for looking at $\mathbb{M}_2$ is its connection with $\mathbb{P}_{FIN}$.

# Chapter 1

# Preliminaries

## 1.1 Set-theoretic notation

We work in $ZFC$, the standard Zermelo-Fraenkel set theory with the Axiom of Choice. About set theoretical matters we follow Kunen's book [16]. Any ordinal number $\alpha$ is the set of all those ordinal numbers which are smaller than $\alpha$. Natural numbers are the finite ordinal numbers. A cardinal number is an ordinal which is not equipotent with any smaller ordinal. The notation ${}^{Y}X$, ${}^{<\kappa}X$, $[X]^{\kappa}$ and $[X]^{<\kappa}$, where $X$ and $Y$ are sets and $\kappa$ is a cardinal number, respectively mean: the set of all functions from $Y$ into $X$, the set of all functions from some $\alpha < \kappa$ into $X$, the set of all subsets of $X$ with cardinality $\kappa$ and the set of all subsets of $X$ with cardinality smaller than $\kappa$. Given a function $f$ and a set $A$, $f[A]$ denotes the set $\{f(x) : x \in A\}$ and $ran(f) := \{y : (\exists x)(f(x) = y)\}$. If $s$ and $t$ are sequences of elements of a set $X$, then by $s^\frown t$ we denote the concatenation of $s$ and $t$. And if $t \in X$, then $s^\frown t$ denotes $s^\frown \langle t \rangle$.

Given any sets $A, B$ and $C$ we express that $A$ is partitioned into the two pieces $B$ and $C$, i.e., that $A = B \cup C$ and $B \cap C = \emptyset$ by writing: $A = B \dot{\cup} C$.

For $X \in [\omega]^{\omega}$, we write $X = \{x_i : i \in \omega\}$ to mean that the sequence $\langle x_i \rangle_{i \in \omega}$ is the increasing enumeration of the elements of $X$.

### 1.1.1 Filters

**Definition 1.1.** A filter *on a nonempty set* $S$ *is a collection* $\mathscr{F}$ *of subsets of* $S$ *such that*

- $S \in \mathscr{F}$ *and* $\emptyset \notin \mathscr{F}$,
- *if* $X \in \mathscr{F}$ *and* $Y \in \mathscr{F}$, *then* $X \cap Y \in \mathscr{F}$,
- *if* $X, Y \subseteq S$, $X \in \mathscr{F}$, *and* $X \subseteq Y$, *then* $Y \in \mathscr{F}$.

*A filter* $\mathscr{F}$ *on a set* $S$ is an ultrafilter *if for every* $X \subseteq S$, *either*

$$X \in \mathscr{F} \text{ or } S \setminus X \in \mathscr{F}.$$

Some examples of filters:

1. The trivial filter: $\mathscr{F} = \{S\}$.
2. A principal filter. Let $X_0$ be a nonempty subset of $S$. The filter

$$\mathscr{F} = \{X \subseteq S : X \supseteq X_0\}$$

is a principal filter. Note that every filter on a finite set is principal.

3. The Fréchet filter. Let $S$ be an infinite set. The filter

$$\mathscr{F} = \{X \subseteq S : S \setminus X \text{ is finite }\}$$

is called Fréchet filter on $S$. Note that the Fréchet filter is not principal.

**Definition 1.2.** *A filter* $\mathscr{F} \subseteq [\omega]^\omega$ *is called a* free filter *if it contains the Fréchet filter.*

For a filter $\mathscr{F} \subseteq [\omega]^\omega$, $\mathscr{F}^+$ denotes the collection of all subsets $x \subseteq \omega$ such $x^c \notin \mathscr{F}$.

**Definition 1.3.** *A family* $\mathscr{E}$ *of subsets of* $\omega$ *is called a* free family *if there is a free filter* $\mathscr{F} \subseteq [\omega]^\omega$ *such that* $\mathscr{E} = \mathscr{F}^+$.

In particular, $[\omega]^\omega$ and all ultrafilters are free families.

The following definitions are Definition 4.4.1, pag. 224 and Definition 4.5.1, pag. 235, respectively, in book [1]. But in fact, these notions are older, definition of $p-$point ultrafilters goes at least back to Rudin in [20].

**Definition 1.4.** *A filter* $\mathscr{F}$ is called a $p-$filter *if for every family*

$$\{X_n : n \in \omega\} \subseteq \mathscr{F}$$

*there exists* $X \in \mathscr{F}$ *such that* $X \setminus X_n$ *is finite, which we denote by* $X \subseteq^* X_n$*, for* $n \in \omega$*. If* $\mathscr{F}$ *is an ultrafilter and a* $p-$*filter, then we call it* a $p-$point.

**Definition 1.5.** *An ultrafilter* $\mathscr{F}$ is called a $q-$point *if for every infinite partition of* $\omega$ *into finite pieces* $\{I_n : n \in \omega\}$ *there exists* $X \in F$ *such that* $|X \cap I_n| \leqslant 1$ *for* $n \in \omega$*.*

Lemma 1.6 is Lemma 4.4.3 in [1], page 224.

**Lemma 1.6.** *Let* $\mathscr{F}$ *be an ultrafilter on* $\omega$*. The following conditions are equivalent:*

1. $\mathscr{F}$ *is a* $p-$*point, and*
2. *for every partition of* $\omega$*,* $\{Y_n : n \in \omega\}$*, either there exists* $n \in \omega$ *such that* $Y_n \in \mathscr{F}$ *or there exists* $X \in \mathscr{F}$ *such that* $X \cap Y_n$ *is finite for* $n \in \omega$*.*

### 1.1.2 Ramsey filters

**Definition 1.7** ([18])**.** *A filter* $\mathscr{F}$ on $\omega$ is called Ramsey *if for every* $\subseteq$*-descending sequence*

$$\{X_n : n \in \omega\} \subseteq \mathscr{F}$$

*there exists a set* $\{x_n : n \in \omega\} \in \mathscr{F}$ *such that* $x_n \in X_n$*, for all* $n \in \omega$*.*

A well known characterization of Ramsey filters is Theorem 4.5.2 in [1].

**Theorem 1.8.** *Let* $\mathscr{F}$ *be an ultrafilter on* $\omega$*. The following are equivalent:*

1. *$\mathscr{F}$ is Ramsey.*

2. *For every partition of $\{Y_n : n \in \omega\}$ of $\omega$, either $Y_n \in \mathscr{F}$ for some $n \in \omega$ or there exists $X \in \mathscr{F}$ such that $|X \cap Y_n| \leqslant 1$ for all $n \in \omega$.*

3. *For every set $A \subseteq [\omega]^2$ there exists $X \in \mathscr{F}$ such that $[X]^2 \subseteq A$ or $[X]^2 \cap A = \emptyset$.*

4. *$\mathscr{F}$ is a $p-$point and a $q-$point.*

## 1.2 Forcing

We refer the reader to Kunen's book [16] for the elementary theory of forcing.

**Definition 1.9.** *Let $M$ be a countable transitive model for ZFC and let $\langle \mathbb{P}, \leqslant \rangle$ be a partial order such that $\langle \mathbb{P}, \leqslant \rangle \in M$. $G$* is $\mathbb{P}$-generic over M *if and only if $G$ is a filter on $\mathbb{P}$ and for all dense $D \subseteq \mathbb{P}$ with $D \in M$, $D \cap G \neq \emptyset$.*

**Theorem 1.10.** *Let $M$ be a countable model and $p \in \mathbb{P}$, then there is a $G$ which is $\mathbb{P}$-generic over $M$ such that $p \in G$.*

**Definition 1.11.** *Let $\mathbb{P}$ be a forcing notion,* $\mathbb{P}$ is $\kappa-$closed *if for every $\lambda < \kappa$, every descending sequence $p_0 \geqslant p_1 \geqslant \ldots \geqslant p_\alpha \geqslant \ldots$ $(\alpha < \lambda)$ has a lower bound.*

**Definition 1.12.** *A partial order* $\mathbb{P}$ is $\sigma-$centered *if it can be partitioned into countably-many pieces, each one of them consisting of finitewise-compatible elements.*

**Definition 1.13.** *For two functions $f, g \in {}^\omega\omega$ we say that $g$* is dominated by $f$*, denoted by $g <^* f$, if there is an $n \in \omega$ such that for all $k \geqslant n$ we have $g(k) < f(k)$.*

**Definition 1.14.** *For two sets $x, y \in [\omega]^\omega$ we say that $x$* splits $y$ *if both sets $y \cap x$ and $y \setminus x$ are infinite.*

Let $V$ be any model of $ZFC$ and let $V[G]$ be a generic extension (with respect to some forcing notion $\mathbb{P}$).

**Definition 1.15.** *A function $f \in {}^{\omega}\omega$ in $V[G]$ is called a* dominating real *over V if each function $g \in {}^{\omega}\omega \cap V$ is dominated by $f$.*

*A set $x \in [\omega]^{\omega}$ in $V[G]$ is called a* splitting real *over V if it splits each $y \in [\omega]^{\omega} \cap V$.*

**Proposition 1.16.** *If $V[G]$ contains a dominating real, then it also contains a splitting real.*

*Proof.* See proof of Proposition 3 in [11]. □

**Definition 1.17.** *Given two partial orders $\mathbb{P}$ and $\mathbb{Q}$ we say that $\mathbb{P}$* is a projection of $\mathbb{Q}$ *if there is a function $\pi : \mathbb{Q} \to \mathbb{P}$ such that $\pi$ preserves order, i.e., if $q \leqslant_{\mathbb{Q}} q'$, then $\pi(q) \leqslant_{\mathbb{P}} \pi(q')$, and for all $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ if $p \leqslant_{\mathbb{P}} \pi(q)$, there is $q' \leqslant_{\mathbb{Q}} q$ such that $\pi(q') \leqslant_{\mathbb{P}} p$.*

The next lemma is Lemma 15.45 in Jech's book [15].

**Lemma 1.18.** *If $\mathbb{P}$ is a projection of $\mathbb{Q}$ and $G$ is a $\mathbb{Q}$-generic filter over $V$ then $\pi[G]$ is a base for a $\mathbb{P}$-generic filter over $V$, where $\pi$ is a function that witnesses that $\mathbb{P}$ is a projection of $\mathbb{Q}$.*

*Proof.* Let

$$U = \{p \in \mathbb{P} : \exists r \in G \text{ such that } \pi(r) \leqslant_{\mathbb{P}} p\}.$$

We shall prove that $U$ is a $\mathbb{P}$-generic filter.

Note that $U \neq \emptyset$. And $U$ is upwards closed: If $p \in \mathbb{P}$ and $p' \in U$ are such that $p' \leqslant_{\mathbb{P}} p$ then there is $r \in G$ such that $\pi(r) \leqslant_{\mathbb{P}} p'$, so $\pi(r) \leqslant_{\mathbb{P}} p$, therefore $p \in U$.

Now let $p$ and $p'$ in $U$. There exist $r$ and $r'$ in $G$ such that $\pi(r) \leqslant_{\mathbb{P}} p$ and $\pi(r') \leqslant p'$. Since $G$ is a filter, there is $s \in G$ such that $s \leqslant_{\mathbb{Q}} r$ and $s \leqslant_{\mathbb{Q}} r'$, using the fact that $\pi$ preserves order, we have that $\pi(s) \leqslant_{\mathbb{P}} p$ and $\pi(s) \leqslant_{\mathbb{P}} p'$. Thus, $p$ and $p'$ are compatible in $U$. Hence $U$ is a filter.

Let $D \subseteq \mathbb{P}$ be a dense set in $V$ and define

$$D' := \{q \in \mathbb{Q} : \exists d \in D \text{ such that } \pi(q) \leqslant_{\mathbb{P}} d\}.$$

We shall prove that $D'$ is a dense subset of $\mathbb{Q}$. Let $q \in \mathbb{Q}$, then $\pi(q) \in \mathbb{P}$. Since $D$ is dense, there is $d \in D$ such that $d \leqslant_{\mathbb{P}} \pi(q)$, by definition of projection then there is $q' \leqslant_{\mathbb{Q}} q$ such that $\pi(q') \leqslant_{\mathbb{P}} d$. So $q' \in D'$, which shows that $D'$ is dense. Since $G$ is a $\mathbb{Q}$-generic filter $G \cap D' \neq \emptyset$. Let $q \in D' \cap G$, then there exists $d \in D$ such that $\pi(q) \leqslant_{\mathbb{P}} d$, i.e., $d \in U$. So $D \cap U \neq \emptyset$. □

**Definition 1.19** (Baumgartner [3])**.** *A partial ordering $\mathbb{P} = (P, \leqslant)$ satisfies* Axiom A *if and only if there exist partial ordering relations $\langle \leqslant_n : n \in \omega \rangle$ on $P$ such that*

*(1) For all $p, q \in P$, $p \leqslant_0 q$ if and only if $p \leqslant q$,*

*(2) For all $p, q \in P$ if $p \leqslant_{n+1} q$, then $p \leqslant_n q$ for all $n \in \omega$,*

*(3) if $\langle p_n : n \in \omega \rangle$ is a sequence of elements of $P$ such that $p_{n+1} \leqslant_n p_n$ for all $n \in \omega$, then there is a condition $q \in P$ such that for all $n \in \omega$ $q \leqslant_n p_n$,*

*(4) if $I \subseteq P$ is pairwise incompatible, then for all $p \in P$ and for all $n \in \omega$ there is $q \in P$ such that $q \leqslant_n p$ and*

$$\{r \in I : q \text{ is compatible with } r\}$$

*is countable.*

**Lemma 1.20** (Baumgartner [3])**.** *Condition* (4) *may be rephrased in terms of forcing as follows:*

*(4') For all $p \in P$ and for all $n \in \omega$, if $p \Vdash$ "$\tau \in V$", then there is a countable set $x$ and $q \leqslant_n p$ such that $q \Vdash \tau \in \check{x}$.*

*Proof.* Assume (4). Let $p \in P$ be such that $p \Vdash$ "$\tau \in V$". Consider the set

$$D := \{r \in P : r \leqslant p \text{ and } (\exists a \in V) r \Vdash \tau = \check{a}\}.$$

Since for all $q \leqslant p$ there is $r \leqslant q$ and $a \in V$ such that $r \Vdash \tau = \check{a}$, so $D$ is not empty. Note that such $a$ is unique and denoted by $a_r$.

Pick a maximal subset $I$ of pairwise incompatible elements of $D$. By (4) of the Definition 1.19, there is $q \leqslant_n p$ such that the set

$$\{r \in I : q \text{ is compatible with } r\}$$

is countable.

For each $r \in I$, pick $a_r \in V$ such that $r \Vdash \tau = \dot{a}_r$. Let $x := \{a_r : r \in I\}$. So, $x$ is a countable set.

**Claim 1.21.** *$q \Vdash \tau \in \check{x}$.*

*Proof of Claim:* If $q' \leqslant q$ and $V$ is such that $q' \Vdash \tau = a$, then $q' \in D$, and so it is compatible with some $r \in I$. But then $a = a_r$ and so $r \Vdash \tau \in \check{x}$. This shows that the set of conditions that force "$\tau \in \check{x}$" is dense below $q$. So, $q \Vdash$ "$\tau \in \check{x}$". □

Assume (4′). Let $I$ be a pairwise incompatible subset of $P$. Let $p \in P$ and $n \in \omega$. Without loss of generality we can assume that $I$ is maximal.

Let $\tau = \{(\check{x}, p) : p \in I \text{ and } x \in p\}$. Note that $1 \Vdash \tau \in \Gamma \cap \check{I}$, where $\Gamma$ is the canonical name for the filter $G$. So $p \Vdash$ "$\tau \in V$". Hence by (4′) there is $q \leqslant_n p$ and a countable set $x$ such that $q \Vdash \tau \in \check{x}$.

We claim that $\{r \in I : q \text{ is compatible with } r\} \subseteq x$, and so it is countable. Let $r \in I$ be such that $q$ and $r$ are compatible. So, there exists $s \leqslant q$ and $s \leqslant r$. We have that $s \Vdash \tau \in \check{x}$. But $s \Vdash$ "$\tau = \check{r}$". □

**Cohen forcing**

Cohen forcing, denoted by $\mathbb{C}$, is the set $2^{<\omega}$ of all finite binary sequences ordered by reversed inclusion.

**The forcing $\mathscr{P}(\omega)/fin$**

Let $fin = [\omega]^{<\omega}$ be the ideal of finite sets and let $\mathbb{U} := \langle \mathscr{P}(\omega)/fin, \leqslant \rangle$ be the partial order whose conditions are *infinite* subsets of $\omega$, ordered by $p \leqslant q$ if and only if $p \subseteq^* q$, that is, if and only if $p$ is almost contained in $q$.

**Lemma 1.22** (Mathias [18]). *Let $\mathscr{V}$ be a $\mathbb{U}$-generic over $V$, then $\mathscr{V}$ is a Ramsey ultrafilter in $V[\mathscr{V}]$.*

*Proof.* We reproduce the proof of the Fact 3.4 in Halbeisen's article [12].

First note that $\mathbb{U}$ is $\sigma-$closed, hence adds no new reals to $V$. Let $\pi \in 2^{[\omega]^2}$, by the Ramsey theorem for each $p \in [\omega]^\omega$ there exists a $q \subseteq^* p$ such that $\pi$ is constant on $[q]^2$. Therefore

$$H_\pi = \{q \in [\omega]^\omega : \pi \restriction [q]^2 \text{ is constant}\}$$

is dense in $\mathbb{U}$, hence $H_\pi \cap \mathscr{V} \neq \emptyset$. □

### 1.2.1 Mathias forcing

The following notion of forcing is due to Adrian Mathias [18].

Mathias forcing, denoted by $\mathbb{M}$, is the set of pairs $(a, A)$ where $a \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ and such that $\max(a) < \min A$. First part $a$ is called the "stem" of the condition $(a, A)$. And the ordering is given by:

$$(a, A) \leqslant (b, B) \Leftrightarrow b \text{ is an initial segment of } a, A \subseteq B$$

$$\text{and } \forall i \in a \setminus b (i \in B)$$

Let $\mathscr{E}$ be an arbitrary free family, Definition 1.3. *Mathias forcing restricted to $\mathscr{E}$*, denoted $\mathbb{M}(\mathscr{E}) = (M_\mathscr{E}, \leqslant)$, is defined as follows:

$$M_\mathscr{E} = \{(a, x) : a \in [\omega]^{<\omega}, x \in \mathscr{E}, \max(a) < \min x\}$$

and $\leqslant$ as in Mathias forcing.

If $G$ is a generic filter for the Mathias forcing, over the ground model $M$, let $x_G$ be the set

$$x_G = \bigcup \{s : (s, A) \in G \text{ for some } A\}.$$

By standard density arguments, $x_G$ is an infinite subset of $\omega$. $x_G$ is called a Mathias real (over $M$). The filter $G$ is determined by $x = x_G$, as

$$G = \{(s, A) : s \subseteq x \subseteq s \cup A\}.$$

Mathias forcing satisfies some interesting properties, which are listed below with references to their proofs.

**Lemma 1.23.** *If $x$ is a Mathias real over $V$, then $x$ is a dominating real over $V$.*

*Proof.* See proof of the Lemma 1.15 in [14]. □

Given a condition $(a, B) \in \mathbb{M}$, and a sentence $\phi$ of the forcing language, we say $(a, B)$ *decides* $\phi$ if $(a, B) \Vdash \phi$ or $(a, B) \Vdash \neg\phi$.

Next lemma is known as the *Pure decision property* for $\mathbb{M}$. It says that for any condition $(a, A)$ every sentence can be decided by strengthening $(a, A)$ while keeping the same stem.

**Lemma 1.24** (Mathias)**.** *Let $\phi$ be a sentence of the forcing language $\mathbb{M}$ and let $(a, A)$ be a condition. Then there exists and infinite set $B \subseteq A$, such that $(a, B) \Vdash \phi$ or $(a, B) \Vdash \neg\phi$.*

Next theorem is due to Mathias (see also [15], Theorem 26.38).

**Theorem 1.25** (Mathias)**.** *Let $M$ be a transitive model of ZFC. An infinite set $x \subseteq \omega$ is a Mathias real over $M$ if and only if for every maximal almost disjoint family $\mathscr{A} \in M$ of subsets of $\omega$, there exists an $X \in \mathscr{A}$ such that $x - X$ is finite.*

*Proof.* See Theorem 26.35 in [15]. □

**Corollary 1.26** (Mathias)**.** *If $x$ is a Mathias real over $M$ and $y \subseteq x$ is infinite, then $y$ is a Mathias real over $M$.*

**Theorem 1.27** (Baumgartner)**.** *Mathias forcing satisfies Axiom A.*

*Proof.* See the proof of Corollary 26.38 in [15]. □

**Definition 1.28** ([18])**.** *For $s \in [\omega]^{<\omega}$, let $\bar{s}^+ := (\max s) + 1$. A set $x \subseteq \omega$ is said to* diagonalize *the set $\{x_s : s \in [\omega]^{<\omega}\} \subseteq [\omega]^\omega$, if $x \subseteq x_\emptyset$ and for all $s \in [\omega]^{<\omega}$, if $\max s \in x$, then $x \setminus \bar{s}^+ \subseteq x_s$.*

For $\mathscr{A} \subseteq [\omega]^\omega$ we write $fil\mathscr{A}$ for the filter generated by the members of $\mathscr{A}$, i.e., $fil\mathscr{A}$ consists of all subsets of $\omega$ which are supersets of intersections of finitely many members of $\mathscr{A}$.

**Definition 1.29** ([18])**.** *A free family $\mathscr{E}$ is called a* happy family *if whenever $fil\{x_s : s \in [\omega]^{<\omega}\} \subseteq \mathscr{E}$, then there is an $x \in \mathscr{E}$ which diagonalizes the set $\{x_s : s \in [\omega]^{<\omega}\}$.*

**Theorem 1.30** ([18])**.** *If $\mathscr{A}$ is a happy family, then the forcing notion $\mathbb{M}(\mathscr{A})$ adds generically a Ramsey ultrafilter $\dot{\mathscr{U}}$ and $\mathbb{M}(\mathscr{A}) \approx \mathscr{A} * \mathbb{M}(\dot{\mathscr{U}})$. A special case: $\mathbb{M} \approx \mathscr{P}(\omega)/fin * \mathbb{M}(\dot{\mathscr{U}})$*

*Proof.* See Lema 3.5 in [12]. □

### 1.2.2 The forcing $\mathbb{P}^*$

We denote by $FIN$ the set of all finite non empty subsets of $\omega$. For $s$ and $t$ in $FIN$, we write $s < t$ if $\max(s) < \min(t)$.

If $X$ is a subset of $FIN$, then we write $FU(X)$ for the set of all finite unions of members of $X$, excluding the empty union.

**Definition 1.31.** *Let $I$ be a natural number or $I = \omega$.* A finite (an infinite) block sequence *is a sequence $\mathscr{D} = \langle d_i \rangle_{i \in I}$ of finite subsets of $\mathbb{N}$ such that $d_i < d_{i+1}$ for all $i \in I$. The set $(FIN)^\omega$ is the collection all infinite block sequences of elements of $FIN$.*

Given an infinite block sequence $\mathscr{D}$, we define $FU(\mathscr{D})$ as before, i.e., viewing $\mathscr{D}$ as a collection of finite subsets of natural numbers. If $\mathscr{D}$ is the empty sequence, then $FU(\mathscr{D}) = \emptyset$.

Given $\mathscr{D}$ and $\mathscr{E}$ in $(FIN)^\omega$, we say that $\mathscr{D}$ is a *condensation* of $\mathscr{E}$, written $\mathscr{D} \sqsubseteq \mathscr{E}$, if $\mathscr{D} \subseteq FU(\mathscr{E})$. If $s \in FIN$ and $\mathscr{D} \in (FIN)^\omega$ we denote

$$\mathscr{D} \setminus s := \{t \in \mathscr{D} : \min(t) > \max(s)\}.$$

If $s$ is a singleton $\{n\}$ then we simply write $\mathscr{D} \setminus n$, instead of $\mathscr{D} \setminus \{n\}$.

**Definition 1.32.** *The partial ordering $\mathbb{P}^*$ is $(FIN)^\omega$ with the ordering $\sqsubseteq^*$ defined as follows: if $\mathscr{D}$ and $\mathscr{E}$ are in $(FIN)^\omega$, then $\mathscr{D} \sqsubseteq^* \mathscr{E}$ iff there is an $n$ such that $\mathscr{D} \setminus n$ is a condensation of $\mathscr{E}$.*

**Proposition 1.33.** *The partial ordering $\mathbb{P}^*$ is $\sigma$-closed.*

*Proof.* Let $\{\mathscr{D}_n : n \in \omega\} \subseteq (FIN)^\omega$ such that $\mathscr{D}_{n+1} \sqsubseteq^* \mathscr{D}_n$, for each $n$. We shall inductively construct $\mathscr{B} \in (FIN)^\omega$ such that $\mathscr{B}$ is a condensation of $\mathscr{D}_n$ for all $n \in \omega$.

We have $\mathscr{D}_0 = \langle d_j^0 \rangle_{j\in\omega}$. Let $b_0 = d_0^0$, since $\mathscr{D}_1 \sqsubseteq^* \mathscr{D}_0$ there are $m_1' \in \omega$ such that $\mathscr{D}_1 \setminus m_1 \sqsubseteq \mathscr{D}_0$.

Assume we have $b_0 < b_1 < \ldots < b_n$ such that $t_j \in \mathscr{D}_j$ $j \leqslant n$ and $t_j \in FU(\mathscr{D}_{j-1})$ for all $j \in \{1, \ldots, n\}$. Since $\mathscr{D}_{n+1} \sqsubseteq^* \mathscr{D}_n$, there is $m_{n+1}' \in \omega$ such that $\mathscr{D}_{n+1} \setminus m_{n+1}' \sqsubseteq \mathscr{D}_n$. Let $m_{n+1} = \max\{m_{n+1}', \max t_n\}$, we take $b_{n+1} \in \mathscr{D}_{n+1} \setminus m_{n+1}$, then $b_n < b_{n+1}$ and $b_{n+1} \in FU(\mathscr{D}_n)$. Let $\mathscr{B} = \langle b_n \rangle_{n\in\omega}$. It is clear that $\mathscr{B} \in (FIN)^\omega$.

**Claim 1.34.** *$\mathscr{B} \sqsubseteq^* \mathscr{D}_n$ for all $n \in \omega$.*

*Proof of Claim:* Note that $\mathscr{B} \setminus m_j \sqsubseteq \mathscr{D}_j \setminus m_j \sqsubseteq \mathscr{D}_{j-1}$ for all $j \geqslant 1$.

Thus $\mathscr{B} \sqsubseteq^* \mathscr{D}_n$ for all $n \in \omega$. We have proved the claim. □

□

### 1.2.3 Matet forcing

**Definition 1.35** (Blass [5])**.**

1. *A filter $\mathscr{F}$ on $FIN$ is said to be an* ordered-union filter *if it has a basis of the form $FU(\mathscr{D})$ for $\mathscr{D} \in (FIN)^\omega$.*

2. *An ordered-union filter is said to be* stable *if, whenever it contains $FU(\mathscr{D}_n)$ for each of countably many sets $\mathscr{D}_n \in (FIN)^\omega$, it also contains $FU(\mathscr{E})$ for some $\mathscr{E} \in (FIN)^\omega$ that is a condensation of each $\mathscr{D}_n$.*

3. *If* $\mathscr{F}$ *is an ordered-union filter, we let* $\mathscr{F} \upharpoonright (FIN)^\omega$ *be the set of all* $\mathscr{D} \in (FIN)^\omega$ *such that* $FU(\mathscr{D}) \in \mathscr{F}$.

**Definition 1.36** ([8]). *We say that a set* $\mathscr{H} \subseteq (FIN)^\omega$ is Matet-adequate *if:*

1. $\mathscr{H}$ *is closed under finite changes, i.e., if* $A \in \mathscr{H}$ *and* $A \triangle B$ *is finite, then* $B \in \mathscr{H}$.
2. $\mathscr{H}$ *is closed upwards:* $A \in \mathscr{H}$ *and* $A \sqsubseteq^* B$ *implies* $B \in \mathscr{H}$.
3. $(\mathscr{H}, \sqsubseteq^*)$ *is countably closed, i.e., if* $\{A_n : n \in \omega\} \subseteq \mathscr{H}$ *and, for all* $n$, $A_{n+1} \sqsubseteq^* A_n$ *then there is a* $B \in \mathscr{H}$ *such that* $B \sqsubseteq^* A_n$ *for each* $n$.
4. *If* $A \in \mathscr{H}$ *and* $FU(A)$ *is partitioned into* 2 *pieces, then there is a* $B \sqsubseteq A$ *in* $\mathscr{H}$ *so that* $FU(B)$ *is included in a single piece of the partition.*

*We refer to condition 4 as* Hindman property.

The most obvious example of a Matet-adequate family is $(FIN)^\omega$ itself. If $\mathscr{U}$ is a stable ordered-union ultrafilter, then

$$\mathscr{U} \upharpoonright (FIN)^\omega = \{\mathscr{D} \in (FIN)^\omega : FU(\mathscr{D}) \in \mathscr{U}\}$$

is another example of a Matet-adequate family.

The following notion of forcing is due to Pierre Matet [17]. We use Eisworth's formulation in [8]. Note that stable ordered-union ultrafilters are Matet's Milliken-Taylor ultrafilters.

**Definition 1.37** ([8]). *Let* $\mathscr{H}$ *be a Matet-adequate family. We define a notion of forcing* $\mathbb{MT}(\mathscr{H})$, Matet forcing with respect to $\mathscr{H}$, *as follows: A condition* $p$ *is a pair* $(s, D)$ *where* $s \in FIN$, $D \in \mathscr{H}$, *and* $\max s < \min d$ *for* $d \in D$. *A condition* $(s, D)$ *extends* $(t, E)$ *if* $s \supseteq t$, $D \sqsubseteq E$, *and* $s \setminus t \in FU(E)$.

If $G$ is any generic subset of $\mathbb{MT}(\mathscr{H})$, then

$$\bigcup\{s : \text{ for some } D \in \mathscr{H}, (s, D) \in G\}$$

is a subset of $\omega$ that we call a *Matet real*. A condition $(s, D) \in \mathbb{MT}(\mathscr{H})$ should be thought of a promise that the Matet real will have $s$ as initial segment and will consist of unions of elements of $D$.

Given $(s, D) \in \mathbb{MT}(\mathscr{H})$, we define

$$[s, D] := \{x \in [\omega]^\omega : x = s \cup \bigcup_{i \in I} d_i \text{ for some } I \in [\omega]^\omega\}$$

where $D = \langle d_i \rangle_{i \in \omega}$.

We are going to consider two special cases:

- If $\mathscr{H} = (FIN)^\omega$, we denote $\mathbb{MT}(\mathscr{H})$ by $\mathbb{MT}$.

- If $\mathscr{H} = \mathscr{U} \upharpoonright (FIN)^\omega$, where $\mathscr{U}$ is a stable ordered-union ultrafilter, we denote $\mathbb{MT}(\mathscr{H})$ by $\mathbb{MT}(\mathscr{U})$.

The next results are essentially the same as those obtained by Blass in his investigation of Matet forcing (see also Eisworth's article [8]).

**Proposition 1.38** ([8]). *Let $\mathscr{H}$ be Matet-adequate family. If $G$ is any generic subset of $\mathscr{H}$, then in the generic extension $V[G]$, the set*

$$\mathscr{U}_G = \{X \subseteq FIN : X \supseteq FU(A) \text{ for some } A \in G\}$$

*is a stable ordered-union ultrafilter.*

*Proof.* See the proof of Proposition 3.2 in [8]. □

**Theorem 1.39** ([8]). *If $\mathscr{H}$ is a Matet-adequate family and $\dot{\mathscr{U}}$ is the canonical $\mathscr{H}-$name for the generic stable ordered-union ultrafilter, then*

$$\mathbb{MT}(\mathscr{H}) \approx \mathscr{H} * \mathbb{MT}(\dot{\mathscr{U}}).$$

*In the special case that $\mathscr{H} = (FIN)^\omega$, we have $\mathbb{MT} \approx \mathbb{P}^* * \mathbb{MT}(\dot{\mathscr{U}})$.*

*Proof.* See the proof in [8], page 458. □

**Lemma 1.40.** *Let $\mathscr{U}$ be a stable ordered-union ultrafilter. If $(s, \mathscr{D}) \in \mathbb{MT}(\mathscr{U})$ and $\theta$ is a sentence of the forcing languaje, then $(s, \mathscr{D})$ has a pure extension deciding $\theta$, i.e., there is a condensation $\mathscr{E}$ of $\mathscr{D}$ in $\mathscr{U} \restriction (FIN)^\omega$ such that either*

$$(s, \mathscr{E}) \Vdash \theta \text{ or } (s, \mathscr{E}) \Vdash \neg\theta.$$

*Proof.* See proof of Lemma 2.6 in [8]. □

Both Andreas Blass and Claude Laflamme proved (independently and unpublished) that "traditional" Matet forcing (i.e., no ultrafilters involved) preserves $p$-points ultrafilters in the ground model, i.e., if $\mathscr{V}$ is a $p$-point, then in the generic extension every subset of $\omega$ either contains or is disjoint from a set in $\mathscr{V}$. Their result can be recovered from Theorem 4 in Eisworth's article [8] by viewing traditional Matet forcing as a two-step iteration, where one first adjoins a generic stable-ordered ultrafilter $\mathscr{U}$, and then forces with $\mathbb{MT}(\mathscr{U})$.

**Definition 1.41.** *Let $\mathscr{U}$ be a stable ordered-union ultrafilter. The core of $\mathscr{U}$, denoted by $\Phi(\mathscr{U})$ is defined by*

$$X \in \Phi(\mathscr{U}) \text{ if and only if } \exists Y \in \mathscr{U} \text{ with } \bigcup Y \subseteq X.$$

The following proposition summarizes some facts about $\Phi(\mathscr{U})$.

**Proposition 1.42** (Eisworth [8])**.**

1. $\Phi(\mathscr{U}) = \{X \in \omega : [X]^{<\omega} \in \mathscr{U}\}$.
2. *$\Phi(\mathscr{U})$ is a p-filter.*
3. *$\Phi(\mathscr{U})$ is not diagonalized, i.e., there is no infinite $Z \subseteq \omega$ such that $Z$ is almost included in each member of $\Phi(\mathscr{U})$.*
4. *Forcing with $\mathbb{MT}(\mathscr{U})$ adjoins a set that diagonalizes $\Phi(\mathscr{U})$.*

*Proof.* See the proof of the Proposition 2.3 in [8]. □

**Proposition 1.43** (Eisworth)**.** *Let $\mathscr{V}$ be an ultrafilter on $\omega$, and suppose there is a finite-to-one function $f$ for which $f(\Phi(\mathscr{U})) \subseteq \mathscr{V}$. Then forcing with $\mathbb{MT}(\mathscr{U})$ destroys $\mathscr{V}$.*

*Proof.* See the proof of the Proposition 2.4 in [8]. □

**Definition 1.44** (Blass [7])**.** *If $\mathscr{F}$ is a filter on a set $I$ and $f$ is a function from $I$ to some set $J$, then $f(\mathscr{F})$ is the filter on $J$ defined by*

$$X \in f(\mathscr{F}) \text{ if and only if } f^{-1}(X) \in \mathscr{F}.$$

*If $\mathscr{F}$ and $\mathscr{G}$ are two filters on $\omega$, we say that $\mathscr{G}$ lies below $\mathscr{F}$ in the Rudin-Blass ordering, written $\mathscr{G} \leqslant_{RB} \mathscr{F}$, if there is a finite-to-one function $f$ with*

$$f(\mathscr{G}) \subseteq f(\mathscr{F}).$$

**Corollary 1.45.** *If $\mathscr{V}$ is an ultrafilter above $\Phi(\mathscr{U})$ in the Rudin-Blass ordering, then forcing with $\mathbb{MT}(\mathscr{U})$ destroys $\mathscr{V}$.*

*Proof.* See the proof of the Corollary 2.5 in [8]. □

**Theorem 1.46** (Eisworth [8])**.** *If $\mathscr{V}$ is a p-point that is not above $\Phi(\mathscr{U})$ in the Rudin-Blass ordering, then continues to generate an ultrafilter after we force with $\mathbb{MT}(\mathscr{U})$.*

*Proof.* See proof of the Theorem 4 in [8]. □

**Corollary 1.47** (Eisworth [8])**.** *Matet forcing preserves p-points.*

*Proof.* By the Theorem 1.39, we have $\mathbb{MT} \approx \mathbb{P}^* * \mathbb{MT}(\dot{\mathscr{U}})$. Let $\mathscr{V}$ a $p$-point ultrafilter in $V$, then $\mathscr{V}$ is $p$-point ultrafilter in intermediate extension, because $\mathbb{P}^*$ is $\sigma$-closed (Proposition 1.33), so we are not adding new reals. By a forcing argument we have that $\mathscr{V}$ is not above $\Phi(\mathscr{U})$ in the Rudin-Blass ordering, so by Corollary 1.45 and Theorem 1.46, $\mathbb{MT}(\mathscr{U})$ does not destroy $\mathscr{V}$. □

**Corollary 1.48.** *Matet forcing does not add splitting reals over $V$.*

*Proof.* Assume towards a contradiction that there exists a splitting real over $V$ in a $\mathbb{MT}$-generic extension of $V$. Let $q \in \mathbb{MT}$ be such that $q \Vdash$ "$\tau$ is splitting real over $V$". Let $G$ be a $Col(\omega_1, < \mathfrak{c})$-generic filter. We have that $CH$ holds in $V[G]$, so by Theorem 7.8 in [15] there exists a $p$-point filter. Let $\mathscr{V}$ a $p$-point filter in $V[G]$. Since $Col(\omega_1, < \mathfrak{c})$ is $\sigma$-closed we are not adding new reals, hence $\mathscr{P}(\omega)^V = \mathscr{P}(\omega)^{V[G]}$. Let $H$ be a $\mathbb{MT}$-generic filter over $V[G]$ such that $q \in H$. Note that in $V[G][H]$ the interpretation $\tau_H$ of $\tau$ is a splitting real over $V$. By Theorem 1.46, there exists $v \in \mathscr{V}$ such that $v \subseteq \tau_H$ or $v \cap \tau_H = \emptyset$, which is a contradiction to the definition of splitting real over $V$. □

## 1.3 Proper forcing

Proper forcing was introduced by Saharon Shelah, who isolated properness as the property of forcing that is common to many standard examples of forcing notions and that is preserved under countable support iteration, see [21] and [22].

A partial order $\mathbb{P}$ is said to be *proper* if and only if for any uncountable set $X$ and every stationary set $S \subseteq [X]^\omega$, $\Vdash_{\mathbb{P}}$ "$\check{S}$ is stationary subset of $[\check{X}]^\omega$".

An equivalent characterization which is easier to work with is the following. If $\lambda$ is a regular uncountable cardinal and $N$ is a countable substructure of $H(\lambda)$ with $\mathbb{P} \in N$, then we will say that $p \in \mathbb{P}$ is a $(N, \mathbb{P})-$*generic* condition if and only if for every maximal antichain $A$ of $\mathbb{P}$ that belongs to $N$ we have that $A \cap N$ is predense below $p$, i.e., for every $q \leqslant p$ there exists $r \in A \cap N$ such that $q$ and $r$ are compatible.

**Definition 1.49.** *$\mathbb{P}$ is proper if and only if for every $\lambda > 2^{|\mathbb{P}|}$ and every countable elementary submodel $N$ of $H(\lambda)$ with $\mathbb{P} \in N$, for every $p \in \mathbb{P} \cap N$ there is a $(N, \mathbb{P})$-generic condition stronger than $p$.*

**Proposition 1.50.** *The following are equivalent:*

*a) $p$ is $(N, P)$-generic.*

*b) p forces that $\dot{G} \cap N$ is $P$-generic filter over $N$.*

**Theorem 1.51.** *If $\mathbb{P}$ satisfies Axiom A, then $\mathbb{P}$ is proper.*

*Proof.* See Lemma 31.11 in Jech's book [15]. □

# Chapter 2

# Ramsey's Theorem

We are interested in functions $h : [A]^n \to \lambda$, where $n$ is a natural number and $\lambda$ is a cardinal. We often refer to any such function $h$ as a partition of $[A]^n$ (into $\leqslant \lambda$ classes), or a coloring of $[A]^n$ into $\leqslant \lambda$ colors.

If $h : [A]^n \to \lambda$, a subset $H$ of $A$ is called *homogeneous* for $h$, or $h-$*homogeneous*, if and only if $h$ is constant on $[H]^n$, i.e., if and only if $h(X) = h(Y)$ for all $X, Y \in [H]^n$. The notation $(\kappa) \to (\alpha)^n_\lambda$, for $\kappa$ and $\lambda$ (finite or infinite) cardinals, $\alpha$ an ordinal, and $n$ a natural number, means that for every partition $h$ of $[\kappa]^n$ into $\leqslant \lambda$ classes there is an $h-$homogeneous set of order-type $\alpha$.

Notice that if $\kappa$ is an infinite cardinal, $A$ is a subset of $\kappa$ of cardinality $\kappa$ and $\kappa \to (\alpha)^n_\lambda$ holds, then every partition $f : [A]^n \to \lambda$ has an $f-$homogenous set of order-type $\alpha$.

We will give a proof of Ramsey's Theorem using forcing arguments.

**Theorem 2.1** (Ramsey [19])**.** *For every $n, m > 0$, $\omega \to (\omega)^n_m$.*

We will take care only of the case $m = 2$, the general case can easily be proved by induction on $m$.

*Proof.* We proceed by induction on $n \geqslant 1$. By the Pigeonhole principle we have it for $n = 1$. Given $n \geqslant 1$ and given $g : [\omega]^{n+1} \to 2$, we must conclude that there is an infinite $g-$homogeneous set.

Assume that there is no $H \subseteq \omega$ infinite such that $H$ is $g-$homogeneous of color 0. We will produce an infinite $g-$homogeneous set of color 1.

We define a partial order $\mathbb{P}_{g,1} = \langle P, \leqslant^* \rangle$ as follows: the elements of $P$ are of the form $(a, A)$ where $a \in [\omega]^{<\omega}$, $A \in [\omega]^{\omega}$, $a < A$, which means $\max(a) < \min(A)$, $g(x) = 1$ for all $x \in [a]^{n+1}$, and for every $j \in \{1, \ldots, n\}$, every $y \in [a]^j$, and every $x \in [A]^{n+1-j}$, $g(y \cup x) = 1$.

Given two elements of $(a, A), (b, B) \in P$, we define $(b, B) \leqslant^* (a, A)$, as in Mathias forcing, if and only if $a$ is an initial segment of $b$, $B \subseteq A$ and for all $x \in (b \setminus a)$ $(x \in A)$.

We have that $P$ is not empty because $(\emptyset, \omega) \in P$, and $\leqslant^*$ is clearly reflexive and transitive.

**Claim 2.2.** *Given a condition $(a, A) \in P$ we can extend it, i.e., there exists $m \in A$ and there exists $B \subseteq A$ infinite with $k > m$ for all $k \in B$ such that $g(\{m\} \cup y) = 1$ for all $y \in [B]^n$. Note that then $(b, B) \leqslant^* (a, A)$ where $b = a \cup \{m\}$.*

*Proof of Claim:* Assume, towards a contradiction, that for all $m \in A$ and for all $B \subseteq A$ if $g(\{m\} \cup y) = 1$ for all $y \in [B]^n$, then $B$ is finite.

Let $m_0 = \min A$, and let $g_{m_0} : [A \setminus \{m_0\}]^n \to 2$ be defined as $g_{m_0}(y) = g(\{m_0\} \cup y)$. By inductive hypothesis there is $B_0 \subseteq A \setminus \{m_0\}$ infinite such that $g_{m_0} \upharpoonright [B_0]^n$ is constant. By our assumption $g_{m_0} \upharpoonright [B_0]^n$ is constant with value 0.

Assume that we have elements $m_0 < m_1 < \ldots < m_k$ in $A$ and

$$B_k \subseteq \ldots \subseteq B_0 \subseteq A$$

are such that $m_j = \min B_{j-1}$ for all $j \in \{1, \ldots, k\}$, the function

$$g_{m_j} : [B_{j-1} \setminus \{m_j\}]^n \to 2$$

is defined as $g_{m_j}(y) = g(\{m_j\} \cup y)$, and $g_{m_j} \upharpoonright [B_j]^n$ is constant 0 for all $j \in \{1, \ldots, k\}$.

Then let $m_{k+1} = \min B_k$, and let $g_{m_{k+1}} : [B_k \setminus \{m_{k+1}\}]^n \to 2$ be defined as

$$g_{m_{k+1}}(y) = g(\{m_{k+1}\} \cup y).$$

By the inductive hypothesis and our assumption, there exists

$$B_{k+1} \subseteq B_k \setminus \{m_{k+1}\}$$

infinite such that $g_{m_{k+1}} \restriction [B_{k+1}]^n$ is constant 0.

Inductively we have constructed $H = \{m_i : i \in \omega\}$. We claim that $g \restriction [H]^{n+1}$ is constant 0, which yielding a contradiction to our initial assumption.

For $y \in [H]^{n+1}$, the least element of $y$ is $m_j$, for some $j \in \omega$. Then

$$g(y) = g(\{m_j\} \cup (y \setminus \{m_j\})) = 0$$

because $y \setminus \{m_j\} \in [B_{j+1}]^n$. This proves the claim. □

For every $n \in \omega$ we define $D_n = \{(a, A) \in P : |a| \geqslant n\}$. Note that $D_n$ is a dense set for all $n \in \omega$ and consider $\mathscr{D} = \{D_n : n \in \omega\}$. Let $G$ a $\mathscr{D}-$generic filter in $\mathbb{P}_{g,1}$ and

$$S := \bigcup \{a \in [\omega]^{<\omega} : \exists A \in [\omega]^\omega \text{ such that } (a, A) \in G\}.$$

By the claim we have that $S$ is infinite.

We shall prove that and $g \restriction [S]^{n+1}$ is constant 1. Let $y \in [S]^{n+1}$, where $y = \{y_0, \ldots, y_n\}$, then there exist $(a_0, A_0), \ldots, (a_n, A_n)$ in $G$ such that every element $y_j \in a_j$ for all $j \leqslant n$. Since $G$ is a filter there is $(b, B) \in G$ such that $(b, B)$ extends $(a_j, A_j)$ for all $j \leqslant n$. Then $y \in [b]^{n+1}$, and so $g(y) = 1$. Hence $S$ is $g-$homogeneous with color 1.

□

Given a partition $g : [\omega]^2 \to 2$, by Ramsey's Theorem there exist $X \in [\omega]^\omega$ and some $i \in \{0, 1\}$ such that $X$ is $g-$homogeneous with color $i$. We define

$$\mathbb{M}_X = \{(s, A) : s \in [X]^{<\omega}, A \in [X]^\omega \text{ and } \max s < \min A\}$$

and we define the ordering relation between elements in $\mathbb{M}_X$ as in Mathias forcing. Then $\mathbb{M}_X$ order is isomorphic to Mathias forcing.

If $\mathbb{P}$ is a partial order and $p \in \mathbb{P}$, $\mathbb{P} \restriction p$ is the suborder of $\mathbb{P}$ whose elements are in $\mathbb{P}$ below $p$.

**Theorem 2.3.** *The following statements are equivalent:*

1. *$\mathbb{P}_{g,i}$ is non trivial, i.e., every condition can be extended in the finite part.*

2. *Every infinite set of natural numbers has an infinite set $X$ such that $\mathbb{P}_{g,i} \restriction (\emptyset, X) = \mathbb{M}_X$.*

3. *Every infinite set of natural numbers has an infinite set $X$ such that $X$ is a $g-$homogeneous set of color $i$.*

*Proof.* 1→2. Assume that $\mathbb{P}_{g,i}$ is non trivial. Let $Y$ be an infinite set of natural numbers. Consider the condition $(\emptyset, Y) \in \mathbb{P}_{g,i}$ and let $G$ be a $\mathbb{P}_{g,i}$-generic filter such that $(\emptyset, Y) \in \mathbb{P}_{g,i}$. Then there exists $X \in [Y]^\omega$ such that $g \restriction [X]^2$ is constant with value $i$.

Consider the partial order $\mathbb{P}_{g,i} \restriction (\emptyset, X)$, i.e., all elements in $\mathbb{P}_{g,i}$ below condition $(\emptyset, X)$. Then the *id* function is an isomorphism between the partial orders $(\mathbb{P}_{g,i} \restriction (\emptyset, X), \leqslant^*)$ and $(\mathbb{M}_X, \leqslant)$.

2→3. Let $Y$ be a infinite set of natural numbers, by assumption there exists $X \in [Y]^2$ such that $\mathbb{P}_{g,i} \restriction (\emptyset, X) = \mathbb{M}_X$. The *id* function is an isomorphism between the partial orders $(\mathbb{P}_{g,i} \restriction (\emptyset, X), \leqslant^*)$ and $(\mathbb{M}_X, \leqslant)$. Let $\{n, m\} \in [X]^2$. Assume that $n < m$ and let

$$A := \{x \in X : x > n\} \text{ and } B := \{x \in X : x > m\}.$$

Then the conditions $(\{n\}, A)$, $(\{n, m\}, B)$ belong to $\mathbb{M}(X)$ and $(\{n, m\}, B) \leqslant (\{n\}, A)$. By the assumption $(\{n, m\}, B) \leqslant^* (\{n\}, A)$, in particular $g(\{n, m\}) = i$. Hence $g \restriction [X]^2$ is contant $i$.

3→1. It is trivial. □

# Chapter 3

# Hindman's Theorem

In this chapter we will define a partial order associated to Hindman's theorem ([13]) and we will give a proof of Hindman's theorem using forcing arguments relative to this partial ordering. Our proof uses some lemmas from Baumgartner's proof of the theorem, in [2].

**Definition 3.1.** *Let* $H \subseteq \omega$. $FS(H) = \{\sum_{n\in a} n : a \in [H]^{<\omega}$ and $a \neq \emptyset\}$, $FS(H)$ *is called the* sum-set *of* $H$. *For example:*

$$FS(\{2, 3, 7\}) = \{2, 3, 5, 7, 9, 10, 12\}.$$

**Theorem 3.2** (Hindman [13])**.** *If* $\omega$ *is finitely colored, then there exists* $H$ *an infinite subset of* $\omega$*, such that* $FS(H)$ *is monochromatic.*

Call $\mathscr{D}$ a *disjoint collection* if $\mathscr{D}$ is an infinite set of pairwise disjoint finite subsets of natural numbers.

**Theorem 3.3** (Baumgartner [2])**.** *Let* $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_k$. *Then there exist* $0 \leqslant i \leqslant k$ *and a disjoint collection* $\mathscr{D}$ *with* $FU(\mathscr{D}) \subseteq \mathscr{C}_i$.

**Lemma 3.4.** *Theorem 3.3 implies Hindman's Theorem.*

*Proof.* Let $k \geqslant 1$ be a natural number and let $h : \omega \to k$ be a coloring of $\omega$ with $k$ colors. Consider the canonical bijection $g : [\omega]^{<\omega} \to \omega$, that assings each $s \in [\omega]^{<\omega}$ to $n_s = \sum_{i\in s} 2^i$. Then $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_{k-1}$ where

$$\mathscr{C}_i = \{s \in [\omega]^{<\omega} : h(g(s)) = i\}$$

for $i \in \{0, \dots, k-1\}$. By Theorem 3.3 there exists $0 \leqslant i < k$ and a disjoint collection $\mathscr{D}$ with $FU(\mathscr{D}) \subseteq \mathscr{C}_i$.

Let $H := \{g(d) : d \in \mathscr{D}\}$. It is clear that $H \subseteq \omega$ is infinite. Let $s \in [H]^{<\omega}$ with $s \neq \emptyset$. Then $s = \{a_0, \dots, a_m\}$, where $a_0 = g(d_{j_0}), \dots, a_m = g(d_{j_m})$.

We have:

$$h(a_0 + \ldots + a_m) = h(g(d_{j_0}) + \ldots + g(d_{j_m})) = h(\sum_{l \in d_{j_0}} 2^l + \ldots + \sum_{l \in d_{j_m}} 2^l) =$$

$$h(\sum_{l' \in \bigcup_{l=0}^{l=m} d_{j_l}} 2^{l'}) = h(g(\bigcup_{l=0}^{l=m} d_{j_l})) = i$$

Hence $h \restriction FS(H)$ is monochromatic. □

On the class of disjoint collections of finite subsets of natural numbers, we define a partial order $\sqsubseteq$ by $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ if and only if $\mathscr{D}_1 \subseteq FU(\mathscr{D})$.

**Definition 3.5.** *Given a collection of finite subsets of natural numbers $\mathscr{C}$, we say $\mathscr{C}$* is large for $\mathscr{D}$ *if $\mathscr{C} \cap FU(\mathscr{D}_1) \neq \emptyset$ for all $\mathscr{D}_1 \sqsubseteq \mathscr{D}$.*

**Lemma 3.6** (Decomposition Lemma, Baumgartner [2])**.** *Assume that $\mathscr{C}$ is large for $\mathscr{D}$ and $\mathscr{C} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_k$. Then there exists $0 \leqslant i \leqslant k$ and $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ such that $\mathscr{C}_i$ is large for $\mathscr{D}_1$.*

*Proof.* By induction on $k$.

Let $k = 1$. If $\mathscr{C} = \mathscr{C}_0 \dot\cup \mathscr{C}_1$ and $\mathscr{C}_0$ is not large for $\mathscr{D}$, then $\mathscr{C}_0 \cap FU(\mathscr{D}_1) = \emptyset$ for some $\mathscr{D}_1 \sqsubseteq \mathscr{D}$. Let $\mathscr{D}_2 \sqsubseteq \mathscr{D}_1$. Since $\mathscr{C}$ is large for $\mathscr{D}$, $\mathscr{C} \cap FU(\mathscr{D}_2) \neq \emptyset$, so $\mathscr{C}_1 \cap FU(\mathscr{D}_2) \neq \emptyset$ (because $FU(\mathscr{D}_2) \subseteq FU(\mathscr{D}_1)$ and $\mathscr{C}_0 \cap FU(\mathscr{D}_1) = \emptyset$). Hence $\mathscr{C}_1$ is large for $\mathscr{D}_1$.

Assume now that the statement is true for $k$.

Let $\mathscr{C} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_{k+1}$. Assume that $\mathscr{C}_0$ is not large for $\mathscr{D}$. Then there is $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ that is large for $\mathscr{C}_1 \dot\cup \ldots \dot\cup \mathscr{C}_{k+1}$. By the inductive hypothesis, there is $\mathscr{D}_2 \sqsubseteq \mathscr{D}_1$ and $i \in \{1, \dots, k+1\}$ such that $\mathscr{D}_2$ is large for $\mathscr{C}_i$. □

Define $\mathscr{C} - s := \{c \in \mathscr{C} : c \cap s = \emptyset\}$.

**Lemma 3.7** (Baumgartner)**.** *If $\mathscr{C}$ is large for $\mathscr{D}$ and $s$ is a finite subset of $\omega$, then $\mathscr{C} - s$ is large for $\mathscr{D}$.*

*Proof.* Suppose that there is $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ such that $(\mathscr{C} - s) \cap FU(\mathscr{D}_1) = \emptyset$. Let $\mathscr{D}_2 = \{d \in \mathscr{D}_1 : d \cap s = \emptyset\}$. Note that $\mathscr{D}_2$ is infinite since $s$ is finite. Then $\mathscr{C} \cap FU(\mathscr{D}_2) = \emptyset$, but $\mathscr{D}_2 \sqsubseteq \mathscr{D}$, and we reach a contradiction.

□

**Lemma 3.8** (Baumgartner)**.** *If $\mathscr{C}$ is large for $\mathscr{D}$, there exist $s \in FU(\mathscr{D})$ and $\mathscr{D}_1 \sqsubseteq \mathscr{D} - s$ such that $\mathscr{C}_1 = \{t \in \mathscr{C} - s : t \cup s \in \mathscr{C}\}$ is large for $\mathscr{D}_1$.*

*Proof.* Let us first prove the following.

**Claim 3.9.** *There exist $n$ and $d_1, \ldots, d_n \in \mathscr{D}$ such that, for every $d_{n+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_n$, there exists non-empty $I \subseteq \{1, \ldots, n\}$ such that $d_{n+1} \cup \bigcup_{i \in I} d_i \in \mathscr{C}$.*

*Proof of Claim:* Suppose, otherwise. If $I \subseteq \{1, \ldots, n\}$, let us write $d_I$ for $\bigcup_{i \in I} d_i$. Thus, for all $n \in \omega$ and for all $d_1, \ldots, d_n \in \mathscr{D}$ there is $d_{n+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_n$ such that $d_{n+1} \cup d_I \notin \mathscr{C}$ for all $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$.

Suppose that we have $d_1, \ldots, d_k$, elements of $FU(\mathscr{D})$, that are pairwise disjoint and such that every finite union of them does not belong to $\mathscr{C}$. By assumption, there is $d_{k+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_k$ such that $d_{k+1} \cup d_I \notin \mathscr{C}$ for all $I \subseteq \{1, \ldots, k\}$, $I \neq \emptyset$. In this way, we construct $\mathscr{D}' = \{d_1, d_2, \ldots\} \sqsubseteq \mathscr{D}$ such that $\mathscr{C} \cap FU(\mathscr{D}') = \emptyset$, which is a contradiction.

□

Continuing with the proof of the Lemma, fix $d_1, \ldots, d_n \in \mathscr{D}$ and write $d^*$ for $d_1 \cup \ldots \cup d_n$. For $\emptyset \neq I \subseteq \{1, \ldots, n\}$ we let

$$\mathscr{C}_I = \{c \in \mathscr{C} : c \cap d^* = \emptyset, c \cup d_I \in \mathscr{C}\}.$$

We claim that $\mathscr{C}_{I_1} \dot\cup \ldots \dot\cup \mathscr{C}_{I_k}$ is large for $\mathscr{D}$, where $\{I_1, \ldots, I_k\}$ is a list of all nonempty subsets of $\{i, \ldots, n\}$. For if $\mathscr{D}' \sqsubseteq \mathscr{D}$, then define

$$\mathscr{D}^* := \{d \in \mathscr{D}' : d > d^*\}.$$

So, $\mathscr{D}^* \sqsubseteq \mathscr{D}$ and since $\mathscr{C}$ is large for $\mathscr{D}$ there exists $d \in FU(\mathscr{D}^*) \cap \mathscr{C}$. In particular, $d \in FU(\mathscr{D}') \cap FU(\mathscr{D})$ and $d$ is disjoint from $d^*$ so $d \cup d_{I_i} \in \mathscr{C}$ for some $i \in \{1, \ldots, k\}$. Thus,

$$d \in \{c \in \mathscr{C} : c \cap d^* = \emptyset \text{ and } c \cup d_{I_i} \in \mathscr{C}\} = \mathscr{C}_{I_i}.$$

Since $\mathscr{D} - d^* \sqsubseteq \mathscr{D}$,

$$\mathscr{C}_{I_1} \dot{\cup} \ldots \dot{\cup} \mathscr{C}_{I_k}$$

is also large for $\mathscr{D} - d^*$, so by Lemma 3.6 there is $\mathscr{C}_{I_i}$ large for some $\mathscr{D}' \sqsubseteq \mathscr{D} - d^*$. And this proves the lemma with $s = d_{I_i}$, because $\mathscr{C}_{I_i} \subseteq \mathscr{C}_1$. □

**Lemma 3.10** (Baumgartner)**.** *If $\mathscr{C}$ is large for $\mathscr{D}$, then there exist $s' \in FU(\mathscr{D}) \cap \mathscr{C}$ and $\mathscr{D}' \sqsubseteq \mathscr{D} - s'$, such that*

$$\mathscr{C}' = \{t \in \mathscr{C} : t \cap s' = \emptyset \textit{ and } t \cup s' \in \mathscr{C}\}$$

*is large for $\mathscr{D}'$.*

*Proof.* Notice that only the requirement $s' \in \mathscr{C}$ distinguishes Lemma 3.10 from Lemma 3.8. We apply Lemma 3.8 repeatedly. Beginning with $\mathscr{C}_0 = \mathscr{C}$, $\mathscr{D}_0 = \mathscr{D}$, we find, for $i \geqslant 1$, $s_i$, $\mathscr{C}_i$, $\mathscr{D}_i$ with $s_{i+1} \in FU(\mathscr{D}_i)$ so that

$$\mathscr{C}_{i+1} = \{T \in \mathscr{C}_i : T \cap s_{i+1} = \emptyset, T \cup s_{i+1} \in \mathscr{C}_i\}$$

is large for $\mathscr{D}_{i+1} \sqsubseteq \mathscr{D}_i$ and $D \cap \bigcup_{j=1}^{i+1} s_j = \emptyset$, for all $D \in FU(\mathscr{D}_{i+1})$.

Note that $\mathscr{C}_{i+1} \subseteq \mathscr{C}_i$ for all $i \in \omega$, and if $T \in \mathscr{C}_{i+1}$ then $T \cup s \in \mathscr{C}$ and $T \cap s = \emptyset$ for all partial unions $s$ of the $s_1, \ldots, s_{i+1}$.

We define $\mathscr{D}^* := \{s_i : i \geqslant 1\}$. So, $\mathscr{D}^*$ is a disjoint collection and $\mathscr{D}^* \sqsubseteq \mathscr{D}$. Since $FU(\mathscr{D}^*) \cap \mathscr{C} \neq \emptyset$, we can find $i_1 < \ldots < i_k$ such that

$$s' = s_{i_1} \cup \ldots \cup s_{i_k} \in \mathscr{C}.$$

If $t \in \mathscr{C}_{i_k}$, then $t \in \mathscr{C}'$ and Lemma 3.10 holds with $\mathscr{D}' = \mathscr{D}_{i_k}$, as $\mathscr{C}_{i_k} \subseteq \mathscr{C}'$. □

Notice that given a disjoint collection $\mathscr{D}$ we can obtain an infinite block sequence from it, recall Definition 1.31, in fact, some $\mathscr{D}' \subseteq \mathscr{D}$ is an infinite block sequence. By convenience when we say $x$ is an element of a finite (an infinite) block sequence, means that $x$ is equal to some element of the range of the finite (infinite) block sequence.

**Theorem 3.11.** *If $\mathscr{C}$ is large for $\mathscr{D}'$, then there exists $\mathscr{E} \sqsubseteq \mathscr{D}'$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}$.*

*Proof.* Assume that $\mathscr{C}$ is large for $\mathscr{D}'$. We are going to define a partial order and by a forcing argument we shall obtain $\mathscr{E}$ with the desired property.

We define

$$\mathbb{P}_{\mathscr{C},\mathscr{D}'} = \langle P, \leqslant_{\mathscr{C}} \rangle \tag{3.0.1}$$

as follows: the elements of $P$ are of the form $(A, \mathscr{D})$, where $A = \langle x_0, \ldots, x_m \rangle$ is a finite block sequence of finite subsets of natural numbers such that $FU(A) \subseteq \mathscr{C}$, $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$ is an infinite block sequence such that $\mathscr{D} \sqsubseteq \mathscr{D}'$ and $A < \mathscr{D}$, i.e., $\max(x_m) < \min(d_0)$, and

$$\mathscr{C}^* = \{y \in FU(\mathscr{D}) \cap \mathscr{C} : \forall x \in FU(A)(x \cup y \in \mathscr{C})\}$$

is large for $\mathscr{D}$.

Given two elements in $P$, $(A, \mathscr{D})$ and $(B, \mathscr{B})$, we let $(B, \mathscr{B}) \leqslant_{\mathscr{C}} (A, \mathscr{D})$ if and only if $A$ is an initial subsequence of $B$, in this context we only write $B \supseteq A$, $\mathscr{B} \sqsubseteq \mathscr{D}$ and $\forall x \in B \setminus A$ $(x \in FU(\mathscr{D}))$.

Note that $(\langle\rangle, \mathscr{D}') \in P$ and the ordering relation $\leqslant_{\mathscr{C}}$ is reflexive and transitive.

**Claim 3.12.** *Every condition in $P$ can be extended in the finite part.*

*Proof of Claim:* Let $(A, \mathscr{D})$ be a condition in $P$, with $A = \langle x_0, \ldots, x_m \rangle$. We have, in particular, that $\mathscr{C}^*$ is large for $\mathscr{D}$.

By Lemma 3.10 there are $s \in \mathscr{C}^* \cap FU(\mathscr{D})$ and $\mathscr{E} \sqsubseteq \mathscr{D} - s$ such that

$$\mathscr{C}' = \{z \in \mathscr{C}^* : s \cap z = \emptyset \text{ and } z \cup s \in \mathscr{C}^*\}$$

is large for $\mathscr{E}$.

Since $s \in \mathscr{C}^*$, $s \in FU(\mathscr{D}) \cap \mathscr{C}$, and for all $x \in FU(A)$ we have $x \cup s \in \mathscr{C}$, so $FU(A^\frown\langle s\rangle) \subseteq \mathscr{C}$.

Let $\mathscr{D}^* = \{d \in \mathscr{E} : d > s\}$.

We shall prove that the set

$$\mathscr{F} := \{y \in FU(\mathscr{D}^*) \cap \mathscr{C} : \forall x \in FU(A^\frown\langle s\rangle)(x \cup y \in \mathscr{C})\}$$

is large for $\mathscr{D}^*$. Let $\mathscr{D}'' \sqsubseteq \mathscr{D}^*$. Since $\mathscr{C}'$ is large for $\mathscr{E}$, $FU(\mathscr{D}'') \cap \mathscr{C}' \neq \emptyset$. So there exists $z \in FU(\mathscr{D}'')$ such that $z \in \mathscr{C}^*$, $z > s$, and $z \cup s \in \mathscr{C}^*$. Thus, $z \in \mathscr{F}$.

Hence $(A^\frown\langle s\rangle, \mathscr{D}^*) \in P$ and $(A^\frown\langle s\rangle, \mathscr{D}^*) \leqslant_{\mathscr{C}} (A, \mathscr{D})$. We have proved the Claim. □

For every $n \in \omega$ we define $D_n := \{(A, \mathscr{D}) \in P : |A| \geqslant n\}$. Note that $D_n$ is a dense set for all $n \in \omega$ and consider $D = \{D_n : n \in \omega\}$. Let $G$ a $D$-generic filter in $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$ and let

$$\mathscr{E} := \bigcup\{A : \exists \mathscr{D} \text{ such that } (A, \mathscr{D}) \in G\}.$$

It is clear, by standard density arguments, that $\mathscr{E}$ is infinite. Moreover, $FU(\mathscr{E}) \subseteq \mathscr{C}$, because if $x \in FU(\mathscr{E})$, then there exist $x_0, \ldots, x_m \in \mathscr{E}$ such that $x = \bigcup_{j=0}^{j=m} x_j$, hence there are $(A_j, \mathscr{D}_j) \in G$ such that $x_j$ is an element of the finite block sequence $A_j$. Since $G$ is a filter there exists $(B, \mathscr{D}) \in G$ such that $(B, \mathscr{D}) \leqslant_{\mathscr{C}} (A_j, \mathscr{D}_j)$ for all $j \in \{0, \ldots, m\}$. By definition of the partial order, $FU(B) \subseteq \mathscr{C}$, and so $x \in \mathscr{C}$.

□

**Corollary 3.13.** *If $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \mathscr{C}_1$, then there exists an infinite block sequence $\mathscr{E}$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}_i$ for some $i \in \{0, 1\}$.*

*Proof.* Since $[\omega]^{<\omega}$ is large for $\langle\{i\}\rangle_{i\in\omega}$, by the Decomposition Lemma 3.6, there is $\mathscr{D}' \sqsubseteq \langle\{i\}\rangle_{i\in\omega}$ such that $\mathscr{C}_i$ is large for $\mathscr{D}'$ for some $i \in \{0, 1\}$. By the Theorem 3.11, there exists $\mathscr{E} \sqsubseteq \mathscr{D}'$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}_i$. □

**Remark 3.14.** *The Corollary above remains true if, instead of partitioning* $[\omega]^{<\omega}$ *one partitions* $FU(\mathscr{D})$ *where* $\mathscr{D}$ *is a block sequence. Then the homogeneous set* $\mathscr{E}$ *given by the theorem is such that* $\mathscr{E} \sqsubseteq \mathscr{D}$.

Note that Corollary 3.13 implies Theorem 3.3, by Remark 3.14.

# Chapter 4

# The $\mathbb{P}_{FIN}$ forcing and its properties

Let $\mathscr{C}$ be a family of finite subsets of natural numbers and $\mathscr{D}$ a infinite block sequence such that $\mathscr{C}$ is large for $\mathscr{D}$. Consider $\mathbb{P}_{\mathscr{C},\mathscr{D}}$ the partial order that was defined for proving Theorem 3.11, (3.0.1).

Let $\mathbb{P}_{FIN} = \mathbb{P}_{FIN,\langle\{i\}\rangle_{i\in\omega}}$. Note that the elements of $\mathbb{P}_{FIN}$ are all pairs of the form $(A,\mathscr{D})$, where $A = \langle x_0,\ldots,x_m\rangle$ is a finite block sequence, and $\mathscr{D} = \langle d_i\rangle_{i\in\omega}$ is an infinite block sequence such that $A < \mathscr{D}$. And the ordering is denoted by $\leqslant_{FIN}$ and defined as in (3.0.1): given two elements $(A,\mathscr{D})$ and $(B,\mathscr{B})$ in $\mathbb{P}_{FIN}$, we let $(B,\mathscr{B}) \leqslant_{FIN} (A,\mathscr{D})$ if and only if $A$ is an initial subsequence of $B$, $\mathscr{B} \sqsubseteq \mathscr{D}$ and $\forall x \in B\setminus A\ (x \in FU(\mathscr{D}))$.

From now on, $\mathscr{C}$ is a fixed set that is large for $\langle\{i\}\rangle_{i\in\omega}$, and $\mathbb{P}_{\mathscr{C}} := \mathbb{P}_{\mathscr{C},\langle\{i\}\rangle_{i\in\omega}}$.

**Definition 4.1.** *Let $\Delta = \Delta_{\mathscr{C}}$ be the set of all pairs $(A,\mathscr{D})$ such that $A$ is a finite block sequence and $\mathscr{D}$ is an infinite block sequence with $A < \mathscr{D}$ and $FU(A^\frown\mathscr{D}) \subseteq \mathscr{C}$. Thus, $\Delta \subseteq \mathbb{P}_{\mathscr{C}}$.*

**Lemma 4.2.** *$\Delta$ is a dense subset of $\mathbb{P}_{\mathscr{C}}$.*

*Proof.* Given $(A,\mathscr{D}) \in \mathbb{P}_{\mathscr{C}}$, let

$$\mathscr{C}' = \{y \in FU(\mathscr{D}) \cap \mathscr{C} : \forall x \in FU(A)(x \cup y \in \mathscr{C})\}.$$

By definition of $\mathbb{P}_{\mathscr{C}}$, $\mathscr{C}'$ is large for $\mathscr{D}$, and by the Theorem 3.11 there exists $\mathscr{D}' \sqsubseteq \mathscr{D}$ such that $A < \mathscr{D}'$ and $FU(\mathscr{D}') \subseteq \mathscr{C}'$. Clearly, $(A, \mathscr{D}') \leqslant_{\mathscr{C}} (A, \mathscr{D})$.

We check that $(A, \mathscr{D}') \in \Delta$. Note that $FU(A) \subseteq \mathscr{C}$, by definition of $\mathbb{P}_{\mathscr{C}}$. And $FU(\mathscr{D}') \subseteq \mathscr{C}' \subseteq \mathscr{C}$. If $x \in FU(A)$ and $y \in FU(\mathscr{D}')$, then since $FU(\mathscr{D}') \subseteq \mathscr{C}'$, we have that $x \cup y \in \mathscr{C}$. That is, $FU(A^\frown \mathscr{D}') \subseteq \mathscr{C}$.

□

**Lemma 4.3.** *If $(A, \mathscr{D}) \in \Delta$ and $(A', \mathscr{D}') \leqslant_{FIN} (A, \mathscr{D})$, then $(A', \mathscr{D}') \in \Delta$.*

*Proof.* Suppose $(A, \mathscr{D}) \in \Delta$ and $(A', \mathscr{D}') \leqslant_{FIN} (A, \mathscr{D})$.

Let $x \in FU(A')$ and $y \in FU(\mathscr{D}')$. We must conclude that $x \cup y \in \mathscr{C}$. Since $x \in FU(A')$ there are $z \in FU(A)$ and $u \in FU(\mathscr{D})$ such that $x = z \cup u$. Thus $u \cup y \in FU(\mathscr{D})$. But then, since $(A, \mathscr{D}) \in \Delta$ and $z \in FU(A)$, $z \cup (u \cup y) = x \cup y \in \mathscr{C}$, as desired.

□

**Proposition 4.4.** *If $(A, \mathscr{D}) \in \Delta$ and $(A', \mathscr{D}') \leqslant_{FIN} (A, \mathscr{D})$, then*

$$(A', \mathscr{D}') \leqslant_{\mathscr{C}} (A, \mathscr{D}).$$

*Proof.* Assume that $(A, \mathscr{D}) \in \Delta$ and $(A', \mathscr{D}') \leqslant_{FIN} (A, \mathscr{D})$. Since $\Delta \subseteq \mathbb{P}_{\mathscr{C}}$, to show that $(A', \mathscr{D}') \leqslant_{\mathscr{C}} (A, \mathscr{D})$ it is enough to see that $(A', \mathscr{D}') \in \Delta$, which is the case by Lemma 4.3. □

**Corollary 4.5.** *If $(A, \mathscr{D}) \in \Delta$ then $\mathbb{P}_{FIN} \restriction (A, \mathscr{D}) = \mathbb{P}_{\mathscr{C}} \restriction (A, \mathscr{D})$.*

*Proof.* By Proposition 4.4, owing to the fact that the identity is an embedding of $\mathbb{P}_{\mathscr{C}}$ into $\mathbb{P}_{FIN}$.

□

**Remark 4.6.** *Suppose $G$ is a $\mathbb{P}_{\mathscr{C}}$-generic filter over $V$, and let $(A, \mathscr{D}) \in \Delta \cap G$ (there is such $(A, \mathscr{D})$ by Lemma 4.2). Thus $G \restriction (A, \mathscr{D})$ is a $\mathbb{P}_{\mathscr{C}} \restriction (A, \mathscr{D})$-generic filter over $V$ and $G$ is the filter generated by $G \restriction (A, \mathscr{D})$. By corollary 4.5, $G \restriction (A, \mathscr{D})$ is a $\mathbb{P}_{FIN} \restriction (A, \mathscr{D})$-generic filter over $V$. Let $H$ the filter on $\mathbb{P}_{FIN}$ generated by $G \restriction (A, \mathscr{D})$. Then, $H$ is a $\mathbb{P}_{FIN}$-generic filter over $V$ and $H \restriction (A, \mathscr{D}) = G \restriction (A, \mathscr{D})$, hence $V[H] = V[G]$.*

By the preceding remark for a dense subset of conditions $(A', \mathscr{D}')$ of $\mathbb{P}_{\mathscr{C},\mathscr{D}}$ we have $\mathbb{P}_{\mathscr{C},\mathscr{D}} \restriction (A', \mathscr{D}') \approx \mathbb{P}_{FIN} \restriction (A', \mathscr{D}')$, thus studying $\mathbb{P}_{FIN}$ allow us to know $\mathbb{P}_{\mathscr{C},\mathscr{D}}$, where $\mathscr{C}$ is large for $\mathscr{D}$, because of this, from now on we restrict our attention to $\mathbb{P}_{FIN}$.

The first question that we ask is: What is the relation between Mathias forcing and $\mathbb{P}_{FIN}$? Are they equivalent? If we look carefully at the definition of $\mathbb{P}_{FIN}$ we may also ask about the relation between Matet forcing and $\mathbb{P}_{FIN}$. Are they equivalent? In order to answer these questions, we need to discover what properties does $\mathbb{P}_{FIN}$ have.

Let $G$ be a $\mathbb{P}_{FIN}$ generic filter over some ground model. Then

$$\mathscr{D}^*_G = \bigcup\{A : \exists \mathscr{D}(A, \mathscr{D}) \in G\}$$

is an infinite block sequence $\langle a^G_i \rangle_{i \in \omega}$.

Like the Mathias real, the generic block sequence $\langle a^G_i \rangle_{i \in \omega}$ reconstructs the generic filter.

Given a finite block sequence $A$ and an infinite block sequence $\mathscr{D}$, we define $\mathscr{D} - A$ as the infinite block sequence whose elements $d$ are such that for all $a \in A$, $a < d$.

**Lemma 4.7.** *The filter $G$ is determined by $\mathscr{D}^* = \mathscr{D}^*_G$, as $G$ is precisely the set*

$$G_{\mathscr{D}^*} := \{(A, \mathscr{D}) \in \mathbb{P}_{FIN} : A \text{ is an initial block sequence of } \mathscr{D}^* \text{ and } \mathscr{D}^* - A \sqsubseteq \mathscr{D}\}.$$

*Proof.* It is clear that $G_{\mathscr{D}^*}$ is a filter.

So let's see that $G = G_{\mathscr{D}^*}$ .

Let $(A, \mathscr{D}) \in G$. Since $\mathscr{D}^* = \bigcup\{A : \exists \mathscr{D}(A, \mathscr{D}) \in G\}$, we have $A$ is a finite initial part of $\mathscr{D}^*$. Let $x \in \mathscr{D}^* - A$. There is $(B, \mathscr{B}) \in G$ such that $x \in B$. Since $G_{\mathscr{D}^*}$ is a filter, there is $(C, \mathscr{C}) \leqslant (A, \mathscr{D}), (B, \mathscr{B})$ so $x \in C$ and on the other hand $A < x$ which implies $x \in C \setminus A$. Thus $x \in FU(\mathscr{D})$ and $(A, \mathscr{D}) \in G_{\mathscr{D}^*}$.

Let $(A, \mathscr{D}) \in G_{\mathscr{D}^*}$. We will show that $(A, \mathscr{D})$ is compatible with all $(B, \mathscr{D}')$ in $G$. Let $(B, \mathscr{D}') \in G$, in particular we have that $B$ is an initial part of $\mathscr{D}^*$, so $A \subseteq B$ or $B \subseteq A$. Assume that $A \subseteq B$. $\mathscr{D}^* - A \sqsubseteq \mathscr{D}$ and since $(B, \mathscr{D}') \in G$ we obtain $\mathscr{D}^* - B \subseteq FU(\mathscr{D}')$, in particular $FU(\mathscr{D}) \cap FU(\mathscr{D}')$ is infinite. We can obtain an infinite block sequence of elements in $FU(\mathscr{D}) \cap FU(\mathscr{D}')$, $\mathscr{E}$, so $\mathscr{E} \sqsubseteq \mathscr{D}, \mathscr{D}'$ then $(B, \mathscr{E}) \leqslant (A, \mathscr{D}), (B, \mathscr{D}')$. Therefore $(A, \mathscr{D}) \in G$. We have proved the claim.

□

## 4.1 $\mathbb{P}_{FIN}$ is not equivalent to Matet forcing

$\mathbb{P}_{FIN}$ and Matet forcing are not equivalent forcing notions. This is a consequence of the following.

**Lemma 4.8.** *Let $f$ be a function from $[\omega]^{<\omega}$ to $\omega$ in $V$ such that $f[\mathscr{D}]$ is infinite for all $\mathscr{D} \in (FIN)^\omega$ and such that for all $\mathscr{D} \in (FIN)^\omega$, $f[FU(\mathscr{D})] \subseteq f[\mathscr{D}]$. Suppose $\mathscr{D}^*$ is a block sequence that is $\mathbb{P}_{FIN}$-generic over $V$. Then $x = f[\mathscr{D}^*]$ is a Mathias real over $V$.*

*Proof.* Let $\mathscr{A} \subseteq [\omega]^\omega$ be a maximal almost disjoint family in $V$ and suppose $G$ is a $\mathbb{P}_{FIN}$-generic filter over $V$. Let

$$D = \{(A, \mathscr{D}) \in \mathbb{P}_{FIN} : \exists a \in \mathscr{A} \text{ such that } f[\mathscr{D}] \subseteq a\}.$$

**Claim 4.9.** *$D$ is a dense subset of $\mathbb{P}_{FIN}$.*

*Proof of Claim:* Let $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$. Since $|f[\mathscr{D}]| = \aleph_0$, there is $a \in \mathscr{A}$ such that $|a \cap f[\mathscr{D}]| = \aleph_0$. Consider $\mathscr{E} \sqsubseteq \mathscr{D}$ such that $f[\mathscr{E}] \subseteq a$, then $(A, \mathscr{E}) \leqslant (A, \mathscr{D})$ and $(A, \mathscr{E}) \in D$. This proves the claim. □

Thus $G \cap D \neq \emptyset$, and so there is $(A, \mathscr{D}) \in G \cap D$, which implies the existence of an element $b \in \mathscr{A}$ such that $f[\mathscr{D}] \subseteq b$.

Assume $|A| = m$. For all $k > m$, $d_k^G \in FU(\mathscr{D})$ where $\mathscr{D}^* = \langle d_i^G \rangle_{i \in \omega}$ is the $\mathbb{P}_{FIN}$-generic block sequence added by $G$. We have $f(d_k^G) \in f[\mathscr{D}] \subseteq b$.

Letting $x = f[\mathscr{D}^*]$, we have that $x \subseteq^* b$. Thus $x$ is a Mathias real by Theorem 1.25. $\square$

**Corollary 4.10.** *If $\mathscr{D}^*$ is the generic infinite block sequence added by $\mathbb{P}_{FIN}$, over $V$, then the images of $\mathscr{D}^*$ under the functions* min *and* max *are Mathias reals over $V$.*

**Corollary 4.11.** *$\mathbb{P}_{FIN}$ is not equivalent to Matet forcing.*

*Proof.* Since $\mathbb{P}_{FIN}$ adds a Mathias real, we have that $\mathbb{P}_{FIN}$ adds a dominating real by Lemma 1.23. Thus it adds a splitting real, see Proposition 3 in [11], but Matet forcing does not add splitting reals, by Corollary 1.48.

$\square$

Note that given a natural number $n \geqslant 1$, the set $D_n$ defined as

$$\{(A, \mathscr{D}) \in \mathbb{P}_{FIN} : \forall x \in \mathscr{D}(|x| \geqslant n)\} \tag{4.1.1}$$

is a dense set.

Let $k \in \omega$. We define a function $f_k : FIN \to \omega$ as follows: for $x \in FIN$, $f_k(x)$ is equal to the $k+1$-th element of $x$, if $|x| > k$, or equal to 0, otherwise.

**Lemma 4.12.** *Let $k \in \omega$. Suppose $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ is such that $|x| > k$ for each $x \in \mathscr{D}$. Suppose $G$ is a $\mathbb{P}_{FIN}$-generic filter over $V$ is such that $(A, \mathscr{D}) \in G$. Let $m_0$ be the Mathias-generic real over $V$ obtained by applying the function $f_0$ to $\mathscr{D}^*$, where $\mathscr{D}^*$ is the $\mathbb{P}_{FIN}$-generic block sequence added by $G$, and let $m_k$ be the Mathias real obtained by applying the function $f_k$ to $\mathscr{D}^*$. Then $V[m_0] = V[m_k]$.*

*Proof.* Suppose $(A, \mathscr{D}) \in G$, where $A = \langle a_0, \dots, a_m \rangle$ and $m_0 = \langle x_i \rangle_{i \in \omega}$. Then we define the sequence $\langle y_i \rangle_{i \in \omega}$ as follows: $y_j = f_k(a_j)$ for all $j \leqslant m$, and for $j > m$, $y_j$ is the $k$-th element of the unique element $d \in \mathscr{D}$ such that $x_j \in d$. Then $m_k = \langle y_i \rangle_{i \in \omega} \in V[m_0]$, so $V[m_k] \subseteq V[m_0]$.

Assume that we have $m_k = \langle y_i \rangle_{i \in \omega}$. Then we define the sequence $\langle x_i \rangle_{i \in \omega}$ as follows: $x_i = f_0(a_i)$ for all $i \leqslant m$, and for all $i > m$, $x_i$ is the minimum element of the unique set $d \in \mathscr{D}$ such that $y_i \in d$. Then $m_0 = \langle x_i \rangle_{i\in\omega} \in V[m_k]$. So $V[m_0] \subseteq V[m_k]$. □

**Lemma 4.13.** *If $G$ is a $\mathbb{P}_{FIN}$ generic filter over $V$, then $V[G] = V[\langle m_i \rangle_{i\in\omega}]$, where for each $i \in \omega$, $m_i$ is the image of $\mathscr{D}^*$, the generic block sequence, under the function $f_i$.*

*Proof.* Let $\mathscr{D}^* = \langle d_i^G \rangle_{i\in\omega}$ be the infinite block sequence added by $G$. For each $k \in \omega$ we apply the function $f_k : [\omega]^{<\omega} \to \omega$ to each element of $\mathscr{D}^*$, thus obtaining a sequence $m_k = \langle m_j^k \rangle_{j\in\omega}$, where $m_j^k = f_k(d_j^G)$ for all $j \in \omega$. Clearly $\langle m_k \rangle_{k\in\omega} \in V[G]$, and so $V[\langle m_k \rangle_{k\in\omega}] \subseteq V[G]$.

We shall show that $G \in V[\langle m_i \rangle_{i\in\omega}]$.

Define, for $k \in \omega$,

$$d'_k := \{m_k^i : i \in \omega\}.$$

Note that $|d'_k| < \aleph_0$ for all $k \in \omega$. For $k \in \omega$, let $d_k = d'_k \setminus \{0\}$. We have that for all $k \in \omega$, $d_k < d_{k+1}$, i.e, $\max d_k < \min d_{k+1}$. Put $\mathscr{D}^* = \langle d_k \rangle_{k\in\omega}$. We have that $\mathscr{D}^* \in V[\langle m_i \rangle_{i\in\omega}]$ and $G = G_{\mathscr{D}^*}$, by Lemma 4.7 $G \in [V\langle m_i \rangle_{i\in\omega}]$ follows. □

**Remark 4.14.** *Note that by Lemma 4.12 all $V[\langle m_i \rangle_{i<k}]$ (for arbitrary $k$) are the same Mathias extension. By Lemma 4.13 $V[\langle m_i \rangle_{i\in\omega}]$ is the $\mathbb{P}_{FIN}$ extension. We will see later (in section 5.2), that the two are not equivalent, i.e., any $V[\langle m_i \rangle_{i<k}]$ is strictly contained in $V[\langle m_i \rangle_{i\in\omega}]$.*

## 4.2 $\mathbb{P}_{FIN}$ decomposed as a two-step iteration

In [5] Andreas Blass defines the notion of stable ordered-union filter, and in [8] Tod Eisworth defines the notion of Matet-adequate family. Both of them prove that forcing with this kind of families adjoins a stable-ordered union ultrafilter $\mathscr{U}$. See Definition 1.35, Definition 1.36 and Proposition 1.38 in the Preliminaries section above.

**Definition 4.15.** *Suppose $\mathscr{U}$ is a stable ordered-union ultrafilter. We define a partial order $\mathbb{P}_{\mathscr{U}}$ as follows: the conditions are pairs $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ such that $FU(\mathscr{D}) \in \mathscr{U}$. The ordering relation on $\mathbb{P}_{\mathscr{U}}$ is the same as in $\mathbb{P}_{FIN}$.*

Notice that any two conditions with the same first coordinate are compatible, so $\mathbb{P}_{\mathscr{U}}$ is $\sigma-$centered.

In 1.2.2 of the Preliminaries section we defined the partial ordering $\mathbb{P}^* = \langle (FIN)^\omega, \sqsubseteq^* \rangle$.

$\mathbb{P}^* = \langle (FIN)^\omega, \sqsubseteq^* \rangle$ is a Matet-adequate family, see Definition 1.36. Let $G$ be a $\mathbb{P}^*$-generic filter, by Proposition 1.38 we add $\mathscr{U}$ a stable ordered-union ultrafilter in the generic extension.

**Lemma 4.16.** *Let $\dot{\mathscr{U}}$ be the canonical $\mathbb{P}^*$-name for the $\mathbb{P}^*$-generic object, then $\mathbb{P}_{FIN} \approx \mathbb{P}^* * \mathbb{P}_{\dot{\mathscr{U}}}$.*

*Proof.* We have that

$$\mathbb{P}^* * \mathbb{P}_{\dot{\mathscr{U}}} = \{(\mathscr{E}, (\dot{A}, \dot{\mathscr{D}})) : \mathscr{E} \in \mathbb{P}^* \text{ and } \mathscr{E} \Vdash_{\mathbb{P}^*} (\dot{A}, \dot{\mathscr{D}}) \in \mathbb{P}_{\dot{\mathscr{U}}}\}.$$

Let $h : \mathbb{P}_{FIN} \to \mathbb{P}^* * \mathbb{P}_{\dot{\mathscr{U}}}$ be defined by: $h(A, \mathscr{D}) = (\mathscr{D}, (\check{A}, \check{\mathscr{D}}))$.

We shall prove that $h$ is a dense embedding.

It is clear that $h$ preserves $\leqslant$.

Let $(\mathscr{E}, (\dot{A}, \dot{\mathscr{D}})) \in \mathbb{P}^* * \mathbb{P}_{\dot{\mathscr{U}}}$. Since $\mathbb{P}^*$ is $\sigma$-closed (Proposition 1.33), there is $\mathscr{E}' \sqsubseteq \mathscr{E}$ such that

$$\mathscr{E}' \Vdash \text{“}\dot{A} = \check{A} \text{ and } \dot{\mathscr{D}} = \check{\mathscr{D}}\text{” for some } A \text{ and } \mathscr{D}.$$

We have $(\mathscr{E}', (\check{A}, \check{\mathscr{D}})) \in \mathbb{P}^* * \mathbb{P}_{\dot{\mathscr{U}}}$, and $(\mathscr{E}', (\check{A}, \check{\mathscr{D}})) \leqslant (\mathscr{E}, (\dot{A}, \dot{\mathscr{D}}))$. If $G$ is a $\mathbb{P}^*$-generic filter over $V$ such that $\mathscr{E}' \in G$ then $(A, \mathscr{D}) \in \mathbb{P}_{\mathscr{U}}$. So, $FU(\mathscr{D}) \in \mathscr{U}$, and there is $\mathscr{B}$ such that $\mathscr{B} \sqsubseteq \mathscr{E}'$ and $\mathscr{B} \sqsubseteq \mathscr{D}$. We have

$$h((A, \mathscr{B})) = (\mathscr{B}, (\check{A}, \check{\mathscr{B}})) \leqslant (\mathscr{E}, (\dot{A}, \dot{\mathscr{D}})).$$

So, $h$ is a dense embedding. □

**Proposition 4.17.** *Let $G$ be a $\mathbb{P}^*$-generic filter over $V$. In $V[G]$ we define*

$$\mathscr{V} = \{v_{\mathscr{D}} \subseteq \omega : v_{\mathscr{D}} = \{\min d_i : i \in \omega\} \text{ for some } \mathscr{D} = \langle d_i \rangle_{i\in\omega} \text{ in } G\}.$$

*Then $\mathscr{V}$ is an ultrafilter on $\omega$.*

*Proof.* We shall prove first that $\mathscr{V}$ is upward closed. Let $v_{\mathscr{D}} \in \mathscr{V}$ for some $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$ in $G$ and suppose $Y \supseteq v_{\mathscr{D}}$. We shall find a $\mathscr{D}' = \langle d'_i \rangle_{i\in\omega}$ such that $\mathscr{D} \sqsubseteq^* \mathscr{D}'$ and $Y = v_{\mathscr{D}'}$.

Let $Y \in [\omega]^\omega$ such that $v_{\mathscr{D}} \subseteq Y$.

- If $y \in Y$ and $y \notin \cup\mathscr{D}$, then we define $d_y := \{y\}$.
- If $y$ is such that $Y \cap d = \{y\}$ for some $d$ element of $\mathscr{D}$, then $d_y := d$.
- If $Y \cap d = \{y_0, \ldots, y_k\}$ for some $d$ element of $\mathscr{D}$, where $k \geqslant 1$, then let $d_y := \{x \in d : y_i \leqslant x < y_{i+1}\}$ if $y = y_i$ and $i < k$ and $d_y = \{x \in d : y_k \leqslant x\}$ if $y = y_k$.

Note that $y = \min d_y$, for all $y \in Y$.

Let $\mathscr{D}' = \langle d_y \rangle_{y\in Y}$. Thus, $\mathscr{D} \sqsubseteq^* \mathscr{D}'$. Since $G$ is a filter, $\mathscr{D}' \in G$, and

$$Y = \{\min d_y : y \in Y\} = v_{\mathscr{D}'}.$$

Let us now check that $G$ is pairwise compatible. So, let $v_{\mathscr{D}}$ and $v_{\mathscr{D}'}$ in $\mathscr{V}$. Since $\mathscr{D}$ and $\mathscr{D}'$ are in $G$ there is $\mathscr{E} \in G$ that extends both $\mathscr{D}$ and $\mathscr{D}'$. Let $\mathscr{E} = \langle e_i \rangle_{i\in\omega}$. Thus, $v_{\mathscr{E}} = \{\min e_i : i \in \omega\} \in \mathscr{V}$. Since $v_{\mathscr{E}} \subseteq v_{\mathscr{D}} \cap v_{\mathscr{D}'}$, we have $v_{\mathscr{D}} \cap v_{\mathscr{D}'} \in \mathscr{V}$. Hence $\mathscr{V}$ is a filter on $\omega$.

Let $Y \in [\omega]^\omega$ and

$$X = \{d \in FIN : \min(d) \in Y\}.$$

Since $\mathscr{U}_G$ is an ultrafilter we have that $X \in \mathscr{U}_G$ or $(FIN \setminus X) \in \mathscr{U}_G$. If $X \in \mathscr{U}_G$, then there is $\mathscr{D}' \in G$ such that $FU(\mathscr{D}') \subseteq X$. Then

$$v_{\mathscr{D}'} = \{\min d'_i : i \in \omega\} \subseteq Y$$

and so $Y \in \mathscr{V}$. The case $FIN \setminus X \in \mathscr{U}_G$ is similar. Hence $\mathscr{V}$ is an ultrafilter on $\omega$.

□

Notice that since $\mathbb{P}^*$ is $\sigma$-closed, forcing with $\mathbb{P}^*$ does not add any new subsets of $\omega$.

**Proposition 4.18.** *Let $\mathbb{U} = \langle \mathscr{P}(\omega)/fin, \leqslant \rangle$ be defined as in Section 1.2. Then $\mathscr{V}$ as defined above, is a $\mathbb{U}$-generic filter over $V$.*

*Proof.* We already proved that it is an ultrafilter. So, it only remains to prove that $\mathscr{V}$ is $\mathbb{U}$-generic. Let $D \subseteq \mathbb{U}$ be an open and dense set and consider

$$D' = \{\mathscr{D} = \langle d_i \rangle_{i \in \omega} \in (FIN)^\omega : \{\min d_i : i \in \omega\} \in D\}.$$

**Claim 4.19.** *$D'$ is dense a subset of $(FIN)^\omega$.*

*Proof of claim:* Let $\mathscr{D} = \langle d_i \rangle_{i \in \omega} \in (FIN)^\omega$, and let

$$x = \{\min d_i : i \in \omega\} \in [\omega]^\omega.$$

Since $D$ is dense there is $A \in D$ such that $A \leqslant x$, i.e., $A \subseteq^* x$. Note that $A \setminus (A \setminus x) \in D$ because $D$ is open. Let $\mathscr{E} = \langle d_i \rangle_{i \in I}$ where $I = A \cap x$. Then we have $\mathscr{E} \sqsubseteq^* \mathscr{D}$ and $\mathscr{E} \in D'$. Hence $D'$ is dense. □

We have $D' \cap G \neq \emptyset$, i.e., there is $\mathscr{D} = \langle d_i \rangle_{i \in \omega} \in D'$ such that $\mathscr{D} \in G$. Let

$$v_{\mathscr{D}} = \{\min d_i : i \in \omega\}.$$

So $v_{\mathscr{D}} \in D \cap \mathscr{V}$. Therefore, $\mathscr{V}$ is an $\mathbb{U}$-generic filter. □

**Corollary 4.20.** *$\mathscr{V}$ is a selective ultrafilter on $\mathbb{U}$.*

*Proof.* All $\mathbb{U}$-generic filters are selective, by Lemma 1.22 in Section 1.2 . □

## 4.3 $\mathbb{P}_{FIN}$ satisfies Axiom A

**Lemma 4.21.** *For every $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ such that*

$$(A, \mathscr{D}) \Vdash \text{“}\tau \in V\text{”}$$

*there exists* $\mathscr{D}^* \sqsubseteq \mathscr{D}$ *and* $x \in V$ *countable such that*

$$(A, \mathscr{D}^*) \Vdash \tau \in \check{x}.$$

*Moreover,* $\mathscr{D}^*$ *may be chosen so that if* $(B, \mathscr{B}) \leqslant (A, \mathscr{D}^*)$*,* $a \in V$ *and* $(B, \mathscr{B}) \Vdash \tau = \check{a}$*, then* $(B, \mathscr{D}^* - B) \Vdash \tau = \check{a}$*.*

*Proof.* We will construct, by induction, a sequence $\langle \mathscr{D}_i \rangle_{i \in \omega}$ such that $\mathscr{D}_{i+1} \sqsubseteq \mathscr{D}_i$, for all $i \in \omega$, and $e_i < e_{i+1}$, for all $i \in \omega$, where $e_i$ is the first element of the block sequence $\mathscr{D}_i$. Let $\mathscr{D}_0 = \mathscr{D}$. Given $\mathscr{D}_n$ consider all finite block sequences $s_0, \ldots, s_k$ of elements of $FU(\{e_i : i < n\})$. We shall obtain a sequence

$$\mathscr{D}_0^n \sqsupseteq \ldots \sqsupseteq \mathscr{D}_{k+1}^n$$

as follows:

Let $\mathscr{D}_0^n = \mathscr{D}_n$. Given $\mathscr{D}_j^n$, if there is $\mathscr{E} \sqsubseteq \mathscr{D}_j^n$ and $a \in V$ such that

$$(A^\frown s_j, \mathscr{E}) \Vdash \tau = \check{a}$$

then $\mathscr{D}_{j+1}^n = \mathscr{E}$. If there is none, then $\mathscr{D}_{j+1}^n = \mathscr{D}_j^n$. Let $e_n$ the first element of the block sequence $\mathscr{D}_{k+1}^n$ and let $\mathscr{D}_{n+1} := \mathscr{D}_{k+1}^n - \langle e_n \rangle$.

Consider the infinite block sequence $\mathscr{D}^* = \langle e_i \rangle_{i \in \omega}$. We define

$$x = \{a \in V : \exists Y \text{ finite block sequence of elements in } FU(\mathscr{D}^*) \text{ such that}$$

$$(A^\frown Y, \mathscr{D}^* - Y) \Vdash \tau = \check{a}\}.$$

It is clear that $x$ is countable.

**Claim 4.22.** $(A, \mathscr{D}^*) \Vdash \tau \in \check{x}$.

*Proof of Claim:* Note that $(A, \mathscr{D}^*) \leqslant (A, \mathscr{D})$. Since

$$(A, \mathscr{D}) \Vdash \text{“}\tau \in V\text{”}$$

there is $(B, \mathscr{D}') \leqslant (A, \mathscr{D}^*)$ and $a \in V$ such that

$$(B, \mathscr{D}') \Vdash \tau = \check{a}.$$

Let $Y = B - A$. Thus, $Y$ is a finite block sequence of elements in $FU(\mathscr{D}^*)$. Suppose that $Y$ is a block sequence of elements in $FU(\{e_i : i < n\})$, then $Y = s_j$ for some $j$ in the $n$-th step. We have that

$$(A^\frown s_j, \mathscr{D}^n_{j+1}) \Vdash \tau = \check{a}'$$

for some $a' \in V$. Since $(A^\frown Y, \mathscr{D}') \leqslant (A^\frown s_j, \mathscr{D}^n_{j+1})$, we have $a = a'$ and

$$(A^\frown Y, \mathscr{D}^* - Y) \leqslant (A^\frown s_j, \mathscr{D}^n_{j+1})$$

so $a \in x$. Thus $(A, \mathscr{D}^*) \Vdash \tau \in \check{x}$. We have proved the claim. □

We shall now prove the last part of the lemma. Let $(B, \mathscr{B}) \leqslant (A, \mathscr{D}^*)$ and $a \in V$ be such that $(b, \mathscr{B}) \Vdash \tau = \check{a}$. We proceed as in the last part of the proof of Claim 4.22. Letting $Y = B - A$, we obtain $(A^\frown Y, \mathscr{D}^* - Y) \Vdash \tau = \check{a}$. □

**Theorem 4.23.** *The partial ordering $\mathbb{P}_{FIN}$ satisfies Axiom A.*

*Proof.* Let $n \geqslant 1$. We define $(B, \mathscr{D}') \leqslant_n (A, \mathscr{D})$ if and only if $(B, \mathscr{D}') \leqslant (A, \mathscr{D})$, $A = B$ and the infinite block sequences $\mathscr{D}$ and $\mathscr{D}'$ have the same initial block sequence of size $n$. We define $(B, \mathscr{D}') \leqslant_0 (A, \mathscr{D})$ if and only if $(B, \mathscr{D}') \leqslant (A, \mathscr{D})$.

It is clear that $\mathbb{P}_{FIN}$ satisfies:

(1) for all $p, q \in P$, $p \leqslant_0 q$ if and only if $p \leqslant q$ and

(2) for all $p, q \in P$ if $p \leqslant_{n+1} q$, then $p \leqslant_n q$ for all $n \in \omega$

of the definition of Axiom A (Definition 1.19 in Section 1.2).

We now show that $\mathbb{P}_{FIN}$ satisfies condition (3). Let $\langle (A_i, \mathscr{D}_i) : i \in \omega \rangle$ be a sequence of elements in $\mathbb{P}_{FIN}$ such that

$$(A_{i+1}, \mathscr{D}_{i+1}) \leqslant_i (A_i, \mathscr{D}_i)$$

for all $i \in \omega$, where $\mathscr{D}_i = \langle d^i_j \rangle_{j \in \omega}$ for all $i \in \omega$.

We define $A = A_1$ and $\mathscr{D} = \langle d^j_{j-1} \rangle_{1 \leqslant j}$. Note that $d^1_0$ is the first element of $\mathscr{D}_1$. Proceeding inductively, if $d^i_0, \ldots, d^i_{i-1}$ are the first $i$ elements of $\mathscr{D}_i$, then $d^1_0, \ldots, d^{i+1}_i$ are the first $i + 1$ elements of $\mathscr{D}_{i+1}$.

$$\begin{array}{ccccc}
d_0^1 & & & & \\
\| & & & & \\
d_0^2 & d_1^2 & & & \\
\| & \| & & & \\
d_0^3 & d_1^3 & d_2^3 & & \\
\| & \| & \| & & \\
d_0^4 & d_1^4 & d_2^4 & d_3^4 & \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Then, for each $i$, since $\mathscr{D} \sqsubseteq \mathscr{D}_i$, we have that $\mathscr{D} \sqsubseteq_i \mathscr{D}_i$ and $(A, \mathscr{D}) \leqslant_i (A_i, \mathscr{D}_i)$, for all $i \in \omega$. Thus $\mathbb{P}_{\mathscr{C}}$ satisfies (3) of the definition of Axiom A.

We finally show that $\mathbb{P}_{FIN}$ satisfies $(4')$. Let $n \in \omega$ and $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ be such that

$$(A, \mathscr{D}) \Vdash \text{“}\tau \in V\text{”}.$$

Let $Y = \langle d_0, \ldots, d_{n-1} \rangle$, the initial block sequence of size $n$ of $\mathscr{D}$, and let $s_0, \ldots, s_k$ be an enumeration of all block sequences that we can form with elements of the set $FU(\{d_0, \ldots, d_{n-1}\})$. We let $s_0$ to be the empty sequence.

Let $\mathscr{D}_0 = \mathscr{D} - Y$. Given $\mathscr{D}_j$, since

$$(A, \mathscr{D}) \Vdash \text{“}\tau \in V\text{”}$$

we have $(A^\frown s_j, \mathscr{D}_j) \Vdash$ “$\tau \in V$”. By Lemma 4.21 there exist $\mathscr{D}_{j+1} \sqsubseteq \mathscr{D}_j$ and $x_j \in V$ countable such that

$$(A^\frown s_j, \mathscr{D}_{j+1}) \Vdash \tau \in \check{x}_j.$$

Let $x = x_0 \cup \ldots \cup x_k$ and $\mathscr{D}^* = Y^\frown \mathscr{D}_{k+1}$. It is clear that $(A, \mathscr{D}^*) \leqslant_n (A, \mathscr{D})$.

**Claim 4.24.** *$(A, \mathscr{D}^*) \Vdash \tau \in \check{x}$.*

*Proof of Claim:* Let $(B, \mathscr{D}'') \leqslant (A, \mathscr{D}^*)$. Then, $Z := B - A$ is a finite block sequence of elements in $FU(\mathscr{D}^*)$. If $Z$ is a block sequence of elements in the set

$$FU(\{d_0, \ldots, d_{n-1}\})$$

then $Z = s_j$ for some $j \in \{0, \dots, k\}$. Then $(A^\frown Z, \mathscr{D}'') \leqslant (A^\frown s_j, \mathscr{D}_{j+1})$. Since

$$(A^\frown s_j, \mathscr{D}_{j+1}) \Vdash \tau \in \check{x}_j$$

we have $(A^\frown Z, \mathscr{D}'') \Vdash \tau \in \check{x}$. Similarly, if $Z'$ is an initial subsequence of $Z$ and $Z' = s_j$ for some $j$. If $Z$ is a finite block sequence with elements in $FU(\mathscr{D}^* - Y)$, then $(A^\frown Z, \mathscr{D}'') \leqslant (A^\frown s_0, \mathscr{D}_1)$ and $(A^\frown s_0, \mathscr{D}_1) \Vdash \tau \in \check{x}_0$, then $(A^\frown Z, \mathscr{D}'') \Vdash \tau \in \check{x}$. □

This proves the claim and the lemma. □

## 4.4 $\mathbb{P}_{FIN}$ satisfies the Pure Decision Property

**Theorem 4.25.** *Let $\varphi$ be a sentence of the language of forcing with $\mathbb{P}_{FIN}$. For any $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ there is $\mathscr{D}^* \sqsubseteq \mathscr{D}$ such that $(A, \mathscr{D}^*) \Vdash \varphi$ or $(A, \mathscr{D}^*) \Vdash \neg\varphi$.*

*Proof.* Let $\tau$ be a name such that if $G$ is a $\mathbb{P}_{FIN}$-generic filter, then $\tau_G = 0$ if $\varphi$ holds in $V[G]$ and $\tau_G = 1$ if $\neg\varphi$ holds in $V[G]$, for example

$$\tau = \{(\check{0}, (B, \mathscr{B})) : (B, \mathscr{B}) \Vdash \neg\varphi\}.$$

Note that $\Vdash \tau \in \{0, 1\}\check{}$ and for all $(B, \mathscr{B}) \in \mathbb{P}_{FIN}$, $(B, \mathscr{B}) \Vdash \tau = \check{0}$ if and only if $(B, \mathscr{B}) \Vdash \varphi$. And $(B, \mathscr{B}) \Vdash \tau = \check{1}$ if and only if $(B, \mathscr{B}) \Vdash \neg\varphi$.

By Theorem 4.21, there is $\mathscr{D}'$ such that if $(E, \mathscr{E}) \leqslant (A, \mathscr{D}')$ and

$$(E, \mathscr{E}) \Vdash \tau = \check{0}$$

then $(E, \mathscr{D}' - E) \Vdash \tau = \check{0}$, so $(E, \mathscr{D}' - E) \Vdash \varphi$, respectively $\neg\varphi$.

For the rest of the proof, if $Y$ is a finite block sequence of elements in $FU(\mathscr{D}')$, let $Y \Vdash \phi$ abbreviate $(A^\frown Y, \mathscr{D}' - Y) \Vdash \varphi$, and similarly for $\neg\varphi$.

We are going to construct elements $e_0 < e_1 < \dots$ of $FU(\mathscr{D}')$ and block sequences $\mathscr{D}_0 \sqsupseteq \mathscr{D}_1 \sqsupseteq \dots$ such that $\mathscr{D}_j \sqsubseteq \mathscr{D}'$ by induction, as follows:

Let $\mathscr{D}_0 = \mathscr{D}'$. Given $\mathscr{D}_n$ find $\mathscr{D}'_{n+1} \sqsubseteq \mathscr{D}_n$ such that $FU(\mathscr{D}'_{n+1}) \subseteq \mathscr{C}_i$ for some $i \in \{0, 1, 2\}$ where:

1. $\mathscr{C}_0 = \{d \in FU(\mathscr{D}_n) : \forall Y \subseteq FU(\{e_i : i < n\})\ Y^\frown\langle d\rangle \Vdash \varphi\}$

2. $\mathscr{C}_1 = \{d \in FU(\mathscr{D}_n) : \forall Y \subseteq FU(\{e_i : i < n\})\ Y^\frown\langle d\rangle \Vdash \neg\varphi\}$

3. $\mathscr{C}_2 = FU(\mathscr{D}_n) \setminus (\mathscr{C}_0 \cup \mathscr{C}_1)$

We are using Hindman's theorem for finding such $\mathscr{D}'_{n+1}$.

Let $e_n$ be the first element of $\mathscr{D}'_{n+1}$ and let $\mathscr{D}_{n+1} := \mathscr{D}'_{n+1} - \langle e_n\rangle$.

Letting $\mathscr{D}^* = \langle e_j\rangle_{j\in\omega}$, we have $\mathscr{D}^* \sqsubseteq \mathscr{D}_j$ for all $j \in \omega$.

Now, suppose that $(E, \mathscr{E}) \leqslant (A, \mathscr{D}^*)$ and $(E, \mathscr{E}) \Vdash \varphi$ (the case for $\neg\phi$ is similar).

Let $|E|$ be minimal such that $(E, \mathscr{E}) \Vdash \phi$. If $|E| = |A|$, then $E = A$ and by assumption on $\mathscr{D}'$, $(A, \mathscr{D}') \Vdash \varphi$. And since $\mathscr{D}^* \sqsubseteq \mathscr{D}'$, we must have $(A, \mathscr{D}^*) \Vdash \varphi$.

If $|E| > |A|$, then $E = A^\frown Y$, where $Y = \langle x_0, \ldots, x_m\rangle$ and $x_j \in FU(\mathscr{D}^*)$ for all $j \in \{0, \ldots, m\}$. We have that $x_m = e_{n_0} \cup \ldots \cup e_{n_l}$. Let

$$n = \min\{n_0, \ldots, n_l\}.$$

Then at the stage $n$ we must have had for $Y' = Y - \langle x_m\rangle$, $Y'^\frown\langle d\rangle \Vdash \varphi$ for all $d \in FU(\mathscr{D}'_{n+1})$.

**Claim 4.26.** *$(A^\frown Y', \mathscr{D}_{n+1}) \Vdash \varphi$.*

*Proof of claim:* Let $(B, \mathscr{B}) \leqslant (A^\frown Y', \mathscr{D}_{n+1})$. Then $B = A^\frown Y'^\frown Z$, where $Z$ is a finite block sequence of elements in $FU(\mathscr{D}_{n+1})$. Note that the first element $z_0$ of $Z$ belongs to $FU(\mathscr{D}'_{n+1})$, so $Y'^\frown\langle z_0\rangle \Vdash \varphi$. Since

$$(B, \mathscr{B}) \leqslant (A^\frown Y'^\frown\langle z_0\rangle, \mathscr{D}' - (A^\frown Y'^\frown\langle z_0\rangle)$$

we have $(B, \mathscr{B}) \Vdash \varphi$. □

Since $\mathscr{E} \sqsubseteq \mathscr{D}_{n+1}$, we have $(A^\frown Y', \mathscr{E}) \Vdash \varphi$, contradicting the minimality of $|E|$. □

## 4.5 $\mathbb{P}_{FIN}$ does not add Cohen reals

**Theorem 4.27.** *$\mathbb{P}_{FIN}$ does not add Cohen reals.*

*Proof.* Let $\dot{f}$ be a $\mathbb{P}_{FIN}$-name for a function from $\omega$ to 2, i.e.,

$$1_{\mathbb{P}_{\mathscr{C}}} \Vdash \dot{f} : \check{\omega} \to \check{2}.$$

We will show that there is a dense set $D$ of $\mathbb{C}$ such that for all $p \in D$,

$$1_{\mathbb{P}_{\mathscr{C}}} \Vdash \dot{f} \restriction dom(p) \neq p.$$

Since $1_{\mathbb{P}_{FIN}} \Vdash \dot{f} : \check{\omega} \to \check{2}$, by the pure decision property for every $(A, \mathscr{D}) \in \mathbb{P}_{\mathscr{C}}$ and every $k \in \omega$, there is $\mathscr{D}' \sqsubseteq \mathscr{D}$ such that $(A, \mathscr{D}')$ decides $\dot{f} \restriction j$ for all $j \leqslant k$.

We can define $\langle \mathscr{D}_n : n \in \omega \rangle$ and $\langle a_n : n \in \omega \rangle$ such that $\mathscr{D}_n$ is an inifinite block sequence, $a_n = \min \mathscr{D}_n$, $\mathscr{D}_{n+1} \sqsubseteq \mathscr{D}_n$, $a_n < a_{n+1}$ and for each finite block sequence $t$ that we can form from the elements of the set

$$FU(\{a_i : i < n\})$$

$(t, \mathscr{D}_n)$ decides $\dot{f} \restriction 2^{b_n}$, where $b_n = 2^n$.

Let $p^n_t$ be such that

$$(t, \mathscr{D}_n) \Vdash \dot{f} \restriction 2^{b_n} = p^n_t.$$

Let $\mathscr{D}^* = \langle a_i \rangle_{i \in \omega}$ and let

$$C := \{q \in \mathbb{C} : \exists n \in \omega \exists t \subseteq FU(\mathscr{D}^*)(q \subseteq p^n_t)\}.$$

Let $D = \mathbb{C} \setminus C$. We will show that $D$ is dense.

Suppose $m < n$ and $t$ is a finite block sequence with elements in

$$FU(\{a_i : i < n\})$$

and $s$ is a block sequence with elements in

$$FU(\{a_i : i < m\})$$

such that $t = s^\frown r$ for some $r$. Since $(t, \mathscr{D}_n) \leqslant (s, \mathscr{D}_m)$, we have $(t, \mathscr{D}_n)$ and $(s, \mathscr{D}_m)$ force the same value for $\dot{f}(\check{k})$, for all $k < 2^{b_m}$. Thus,

$$p_t^n \restriction 2^{b_m} = p_s^m.$$

**Claim 4.28.** $|\{q \in C : \mathrm{dom}(q) = 2^{b_l}\}| < 2^{b_l}$.

*Proof of Claim:* If $q \in C$ and $\mathrm{dom}(q) = 2^{b_l}$, then $q = p_t^l$ for some finite block sequence $t$ with elements in $FU(\{a_i : i < l\})$. The number of such $q$'s is less than

$$|\mathscr{P}(FU(\{a_i : i < l\}))| = 2^{b_l}.$$

$\square$

**Claim 4.29.** *If $t = \langle x_0, \ldots, x_m \rangle$ is a finite block sequence of elements in $FU(\mathscr{D}^*)$ and $n \geqslant \max x_m$, then*

$$|\{q \in \mathbb{C} : p_t^n \subseteq q \text{ and } \mathrm{dom}(q) = 2^{b_{n+1}}\}| = 2^{2^{b_{n+1}} - 2^{b_n}}.$$

*Proof of Claim:* Note that $2^{2^{b_{n+1}}}/2^{2^{b_n}} = 2^{2^{b_{n+1}} - 2^{b_n}}$. $\square$

Let us check that $D$ is a dense set. So let $r \in \mathbb{C}$. If $r \notin D$ then $r \in C$, so there is $n \in \omega$ and a finite block sequence $t$ with elements in $FU(\mathscr{D}^*)$ such that $r \subseteq p_t^n$. There is $q$ such that $p_t^n \subseteq q$, $\mathrm{dom}(q) = 2^{b_{n+1}}$ and $q \notin C$, because

$$b_{n+1} < 2^{b_{n+1}} - 2^{b_n}.$$

Therefore $q \in D$ and $q \leqslant r$. $\square$

# Chapter 5

# $\mathbb{P}_{FIN}$ and Mathias forcing

We have proved that $\mathbb{P}_{FIN}$ and Matet forcing are not equivalent in Section 4.1. It remains to answer the question about the relation between Mathias forcing and $\mathbb{P}_{FIN}$.

In this chapter we will prove that $\mathbb{P}_{FIN}$ and Mathias forcing are not equivalent forcing notions. It is known that Matet and Mathias forcing are not equivalent forcing notions, because Mathias forcing adds a dominating real, so a splitting real, see the Lemma 1.23 and the Proposition 1.16, but Matet forcing does not add splitting reals by Corollary 1.48.

First, we are going to prove that $\mathbb{P}_{FIN}$ adds a Matet real, and then we will show that Mathias forcing does no add Matet reals.

In the last part of this chapter we will recall a forcing notion due to Saharon Shelah and Otmar Spinas in [23], denoted by $\mathbb{M}_2$, which is a sort of a product of two copies of Mathias forcing. We are interested in the connection between this forcing notion and $\mathbb{P}_{FIN}$.

## 5.1 $\mathbb{P}_{FIN}$ adds a Matet real

Recall Matet forcing (Definition 1.37) and the notion of projection (Definition 1.17).

**Lemma 5.1.** *Matet forcing, denoted by* $\mathbb{MT}$*, is a projection of* $\mathbb{P}_{FIN}$*.*

*Proof.* Let $\pi : \mathbb{P}_{FIN} \to \mathbb{MT}$ be the function defined by

$$\pi((A, \mathscr{D})) = (\bigcup A, \mathscr{D}).$$

We shall prove that $\pi$ witnesses that $\mathbb{MT}$ is a projection of $\mathbb{P}_{FIN}$.

It is clear that $\pi$ preserves order.

Let $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ and $(b, \mathscr{D}') \in \mathbb{MT}$ be such that $(b, \mathscr{D}') \leqslant \pi(A, \mathscr{D})$. Then $(b, \mathscr{D}') \leqslant (\bigcup A, \mathscr{D})$, i.e., $\bigcup A \subseteq b$, $\mathscr{D}' \sqsubseteq \mathscr{D}$, and $b \setminus \bigcup A \in FU(\mathscr{D})$. Define $(C, \mathscr{D}'') := (A^\frown(b \setminus \bigcup A), \mathscr{D}') \in \mathbb{P}_{FIN}$. Now notice that $(C, \mathscr{D}'') \leqslant (A, \mathscr{D})$ and $\pi((C, \mathscr{D}'')) \leqslant (b, \mathscr{D}')$. □

**Corollary 5.2.** *$\mathbb{P}_{FIN}$ adds a Matet real.*

## 5.2 $\mathbb{P}_{FIN}$ is not equivalent to Mathias forcing

We are going to prove that $\mathbb{P}_{FIN}$ is not equivalent to Mathias forcing. For this we are going to show that Mathias forcing does not a Matet real. Given $\dot{x}$, a $\mathbb{M}-$name for a real, and a Mathias condition $(a, A)$ we will prove that there exists a stronger condition with the same stem that forces $\dot{x}$ is not a Matet generic real in the generic extension.

First we shall prove a lemma that gives us approximations to $\dot{x}$. Given $(a, A)$ we construct an infinite $B \subseteq A$ and sets $X_t$, with $a \subseteq t \subseteq a \cup B$, where $X_t$ is the set of approximations to $\dot{x}$. We are going to make a distinction between finite $X_t$ and infinite $X_t$, because this will be important for the proof that Mathias forcing does not add a Matet real.

**Lemma 5.3.** *If $\dot{x}$ is a name in Mathias forcing $\mathbb{M}$ and $(a, A) \in \mathbb{M}$ is such that $(a, A) \Vdash_{\mathbb{M}}$ "$\dot{x} \in [\omega]^\omega$", then there are $B \subseteq A$, $B = \{b^j : j \in \omega\}$ infinite, and sets $X_t$, for all finite $t$ with $a \subseteq t \subseteq a \cup B$, such that*

*(1)* $X_t = \{n \in \omega : \exists C \subseteq B \setminus \max(t) + 1((t, C) \Vdash \text{"}n \in \dot{x}\text{"})\}$.

*(2) If $X_t$ is finite, then* $(t, B \setminus \max(t) + 1) \Vdash X_t \subseteq \dot{x}$.

(3) *If* $X_t = \{x_t^j : j \in \omega\}$ *is infinite and* $B \setminus \max(t)+1 = \{b^j : j \geqslant k\}$ *for some* $k \in \omega$*, then there is* $(l_i)_{i \geqslant k}$ *a strictly increasing sequence of natural numbers such that for all* $i \geqslant k$

$$(t, \{b^j : j \geqslant i\}) \Vdash \{x_t^j : j \leqslant l_i\} \subseteq \dot{x}.$$

(4) $(t, B \setminus \max(t)+1) \Vdash \dot{x} \cap (\max(t)+1) = X_t \cap (\max(t)+1)$.

*Proof.* We are going to define recursively an infinite set $B \subseteq A$,

$$B = \{b^j : j \in \omega\}$$

that satisfies the four statements above.

This is a fusion argument, which uses the pure decision property (pdp) for $\mathbb{M}$.

Given $(a, A) \in \mathbb{M}$ and $s_0 := \emptyset$, by the pdp there exists an infinite subset $C^0_{a \cup s_0} \subseteq A$ such that $(a \cup s_0, C^0_{a \cup s_0})$ decides "$0 \in \dot{x}$", that is, either

$$(a \cup s_0, C^0_{a \cup s_0}) \Vdash \text{“}0 \in \dot{x}\text{” or } (a \cup s_0, C^0_{a \cup s_0}) \Vdash \text{“}0 \notin \dot{x}\text{”}.$$

Assume that for all $i \leqslant k$ we have that $(a \cup s_0, C^i_{a \cup s_0})$ decides "$i \in \dot{x}$" and $C^i_{a \cup s_0} \supseteq C^{i+1}_{a \cup s_0}$, for all $i < k$. By the pdp there exists an infinite $C^{k+1}_{a \cup s_0} \subseteq C^k_{a \cup s_0}$ such that $(a \cup s_0, C^{k+1}_{a \cup s_0})$ decides "$k+1 \in \dot{x}$".

We define

$$X_{a \cup s_0} := \{i \in \omega : (a \cup s_0, C^i_{a \cup s_0}) \Vdash i \in \dot{x}\}.$$

- If $|X_{a \cup s_0}| < \aleph_0$, then let $k' := \max X_{a \cup s_0}$ and let $k := \max\{k', \max(a \cup s_0)+1\}$. Choose $A_0 \subseteq C^k_{a \cup s_0}$ such that $A_0 \subseteq^* C^i_{a \cup s_0}$ for all $i > k$. Notice that

$$(a \cup s_0, A_0) \Vdash X_{a \cup s_0} \subseteq \dot{x}.$$

- If $|X_{a \cup s_0}| = \aleph_0$, say $X_{a \cup s_0} = \{x^j_{a \cup s_0} : j \in \omega\}$, then let $l \in \omega$ be the least such that $\max(a \cup s_0)+1 < x^l_{a \cup s_0}$ and let

$$A_0 := \{c_i : i \in \omega\}$$

where the $c_i$ are defined as follows: $c_0 = \min C^{j_0}_{a\cup s_0}$, where $j_0 = x^l_{a\cup s_0}$. Given $c_k$ such that $c_k \in C^{j_k}_{a\cup s_0}$, where $j_k = x^{l+k}_{a\cup s_0}$, take $c_{k+1} \in C^{j_{k+1}}_{a\cup s_0}$ such that $c_{k+1} > c_k$. Notice that $A_0 \subseteq^* C^j_{a\cup s_0}$ for all $j \in X_{a\cup s_0}$, in particular, $A_0 \subseteq C^{j_0}_{a\cup s_0}$. Moreover,

$$(a \cup s_0, A_0) \Vdash \{x^j_{a\cup s_0} : j \leqslant l\} \subseteq \dot{x},$$

and for all $k \in \omega$ we have

$$(a \cup s_0, A_0 \setminus c_k + 1) \Vdash \{x^j_{a\cup s_0} : j \leqslant l + k + 1\} \subseteq \dot{x}.$$

Let $b^0 = \min A_0$.

Suppose that we already have $b^i = \min A_i$ for all $i \leqslant k$, such that $b^i < b^{i+1}$ and $A_{i+1} \subseteq A_i$, for all $i \leqslant k - 1$.

Let $s_0, \dots, s_{m-1}$ be an enumeration of all the subsets of $\{b^i : i \leqslant k\}$. We are going to define $A_{k+1}$ and $b^{k+1}$.

Let $A^{k+1}_0 = A_k \setminus \{b^k\}$. Given $A^{k+1}_j$ with $j < m$ consider the condition $(a \cup s_j, A^{k+1}_j)$. By the pdp there exists an infinite subset $C^0_{a\cup s_j}$ of $A^{k+1}_j$ such that $(a \cup s_j, C^0_{a\cup s_j})$ decides "$0 \in \dot{x}$".

In this way, we obtain an infinite decreasing sequence of infinite subsets $C^i_{a\cup s_j}$ so that $(a \cup s_j, C^i_{a\cup s_j})$ decides "$i \in \dot{x}$".

As before we form the set

$$X_{a\cup s_j} := \{i \in \omega : (a \cup s_j, C^i_{a\cup s_j}) \Vdash i \in \dot{x}\}.$$

- If $|X_{a\cup s_j}| < \aleph_0$ then let $l' := \max X_{a\cup s_j}$ and let $l := \max\{l', \max(a \cup s_j) + 1\}$. Choose $A^{k+1}_{j+1} \subseteq C^l_{a\cup s_j}$ and $A^{k+1}_{j+1} \subseteq^* C^i_{a\cup s_j}$ for all $i > l$. Note that $(a \cup s_j, A^{k+1}_{j+1}) \Vdash X_{a\cup s_j} \subseteq \dot{x}$.

- If $|X_{a\cup s_j}| = \aleph_0$, then let $X_{a\cup s_j} = \{x^i_{a\cup s_j} : i \in \omega\}$ and let $r$ be the least such that $\max(a \cup s_j) + 1 < x^r_{a\cup s_j}$. Let

$$A^{k+1}_{j+1} = \{c_i : i \in \omega\}$$

where $c_0 := \min C^{j_0}_{a\cup s_j}$, $j_0 = x^r_{a\cup s_j}$, and where the elements $c_i \in C^{j_i}_{a\cup s_j}$ are chosen so that $c_i < c_{i+1}$ for all $i \in \omega$ and $j_i = x^{r+i}_{a\cup s_j}$ for all $i \geqslant 1$.

Notice that $A^{k+1}_{j+1} \subseteq^* C^l_{a\cup s_j}$ for all $l \in \omega$, in particular, $A^{k+1}_{j+1} \subseteq C^{j_0}_{a\cup s_j}$ and

$$(a \cup s_j, A^{k+1}_{j+1}) \Vdash \{x^i_{a\cup s_j} : i \leqslant r\} \subseteq \dot{x}.$$

For all $i \in \omega$ we have

$$(a \cup s_j, A^{k+1}_{j+1} \setminus c_i + 1) \Vdash \{x^l_{a\cup s_j} : l \leqslant r + i + 1\} \subseteq \dot{x}. \tag{5.2.1}$$

Let $A_{k+1} := A^{k+1}_m$. Clearly $A_{k+1} \subseteq A_k$.

Let $b^{k+1} := \min A_{k+1}$. This completes the definition of $B = \{b^i : i \in \omega\}$.

Clearly $(a, B) \leqslant (a, A)$. Note that $B \subseteq^* C^i_t$ for all $i \in \omega$ and $a \subseteq t \subseteq B$, where the $C^i_t$ are the sets obtained in the proof.

We shall prove 1-4.

**Claim 5.4.** *Let $t$ be a finite subset such that $a \subseteq t \subseteq a \cup B$. For all $n \in \omega$ there exists $C \subseteq B \setminus \max(t) + 1$ such that $(t, C) \Vdash$ "$n \in \dot{x}$" if and only if $n \in X_t$.*

*Proof of Claim:* Let $t$ be a finite subset such that $a \subseteq t \subseteq a \cup B$ and let $n \in \omega$. Suppose that $n \in X_t$. Then $(t, C^n_t) \Vdash n \in \dot{x}$. Since $B \subseteq^* C^n_t$, there exits $m \in \omega$ such that $B \setminus m \subseteq C^n_t$. Let $k' = \max\{m, \max(t) + 1\}$ and let $C := B \setminus k'$. We have that $(t, C) \leqslant (t, C^n_t)$, and so $(t, C) \Vdash n \in \dot{x}$.

For the other direction, assume towards a contradiction that there are $n \in \omega$ and $C \subseteq B \setminus \max(t) + 1$ such that $(t, C) \Vdash n \in \dot{x}$ and $n \notin X_t$.

Since $n \notin X_t$, we have $(t, C^n_t) \Vdash n \notin \dot{x}$. Let $m \in \omega$ be such that $B \setminus m \subseteq C^n_t$. Let $k = \max\{m, \max(t)+1\}$. Then, $C \setminus k \subseteq C^n_t$. Thus $(t, C \setminus k) \leqslant (t, C^n_t)$ and $(t, C \setminus k) \Vdash n \notin \dot{x}$, which yields a contradiction. □

**Claim 5.5.** *Let $t$ be a finite subset such that $a \subseteq t \subseteq a \cup B$. If $X_t$ is finite, then $(t, B \setminus \max(t) + 1) \Vdash X_t \subseteq \dot{x}$.*

*Proof of the claim:* Let $t$ be a finite subset such that $a \subseteq t \subseteq a \cup B$, then $t = a \cup s$. Let $k \in \omega$ the least natural number such that $s \subseteq \{b^0, \ldots, b^k\}$, then in the step $k + 1$ of the construction

$$(t, A_{k+1}) \Vdash X_t \subseteq \dot{x}$$

then

$$(t, B \setminus \max(t) + 1) \Vdash X_t \subseteq \dot{x}.$$

We have proved the claim. □

**Claim 5.6.** *Let $t$ be a finite subset such that $a \subseteq t \subseteq a \cup B$. If $X_t = \{x_t^j : j \in \omega\}$ is infinite and $B \setminus \max(t) + 1 = \{b^j : j \geqslant k\}$ for some $k \in \omega$, then there is $(l_i)_{i \geqslant k}$ a strictly increasing sequence of natural numbers such that for all $i \geqslant k$*

$$(t, \{b^j : j \geqslant i\}) \Vdash \{x_t^j : j \leqslant l_i\} \subseteq \dot{x}.$$

*Proof of Claim:* We shall prove by induction on $i \geqslant k$ that

$$(t, \{b^j : j \geqslant i\}) \Vdash \{x_t^j : j \leqslant l_i\} \subseteq \dot{x}$$

for some increasing sequence $(l_i)_{i \geqslant k}$.

Suppose $i = k$. We are going to prove that

$$(t, B \setminus \max(t) + 1) \Vdash \{x_t^j : j \leqslant l_k\} \subseteq \dot{x}$$

for some $l_k$.

We have that $\max(t) = b^{k-1}$.

Let $t = a \cup s_j$, where $s_j \subseteq \{b^0, \ldots, b^{k-1}\}$. In the $k$-th step we have that $A_k \subseteq^* C_t^j$ for all $j \in X_t$. Let $l$ be the least such that $\max(t) + 1 < x_t^l$. By the construction we have that $(t, A_k) \Vdash \{x_t^j : j \leqslant l\} \subseteq \dot{x}$. Let $l_k = l$.

Since $B \setminus \max(t) + 1 \subseteq A_k$, we obtain

$$(t, B \setminus \max(t) + 1) \Vdash \{x_t^j : j \leqslant l_k\} \subseteq \dot{x}.$$

Assume that for $k' \geqslant k$, we have $(t, \{b^j : j \geqslant k'\}) \Vdash \{x_t^j : j \leqslant l_{k'}\} \subseteq \dot{x}$ and $l_{k'} = l + r' + 1$, where $b^{k'-1} = c_{r'}$, similarly as in (5.2.1). We shall prove that $(t, \{b^j : j \geqslant k' + 1\}) \Vdash \left\{x_t^j : j \leqslant l_{k'+1}\right\}$ for some $l_{k'+1} > l_{k'}$.

Then $b^{k'} = c_r$, for some $c_r \in C_t^{j_r}$, and $j_r = x_t^{l+r}$. Since $b^{k'-1} < b^{k'}$, we have that $r' < r$. By the construction we have

$$(t, A_k \setminus c_r + 1) \Vdash \{x_t^j : j \leqslant l + r + 1\} \subseteq \dot{x}.$$

Let $l_{k'+1} = l+r+1$. Notice that $l_{k'+1} > l_{k'}$. Since $\{b^j : j \geqslant k'+1\} \subseteq A_k \setminus b^{k'}+1$,

$$(t, \{b^j : j \geqslant k'+1\}) \Vdash \{x_t^j : j \leqslant l_{k'+1}\} \subseteq \dot{x}.$$

This finishes the proof of the Claim. □

**Claim 5.7.** *For all $t$ with $a \subseteq t \subseteq a \cup B$,*

$$(t, B \setminus \max(t)+1) \Vdash \dot{x} \cap (\max(t)+1) = X_t \cap (\max(t)+1).$$

*Proof of Claim:* Let $t$ be finite and such that $a \subseteq t \subseteq a \cup B$, where $\max(t) = b^m$.

Suppose first that $|X_t| < \aleph_0$. By the construction $(t, A_{m+1}) \Vdash X_t \subseteq \dot{x}$. Since $B \setminus \max(t)+1 \subseteq A_{m+1}$, we have $(t, B \setminus \max(t)+1) \Vdash X_t \subseteq \dot{x}$. If $n \in \max(t)+1$ and $(t, C_t^n) \Vdash n \in \dot{x}$, then by Claim 5.4 we have $n \in X_t$. Therefore

$$(t, B \setminus \max(t)+1) \Vdash \dot{x} \cap (\max(t)+1) = X_t \cap (\max(t)+1).$$

In fact,

$$(t, B \setminus \max(t)+1) \Vdash \dot{x} \cap (\max(t)+1) = X_t.$$

Now, suppose that $|X_t| = \aleph_0$ and $X_t = \{x_t^j : j \in \omega\}$. We have that for the least $k$ such that $b^m < x_t^k$, $A_{m+1} \subseteq C_t^l$, where $l = x_t^k$, and

$$(t, C_t^l) \Vdash \{x_t^j : j \leqslant k\} \subseteq \dot{x}.$$

Since $B \setminus \max(t)+1 \subseteq A_{m+1}$,

$$(t, B \setminus \max(t)+1) \Vdash \{x_t^j : j \leqslant k\} \subseteq \dot{x}.$$

By Claim 5.4 we have $(t, B \setminus \max(t)+1) \Vdash \dot{x} \cap (\max(t)+1) = X_t \cap (\max(t)+1)$. This proves the Claim. □

This finishes the proof of Lemma 5.3. □

**Lemma 5.8.** *Suppose that for all $t$ such that $a \subseteq t \subseteq a \cup B$, $X_t$ is finite. Then for every $a \subseteq t \subseteq a \cup B$, letting $B \setminus \max(t)+1 = \{b^i : i \in \omega\}$, we have that $\min(X_{t \cup \{b^i\}} \setminus X_t) \to \infty$ as $i \to \infty$ (where we set $\min(X_{t \cup \{b^i\}} \setminus X_t) = \infty$ if $X_{t \cup \{b^i\}} \setminus X_t = \emptyset$).*

*Proof.* Notice that if $t \subseteq t'$, then $X_t \subseteq X_{t'}$. Assume towards a contradiction that there are $k \in \omega$ and $M \in [\omega]^\omega$ such that for all $m \in M$

$$\min(X_{t\cup\{b^m\}} \setminus X_t) = k.$$

Consider $B' = \{b^i : i \in M\}$.

**Claim 5.9.** *$(t, B') \Vdash k \in \dot{x}$.*

*Proof of Claim:* We need to show that the set of conditions that force $k \in \dot{x}$ is dense below $(t, B')$. Let $(s, C) \leqslant (t, B')$. By going to stronger condition, if necessary, we may assume that $t \setminus s \neq \emptyset$. So, we have

$$s = t \cup \{b^{i_0}, \dots, b^{i_m}\}$$

with $\emptyset \neq \{i_0, \dots, i_m\} \subseteq M$. Notice that $k \in X_{t\cup\{b^{i_0}\}} \setminus X_t$. By Lemma 5.3 (2), we have

$$(t \cup \{b^{i_0}\}, B \setminus \max(t \cup \{b^{i_0}\}) + 1) \Vdash k \in \dot{x}.$$

Since $(s, C) \leqslant (t \cup \{b^{i_0}\}, B \setminus \max(t \cup \{b^{i_0}\}) + 1)$, we have $(s, C) \Vdash k \in \dot{x}$. We have proved the claim. □

By (1) of Lemma 5.3, we now have that $k \in X_t$, which is a contradiction. □

We are going to use following notation: $Y_{t\cup\{i\}} := X_{t\cup\{i\}} \setminus X_t$, whenever $a \subseteq t \subseteq a \cup B$ and $i \in B \setminus \max(t) + 1$.

We define $Y_a := X_a$.

**Lemma 5.10.** *Suppose that for all $t$ such that $a \subseteq t \subseteq a \cup B$, $X_t$ is finite. Then there is an infinite $B' \subseteq B$ such that for any $t$ and $t'$ with $a \subseteq t \subseteq a \cup B'$, $a \subseteq t' \subseteq a \cup B'$, and $\max(t) < \max(t')$, we have $\max Y_t < \min Y_{t'}$.*

*Proof.* We are going to prove the lemma using lemma 5.8 and some pruning.

Let $t = a \cup \{b^0\}$ and assume that $Y_t \neq \emptyset$.

Since $\min(Y_{a\cup\{b^i\}}) \to \infty$, as $i \to \infty$, there exists $i_0 > 0$ such that for all $i \geqslant i_0$, $\max(Y_{a\cup\{b^0\}}) < \min(Y_{a\cup\{b^i\}})$.

Consider now $\{b^0, b^{i_0}\}$ and all subsets $r_j$ such that $a \subseteq r_j \subseteq a \cup \{b^0, b^{i_0}\}$. Say $r_0 = a$, $r_1 = a \cup \{b^0\}$, $r_2 = a \cup \{b^{i_0}\}$ and $r_3 = a \cup \{b^0, b^{i_0}\}$.

Since $\min(Y_{r_k \cup \{b^i\}}) \to \infty$ whenever $i \to \infty$, there is $i_1 > i_0$ such that for all $i \geqslant i_1$ $\min(Y_{r_k \cup \{b^i\}}) > \max\{\max Y_{r_2}, \max Y_{r_3}\}$, for all $k \in 4$.

In general, assume that we have $\{b^0, b^{i_0}, \dots, b^{i_n}\}$ with the property that for any two subsets $s, r$ such that $a \subseteq s, r \subseteq a \cup \{b^0, b^{i_0}, \dots, b^{i_n}\}$ and $\max s < \max r$ we have $\max Y_s < \min Y_r$.

We are going to define $b^{i_{n+1}}$.

Let $\{r_j : j < 2^{n+2}\}$ be an enumeration of all $r$, where $a \subseteq r \subseteq a \cup \{b^0, b^{i_0}, \dots, b^{i_n}\}$.

Since $\min(Y_{r_k \cup \{b^i\}}) \to \infty$ whenever $i \to \infty$, for all $k \in 2^{n+2}$, there is $i_{n+1} > i_n$ such that for all $i \geqslant i_{n+1}$, $\min(Y_{r_k \cup \{b^i\}}) > \max\{\max Y_{r_j} : \max r_j = b^{i_n}\}$ for all $k \in 2^{n+2}$.

Let $B' = \{b^0\} \cup \{b^{i_m} : m \in \omega\}$. It is clear that $B' \subseteq B$, and for any $t$ and $t'$ such that $\max t < \max t'$, we have that $\max Y_t < \min Y_{t'}$. $\square$

**Corollary 5.11.** *Suppose that $B$ is an infinite set of natural numbers that satisfies lemma 5.10, i.e., for any $t$ and $t'$ with $a \subseteq t \subseteq a \cup B$, $a \subseteq t' \subseteq a \cup B$, if $\max(t) < \max(t')$, then $\max Y_t < \min Y_{t'}$. Then for all $i, j \in B$ such that $i < j$,*

$$\max \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\} <$$
$$\min \bigcup \{Y'_t : \max t' = j, a \subseteq t' \subseteq a \cup B\}.$$

*Proof.* Suppose $t'$ is such that $a \subseteq t' \subseteq a \cup B$ and $\max t' = j$. If $t$ is any finite set such that $a \subseteq t \subseteq a \cup B$ and with $\max t = i$, then $\max Y_t < \min Y_{t'}$. So

$$\max \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\} < \min Y_{t'}.$$

Since this holds, for all $t'$ such that $a \subseteq t' \subseteq a \cup B$ and $\max t' = j$, we have

$$\max \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\} <$$

$$\min \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\}.$$

$\square$

**Lemma 5.12.** *Assume that $B$ is an infinite subset of the natural numbers that satisfies lemma 5.10. Let $d$ be a finite set of natural numbers such that*

$$\min d < \max \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\} < \max d$$

*where $j$ is some fixed element of $B$. Then*

$$d \not\subseteq \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}$$

*for all $i \in B$.*

*Proof.* Let $i$ be an element in $B$. We are going to prove the lemma by cases:

- if $i = j$, then $\max d \notin \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}$ because by hypothesis

$$\max \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\} < \max d.$$

- if $j < i$, then by Corollary 5.11, we have

$$\max \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\} <$$
$$\min \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}.$$

  Then,

$$\min d \notin \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}.$$

- if $i < j$, then by Corollary 5.11, we have

$$\max \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\} <$$
$$\min \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\}.$$

  Since

$$\min \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\} \leqslant$$
$$\max \bigcup \{Y_{t'} : \max t' = j, a \subseteq t' \subseteq a \cup B\}$$

  it follows

$$\max \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\} < \max d.$$

□

**Theorem 5.13.** *Mathias forcing does not add a Matet real.*

*Proof.* Let $\dot{x}$ be a $\mathbb{M}$-name for a real (i.e., an element of $[\omega]^\omega$) which is not in the ground model $V$. We want to prove that $\dot{x}$ is not a $\mathbb{M}$-name for a Matet real over $V$.

If $m$ is a Matet real over $V$, then for all $D \subseteq \mathbb{MT}$ dense, $D \in V$, there is $(b, \mathscr{D}) \in D$, where $\mathscr{D} = \langle d_i \rangle_{i \in \omega}$, such that

$$m \in [b, \mathscr{D}] = \{x \in [\omega]^\omega : x = b \cup \bigcup_{i \in I} d_i \text{ for some } I \in [\omega]^\omega\}.$$

So, we shall prove that, in $V$, for all $(a, A) \in \mathbb{M}$, there are $(a, B) \leqslant (a, A)$ and $D \subseteq \mathbb{MT}$ dense such that for all $(b, \mathscr{D}) \in D$, $(a, B) \Vdash \dot{x} \notin [b, \mathscr{D}]$. This implies that $\dot{x}$ is not an $\mathbb{M}$-name for a Matet real over $V$.

Let $(a, A)$ be an element in Mathias forcing $\mathbb{M}$. By lemma 5.3 there is $(a, B) \leqslant (a, A)$ such that $B = \{b^j : j \in \omega\}$ is an increasing enumeration of $B$ that satisfies properties (1)-(4).

We have two cases:

1. For all $(a', B') \leqslant (a, B)$ there is $t$ finite with $a' \subseteq t \subseteq a' \cup B'$ such that $X_t$ is infinite, or

2. there exists $(a', B') \leqslant (a, B)$ such that for all $t$ with $a' \subseteq t \subseteq a' \cup B'$, $X_t$ is finite.

Assume that we are in case 1. This is the easy case.

**Claim 5.14.** *Given a family* $\{X_m : m \in \omega\} \subseteq [\omega]^\omega$ *and a condition* $(b', \mathscr{D}') \in \mathbb{MT}$ *there is* $(b, \mathscr{D}) \leqslant (b', \mathscr{D}')$ *with* $b = b'$ *and* $\mathscr{D} = \langle d_i \rangle_{i \in \omega}$ *such that* $X_m \setminus \bigcup_{i \in \omega} d_i$ *is infinite for all* $m \in \omega$.

*Proof of Claim:* View each $X_m$ as an increasing sequence $\langle x_j^m \rangle_{j \in \omega}$.

Let $\mathscr{D}' = \langle d_i' \rangle_{i \in \omega}$. We are going to construct $\mathscr{D} = \langle d_i \rangle_{i \in \omega}$ such that $\mathscr{D} \sqsubseteq \mathscr{D}'$ and a family

$$\{Z_m : m \in \omega\}$$

where $Z_m \subseteq X_m \setminus \bigcup_{i\in\omega} d_i$ for all $m \in \omega$.

Let $d_0 = d'_0$. Since $|d_0 \cap X_0| < \aleph_0$ let $z^0_0 \in X_0$ larger than $\max(d_0)$.

Let $d_1 = d'_{l_0}$ be such that $z^0_0 < \min d'_{l_0}$, and let $z^1_0 \in X_1$ be larger than $\max(d'_{l_0})$.

Let $z^0_1 \in X_0$ be such that $z^0_1 > z^1_0$.

Assume that we have $z^0_k$ such that $\max d_0 < z^0_0 < \min d_1 \leqslant \max d_1 < z^1_0 < z^0_1 < \min d_2 \leqslant \max d_2 < z^2_0 < z^1_1 < z^0_2 < \ldots < \min d_k \leqslant \max d_k < z^k_0 < \ldots < z^0_k$.

Let $d_{k+1} = d'_{l_{k+1}}$ be such that $z^0_k < \min d'_{l_{k+1}}$. Since $|d'_{l_{k+1}} \cap X_{k+1}| < \aleph_0$, we take $z^{k+1}_0 \in X_{k+1}$ such that $z^{k+1}_0 > \max(d'_{l_{k+1}} \cap X_{k+1})$. Choose $z^k_1 \in X_k$ such that $z^k_1 > z^{k+1}_0$, and finally let $z^0_{k+1} \in X_0$ be such that $z^0_{k+1} > z^1_k > \ldots > z^k_1 > z^{k+1}_0$.

We have obtained $Z_m = \langle z^m_j \rangle_{j\in\omega}$ for all $m \in \omega$ and $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$ such that $Z_m \subseteq X_m \setminus \bigcup_{i\in\omega} d_i$ is infinite for all $m \in \omega$. We have thus proved the claim. $\square$

We shall define a dense $D \subseteq \mathbb{MT}$. Let

$$D := \{(b, \mathscr{D}) \in \mathbb{MT} : \text{ for all } a \subseteq t \subseteq a \cup B \text{ such that } |X_t| = \aleph_0$$

$$\text{we have } |X_t \setminus \bigcup_{n\in\omega} d_n| = \aleph_0, \text{ where } \mathscr{D} = \langle d_i \rangle_{i\in\omega}\}.$$

**Claim 5.15.** *$D$ is dense.*

*Proof of Claim:* Given any element $(b', \mathscr{D}') \in \mathbb{MT}$, by Claim 5.14 there is $(b, \mathscr{D}) \leqslant (b', \mathscr{D}')$ such that $X_t \setminus \bigcup_{i\in\omega} d_i$ is infinite for all $t$ with $a \subseteq t \subseteq a \cup B$ such that $X_t$ infinite. Therefore $D$ is dense. $\square$

**Claim 5.16.** *For all $(b, \mathscr{D}) \in D$, where $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$,*

$$(a, B) \Vdash \text{“}\exists n \in \omega (n \in \dot{x} \textit{ and } n \notin \bigcup_{i\in\omega} d_i)\text{”}.$$

*Therefore $(a, B) \Vdash \dot{x} \notin [b, \mathscr{D}]$.*

*Proof of Claim:* Let $(b, \mathscr{D})$ be an element in $D$, and let $(s, C) \leqslant (a, B)$. Then there is some $t$ with $s \subseteq t \subseteq s \cup C$ such that $X_t$ is infinite.

Let $n \in X_t \setminus \bigcup_{i\in\omega} d_i$ be such that $n > \max(b)$. Since

$$n \in X_t = \{x_t^i : i \in \omega\}$$

we have that $n = x_t^k$, for some $k \in \omega$. Consider $i' \in \omega$ such that $k < l_{i'}$, where $l_{i'}$ is an element of the increasing sequence $(l_i)_{i\in\omega}$ in (4) of Lemma 5.3. Let $C' := \{b^j \in C : j \geqslant i'\}$. Then

$$(t, \{b^j : j \geqslant i'\}) \Vdash \{x_t^j : j \leqslant l_{i'}\} \subseteq \dot{x}$$

by (3) of Lemma 5.3. Since $(t, C') \leqslant (t, \{b^j : j \geqslant i'\})$,

$$(t, C') \Vdash \{x_t^j : j \leqslant l_{i'}\} \subseteq \dot{x},$$

and so

$$(t, C') \Vdash \text{“}n \in \dot{x} \text{ and } n \notin b \cup \bigcup_{i\in\omega} d_i\text{”}.$$

Thus $(t, C') \Vdash \dot{x} \notin [b, \mathscr{D}]$. We have shown that the set of conditions that force “$\dot{x} \notin [b, \mathscr{D}]$” is dense below $(a, B)$. This implies $(a, B) \Vdash$ “$\dot{x} \notin [b, \mathscr{D}]$”. $\square$

Suppose now that we are in the case 2. So, suppose there is $(a', B') \leqslant (a, B)$ such that for all $t$, where $a' \subseteq t \subseteq a' \cup B'$, $X_t$ is finite. For simplicity of notation we may take $(a', B') = (a, B)$.

We assume that the condition $(a, B)$ satisfies the property of lemma 5.10, namely, for any $t$ and $t'$ with $a \subseteq t \subseteq a \cup B$, $a \subseteq t' \subseteq a \cup B$ and $\max(t) < \max(t')$, we have $\max Y_t < \min Y_{t'}$, where we recall the notation, $Y_{t\cup\{i\}} = X_{t\cup\{i\}} \setminus X_t$, whenever $a \subseteq t \subseteq a \cup B$ and $i \in B \setminus \max(t) + 1$.

We shall define in this case a dense $D \subseteq \mathbb{MT}$ such that for all $(b, \mathscr{D}) \in D$,

$$(a, B) \Vdash \text{“}\exists n_0 \in \omega \forall n \geqslant n_0 (d_n \not\subseteq \dot{x})\text{”}$$

where $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$. Let

$$D = \{(b, \mathscr{D}) \in \mathbb{MT} : \forall n \in \omega \forall i \in B (d_n \not\subseteq \bigcup\{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\})\}.$$

**Claim 5.17.** *$D$ is a dense subset of Matet forcing.*

*Proof of Claim:* Let $(b, \mathscr{D}')$ be an element of Matet forcing, where $\mathscr{D}' = \langle d'_i \rangle_{i\in\omega}$. We are going to define $\mathscr{D} \sqsubseteq \mathscr{D}'$ such that $(b, \mathscr{D}) \in D$.

Let $j \in B$ be minimal such that

$$\min d'_0 < \max \bigcup \{Y_t : \max t = j, a \subseteq t \subseteq a \cup B\}.$$

Note that such $j$ exists, because by lemma 5.8 there exists $j \in B$ such that for all $i' \geqslant j$

$$\min d'_0 < \min Y_{a\cup\{i'\}} \leqslant \max Y_{a\cup\{i'\}}$$

and so, in particular, $\min d'_0 < \max Y_{a\cup\{j\}}$.

Let $k \geqslant 0$ be minimal such that

$$\max d'_k > \max \bigcup \{Y_t : \max t = j, a \subseteq t \subseteq a \cup B\}.$$

Let $d_0 = d'_0 \cup d'_k$. Since $d_0$ satisfies the hypothesis of lemma 5.12,

$$d_0 \not\subseteq \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}$$

for all $i \in B$.

Suppose now that we have a finite sequence $d_0 < d_1 < \ldots < d_l$ such that for all $m \in \{0, \ldots, l\}$, $d_m \in FU(\mathscr{D}')$, and

$$d_m \not\subseteq \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}$$

for all $i \in B$.

We are going to obtain $d_{l+1}$ with the last property.

Let $d'_r$ be such that $\max d_l < \min d'_r$, and $j'$ is minimal such that

$$\min d'_r < \max \bigcup \{Y_t : \max t = j', a \subseteq t \subseteq a \cup B\}.$$

Let $s \geqslant r$ be minimal such that

$$\max d'_s > \max \bigcup \{Y_t : \max t = j', a \subseteq t \subseteq a \cup B\}.$$

Define $d_{l+1} := d'_r \cup d'_s$. Notice that $d_{l+1}$ satisfies the hypothesis of lemma 5.12, so

$$d_{l+1} \not\subseteq \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}.$$

We define $\mathscr{D} := (d_i)_{i \in \omega}$. It is clear that $\mathscr{D} \sqsubseteq \mathscr{D}'$ and $\mathscr{D}$ satisfies the property that for all $n \in \omega$ and for all $i \in B$

$$d_n \not\subseteq \bigcup \{Y_t : \max t = i, a \subseteq t \subseteq a \cup B\}.$$

Hence $D$ is a dense subset of Matet forcing. We have proved the claim. $\square$

**Claim 5.18.** *For all $(b, \mathscr{D}) \in D$,*

$$(a, B) \Vdash \text{“}\exists n_0 \in \omega \forall n \geqslant n_0 (d_n \not\subseteq \dot{x})\text{”}$$

*and therefore $(a, B) \Vdash \dot{x} \notin [b, \mathscr{D}]$.*

*Proof of Claim:* Let $(a', B') \leqslant (a, B)$. We shall find $(t, C) \leqslant (a', B')$ and $n_0 \in \omega$ such that for all $n \geqslant n_0$

$$d_n \not\subseteq \bigcup \{Y_{t'} : t \subseteq t' \subseteq t \cup C\}.$$

Let $t = a'$. Let $Y_{a'}$ and $n_0 \in \omega$ be such that $\max Y_{a'} < \min d_{n_0}$. We are going to obtain inductively $C = \{j_l : l \in \omega\} \subseteq B'$.

There exists $j_0 \in B'$ such that $\max d_{n_0} < \min Y_{a' \cup \{j\}}$, for all $j \geqslant j_0$.

Let $B_0 = \{j \in B' : j \geqslant j_0\}$. Notice that

$$d_{n_0} \not\subseteq \bigcup \{Y_{t'} : a' \subseteq t' \subseteq a' \cup B_0\}$$

and that for any $C \subseteq B_0$ infinite we still have

$$d_{n_0} \not\subseteq \bigcup \{Y_{t'} : a' \subseteq t' \subseteq a' \cup C\}.$$

Choose $n_1 > n_0$ minimal such that

$$\min d_{n_1} > \max \bigcup \{Y_{t'} : \max t' = j_0, a' \subseteq t' \subseteq a' \cup B_0\}.$$

Now, choose $j_1 \in B_0$ such that $j_1 > j_0$ and

$$\max d_{n_1} < \min \bigcup \{Y_{t'} : \max t' = j_1, a' \subseteq t' \subseteq a' \cup B_0\}.$$

Such $j_1$ exists by corollary 5.11.

Let $B_1 := \{j \in B_0 : j \geqslant j_1\}$.

**Subclaim 5.19.** *For all $m$ such that $n_0 + 1 \leqslant m \leqslant n_1$,*

$$d_m \not\subseteq \bigcup\{Y_{t'} : a' \subseteq t' \subseteq a' \cup \{j_0\} \cup B_1\}.$$

*Proof of Subclaim:* We shall prove the statement by cases:

- If $m = n_1$, then by selection of $n_1$ and $j_1$, we have $d_m \cap Y_{t'} = \emptyset$ for all $t'$ such that $a' \subseteq t' \subseteq a' \cup \{j_0\} \cup B_1$.
- If $m$ is such that $n_0 + 1 \leqslant m < n_1$ then
  - $d_m \cap Y_{a'} = \emptyset$ because $\max Y_{a'} < \min d_{n_0} < \min d_m$.
  - $d_m \cap Y_{t'} = \emptyset$ for all $t'$ such that $a' \subseteq t' \subseteq a' \cup B_0$ and $\max t' \geqslant j_1$ because

    $$\max d_m < \min d_{n_1} < \min \bigcup\{Y_{t'} : \max t' \geqslant j_1, a' \subseteq t' \subseteq a' \cup B_0\},$$

  - and finally $d_m \not\subseteq \bigcup\{Y_{t'} : \max t' = j_0, a' \subseteq t' \subseteq a' \cup B_0\}$ by definition of $D$.

We have proved the subclaim. □

Note that the claim above is true for any infinite subset $C$ of $B_1$, i.e., for all $m$ such that $n_0 + 1 \leqslant m \leqslant n_1$ we have

$$d_m \not\subseteq \bigcup\{Y_{t'} : a' \subseteq t' \subseteq a' \cup \{j_0\} \cup C\}.$$

Assume that we have $j_0 < j_1 < \ldots < j_k$ elements in $B'$, infinite subsets $B_k \subseteq \ldots \subseteq B_0 \subseteq B'$ such that $\min B_l = j_l$ for all $l \in \{0, \ldots, k\}$, and natural numbers $n_1 < \ldots < n_k$ such that

$$d_l \not\subseteq \bigcup\{Y_{t'} : a' \subseteq t' \subseteq a' \cup \{j_0, \ldots, j_{k-1}\} \cup B_k\}$$

for all $l$ with $n_0 \leqslant l \leqslant n_k$.

We shall obtain $j_{k+1}$, $B_{k+1} \subseteq B_k$ and $n_{k+1} > n_k$ such that

$$d_m \not\subseteq \bigcup\{Y_{t'} : a' \subseteq t' \subseteq a' \cup \{j_0, \ldots, j_k\} \cup B_{k+1}\}$$

for all $m$ such that $n_k \leqslant m \leqslant n_{k+1}$.

Choose $n_{k+1} > n_k$ minimal such that

$$\min d_{n_{k+1}} > \max \bigcup \{Y_{t'} : \max t' = j_k, a' \subseteq t' \subseteq a' \cup \{j_0, \ldots, j_{k-1}\} \cup B_k\}.$$

Choose $j_{k+1} \in B_k$ such that $j_{k+1} > j_k$ and

$$\max d_{n_{k+1}} < \min \bigcup \{Y_{t'} : \max t' = j_{k+1}, a' \subseteq t' \subseteq a' \cup \{j_0, \ldots, j_{k-1}\} \cup B_k\}.$$

Let $B_{k+1} = \{j \in B_k : j \geqslant j_{k+1}\}$.

**Subclaim 5.20.** *For all $m$ such that $n_k \leqslant m \leqslant n_{k+1}$*

$$d_m \nsubseteq \bigcup \{Y_{t'} : a' \subseteq t' \subseteq a' \cup \{j_0, \ldots, j_k\} \cup B_{k+1}\}$$

*Proof of subclaim:* Exactly like the proof of subclaim 5.19. □

We define $C = \{j_l : l \in \omega\}$, then $(a', C) \leqslant (a', B')$ and for all $n \geqslant n_0$

$$d_n \nsubseteq \bigcup \{Y_t : a' \subseteq t \subseteq a' \cup C\}.$$

Hence $(a', C) \Vdash \exists n_0 \forall n \geqslant n_0 (d_n \nsubseteq \dot{x})$, and thus $(a, B) \Vdash \dot{x} \notin [b, \mathscr{D}]$. We have proved the Claim 5.18 and the Theorem. □

□

**Corollary 5.21.** $\mathbb{P}_{FIN}$ *is not equivalent to Mathias forcing.*

*Proof.* By the Corollary 5.2, we have that $\mathbb{P}_{FIN}$ adds a Matet real. □

## 5.3 A "product" of two copies of Mathias forcing.

We are going to consider a "product" of two copies of Mathias forcing, denoted by $\mathbb{M}_2$, that is the same as the forcing notion considered by Shelah and Spinas in [23]. The reason for looking at $\mathbb{M}_2$ is its connection with $\mathbb{P}_{FIN}$. We shall prove that $\mathbb{M}_2$ is a projection of $\mathbb{P}_{FIN}$, hence adding a generic for $\mathbb{P}_{FIN}$ adds a generic for $\mathbb{M}_2$.

**Definition 5.22.** *Conditions in* $\mathbb{M}_2$ *are pairs* $((s, A), (t, B))$ *such that* $s, t \in [\omega]^{<\omega}$, $A, B \in [\omega]^{\omega}$, $|s| = |t|$, $\max s < \min A$, $\max t < \min B$, *and if*

$$s = \{s_i : i < n\}$$

$$t = \{t_i : i < n\}$$

*are the increasing enumerations of* $s$ *and* $t$*, respectively, then*

$$s_0 \leqslant t_0 \leqslant s_1 \leqslant \ldots \leqslant s_k \leqslant t_k \leqslant \ldots \leqslant s_{n-1} \leqslant t_{n-1}.$$

*For any elements* $((s', A'), (t', B'))$,$((s, A), (t, B))$ *of* $\mathbb{M}_2$*, the ordering is given by*

$$((s', A'), (t', B')) \leqslant_{\mathbb{M}_2} ((s, A), (t, B))$$

*if and only if* $(s', A') \leqslant_{\mathbb{M}} (s, A)$ *and* $(t', B') \leqslant_{\mathbb{M}} (t, B)$.

**Remark 5.23.** *Let* $D$ *be the set of all pairs* $((s, A), (t, B)) \in \mathbb{M}_2$ *such that whenever* $A = \{a_i : i \in \omega\}$ *and* $B = \{b_i : i \in \omega\}$ *are increasing enumerations of* $A$ *and* $B$*, we have* $a_i < b_i < a_{i+1}$ *for all* $i \in \omega$*. It is clear that* $D$ *is dense in* $\mathbb{M}_2$.

Note that $Q^2$, in Definition 1.3 [23], is the same as the set $D$ defined in previous remark.

Notice that if $G$ is $\mathbb{M}_2$-generic filter over $V$, then

$$H_0 := \{(a, A) \in \mathbb{M} : \exists (b, B) \in \mathbb{M}((a, A), (b, B)) \in G\}$$

is an $\mathbb{M}$-generic filter over $V$. Similarly,

$$H_1 := \{(b, B) \in \mathbb{M} : \exists (a, A) \in \mathbb{M}((a, A), (b, B)) \in G\}$$

is an $\mathbb{M}$-generic filter over $V$. Thus $\mathbb{M}_2$ adds a pair of Mathias reals $(n, m)$. By Remark 5.23, we have that for some $k \in \omega$, $n_j < m_j < n_{j+1}$ for all $j \geqslant k$, where $n = \langle n_i \rangle_{i \in \omega}$ and $m = \langle m_i \rangle_{i \in \omega}$.

Recall definition of dense sets $D_n$ of $\mathbb{P}_{FIN}$ for all $n \in \omega$, as in (4.1.1) of Chapter 4:

$$D_n := \{(A, \mathscr{D}) \in \mathbb{P}_{FIN} : \forall x \in \mathscr{D}(|x| \geqslant n)\}.$$

**Lemma 5.24.** $\mathbb{M}_2$ *is a projection of* $D_2$*. Hence adding a generic for* $\mathbb{P}_{FIN}$ *adds a generic for* $\mathbb{M}_2$*.*

*Proof.* Given $(A, \mathscr{D}) \in D_2$, where $A = \langle x_0, \ldots, x_m \rangle$ and $\mathscr{D} = \langle d_i \rangle_{i \in \omega}$ such that $|d_i| \geqslant 2$ for all $n \in \omega$. We define $\pi((A, \mathscr{D})) \in \mathbb{M}_2$ as

$$((\{\min x_j : j \leqslant m\}, \{\min d_i : i \in \omega\}), (\{\max x_j : j \leqslant m\}, \{\max d_i : i \in \omega\})).$$

- $\pi$ preserves order.
- For all $((s, S), (t, T)) \in \mathbb{M}_2$ and for all $(A, \mathscr{D}) \in \mathbb{P}_{FIN}$ such that

  $$((s, S), (t, T)) \leqslant \pi((A, \mathscr{D}))$$

  there is $(A', \mathscr{D}') \leqslant (A, \mathscr{D})$ such that $\pi((A', \mathscr{D}')) \leqslant ((s, S), (t, T))$.

Order preservation is trivial. We shall prove the last statement.

Let $((s, S), (t, T)) \in \mathbb{M}_2$ and let $(A, \mathscr{D}) \in D_2$ where $A = \langle x_0, \ldots, x_m \rangle$, $\mathscr{D} = \langle d_i \rangle_{i \in \omega}$ such that $|d_i| \geqslant 2$ for all $n \in \omega$ and

$$((s, S), (t, T)) \leqslant_{\mathbb{M}_2} \pi((A, \mathscr{D})).$$

By Remark 5.23, there exists $((u, S'), (v, T'))$ in $D$ such that

$$((u, S'), (v, T')) \leqslant_{\mathbb{M}_2} ((s, S), (t, T)) \leqslant_{\mathbb{M}_2} \pi((A, \mathscr{D})).$$

Since $((u, S'), (v, T'))$ belongs to $D$,

$$S' = \{s'_i : i \in \omega\} \text{ and } T' = \{t'_i : i \in \omega\}$$

are increasing enumerations such that $s'_i < t'_i < s'_{i+1}$, for all $i \in \omega$.

We have

$$S' \subseteq S \subseteq \{\min d_i : i \in \omega\} \text{ and } T' \subseteq T \subseteq \{\max d_i : i \in \omega\}.$$

Define $\mathscr{D}' = \langle d'_l \rangle_{l \in \omega}$ as follows: for $l \in \omega$, $d'_l = d_{i_l} \cup d_{j_l}$, where $s'_l = \min d_{i_l}$ and $t'_l = \max d_{j_l}$. It is clear that $\mathscr{D}' \sqsubseteq \mathscr{D}$.

Let $l := |u| = |v|$. Since $((u, S'), (v, T')) \in \mathbb{M}_2$, we have

$$u_0 \leqslant v_0 \leqslant u_1 \leqslant v_1 \leqslant \ldots \leqslant u_{l-1} \leqslant v_{l-1}.$$

Notice that $u \supseteq \{\min x_j : j \leqslant m\}$ and if $x \in u \backslash \{\min x_j : j \leqslant m\}$, then $x \in \{\min d_i : i \in \omega\}$. Similary, for all $y \in v \backslash \{\max x_j : j \leqslant m\}$, $y \in \{\max d_i : i \in \omega\}$. Since $(A, \mathscr{D}) \in D_2$, we have that for all $j$ such that $m < j < l$, $u_j < v_j < u_{j+1}$.

For $m < k < l$, we have $u_k = \min d_{i_k}$ and $v_k = \max d_{j_k}$. Define $x_k := d_{i_k} \cup d_{j_k}$. Put $A' = A^\frown \langle x_{m+1}, \ldots, x_{l-1} \rangle$

Now

$$(A', \mathscr{D}') \leqslant_{\mathbb{P}_{FIN}} (A, \mathscr{D})$$

and

$$\pi((A', \mathscr{D}')) = ((u, S'), (v, T')) \leqslant_{\mathbb{M}_2} ((s, S), (t, T)).$$

Thus $\mathbb{P}_{FIN}$ adds a generic for $\mathbb{M}_2$. If $\mathscr{D}^*$ is a $\mathbb{P}_{FIN}$-generic block sequence over $V$, then

$$(\{\min d : d \in \mathscr{D}^*\}, \{\max d : d \in \mathscr{D}^*\})$$

is an $\mathbb{M}_2$-generic over $V$. $\square$

**Proposition 5.25.** $\mathbb{M}_2$ *is not equivalent to the usual product of two copies of Mathias forcing* $\mathbb{M} \times \mathbb{M}$.

*Proof.* We are going to prove that $\mathbb{M} \times \mathbb{M}$ adds a Cohen real. Let $(m_0, m_1)$ be a pair of Mathias reals $\mathbb{M} \times \mathbb{M}$-generic over $V$. We define $c \in 2^\omega$ as follows

$$c(i) = \begin{cases} 0 & \text{if } m_0(i) \leqslant m_1(i) \\ 1 & \text{if } m_0(i) > m_1(i) \end{cases}$$

We shall show that $c$ is a Cohen real. Let $D \subseteq \mathbb{C}$ be a dense subset.

Given $(a, b) \in [\omega]^{<\omega} \times [\omega]^{<\omega}$ such that $|a| = |b| = k$, and $a = \{a_i : i < k\}$ and $b = \{b_i : i < k\}$ are increasing enumerations of $a$ and $b$, respectively, we define a function $q_{(a,b)} : k \to 2$ as follows:

$$q_{(a,b)}(i) = \begin{cases} 0 & \text{if } a_i \leqslant b_i \\ 1 & \text{if } a_i > b_i \end{cases}$$

Define

$$D' := \{((a, A), (b, B)) \in \mathbb{M} \times \mathbb{M} : |a| = |b|, \exists q \in D \text{ such that } q = q_{(a,b)}\}.$$

**Claim 5.26.** *$D'$ is a dense subset of $\mathbb{M} \times \mathbb{M}$.*

*Proof of Claim:* Let $((a', A'), (b', B')) \in \mathbb{M}\times\mathbb{M}$. Assume that $|a'| \leqslant |b'|$, there exists a condition $(a'', A'') \in \mathbb{M}$ such that $(a'', A'') \leqslant (a', A')$ and $|a''| = |b'| = k$. Consider $q_{(a'',b')}$ defined as before. Since $D$ is dense, there is $q \in D$ such that $q \leqslant q_{(a'',b')}$, i.e., $q \restriction k = q_{(a'',b')}$. Hence, $((a'', A''), (b', B')) \leqslant ((a', A'), (b', B'))$ and $((a'', A''), (b', B')) \in D'$. So, $D'$ is a dense subset of $\mathbb{M} \times \mathbb{M}$. □

We have that $(G_{m_0} \times G_{m_1}) \cap D' \neq \emptyset$, there is $((a, A), (b, B)) \in G_{m_0} \times G_{m_1}$ and there is $q \in D$ such that $q = q_{(a,b)}$, where $|a| = |b| = k$. By definition of function $c$ we obtain that $c \restriction k = q_{(a,b)} = q$. Hence $c$ is a Cohen real.

Since $\mathbb{M}_2$ is a projection of $\mathbb{P}_{FIN}$ and $\mathbb{P}_{FIN}$ does not add Cohen reals by Theorem 4.27, we have that $\mathbb{M}_2$ does not add a Cohen real. Hence, $\mathbb{M}_2$ is different from $\mathbb{M} \times \mathbb{M}$. □

$\mathbb{P}_{FIN}$ does not add Cohen reals (Theorem 4.27), so it does not add a generic for $\mathbb{M} \times \mathbb{M}$.

In 1.2.2 of the Preliminaries section we defined the partial ordering $\mathbb{P}^* = \langle (FIN)^\omega, \sqsubseteq^* \rangle$. In Section 1.2, we defined $\mathbb{U} = \langle \mathscr{P}(\omega)/fin, \leqslant \rangle$ as the partial order whose elements are infinite subsets of $\omega$ ordered by $\subseteq^*$.

**Lemma 5.27.** *The partial order $\mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$ with the usual product is a projection of $\mathbb{P}^*$.*

*Proof.* Define $\pi : \mathbb{P}^* \to \mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$ as follows.

For $\mathscr{D} \in \mathbb{P}^*$, where $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$. We define

$$\pi(\mathscr{D}) := (\{\min d_i : i \in \omega\}, \{\max d_i : i \in \omega\}).$$

- $\pi$ preserves order.

Let $\mathscr{D} = \langle d_i \rangle_{i\in\omega}$ and $\mathscr{D}' = \langle d'_i \rangle_{i\in\omega}$ be elements of $(FIN)^\omega$ such that $\mathscr{D} \sqsubseteq^* \mathscr{D}'$. Then there exists $n \in \omega$ such that $\mathscr{D} \setminus n \sqsubseteq \mathscr{D}'$, i.e., $\mathscr{D} \setminus n \subseteq FU(\mathscr{D}')$.

Then

$$\pi(\mathscr{D}) = (\{\min d_i : i \in \omega\}, \{\max d_i : i \in \omega\}) \leqslant$$

$$\pi(\mathscr{D}') = (\{\min d'_i : i \in \omega\}, \{\max d'_i : i \in \omega\})$$

because

$$\{\min d_i : i \in \omega\} \setminus \{\min d'_i : i \in \omega\} \in [\omega]^{<\omega}$$

and

$$\{\max d_i : i \in \omega\} \setminus \{\max d'_i : i \in \omega\} \in [\omega]^{<\omega}.$$

- For all $(A, B) \in \mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$ and for all $\mathscr{D} = \langle d_i \rangle_{i\in\omega} \in \mathbb{P}^*$ such that $(A, B) \leqslant \pi(\mathscr{D})$, there is $\mathscr{D}' \sqsubseteq^* \mathscr{D}$ such that $\pi(\mathscr{D}') \leqslant (A, B)$.

  Let $(A, B) \in \mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$ and $\mathscr{D} = \langle d_i \rangle_{i\in\omega} \in \mathbb{P}^*$ be such that $(A, B) \leqslant \pi(\mathscr{D})$.

  Since the set

$$D = \{(A', B') : \text{ if } A' = \{a'_i : i \in \omega\} \text{ and } B' = \{b'_i : i \in \omega\}, \text{ then}$$

$$a'_i < b'_i < a'_{i+1}, \text{ for all } i \in \omega\}$$

  is a dense subset of $\mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$, there is some $(A', B') \in D$ such that $(A', B') \leqslant (A, B)$, and

$$A' \subseteq^* \{\min d_i : i \in \omega\} \text{ and } B' \subseteq^* \{\max d_i : i \in \omega\}$$

  There exists $n \in \omega$ such that $a'_i = \min d_{j_i}$ for all $i \geqslant n$ and there exists $m \in \omega$ such that $b'_i = \max d_{l_i}$ for all $i \geqslant m$. Letting $k := \max\{n, m\}$, we define $\mathscr{D}'$ as follows: let $d'_i = d_{j_i} \cup d_{l_i}$ for all $i \geqslant k$.

  It is clear that $\mathscr{D}' \sqsubseteq^* \mathscr{D}$ and

$$\pi(\mathscr{D}') = (\{\min d'_i : i \in \omega\}, \{\max d'_i : i \in \omega\}) \leqslant (A', B') \leqslant (A, B).$$

Hence $\mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$ is a projection of $\mathbb{P}^*$. $\square$

Propositions 4.17 and 4.18 proved that $\mathbb{P}^*$ adds a $\mathscr{P}(\omega)/fin$-generic filter over $V$ and the last lemma is a generalization of this fact because we are proving that $\mathbb{P}^*$ adds a generic for $\mathscr{P}(\omega)/fin \times \mathscr{P}(\omega)/fin$.

# Chapter 6

# Conclusion

We have worked with three forcing notions, namely, Mathias forcing, which we denote by $\mathbb{M}$, Matet forcing, which we denote by $\mathbb{MT}$, and $\mathbb{P}_{FIN}$. The partial orders $\mathbb{M}$ and $\mathbb{P}_{FIN}$ arose in our forcing proofs of the theorems of Ramsey and Hindman, respectively. As to $\mathbb{MT}$, we considered it by its similarity both to Mathias forcing and to $\mathbb{P}_{FIN}$, and it is by means of it that we managed to prove the non-equivalence of these last two forcing notions. In fact, none of the three partial orders mentioned is equivalent to any other.

In order to compare $\mathbb{P}_{FIN}$ with $\mathbb{M}$ and $\mathbb{MT}$ we have examined what properties $\mathbb{P}_{FIN}$ has. In particular, we have shown that, like the Mathias real in Mathias forcing, the generic block sequence added by $\mathbb{P}_{FIN}$ allows us to reconstruct the generic filter. As in the case of $\mathbb{M}$ and $\mathbb{MT}$, $\mathbb{P}_{FIN}$ can be put into the form of a two-step iteration of a $\sigma$-closed and a $\sigma$-centered partial orders. Again, also as $\mathbb{M}$ and $\mathbb{MT}$, $\mathbb{P}_{FIN}$ satisfies Axiom A and has the pure decision property. $\mathbb{P}_{FIN}$ adds no Cohen reals, either.

Finally, we considered a partial order defined by Shelah and Spinas, which we denote by $\mathbb{M}_2$, and have shown that $\mathbb{P}_{FIN}$ adds a generic filter for it.

There is more work to do. On the one hand, on the line of what we did regarding Ramsey's and Hindman's theorems, we are seeking a partial order suitable for a proof of Gowers' Theorem on the existence of homogeneous sets for partitions of $\mathrm{FIN}_k$. On the other hand, we want to keep on studying

$\mathbb{P}_{FIN}$, by answering such natural questions as

1. What are the cardinal invariants associated with $\mathbb{P}_{FIN}$?
2. Does $\mathbb{P}_{FIN}$ add Sacks, Silver, or Laver reals?
3. Is there an extrinsic characterization of the generic block sequences of $\mathbb{P}_{FIN}$, in a way similar to Mathias reals being characterized in terms of almost disjoint families of the ground model?
4. Is it true, again as in the case of Mathias forcing, that every condensation of a $\mathbb{P}_{FIN}$-generic block sequence is also a generic block sequence?

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