Random zero sets of analytic functions and traces of functions in Fock spaces

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Jeremiah Anthony Buckley

Certifiquem que la present memòria ha estat realitzada per
Jeremiah Anthony Buckley i dirigida per nosaltres,

Xavier Massaneda, Codirector i tutor.	Joaquim Ortega-Cerdà, Codirector.

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Contents

Resum

Els objectes d'interès principal d'aquesta monografia són funcions associades a certs espais de Hilbert de funcions holomorfes. Considerem dos problemes principals que són essencialment diferents, tot i que fan ús de les mateixes tècniques. En el primer problema estudiem els valors que prenen les funcions de l'espai en certes successions 'crítiques' (definció que precisarem an breu). El segon problema té natura probabilística, estudiem el conjunt de zeros de combinacions lineals aleatòries de funcions triades en aquests espais.

Traces

Comencem descrivint el primer d'aquests problemes en l'espai de Bargmann-Fock clàssic, on les idees són més fàcilment assimilables. Aquest espai es defineix com

$$\mathcal{F}^2 = \{f \in H(\mathbb{C}) : \|f\|^2_{\mathcal{F}^2} = \int_{\mathbb{C}} |f(z)|^2 e^{-2|z|^2}\, dm(z) < +\infty\}$$

on m denota la mesura de Lebesgue del plà. Aquest és un espai de Hilbert amb nucli reproductor $K(z,w) = 2e^{2z\overline{w}}/\pi$. Denotem els nuclis reproductors normalitzats per

$$k_w(z) = \frac{K(z,w)}{\|K(\cdot,w)\|_{\mathcal{F}^2}} = \sqrt{\frac{2}{\pi}} e^{2z\overline{w} - |w|^2}.$$

Ens interessen els dos conceptes següents.

Definició. Una successió $\Lambda \subseteq \mathbb{C}$ és d'*interpolació* per a $\mathcal{F}^2$ si per a tota successió de valors $c = (c_\lambda)_{\lambda\in\Lambda}$ tal que

$$\sum_{\lambda\in\Lambda} |c_\lambda|^2 e^{-2|\lambda|^2} < +\infty$$

existeix $f \in \mathcal{F}^2$ tal que $f|\Lambda = c$.

Una successió $\Lambda \subseteq \mathbb{C}$ és de *mostreig* per a $\mathcal{F}^2$ si existeix una constant $C > 0$ tal que

$$C^{-1} \sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-2|\lambda|^2} \le \|f\|^2_{\mathcal{F}^2} \le C \sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-2|\lambda|^2}$$

per a tota $f \in \mathcal{F}^2$.

Així doncs les successions d'interpolació són aquelles en les que els valors de les funcions de l'espai a la successió es poden descriure en termes purament de creixement, mentre que les successions de mostreig són aquelles que permeten recuperar la funció a partir de les seves mostres. Interpolació i mostreig han de ser per tant conceptes enfrontats; les successions d'interpolació han d'ésser disperses mentre que les de mostreig han de ser denses. Seip i Wallstén ([Sei92] i [SW92]) van caracteritzar completament els conjunts de mostreig i/o interpolació en termes de densitats, la qual cosa fa precisa aquesta idea.

Teorema ([Sei92, Theorem 2.1, Theorem 2.2; SW92, Theorem 1.1, Theorem 1.2])**.** *Una successió Λ és d'interpolació per a $\mathcal{F}^2$ si i només si*

- *Λ és uniformement separada, és a dir, $\inf_{\lambda\neq\lambda'}|\lambda-\lambda'|>0$ i*
- $\mathcal{D}^+(\Lambda)=\limsup_{r\to\infty}\sup_{z\in\mathbb{C}}\dfrac{\#\left(\Lambda\cap\overline{D(z,r)}\right)}{\pi r^2}<2/\pi.$

Una successió Λ és de mostreig per a $\mathcal{F}^2$ si i només si

- *Λ és unió finita de successions uniformement separades, i*
- *existeix una successió uniformement separada $\Lambda'\subseteq\Lambda$ tal que*

$$\mathcal{D}^-(\Lambda')=\liminf_{r\to\infty}\inf_{z\in\mathbb{C}}\frac{\#\left(\Lambda'\cap\overline{D(z,r)}\right)}{\pi r^2}>2/\pi.$$

En particular, no hi ha successions que siguin simultàniament d'interpolació i mostreig.

Aquests conceptes es poden pensar també en termes de l'operador restricció. Definim

$$\begin{aligned} R_\Lambda:\mathcal{F}^2&\to\ell^2(\Lambda)\\ f&\mapsto f(\lambda)e^{-|\lambda|^2}. \end{aligned}$$

Aleshores Λ és d'interpolació si i només si R_Λ és exhaustiu, mentre que Λ és de mostreig si i només si R_Λ és acotat i injectiu. Considerem el problema de descriure el conjunt de valors $c=(c_\lambda)_{\lambda\in\Lambda}$ per als que existeix alguna funció f en un espai de Fock amb la condició $f|\Lambda=c$, on Λ és una successió tal que $\mathcal{D}^+(\Lambda)=\mathcal{D}^-(\Lambda)=2/\pi$. De manera equivalent, volem descriure el conjunt imatge R_Λ en aquest cas crític.

Un exemple instructiu ve donat per la xarxa entera

$$\Lambda=\alpha(\mathbb{Z}+i\mathbb{Z})$$

que compleix $\mathcal{D}^+(\Lambda)=\mathcal{D}^-(\Lambda)=\alpha^2$. Així doncs Λ és d'interpolació per a $\alpha<\sqrt{\frac{\pi}{2}}$, i de mostreig per a $\alpha>\sqrt{\frac{\pi}{2}}$. Considerem el cas $\alpha=\sqrt{\frac{\pi}{2}}$, que anomenem xarxa entera crítica.

No ens restringim només al context d'espais de Hilbert, sino que també considerem tots els espais

$$\mathcal{F}^p = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2}\, dm(z) < +\infty\}, \text{ for } 1 \leq p < +\infty$$

i

$$\mathcal{F}^\infty = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^\infty} = \sup_{z\in\mathbb{C}} |f(z)|e^{-|z|^2} < +\infty\}$$

per als que existeixen nocions anàlogues de successió d'interpolació i de mostreig. A més estudiem aquest mateix problema en espais més generals, reemplaçant el pes $|z|^2$ per una funció subharmònica ϕ el Laplacià $\Delta\phi$ de la qual és una mesura doblant.

Funcions analítiques Gaussianes

Comencem amb una descripció general de les funcions analítiques Gaussianes; se'n pot trobar un tractament comprensiu a [HKPV09]. Diem que una variable aleatòria a valors complexos és una normal complexa estàndard si la seva densitat de probabilitat és $\frac{1}{\pi}\exp(-|z|^2)$ respecte la mesura de Lebesgue del plà; la denotem $\mathcal{N}_{\mathbb{C}}(0,1)$.

Teorema ([HKPV09, Lemma 2.2.3]). *Sigui $\mathcal{D}$ un domini a $\mathbb{C}$, sigui f_n una successió de funcions analítiques a $\mathcal{D}$ i sigui a_n una successió de variables aleatòries independents i idènticament distribuïdes (iid) amb distribució $\mathcal{N}_{\mathbb{C}}(0,1)$. Suposem a més que $\sum_n |f_n|^2$ convergeix uniformement a compactes de $\mathcal{D}$. Aleshores, gairebé segurament,*

$$f = \sum_n a_n f_n$$

és una funció holomorfa a $\mathcal{D}$. Diem que f és una funció analítica Gaussiana (GAF).

En primer lloc observem que $f(z)$ és una variable normal complexa de mitjana zero, per a cada $z \in \mathcal{D}$. El nucli de covariàncies associat a f ve donat per

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \sum_n f_n(z)\overline{f_n(w)}$$

que és un nucli semidefinit positiu, analític en z i anti-analític en w. A més la distribució de la variable aleatòria f queda determinada pel nucli K.

Ens interessem en l'estudi del conjunt de zeros $\mathcal{Z}(f)$ i una primera observació és que, com que $f(z)$ és una variable normal amb mitjana zero i variància $K(z,z)$, f té un zero determinista z si i només si $K(z,z) = 0$. Per simplificar l'estudi suposarem que això no succeeix, és a dir, suposarem que $K(z,z) \neq 0$ per a tot $z \in \mathcal{D}$.

Estudiem el conjunt de zeros $\mathcal{Z}(f)$ a través de la mesura comptadora

$$n_f = \frac{1}{2\pi}\Delta \log|f|$$

(aquesta igualtat s'ha d'entendre en le sentit de les distribucions). Ens referim a la mesura $\mathbb{E}[n_f]$ com la primera intensitat del GAF. La fórmula de Edelman-Kostlan ([Sod00, Theorem 1] o [HKPV09, Section 2.4]) dóna (com anteriorment m és la mesura de Lebesgue)

$$\mathbb{E}[n_f(z)] = \frac{1}{4\pi}\Delta \log K(z,z)\, m(z).$$

Observem que, essent $K(z,z)$ una funció regular que no s'anul·la, aquest Laplacià és ben definit.

Una propietat sorprenent dels GAFs és la següent, que diu que el nombre mitjà de zeros determina la distribució del conjunt de zeros. Utilitzem la notació $\overset{d}{=}$ per a indicar que dues variables aleatòries tenen la mateixa distribució.

Teorema ([Sod00, Theorem 2; HKPV09, Theorem 2.5.2])**.** *Siguin f_1 i f_2 dos GAFs en un domini $\mathcal{D}$ amb la mateixa primera intensitat. Aleshores existeix una funció determinista que no s'anul·la $g \in H(\mathcal{D})$ tal que $f_1 \overset{d}{=} gf_2$. En particular $\mathcal{Z}(f_1)$ i $\mathcal{Z}(f_2)$ tenen la mateixa distribució (com a mesures no-negatives a valors enters).*

Malauradament la demostració d'aquest resultat no és de cap manera constructiva, donada una primera intensitat no hi ha cap indicació de com construir el GAF corresponent. Tornarem a aquest problema més endavant.

La funció analítica Gaussiana plana

Ens centrem ara en el GAF 'pla'. Aquesta funció és particularment interessant a causa de la propietat remarcable que la distribució del seu conjunt de zeros és invariant per automorfismes del plà, i és l'única funció analítica Gaussiana amb aquesta propietat.

Fixem un paràmetre $L > 0$ i considerem l'espai de Fock [1]

$$\mathcal{F}^2_L = \{f \in H(\mathbb{C}) : \|f\|^2_{\mathcal{F}^2_L} = \frac{L}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-L|z|^2}\, dm(z) < +\infty\}.$$

Observem que $\left(\frac{(\sqrt{L}z)^n}{\sqrt{n!}}\right)_{n=0}^{\infty}$ és una base ortonormal de $\mathcal{F}^2_L$ i per tant

$$K_L(z,w) = \sum_{n=0}^{\infty} \frac{(\sqrt{L}z)^n}{\sqrt{n!}}\frac{(\sqrt{L}\overline{w})^n}{\sqrt{n!}} = \exp(Lz\overline{w})$$

és el nucli reproductor d'aquest espai. Definim

$$f_L(z) = \sum_{n=0}^{\infty} a_n \frac{(\sqrt{L}z)^n}{\sqrt{n!}}$$

[1] Aquí ens convé utilitzar una normalització lleugerament diferent de la utilitzada anteriorment.

on, com abans, $(a_n)_{n=0}^{\infty}$ és una successió de variables aleatòries iid $\mathcal{N}_{\mathbb{C}}(0,1)$. Veiem que f_L és un GAF definit a tot $\mathbb{C}$ amb nucli de covariàncies K_L.

El nombre esperat de zeros, per la fórmula d'Edelman-Kostlan, és

$$\frac{1}{4\pi}\Delta \log K_L(z,z)\, m(z) = \frac{1}{4\pi}\Delta(L|z|^2)\, m(z) = \frac{L}{\pi}m,$$

de manera que podem pensar L és un paràmetre que descriu la 'intensitat' del nombre mitjà de zeros. La mesura de Lebesgue és clarament invariant per automorfismes del plà, i com que la primera intensitat determina la distribució del conjunt de zeros, veiem que aquesta és també invariant. A més, com que els múltiples (per constants) de la mesura de Lebesgue són les úniques mesures invariants del plà, veiem que aquests són (essencialment) els únics GAFs amb aquesta propietat.

Ara discutim alguns dels resultats coneguts sobre el conjunt de zeros del GAF pla. La llista no pretén de cap manera ésser completa.

Una manera de mirar d'entendre el comportament del conjunt de zeros és l'estudi dels 'estadístics lineals regulars'. Donada una funció real regular ψ amb suport compacte (que suposem no idènticament nul·la) definim

$$n(\psi, L) = \frac{1}{L}\int \psi\, dn_L = \frac{1}{L}\sum_{a\in\mathcal{Z}(f_L)} \psi(a).$$

La fórmula d'Edelman-Kostlan dóna immediatament

$$\mathbb{E}[n(\psi, L)] = \frac{1}{\pi}\int \psi\, dm,$$

i la primera qüestió òbvia és el càlcul de la variància, que denotem per $\mathbb{V}$. Sodin i Tsirelson [ST04] provaren que (aquí ζ és la funció zeta de Riemann)

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3)}{16\pi}\frac{1}{L^3}\|\Delta\psi\|_{L^2}(1+o(1)) \text{ as } L\to\infty.$$

També s'ha estudiat, per al conjunt de zeros del GAF pla, la probabilitat que hi hagi grans desviacions de la mitjana. Un esdeveniment interessant és la probabilitat que hi hagi un 'forat' (hole probability), la probabilitat que no hi hagi zeros en una regió del plà complex on el nombre esperat de punts és gran. El decaïment asimptòtic d'aquesta probabilitat per a discs fou calculada a [ST05], i a [Nis10, Theorem 1.1] Nishry va obtenir la versió més precisa

$$\mathbb{P}[n_L(D(z_0,r))=0] = \exp\Big\{-\frac{e^2}{4}L^2r^4(1+o(1))\Big\}$$

quan $Lr^2\to\infty$.

En aquesta tesi construïm nous GAFs considerant bases d'espais de Fock més generals. En particular construïm GAFs amb una primera intensitat que tendeix asimptòticament a una mesura donada. Els GAFs que considerem no tenen invariància per translacions, però molts dels resultats que hem esmentat estenen a aquest context.

La funció analítica Gaussiana hiperbòlica

Un company natural del GAF pla és el GAF hiperbòlic. De nou, fixem un paràmetre $L > 0$ i definim ara

$$f_L(z) = \sum_{n=0}^{\infty} a_n \left(\frac{L(L+1)\cdots(L+n-1)}{n!} \right)^{1/2} z^n$$

per a $z \in \mathbb{D}$, on, com abans, $(a_n)_{n=0}^{\infty}$ és una successió de variables aleatòries normals complexes iid. Per a $L > 1$ la successió $(\left(\frac{L(L+1)\cdots(L+n-1)}{n!}\right)^{1/2} z^n)_{n=1}^{\infty}$ és una base ortonormal per als espais de Bergman amb pesos

$$\mathcal{B}_L^2 = \{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_L^2}^2 = L \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^L \, d\nu(z) < +\infty \}$$

on $d\nu(z) = \frac{dm(z)}{\pi(1-|z|^2)^2}$ denota l'àrea hiperbòlica. Per a $L \leq 1$ les mateixes successions són bases ortonormals de diversos espais de funcions holomorfes al disc, però no farem èmfasi en aquest aspecte, ja que no és important per al que volem fer.

Tenim que f_L és un GAF definit a $\mathbb{D}$ amb nucli de covariàncies associat

$$K_L(z,w) = \mathbb{E}[f_L(z)\overline{f_L(w)}] = (1 - z\overline{w})^{-L}.$$

Denotem la mesura comptadora del conjunt de zeros de f_L per n_L. La primera intensitat del conjunt de zeros ve donada per $L\nu$, que és invariant per automorfismes del disc. Com al cas pla veiem que la distribució del conjunt de zeros és també invariant per automorfismes del disc, i que f_L és essencialment l'únic GAF amb aquesta propietat.

Definim de nou l''estadístic lineal regular' mitjançant

$$n(\psi, L) = \frac{1}{L} \int \psi \, dn_L = \frac{1}{L} \sum_{a \in \mathcal{Z}(f_L)} \psi(a).$$

on ψ és una funció regular (no idènticament nul·la) suportada a un compacte de $\mathbb{D}$. Sodin i Tsirelson [ST04] provaren, utilitzant les mateixes tècniques que al cas pla, que per al GAF hiperbòlic la variància de l'estadístic lineal regular té el mateix decaïment asimptòtic, és a dir,

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3)}{16\pi} \frac{1}{L^3} \|\Delta\psi\|_{L^2} (1 + o(1)) \text{ as } L \to \infty.$$

Pel valor particular $L = 1$ Peres i Virág [PV05, Theorem 2] donaren una descripció completa de la variable aleatòria $n_1(D(0,r))$, en particular mostraren que

$$\mathbb{V}[n_1(D(0,r))] = \frac{r^2}{1-r^4}$$

i calcularen la probabilitat que hi hagi un forat,

$$\mathbb{P}[n_1(D(0,r)) = 0] = \exp\left(-\frac{\pi^2 r^2}{6(1-r^2)} (1 + o(1)) \right)$$

quan $r \to 1^-$. Demostraren aquests resultats provant que el conjunt de zeros corresponent és un procés amb certes propietats especials (anomenat 'determinantal'). Però això només passa per a $L = 1$, i per tant les seves tècniques no funcionen per a cap altre valor de L.

En aquesta tesi calculem el comportament asimptòtic de la variància de $n_L(D(0,r))$ quan r s'acosta a 1 per a tots els valors de L. També calculem el decaïment asimptòtic de la probabilitat que hi hagi un forat per a r fix i valors de L grans.

El procés de Paley-Wiener

Per acabar esmentem un problema lligat directament amb els anteriors. Definim la funció aleatòria

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)},$$

on a_n són variables normals *reals* iid amb mitjana zero i variància 1. És ben sabut que les translacions enteres del sinus cardinal $(\sin \pi(z-n)/\pi(z-n))_{n\in\mathbb{Z}}$ formen una base ortonormal de l'espai de Paley-Wiener. Així f és gairebé segurament una funció entera [HKPV09, Lemma 2.2.3] amb nucli de covariàncies donat per

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \frac{\sin \pi(z-\overline{w})}{\pi(z-\overline{w})},$$

el nucli reproductor de l'espai de Paley-Wiener.

Un cop més denotem per n_f la mesura comptadora del conjunt de zeros de f, i diem que aquest és el procés de Paley-Wiener. Feldheim [Fel10] ha calculat la versió corresponent de la fórmula d'Edelman-Kostlan, que dóna

$$\mathbb{E}[n_f(z)] = S(y)m(z) + \frac{1}{2\sqrt{3}}\mu(x),$$

on $z = x + iy$, m és de nou la mesura de Lebesgue planar, μ és la mesura singular respecte m suportada a $\mathbb{R}$ i idèntica a la mesura de Lebesgue, i S és una funció regular fora de 0 que satisfà $S(y) \leq C|y|$ quan y s'acosta a zero, per a alguna constant $C > 0$. Així gairebé segurament hi ha zeros a la recta real, però són rars a prop de la recta real. A més el conjunt de zeros és en mitjana distribuït uniformement a la recta real.

Demostrem que la probabilitat que hi hagi un interval gran de la recta real sense zeros decau exponencialment en funció de la longitud de l'interval.

. . . et ignotas animum dimittit in artes. . .

Ovid, Metamorphosis VII, 188

All I know is a door into the dark.

Seamus Heaney, The Forge

Chapter 1

Introduction

The primary objects of interest in this monograph are linear combinations of functions chosen from certain Hilbert spaces of holomorphic functions. The functions we consider satisfy some spanning conditions, they may be an orthonormal basis or they may satisfy some weaker notion, for example, they might be a frame (see Definition 1.2).

We consider two main problems which are essentially distinct, although we use many of the same techniques. In the first problem we study the behaviour of sequences of functions that 'just fail' to have these spanning properties. The second problem is of a probabilistic nature, we study the zero sets of random linear combinations of functions with good spanning properties.

1.1 Traces

We begin by describing the first of these problems in the classical Bargmann-Fock space, where the ideas are more easily digestible. This space is defined as

$$\mathcal{F}^2 = \{f \in H(\mathbb{C}) : \|f\|^2_{\mathcal{F}^2} = \int_{\mathbb{C}} |f(z)|^2 e^{-2|z|^2}\, dm(z) < +\infty\}$$

where m denotes the Lebesgue measure on the plane. This is a reproducing kernel Hilbert space with reproducing kernel $K(z,w) = 2e^{2z\overline{w}}/\pi$, that is, for all $f \in \mathcal{F}^2$

$$f(w) = \langle f, K(\cdot, w)\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural inner product on $\mathcal{F}^2$ given by

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2|z|^2}\, dm(z).$$

We denote the normalised reproducing kernels by

$$k_w(z) = \frac{K(z,w)}{\|K(\cdot,w)\|_{\mathcal{F}^2}} = \sqrt{\frac{2}{\pi}} e^{2z\overline{w} - |w|^2}.$$

We are interested in the following two concepts which, as we will soon see, are related to the spanning properties mentioned above.

Definition 1.1. A sequence $\Lambda \subseteq \mathbb{C}$ is *interpolating* for $\mathcal{F}^2$ if for every sequence of values $c = (c_\lambda)_{\lambda\in\Lambda}$ such that

$$\sum_{\lambda\in\Lambda} |c_\lambda|^2 e^{-2|\lambda|^2} < +\infty$$

there exists $f \in \mathcal{F}^2$ such that $f|\Lambda = c$.

A sequence $\Lambda \subseteq \mathbb{C}$ is *sampling* for $\mathcal{F}^2$ if there exists a constant $C > 0$ such that

$$C^{-1}\sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-2|\lambda|^2} \leq \|f\|^2_{\mathcal{F}^2} \leq C\sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-2|\lambda|^2}$$

for every $f \in \mathcal{F}^2$.

To relate these definitions to spanning properties we recall the definitions of a frame and a Riesz sequence.

Definition 1.2. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

A sequence $(x_n)_n$ in X is said to be a ***Riesz sequence*** if there exists a constant $C > 0$ such that

$$C^{-1}\sum_n |a_n|^2 \leq \Big\|\sum_n a_n x_n\Big\|^2 \leq C\sum_n |a_n|^2$$

for any sequence $(a_n)_n \in \ell^2$.

A sequence $(x_n)_n$ in X is said to be a *frame* if there exist $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\|x\|^2$$

for all $x \in X$.

It can be shown that if $(x_n)_n$ is a frame in X then there exists a sequence $(\tilde{x}_n)_n$ in X (the canonical dual frame) such that

$$x = \sum_n \langle x, \tilde{x}_n\rangle x_n$$

and

$$\frac{1}{B}\|x\|^2 \leq \sum_n |\langle x, \tilde{x}_n\rangle|^2 \leq \frac{1}{A}\|x\|^2$$

for all $x \in X$. Thus a frame can be thought of as a generalisation of a basis that retains the 'expansion' properties of a basis although the elements of the frame are not, in general, linearly independent. On the other hand, the elements of a Riesz sequence are clearly linearly independent, although in general not every element of X may be expanded as a linear combination of elements from a Riesz sequence. An orthonormal basis is of course both a Riesz sequence and a frame. For a proof of the above facts and a general introduction to frames and Riesz sequences see, for example, [Chr03].

To relate these two definitions we note that

$$\langle f, k_\lambda \rangle = \frac{f(\lambda)}{\sqrt{K(\lambda, \lambda)}} = \sqrt{\frac{\pi}{2}} f(\lambda) e^{-|\lambda|^2}$$

which means that Λ is a sampling sequence if and only if there exists a constant $C > 0$ such that

$$C^{-1} \sum_{\lambda \in \Lambda} |\langle f, k_\lambda \rangle|^2 \le \|f\|_{\mathcal{F}^2}^2 \le C \sum_{\lambda \in \Lambda} |\langle f, k_\lambda \rangle|^2$$

for every $f \in \mathcal{F}^2$, that is, if and only if $(k_\lambda)_{\lambda \in \Lambda}$ is a frame for $\mathcal{F}^2$. One may also show that Λ is an interpolating sequence if and only if $(k_\lambda)_{\lambda \in \Lambda}$ is a Riesz sequence.

Thus the interpolating sequences are the sequences such that the values functions from the space take on the sequence can be described purely in terms of a natural growth condition, while sampling sequences are the sequences that allow one to recover a function from its samples. Interpolating and sampling should therefore be competing concepts, interpolating sequences should be sparse, in some sense, while sampling sequences should be dense. Seip and Wallstén ([Sei92] and [SW92]) completely characterised sets of sampling and sets of interpolation in terms of densities, which makes this idea precise.

Theorem 1.3 ([Sei92, Theorem 2.1, Theorem 2.2; SW92, Theorem 1.1, Theorem 1.2]). *A sequence Λ is an interpolating sequence for $\mathcal{F}^2$ if and only if*

- *Λ is a uniformly separated sequence, that is, $\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| > 0$, and*
- $\mathcal{D}^+(\Lambda) = \limsup_{r \to \infty} \sup_{z \in \mathbb{C}} \dfrac{\#\left(\Lambda \cap \overline{D(z,r)}\right)}{\pi r^2} < 2/\pi.$

A sequence Λ is a sampling sequence for $\mathcal{F}^2$ if and only if

- *Λ is a finite union of uniformly separated sequences, and*
- *there exists a uniformly separated sequence $\Lambda' \subseteq \Lambda$ such that*

$$\mathcal{D}^-(\Lambda') = \liminf_{r \to \infty} \inf_{z \in \mathbb{C}} \frac{\#\left(\Lambda' \cap \overline{D(z,r)}\right)}{\pi r^2} > 2/\pi.$$

In particular there are no sequences that are simultaneously interpolating and sampling.

We will be interested in sequences Λ such that these 'spanning properties' just fail, in the sense that $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = 2/\pi$. An instructive example is given by the integer lattice

$$\Lambda = \alpha(\mathbb{Z} + i\mathbb{Z})$$

which satisfies $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = \alpha^2$. Thus Λ is interpolating for $\alpha < \sqrt{\frac{\pi}{2}}$, and Λ is sampling for $\alpha > \sqrt{\frac{\pi}{2}}$. We will consider the case $\alpha = \sqrt{\frac{\pi}{2}}$, which we call the critical integer lattice.

Our problem can also be motivated from the perspective of time-frequency analysis. Gabor [Gab46] proposed that any function $f \in L^2(\mathbb{R})$ could be expanded as

$$f(t) = \sum_{n,m\in\mathbb{Z}} a_{mn} g(t-n) e^{2\pi i m t} \tag{1.1}$$

where g is the Gaussian 'window',

$$g(x) = e^{-\frac{\pi}{2}x^2}.$$

The Bargmann transform maps $L^2(\mathbb{R})$ isometrically to the Bargmann-Fock space, and takes the Gabor system $(g(t-n)e^{2\pi i m t})_{n,m\in\mathbb{Z}}$ to the sequence of normalised reproducing kernels for the Fock space $(k_\lambda(z))_{\lambda\in\Lambda}$, where Λ is the critical integer lattice (see, for example, [DG88] for details). If the Gabor system were a Riesz basis of $L^2(\mathbb{R})$ then $(k_\lambda(z))_{\lambda\in\Lambda}$ would be a Riesz basis of $\mathcal{F}^2$ and so the critical lattice would be simultaneously interpolating and sampling, which we know does not hold by Theorem 1.3. This means that the expansion (1.1) is not stable. We will be interested in understanding, in some sense, how far the Gabor system is from being a Riesz basis.

These concepts can also be thought of in terms of the restriction operator. Define

$$R_\Lambda : \mathcal{F}^2 \to \ell^2(\Lambda)$$
$$f \mapsto f(\lambda)e^{-|\lambda|^2}.$$

Then Λ is interpolating if and only if R_Λ is surjective, while Λ is sampling if and only if R_Λ is bounded and injective. We will consider the problem of describing the set of values $c = (c_\lambda)_{\lambda\in\Lambda}$ such that there exists some function f in a Fock space satisfying the condition $f|\Lambda = c$ where Λ is a sequence satisfying $\mathcal{D}^+(\Lambda) = \mathcal{D}^-(\Lambda) = 2/\pi$. Equivalently, we are interested in describing the range of R_Λ in this critical case.

We shall not just restrict ourselves to the Hilbert space context, but rather consider the full range of spaces

$$\mathcal{F}^p = \{f \in H(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p} = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2}\, dm(z) < +\infty\}, \text{ for } 1 \le p < +\infty$$

and

$$\mathcal{F}^\infty = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^\infty} = \sup_{z\in\mathbb{C}} |f(z)|e^{-|z|^2} < +\infty\}$$

where analogous notions of interpolating sequences, sampling sequences and sequences of critical density exist. Moreover we will study this same problem in more general spaces, where the function $|z|^2$ is replaced by a subharmonic function ϕ whose Laplacian $\Delta\phi$ is a doubling measure.

1.2 Gaussian analytic functions

Random point processes are well-studied objects in both mathematics and physics. Many physical phenomena can be modelled by random point processes, for example, the arrival times of people in a queue, the arrangement of stars in a galaxy, and the energy levels of heavy nuclei of atoms. The classical, and most important, example of a random point process is the Poisson point process. The defining characteristic of the Poisson process is that the process is stochastically independent when restricted to disjoint sets. This means that knowing that there is a point of the process at a given location does not affect the probability that there are points nearby. In many physical situations this independence is a natural assumption, but it is obviously unacceptable in others. For example, if one considers negatively charged particles confined in an external field (a 'one-component plasma' in the nomenclature of physics) then the particles naturally repel. If we know that there is a particle at a given point, then it is highly unlikely that there are particles nearby. In contrast, if one studies the outbreak of a contagious disease, then knowing that there is a case in a given location makes it much more likely that there are cases nearby.

For this reason it is of interest to study random point processes that do not satisfy an independence assumption. One way to do this is to consider the zero sets of random analytic functions. By this we mean that we specify the distribution of some coefficients that define a random analytic function. If one then considers the point process in the complex plane given by the zero set of the random analytic function, it is well known that the zero set exhibits 'local repulsion' (see [HKPV09, Chapter 1]). We will consider certain Gaussian analytic functions, which arise by choosing the defining coefficients to be normally distributed. We begin with a general description of Gaussian analytic functions; for a comprehensive treatment we refer the reader to the book [HKPV09], non-technical introductions can also be found in [Sod05], [NS10a] and [NS10b, Part I].

We say that a complex-valued random variable is a standard complex normal if its probability density is $\frac{1}{\pi}\exp(-|z|^2)$ with respect to the Lebesgue measure on the plane; we denote this distribution $\mathcal{N}_{\mathbb{C}}(0,1)$.

Proposition 1.4 ([HKPV09, Lemma 2.2.3]). *Let $\mathcal{D}$ be a domain in $\mathbb{C}$, let f_n be a sequence of analytic functions on $\mathcal{D}$ and let a_n be a sequence of iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables. Suppose further that $\sum_n |f_n|^2$ converges uniformly on compact subsets of $\mathcal{D}$. Then, almost surely,*

$$f = \sum_n a_n f_n$$

is a holomorphic function on $\mathcal{D}$. We say that f is a Gaussian analytic function (GAF).

A brief remark on our notation; we are tacitly assuming here that we are given a probability space $(\Omega, \mathbb{P})$ and we have defined a random variable

$$\begin{aligned} f : \Omega &\to H(\mathcal{D}) \\ \omega &\mapsto f_\omega \end{aligned}$$

outside a set of probability zero. Here, and throughout this monograph, we follow the probabilistic convention of suppressing the dependence on the random parameter ω, and ignoring events of probability zero.

We first note that, since linear combinations of normals are normal, and limits of normals are normal, $f(z)$ is a mean-zero complex normal random variable, for each $z \in \mathcal{D}$. The covariance kernel associated to f is given by

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \sum_n f_n(z)\overline{f_n(w)} \tag{1.2}$$

which is a positive semi-definite kernel, analytic in z and anti-analytic in w. Moreover the distribution of the random variable f is determined by the kernel K.

We are interested in studying the zero set $\mathcal{Z}(f)$ and a first observation is that, since $f(z)$ is a mean-zero normal random variable with variance $K(z,z)$, f has a deterministic zero at z if and only if $K(z,z) = 0$. To simplify matters we will assume this is not the case, that is, we assume that $K(z,z) \neq 0$ for all $z \in \mathcal{D}$. In other words, by (1.2), we are assuming that the functions f_n do not have a common zero. Moreover, since $f(z)$ has mean zero, there are no deterministic solutions to $f(z) = \zeta$ for any non-zero $\zeta \in \mathbb{C}$. Furthermore ([HKPV09, Lemma 2.4.1]) the random zeroes of $f - \zeta$ are almost surely simple, for all $\zeta \in \mathbb{C}$.

We study the zero set $\mathcal{Z}(f)$ through the counting measure

$$n_f = \frac{1}{2\pi}\Delta \log|f|$$

(this equality is to be understood in the distributional sense). We refer to the measure $\mathbb{E}[n_f]$ as the first intensity. The Edelman-Kostlan formula ([Sod00, Theorem 1] or [HKPV09, Section 2.4]) gives (as before m is the Lebesgue measure)

$$\mathbb{E}[n_f(z)] = \frac{1}{4\pi}\Delta \log K(z,z)\, m(z). \tag{1.3}$$

We note that, since $K(z,z)$ is a smooth non-vanishing function, this is well-defined[1].

A surprising property of GAFs is the following, which says that the mean number of zeroes determines the distribution of the zero set. We use the notation $\overset{d}{=}$ to indicate that two random variables have the same distribution.

Theorem 1.5 ([Sod00, Theorem 2; HKPV09, Theorem 2.5.2]). *Suppose f_1 and f_2 are two GAFs on a domain $\mathcal{D}$ with the same first intensity. Then there exists a deterministic, non-vanishing $g \in H(\mathcal{D})$ such that $f_1 \overset{d}{=} gf_2$. In particular $\mathcal{Z}(f_1)$ and $\mathcal{Z}(f_2)$ have the same distribution (as non-negative integer-valued measures).*

[1] If we allow deterministic zeroes, then the measure $E[n_f]$ has an atom at each of these zeroes. These correspond to the distributional Laplacian of $\log K(z,z)$, so we can interpret the Edelman-Kostlan formula in a distributional sense.

It is unfortunate[2] that the proof of this result is in no way constructive, given a first intensity there is no indication of how one might construct 'the' corresponding GAF. We shall return to this problem later.

The flat Gaussian analytic function

We now focus on the 'flat' GAF. This function is particularly interesting because of the remarkable property that the distribution of its zero set is invariant under automorphisms of the plane, and it is the unique (in a certain sense) Gaussian analytic function with this property.

We fix a parameter $L > 0$ and consider the Fock space[3]

$$\mathcal{F}_L^2 = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}_L^2}^2 = \frac{L}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-L|z|^2}\, dm(z) < +\infty\}.$$

We note that $\left(\frac{(\sqrt{L}z)^n}{\sqrt{n!}}\right)_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{F}_L^2$ and so

$$K_L(z,w) = \sum_{n=0}^{\infty} \frac{(\sqrt{L}z)^n}{\sqrt{n!}}\frac{(\sqrt{L}\overline{w})^n}{\sqrt{n!}} = \exp(Lz\overline{w})$$

is the reproducing kernel for this space. We define

$$f_L(z) = \sum_{n=0}^{\infty} a_n \frac{(\sqrt{L}z)^n}{\sqrt{n!}}$$

where, as before, $(a_n)_{n=0}^{\infty}$ is a sequence of iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables. In light of Proposition 1.4 we see that f_L is a GAF defined on all of $\mathbb{C}$ with covariance kernel K_L. Now f_L is defined by random linear combinations of elements of a basis of $\mathcal{F}_L^2$, and since the reproducing kernel does not depend on the choice of basis, we see that a GAF formed from a different basis would have the same covariance kernel, and thus have the same distribution as f_L. An important caveat; it is tempting to conclude that f_L is an element of $\mathcal{F}_L^2$, since it is defined by random linear combinations of elements of a basis. This is almost surely not the case, since the sequence $(a_n)_{n=0}^{\infty}$ is almost surely not in ℓ^2.

The expected number of zeroes, by the Edelman-Kostlan formula, is

$$\frac{1}{4\pi}\Delta \log K_L(z,z)\, m(z) = \frac{1}{4\pi}\Delta(L|z|^2)\, m(z) = \frac{L}{\pi}m,$$

so we may think of L as a parameter that describes the 'intensity' of the average number of zeroes. The Lebesgue measure is clearly invariant under plane automorphisms, and since the first

[2]This is a matter of perspective, if one is looking for topics to study during one's PhD one might suggest that it is fortunate!

[3]For convenience the normalisation we use here is slightly different to the one used earlier.

intensity determines the distribution of the zero set, we see that this too is invariant. Moreover since constant multiples of the Lebesgue measure are the only invariant measures on the plane we see that, in the sense of Theorem 1.5, these are the only GAFs with this property.

In fact, by computing covariances, one can see that for any $z_0 \in \mathbb{C}$ and λ satisfying $|\lambda| = 1$ we have

$$f_L(\lambda z + z_0) \overset{d}{=} f_L(z) e^{L\lambda \overline{z_0} z + L|z_0|^2/2},$$

that is, we can explicitly compute the function g from Theorem 1.5. This means that the distribution of the random potential

$$\log|f_L(z)| - \frac{L}{2}|z|^2$$

is also invariant under plane automorphisms.

It is clear from the definition that we have $f_L(z) = f_1(\sqrt{L}z)$. This means that we could just as well choose $L = 1$ and instead consider dilations of the plane. While this is often what is done in the literature, we shall avoid doing so since this does not generalise well to other settings.

We now discuss some of the known results about the zero set of the flat GAF. Our list is by no means comprehensive.

One way to try to understand the behaviour of the zero set is to study the 'linear statistics'. Given a measurable real-valued function ψ (which we assume is not identically zero) we define

$$n(\psi, L) = \frac{1}{L}\int \psi \, dn_L = \frac{1}{L}\sum_{a \in \mathcal{Z}(f_L)} \psi(a).$$

The Edelman-Kostlan formula (1.3) immediately yields

$$\mathbb{E}[n(\psi, L)] = \frac{1}{\pi}\int \psi \, dm,$$

assuming that the integral converges, and the first obvious question is to compute the variance, which we denote by $\mathbb{V}$.

Forrester and Honner [FH99] arrived at the following results (here ζ is the usual Riemann zeta function): If ψ is a smooth function with compact support then

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3)}{16\pi}\frac{1}{L^3}\|\Delta\psi\|_{L^2}(1 + o(1)) \text{ as } L \to \infty \tag{1.4}$$

while if ψ is the characteristic function of a set D with piecewise smooth boundary (so that $n(\psi, L) = \frac{1}{L}n_L(D)$) then

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3/2)}{8\pi^{3/2}}\frac{1}{L^{3/2}}|\partial D|(1 + o(1)) \text{ as } L \to \infty$$

where $|\partial D|$ is the length of the boundary of D.

Sodin and Tsirelson gave a rigorous derivation of (1.4) in [ST04], Nazarov and Sodin [NS11] computed the variance exactly for $\psi \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$. Asymptotic normality was also shown for the random variable $n(\psi, L)$ in [ST04] under the assumption that ψ is a smooth function with compact support, that is, the random variable

$$\frac{n(\psi, L) - \mathbb{E}[n(\psi, L)]}{\mathbb{V}[n(\psi, L)]^{1/2}}$$

converges in distribution to a (real) normal as $L \to \infty$. The assumptions on ψ such that this holds have since been relaxed, see [NS11] and the references therein for a more complete discussion.

Large fluctuations from the mean have also been studied for the zero set of the flat GAF. One interesting event is the 'hole probability', the probability that there are no zeroes in a region of the complex plane when the expected number of zeroes in that region is large. The asymptotic decay of the hole probability for discs was computed in [ST05], and the more precise version[4]

$$\mathbb{P}[n_L(D(z_0, r)) = 0] = \exp\Big\{ -\frac{e^2}{4} L^2 r^4 (1 + o(1)) \Big\}$$

as $Lr^2 \to \infty$ was obtained by Nishry in [Nis10, Theorem 1.1][5].

An upper bound for the hole probability was established in [ST05] via a large deviations estimate, which is also of interest.

Theorem 1.6 ([ST05, Theorem 2])**.** *Let $\delta > 0$. There exists $c > 0$ depending only on δ such that*

$$\mathbb{P}\left[\left|\frac{n_L(D(z_0, r))}{Lr^2} - 1\right| > \delta\right] \le e^{-cL^2 r^4}$$

for Lr^2 sufficiently large.

We shall construct new GAFs by considering bases from more general Fock spaces. In particular we will try to construct GAFs with a first intensity that closely matches some given measure. The GAFs we consider will no longer have any translation invariance, but we shall see that many of the results we have listed will carry over to this setting.

The hyperbolic Gaussian analytic function

A natural companion of the flat GAF is the hyperbolic GAF. Again we fix a parameter $L > 0$ and now define

$$f_L(z) = \sum_{n=0}^{\infty} a_n \binom{L+n-1}{n}^{1/2} z^n$$

[4]There is an error in the statement of [Nis10, Theorem 1.1], this has been corrected in [Nis12, p. 497].

[5]We note that the author considers $L = 1$ and discs of large radius centred at the origin, however the results are equivalent by re-scaling and translation invariance.

for $z \in \mathbb{D}$, where, as before, $(a_n)_{n=0}^{\infty}$ is a sequence of iid standard complex normal random variables, and

$$\binom{L+n-1}{n} = \frac{\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} = \frac{L(L+1)\cdots(L+n-1)}{n!}.$$

For $L > 1$ the sequence $(\binom{L+n-1}{n}^{1/2} z^n)_{n=1}^{\infty}$ is an orthonormal basis for the weighted Bergman space

$$\mathcal{B}_L^2 = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_L^2}^2 = L\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^L \, d\nu(z) < +\infty\}$$

where $d\nu(z) = \frac{dm(z)}{\pi(1-|z|^2)^2}$ denotes the hyperbolic area. The sequences $(\binom{L+n-1}{n}^{1/2} z^n)_{n=1}^{\infty}$ are also orthonormal bases for various spaces of holomorphic functions on the disc for $L \leq 1$, but we shall not focus on this issue as it will not be important for our purposes.

Proposition 1.4 shows that f_L is a GAF defined on $\mathbb{D}$ with associated covariance kernel

$$K_L(z,w) = \mathbb{E}[f_L(z)\overline{f_L(w)}] = (1 - z\overline{w})^{-L}.$$

We denote the counting measure on the zero set of f_L by n_L. The first intensity of the zero set is given by $L\nu$, which is invariant under disc automorphisms. As in the flat case we see that the distribution of the zero set is also invariant under disc automorphisms, and that f_L is essentially the only GAF with this property. Moreover, for any $z_0 \in \mathbb{D}$ and λ satisfying $|\lambda| = 1$ we have

$$f_L\left(\lambda \frac{z-z_0}{1-\overline{z_0}z}\right) \overset{d}{=} f_L(z)\left(\lambda \frac{1-|z_0|^2}{(1-\overline{z_0}z)^2}\right)^{L/2},$$

and so the distribution of the random potential

$$\log|f_L(z)| - \frac{L}{2}\log\frac{1}{1-|z|^2}$$

is also invariant under disc automorphisms.

We note that there is no simple re-scaling that allows us to study f_L by simply fixing one particular value of L. This leads to a richer theory, since there are two natural parameters, the intensity L and the set where we study the zero set.

We can again define a 'smooth linear statistic' by

$$n(\psi, L) = \frac{1}{L}\int \psi \, dn_L = \frac{1}{L}\sum_{a \in \mathcal{Z}(f_L)} \psi(a).$$

where ψ is a smooth function (which is not identically zero) supported in a compact subset of $\mathbb{D}$. Sodin and Tsirelson [ST04] showed, using the same techniques as in the flat case, that for the hyperbolic GAF the variance of the smooth linear statistics has the same asymptotic decay, that is,

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3)}{16\pi}\frac{1}{L^3}\|\Delta\psi\|_{L^2}(1 + o(1)) \text{ as } L \to \infty,$$

and that the smooth linear statistics are asymptotically normal.

For the particular value $L = 1$ Peres and Virág [PV05, Theorem 2] gave a complete description of the random variable $n_1(D(0,r))$, in particular they showed that

$$\mathbb{V}[n_1(D(0,r))] = \frac{r^2}{1-r^4}$$

and computed the hole probability,

$$\mathbb{P}[n_1(D(0,r)) = 0] = \exp\left(-\frac{\pi^2 r^2}{6(1-r^2)}(1+o(1)\right)$$

as $r \to 1^-$. They proved these results by showing that the corresponding zero set is a so-called determinantal process, but this holds for no other value of L. This means that their techniques do not work for any other values of L.

We shall be interested in understanding the behaviour of the zero set of the hyperbolic GAF for other values of L.

The Paley-Wiener process

We finally mention a closely related problem. We define the random function

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)},$$

where a_n are iid *real* normal random variables with zero mean and unit variance. The analogy with the flat GAF becomes clearer if we consider the Paley-Wiener space

$$PW = \left\{ f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t) e^{izt}\, dt \;:\; \phi \in L^2[-\pi,\pi] \right\},$$

of all entire functions of exponential type at most π that are square integrable on the real line. It is well known that the integer translations of the cardinal sine function $(\sin \pi(z-n)/\pi(z-n))_{n\in\mathbb{Z}}$ constitute an orthonormal basis for the Paley-Wiener space, and so

$$\sum_{n\in\mathbb{Z}} \left| \frac{\sin \pi(z-n)}{\pi(z-n)} \right|^2$$

converges uniformly on compact subsets of the plane. Thus f is almost surely an entire function [HKPV09, Lemma 2.2.3] with covariance kernel given by

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \frac{\sin \pi(z-\overline{w})}{\pi(z-\overline{w})},$$

the reproducing kernel for the Paley-Wiener space.

In fact this function is an example of a 'stationary, symmetric GAF', where stationary means that $f(z) \stackrel{d}{=} f(z+t)$ for all real t, and symmetric means that $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$. Symmetry immediately indicates that the zeroes of this function are quite different in nature to those of the functions previously considered, in particular the non-real zeroes come in conjugate pairs.

We once more denote by n_f the counting measure on the set of zeros of f, we call this the Paley-Wiener process. Feldheim [Fel10] has computed a counterpart of the Edelman-Kostlan formula for general stationary symmetric GAFs, (c.f. the Kac-Rice formula [Kac43]). In our case Feldheim's result reduces to

$$\mathbb{E}[n_f(z)] = S(y)m(z) + \frac{1}{2\sqrt{3}}\mu(x), \tag{1.5}$$

where $z = x+iy$, m is again the planar Lebesgue measure, μ is the singular measure with respect to m supported on $\mathbb{R}$ and identical to Lebesgue measure there, and

$$S\Big(\frac{y}{2\pi}\Big) = \pi \left| \frac{d}{dy} \left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}} \right) \right| .$$

(Here S is defined only for $y \neq 0$, in fact the singular part of (1.5) is the distributional derivative at 0.) We observe that since $S(y) \leq C|y|$ as y approaches zero, for some constant $C > 0$, there are almost surely zeros on the real line, but that they are sparse close to the real line. Moreover the zero set is on average uniformly distributed on the real line.

We will be interested in the 'gap probability', that is, the probability that there are no zeros in a large interval on the real line.

The monograph is structured as follows:

In Chapter 2 we introduce the generalised Fock spaces we shall be studying, and prove some technical results that we will need later. Much of the development will follow [MMO03].

In Chapter 3 we characterise the traces of functions from these Fock spaces on lattices of critical density. This characterisation will be in terms of a cancellation condition that involves discrete versions of the Cauchy and Beurling-Ahlfors transforms. These results first appeared in the article

Jeremiah Buckley, Xavier Massaneda, and Joaquim Ortega-Cerdà, Traces of functions in Fock spaces on lattices of critical density. Bull. Lond. Math. Soc., **44** (2012), no. 2, 222-240.

In Chapter 4 we construct GAFs such that the average distribution of the zero set is close to a given doubling measure. These GAFs are constructed via generalised Fock spaces. We show that the variance is much less than the variance of the corresponding Poisson point process. We prove some asymptotic large deviation estimates for these processes, which in particular allow us to estimate the hole probability. We also show that the smooth linear statistics are asymptotically normal, under an additional regularity hypothesis on the measure. These results are contained in

Jeremiah Buckley, Xavier Massaneda, and Joaquim Ortega-Cerdà, Inhomogenous random zero sets. (Preprint) arXiv:1212.5548 [math.CV].

In Chapter 5 we study the hyperbolic GAF. We compute the asymptotics of the variance of the number of points in a disc of radius r as $r \to 1^-$. We do this for the full range of L. We also compute the asymptotic decay of the hole probability for a fixed hole, as $L \to \infty$.

In Chapter 6 we study the gap probability for the Paley-Wiener process. We show that the asymptotic probability that there is no zero in a bounded interval decays exponentially as a function of the length. This result is from

Jorge Antezana, Jeremiah Buckley, Jordi Marzo and Jan-Fredrik Olsen, Gap probabilities for the cardinal sine. J. Math. Anal. Appl. **396** (2012), no. 2, 466-472.

Throughout this monograph we shall use the following standard notation: The expression $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \le Cg$, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$. We sometimes write $f = O(g)$ to mean $|f| \lesssim g$. We write $f = o(g)$ if the ratio $|f/g|$ can be made arbitrarily small by an appropriate choice of some parameter(s).

Chapter 2

Generalised Fock spaces

In this short chapter we gather many of the technical results that we shall use in Chapters 3 and 4. We recall the definition of a doubling measure on the complex plane, and define generalised Fock spaces using a doubling measure.

2.1 Doubling measures

Definition 2.1. A non-negative Borel measure μ on $\mathbb{C}$ is called *doubling* if there exists $C > 0$ such that

$$\mu(D(z, 2r)) \leq C\mu(D(z, r))$$

for all $z \in \mathbb{C}$ and $r > 0$. We denote by C_μ the infimum of the constants C for which the inequality holds, which is called the *doubling constant* for μ.

Let μ be a doubling measure and let ϕ be a subharmonic function with $\mu = \Delta\phi$. Canonical examples of such functions are given by $\phi(z) = |z|^\alpha$ where $\alpha > 0$ (the value $\alpha = 2$ corresponds of course to the Lebesgue measure). The function $\phi(z) = (\operatorname{Re} z)^2$ gives a non-radial example, and more generally one can take ϕ to be any subharmonic, non-harmonic, (possibly non-radial) polynomial.

For $z \in \mathbb{C}$ we define $\rho_\mu(z)$ to be the radius such that $\mu(D(z, \rho_\mu(z))) = 1$. We shall normally ignore the dependence on μ and simply write $\rho(z)$. Note that all of the constants (including implicit constants) in this section depend only on the doubling constant C_μ, unless explicitly stated otherwise.

Lemma 2.2 ([Chr91, Lemma 2.1])**.** *Let μ be a doubling measure in $\mathbb{C}$. There exists $\gamma > 0$ such that for any discs D, D' of respective radius $r(D) > r(D')$ with $D \cap D' \neq \emptyset$*

$$\left(\frac{\mu(D)}{\mu(D')}\right)^\gamma \lesssim \frac{r(D)}{r(D')} \lesssim \left(\frac{\mu(D)}{\mu(D')}\right)^{1/\gamma}.$$

In particular, the support of μ has positive Hausdorff dimension. We will sometimes require some further regularity on the measure μ, so we make the following definition.

Definition 2.3. We say that a doubling measure μ is *locally flat* if for any disc D of radius $r(D)$ satisfying $\mu(D) = 1$ then for every disc $D' \subseteq D$ of radius $r(D')$ we have

$$\frac{1}{\mu(D')} \simeq \left(\frac{r(D)}{r(D')}\right)^2$$

where the implicit constants depend only on μ.

Trivially $\phi(z) = |z|^2$ gives us a locally flat measure, indeed the condition $0 < c < \Delta\phi < C$ ensures that the measure $\Delta\phi$ is locally flat. Moreover there is always a regularisation of the measure $\Delta\phi$ that is locally flat (see [MMO03, Theorem 14]).

We have the following estimates from [MMO03, p. 869]: There exist $\eta > 0$, $C > 0$ and $\beta \in (0, 1)$ such that

$$C^{-1}|z|^{-\eta} \leq \rho(z) \leq C|z|^{\beta} \text{ for } |z| > 1 \tag{2.1}$$

and

$$|\rho(z) - \rho(\zeta)| \leq |z - \zeta| \text{ for } z, \zeta \in \mathbb{C}. \tag{2.2}$$

Thus ρ is a Lipschitz function and so, in particular, is continuous. We will write

$$D^r(z) = D(z, r\rho(z))$$

and

$$D(z) = D^1(z).$$

A simple consequence of Lemma 2.2 is that $\rho(z) \simeq \rho(\zeta)$ for $\zeta \in D(z)$. We will make use of the following estimate.

Lemma 2.4 ([Chr91, p. 205])**.** *If $\zeta \notin D(z)$ then*

$$\frac{\rho(z)}{\rho(\zeta)} \lesssim \left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^{1-t}$$

for some $t \in (0, 1)$ depending only on the doubling constant, C_μ.

In Chapter 3 the value of the constant t appearing in this lemma will be important, we therefore show that we cannot improve on the given range of t. In the classical example of $\phi(z) = |z|^2$ we may take t to be arbitrarily close to 1. For the other case we consider the function $\phi(z) = |z|^\alpha$ where $\alpha > 0$. We may assume, by normalising appropriately, that $\rho(0) = 1$. Then for $\zeta \notin D(0)$ we have $\rho(\zeta) \simeq |\zeta|^{1-\alpha/2}$. Taking now $z = 0$ in Lemma 2.4 we see that we must have

$$1 \lesssim |\zeta|^{1-(\alpha t/2)}$$

for $\zeta \notin D(0)$ so that $t \leq 2/\alpha$. Thus, for the function $\phi(z) = |z|^\alpha$ when α is large, t will be close to zero.

Throughout this monograph we shall fix one value of t such that the lemma holds. Of course the larger the value of t that we choose, the better the estimate we obtain.

We will need the following estimate.

Lemma 2.5 ([Chr91, Lemma 2.3])**.** *There exists $C > 0$ depending on C_μ such that for any $r > 0$*

$$\int_{D(z,r)} \log\left(\frac{2r}{|z-\zeta|}\right) d\mu(\zeta) \leq C\, \mu(D(z,r)) \qquad z \in \mathbb{C}.$$

We note, as in [Chr91], that ρ^{-2} can be seen as a regularisation of μ. We define d_μ to be the distance induced by the metric $\rho(z)^{-2} dz \otimes d\overline{z}$, that is,

$$d_\mu(z,\zeta) = \inf \int_0^1 |\gamma'(t)| \rho^{-1}(\gamma(t))\, dt,$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = \zeta$. We have the following estimates.

Lemma 2.6 ([MMO03, Lemma 4])**.** *There exists $\delta > 0$ such that for every $r > 0$ there exists $C_r > 0$ such that*

- $C_r^{-1} \dfrac{|z-\zeta|}{\rho(z)} \leq d_\mu(z,\zeta) \leq C_r \dfrac{|z-\zeta|}{\rho(z)}$ *if $|z-\zeta| \leq r\rho(z)$, and*
- $C_r^{-1} \left(\dfrac{|z-\zeta|}{\rho(z)}\right)^\delta \leq d_\mu(z,\zeta) \leq C_r \left(\dfrac{|z-\zeta|}{\rho(z)}\right)^{2-\delta}$ *if $|z-\zeta| > r\rho(z)$.*

Definition 2.7. A sequence Λ is *d_μ-separated* if there exists $\delta > 0$ such that

$$\inf_{\lambda \neq \lambda'} d_\mu(\lambda, \lambda') > \delta.$$

One consequence of Lemma 2.6 is that a sequence Λ is d_μ-separated if and only if there exists $\delta > 0$ such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda')) \qquad \lambda \neq \lambda'.$$

This equivalent condition is often easier to work with.

We shall make repeated use of the following lemma.

Lemma 2.8. *Let Λ be a d_μ-separated sequence. Then for any $\epsilon > 0$ and $k \geq 0$ there exists a constant $C > 0$ depending only on k, ϵ, and C_μ such that*

(a) $\displaystyle\int_{\mathbb{C}} \frac{|z-\zeta|^k}{\exp d_\mu^\epsilon(z,\zeta)} \frac{dm(z)}{\rho(z)^2} \leq C\rho^k(\zeta)$, *and*

(b) $\displaystyle\sum_{\lambda\in\Lambda} \frac{|z-\lambda|^k}{\exp d_\mu^\epsilon(z,\lambda)} \le C\rho^k(\zeta).$

Proof. The proof of (a) is almost identical to the proof of [MO09, Lemma 2.7]. Lemma 2.6 implies that there exists $\varepsilon > 0$ such that

$$\exp d_\mu^\epsilon(z,\zeta) \gtrsim \exp\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\varepsilon.$$

Let $f(x) = x^{\frac{k}{\varepsilon}} - \frac{k}{\varepsilon}x^{\frac{k}{\varepsilon}-1}$ and note that for any $y > 0$

$$\int_y^{+\infty} e^{-x} f(x)\,dx = e^{-y}y^{k/\varepsilon}.$$

Splitting the integral over the regions $D(\zeta)$ and $\mathbb{C}\setminus D(\zeta)$ and using Lemma 2.4 we see that

$$\begin{aligned}\int_{\mathbb{C}} \frac{|z-\zeta|^k}{\exp d_\mu^\epsilon(z,\zeta)} \frac{dm(z)}{\rho(z)^2} &\lesssim \rho^k(\zeta) + \int_{\mathbb{C}\setminus D(\zeta)} \rho^k(\zeta) \int_{\left(\frac{|z-\zeta|}{\rho(\zeta)}\right)^\varepsilon}^{\infty} e^{-x} f(x)\,dx \frac{dm(z)}{\rho(z)^2} \\ &\lesssim \rho^k(\zeta) + \rho^k(\zeta)\int_1^{+\infty} e^{-x} f(x) \int_{D^{x^{1/\varepsilon}}(\zeta)} \frac{dm(z)}{\rho(z)^2} dx \\ &\lesssim \rho^k(\zeta)\left(1 + \int_1^{+\infty} e^{-x} f(x) x^\alpha\,dx\right)\end{aligned}$$

for some positive α.

We may estimate the sum appearing in (b) by the integral in (a) so the result follows. ■

2.2 Interpolation and sampling in Fock spaces

As before we consider a doubling measure μ and a subharmonic function ϕ whose Laplacian satisfies $\Delta\phi = \mu$. The generalised Fock spaces we deal with are defined as (recall that ρ^{-2} can be thought of as a regularisation of μ)

$$\mathcal{F}_\phi^p = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}_\phi^p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} \frac{dm(z)}{\rho(z)^2} < +\infty\}, \text{ for } 1 \le p < +\infty$$

and

$$\mathcal{F}_\phi^\infty = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}_\phi^\infty} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\phi(z)} < +\infty\}.$$

The choice $\phi(z) = |z|^2$ gives the classical Bargmann-Fock spaces defined in the introduction[1] (and corresponds to μ being the Lebesgue measure). It is worth noting, as in [MMO03, p. 863], that there are many spaces of functions which correspond to $\mathcal{F}_\phi^p$ for some ϕ, although this may not be initially apparent.

The following describes the growth of functions in these spaces.

[1] There is another slight change in the normalisation.

Lemma 2.9 ([MMO03, Lemma 19]). *For any $r > 0$ there exists $C = C(r) > 0$ such that for any $f \in \mathcal{H}(\mathbb{C})$ and $z \in \mathbb{C}$*

(a) $|f(z)|^p e^{-p\phi(z)} \leq C \int_{D^r(z)} |f(\zeta)|^p e^{-p\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)}$,

(b) $|\nabla(|f|e^{-\phi})(z)| \leq \frac{C}{\rho(z)} \left(\int_{D^r(z)} |f(\zeta)|^p e^{-p\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)} \right)^{1/p}$, *and*

(c) *if $s > r$ then* $|f(z)|^p e^{-p\phi(z)} \leq C_{r,s} \int_{D^s(z)\setminus D^r(z)} |f(\zeta)|^p e^{-p\phi(\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)}$.

This has an elementary but useful consequence, a Plancherel-Polya-type inequality. Suppose that Λ is a d_μ-separated sequence and that $f \in \mathcal{F}^p_\phi$. Then

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^p e^{-p\phi(\lambda)} \leq C \sum_{\lambda\in\Lambda} \int_{D^{\delta/2}(\lambda)} |f(\zeta)|^p e^{-p\phi(\zeta)} \frac{dm(\zeta)}{\rho(\zeta)^2} \leq C \int_{\mathbb{C}} |f(\zeta)|^p e^{-p\phi(\zeta)} \frac{dm(\zeta)}{\rho(\zeta)^2} < \infty \tag{2.3}$$

where δ is the constant such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda')) \qquad \lambda \neq \lambda'.$$

and $C = C(\delta/2)$.

Moreover, if $f \in \mathcal{F}^p_\phi$ for $1 \leq p < \infty$ then

$$|f(z)|^p e^{-p\phi(z)} \to 0$$

uniformly as $|z| \to \infty$, from which we infer that $\mathcal{F}^p_\phi \subseteq \mathcal{F}^\infty_\phi$.

The Plancherel-Polya-type inequality motivates the following definitions, which are taken verbatim from [MMO03].

Definition 2.10. A sequence $\Lambda \subseteq \mathbb{C}$ is *interpolating* for $\mathcal{F}^p_\phi$, where $1 \leq p < +\infty$, if for every sequence of values $c = (c_\lambda)_{\lambda\in\Lambda}$ such that

$$\sum_{\lambda\in\Lambda} |c_\lambda|^p e^{-p\phi(\lambda)} < +\infty$$

there exists $f \in \mathcal{F}^p_\phi$ such that $f|\Lambda = c$.

Also Λ is *interpolating* for $\mathcal{F}^\infty_\phi$ if for every sequence of values c such that

$$\sup_{\lambda\in\Lambda} |c_\lambda| e^{-\phi(\lambda)} < +\infty$$

there exists $f \in \mathcal{F}^\infty_\phi$ such that $f|\Lambda = c$.

Definition 2.11. A sequence Λ is *sampling* for $\mathcal{F}^p_\phi$, where $1 \leq p < +\infty$, if there exists $C > 0$ such that for every $f \in \mathcal{F}^p_\phi$

$$C^{-1} \sum_{\lambda \in \Lambda} |f(\lambda)|^p e^{-p\phi(\lambda)} \leq \|f\|^p_{\mathcal{F}^p_\phi} \leq C \sum_{\lambda \in \Lambda} |f(\lambda)|^p e^{-p\phi(\lambda)}. \tag{2.4}$$

Also, Λ is *sampling* for $\mathcal{F}^\infty_\phi$ if there exists $C > 0$ such that for every $f \in \mathcal{F}^\infty_\phi$

$$\|f\|_{\mathcal{F}^\infty_\phi} \leq C \sup_{\lambda \in \Lambda} |f(\lambda)| e^{-\phi(\lambda)}.$$

The definitions of sampling and interpolating sequences for $\mathcal{F}^2_\phi$ may also be motivated in terms of Riesz sequences and frames, as we did in Section 1.1. We first note that Lemma 2.9 (a) implies that point evaluation is a bounded linear operator in $\mathcal{F}^p_\phi$, and so $\mathcal{F}^2_\phi$ is a reproducing-kernel Hilbert space. We have the following estimates for the reproducing kernel $\mathcal{K}$.

Proposition 2.12 ([MMO03, Lemma 21; MO09, Theorem 1.1 and Proposition 2.11; CO11, p. 355]). *There exist positive constants C and ϵ (depending only on the doubling constant for μ) such that for any $z, w \in \mathbb{C}$*

(a) $|\mathcal{K}(z, w)| \leq C e^{\phi(z)+\phi(w)} e^{-d^\epsilon_\mu(z,w)}$,

(b) $C^{-1} e^{2\phi(z)} \leq \mathcal{K}(z, z) \leq C e^{2\phi(z)}$, *and*

(c) $C^{-1}/\rho(z)^2 \leq \Delta \log \mathcal{K}(z, z) \leq C/\rho(z)^2$.

(d) *There exists $r > 0$ such that $|\mathcal{K}(z, w)| \geq C e^{\phi(z)+\phi(w)}$ for all $w \in D^r(z)$.*

Remark. The off diagonal decay estimates in Theorem 1.1 and Proposition 2.11 of [MO09] differ from the results just stated by factors involving ρ. This is because the authors study spaces with a different norm; in [MMO03, Section 2.3] it is shown that the class of spaces considered here and in [MO09] is the same. However one can easily verify that minor modifications to the proof in [MO09] give the result just stated in the spaces we are considering.

We define $k_\zeta(z) = \frac{\mathcal{K}(z,\zeta)}{\mathcal{K}(\zeta,\zeta)^{1/2}}$. It is clear from Proposition 2.12 (b) that $|\langle k_\lambda, f\rangle| \simeq |f(\lambda)| e^{-\phi(\lambda)}$ for all $f \in \mathcal{F}^2_\phi$. Thus Λ is a sampling sequence for $\mathcal{F}^2_\phi$ if and only if

$$\|f\|^2_{\mathcal{F}^2_\phi} \simeq \sum_{\lambda \in \Lambda} |\langle k_\lambda, f\rangle|^2 \quad \text{for all } f \in \mathcal{F}^2_\phi,$$

that is, if and only if $(k_\lambda)_{\lambda \in \Lambda}$ is a frame in $\mathcal{F}^2_\phi$ (see Definition 1.2). Similarly Λ is an interpolating sequence for $\mathcal{F}^2_\phi$ if and only if $(k_\lambda)_{\lambda \in \Lambda}$ is a Riesz sequence in $\mathcal{F}^2_\phi$.

Interpolating and sampling sequences in the Fock spaces we consider have been characterised in terms of a Beurling-type density. The following definitions appear in [MMO03].

Definition 2.13. Assume that Λ is a d_μ-separated sequence.

The *upper uniform density* of Λ with respect to μ is

$$\mathcal{D}_\mu^+(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\left(\Lambda \bigcap \overline{D^r(z)}\right)}{\mu(D^r(z))}.$$

The *lower uniform density* of Λ with respect to μ is

$$\mathcal{D}_\mu^-(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\left(\Lambda \bigcap \overline{D^r(z)}\right)}{\mu(D^r(z))}.$$

It should be noted that replacing $\phi(z)$ by $|z|^2$ in this definition does not produce the densities given in Theorem 1.3, but rather a constant multiple of them.

Theorem 2.14 ([MMO03, Theorems A and B]). *A sequence Λ is interpolating for $\mathcal{F}_\phi^p$, where $p \in [1, \infty]$, if and only if Λ is d_μ-separated and $\mathcal{D}_\mu^+(\Lambda) < 1/2\pi$.*

A sequence Λ is sampling for $\mathcal{F}_\phi^p$, where $p \in [1, \infty]$, if and only if Λ is a finite union of d_μ-separated sequences containing a d_μ-separated subsequence Λ' such that $\mathcal{D}_\mu^-(\Lambda') > \frac{1}{2\pi}$.

In Chapter 3 we will fix a sequence Λ such that $\mathcal{D}_\mu^+(\Lambda) = \mathcal{D}_\mu^-(\Lambda) = 1/2\pi$ and describe the set of values $c = (c_\lambda)_{\lambda\in\Lambda}$ such that there exists some function f in a Fock space satisfying the condition $f|\Lambda = c$.

In Chapter 4 we will study GAFs defined by random linear combinations of elements of a basis for $\mathcal{F}_\phi^2$. We will also consider random linear combinations of frames of reproducing kernels, that is, we will study the GAF

$$f(z) = \sum_{\lambda\in\Lambda} a_\lambda k_\lambda(z)$$

where Λ is a sampling sequence and a_λ is a sequence of iid $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables indexed by the sequence Λ. The covariance kernel corresponding to this GAF is given by

$$K(z, w) = \mathbb{E}[f(z)\overline{f(w)}] = \sum_{\lambda\in\Lambda} k_\lambda(z)\overline{k_\lambda(w)}$$

which we study in the next section.

2.3 Kernel estimates

In this section we show that the kernel

$$K(z, w) = \sum_{\lambda\in\Lambda} k_\lambda(z)\overline{k_\lambda(w)},$$

where Λ is a sampling sequence, satisfies similar growth estimates to the reproducing kernel $\mathcal{K}$. We will do this by showing that it is the reproducing kernel for a different (but equivalent) norm on the space $\mathcal{F}^2_\phi$. While the appearance of this kernel may seem mysterious at the moment, we shall provide a motivation for studying it in Chapter 4.

We will prove the following result, which should be compared with Proposition 2.12.

Proposition 2.15. *There exist positive constants C, c and ϵ (depending only on the doubling constant for μ and the sampling constant appearing in* (2.4) *when $p = 2$) such that for any $z, w \in \mathbb{C}$*

(a) $|K(z,w)| \le Ce^{\phi(z)+\phi(w)}e^{-cd^\epsilon_\mu(z,w)}$,

(b) $C^{-1}e^{2\phi(z)} \le K(z,z) \le Ce^{2\phi(z)}$, *and*

(c) $C^{-1}\frac{1}{\rho(z)^2} \le \Delta \log K(z,z) \le C\frac{1}{\rho(z)^2}$.

(d) *There exists $r > 0$ such that $|K(z,w)| \ge Ce^{\phi(z)+\phi(w)}$ for all $w \in D^r(z)$.*

Recall that $k_\zeta(z) = \frac{\mathcal{K}(z,\zeta)}{\mathcal{K}(\zeta,\zeta)^{1/2}}$ and that $(k_\lambda)_{\lambda\in\Lambda}$ is a frame in $\mathcal{F}^2_\phi$. We denote the (canonical) dual frame by $(\tilde{k}_\lambda)_{\lambda\in\Lambda}$, and note that any $f \in \mathcal{F}^2_\phi$ can be expanded as

$$f = \sum_{\lambda\in\Lambda}\langle f, \tilde{k}_\lambda\rangle k_\lambda.$$

We introduce a new inner product on the space $\mathcal{F}^2_\phi$ given by

$$\langle\langle f, g\rangle\rangle = \sum_{\lambda\in\Lambda}\langle f, \tilde{k}_\lambda\rangle\overline{\langle g, \tilde{k}_\lambda\rangle}$$

and note that the norm $|||f||| = \langle\langle f, f\rangle\rangle^{1/2}$ is equivalent to the original norm $\|\cdot\|_{\mathcal{F}^2_\phi}$ (if Λ is sampling).

Proposition 2.16. *The reproducing kernel for the (re-normed) space $(\mathcal{F}^2_\phi, |||\cdot|||)$ is*

$$K(z,w) = \sum_{\lambda\in\Lambda} k_\lambda(z)\overline{k_\lambda(w)}.$$

Proof. It is clear that, for each fixed $w \in \mathbb{C}$, $K(\cdot, w) = K_w$ is in the space, so we need only verify the reproducing property. Note first that $\langle k_{\lambda'}, \tilde{k}_\lambda\rangle = \langle \tilde{k}_{\lambda'}, k_\lambda\rangle$. This follows from the fact that $\tilde{k}_\lambda = S^{-1}k_\lambda$, where S is the frame operator associated to $(k_\lambda)_{\lambda\in\Lambda}$, and S is self adjoint with respect to $\langle\cdot,\cdot\rangle$. Now, for any $f \in \mathcal{F}^2_\phi$,

$$\begin{aligned}\langle\langle f, K_w\rangle\rangle = \sum_{\lambda\in\Lambda}\langle f,\tilde{k}_\lambda\rangle\overline{\langle K_w,\tilde{k}_\lambda\rangle} &= \sum_{\lambda\in\Lambda}\sum_{\lambda'\in\Lambda}\langle f,\tilde{k}_\lambda\rangle\overline{\langle k_{\lambda'},\tilde{k}_\lambda\rangle}k_{\lambda'}(w)\\ &= \sum_{\lambda'\in\Lambda}k_{\lambda'}(w)\sum_{\lambda\in\Lambda}\langle f,\tilde{k}_\lambda\rangle\langle k_\lambda,\tilde{k}_{\lambda'}\rangle\\ &= \sum_{\lambda'\in\Lambda}k_{\lambda'}(w)\left\langle\sum_{\lambda\in\Lambda}\langle f,\tilde{k}_\lambda\rangle k_\lambda,\tilde{k}_{\lambda'}\right\rangle\\ &= \sum_{\lambda'\in\Lambda}k_{\lambda'}(w)\langle f,\tilde{k}_{\lambda'}\rangle = f(w),\end{aligned}$$

which completes the proof. ■

Proof of Proposition 2.15. We have (see [Ber70, p. 26])

$$\begin{aligned}\sqrt{K(z,z)} &= \sup\{|f(z)| : f\in\mathcal{F}^2_\phi\ ,\ \|\!|f|\!\|\le 1\}\\ &\simeq \sup\{|f(z)| : f\in\mathcal{F}^2_\phi\ ,\ \|f\|_{\mathcal{F}^2_\phi}\le 1\} = \sqrt{\mathcal{K}(z,z)}\end{aligned}$$

and so Proposition 2.12 implies (b). Similarly (again see [Ber70, p. 26])

$$\Delta\log K(z,z) = \frac{4\sup\{|f'(z)|^2:\ f\in\mathcal{F}^2_\phi\ ,\ f(z)=0\ ,\ \|\!|f|\!\|\le 1\}}{K(z,z)} \simeq \Delta\log\mathcal{K}(z,z)$$

so that (c) also follows from Proposition 2.12.

We note that for all $w\in D^r(z)$, applying Lemma 2.9 (b),

$$||K(w,z)|e^{-\phi(w)} - |K(z,z)|e^{-\phi(z)}| \lesssim \frac{1}{\rho(z)}\|K(\cdot,z)\|_{\mathcal{F}^2_\phi}|z-w| \lesssim re^{\phi(z)}$$

so that for sufficiently small r, (b) implies (d).

Finally we have, by the estimates in Proposition 2.12,

$$|K(w,z)| \le \sum_{\lambda\in\Lambda}|k_\lambda(z)\overline{k_\lambda(w)}| \lesssim e^{\phi(z)+\phi(w)}\sum_{\lambda\in\Lambda}e^{-d^\epsilon_\mu(z,\lambda)-d^\epsilon_\mu(\lambda,w)}.$$

Now

$$\sum_{\lambda\in\Lambda, d_\mu(z,\lambda)>\frac{1}{2}d_\mu(z,w)} e^{-d^\epsilon_\mu(z,\lambda)-d^\epsilon_\mu(\lambda,w)} \le e^{-2^{-\epsilon}d^\epsilon_\mu(z,w)}\sum_{\lambda\in\Lambda}e^{-d^\epsilon_\mu(\lambda,w)} \lesssim e^{-2^{-\epsilon}d^\epsilon_\mu(z,w)}$$

where we have used Lemma 2.8. The remaining terms satisfy $d_\mu(w,\lambda)\ge\frac{1}{2}d_\mu(z,w)$ and may be treated similarly. ■

Remark. We have used only the fact that $(k_\lambda)_{\lambda\in\Lambda}$ is a frame, and the expression of the reproducing kernel as an extremal problem, to show that $K(z,z)\simeq\mathcal{K}(z,z)$ and $\Delta\log K(z,z)\simeq\Delta\log\mathcal{K}(z,z)$. Our proof therefore carries over to any space where these properties hold.

Chapter 3

Traces of functions in Fock spaces

In this chapter we are interested in describing the set of values $c = (c_\lambda)_{\lambda\in\Lambda}$ such that there exists some function f in a Fock space satisfying the condition $f|\Lambda = c$, where Λ is a 'critical lattice'. We will define this shortly, but we first give an elementary example in the classical Bargmann-Fock spaces.

We recall that the Bargmann-Fock spaces are defined as

$$\mathcal{F}^p = \{f \in H(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p} = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2}\, dm(z) < +\infty\}, \text{ for } 1 \leq p < +\infty$$

and

$$\mathcal{F}^\infty = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^\infty} = \sup_{z\in\mathbb{C}} |f(z)|e^{-|z|^2} < +\infty\}$$

where m denotes the Lebesgue measure on the plane.

We consider again the critical integer lattice

$$\Lambda = \sqrt{\frac{\pi}{2}}(\mathbb{Z} + i\mathbb{Z}),$$

which, by Theorem 2.14 (or indeed Theorem 1.3), is neither an interpolating nor a sampling sequence. The Weierstrass σ-function associated to Λ is defined by

$$\sigma(z) = z \prod_{\lambda\in\Lambda_0} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{1}{2}\frac{z^2}{\lambda^2}},$$

where we use the notation $\Lambda_\lambda = \Lambda \setminus \{\lambda\}$. Note that Λ is the zero set of σ, and that

$$|\sigma(z)| \simeq e^{|z|^2} d(z, \Lambda)$$

for all $z \in \mathbb{C}$ [SW92, p. 108]. Here, d refers to the usual Euclidean distance between a point and a set.

Given a sequence $(a_\lambda)_{\lambda\in\Lambda}$, we define the principal value of its sum to be

$$\text{p.v.}\sum_{\lambda\in\Lambda} a_\lambda = \lim_{R\to\infty}\sum_{|\lambda|<R} a_\lambda.$$

We are ready to state our main result, in this special case.

Theorem 3.1. *Let $\Lambda = \sqrt{\frac{\pi}{2}}(\mathbb{Z}+i\mathbb{Z})$. There exists $f\in\mathcal{F}^1$ satisfying $f|\Lambda = c$ if and only if*

- $\displaystyle\sum_{\lambda\in\Lambda}|c_\lambda|e^{-|\lambda|^2} < +\infty,$
- $\displaystyle\sum_{\lambda'\in\Lambda}\Big|\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{\sigma'(\lambda)(\lambda-\lambda')}\Big| < +\infty,$ *and*
- $\displaystyle\sum_{\lambda'\in\Lambda}\Big|\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{\sigma'(\lambda)(\lambda-\lambda')^2}\Big| < +\infty.$

There exists $f\in\mathcal{F}^p$ for $1<p<\infty$ satisfying $f|\Lambda = c$ if and only if

- $\displaystyle\sum_{\lambda\in\Lambda}|c_\lambda|^p e^{-p|\lambda|^2} < +\infty$ *and*
- $\displaystyle\sum_{\lambda'\in\Lambda}\Big|\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{\sigma'(\lambda)(\lambda-\lambda')}\Big|^p < +\infty.$

There exists $f\in\mathcal{F}^\infty$ satisfying $f|\Lambda = c$ if and only if

- $\displaystyle\sup_{\lambda\in\Lambda}|c_\lambda|e^{-|\lambda|^2} < +\infty,$
- $\displaystyle\sup_{\lambda'\in\Lambda_0}\Big|-\frac{c_0}{\sigma'(0)\lambda'} + \text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}\setminus\{0\}}\frac{c_\lambda}{\sigma'(\lambda)}\Big(\frac{1}{\lambda-\lambda'}-\frac{1}{\lambda}\Big)\Big| < +\infty,$ *and*
- $\displaystyle\sup_{\lambda'\in\Lambda}\Big|\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{\sigma'(\lambda)(\lambda-\lambda')^2}\Big| < +\infty.$

In fact these results hold for any sequence Λ that is the zero set of a function τ with growth similar to the Weierstrass σ-function. Specifically suppose that τ is an entire function such that

- the zero set $\mathcal{Z}(\tau)$ of τ is uniformly separated, that is, $\displaystyle\inf_{\lambda\neq\lambda'}|\lambda-\lambda'| > 0,$
- $\sup_{z\in\mathbb{C}} d(z,\mathcal{Z}(\tau)) < +\infty$, and
- $|\tau(z)| \simeq e^{|z|^2} d(z,\mathcal{Z}(\tau))$ for all $z\in\mathbb{C}$.

Then Theorem 3.1 holds if we replace σ by τ and take $\Lambda = \mathcal{Z}(\tau)$. Such a set is always a set of critical density, in the sense that both the upper and lower densities in Theorem 2.14 have the critical value. The existence of many such functions τ is guaranteed by Theorem 3.2. We will prove a more general version of this result, in the Fock spaces introduced in Chapter 2.

Our methods of attacking this problem are inspired by a similar result due to Levin in the classical Paley-Wiener spaces [Lev96, Lecture 21]. In these spaces, the integers are simultaneously an interpolating sequence and a sampling sequence in almost every situation, however this fails in the two extremes, namely the L^1 and L^∞ cases. Levin completely described the traces of functions in the L^∞ space on the integers, and Ber (see [Lev96, Lecture 21] and also [Ber80]) solved the same problem in the L^1 case. While a discrete version of the Hilbert transform is the key ingredient in these results, we shall see that it is discrete versions of the Cauchy and Beurling-Ahlfors transforms that shall play a similar rôle in the Fock context.

The chapter is structured as follows. In the first section we describe an analogue of the critical integer lattice in generalised Fock spaces. In the second section we give the statements of the main results of this chapter, which generalise Theorem 3.1 to this setting. In the third section we prove two representation formulae for functions in our generalised Fock spaces in terms of the values of the function on a critical lattice. In the fourth section we study a discrete version of the Beurling-Ahlfors transform. In the final section we prove the statements from the second section.

3.1 Generalised lattices

We shall now consider analogues of the critical integer lattice, that will play a similar rôle in generalised Fock spaces. Throughout this chapter μ denotes a fixed doubling measure and ϕ is a fixed subharmonic function with $\Delta\phi = \mu$. We recall the spaces introduced in Section 2.2,

$$\mathcal{F}^p_\phi = \{f \in H(\mathbb{C}) : \|f\|^p_{\mathcal{F}^p_\phi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} \frac{dm(z)}{\rho(z)^2} < +\infty\}, \text{ for } 1 \le p < +\infty$$

and

$$\mathcal{F}^\infty_\phi = \{f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}^\infty_\phi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\phi(z)} < +\infty\}.$$

We shall assume (see [MMO03, Theorem 14]) that $\phi \in \mathcal{C}^\infty(\mathbb{C})$.

We begin with the following result.

Theorem 3.2 ([MMO03, Theorem 17]). *There exists an entire function g such that*

- *the zero set $\mathcal{Z}(g)$ of g is d_μ-separated and $\sup_{z\in\mathbb{C}} d_\mu(z, \mathcal{Z}(g)) < \infty$, and*
- *$|g(z)| \simeq e^{\phi(z)} d_\mu(z, \mathcal{Z}(g))$ for all $z \in \mathbb{C}$.*

The function g can be chosen so that, moreover, it vanishes on a prescribed $z_0 \in \mathbb{C}$.

We say that g is a *multiplier* associated to ϕ. Furthermore, [MMO03, Lemma 37], we have $\mathcal{D}_\mu^+(\mathcal{Z}(g)) = \mathcal{D}_\mu^-(\mathcal{Z}(g)) = 1/2\pi$. Theorem 2.14 shows that $\mathcal{Z}(g)$ is neither an interpolating nor a sampling sequence.

We shall now regard g as fixed and we will say that $\Lambda = \mathcal{Z}(g)$ is a *critical lattice associated to the multiplier* g. The multiplier can be thought of as playing the same rôle in Fock spaces that sine-type functions play in Paley-Wiener spaces.

Suppose now that $f \in \mathcal{F}_\phi^p$, that z is uniformly bounded away from Λ in the distance d_μ and that $\epsilon > 0$ is arbitrary. Then

$$\left|\frac{f(z)}{g(z)}\right|^p \simeq |f(z)|^p e^{-p\phi(z)} < \epsilon$$

uniformly as $|z| \to \infty$, where we have used Theorem 3.2 and Lemma 2.9. In fact, if $(z_n)_n$ is any d_μ-separated sequence that satisfies $d_\mu(z_n, \Lambda) \geq C > 0$ for all n (here C is any positive constant), then Theorem 3.2 and (2.3) imply that

$$\sum_n \left|\frac{f(z_n)}{g(z_n)}\right|^p < \infty. \tag{3.1}$$

For any $\lambda \in \Lambda$, Theorem 3.2 and Lemma 2.6 show that $|g'(\lambda)| \simeq e^{\phi(\lambda)}/\rho(\lambda)$ and we conclude that

$$\sum_{\lambda\in\Lambda} \left|\frac{f(\lambda)}{g'(\lambda)\rho(\lambda)}\right|^p < +\infty \tag{3.2}$$

by again invoking (2.3).

There exists $\delta_1 > 0$ such that $|\lambda - \lambda'| > 2\delta_1 \max\{\rho(\lambda), \rho(\lambda')\}$ for all $\lambda \neq \lambda'$; we will denote this constant by δ_1 throughout this chapter. Recall that $D^r(z) = D(z, r\rho(z))$ and $D(z) = D^1(z)$. We will write

$$Q_\lambda = \{z \in \mathbb{C} : d_\mu(z, \Lambda) = d_\mu(z, \lambda)\}$$

for $\lambda \in \Lambda$. Note that for any $0 < \delta \leq \delta_1$ Lemma 2.6 implies that $D^\delta(\lambda) \subseteq Q_\lambda$ and for some constant $R_1 > 0$ we have $Q_\lambda \subseteq D^{R_1}(\lambda)$. In fact the sets $D^{\delta_1}(\lambda)$ are pairwise disjoint and $\mathbb{C} = \bigcup_{\lambda\in\Lambda} Q_\lambda$. Additionally, we have

$$\int_{Q_\lambda} \frac{dm(z)}{\rho(z)^2} \simeq \frac{1}{\rho(\lambda)^2}\int_{Q_\lambda} dm(z) = \frac{|Q_\lambda|}{\rho(\lambda)^2} \leq \frac{|D^{R_1}(\lambda)|}{\rho(\lambda)^2} = \pi R_1^2 \tag{3.3}$$

where $|A|$ is the Lebesgue measure of the set A.

We shall henceforth assume that $0 \in \Lambda$. This can always be achieved by fixing some $\lambda_0 \in \Lambda$ and translating this point to the origin. This is merely a matter of convenience and will simplify many of our calculations.

Let β and η be as in (2.1) and choose $\gamma > 2 + 2\eta$. Then, for $0 < \delta < \delta_1$,

$$\sum_{|\lambda|>1} \frac{1}{|\lambda|^\gamma} \lesssim \sum_{|\lambda|>1} \frac{\rho(\lambda)^2}{|\lambda|^{\gamma-2\eta}} \simeq \sum_{|\lambda|>1} \int_{D^\delta(\lambda)} \frac{dm(z)}{|z|^{\gamma-2\eta}} \leq \int_{\mathbb{C}\setminus D^\delta(0)} \frac{dm(z)}{|z|^{\gamma-2\eta}} < +\infty$$

so that $\sum_{\lambda\in\Lambda_0} \lambda^{-\gamma}$ is an absolutely convergent sum.

Discrete potentials

In this subsection we shall only assume that Λ is d_μ-separated, although when we apply it later we shall take $\Lambda = \mathcal{Z}(g)$. Given a sequence $(d_\lambda)_{\lambda\in\Lambda}$ such that $(d_\lambda \rho(\lambda)^\gamma)_{\lambda\in\Lambda} \in \ell^p$ where γ is real, we will say that $d \in \ell^p(\rho^\gamma)$. We shall repeatedly need the following result.

Lemma 3.3. *(i) If Λ is d_μ-separated, $1 \le p \le 2$ and $d \in \ell^p(\rho^{-1})$ then*

$$\left(\sum_{\lambda\in\Lambda_{\lambda'}} \frac{d_\lambda}{|\lambda' - \lambda|^3} \right)_{\lambda'\in\Lambda} \in \ell^p(\rho^2).$$

(ii) If Λ is d_μ-separated, $1 \le p \le +\infty$ and $d \in \ell^p(\rho^{-1})$ then

$$\left(\sum_{\lambda\in\Lambda_{\lambda'}} \frac{d_\lambda}{|\lambda' - \lambda|^{N+1}} \right)_{\lambda'\in\Lambda} \in \ell^p(\rho^N)$$

for any integer $N > 1/t$, where t is the constant occurring in Lemma 2.4.

Proof. (i) Define $\tilde{d}_\lambda = d_\lambda/\rho(\lambda)$ and

$$L_{\lambda'}(\tilde{d}) = \sum_{\lambda\in\Lambda_{\lambda'}} \frac{\tilde{d}_\lambda \rho(\lambda)\rho(\lambda')^2}{|\lambda' - \lambda|^3}.$$

We will show that L is in fact a bounded operator from ℓ^p to ℓ^p, which will imply the claimed result. Note first that

$$\sum_{\lambda'\in\Lambda} |L_{\lambda'}| \le \sum_{\lambda\in\Lambda} |\tilde{d}_\lambda|\rho(\lambda) \sum_{\lambda'\in\Lambda_\lambda} \frac{\rho(\lambda')^2}{|\lambda' - \lambda|^3} \lesssim \sum_{\lambda\in\Lambda} |\tilde{d}_\lambda|$$

so that L is a bounded linear operator from ℓ^1 to ℓ^1. Here we have used the fact that

$$\sum_{\lambda'\in\Lambda_\lambda} \frac{\rho(\lambda')^2}{|\lambda' - \lambda|^3} \simeq \sum_{\lambda'\in\Lambda_\lambda} \int_{D^\delta(\lambda')} \frac{dm(z)}{|z - \lambda|^3} \le \int_{\mathbb{C}\setminus D^\delta(\lambda)} \frac{dm(z)}{|z - \lambda|^3} \simeq \rho(\lambda)^{-1}$$

where $0 < \delta < \delta_1$.

We now show that L is a bounded operator from ℓ^2 to ℓ^2, using Schur's test (see, for example, [Wc13]). We consider L as an integral operator with kernel $K(\lambda', \lambda) = \frac{\rho(\lambda)\rho(\lambda')^2}{|\lambda'-\lambda|^3}$ for $\lambda \neq \lambda'$ and $K(\lambda, \lambda) = 0$. Now

$$\sum_{\lambda\in\Lambda} K(\lambda', \lambda)\rho(\lambda) = \rho(\lambda')^2 \sum_{\lambda\in\Lambda_{\lambda'}} \frac{\rho(\lambda)^2}{|\lambda' - \lambda|^3} \lesssim \rho(\lambda') \tag{3.4}$$

and

$$\sum_{\lambda'\in\Lambda} K(\lambda',\lambda)\rho(\lambda') = \rho(\lambda)\sum_{\lambda'\in\Lambda_\lambda}\frac{\rho(\lambda')^3}{|\lambda'-\lambda|^3} \lesssim \rho(\lambda)\int_{\mathbb{C}\setminus D(\lambda')}\frac{\rho(z)}{|\lambda'-z|^3}\,dm(z).$$

Applying Lemma 2.4 we have

$$\int_{\mathbb{C}\setminus D(\lambda')}\frac{\rho(z)}{|\lambda'-z|^3}\,dm(z) \lesssim \rho(\lambda')^t\int_{\mathbb{C}\setminus D(\lambda')}\frac{dm(z)}{|\lambda'-z|^{2+t}} \simeq 1$$

so that

$$\sum_{\lambda'\in\Lambda} K(\lambda',\lambda)\rho(\lambda') \lesssim \rho(\lambda). \tag{3.5}$$

Now (3.4) and (3.5) together imply, by the Schur test, that L is indeed bounded from ℓ^2 to ℓ^2. Applying now the Riesz-Thorin interpolation theorem (see, for example, [Fol84, Theorem 6.27]) completes the proof.

(ii) We use the same notation. Define

$$M_{\lambda'}(\tilde{d}) = \sum_{\lambda\in\Lambda_{\lambda'}}\frac{\tilde{d}_\lambda\rho(\lambda)\rho(\lambda')^N}{|\lambda'-\lambda|^{N+1}}.$$

Since Λ is d_μ-separated and $N \geq 2$ we have

$$\frac{\rho(\lambda')^N}{|\lambda'-\lambda|^{N+1}} \lesssim \frac{\rho(\lambda')^2}{|\lambda'-\lambda|^3}$$

so that (i) shows that M defines a bounded linear operator from ℓ^1 to ℓ^1.

Also

$$\sup_{\lambda'\in\Lambda}|M_{\lambda'}| \leq \|\tilde{d}\|_{\ell^\infty}\sup_{\lambda'\in\Lambda}\rho(\lambda')^N\sum_{\lambda\in\Lambda_{\lambda'}}\frac{\rho(\lambda)}{|\lambda'-\lambda|^{N+1}}.$$

Applying again Lemma 2.4 we have

$$\sum_{\lambda\in\Lambda_{\lambda'}}\frac{\rho(\lambda)}{|\lambda'-\lambda|^{N+1}} \lesssim \int_{\mathbb{C}\setminus D(\lambda')}\frac{dm(z)}{|\lambda'-z|^{N+1}\rho(z)} \lesssim \rho(\lambda')^{-\frac{1}{t}}\int_{\mathbb{C}\setminus D(\lambda')}\frac{dm(z)}{|\lambda'-z|^{2+N-\frac{1}{t}}} \simeq \rho(\lambda')^{-N}$$

since $N > 1/t$. Consequently

$$\sup_{\lambda'\in\Lambda}|M_{\lambda'}| \lesssim \|\tilde{d}\|_{\ell^\infty}$$

so that M defines a bounded linear operator from ℓ^∞ to ℓ^∞. Once more the Riesz-Thorin interpolation theorem completes the proof. ■

3.2 Statements of the main results

We are ready to state our results in full generality. As before Λ is a critical lattice, the zero set of a multiplier, g, associated to ϕ. We begin with the simplest case, which is the Hilbert space $\mathcal{F}^2_\phi$, where we need only slightly modify Theorem 3.1.

Theorem 3.4. *Let Λ be a critical lattice associated to the multiplier g. There exists $f \in \mathcal{F}^2_\phi$ satisfying $f|\Lambda = c$ if and only if*

- $\displaystyle\sum_{\lambda\in\Lambda} |c_\lambda|^2 e^{-2\phi(\lambda)} < +\infty$, *and*
- $\displaystyle\sum_{\lambda'\in\Lambda} \Big| \text{p.v.} \sum_{\lambda\in\Lambda_{\lambda'}} \frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')} \Big|^2 < +\infty.$

Our result in $\mathcal{F}^1_\phi$ is also only a slight modification of Theorem 3.1.

Theorem 3.5. *Let Λ be a critical lattice associated to the multiplier g. There exists $f \in \mathcal{F}^1_\phi$ satisfying $f|\Lambda = c$ if and only if*

- $\displaystyle\sum_{\lambda\in\Lambda} |c_\lambda| e^{-\phi(\lambda)} < +\infty$,
- $\displaystyle\sum_{\lambda'\in\Lambda} \Big| \sum_{\lambda\in\Lambda_{\lambda'}} \frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')} \Big| < +\infty$, *and*
- $\displaystyle\sum_{\lambda'\in\Lambda} \Big| \rho(\lambda') \sum_{\lambda\in\Lambda_{\lambda'}} \frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')^2} \Big| < +\infty.$

For other values of p the situation is slightly more complicated. We begin with the case $1 < p < 2$. Here there are two possibilities, depending on whether or not ρ^{p-2} is a Muckenhoupt A_p weight (see Section 3.4 for the definition). If this additional assumption holds then our result is essentially the same as in the classical case; otherwise we add an additional condition to our result. We also show in Section 3.4 that both of these possibilities can occur.

Theorem 3.6. *Let Λ be a critical lattice associated to the multiplier g and suppose $1 < p < 2$.*

- *If ρ_μ^{p-2} is an A_p weight then there exists $f \in \mathcal{F}^p_\phi$ satisfying $f|\Lambda = c$ if and only if*

  (a) $\displaystyle\sum_{\lambda\in\Lambda} |c_\lambda|^p e^{-p\phi(\lambda)} < +\infty$, *and*

  (b) $\displaystyle\sum_{\lambda'\in\Lambda} \Big| \text{p.v.} \sum_{\lambda\in\Lambda_{\lambda'}} \frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')} \Big|^p < +\infty.$

- *If ρ_μ^{p-2} is not an A_p weight then there exists $f \in \mathcal{F}_\phi^p$ satisfying $f|\Lambda = c$ if and only if (a) and (b) hold, and in addition*

  (c) $$\sum_{\lambda'\in\Lambda}\left|\rho(\lambda')\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')^2}\right|^p < +\infty.$$

Remark. In fact Theorem 3.11 shows that if ρ_μ^{p-2} is an A_p weight then condition (a) implies condition (c), since the inner sum in (c) can be viewed as a discrete version of a Calderón-Zygmund operator. Since the A_p condition is trivially satisfied for $p = 2$, this also explains why there are only two conditions appearing in the statement of Theorem 3.4.

If $2 < p < \infty$ then our result becomes more complicated, depending on the doubling constant.

Theorem 3.7. *Let Λ be a critical lattice associated to the multiplier g and suppose $2 < p < \infty$. Let t be the constant occurring in Lemma 2.4 (which depends on the doubling constant).*

(i) *If $t > 1/2$ and ρ_μ^{p-2} is an A_p weight then there exists $f \in \mathcal{F}_\phi^p$ satisfying $f|\Lambda = c$ if and only if (a) and (b) hold.*

(ii) *If $t > 1/2$ and ρ_μ^{p-2} is not an A_p weight then there exists $f \in \mathcal{F}_\phi^p$ satisfying $f|\Lambda = c$ if and only if (a), (b) and (c) hold.*

(iii) *If $t \le 1/2$ then there exists $f \in \mathcal{F}_\phi^p$ satisfying $f|\Lambda = c$ if and only if (a) holds and*

  (b′) there exists an integer $N > 1/t$ such that, for every $1 \le n \le N$,

$$\sum_{\lambda'\in\Lambda}\left|\rho(\lambda')^{n-1}\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')^n}\right|^p < +\infty.$$

Remark. As in the previous theorem, Theorem 3.11 and Lemma 3.3 show that the 'extra' conditions in part (iii) of this theorem are automatically satisfied in parts (i) and (ii).

We finally state our result for $\mathcal{F}_\phi^\infty$, which again depends on the doubling constant.

Theorem 3.8. *Let Λ be a critical lattice associated to the multiplier g and let t be the constant occurring in Lemma 2.4. There exists $f \in \mathcal{F}_\phi^\infty$ satisfying $f|\Lambda = c$ if and only if*

- $\sup_{\lambda\in\Lambda}|c_\lambda|e^{-\phi(\lambda)} < +\infty$,
- $\sup_{\lambda'\in\Lambda_0}\left|-\frac{c_0}{g'(0)\lambda'} + \text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}\setminus\{0\}}\frac{c_\lambda}{g'(\lambda)}\left(\frac{1}{\lambda-\lambda'}-\frac{1}{\lambda}\right)\right| < +\infty$, *and*
- *there exists an integer $N > 1/t$ such that, for every $2 \le n \le N$,*

$$\sup_{\lambda'\in\Lambda}\left|\rho(\lambda')^{n-1}\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{c_\lambda}{g'(\lambda)(\lambda-\lambda')^n}\right| < +\infty.$$

3.3 Representation formulae

In this section we will prove two representation formulae for functions in generalised Fock spaces in terms of the values of the function on a critical lattice. These formulae are reminiscent of the Lagrange interpolation formula.

Lemma 3.9. *Let Λ be a critical lattice associated to the multiplier g.*
(i) If $f \in \mathcal{F}^\infty_\phi$ then

$$f(z) = g(z)\Big[w_0 + \frac{f(0)}{g'(0)z} + \text{p.v.}\sum_{\lambda\in\Lambda_0} \frac{f(\lambda)}{g'(\lambda)}\left(\frac{1}{z-\lambda} + \frac{1}{\lambda}\right)\Big] \tag{3.6}$$

where $w_0 = \lim_{z\to 0} \frac{d}{dz}\big(\frac{zf(z)}{g(z)}\big) = \frac{f'(0)}{g'(0)} - \frac{f(0)g''(0)}{2g'(0)}$.
(ii) If $f \in \mathcal{F}^p_\phi$ for $1 \le p < +\infty$ then

$$f(z) = g(z)\text{p.v.}\sum_{\lambda\in\Lambda} \frac{f(\lambda)}{g'(\lambda)(z-\lambda)}. \tag{3.7}$$

Proof. (i) We denote $d_\lambda = f(\lambda)/g'(\lambda)$ and note that

$$|d_\lambda/\rho(\lambda)| \simeq |f(\lambda)|e^{-\phi(\lambda)} \le \sup_{z\in\mathbb{C}} |f(z)|e^{-\phi(z)} = \|f\|_{\mathcal{F}^\infty_\phi}$$

so that $(d_\lambda/\rho(\lambda))_{\lambda\in\Lambda} \in \ell^\infty$ and $\|d_\lambda/\rho(\lambda)\|_\infty \lesssim \|f\|_{\mathcal{F}^\infty_\phi}$. Let β and η be as in (2.1) and fix a positive integer $n > 2 + 2\eta + \beta$. We will write

$$f(z) = \sum_{\lambda\in\Lambda} f(\lambda)g_\lambda(z) \tag{3.8}$$

where g_λ are entire functions satisfying $g_\lambda(\lambda') = \delta_{\lambda\lambda'}$. The obvious candidate for $g_\lambda(z)$ is the function $g_\lambda(z) = \frac{g(z)}{g'(\lambda)(z-\lambda)}$, however the resultant series is, in general, not convergent. We shall keep $g_0(z) = \frac{g(z)}{g'(0)z}$, but instead take $g_\lambda(z) = \frac{g(z)}{g'(\lambda)}\big(\frac{1}{z-\lambda} - p_{n-1}(z)\big)$ for $\lambda \ne 0$, where p_{n-1} is the Taylor polynomial of degree $n-1$ of the function $C_\lambda(z) = \frac{1}{z-\lambda}$ expanded around 0. Note that we still have $g_\lambda(\lambda') = \delta_{\lambda\lambda'}$ but now the series is pointwise convergent. In fact

$$\frac{1}{z-\lambda} - p_{n-1}(z) = \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} + \cdots + \frac{z^{n-1}}{\lambda^n} = \frac{z^n}{\lambda^n(z-\lambda)} \tag{3.9}$$

so that if we define

$$G(z) = \frac{d_0}{z} + \sum_{\lambda\in\Lambda_0} d_\lambda\left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} + \cdots + \frac{z^{n-1}}{\lambda^n}\right) = \frac{d_0}{z} + \sum_{\lambda\in\Lambda_0} d_\lambda \frac{z^n}{\lambda^n(z-\lambda)}$$

then for any K a compact subset of $\mathbb{C} \setminus \Lambda$ we have

$$\sum_{\lambda \in \Lambda_0} \Big| d_\lambda \frac{z^n}{\lambda^n (z-\lambda)} \Big| \lesssim \sum_{\lambda \in \Lambda_0} \frac{|d_\lambda|}{|\lambda|^n} \lesssim \sum_{\lambda \in \Lambda_0} \frac{\rho(\lambda)}{|\lambda|^n} \lesssim \sum_{\lambda \in \Lambda_0} \frac{1}{|\lambda|^{n-\beta}} < +\infty$$

for all $z \in K$, since $n - \beta > 2 + 2\eta$. Hence G defines a meromorphic function on $\mathbb{C}$ with a simple pole at each $\lambda \in \Lambda$. Consequently gG is an entire function that agrees with f at each $\lambda \in \Lambda$. This implies that there exists an entire function h such that $f - gG = gh$.

Fix $\epsilon > 0$ and $0 < \delta < \delta_1$, and define $\mathcal{D} = \mathbb{C} \setminus \bigcup_{\lambda \in \Lambda} D^\delta(\lambda)$. Now for each $z \in \mathcal{D}$ we have

$$\left| \frac{f(z)}{g(z)} \right| \simeq \frac{|f(z)| e^{-\phi(z)}}{d_\mu(z, \Lambda)} \simeq |f(z)| e^{-\phi(z)} \leq \|f\|_{\mathcal{F}^\infty_\phi}.$$

Also, when $z \in \mathcal{D}$, we obviously have

$$|G(z)| \leq \frac{|d_0|}{|z|} + |z|^n \sum_{\lambda \in \Lambda_0} \Big| \frac{d_\lambda}{\lambda^n (z-\lambda)} \Big|.$$

We split this sum over two separate ranges. For any $R > 1$,

$$\sum_{|\lambda| > R} \Big| \frac{d_\lambda}{\lambda^n (z-\lambda)} \Big| \lesssim \sum_{|\lambda| > R} \frac{\rho(\lambda)}{|\lambda|^n |z-\lambda|} \lesssim \sum_{|\lambda| > R} \frac{1}{|\lambda|^n} < \epsilon$$

for sufficiently large R. Fixing one such R we then have, for $|z| > 2R$,

$$\sum_{0 < |\lambda| \leq R} \left| \frac{d_\lambda}{\lambda^n (z-\lambda)} \right| \leq \frac{2}{|z|} \sum_{0 < |\lambda| \leq R} \frac{|d_\lambda|}{|\lambda|^n} = \frac{C}{|z|}$$

for some constant C. Hence

$$|G(z)| \leq \left| \frac{d_0}{z} \right| + |z|^n \Big(\frac{C}{|z|} + \epsilon \Big) = o(z^n)$$

for $|z| \geq 2R$. Gathering these estimates we have $h(z) = o(z^n)$ for $z \in \mathcal{D}$ of sufficiently large modulus. Applying now the maximum principle to h on $D^\delta(\lambda)$ for each $\lambda \in \Lambda$ far from the origin we see that this holds for all $z \in \mathbb{C}$ of sufficiently large modulus. We conclude that h is a polynomial of degree less than or equal to $n - 1$.

Note that if we define

$$H(z) = G(z) - \frac{d_0}{z} = \sum_{\lambda \in \Lambda_0} d_\lambda \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} + \cdots + \frac{z^{n-1}}{\lambda^n} \right)$$

then $H^{(j)}(0) = 0$ for $0 \leq j < n$. Since

$$h(z) = \frac{f(z)}{g(z)} - G(z) = \frac{1}{z} \left(\frac{z f(z)}{g(z)} - d_0 \right) - H(z)$$

we may evaluate h by computing the Laurent expansion of f/g around 0. This yields

$$h(z) = \sum_{m=1}^{n} \frac{1}{m!} \lim_{w\to 0} \frac{d^m}{dw^m} \left(\frac{wf(w)}{g(w)} \right) z^{m-1}. \tag{3.10}$$

Fix some $0 < \delta' < \delta_1$ and define $\gamma(R)$ to be the closed curve consisting of the portion of the circle $|z| = R$ for which $|z - \lambda| \geq \delta'\rho(\lambda)$ and of the portions the circles $|z - \lambda| = \delta'\rho(\lambda)$ that intersect the disc $|z| \leq R$ in such a manner that λ is in the domain bounded by $\gamma(R)$ if and only if $|\lambda| < R$. Then the Cauchy residue theorem implies that

$$\frac{1}{2\pi i} \int_{\gamma(R)} \frac{f(w)}{g(w)w^m}\, dw = \frac{1}{m!} \lim_{w\to 0} \frac{d^m}{dw^m} \left(\frac{wf(w)}{g(w)} \right) + \sum_{0<|\lambda|<R} \frac{d_\lambda}{\lambda^m}.$$

Now the length of the contour of integration is comparable to the length of the circle of radius R. Moreover $d_\mu(z, \Lambda)$ is bounded away from 0 for $z \in \gamma(R)$ so that $|f(z)/g(z)|$ is bounded above. This implies that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma(R)} \frac{f(w)}{g(w)w^m}\, dw = 0$$

for $m \geq 2$, whence

$$\frac{1}{m!} \lim_{w\to 0} \frac{d^m}{dw^m} \left(\frac{wf(w)}{g(w)} \right) = -\text{p.v.} \sum_{\lambda\in\Lambda_0} \frac{d_\lambda}{\lambda^m}.$$

Inserting this expression into (3.10) yields

$$h(z) = \lim_{w\to 0} \frac{d}{dw} \left(\frac{wf(w)}{g(w)} \right) - \text{p.v.} \sum_{\lambda\in\Lambda_0} d_\lambda \sum_{m=2}^{n} \frac{z^{m-1}}{\lambda^m}.$$

Computing now $f = g(G + h)$ completes the proof.

(ii) We use the same notation. Since $\mathcal{F}^p_\phi \subseteq \mathcal{F}^\infty_\phi$ we know that (3.6) must hold. But now

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma(R)} \frac{f(w)}{g(w)w}\, dw = 0$$

so that $w_0 = -\text{p.v.} \sum_{\lambda\in\Lambda_0} \frac{d_\lambda}{\lambda}$. ■

Remarks. 1. Given any function $f \in \mathcal{F}^\infty_\phi$ the function $f + Cg$ is also in $\mathcal{F}^\infty_\phi$ for any constant C, and the functions agree at every $\lambda \in \Lambda$. Thus Λ is not a set of uniqueness for this space. Part (i), however, tells us that this is the only possibility, that is, if $f, \tilde{f} \in \mathcal{F}^\infty_\phi$ and $f(\lambda) = \tilde{f}(\lambda)$ for all $\lambda \in \Lambda$ then $f - \tilde{f} = Cg$ for some constant C.

2. On the other hand, (ii) shows that Λ is a set of uniqueness for the spaces $\mathcal{F}^p_\phi$ when $1 \leq p < +\infty$.

3. The representation (3.7) is (3.8) with the obvious choice of g_λ, except that we are taking principal values of the sum. In fact if $p = 1$ then the sum appearing in (3.7) is absolutely convergent, so the principal value may be ignored. In this case the proof may be simplified by taking G to be this sum and estimating similarly. The decay of this function away from the lattice means that we have no need to invoke the Cauchy residue theorem, or to involve principal values.

3.4 The discrete Beurling-Ahlfors transform

It is well known that the Beurling-Ahlfors transform given by[1]

$$T[f](\zeta) = \lim_{\epsilon \to 0} \int_{\mathbb{C} \setminus D(\zeta,\epsilon)} \frac{f(z)}{(\zeta - z)^2} \, dm(z), \tag{3.11}$$

where m denotes the Lebesgue measure on the plane, is a bounded linear operator from $L^p(\mathbb{C})$ to $L^p(\mathbb{C})$ for $1 < p < +\infty$. In fact this also holds if we replace $L^p(\mathbb{C})$ by a more general weighted space. We make use of the following definition.

Definition 3.10 ([Ste93, p. 194]). A weight ω on $\mathbb{R}^n$ is said to be a *Muckenhoupt A_p weight* if it is locally integrable and there exists some constant A such that

$$\left(\frac{1}{|B|} \int_B \omega(x) \, dm(x) \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{q}{p}} \, dm(x) \right)^{\frac{p}{q}} \leq A < \infty \tag{3.12}$$

for all balls B in $\mathbb{R}^n$. Here, m is the Lebesgue measure on $\mathbb{R}^n$, q is the Hölder conjugate exponent of p (that is, $1/p + 1/q = 1$) and $|B|$ is the Lebesgue measure of the ball B. The least constant A for which this holds is called the A_p bound of ω, denoted $A_p(\omega)$

We shall of course be interested in $\mathbb{R}^2$ which we identify with $\mathbb{C}$. Now the corollary to [Ste93, Chapter V, Theorem 2] combined with [Ste93, Chapter V, Section 4.5.2] shows that T is a bounded linear operator from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < +\infty$ for any A_p weight ω. (In fact the proof is given for a much more general class of integral operators, of which T is a special case.) We aim to use this property to study a discrete analogue.

We shall be interested in the case when ρ^{p-2} is an A_p weight. We write[2] $d\nu = dm/\rho^2$. Substituting into (3.12) and re-formulating shows that this is equivalent to saying that there exists some constant A such that

$$\frac{1}{|D|} \left(\int_D \rho(z)^p \, d\nu(z) \right)^{\frac{1}{p}} \left(\int_D \rho(z)^q \, d\nu(z) \right)^{\frac{1}{q}} \leq A \tag{3.13}$$

for all discs D in the plane. We note that this is trivially satisfied if $p = 2$. It is also satisfied for all p if $\rho(z) \simeq 1$, as is the case in the classical Bargmann-Fock space. We now construct an example to show that there are situations where this condition does not hold. As a first observation, since (3.13) is symmetric in p and q, we can assume that $p < 2$. We also note that by (3.12) it suffices to check only discs of large radius.

We will take $\phi(z) = C_\alpha |z|^\alpha$ for some positive constants α and C_α, which means that $\rho(z) \simeq \rho(0)$ for $z \in D(0)$ and $\rho(z) \simeq |z|^{1-\frac{\alpha}{2}}$ for $z \notin D(0)$. By choosing C_α appropriately, we may

[1]This differs from the usual definition by a factor of $-1/\pi$, and it is customary to denote this limit as a principal value. We have avoided doing so to eliminate any possible confusion with the principal value of a sum.

[2]This should not be confused with the hyperbolic area, which we have denoted by ν in other chapters.

assume that $\rho(0) = 2$. We pick $R > \rho(0)$ and take $D = D(0, R)$. Now

$$\begin{aligned}\left(\int_D \rho(z)^p \, d\nu(z)\right)^{\frac{1}{p}} &= \left(\int_{D(0)} \rho(z)^p \, d\nu(z) + \int_{D\setminus D(0)} \rho(z)^p \, d\nu(z)\right)^{\frac{1}{p}} \\ &\simeq \left(\rho(0)^{p-2}|D(0)| + \int_{\rho(0)}^{R} r^{(1-\frac{\alpha}{2})(p-2)} r \, dr\right)^{\frac{1}{p}} \\ &\simeq \left(\rho(0)^p + \frac{R^{p-\frac{\alpha p}{2}+\alpha} - \rho(0)^{p-\frac{\alpha p}{2}+\alpha}}{p - \frac{\alpha p}{2} + \alpha}\right)^{\frac{1}{p}} \\ &\simeq \left(R^{p-\frac{\alpha p}{2}+\alpha}\right)^{\frac{1}{p}} = R^{1-\frac{\alpha}{2}+\frac{\alpha}{p}}\end{aligned}$$

since $p - \frac{\alpha p}{2} + \alpha > 0$ for $p < 2$. We now choose some α such that $q - \frac{\alpha q}{2} + \alpha < 0$. Then an identical computation gives

$$\left(\int_D \rho(z)^q \, d\nu(z)\right)^{\frac{1}{q}} \simeq \left(\rho(0)^q + \frac{R^{q-\frac{\alpha q}{2}+\alpha} - \rho(0)^{q-\frac{\alpha q}{2}+\alpha}}{q - \frac{\alpha q}{2} + \alpha}\right)^{\frac{1}{q}} \simeq \rho(0) = 2.$$

Therefore

$$\frac{1}{|D|}\left(\int_D \rho(z)^p d\nu(z)\right)^{\frac{1}{q}} \left(\int_D \rho(z)^q d\nu(z)\right)^{\frac{1}{q}} \simeq \frac{2}{R^2} R^{1-\frac{\alpha}{2}+\frac{\alpha}{p}} \simeq R^{-1-\frac{\alpha}{2}+\frac{\alpha}{p}}$$

which is only uniformly bounded if $-1 - \frac{\alpha}{2} + \frac{\alpha}{p} < 0$. However

$$-1 - \frac{\alpha}{2} + \frac{\alpha}{p} = -1 - \frac{\alpha}{2} + \alpha(1 - \frac{1}{q}) = -1 + \frac{\alpha}{2} - \frac{\alpha}{q}$$

which we have assumed to be positive. This shows that there exist situations where ρ^{p-2} is not an A_p weight.

As before $\Lambda = \mathcal{Z}(g)$ will be the irregular lattice we are considering. Given a sequence $d \in \ell^p(\rho^{-1})$ we define, for each $\lambda' \in \Lambda$,

$$B_{\lambda'}(d) = \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{(\lambda' - \lambda)^2} \tag{3.14}$$

which we shall normally write as $B_{\lambda'}$, suppressing the dependence on d. It is clear that this is the discrete analogue of (3.11). Lemma 3.3 shows that, for $1 \leq p \leq 2$, this sum converges absolutely for each $\lambda' \in \Lambda$. Also, by Lemma 3.3, this sum converges for $2 < p < \infty$ if $t > 1/2$ where t is the constant occurring in Lemma 2.4. The main result in this section is the following, which is proved using the boundedness of (3.11).

Theorem 3.11. *Fix $1 < p < +\infty$ and suppose that ρ^{p-2} is an A_p weight. Define the operator*

$$\begin{aligned} B : \ell^p(\rho^{-1}) &\to \mathbb{C}^\Lambda \\ d &\mapsto (B_{\lambda'})_{\lambda' \in \Lambda} \end{aligned}$$

where $B_{\lambda'}$ is given by (3.14). *Then B is a bounded linear operator from $\ell^p(\rho^{-1})$ to $\ell^p(\rho)$ for $1 < p \leq 2$. If in addition $t > 1/2$ then the result also holds for $2 < p < +\infty$. Here t is the constant occurring in Lemma 2.4.*

Proof. We first note that it is obvious that B is linear, we are interested in showing that it indeed maps $\ell^p(\rho^{-1})$ to $\ell^p(\rho)$, and is a bounded operator. Recall that the sets $D^{\delta_1}(\lambda)$ are pairwise disjoint. Suppose that $d \in \ell^p(\rho^{-1})$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \frac{1}{\pi\delta_1^2} \sum_{\lambda\in\Lambda} \frac{d_\lambda}{\rho(\lambda)^2} \chi_{D^{\delta_1}(\lambda)}(z)$$

where χ_D is the characteristic function of the set D. Then clearly $f \in L^p(\rho^{p-2})$. In fact $\|f\|^p_{L^p(\rho^{p-2})} \simeq \frac{\pi\delta_1^2}{\pi^p\delta_1^{2p}} \sum_{\lambda\in\Lambda} \left|\frac{d_\lambda}{\rho(\lambda)}\right|^p$ so that, by our A_p assumption, $T[f] \in L^p(\rho^{p-2})$ and indeed

$$\|T[f]\|_{L^p(\rho^{p-2})} \leq \|T\|\|f\|_{L^p(\rho^{p-2})} \simeq \|T\|\|d_\lambda\|_{\ell^p(\rho^{-1})}.$$

Now

$$\begin{aligned}
T[f](\lambda') &= \lim_{\epsilon\to 0} \int_{\mathbb{C}\setminus D(\lambda',\epsilon)} \frac{f(z)}{(\lambda'-z)^2}\, dm(z) \\
&= \sum_{\lambda\in\Lambda_{\lambda'}} \int_{D^{\delta_1}(\lambda)} \frac{f(z)}{(\lambda'-z)^2}\, dm(z) + \lim_{\epsilon\to 0} \int_{D^{\delta_1}(\lambda')\setminus D(\lambda',\epsilon)} \frac{f(z)}{(\lambda'-z)^2}\, dm(z) \\
&= \sum_{\lambda\in\Lambda_{\lambda'}} \int_{D^{\delta_1}(\lambda)} \frac{d_\lambda}{\pi\delta_1^2\rho(\lambda)^2} \frac{1}{(\lambda'-z)^2}\, dm(z) \\
&= \sum_{\lambda\in\Lambda_{\lambda'}} \frac{d_\lambda}{(\lambda'-\lambda)^2} = B_{\lambda'},
\end{aligned}$$

since the value of f is constant on $D^{\delta_1}(\lambda)$ for each $\lambda \in \Lambda$ and the average value of a harmonic function on a disk is the value at the centre. Fix $0 < \delta < \delta_1$. It is obvious that

$$\begin{aligned}
|\rho(\lambda')B_{\lambda'}|^p &= \rho(\lambda')^p |T[f](\lambda')|^p \\
&\lesssim \rho(\lambda')^p \left| T[f](\lambda') - \frac{1}{\pi\delta^2\rho(\lambda')^2} \int_{D^\delta(\lambda')} T[f](\zeta)\, dm(\zeta) \right|^p \\
&\quad + \rho(\lambda')^p \left| \frac{1}{\pi\delta^2\rho(\lambda')^2} \int_{D^\delta(\lambda')} T[f](\zeta)\, dm(\zeta) \right|^p
\end{aligned} \tag{3.15}$$

and we shall estimate these terms separately. The second term is especially easy to bound since, by Jensen's inequality,

$$\begin{aligned}
\rho(\lambda')^p \left| \frac{1}{\pi\delta^2\rho(\lambda')^2} \int_{D^\delta(\lambda')} T[f](\zeta)\, dm(\zeta) \right|^p &\leq \rho(\lambda')^p \frac{1}{\pi\delta^2\rho(\lambda')^2} \int_{D^\delta(\lambda')} |T[f](\zeta)|^p\, dm(\zeta) \\
&\simeq \frac{1}{\pi\delta^2} \int_{D^\delta(\lambda')} |T[f](\zeta)|^p \frac{dm(\zeta)}{\rho(\zeta)^{2-p}}.
\end{aligned} \tag{3.16}$$

We now estimate the first term. Applying the definitions and computing gives

$$
\begin{aligned}
&T[f](\lambda') - \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')} T[f](\zeta)\,dm(\zeta) = \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')} T[f](\lambda') - T[f](\zeta)\,dm(\zeta)\\
&= \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')}\Big(\lim_{\epsilon\to 0}\int_{\mathbb{C}\setminus D(\lambda',\epsilon)}\frac{f(z)}{(\lambda'-z)^2}\,dm(z) - \lim_{\epsilon\to 0}\int_{\mathbb{C}\setminus D(\zeta,\epsilon)}\frac{f(z)}{(\zeta-z)^2}\,dm(z)\Big)\,dm(\zeta)\\
&= \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')}\Big\{\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')} f(z)\Big[\frac{1}{(\lambda'-z)^2} - \frac{1}{(\zeta-z)^2}\Big]\,dm(z)\\
&\qquad + \lim_{\epsilon\to 0}\int_{D^{\delta_1}(\lambda')\setminus D(\lambda',\epsilon)}\frac{f(z)}{(\lambda'-z)^2}\,dm(z) - \lim_{\epsilon\to 0}\int_{D^{\delta_1}(\lambda')\setminus D(\zeta,\epsilon)}\frac{f(z)}{(\zeta-z)^2}\,dm(z)\Big\}\,dm(\zeta).
\end{aligned}
$$

We shall bound each of these three terms separately. First note that, by symmetry,

$$
\lim_{\epsilon\to 0}\int_{D^{\delta_1}(\lambda')\setminus D(\lambda',\epsilon)}\frac{f(z)}{(\lambda'-z)^2}\,dm(z) = 0.
$$

Note also that if

$$
\int_{D^\delta(\lambda')}\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')} f(z)\Big[\frac{1}{(\lambda'-z)^2} - \frac{1}{(\zeta-z)^2}\Big]\,dm(z)dm(\zeta)
$$

is absolutely convergent then it vanishes similarly, since we may apply Fubini's theorem. But

$$
\begin{aligned}
&\int_{D^\delta(\lambda')}\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')}\Big|f(z)\Big[\frac{1}{(\lambda'-z)^2} - \frac{1}{(\zeta-z)^2}\Big]\Big|\,dm(z)dm(\zeta)\\
&\qquad = \int_{D^\delta(\lambda')}\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')}\Big|f(z)\Big[\frac{(\zeta+\lambda'-2z)(\zeta-\lambda')}{(\lambda'-z)^2(\zeta-z)^2}\Big]\Big|\,dm(z)dm(\zeta)\\
&\qquad \lesssim \int_{D^\delta(\lambda')}|\zeta-\lambda'|\,dm(\zeta)\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')}\frac{|f(z)|}{|\lambda'-z|^3}\,dm(z),
\end{aligned}
$$

since for $\zeta\in D^\delta(\lambda')$ and $z\in\mathbb{C}\setminus D^{\delta_1}(\lambda')$ we have $|\zeta-z|\simeq|\lambda'-z|$. The integral in ζ is clearly finite. It remains only to estimate

$$
\int_{\mathbb{C}\setminus D^{\delta_1}(\lambda')}\frac{|f(z)|}{|\lambda'-z|^3}\,dm(z) = \sum_{\lambda\in\Lambda_{\lambda'}}\frac{|d_\lambda|}{\pi\delta_1^2\rho(\lambda)^2}\int_{D^{\delta_1}(\lambda)}\frac{dm(z)}{|\lambda'-z|^3} \simeq \sum_{\lambda\in\Lambda_{\lambda'}}\frac{|d_\lambda|}{|\lambda'-\lambda|^3}
$$

which we have already seen is finite under our hypothesis, in Lemma 3.3. We consequently have

$$
\begin{aligned}
&T[f](\lambda') - \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')} T[f](\zeta)\,dm(\zeta)\\
&\qquad = \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')} -\lim_{\epsilon\to 0}\int_{D^{\delta_1}(\lambda')\setminus D(\zeta,\epsilon)}\frac{f(z)}{(\zeta-z)^2}\,dm(z)dm(\zeta)\\
&\qquad = -\frac{d_{\lambda'}}{\pi^2\delta^2\delta_1^2\rho(\lambda')^4}\int_{D^\delta(\lambda')}\lim_{\epsilon\to 0}\int_{D^{\delta_1}(\lambda')\setminus D(\zeta,\epsilon)}\frac{dm(z)}{(\zeta-z)^2}dm(\zeta)
\end{aligned}
$$

Now the inner integral does not change in value for $\epsilon \le (\delta_1 - \delta)\rho(\lambda')$. We therefore have

$$\begin{aligned}
&\Big|T[f](\lambda') - \frac{1}{\pi\delta^2\rho(\lambda')^2}\int_{D^\delta(\lambda')} T[f](\zeta)\,dm(\zeta)\Big| \\
&\le \frac{|d_{\lambda'}|}{\pi^2\delta^2\delta_1^2\rho(\lambda')^4}\int_{D^\delta(\lambda')}\int_{D^{\delta_1}(\lambda')\setminus D(\zeta,(\delta_1-\delta)\rho(\lambda'))}\frac{dm(z)}{|\zeta - z|^2}dm(\zeta) \\
&\le \frac{|d_{\lambda'}|}{\pi^2\delta^2\delta_1^2\rho(\lambda')^4}\frac{1}{(\delta_1-\delta)^2\rho(\lambda')^2}|D^\delta(\lambda')||D^{\delta_1}(\lambda')\setminus D(\zeta,(\delta_1-\delta)\rho(\lambda'))| \\
&\simeq \frac{|d_{\lambda'}|}{\rho(\lambda')^2}. \qquad (3.17)
\end{aligned}$$

Inserting (3.16) and (3.17) into (3.15) gives finally that

$$\begin{aligned}
\sum_{\lambda'\in\Lambda}|\rho(\lambda')B_{\lambda'}|^p &\lesssim \sum_{\lambda'\in\Lambda}\left(\int_{D^\delta(\lambda')}|T[f](\zeta)|^p\frac{dm(\zeta)}{\rho(\zeta)^{2-p}} + \frac{|d_{\lambda'}|^p}{\rho(\lambda')^p}\right) \\
&\le \int_{\mathbb{C}}|T[f](\zeta)|^p\frac{dm(\zeta)}{\rho(\zeta)^{2-p}} + \sum_{\lambda'\in\Lambda}\frac{|d_{\lambda'}|^p}{\rho(\lambda')^p} \\
&= \|T[f]\|^p_{L^p(\rho^{p-2})} + \|d\|^p_{\ell^p(\rho^{-1})} \lesssim (1 + \|T\|^p)\|d\|^p_{\ell^p(\rho^{-1})}
\end{aligned}$$

so that B is indeed a bounded operator as claimed. ■

3.5 Proofs

We are essentially going to give a unified proof of Theorems 3.4, 3.5, 3.6 and 3.7. We shall refer to an integer N which should be thought of as 2 in the cases of Theorems 3.4, 3.5, 3.6 and 3.7 (i) and (ii), but to be the integer N appearing in the statement of Theorem 3.7(iii). We also note that if $N = 2$ and ρ^{p-2} satisfies the A_p condition then we may apply Theorem 3.11. We begin by showing the necessity of the stated results. We shall use the same notation as before. We write $d_\lambda = f(\lambda)/g'(\lambda)$ which, by virtue of the growth conditions on g, satisfies $(d_\lambda/\rho(\lambda))_{\lambda\in\Lambda} \in \ell^p$.

Proof of the necessity. We have already remarked in (3.2) that (a) follows from the Plancherel-Polya-type estimate. We define $\gamma(R)$ as in the proof of Lemma 3.9. Computing, for any $\lambda' \in \Lambda$,

$$\frac{1}{2\pi i}\int_{\gamma(R)}\frac{f(w)}{g(w)(w-\lambda')^n}\,dw$$

in exactly the same manner as in the proof of Lemma 3.9, where $1 \le n \le N$, shows that

$$\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{d_\lambda}{(\lambda - \lambda')^n} \qquad (3.18)$$

is well-defined. Fix some $0 < \delta < \delta_1$ and some integer $0 \le k < N$. Define $\omega_k = e^{2\pi i k/N}$ and $z_{\lambda'}^k = \lambda' + \delta\omega_k\rho(\lambda')$. Then, for each k, $(z_{\lambda'}^k)_{\lambda'\in\Lambda}$ is a d_μ-separated sequence that is bounded away from Λ in the distance d_μ. Now (3.1) implies that

$$\sum_{\lambda'\in\Lambda}\left|\frac{f(z_{\lambda'}^k)}{g(z_{\lambda'}^k)}\right|^p < +\infty.$$

Replacing z by $\delta\omega_k\rho(\lambda')$ and λ by $\lambda-\lambda'$ in Identity (3.9) yields

$$\frac{1}{z_{\lambda'}^k-\lambda}+\frac{1}{\lambda-\lambda'}+\frac{\delta\omega_k\rho(\lambda')}{(\lambda-\lambda')^2}+\cdots+\frac{(\delta\omega_k\rho(\lambda'))^{n-1}}{(\lambda-\lambda')^n}=\frac{(\delta\omega_k\rho(\lambda'))^n}{(\lambda-\lambda')^n(z_{\lambda'}^k-\lambda)}.$$

Consequently, invoking (3.7), we compute that

$$\begin{aligned}\frac{f(z_{\lambda'}^k)}{g(z_{\lambda'}^k)}&+\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}d_\lambda\left(\frac{1}{\lambda-\lambda'}+\frac{\delta\omega_k\rho(\lambda')}{(\lambda-\lambda')^2}+\cdots+\frac{(\delta\omega_k\rho(\lambda'))^{N-1}}{(\lambda-\lambda')^N}\right)\\ &=\frac{d_{\lambda'}}{\delta\omega_k\rho(\lambda')}+\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{d_\lambda(\delta\omega_k\rho(\lambda'))^N}{(\lambda-\lambda')^N(z_{\lambda'}^k-\lambda)}.\end{aligned}$$

(We are allowed to add the principal values because they are finite by virtue of (3.7) and (3.18).) Hence

$$\begin{aligned}&\sum_{\lambda'\in\Lambda}\Big|\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}d_\lambda\Big(\frac{1}{\lambda-\lambda'}+\frac{\delta\omega_k\rho(\lambda')}{(\lambda-\lambda')^2}+\cdots+\frac{(\delta\omega_k\rho(\lambda'))^{N-1}}{(\lambda-\lambda')^N}\Big)\Big|^p\\ &\qquad\lesssim\sum_{\lambda'\in\Lambda}\Big\{\Big(\delta^N\rho(\lambda')^N\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{|d_\lambda|}{|\lambda-\lambda'|^N|z_{\lambda'}^k-\lambda|}\Big)^p+\left|\frac{d_{\lambda'}}{\delta\rho(\lambda')}\right|^p+\left|\frac{f(z_{\lambda'}^k)}{g(z_{\lambda'}^k)}\right|^p\Big\}.\end{aligned}$$

We know that the second and third terms on the right-hand side are summable, it remains only to estimate the first. All of the terms in this sum are positive, so we may ignore the principal value. Moreover $|z_{\lambda'}^k-\lambda|\simeq|\lambda-\lambda'|$, so that Lemma 3.3 shows that this double sum is convergent. (It is here that the value of N is important.) Taking now linear combinations over different k completes the proof, for example,

$$\begin{aligned}&\sum_{\lambda'\in\Lambda}\Big|\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}\frac{d_\lambda}{\lambda-\lambda'}\Big|^p\\ &\quad=\sum_{\lambda'\in\Lambda}\Big|\text{p.v.}\sum_{\lambda\in\Lambda_{\lambda'}}d_\lambda\frac{1}{N}\sum_{k=0}^{N-1}\Big(\frac{1}{\lambda-\lambda'}+\frac{\delta\omega_k\rho(\lambda')}{(\lambda-\lambda')^2}+\cdots+\frac{(\delta\omega_k\rho(\lambda'))^{N-1}}{(\lambda-\lambda')^N}\Big)\Big|^p<+\infty.\end{aligned}$$

■

We now turn to the proof of the sufficiency, which is similar. We use the notation $d_\lambda = c_\lambda/g'(\lambda)$.

Proof of the sufficiency. We wish to construct a function that solves the interpolation problem $f|\Lambda = c$. As in the proof of Lemma 3.9, the naïve attempt at Lagrange interpolation is not, in general, convergent. We modify in the exact same manner, and a similar argument shows that

$$G(z) = \frac{d_0}{z} + \sum_{\lambda \in \Lambda_0} d_\lambda \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} + \cdots + \frac{z^{N-1}}{\lambda^N} \right)$$

defines a meromorphic function on $\mathbb{C}$. (Here we invoke Lemma 3.3 to see the series is convergent, which once more determines the value of N.) Hence

$$G(z) - \sum_{k=1}^{N} z^{k-1} \text{p.v.} \sum_{\lambda \in \Lambda_0} \frac{d_\lambda}{\lambda^k} = \text{p.v.} \sum_{\lambda \in \Lambda} \frac{d_\lambda}{z-\lambda}$$

is a well-defined meromorphic function. It follows that $f(z) = g(z)\text{p.v.} \sum_{\lambda \in \Lambda} \frac{d_\lambda}{z-\lambda}$ is an entire function satisfying $f(\lambda) = c_\lambda$. It remains to show that $f \in \mathcal{F}^p_\phi$. We must show that the following integral is finite (recall that $Q_\lambda = \{z \in \mathbb{C} : d_\mu(z, \Lambda) = d_\mu(z, \lambda)\}$):

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(z)} \frac{dm(z)}{\rho(z)^2} &= \sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} |f(z)|^p e^{-p\phi(z)} \frac{dm(z)}{\rho(z)^2} \\
&= \sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big| g(z) e^{-\phi(z)} \text{p.v.} \sum_{\lambda \in \Lambda} \frac{d_\lambda}{z-\lambda} \Big|^p \frac{dm(z)}{\rho(z)^2} \\
&\simeq \sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big| d_\mu(z, \lambda') \Big(\frac{d_{\lambda'}}{z-\lambda'} + \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{z-\lambda} \Big) \Big|^p \frac{dm(z)}{\rho(z)^2} \\
&\lesssim \sum_{\lambda' \in \Lambda} \Big| \frac{d_{\lambda'}}{\rho(\lambda')} \Big|^p + \sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big| \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{z-\lambda} \Big|^p \frac{dm(z)}{\rho(z)^2},
\end{aligned}$$

where we have used the fact that $d_\mu(z, \lambda') \simeq |z - \lambda'|/\rho(\lambda') \lesssim 1$ for $z \in Q_{\lambda'}$, which follows from Lemma 2.6. The first term is finite by hypothesis, so we need only bound the second. Once more we use Identity (3.9) which suitably modified yields

$$\frac{1}{z-\lambda} = \frac{(z-\lambda')^N}{(z-\lambda)(\lambda-\lambda')^N} - \frac{1}{\lambda-\lambda'} - \frac{z-\lambda'}{(\lambda-\lambda')^2} - \cdots - \frac{(z-\lambda')^{N-1}}{(\lambda-\lambda')^N}$$

whence

$$\begin{aligned}
\sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big| \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{z-\lambda} \Big|^p \frac{dm(z)}{\rho(z)^2} \lesssim \sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big| \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda (z-\lambda')^N}{(z-\lambda)(\lambda-\lambda')^N} \Big|^p \\
+ \sum_{n=1}^{N} \Big| \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda (z-\lambda')^{n-1}}{(\lambda-\lambda')^n} \Big|^p \frac{dm(z)}{\rho(z)^2}.
\end{aligned}$$

Now for $z \in Q_{\lambda'}$ we have $|z-\lambda'| \lesssim \rho(\lambda')$ and $|z-\lambda| \simeq |\lambda'-\lambda|$. Hence, by (3.3),

$$\sum_{\lambda' \in \Lambda} \int_{Q_{\lambda'}} \Big|\text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{z-\lambda}\Big|^p \frac{dm(z)}{\rho(z)^2} \lesssim \sum_{\lambda' \in \Lambda} \Big(\rho(\lambda')^N \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{|d_\lambda|}{|\lambda-\lambda'|^{N+1}}\Big)^p$$

$$+ \sum_{n=1}^{N} \sum_{\lambda' \in \Lambda} \Big|\rho(\lambda')^{n-1} \text{p.v.} \sum_{\lambda \in \Lambda_{\lambda'}} \frac{d_\lambda}{(\lambda-\lambda')^n}\Big|^p.$$

The first term is finite by Lemma 3.3 (again the value of N is important here), the remainder by hypothesis. This completes the proof. ■

The proof of Theorem 3.8 is similar and omitted.

Chapter 4

Inhomogeneous random zero sets

In this chapter we are interested in random point processes that mimic a given σ-finite measure μ on the complex plane. The classical example is the inhomogeneous Poisson point process, which we consider in the following manner. Fix a parameter $L > 0$ and let N_L be the Poisson random measure on $\mathbb{C}$ with intensity $L\mu$, that is,

- N_L is a random measure on $\mathbb{C}$,
- for every measurable $A \subset \mathbb{C}$, $N_L(A)$ is a Poisson random variable with mean $L\mu(A)$, and
- if A and B are disjoint then $N_L(A)$ and $N_L(B)$ are independent.

Such an N_L always exists, see, for example, [Sat99, Proposition 19.4]. Suppose that $\psi \in L^1(\mu) \cap L^2(\mu)$ and define

$$N(\psi, L) = \frac{1}{L}\int_{\mathbb{C}} \psi(z)\, dN_L(z).$$

Then (see [Sat99, Proposition 19.5])

$$\mathbb{E}\left[N(\psi, L)\right] = \int_{\mathbb{C}} \psi\, d\mu \tag{4.1}$$

and

$$\mathbb{V}[N(\psi, L)] = \frac{1}{L}\int_{\mathbb{C}} |\psi|^2\, d\mu. \tag{4.2}$$

In contrast to the Poisson point process, the zero sets of random analytic functions are known to be more 'rigid' processes, in particular these processes exhibit 'local repulsion' (see [HKPV09, Chapter 1]). We will construct a random zero set (the zero set of a GAF) such that (4.1) continues to hold (at least for smooth ψ in the limit $L \to \infty$, see Theorem 4.1) but with a variance that decays faster than L^{-2}, in contrast to (4.2) (Theorem 4.2). In fact we will also have

$$N(\psi, L) \to \int_{\mathbb{C}} \psi\, d\mu \quad \text{as} \quad L \to \infty$$

almost surely, as well as being true in mean (Theorem 4.1).

As a further measure of the 'rigidity' of our process we note that the 'hole probability' for the Poisson point process is, by definition,

$$\mathbb{P}[N_L(A) = 0] = e^{-L\mu(A)}$$

for any $A \subset \mathbb{C}$ whereas we shall see that the 'hole probability' for the zero sets we construct decays at least like e^{-cL^2} for some $c > 0$.

When μ is the Lebesgue measure on the plane, our construction will correspond to the flat GAF considered in the introduction, whose zero-set is invariant in distribution under plane isometries. We are interested in generalising this construction to other measures, where we cannot expect any such invariance to hold.

The chapter is structured as follows. In Section 1 we define two GAFs via generalised Fock spaces, and state our main results about their zero sets. We also include some simple technical results on re-scaled weights that shall be used throughout the chapter. In Section 2 we prove Theorems 4.1 and 4.2, that compute the mean and the variance of the smooth linear statistics. In Section 3 we show that the smooth linear statistics are asymptotically normal, under some extra regularity assumptions (Theorem 4.5). In Section 4 we prove some large deviations estimates, Theorem 4.3 and Corollary 4.4. In Section 5 we give an upper bound for the hole probability for the zero set of one of the GAFs defined in Section 1, Theorem 4.6. Finally in Section 6 we compute the asymptotic hole probability for the zero set of the other GAF defined in Section 1, Theorem 4.7. In an appendix we compute the L^2 decay of the covariance kernel for the GAFs corresponding to certain radial measures.

4.1 Definitions and statements of our results

Throughout this chapter μ will be a doubling measure and ϕ will be a subharmonic function with $\mu = \Delta\phi$. We consider the generalised Fock space introduced in Chapter 2

$$\mathcal{F}^2_\phi = \{f \in H(\mathbb{C}) : \|f\|^2_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)} \frac{dm(z)}{\rho(z)^2} < +\infty\}$$

where m is the Lebesgue measure on the plane. Let $(e_n)_n$ be an orthonormal basis for the space $\mathcal{F}^2_\phi$ and let $(a_n)_n$ be a sequence of iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables. Consider the GAF (see Section 1.2 for the definition) defined by

$$g(z) = \sum_n a_n e_n(z).$$

This sum almost surely defines an entire function with associated covariance kernel

$$\mathcal{K}(z,w) = \mathbb{E}[g(z)\overline{g(w)}] = \sum_n e_n(z)\overline{e_n(w)},$$

which is the reproducing kernel for the space $\mathcal{F}^2_\phi$. Moreover the distribution of the random analytic function g is determined by the kernel $\mathcal{K}$ so it does not matter which basis we choose.

We are interested in studying the zero set $\mathcal{Z}(g)$, and a first observation is that since $\mathcal{K}(z,z) \neq 0$ (see Proposition 2.12), g has no deterministic zeroes. We note also that the first intensity is given by $\Delta \log \mathcal{K}(z,z) \simeq \frac{1}{\rho(z)^2}$ (again see Proposition 2.12) which, as we noted in Chapter 2, can be viewed as a regularisation of the measure μ.

We will modify this construction by re-scaling the weight ϕ, so that the zeroes will be even better distributed. Specifically, let L be a (large) positive parameter and consider instead the weight $\phi_L = L\phi$ (and $\rho_L = \rho_{L\mu}$). For each L we take a basis $(e^L_n)_n$ for the space[1] $\mathcal{F}^2_L = \mathcal{F}^2_{\phi_L}$ and define

$$g_L(z) = \sum_n a_n e^L_n(z) \tag{4.3}$$

and

$$\mathcal{K}_L(z,w) = \mathbb{E}[g_L(z)\overline{g_L(w)}] = \sum_n e^L_n(z)\overline{e^L_n(w)}.$$

The following result states that the corresponding zero set, suitably scaled, is well distributed with respect to the measure μ for large values of L.

Theorem 4.1. *Let ψ be a smooth real-valued function with compact support in $\mathbb{C}$ (which we always assume is not identically zero), let n_L be the counting measure on the zero set of g_L and define the random variable $n(\psi, L) = \frac{1}{L}\int \psi \, dn_L$.*

(a)

$$\left| \mathbb{E}\left[n(\psi,L)\right] - \frac{1}{2\pi}\int \psi \, d\mu \right| \lesssim \frac{1}{L}\int_{\mathbb{C}} |\Delta\psi(z)| \, dm(z),$$

where the implicit constant depends only on the doubling constant of the measure μ.

(b) If we restrict L to taking integer values then, almost surely,

$$n(\psi,L) \to \frac{1}{2\pi}\int \psi \, d\mu$$

as $L \to \infty$.

The proof of part (b) of this result uses an estimate on the the decay of the variance of $n(\psi, L)$ which is interesting by itself.

Theorem 4.2. *For any smooth function ψ with compact support in $\mathbb{C}$*

$$\mathbb{V}[n(\psi,L)] \simeq \frac{1}{L^2}\int_{\mathbb{C}} (\Delta\psi(z))^2 \rho_L(z)^2 \, dm(z).$$

[1] If $\phi(z) = |z|^2/2$ then this corresponds to the space $\mathcal{F}^2_L$ defined in the introduction, with yet another change of normalisation.

Remark. We may estimate the dependence on L using (4.6) to see that the integral decays polynomially in L. If the measure μ is locally flat (see Definition 2.3) then, using (4.7), we see that the variance decays as L^{-3}, just as in [ST04].

In the special case $\phi(z) = |z|^2/2$ it is easy to see that the set[2] $(\frac{1}{\pi\sqrt{2}}\frac{(\sqrt{L}z)^n}{\sqrt{n!}})_{n=0}^{\infty}$ is an orthonormal basis for the corresponding Fock space, so that the construction just given corresponds to the flat GAF from the introduction. More generally if $\phi(z) = |z|^\alpha/2$ and $\alpha > 0$ then the set $(\frac{(L^{1/\alpha}z)^n}{c_{\alpha n}})_{n=0}^{\infty}$ is an orthonormal basis for the corresponding Fock space, for some $c_{\alpha n} \simeq \Gamma(\frac{2}{\alpha}n+1)^{1/2}$ (actually $c_{\alpha n} = c_{\alpha n}^{(L)}$ but the implicit constants depend only on α, see the appendix to this chapter).

However, besides these special cases, we have very little information about the behaviour of an orthonormal basis for $\mathcal{F}_L^2$. For this reason we also study random functions that are constructed via frames. We will consider frames for $\mathcal{F}_L^2$ consisting of normalised reproducing kernels, $(k_\lambda)_{\lambda\in\Lambda_L}$, where the index set $\Lambda_L \subset \mathbb{C}$ is a sampling sequence (this has been defined in Chapter 2). We shall require the sampling constant for Λ_L to be uniform in L (see pp. 51-52 for a precise statement of the assumptions on Λ_L and a proof that such a sequence always exists).

The advantage of this approach is that we have estimates for the size of the reproducing kernel (Theorem 2.12), and so we also have estimates for the size of the frame elements. We now define

$$f_L(z) = \sum_{\lambda\in\Lambda_L} a_\lambda k_\lambda(z) \tag{4.4}$$

where a_λ is a sequence of iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables indexed by the sequence Λ_L. The covariance kernel for f_L is given by

$$K_L(z,w) = \mathbb{E}[f_L(z)\overline{f_L(w)}] = \sum_{\lambda\in\Lambda_L} k_\lambda(z)\overline{k_\lambda(w)}$$

which satisfies similar estimates to $\mathcal{K}_L$ (see Proposition 2.15).

Since the proof of Theorem 4.1 uses only estimates for the size of the covariance kernel we may state an identical theorem for the GAF defined via frames. However in this case we also have the following stronger result.

Theorem 4.3. *Let n_L be the counting measure on the zero set of the GAF f_L defined via frames* (4.4)*, let ψ be a smooth real-valued function with compact support in $\mathbb{C}$, and let $n(\psi, L) = \frac{1}{L}\int \psi\, dn_L$.*

(a)

$$\left|\mathbb{E}\left[n(\psi,L)\right] - \frac{1}{2\pi}\int \psi\, d\mu\right| \lesssim \frac{1}{L}\int_{\mathbb{C}} |\Delta\psi(z)|\, dm(z),$$

where the implicit constant depends only on the doubling constant of the measure μ.

[2] The constant $1/(\pi\sqrt{2})$ arises from the change in normalisation.

(b) Let $\delta > 0$. There exists $c > 0$ depending only on δ, ψ and μ such that

$$\mathbb{P}\left[\left|\frac{n(\psi, L)}{\frac{1}{2\pi}\int \psi \, d\mu} - 1\right| > \delta\right] \leq e^{-cL^2}. \tag{4.5}$$

as $L \to \infty$.

The proof of part (a) is identical to the proof of Theorem 4.1 (a) (using the appropriate estimates for the covariance kernel of f_L). It is also easy to see, by an appeal to the first Borel-Cantelli Lemma, that the large deviations estimate (4.5) implies that in this case we also have almost sure convergence exactly as stated in Theorem 4.1 (b). This result has an obvious corollary.

Corollary 4.4. *Suppose that n_L is the counting measure on the zero set of the GAF f_L defined via frames* (4.4) *and that U is an open bounded subset of the complex plane.*

(a)

$$\mathbb{E}\left[\frac{1}{L}n_L(U)\right] \to \frac{1}{2\pi}\mu(U)$$

as $L \to \infty$.

(b) Let $\delta > 0$. There exists $c > 0$ depending only on δ, U and μ such that for sufficiently large values of L

$$\mathbb{P}\left[\left|\frac{\frac{1}{L}n_L(U)}{\frac{1}{2\pi}\mu(U)} - 1\right| > \delta\right] \leq e^{-cL^2}.$$

Remark. As before, the large deviations estimate combined with the first Borel-Cantelli Lemma implies that $\frac{1}{L}n_L(U) \to \frac{1}{2\pi}\mu(U)$ almost surely when L is restricted to integer values.

It is well known that the linear statistics are asymptotically normal for the Poisson point process, and we also show that the smooth linear statistics for our zero sets are asymptotically normal, for large values of L, if the measure μ is locally flat. We shall state and prove this result only for the GAF defined via frames (that is (4.4)) but it is easy to verify that the proof works equally well for the GAF defined via bases (4.3) since it relies only on estimates for the size of the covariance kernel.

Theorem 4.5. *Let ψ be a smooth function with compact support in $\mathbb{C}$, let n_L be the counting measure on the zero set of the GAF defined via frames* (4.4)*, and suppose that the measure μ is locally flat (see Definition 2.3). Define $n(\psi, L) = \frac{1}{L}\int \psi dn_L$ as before. Then the random variable*

$$\frac{n(\psi, L) - \mathbb{E}[n(\psi, L)]}{\mathbb{V}[n(\psi, L)]^{1/2}}$$

converges in distribution to $\mathcal{N}(0, 1)$ (the real standard normal) as $L \to \infty$.

We are also interested in the 'hole probability', the probability that there are no zeroes in a region of the complex plane. When we take $\phi(z) = |z|^2/2$ then, as we saw in the introduction, we have [Nis10, Theorem 1.1]

$$\mathbb{P}[n_{g_L}(D(z_0, r)) = 0] = \exp\Big\{ -\frac{e^2}{4} L^2 r^4 (1 + o(1)) \Big\}$$

as $L \to \infty$. If $\phi(z) = |z|^\alpha/2$ and we consider the random function g_L generated by the basis $(\frac{(L^{1/\alpha} z)^n}{c_{\alpha n}})_{n=0}^\infty$ then we can use [Nis11, Theorem 1] to see that

$$\mathbb{P}[n_{g_1}(D(0, rL^{1/\alpha})) = 0] = \exp\Big\{ -\frac{\alpha e^2}{8} r^{2\alpha} L^2 (1 + o(1)) \Big\}$$

as $L \to \infty$ and, by an identical computation, that

$$\mathbb{P}[n_{g_L}(D(0, r)) = 0] = \exp\Big\{ -\frac{\alpha e^2}{8} r^{2\alpha} L^2 (1 + o(1)) \Big\}$$

as $L \to \infty$ (we omit the details), however we no longer have translation invariance.

Our first result says that we always have an upper bound of the form e^{-cL^2}, however we have no estimate for the lower bound in general.

Theorem 4.6. *Suppose that n_L is the counting measure on the zero set of the GAF g_L defined via bases* (4.3)*. Let U be a bounded open subset of the complex plane. There exists $c > 0$ depending only on U and μ such that for sufficiently large values of L*

$$\mathbb{P}[n_L(U) = 0] \le e^{-cL^2}.$$

When we work with frames, because we have estimates for the pointwise decay of the reproducing kernel, and consequently for the frame elements, we can prove much more. In this case we show that we have the same upper bound (with a different constant) and that the upper bound is sharp (up to constants) under additional assumptions on the decay of the kernel $\mathcal{K}_L$.

Theorem 4.7. *Suppose that n_L is the counting measure on the zero set of the GAF f_L defined via frames* (4.4)*. Let U be an open bounded subset of the complex plane.*

(a) There exist $c, C > 0$ depending on U and μ, and $\tau \ge 2$ depending only on μ, such that for sufficiently large values of L

$$e^{-CL^\tau} \le \mathbb{P}[n_L(U) = 0] \le e^{-cL^2}.$$

(b) If the reproducing kernel $\mathcal{K}_L$ has fast L^2 off-diagonal decay (Definition 4.8) then we have $\tau = 2$ in (a).

Remarks. 1. The upper bound in this theorem follows directly from Corollary 4.4 (b)

2. In proving this result we will give upper bounds for the value τ when we do not have fast L^2 off-diagonal decay.

3. The kernel corresponding to $\phi(z) = |z|^\alpha/2$ has fast L^2 off-diagonal decay (see the appendix).

While we have stressed heretofore that these results generalise the known cases in the complex plane, we should point out that these ideas have also been studied on manifolds. For example in [SZ99], [SZ08] and [SZZ08] the authors study the distribution of zeroes of random holomorphic sections of (large) powers of a positive holomorphic line bundle L over a compact complex manifold M. Theorem 4.1, for example, is completely analogous to [SZ99, Theorem 1.1], although our proof is less technical. We have also used many of the ideas from [SZZ08] in our proof of Theorem 4.4, where the authors also deal with the problem of having no information about a basis. A key difference between the two settings is the compactness of the manifold M, which means that the spaces of sections considered are finite dimensional with a control on the growth of the dimension. There are also some recent results in a non-compact setting, see [DMS12], however the spaces considered are still assumed to be finite dimensional.

Scaled weights

We shall scale the measure μ by a (large) parameter $L \geq 1$. We shall write $\phi_L = L\phi$, $\rho_L = \rho_{L\mu}$, $d_L = d_{L\mu}$ and $D_L^r(z) = D(z, r\rho_L(z))$. Note that the measures μ and $L\mu$ have the same doubling constant, so we may apply most of the results of Chapter 2 *mutatis mutandis* to the measure $L\mu$ without changing the constants. It is clear from the definition

$$L\mu(D(z, \rho_L(z))) = 1$$

that $\rho_L(z) < \rho(z)$ for $L > 1$. Thus, by Lemma 2.2, we have

$$L^\gamma \lesssim \frac{\rho(z)}{\rho_L(z)} \lesssim L^{1/\gamma} \tag{4.6}$$

and

$$L^{-1/\gamma} \lesssim \frac{d(z,w)}{d_L(z,w)} \lesssim L^{-\gamma}$$

for some $\gamma \leq 1$, where the implicit constants are uniform in z.

If the measure μ is locally flat then we see that

$$\frac{\rho(z)}{\rho_L(z)} \simeq \sqrt{L} \tag{4.7}$$

and

$$\frac{d(z,w)}{d_L(z,w)} \simeq \frac{1}{\sqrt{L}}.$$

To apply Proposition 2.15 to the kernel K_L, it is important that the constants in the relation $\sum_{\lambda\in\Lambda_L} |f(\lambda)|^2 e^{-2\phi(\lambda)} \simeq \|f\|_{\mathcal{F}_L^2}$ are uniform in L, so that the constant C appearing in the conclusion can be taken to be uniform in L. This is the assumption we referred to when we chose the sequence Λ_L. It is not difficult to see that we can always do this. For each L we choose a sequence Λ_L and constants $\delta_0 < R_0$ which do not depend on L such that

- the discs $(D_L^{\delta_0}(\lambda))_{\lambda\in\Lambda_L}$ are pairwise disjoint,
- $\mathbb{C} = \cup_{\lambda\in\Lambda_L} D_L^{R_0}(\lambda)$, and
- each $z \in \mathbb{C}$ is contained in at most N_0 discs of the form $D_L^{R_0+1}(\lambda)$ where N_0 does not depend on z or L.

Applying Lemma 2.9 (b) one can show that if R_0 is sufficiently small then

$$\sum_{\lambda\in\Lambda_L} |f(\lambda)|^2 e^{-2\phi(\lambda)} \simeq \|f\|_{\mathcal{F}_L^2}$$

where the implicit constants are uniform in L.

We will sometimes be able to prove sharper results if we assume some extra off-diagonal decay on the kernel $\mathcal{K}_L$. The condition we will use is the following.

Definition 4.8. The kernel $\mathcal{K}_L$ has *fast L^2 off-diagonal decay* if, given $C, r > 0$ there exists $R > 0$ (independent of L) such that

$$\sup_{z\in D^r(z_0)} e^{-2\phi_L(z)} \int_{\mathbb{C}\setminus D^R(z_0)} |\mathcal{K}_L(z,\zeta)|^2 e^{-2\phi_L(\zeta)} \frac{dm(\zeta)}{\rho_L(\zeta)^2} \le e^{-CL} \tag{4.8}$$

for all $z_0 \in \mathbb{C}$ and L sufficiently large.

Remarks. 1. If $\phi(z) = |z|^2/2$ then since $\mathcal{K}_L(z,\zeta) = e^{Lz\bar{\zeta}}/2\pi^2$ it is easy to see that the $\mathcal{K}_L$ has fast L^2 off-diagonal decay. More generally if $\phi(z) = |z|^\alpha/2$ it can also be seen that $\mathcal{K}_L$ has fast L^2 off-diagonal decay but we postpone the proof to an appendix since it is long and tedious.

2. We also note that [Chr91, Proposition 1.18] shows that there exist ϕ with $0 < c < \Delta\phi < C$ that do not satisfy (4.8), so that local flatness does not imply fast L^2 decay.

4.2 Proof of Theorems 4.1 and 4.2

In this section we will prove Theorems 4.1 and 4.2. We will follow the scheme of the proof of [SZ99, Theorem 1.1]. We begin by proving Theorem 4.1 (a). Recall that n_L is the counting measure on the zero set of the GAF defined via bases, (4.3).

Proof of Theorem 4.1(a). Let ψ be a smooth function with compact support in $\mathbb{C}$. The Edelman-Kostlan formula gives

$$\mathbb{E}\left[n(\psi, L)\right] = \frac{1}{4\pi L}\int_{\mathbb{C}} \psi(z)\Delta \log \mathcal{K}_L(z,z) dm(z)$$

so that, by Proposition 2.12,

$$\begin{aligned}\left|\mathbb{E}\left[n(\psi, L)\right] - \frac{1}{2\pi}\int \psi d\mu\right| &= \frac{1}{4\pi L}\left|\int_{\mathbb{C}} \Delta\psi(z)\left(\log \mathcal{K}_L(z,z) - 2\phi_L(z)\right) dm(z)\right| \\ &\lesssim \frac{1}{L}\int_{\mathbb{C}} |\Delta\psi(z)| dm(z).\end{aligned}$$

∎

Proof of Theorem 4.2. We recall that the dilogarithm

$$\mathrm{Li}_2(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n^2}$$

satisfies

$$\mathrm{Li}_2(|\zeta|) \simeq |\zeta|$$

for $|\zeta| \leq 1$. We have (see [SZ08, Theorem 3.1] or [NS11, Lemma 2.3])

$$\mathbb{V}[n(\psi, L)] = \frac{1}{16\pi^2 L^2}\int_{\mathbb{C}}\int_{\mathbb{C}} \Delta\psi(z)\Delta\psi(w) J_L(z,w) dm(z) dm(w).$$

where

$$J_L(z,w) = \mathrm{Li}_2\left(\frac{|\mathcal{K}_L(z,w)|^2}{\mathcal{K}_L(z,z)\mathcal{K}_L(w,w)}\right).$$

For completeness, we will sketch a proof of this, for general GAFs: Detailed computations can be found in [SZ08, Section 3], [NS11, Section 2.1] or [HKPV09, Section 3.5]. Let f be a GAF with covariance kernel K, and write n_f for the counting measure on the zero set and

$$\hat{f}(z) = \frac{f(z)}{K(z,z)^{1/2}}.$$

For any smooth ψ with compact support in $\mathbb{C}$, Green's formula implies that

$$\int_{\mathbb{C}} \psi \, dn_f = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi \log|f| \, dm$$

which combined with the Edelman-Kostlan formula gives

$$\int_{\mathbb{C}} \psi \, dn_f - \mathbb{E}\left[\int_{\mathbb{C}} \psi \, dn_f\right] = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi \log|\hat{f}| \, dm.$$

Thus

$$\left(\int_{\mathbb{C}} \psi \, dn_f - \mathbb{E}\left[\int_{\mathbb{C}} \psi \, dn_f\right]\right)^2 = \frac{1}{4\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}} \Delta\psi(z)\Delta\psi(w) \log|\hat{f}(z)| \log|\hat{f}(w)| \, dm(z) dm(w).$$

Taking expectations, and applying Fubini's Theorem (assuming absolute convergence) we have

$$\mathbb{V}\left[\int_{\mathbb{C}}\psi\,dn_f\right]=\frac{1}{4\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\Delta\psi(z)\Delta\psi(w)\mathbb{E}\left[\log|\hat{f}(z)|\log|\hat{f}(w)|\right]\,dm(z)dm(w).$$

It remains only to compute the expectation. Note that $\hat{f}(z)$ is a $\mathcal{N}_{\mathbb{C}}(0,1)$ random variable for each $z\in\mathbb{C}$. Thus $\mathbb{E}[\hat{f}(z)]$ is independent of z and so

$$\int_{\mathbb{C}}\Delta\psi(z)\mathbb{E}[\hat{f}(z)]\,dm(z)=\int_{\mathbb{C}}\psi(z)\Delta\mathbb{E}[\hat{f}(z)]\,dm(z)=0.$$

Thus

$$\mathbb{V}\left[\int_{\mathbb{C}}\psi\,dn_f\right]=\frac{1}{4\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\Delta\psi(z)\Delta\psi(w)\,\mathrm{Cov}\left[\log|\hat{f}(z)|,\log|\hat{f}(w)|\right]\,dm(z)dm(w)$$

where Cov indicates covariance. We may therefore apply the following lemma.

Lemma 4.9 ([SZ08, Lemma 3.3; NS11, Lemma 2.2; HKPV09, Lemma 3.5.2])**.** *If ζ_1 and ζ_2 are $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables with $\mathbb{E}[\zeta_1\bar{\zeta_2}]=\theta$ then*

$$\mathrm{Cov}[\log|\zeta_1|,\log|\zeta_1|]=\frac{1}{4}\,\mathrm{Li}_2(|\theta|^2)$$

Noting that

$$\mathbb{E}\left[\hat{f}(z)\overline{\hat{f}(w)}\right]=\mathbb{E}\left[\frac{f(z)}{K(z,z)^{1/2}}\frac{\overline{f(w)}}{K(w,w)^{1/2}}\right]=\frac{K(z,w)}{K(z,z)^{1/2}K(w,w)^{1/2}}$$

and applying the lemma we get

$$\mathbb{V}\left[\int_{\mathbb{C}}\psi\,dn_f\right]=\frac{1}{16\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}}\Delta\psi(z)\Delta\psi(w)\,\mathrm{Li}_2\left(\frac{|K(z,w)|^2}{K(z,z)K(w,w)}\right)\,dm(z)dm(w).$$

Fix $z\in\operatorname{supp}\psi$, choose $\alpha>2/\gamma$ where γ is the constant appearing in (4.6), and let ϵ be the constant from Proposition 2.12. Write

$$I_1=\int_{d_L(z,w)\geq(\alpha\log L)^{1/\epsilon}}\Delta\psi(w)J_L(z,w)dm(w),$$

$$I_2=\int_{d_L(z,w)<(\alpha\log L)^{1/\epsilon}}\left(\Delta\psi(w)-\Delta\psi(z)\right)J_L(z,w)dm(w),$$

$$I_3=\int_{d_L(z,w)<(\alpha\log L)^{1/\epsilon}}J_L(z,w)dm(w).$$

and note that

$$\int_{\mathbb{C}}\Delta\psi(w)J_L(z,w)dm(w)=I_1+I_2+\Delta\psi(z)I_3.$$

Now, by Proposition 2.12, $J_L(z,w) \lesssim e^{-d_L^\epsilon(z,w)} \le L^{-\alpha}$ when $d_L(z,w) \ge (\alpha \log L)^{1/\epsilon}$ and so

$$|I_1| \lesssim L^{-\alpha} \int_{d_L(z,w)\ge(\alpha\log L)^{1/\epsilon}} |\Delta\psi(w)| dm(w) \le L^{-\alpha} \|\Delta\psi\|_{L^1(\mathbb{C})}.$$

Also, since $d_L(z,w) \gtrsim L^\gamma d_\mu(z,w)$, we see that if z and w satisfy $d_L(z,w) < (\alpha \log L)^{1/\epsilon}$ then

$$\Delta\psi(w) - \Delta\psi(z) \to 0 \text{ as } L \to \infty,$$

and so

$$|I_2| \le \sup_{d_L(z,w)<(\alpha\log L)^{1/\epsilon}} |\Delta\psi(w) - \Delta\psi(z)| I_3 = o(I_3).$$

Finally, using Proposition 2.12 and Lemma 2.6, we see that

$$I_3 \lesssim \int_{\mathbb{C}} e^{-d_L^\epsilon(z,w)} dm(w) \lesssim \int_{\mathbb{C}} e^{-c\left(\frac{|z-w|}{\rho_L(z)}\right)^{\epsilon'}} dm(w) = \rho_L(z)^2 \int_{\mathbb{C}} e^{-c'|\zeta|^{\epsilon'}} dm(\zeta).$$

Similarly, for r sufficiently small,

$$I_3 \gtrsim \int_{D_L^r(z)} dm(w) = \pi r^2 \rho_L(z)^2,$$

that is, $I_3 \simeq \rho_L(z)^2$. We thus conclude that (note that $\rho_L(z)^2 \gtrsim L^{-2/\gamma}\rho(z)^2$ and $\alpha > 2/\gamma$)

$$\mathbb{V}[n(\psi, L)] = \frac{1}{L^2} \int_{\mathbb{C}} \Delta\psi(z) \left(I_1 + I_2 + \Delta\psi(z) I_3\right) dm(z) \simeq \frac{1}{L^2} \int_{\mathbb{C}} \Delta\psi(z)^2 \rho_L(z)^2 dm(z)$$

which completes the proof. ∎

We will now use the results we have just proved for the mean and the variance of the 'smooth linear statistics' to prove Theorem 4.1 (b).

Proof of Theorem 4.1 (b). First note that

$$\mathbb{E}\left[\left(n(\psi, L) - \frac{1}{2\pi}\int \psi d\mu\right)^2\right] \lesssim \mathbb{E}\left[\Big(n(\psi, L) - \mathbb{E}[n(\psi, L)]\Big)^2\right] + \left(\mathbb{E}[n(\psi, L)] - \frac{1}{2\pi}\int \psi d\mu\right)^2.$$

Now Theorem 4.2 implies that

$$\begin{aligned}\mathbb{E}\Big[\big(n(\psi, L) - \mathbb{E}[n(\psi, L)]\big)^2\Big] &= \mathbb{V}[n(\psi, L)] \\ &\simeq L^{-2} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho_L(z)^2 dm(z) \\ &\lesssim L^{-2(1+\gamma)} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho(z)^2 dm(z)\end{aligned}$$

while (a) implies that

$$\left|\mathbb{E}[n(\psi, L)] - \frac{1}{2\pi}\int \psi d\mu\right| = O(L^{-1}).$$

We thus infer that

$$\mathbb{E}\left[\left(n(\psi, L) - \frac{1}{2\pi}\int \psi d\mu\right)^2\right] \lesssim L^{-2}$$

which implies that

$$\mathbb{E}\left[\sum_{L=1}^{\infty}\left(n(\psi, L) - \frac{1}{2\pi}\int \psi\, d\mu\right)^2\right] = \sum_{L=1}^{\infty}\mathbb{E}\left[\left(n(\psi, L) - \frac{1}{2\pi}\int \psi\, d\mu\right)^2\right] < +\infty.$$

This means that

$$n(\psi, L) - \frac{1}{2\pi}\int \psi\, d\mu \to 0$$

almost surely, as claimed. ∎

4.3 Asymptotic Normality

This section consists of the proof of Theorem 4.5. As we have previously noted, we shall consider only the GAF defined via frames (4.4). All of the results stated here apply equally well to the GAF defined via bases (4.3), and the proofs are identical except that the estimates from Proposition 2.15 should be replaced by the estimates from Proposition 2.12. Our proof of Theorem 4.5 is based entirely on the following result which was used to prove asymptotic normality in the case $\phi(z) = |z|^2/2$ ([ST04, Main Theorem]).

Theorem 4.10 ([ST04, Theorem 2.2]). *Suppose that for each natural number m, f_m is a Gaussian process with covariance kernel Ξ_m satisfying $\Xi_m(z, z) = 1$ and let n_m be the counting measure on the set of zeroes of f_m. Let ν be a measure on $\mathbb{C}$ satisfying $\nu(\mathbb{C}) = 1$ and suppose that $\Theta : \mathbb{C} \to \mathbb{R}$ is a bounded measurable function. Define $Z_m = \int_{\mathbb{C}} \log(|f_m(z)|)\Theta(z)d\nu(z)$ and suppose that*

$$\liminf_{m\to\infty} \frac{\iint_{\mathbb{C}^2} |\Xi_m(z, w)|^2\Theta(z)\Theta(w)\, d\nu(z)d\nu(w)}{\sup_{z\in\mathbb{C}} \int_{\mathbb{C}}|\Xi_m(z, w)|^2\, d\nu(z)} > 0, \tag{4.9}$$

and that

$$\lim_{m\to\infty} \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\Xi_m(z, w)|\, d\nu(w) = 0. \tag{4.10}$$

Then the distributions of the random variables

$$\frac{Z_m - \mathbb{E}[Z_m]}{\sqrt{\mathbb{V}[Z_m]}}$$

converge weakly to $\mathcal{N}(0, 1)$ as $m \to \infty$.

Remark. In fact in [ST04] the authors prove a more general result, but we shall only require the form we have stated. We have also slightly modified the denominator in condition (4.9), but it is easy to verify that this does not affect the proof (cf. [ST04, Section 2.5]).

Proof of Theorem 4.5. We consider instead the random variable $n(\psi, L) = \int \psi dn_L$ since it is clear that the factor L^{-1} is unimportant. We first note that Green's formula implies that

$$n(\psi, L) = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi(z) \log |f_L(z)| \, dm(z)$$

which combined with the Edelman-Kostlan formula gives

$$Z_L(\psi) = n(\psi, L) - \mathbb{E}[n(\psi, L)] = \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi(z) \log \frac{|f_L(z)|}{K_L(z,z)^{1/2}} \, dm(z).$$

Write $\hat{f}_L(z) = \frac{f_L(z)}{K_L(z,z)^{1/2}}$, $\Theta(z) = \frac{c}{2\pi}\Delta\psi(z)\rho(z)^2$ and $d\nu(z) = \frac{1}{c}\chi_{\operatorname{supp}\psi}(z)\frac{dm(z)}{\rho(z)^2}$ where the constant c is chosen so that $\nu(\mathbb{C}) = 1$. Note that

$$Z_L(\psi) = \int_{\mathbb{C}} \log |\hat{f}_L(z)| \Theta(z) \, d\nu(z)$$

so we need only check that conditions (4.9) and (4.10) hold to show asymptotic normality. Here $\Xi_L(z, w) = \frac{K_L(z,w)}{K_L(z,z)^{1/2}K_L(w,w)^{1/2}}$. Now, by the estimates of Proposition 2.15 and (4.7),

$$\begin{aligned}
\int_{\mathbb{C}} |\Xi_L(z, w)| \, d\nu(w) &\simeq e^{-\phi_L(z)} \int_{\mathbb{C}} |K_L(z, w)| e^{-\phi_L(w)} \, d\nu(w) \\
&\leq e^{-\phi_L(z)} \left(\int_{\mathbb{C}} |K_L(z, w)|^2 e^{-2\phi_L(w)} \frac{dm(w)}{\rho(w)^2} \right)^{1/2} \nu(\mathbb{C})^{1/2} \\
&\overset{(*)}{\simeq} L^{-\frac{1}{2}} e^{-\phi_L(z)} \left(\int_{\mathbb{C}} |K_L(z, w)|^2 e^{-2\phi_L(w)} \frac{dm(w)}{\rho_L(w)^2} \right)^{1/2} \simeq L^{-\frac{1}{2}},
\end{aligned}$$

where we have used local flatness (4.7) for the estimate $(*)$, so (4.10) holds. (In fact to prove (4.10) it suffices to use the estimate (4.6).) Similarly

$$\int_{\mathbb{C}} |\Xi_L(z, w)|^2 \, d\nu(w) \simeq L^{-1}.$$

By a computation almost identical to that in the proof of Theorem 4.2 we also have

$$\begin{aligned}
\iint_{\mathbb{C}^2} |\Xi_L(z, w)|^2 \Theta(z)\Theta(w) \, d\nu(z) d\nu(w) &\simeq \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho_L(z)^2 \, dm(z) \\
&\simeq L^{-1} \int_{\mathbb{C}} (\Delta\psi(z))^2 \rho(z)^2 \, dm(z)
\end{aligned}$$

which verifies (4.9). (In both of these estimates we use (4.7) since the estimate (4.6) is not enough, it is here that our local flatness assumption is important.) ■

4.4 Large deviations

In this section we prove Theorem 4.3 and Corollary 4.4. We borrow many of the ideas used here from [ST05] and [SZZ08], but some modifications are necessary to deal with the fact that ϕ is non-radial and that we are in a non-compact setting. The key ingredient in the proof of Theorem 4.3 is the following lemma.

Lemma 4.11. *For any disc $D = D^r(z_0)$ and any $\delta > 0$ there exists $c > 0$ depending only on δ, D and μ such that, for sufficiently large L,*

$$\int_D |\log|f_L(z)| - \phi_L(z)|\, dm(z) \leq \delta L$$

outside an exceptional set of probability at most e^{-cL^2}.

We begin with the following lemma.

Lemma 4.12. *Given a disc $D = D^r(z_0)$ and $\delta > 0$ there exists $c > 0$ depending only on the doubling constant such that, for sufficiently large L,*

$$\left|\max_{z\in\overline{D}} \big(\log|f_L(z)| - \phi_L(z)\big)\right| \leq \delta L$$

outside an exceptional set of probability at most $e^{-c\delta\mu(D)L^2}$.

Proof. Define $\hat{f}_L(z) = \frac{f_L(z)}{K_L(z,z)^{1/2}}$. We will show that

$$\mathbb{P}\left[\left|\max_{z\in\overline{D}} \log|\hat{f}_L(z)|\right| \geq \delta L\right] \leq e^{-c\delta\mu(D)L^2}$$

for L sufficiently large, which will imply the claimed result by Proposition 2.15 (b). We divide the proof in two parts.

1. We first show that

$$\mathbb{P}\left[\max_{z\in\overline{D}} |\hat{f}_L(z)| \leq e^{-\delta L}\right] \leq e^{-c\delta\mu(D)L^2}.$$

For each L define S_L to be a d_L-separated sequence with the constant

$$R = \inf\{d_L(s,t)\colon\ s \neq t \text{ and } s,t \in S_L\}$$

to be chosen (large but uniform in L). Moreover we assume that

$$\sup_{z\in\mathbb{C}} d_L(z, S_L) < \infty,$$

uniformly in L once more. Trivially

$$\mathbb{P}\left[\max_{z\in\overline{D}} |\hat{f}_L(z)| \leq e^{-\delta L}\right] \leq \mathbb{P}\left[|\hat{f}_L(s)| \leq e^{-\delta L} \text{ for all } s \in \overline{D} \cap S_L\right]$$

and we now estimate the probability of this event. We write

$$\overline{D} \cap S_L = \{s_1, \ldots, s_N\}.$$

Note that for R_1 sufficiently small

$$L\mu(D^{2r}(z_0)) \leq \sum_{j=1}^{N} L\mu(D_L^{R_1}(s_j)) \lesssim N$$

while for R_2 large enough

$$L\mu(D^{r}(z_0)) \geq \sum_{j=1}^{N} L\mu(D_L^{R_2}(s_j)) \gtrsim N$$

so that $N \simeq L\mu(D)$. Consider the vector

$$\xi = \begin{pmatrix} \hat{f}_L(s_1) \\ \vdots \\ \hat{f}_L(s_N) \end{pmatrix}$$

which is a mean-zero N-dimensional complex normal with covariance matrix σ given by

$$\sigma_{mn} = \frac{K_L(s_m, s_n)}{K_L(s_m, s_m)^{1/2} K_L(s_n, s_n)^{1/2}}.$$

Note that $\sigma_{nn} = 1$ and $|\sigma_{mn}| \lesssim e^{-d_L^{\epsilon}(s_n, s_m)}$ so that if the sequence S_L is chosen to be sufficiently separated then the components of the vector ξ will be 'almost independent'. We write $\sigma = I + A$ and note that

$$\begin{aligned} \max_n \left| \sum_{m \neq n} \sigma_{mn} \right| \lesssim \max_n \sum_{m \neq n} e^{-d_L^{\epsilon}(s_n, s_m)} &\lesssim \max_n \int_{\mathbb{C} \setminus B_L(s_n, R)} e^{-d_L^{\epsilon}(s_n, w)} \frac{dm(w)}{\rho_L(w)^2} \\ &\lesssim \int_{R^{\epsilon'}}^{\infty} x^{\alpha} e^{-x} dx \end{aligned}$$

for some $\alpha, \epsilon' > 0$ by an argument identical to that given in the proof of Lemma 2.8. Thus by choosing R sufficiently large we have $\|A\|_{\infty} \leq \frac{1}{2}$ and so for any $v \in \mathbb{C}^N$

$$\|\sigma v\|_{\infty} \geq \frac{1}{2} \|v\|_{\infty}.$$

Thus the eigenvalues of σ are bounded below by $\frac{1}{2}$ and so for any B with $BB^* = \sigma$ we have

$$\|B^{-1}\|_2 \leq \sqrt{2}.$$

Now the components of the vector $\zeta = B^{-1}\xi$ are iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables, which we denote ζ_j, and moreover
$$\|\zeta\|_\infty \le \|B^{-1}\xi\|_2 \le \sqrt{2}\|\xi\|_2 \le \sqrt{2N}\|\xi\|_\infty.$$
This means that
$$\begin{aligned}\mathbb{P}\left[|\hat{f}_L(s)| \le e^{-\delta L} \text{ for all } s \in \overline{D} \cap S_L\right] &\le \mathbb{P}\left[|\zeta_j| \le \sqrt{2N}e^{-\delta L} \text{ for all } 1 \le j \le N\right] \\ &= \left(1 - \exp(-2Ne^{-2\delta L})\right)^N \le e^{-c\delta\mu(D)L^2}\end{aligned}$$
for L sufficiently large, where c depends only on the doubling constant (and $\sup_{z\in\mathbb{C}} d_L(z, S_L)$), as claimed.

Here we have used the fact that if ζ is a $\mathcal{N}_{\mathbb{C}}(0,1)$ random variable then
$$\mathbb{P}[|\zeta| \le \lambda] = \frac{1}{\pi}\int_{|z|\le\lambda} e^{-|z|^2}\, dm(z) = 1 - e^{-\lambda^2}$$
for $\lambda \ge 0$. We shall use this fact repeatedly throughout the rest of this monograph.

2. For the second part of the proof we must estimate
$$\mathbb{P}\left[\max_{z\in\overline{D}} |\hat{f}_L(z)| \ge e^{\delta L}\right]$$
and so we define the event
$$\mathcal{E} = \{\max_{z\in\overline{D}} |\hat{f}_L(z)| \ge e^{\delta L}\}.$$
We write $\tilde{\Lambda}_L = \Lambda_L \cap D^{2r}(z_0)$ and $\tilde{f}_L = \sum_{\lambda\in\tilde{\Lambda}_L} a_\lambda k_\lambda(z)$, and note that $\#\tilde{\Lambda}_L \simeq L\mu(D)$ as in the first part of the proof. Consider the event
$$\mathcal{A} = \left\{|a_\lambda| \le L\frac{|\lambda - z_0|}{\rho_L(z_0)} \text{ for } \lambda \in \Lambda_L \setminus \tilde{\Lambda}_L\right\}.$$
If the event $\mathcal{A}$ occurs and $z \in \overline{D}$ then, since $d_L(\lambda, z) \ge C_r d_L(\lambda, z_0)$ for some $C_r > 0$, we have, by Proposition 2.15 (a),
$$\begin{aligned}\left|\frac{f_L(z) - \tilde{f}_L(z)}{K_L(z,z)^{1/2}}\right| &\lesssim \sum_{\lambda\in\Lambda_L\setminus\tilde{\Lambda}_L} |a_\lambda| e^{-d_L^\epsilon(\lambda,z)} \\ &\le \frac{L}{\rho_L(z_0)} \sum_{\lambda\in\Lambda_L\setminus\tilde{\Lambda}_L} |\lambda - z_0| e^{-C_r^\epsilon d_L^\epsilon(\lambda,z_0)} \\ &\lesssim \frac{L}{\rho_L(z_0)} \int_{\mathbb{C}\setminus D^r(z_0)} |\zeta - z_0| e^{-C_r^\epsilon d_L^\epsilon(\zeta,z_0)} \frac{dm(\zeta)}{\rho_L(\zeta)^2} \lesssim L\end{aligned}$$

where the final estimate comes from an argument similar to that used in the proof of Lemma 2.8 and the implicit constant depends on r. Hence the event $\mathcal{A} \cap \mathcal{E}$ implies that

$$\max_{z \in \overline{D}} \left| \frac{\tilde{f}_L(z)}{K_L(z,z)^{1/2}} \right| \geq e^{\delta L} - C_r' L \geq e^{\frac{\delta L}{2}}$$

for L sufficiently large, where C_r' is another positive constant. Now a simple application of the Cauchy-Schwartz inequality shows that

$$|\tilde{f}_L(z)| \leq \left(\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \right)^{1/2} \left(\sum_{\lambda \in \tilde{\Lambda}_L} |k_\lambda(z)|^2 \right)^{1/2} \leq \left(\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \right)^{1/2} K_L(z,z)^{1/2}$$

and so

$$\begin{aligned}
\mathbb{P}[\mathcal{A} \cap \mathcal{E}] &\leq \mathbb{P}\left[\max_{z \in \overline{D}} \left| \frac{\tilde{f}_L(z)}{K_L(z,z)^{1/2}} \right| \geq e^{\frac{\delta L}{2}} \right] \\
&\leq \mathbb{P}\left[\sum_{\lambda \in \tilde{\Lambda}_L} |a_\lambda|^2 \geq e^{\delta L} \right] \\
&\leq \mathbb{P}\left[|a_\lambda|^2 \geq \frac{e^{\delta L}}{\#\tilde{\Lambda}_L} \text{ for some } \lambda \in \tilde{\Lambda}_L \right] \leq \#\tilde{\Lambda}_L \left(\exp - \frac{e^{\delta L}}{\#\tilde{\Lambda}_L} \right) = e^{-e^{\delta L/2}}.
\end{aligned}$$

We finally estimate the probability of the event $\mathcal{A}$; using (4.6) and (2.1) we see that

$$\begin{aligned}
\log \mathbb{P}[\mathcal{A}] &= \log \prod_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} \left(1 - \exp\left(-L^2 \frac{|\lambda - z_0|^2}{\rho_L^2(z_0)} \right) \right) \\
&\simeq - \sum_{\lambda \in \Lambda_L \setminus \tilde{\Lambda}_L} \exp\left(-L^2 \frac{|\lambda - z_0|^2}{\rho_L^2(z_0)} \right) \\
&\gtrsim - \int_{\mathbb{C} \setminus D} \exp\left(-L^2 \frac{|\zeta - z_0|^2}{\rho_L^2(z_0)} \right) \frac{dm(\zeta)}{\rho_L(\zeta)^2} \\
&\gtrsim -L^{2/\gamma} \int_{\mathbb{C} \setminus D} \exp\left(-C L^{2+2/\gamma} \frac{|\zeta - z_0|^2}{\rho^2(z_0)} \right) \frac{dm(\zeta)}{\rho(\zeta)^2} \\
&\gtrsim -C_0 L^{2/\gamma} e^{-C_1 L^{2+2/\gamma}} \qquad (4.11)
\end{aligned}$$

where C_0 and C_1 depend on r and the doubling constant, and the final estimate uses an argument similar to that given in the proof of Lemma 2.8. We finally compute that

$$\mathbb{P}[\mathcal{E}] \leq \mathbb{P}[\mathcal{E} \cap \mathcal{A}] + \mathbb{P}[\mathcal{A}^c] \leq e^{-e^{\delta L/2}} + (1 - \exp\{-C_0 L^{2/\gamma} e^{-C_1 L^{2+2/\gamma}}\}) \leq e^{-cL^2}$$

for L sufficiently large and for any positive c. ∎

Lemma 4.13. *Given a disc $D = D^r(z_0)$ there exist $c, C > 0$ depending only on the doubling constant such that, for L sufficiently large,*

$$\int_D |\log|f_L(z)| - \phi_L(z)|\, dm(z) \le Cr^2\rho(z_0)^2\mu(D)L$$

outside of an exceptional set of probability at most $e^{-c\mu(D)^2L^2}$.

We will use the following result to prove this lemma.

Theorem 4.14 ([Pas88, Chapter 1, Lemma 7; AČ96, Theorem 1])**.** *If u is a subharmonic function on $\overline{\mathbb{D}}$ then, for all $\zeta \in \mathbb{D}$,*

$$u(\zeta) = \int_{\mathbb{D}} \widetilde{P}(\zeta, z)u(z)dm(z) - \int_{\mathbb{D}} \widetilde{G}(\zeta, z)\Delta u(z)$$

where

$$\widetilde{P}(\zeta, z) = \frac{1}{\pi}\frac{(1-|\zeta|^2)^2}{|1-\overline{z}\zeta|^4}$$

and

$$\widetilde{G}(\zeta, z) = \frac{1}{4\pi}\left(\log\left|\frac{1-\overline{\zeta}z}{\zeta - z}\right|^2 - \left(1 - \left|\frac{\zeta - z}{1-\overline{\zeta}z}\right|^2\right)\right)$$

Proof of Lemma 4.13. Applying Lemma 4.12 we see that outside of an exceptional set, we may find $\zeta \in D^{r/2}(z_0)$ such that

$$-L\mu(D) \le \log|f_L(\zeta)| - \phi_L(\zeta).$$

Making the appropriate change of variables in Theorem 4.14 and applying the resulting decomposition to the subharmonic functions $\log|f_L|$ and ϕ_L on D we see that

$$\begin{aligned}
\log|f_L(\zeta)| - \phi_L(\zeta) &= \int_D \widetilde{P}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)(\log|f_L(z)| - \phi_L(z))\frac{dm(z)}{r^2\rho(z_0)^2} \\
&\quad - \int_D \widetilde{G}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)(2\pi dn_L(z) - \Delta\phi_L(z)) \\
&\le \int_D \widetilde{P}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)(\log|f_L(z)| - \phi_L(z))\frac{dm(z)}{r^2\rho(z_0)^2} \\
&\quad + \int_D \widetilde{G}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)\Delta\phi_L(z)
\end{aligned}$$

since $\widetilde{G}$ is always positive. Now, since $\zeta \in D^{r/2}(z_0)$, we have by Lemma 2.5

$$\begin{aligned}\int_D \widetilde{G}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)\Delta\phi_L(z) &\le \int_{D^{r/2}(\zeta)} \widetilde{G}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)\Delta\phi_L(z) \\ &\quad + \int_{D\setminus D^{r/2}(\zeta)} \widetilde{G}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{z - z_0}{r\rho(z_0)}\right)\Delta\phi_L(z) \\ &\lesssim L\int_{D^{r/2}(\zeta)} \log\left(\frac{\frac{3}{2}r}{|\zeta - z|}\right) d\mu(z) + L\mu(D) \\ &\lesssim L\mu(D)\end{aligned}$$

and so

$$0 \le \int_D \widetilde{P}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{w - z_0}{r\rho(z_0)}\right)(\log|f_L(w)| - \phi_L(w))\frac{dm(w)}{r^2\rho(z_0)^2} + CL\mu(D)$$

for some positive C depending only on the doubling constant. Noting that $\widetilde{P}$ is also positive and satisfies

$$\widetilde{P}\left(\frac{\zeta - z_0}{r\rho(z_0)}, \frac{w - z_0}{r\rho(z_0)}\right) \simeq 1$$

for $w \in D$ and $\zeta \in D^{r/2}(z_0)$ we see that

$$\int_D \log^-(|f_L(w)|e^{-\phi_L(w)})\frac{dm(w)}{r^2\rho(z_0)^2} \lesssim \int_D \log^+(|f_L(w)|e^{-\phi_L(w)})\frac{dm(w)}{r^2\rho(z_0)^2} + L\mu(D)$$

and so

$$\int_D |\log|f_L(w)| - \phi_L(w)|\frac{dm(w)}{r^2\rho(z_0)^2} \lesssim \int_D \log^+(|f_L(w)|e^{-\phi_L(w)})\frac{dm(w)}{r^2\rho(z_0)^2} + L\mu(D).$$

Applying Lemma 4.12 once more we see that outside of another exceptional set

$$\int_D \log^+(|f_L(w)|e^{-\phi_L(w)})\frac{dm(w)}{r^2\rho(z_0)^2} \lesssim L\mu(D)$$

which completes the proof. ■

We are now ready to prove Lemma 4.11

Proof of Lemma 4.11. It suffices to consider only small values of δ. Given $\delta > 0$ we may cover D with discs $(D^{r_j}(z_j))_{j=1}^N$ such that $z_j \in D$ and $\mu(D^{r_j}(z_j)) = \delta$. The Vitali covering lemma implies that we may assume that $N \lesssim \mu(D)/\delta$. Now, applying Lemma 4.13 we see that outside of an exceptional set

$$\int_U \big|\log|f_L(z)| - \phi_L(z)\big|dm(z) \le \delta L\sum_{j=1}^N r_j^2\rho(z_j)^2.$$

We finally note that $\rho(z_j) \simeq \rho(z_0)$ and that Lemma 2.4 implies that

$$r_j \lesssim \delta^\gamma$$

for all j. Thus

$$\int_U |\log|f_L(z)| - \phi_L(z)|\, dm(z) \lesssim \delta L N \delta^{2\gamma} \lesssim L\delta^{2\gamma}.$$

Appropriately changing the value of δ completes the proof. ■

Proof of Theorem 4.3. We have already noted that the proof of (a) is identical to the proof of Theorem 4.1 (a). It remains to show the large deviations estimate (b). We first note that

$$\begin{aligned}\left|n(\psi, L) - \frac{1}{2\pi}\int \psi d\mu\right| &= \frac{1}{2\pi L}\left|\int_{\mathbb{C}} \Delta\psi(z)(\log|f_L(z)| - \phi_L(z))dm(z)\right| \\ &\leq \frac{1}{2\pi L}\max_{z\in\mathbb{C}}|\Delta\psi(z)|\int_{\operatorname{supp}\psi} |\log|f_L| - \phi_L| dm\end{aligned}$$

and so applying Lemma 4.11 with $\delta' = \delta|\int \psi d\mu|/\|\Delta\psi\|_\infty$ we see that

$$\left|n(\psi, L) - \frac{1}{2\pi}\int \psi d\mu\right| \leq \delta \left|\frac{1}{2\pi}\int \psi d\mu\right|$$

outside an exceptional set of probability at most e^{-cL^2}, as claimed. ■

Proof of Corollary 4.4. Let $\delta > 0$ and choose smooth, compactly supported ψ_1 and ψ_2 satisfying

$$0 \leq \psi_1 \leq \chi_U \leq \psi_2 \leq 1,$$

$$\int_{\mathbb{C}} \psi_1 d\mu \geq \mu(U)(1-\delta)$$

and

$$\int_{\mathbb{C}} \psi_2 d\mu \leq \mu(U)(1+\delta).$$

(a) Applying Theorem 4.3 (a) we see that, for L sufficiently large,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{L}n_L(U)\right] - \frac{1}{2\pi}\mu(U) &\leq \mathbb{E}\left[\frac{1}{L}\int \psi_2 dn_L\right] - \frac{1}{2\pi}\mu(U) \\ &\leq \frac{1}{2\pi}\int \psi d\mu + \frac{C}{L}\int_{\mathbb{C}} |\Delta\psi_2(z)| dm(z) - \frac{1}{2\pi}\mu(U) \\ &\leq \frac{\delta}{2\pi}\mu(U) + \frac{C}{L}\int_{\mathbb{C}} |\Delta\psi_2(z)| dm(z).\end{aligned}$$

Similarly, using ψ_1, we see that

$$\mathbb{E}\left[\frac{1}{L}n_L(U)\right] - \frac{1}{2\pi}\mu(U) \geq -\frac{\delta}{2\pi}\mu(U) - \frac{C}{L}\int_{\mathbb{C}} |\Delta\psi_2(z)| dm(z).$$

Choosing first δ small and then L large (depending on δ) completes the proof of (a).

(b) Outside an exceptional set of probability e^{-cL^2} we have, by Theorem 4.3 (b)

$$\frac{1}{L}n(\psi_2, L) \leq (1+\delta)\frac{1}{2\pi}\int_{\mathbb{C}}\psi_2 d\mu.$$

We see that

$$\frac{1}{L}n_L(U) \leq \frac{1}{L}n(\psi_2, L) \leq (1+\delta)\frac{1}{2\pi}\int_{\mathbb{C}}\psi_2 d\mu \leq (1+\delta)^2\frac{\mu(U)}{2\pi}$$

whence

$$\frac{\frac{1}{L}n_L(U)}{\frac{\mu(U)}{2\pi}} - 1 \lesssim \delta.$$

Similarly, using ψ_1, we have

$$\frac{\frac{1}{L}n_L(U)}{\frac{\mu(U)}{2\pi}} - 1 \gtrsim -\delta.$$

outside another exceptional set of probability e^{-cL^2}, which after appropriately changing the value of δ completes the proof. ■

4.5 Proof of Theorem 4.6

We will use some of the ideas from the proof of Theorem 4.4 here. We begin with a lemma that is very similar to Lemma 4.12. It is clear that if we could prove an exact analogue of Lemma 4.12 then we could prove a large deviations theorem, since it is only in the proof of this lemma that we use the decay estimates for the frame elements. Unfortunately we are unable to prove such a result, but the following result will be enough to prove a hole theorem. Recall that we write $D = D^r(z_0)$.

Lemma 4.15. *Given $z_0 \in \mathbb{C}$ and $\delta, r > 0$ there exists $c > 0$ depending only on the doubling constant such that*

$$\max_{z\in\overline{D}}\big(\log|g_L(z)| - \phi_L(z)\big) \geq -\delta L$$

outside an exceptional set of probability at most $e^{-c\delta\mu(D)L^2}$, for L sufficiently large. Moreover there exists $C' > 0$ depending on ϕ and D such that for all $C > C'$ and sufficiently large L

$$\max_{z\in\overline{D}}\big(\log|g_L(z)| - \phi_L(z)\big) \leq CL$$

outside an exceptional set of probability at most e^{-cL^2}.

Proof. The proof that

$$\mathbb{P}\left[\max_{z\in\overline{D}}\big(\log|g_L(z)| - \phi_L(z)\big) \leq -\delta L\right] \leq e^{-c\delta\mu(D)L^2}$$

is identical to the proof of the first part of Lemma 4.12, we omit the details.

To prove the second estimate we use the following result, which is simply [HKPV09, Lemma 2.4.4] translated and re-scaled.

Lemma 4.16. *Let f be a Gaussian analytic function in a neighbourhood of the disc $D(z_0, R)$ with covariance kernel K. Then for $r < R/2$ we have*

$$\mathbb{P}\left[\max_{z\in\overline{D(z_0,r)}} |f(z)| > t\right] \le 2e^{-t^2/8\sigma_{2r}^2}$$

where $\sigma_{2r}^2 = \max\{K(z,z) : z \in \overline{D(z_0, 2r)}\}$.

Let $C_1 = \min\{\phi(z) : z \in \overline{D}\}$ and $C_2 = \max\{\phi(z) : z \in \overline{D^{2r}(z_0)}\}$. Note that

$$\max\{\mathcal{K}_L(z,z) : z \in D(z_0, 2r)\} \lesssim e^{2C_2 L}.$$

Hence

$$\begin{aligned}\mathbb{P}\left[\max_{z\in\overline{D}} \big(\log|g_L(z)| - \phi_L(z)\big) \ge CL\right] &\le \mathbb{P}\left[\max_{z\in\overline{D}} |g_L(z)| \gtrsim e^{(C+C_1)L}\right] \\ &\le 2\exp\{-c' e^{2(C+C_1-C_2)L}\} \le e^{-cL^2}\end{aligned}$$

for any $c > 0$ if $C + C_1 - C_2 > 0$. ■

We may now immediately infer the following lemma. All integrals over circles are understood to be with respect to normalised Lebesgue measure on the circle.

Lemma 4.17. *For any $z_0 \in \mathbb{C}$ and any $\delta, r > 0$ there exists $c > 0$ depending only on δ, $\mu(D)$ and the doubling constant such that, for sufficiently large L,*

$$\int_{\partial D} \big(\log|g_L(z)| - \phi_L(z)\big) \ge -\delta L$$

outside an exceptional set of probability at most e^{-cL^2}.

Proof. It suffices to show this for small δ. Put $\kappa = 1 - \delta^{1/4}$, $N = [2\pi\delta^{-1}]$ and define $z_j = z_0 + \kappa r\rho(z_0)\exp(\frac{2\pi i j}{N})$ and $D_j = D(z_j, \delta r\rho(z_0))$ for $j = 1, \ldots, N$. Lemma 4.15 implies that outside an exceptional set of probability at most $Ne^{-c\delta\mu(D_j)L^2} \le e^{-c'L^2}$ (where c' depends on δ, $\mu(D)$ and the doubling constant) there exist $\zeta_j \in \overline{D_j}$ such that

$$\log|g_L(\zeta_j)| - \phi_L(\zeta_j) \ge -\delta L.$$

Let $P(\zeta, z)$ and $G(\zeta, z)$ be, respectively, the Poisson kernel and the Green function for D where we use the convention that the Green function is positive. Applying the Riesz decomposition to the subharmonic functions $\log|g_L|$ and ϕ_L on the disc D implies that

$$\begin{aligned}
-\delta L &\le \frac{1}{N}\sum_{j=1}^{N}\big(\log|g_L(\zeta_j)| - \phi_L(\zeta_j)\big) \\
&= \int_{\partial D}\big(\log|g_L(z)| - \phi_L(z)\big) + \int_{\partial D}\Big(\frac{1}{N}\sum_{j=1}^{N}P(\zeta_j,z)-1\Big)\big(\log|g_L(z)| - \phi_L(z)\big) \\
&\quad - \int_D \frac{1}{N}\sum_{j=1}^{N}G(\zeta_j,z)\big(2\pi\, dn_L(z) - \Delta\phi_L(z)\big) \\
&\le \int_{\partial D}\big(\log|g_L(z)| - \phi_L(z)\big) + \int_{\partial D}\Big(\frac{1}{N}\sum_{j=1}^{N}P(\zeta_j,z)-1\Big)\big(\log|g_L(z)| - \phi_L(z)\big) \\
&\quad + \int_D \frac{1}{N}\sum_{j=1}^{N}G(\zeta_j,z)\,\Delta\phi_L(z).
\end{aligned}$$

Claim 4.18. *There exists $\widetilde{C} > 0$ such that*

$$\int_{\partial D}\big|\log|g_L(z)| - \phi_L(z)\big| \le \widetilde{C}\mu(D)L$$

outside of an exceptional set of probability at most e^{-cL^2}.

Claim 4.19 ([ST05, Claim 2])**.** *There exists $C_0 > 0$ such that*

$$\max_{z\in\partial D}\left|\frac{1}{N}\sum_{j=1}^{N}P(\zeta_j,z)-1\right| \le C_0\delta^{1/2}$$

Claim 4.20. *There exists $C_1 > 0$ and $0 < \alpha < 1/4$ depending only on the doubling constant and $\mu(D)$ such that*

$$\int_D G(\zeta_j,z)\,\Delta\phi_L(z) \le C_1\delta^\alpha L$$

for δ sufficiently small.

Applying Claims 4.18 and 4.19 we see that outside another exceptional set we have

$$\left|\int_{\partial D}\Big(\frac{1}{N}\sum_{j=1}^{N}P(\zeta_j,z)-1\Big)\big(\log|g_L(z)| - \phi_L(z)\big)\right| \lesssim \delta^{1/2}L$$

while Claim 4.20 implies that

$$\int_D \frac{1}{N}\sum_{j=1}^{N}G(\zeta_j,z)\,\Delta\phi_L(z) \le C_1\delta^\alpha L.$$

Hence
$$\int_{\partial D} (\log |g_L(z)| - \phi_L(z)) \geq -(\delta + C_0\delta^{3/2} + C_1\delta^{\alpha})L \gtrsim -\delta^{\alpha} L$$
outside an exceptional set, and so the lemma follows. ∎

Proof of Claim 4.18. We use the same notation. Lemma 4.15 implies that outside an exceptional set of probability at most e^{-cL^2} there exists $\zeta_0 \in \overline{D^{r/2}(z_0)}$ such that
$$\log |g_L(\zeta_j)| - \phi_L(\zeta_j) \geq -\mu(D)L.$$
Another application of the Riesz decomposition to the subharmonic functions $\log |g_L|$ and ϕ_L on the disc D implies that
$$\begin{aligned}
-\mu(D)L &\leq \log |g_L(\zeta_0)| - \phi_L(\zeta_0) \\
&= \int_{\partial D} P(\zeta_0, z)\big(\log |g_L(z)| - \phi_L(z)\big) - \int_D G(\zeta_0, z)\big(2\pi\, dn_L(z) - \Delta\phi_L(z)\big) \\
&\leq \int_{\partial D} P(\zeta_0, z)\big(\log |g_L(z)| - \phi_L(z)\big) + L\int_D G(\zeta_0, z)\,\Delta\phi(z).
\end{aligned}$$
Now since $\zeta_0 \in \overline{D^{r/2}(z_0)}$, we have, by Lemma 2.5,
$$\begin{aligned}
\int_D G(\zeta_0, z)\,\Delta\phi(z) &\leq \int_{D^{r/2}(\zeta_0)} G(\zeta_0, z)\,\Delta\phi(z) + \int_{D\setminus D^{r/2}(\zeta_0)} G(\zeta_0, z)\,\Delta\phi(z) \\
&\lesssim \int_{D^{r/2}(\zeta_0)} \log\left(\frac{\frac{3}{2}r}{|\zeta_- z|}\right) d\mu(z) + \mu(D) \\
&\lesssim \mu(D)
\end{aligned}$$
and so
$$0 \leq \int_{\partial D} P(\zeta_0, z)\big(\log |g_L(z)| - \phi_L(z)\big) + CL\mu(D)$$
for some positive C depending only on the doubling constant. Thus
$$\int_{\partial D} P(\zeta_0, z)\log^-(|g_L(z)|e^{-\phi_L(z)}) \leq \int_{\partial D} P(\zeta_0, z)\log^+(|g_L(z)|e^{-\phi_L(z)}) + CL\mu(D).$$
We note that for $z \in \partial D$ and $\zeta_0 \in \overline{D^{r/2}(z_0)}$ we have
$$\frac{1}{3} \leq P(\zeta_0, z) \leq 3$$
and so
$$\int_{\partial D} |\log |g_L(z)| - \phi_L(z)| \lesssim \int_{\partial D} \log^+(|g_L(z)|e^{-\phi_L(z)}) + L\mu(D).$$
Applying Lemma 4.12 once more we see that outside of another exceptional set
$$\int_{\partial D} \log^+(|g_L(z)|e^{-\phi_L(z)}) \lesssim L\mu(D),$$
which completes the proof. ∎

Proof of Claim 4.20. To simplify the notation we move to the unit disc. We write

$$\varphi(w) = \phi(z_0 + r\rho(z_0)w)$$

for $w \in \mathbb{D}$ and $w_j = (\zeta_j - z_0)/r\rho(z_0)$ and note that $1 - |w_j| \lesssim \delta^{1/4}$. We see that

$$\int_D G(\zeta_j, z)\Delta\phi_L(z) = \frac{L}{2\pi}\int_{\mathbb{D}} \log\left|\frac{1-\overline{w_j}w}{w-w_j}\right|\Delta\varphi(w)$$

and we write

$$B_j(r) = \left\{w \in \mathbb{D} : \left|\frac{w-w_j}{1-\overline{w_j}w}\right| \le r\right\} = D\Big(\frac{1-r^2}{1-r^2|w_j|^2}w_j, \frac{1-|w_j|^2}{1-r^2|w_j|^2}r\Big)$$

for the hyperbolic discs of centre w_j and radius r. Fix some $\beta < 1/4$ and note that

$$\int_{\mathbb{D}\setminus B_j(1-\delta^\beta)} \log\left|\frac{1-\overline{w_j}w}{w-w_j}\right|\Delta\varphi(w) \le -\log(1-\delta^\beta)\Delta\varphi(\mathbb{D}) \le 2\delta^\beta\mu(D).$$

Also, using the distribution function, we see that

$$\begin{aligned}\int_{B_j(1-\delta^\beta)} \log\left|\frac{1-\overline{w_j}w}{w-w_j}\right|\Delta\varphi(w) &= \int_0^\infty \Delta\varphi(B_j(1-\delta^\beta)\cap B_j(e^{-x}))dx \\ &= \int_0^{-\log(1-\delta^\beta)} \Delta\varphi(B_j(1-\delta^\beta))dx \\ &\quad + \int_{-\log(1-\delta^\beta)}^\infty \Delta\varphi(B_j(e^{-x}))dx \\ &\le 2\delta^\beta\mu(D) + \int_{-\log(1-\delta^\beta)}^\infty \Delta\varphi(B_j(e^{-x}))dx.\end{aligned}$$

Now the Euclidean radius of the disc $B_j(e^{-x})$ is given by

$$\frac{1-|w_j|^2}{1-e^{-2x}|w_j|^2}e^{-x} \lesssim \frac{1-|w_j|}{1-e^{-x}|w_j|} \lesssim \frac{\delta^{1/4}}{\delta^\beta}$$

which gets arbitrarily small, while $\rho_{\Delta\varphi}(w) \simeq \rho_{\Delta\varphi}(0)$ for all $w \in \mathbb{D}$. Applying Lemma 2.4 to the doubling measure $\Delta\varphi$ we see that there exists $0 < \gamma < 1$ such that

$$\begin{aligned}\int_{-\log(1-\delta^\beta)}^\infty \Delta\varphi(B_j(e^{-x}))dx &\lesssim \int_{\delta^\beta}^\infty \left(\frac{1-|w_j|^2}{1-e^{-2x}|w_j|^2}e^{-x}\right)^\gamma dx \\ &\lesssim (1-|w_j|)^\gamma \int_{\delta^\beta}^\infty \frac{e^{-\gamma x}}{(1-e^{-x}|w_j|)^\gamma}dx \\ &\le (1-|w_j|)^\gamma |w_j|^{-\gamma}\int_0^1 \frac{(1-u)^{\gamma-1}}{u^\gamma}du \le C_\gamma\delta^{\gamma/4}.\end{aligned}$$

where we have made the change of variables $u = 1 - e^{-x}|w_j|$. We therefore have

$$\int_D G(\zeta_j, z)\Delta\phi_L(z) \lesssim (\delta^{\gamma/4} + \delta^{\beta})L$$

and the claim follows by choosing $\alpha = \min\{\gamma/4, \beta\}$. ■

We are now ready to prove Theorem 4.6. Since we do not have any control on the dependence of the constants on the bounded set U, we assume that U is the disc D.

Proof of Theorem 4.6. Suppose that g_L has no zeroes in D. Recall that we use $G(\zeta, z)$ to denote the Green function for D. Applying Jensen's formula to g_L and the Riesz decomposition to the subharmonic function ϕ_L on the disc D we see that

$$\log|g_L(z_0)| - \phi_L(z_0) = \int_{\partial D} (\log|g_L(z)| - \phi_L(z)) + L\int_D G(z_0, z)\Delta\phi(z)dm(z).$$

Choosing $\delta = \int_D G(z_0, z)\Delta\phi(z)/2$ in Lemma 4.17 shows that outside an exceptional set of probability at most e^{-cL^2} we have

$$\log|g_L(z_0)| - \phi_L(z_0) \geq \delta L.$$

Now Proposition 2.12 shows that

$$\mathbb{P}[\log|g_L(z_0)| - \phi_L(z_0) \geq \delta L] \leq \mathbb{P}\Big[\frac{|g_L(\zeta_0)|}{\mathcal{K}_L(z_0, z_0)^{1/2}} \gtrsim e^{\delta L}\Big] \leq \exp\{-Ce^{2\delta L}\} \leq e^{-cL^2}$$

and so

$$\mathbb{P}[n_L(D) = 0] \leq e^{-cL^2},$$

which completes the proof. ■

4.6 Proof of Theorem 4.7

We have previously remarked that the upper bound in Theorem 4.7 is a simple consequence of Theorem 4.4, we now prove the lower bounds. We will do this by first finding a deterministic function h_L that does not vanish in the hole and then constructing an event that ensures the GAF f_L is 'close' to h_L. Since we can always find a disc $D = D^r(z_0)$ contained in U, and we do not have any control on the dependence of the constants on U, we will prove the theorem only in the case $U = D$. We begin by constructing the function h_L.

Lemma 4.21. *There exists an entire function h_L such that*

- $\|h_L\|_{\mathcal{F}^2_L} = 1$, *and*

- *there exists $C_0 > 0$ depending on $\mu(D)$ and the doubling constant such that*

$$|h_L(z)|e^{-\phi_L(z)} \geq e^{-C_0 L}$$

for all $z \in D$.

Remark. In the case $\phi(z) = |z|^2/2$ we may take h_L to be constant. More generally, if

$$\int_{\mathbb{C}} e^{-2\phi_L} \frac{dm}{\rho_L^2} \leq C^L$$

then we can take $h_L = C^{-L}$. In general, however, it may not even be the case that

$$\int_{\mathbb{C}} e^{-2\phi} \frac{dm}{\rho^2}$$

is finite (consider $\phi(z) = (\operatorname{Re} z)^2$).

Proof. Let $\mathcal{K}_\delta(z, w)$ be the reproducing kernel for the space $\mathcal{F}^2_{\delta\phi}$ and consider the normalised reproducing kernel

$$k_\delta(z) = \frac{\mathcal{K}_\delta(z, z_0)}{\mathcal{K}_\delta(z_0, z_0)^{1/2}}.$$

Now since $\rho_{\delta\mu}(z_0) \to \infty$ as $\delta \to 0$, Proposition 2.12 shows that there exists δ_0 and $C > 0$ (depending only on r and the doubling constant) such that

$$|k_\delta(z)|e^{-\delta\phi(z)} \geq C \tag{4.12}$$

for all $z \in D$ and all $\delta < \delta_0$. Given any L sufficiently large we can find $\delta \in [\delta_0/2, \delta_0]$ and an integer N such that $L = N\delta$. We note that $\rho_{\delta\mu}(z) \simeq \rho_\mu(z)$ for all δ in this range (where the implicit constants depend on δ_0) and so applying Proposition 2.12 and (4.6) gives

$$\begin{aligned}
\int_{\mathbb{C}} |k_\delta(z)|^{2N} e^{-2\phi_L(z)} \frac{dm(z)}{\rho_L(z)^2} &\lesssim L^{2/\gamma} \int_{\mathbb{C}} (|k_\delta(z)|e^{-\delta\phi(z)})^{2N} \frac{dm(z)}{\rho_{\delta\mu}(z)^2} \\
&\lesssim L^{2/\gamma} \int_{\mathbb{C}} e^{-d^\epsilon_{\delta\phi}(z,z_0)} \frac{dm(z)}{\rho_{\delta\mu}(z)^2} \lesssim L^{2/\gamma}.
\end{aligned}$$

Hence k_δ^N is an entire function in $\mathcal{F}^2_L$ and we define $h_L = k_\delta^N/\|k_\delta^N\|_{\mathcal{F}^2_{\phi_L}}$. We finally note that (4.12) implies that for all $z \in D$

$$|h_L(z)|e^{-\phi_L(z)} = (|k_\delta(z)|e^{-\delta\phi(z)})^N/\|k_\delta^N\|_{\mathcal{F}^2_{\phi_L}} \gtrsim C^N L^{1/\gamma} \geq e^{-C_0 L}$$

where C_0 depends on δ_0 and the doubling constant. ■

Proof of the upper bounds in Theorem 4.7. (a) Recall that $(\tilde{k}_\lambda)_{\lambda\in\Lambda_L}$ is the (canonical) dual frame associated to the frame $(k_\lambda)_{\lambda\in\Lambda_L}$. Since $h_L \in \mathcal{F}^2_L$ we may write $h_L = \sum_{\lambda\in\Lambda}\langle h_L, \tilde{k}_\lambda\rangle k_\lambda = \sum_{\lambda\in\Lambda} c_\lambda k_\lambda$ where we define $c_\lambda = \langle h_L, \tilde{k}_\lambda\rangle$ (and we ignore the dependence on L to simplify the notation). Note that, for any $z\in D$, we have

$$|f_L(z)|e^{-\phi_L(z)} = \left|h_L(z) + \sum_{\lambda\in\Lambda}(a_\lambda - c_\lambda)\tilde{k}_\lambda(z)\right|e^{-\phi_L(z)} \geq e^{-C_0 L} - \sum_{\lambda\in\Lambda}|a_\lambda - c_\lambda||\tilde{k}_\lambda(z)|e^{-\phi_L(z)}.$$

We therefore have

$$\mathbb{P}[n_L(D)=0] \geq \mathbb{P}\left[\max_{z\in D}\sum_{\lambda\in\Lambda}|a_\lambda - c_\lambda||\tilde{k}_\lambda(z)|e^{-\phi_L(z)} < e^{-C_0 L}\right]$$

and we now estimate the probability of this event. First define

$$\alpha = \max\left\{0, \frac{1}{\delta}\left(\frac{1}{\epsilon} - \gamma\right)\right\}$$

where ϵ, γ and δ are the constants appearing in Proposition 2.12, (4.6) and Lemma 2.6 respectively. Fix a large positive constant C_1 to be specified, write

$$D_L = D^{C_1 L^\alpha r}(z_0)$$

and define the event

$$\mathcal{E}_1 = \left\{|a_\lambda - c_\lambda| \leq L\frac{|\lambda - z_0|}{\rho_L(z_0)} : \lambda \in \Lambda_L \setminus D_L\right\}.$$

If $\mathcal{E}_1$ occurs and $z\in D$ then, using an argument identical to that given in the proof of Lemma 2.8, we see that

$$\begin{aligned}
\sum_{\lambda\in\Lambda_L\setminus D_L}|a_\lambda - c_\lambda||k_\lambda(z)|e^{-\phi_L(z)} &\lesssim L\sum_{\lambda\in\Lambda_L\setminus D_L}\frac{|\lambda - z_0|}{\rho_L(z_0)}e^{-d^\epsilon_L(z,\lambda)} \\
&\lesssim L^{1+1/\gamma}\sum_{\lambda\in\Lambda_L\setminus D_L}\frac{|\lambda - z_0|}{\rho(z_0)}e^{-c' L^{\epsilon\gamma} d^\epsilon_\phi(z_0,\lambda)} \\
&\lesssim L^{1+1/\gamma}\int_{\mathbb{C}\setminus D_L}\frac{|\zeta - z_0|}{\rho(z_0)}e^{-c' L^{\epsilon\gamma}\left(\frac{|\zeta - z_0|}{\rho(z_0)}\right)^{\epsilon\delta}}\frac{dm(\zeta)}{\rho_L(\zeta)^2} \\
&\lesssim L^{\beta_0}\int_{c' C_1^{\delta\epsilon} L^{\alpha'}}^{+\infty} e^{-t} t^{\beta_1}\, dt \\
&\leq \frac{1}{2}e^{-C_0 L}
\end{aligned}$$

for C_1 sufficiently large, where $\alpha' = \max\{1, \epsilon\gamma\}$, and β_0 and $\beta_1 > 0$ are some exponents that depend on the doubling constant.

We define the event

$$\mathcal{E}_2 = \left\{ |a_\lambda - c_\lambda| \le \frac{e^{-C_0 L}}{C_2\sqrt{\#\Lambda_L \cap D_L}} : \lambda \in \Lambda_L \cap D_L \right\},$$

where C_2 is a positive constant to be chosen. Note that for all $z \in \mathbb{C}$, $\mathcal{E}_2$ implies that by choosing C_2 sufficiently large

$$\begin{aligned}\sum_{\lambda\in\Lambda_L\cap D_L} |a_\lambda - c_\lambda||k_\lambda(z)|e^{-\phi_L(z)} &\le \Big(\sum_{\lambda\in\Lambda_L\cap D_L} |a_\lambda - c_\lambda|^2\Big)^{1/2}\Big(\sum_{\lambda\in\Lambda_L\cap D_L} |k_\lambda(z)|^2\Big)^{1/2} e^{-\phi_L(z)} \\ &\le \frac{e^{-C_0 L}}{C_2} K_L(z,z)^{1/2} e^{-\phi_L(z)} < \frac{1}{2} e^{-C_0 L}.\end{aligned}$$

Hence

$$\mathbb{P}[n_L(D) = 0] \ge \mathbb{P}[\mathcal{E}_1]\mathbb{P}[\mathcal{E}_2].$$

Recalling the definition of the coefficients c_λ we note that

$$\sum_{\lambda\in\Lambda_L} |c_\lambda|^2 \simeq \|h_L\|^2_{\mathcal{F}^2_{\phi_L}} = 1$$

and so the coefficients c_λ are bounded. This means that

$$\mathbb{P}\Big[|a_\lambda - c_\lambda| \le L\frac{|\lambda - z_0|}{\rho_L(z_0)}\Big] \ge \mathbb{P}\Big[|a_\lambda| \le L\frac{|\lambda - z_0|}{2\rho_L(z_0)}\Big]$$

when $\lambda \in \Lambda_L \setminus D_L$ and L is large. We may therefore estimate $\mathbb{P}[\mathcal{E}_1]$ similarly to (4.11) in the proof of Lemma 4.12. This yields $\mathbb{P}[\mathcal{E}_1] \ge 1/2$ for large L.

Finally since $\#\Lambda_L \cap D_L \simeq L\mu(D_L) \lesssim L^{1+\alpha/\gamma}$ we have

$$\mathbb{P}[\mathcal{E}_2] = \prod_{\lambda\in\Lambda_L\cap D_L} \mathbb{P}\Big[|a_\lambda - c_\lambda| \le \frac{e^{-C_0 L}}{C_2\sqrt{\#\Lambda_L \cap D_L}}\Big] \ge \Big(C\frac{e^{-2C_0 L}}{\#\Lambda_L \cap D_L}\Big)^{\#\Lambda_L\cap D_L} \ge e^{-cL^{2+\alpha/\gamma}}$$

for some positive constants C and c. Considering the two possible values of α completes the proof of the lower bounds in Theorem 4.7, where $\tau = 2 + \max\{0, \frac{1}{\delta}(\frac{1}{\epsilon\gamma} - 1)\}$.

(b) We assume that the reproducing kernel $\mathcal{K}_L$ satisfies the estimate (4.8). We will use the same notation as before. Let C_3 and C_4 be constants to be chosen and define the events

$$\mathcal{A}_1 = \{|a_\lambda - c_\lambda| \le L\frac{|\lambda - z_0|}{\rho_L(z_0)} : \lambda \in \Lambda_L \setminus D^{C_3 r}(z_0)\}$$

and

$$\mathcal{A}_2 = \{|c_\lambda - a_\lambda| \le \frac{e^{-C_0 L}}{C_4\sqrt{\#\Lambda_L \cap D^{C_3 r}(z_0)}} : \lambda \in \Lambda_L \cap D^{C_3 r}(z_0)\}.$$

We have already seen that the event $\mathcal{A}_1$ implies that

$$\Big|\sum_{\lambda\in\Lambda_L\setminus D_L}(a_\lambda-c_\lambda)\tilde{k}_\lambda(z)\Big|e^{-\phi_L(z)}\leq\frac{1}{2}e^{-C_0L}$$

for $z\in D$. We write $\tilde{\Lambda}_L=\Lambda_L\cap(D_L\setminus D^{C_3r}(z_0))$. Note that $\mathcal{A}_1$ and (4.8) imply that, for $z\in D$,

$$\begin{aligned}&\sum_{\lambda\in\tilde{\Lambda}_L}|a_\lambda-c_\lambda||k_\lambda(z)|e^{-\phi_L(z)}\\&\leq\Big(\sum_{\lambda\in\tilde{\Lambda}_L}|a_\lambda-c_\lambda|^2\Big)^{1/2}\Big(\sum_{\lambda\in\tilde{\Lambda}_L}|k_\lambda(z)|^2\Big)^{1/2}e^{-\phi_L(z)}\\&\lesssim L^{1+\alpha+1/\gamma}\sqrt{\#\tilde{\Lambda}_L}\left(\int_{\mathbb{C}\setminus D^{C_3r}(z_0))}|\mathcal{K}_L(z,\zeta)|e^{-2\phi_L(\zeta)}\frac{dm(z)}{\rho_L(z)^2}\right)^{1/2}e^{-\phi_L(z)}\\&<\frac{1}{4}e^{-C_0L}\end{aligned}$$

for an appropriately large choice of C_3 and for all large L. By an identical computation to before we see that $\mathcal{A}_2$ implies that for all $z\in\mathbb{C}$

$$\sum_{\lambda\in\Lambda_L\cap D^{C_3r}(z_0)}|a_\lambda-c_\lambda||k_\lambda(z)|e^{-\phi_L(z)}<\frac{1}{4}e^{-C_0L}$$

by choosing C_4 sufficiently large. It remains only to estimate the probabilities of the events $\mathcal{A}_1$ and $\mathcal{A}_2$, which are again identical to the previous computation. This completes the proof. ■

Appendix: The case $|z|^\alpha/2$

We consider the space $\mathcal{F}^2_L$ when $\phi(z)=|z|^\alpha/2$ and we first note that for $|z|\leq 1$

$$\rho(z)\simeq 1$$

and that

$$\rho(z)\simeq|z|^{1-\alpha/2}$$

otherwise. We begin by showing that the set $(\frac{(L^{1/\alpha}z)^n}{c_{\alpha n}})_{n=0}^\infty$ is an orthonormal basis for some choice of $c_{\alpha n}\simeq\Gamma(\frac{2}{\alpha}n+1)^{1/2}$. It is clear that the functions z^n are orthogonal because ϕ_L (and therefore ρ_L) are radial, and so we need only compute the appropriate normalising constants

$$\|z^n\|^2_{\mathcal{F}^2_L}=\int_{\mathbb{C}}|z|^{2n}e^{-L|z|^\alpha}\frac{dm(z)}{\rho_L(z)^2}.$$

Now it is easy to see that for $|z|\leq\rho_L(0)$

$$\rho_L(z)\simeq\rho_L(0)\simeq L^{-1/\alpha}$$

and that

$$\rho_L(z) \simeq L^{-1/2}|z|^{1-\alpha/2}$$

otherwise. Hence, using the fact that $L\rho_L(0)^\alpha \simeq 1$, we have

$$\begin{aligned}
\|z^n\|^2_{\mathcal{F}^2_L} &= \int_{\mathbb{C}} |z|^{2n} e^{-L|z|^\alpha} \frac{dm(z)}{\rho_L(z)^2} \\
&\simeq L^{2/\alpha} \int_{D^L(0)} |z|^{2n} e^{-L|z|^\alpha} dm(z) + L \int_{\mathbb{C}\setminus D^L(0)} |z|^{2n} e^{-L|z|^\alpha} |z|^{\alpha-2} dm(z) \\
&\simeq L^{-2n/\alpha} \left(\int_0^{L\rho_L(0)^\alpha} u^{1+(2n+2)/\alpha} e^{-u} du + \int_{L\rho_L(0)^\alpha}^\infty u^{2n/\alpha} e^{-u} du \right) \\
&\simeq L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha} n + 1\right).
\end{aligned}$$

It follows that, for some coefficients $c_{\alpha n} \simeq \Gamma(\frac{2}{\alpha}n+1)^{1/2}$, the set $(\frac{(L^{1/\alpha}z)^n}{c_{\alpha n}})_{n=0}^\infty$ is an orthonormal basis for $\mathcal{F}^2_L$ and the reproducing kernel for this space is then given by

$$\mathcal{K}_L(z,w) = \sum_{n=0}^\infty \frac{(L^{2/\alpha} z\overline{w})^n}{c^2_{\alpha n}}.$$

We recall that for positive a the Mittag-Leffler function

$$E_{a,1}(\zeta) = \sum_{n=0}^\infty \frac{\zeta^n}{\Gamma(an+1)}$$

is an entire function of order $1/a$ satisfying

$$E_{a,1}(x) \lesssim e^{x^{1/a}}$$

for all real positive x.

We now show that $\mathcal{K}_L$ has fast L^2 off-diagonal decay, that is, given $C, r > 0$ there exists $R > 0$ (independent of L) such that

$$\sup_{z \in D^r(z_0)} e^{-L|z|^\alpha} \int_{\mathbb{C}\setminus D(z_0,2R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \le e^{-CL}$$

for all $z_0 \in \mathbb{C}$ and L sufficiently large (we have replaced $D^R(z_0)$ by $D(z_0, 2R)$ to simplify the notation in what follows). Choosing R sufficiently large we have

$$\int_{\mathbb{C}\setminus D(z_0,2R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \le \int_{\mathbb{C}\setminus D(0,R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2}$$

and we note again that ϕ_L and ρ_L are radial. Thus, for any positive integers n and m,

$$\begin{aligned}\int_{\mathbb{C}\setminus D(0,R)} w^n \overline{w}^m e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} &\simeq \delta_{nm} \int_{\mathbb{C}\setminus D(0,R)} |w|^{2n} e^{-L|w|^\alpha} L|w|^{\alpha-2}\, dm(w) \\ &= 2\pi\delta_{nm} \int_R^\infty r^{2n} e^{-Lr^\alpha} Lr^{\alpha-1} dr \\ &= \frac{2\pi}{\alpha}\delta_{nm} L^{-2n/\alpha} \int_{LR^\alpha}^\infty u^{2n/\alpha} e^{-u} du \\ &= \frac{2\pi}{\alpha}\delta_{nm} L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha}n+1, LR^\alpha\right)\end{aligned}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ denotes the incomplete Gamma function. Now, recalling the expression for the kernel $\mathcal{K}_L$ we see that

$$|\mathcal{K}_L(z,w)|^2 = \sum_{m,n=0}^\infty \frac{L^{2(m+n)/\alpha}}{c_{\alpha m} c_{\alpha n}} z^m \overline{z}^n \overline{w}^m w^n$$

and so

$$\begin{aligned}\int_{\mathbb{C}\setminus D(0,R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} &= \sum_{m,n=0}^\infty \frac{L^{2(m+n)/\alpha}}{c_{\alpha m} c_{\alpha n}} z^m \overline{z}^n \int_{\mathbb{C}\setminus D(0,R)} w^n \overline{w}^m e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \\ &\simeq \sum_{n=0}^\infty \frac{L^{4n/\alpha}}{\Gamma(\frac{2}{\alpha}n+1)^2} |z|^{2n} L^{-2n/\alpha} \Gamma\left(\frac{2}{\alpha}n+1, LR^\alpha\right) \\ &= \sum_{n=0}^\infty \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n+1)} \frac{\Gamma(\frac{2}{\alpha}n+1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n+1)}.\end{aligned}$$

We split this sum in two parts. Choose $N = [\frac{\alpha}{4}LR^\alpha]$ and note that for $n \le N$ we have, by standard estimates for the incomplete Gamma function,

$$\Gamma\left(\frac{2}{\alpha}n+1, LR^\alpha\right) \simeq (LR^\alpha)^{2n/\alpha} e^{-LR^\alpha}$$

as $R \to \infty$. Now Stirling's approximation shows that

$$\begin{aligned}\frac{(LR^\alpha)^{\frac{2}{\alpha}n}}{\Gamma(\frac{2}{\alpha}n+1)} &\le \frac{(LR^\alpha)^{\frac{2}{\alpha}N}}{\Gamma(\frac{2}{\alpha}N+1)} \\ &\simeq \left(\frac{4}{\alpha}N\right)^{\frac{2}{\alpha}N} \left(\frac{2}{\alpha}N+1\right)^{1/2} \left(\frac{e}{\frac{2}{\alpha}N+1}\right)^{\frac{2}{\alpha}N+1} \\ &\lesssim \left(\frac{\frac{4}{\alpha}N}{\frac{2}{\alpha}N+1}\right)^{\frac{2}{\alpha}N} e^{\frac{2}{\alpha}N} \\ &\lesssim 2^{\frac{2}{\alpha}N} e^{\frac{2}{\alpha}N} \\ &\le e^{LR^\alpha(1+\log 2)/2}\end{aligned}$$

and so

$$\frac{\Gamma(\frac{2}{\alpha}n+1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n+1)} \lesssim e^{-cLR^\alpha}.$$

It follows that

$$\begin{aligned}\sum_{n=0}^{N} \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n+1)} \frac{\Gamma(\frac{2}{\alpha}n+1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n+1)} &\lesssim e^{-cLR^\alpha} \sum_{n=0}^{\infty} \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n+1)} \\ &= e^{-cLR^\alpha} E_{\frac{2}{\alpha},1}(L^{2/\alpha}|z|^2) \\ &\lesssim e^{-cLR^\alpha} e^{L|z|^\alpha}.\end{aligned}$$

To deal with the remaining terms we first note that

$$\frac{\Gamma(\frac{2}{\alpha}n+1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n+1)} \leq 1$$

for all n. We now choose R so large that

$$L^{2/\alpha}|z|^2 < 4^{-2/\alpha} e^{-4/\alpha} L^{2/\alpha} R^2 < e^{-4/\alpha} \left(\frac{2}{\alpha}N+1\right)^{2/\alpha}$$

for $z \in D^r(z_0)$. Note that another application of Stirling's approximation yields, for any $n > N$,

$$\Gamma\left(\frac{2}{\alpha}n+1\right) \gtrsim \Gamma\left(\frac{2}{\alpha}N+1\right) \left(\frac{2}{\alpha}N+1\right)^{2(n-N)/\alpha}.$$

We conclude that for $z \in D^r(z_0)$ and R sufficiently large we have

$$\begin{aligned}\sum_{n>N} \frac{(L^{2/\alpha}|z|^2)^n}{\Gamma(\frac{2}{\alpha}n+1)} \frac{\Gamma(\frac{2}{\alpha}n+1, LR^\alpha)}{\Gamma(\frac{2}{\alpha}n+1)} &\lesssim \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N+1)} \sum_{n=0}^{\infty} \frac{(L^{2/\alpha}|z|^2)^n}{(\frac{2}{\alpha}N+1)^{2n/\alpha}} \\ &= \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N+1)} \left(1 - \frac{L^{2/\alpha}|z|^2}{(\frac{2}{\alpha}N+1)^{2/\alpha}}\right)^{-1} \\ &\simeq \frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N+1)}.\end{aligned}$$

A final appeal to Stirling's approximation yields

$$\begin{aligned}\frac{(L^{2/\alpha}|z|^2)^N}{\Gamma(\frac{2}{\alpha}N+1)} &\simeq \left(L^{2/\alpha}|z|^2\right)^N \left(\frac{2}{\alpha}N+1\right)^{1/2} \left(\frac{e}{\frac{2}{\alpha}N+1}\right)^{\frac{2}{\alpha}N+1} \\ &\simeq \left(\frac{L^{2/\alpha}|z|^2}{(\frac{2}{\alpha}N+1)^{2/\alpha}}\right)^N \left(\frac{2}{\alpha}N+1\right)^{-1/2} e^{2N/\alpha} \\ &\lesssim e^{-4N/\alpha} e^{2N/\alpha} \\ &= e^{-LR^\alpha/2}.\end{aligned}$$

Retracing our footsteps we see that we have shown that

$$\int_{\mathbb{C}\setminus D(z_0,2R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \lesssim e^{-cLR^\alpha}(1+e^{L|z|^\alpha})$$

for all $z \in D^r(z_0)$ and R sufficiently large. Hence

$$\sup_{z\in D^r(z_0)} e^{-L|z|^\alpha} \int_{\mathbb{C}\setminus D(z_0,2R)} |\mathcal{K}_L(z,w)|^2 e^{-L|w|^\alpha} \frac{dm(w)}{\rho_L(w)^2} \leq e^{-CL}$$

for an appropriately large R, as claimed.

Chapter 5

The hyperbolic GAF

In this chapter we discuss the hyperbolic GAF, whose zero set is invariant under disc automorphisms. While the hyperbolic GAF is in some sense quite similar to the flat GAF, many of its properties are less well understood. We begin by recalling the definition and some of the basic properties introduced earlier. We refer the reader to [HKPV09, Chapter 2] for proofs of the results given below, and for a more thorough comparison with the flat GAF (and the spherical GAF, defined on the Riemann sphere and invariant with respect to the spherical metric). Recall that

$$f_L(z) = \sum_{n=0}^{\infty} a_n \binom{L+n-1}{n}^{1/2} z^n$$

for $z \in \mathbb{D}$, where $L > 0$ and $(a_n)_{n=0}^{\infty}$ is a sequence of iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables. This sum almost surely defines a holomorphic function in the unit disc with associated covariance kernel

$$K_L(z,w) = \mathbb{E}[f_L(z)\overline{f_L(w)}] = (1 - z\overline{w})^{-L}.$$

In fact the disc is the natural domain of definition of f_L, almost surely it does not extend analytically to any larger domain.

We denote the counting measure on the zero set of f_L by n_L. To simplify the notation we also write $n_L(r) = n_L(D(0,r))$. The first intensity of the zero set is given by $L\nu$, where ν is the hyperbolic area $d\nu(z) = \frac{dm(z)}{\pi(1-|z|^2)^2}$, so in particular

$$\mathbb{E}[n_L(r)] = L\nu(D(0,r)) = \frac{Lr^2}{1-r^2}.$$

Moreover the distribution of the zero set is invariant under disc automorphisms, as is the distribution of the random potential

$$\log|f_L(z)| - \frac{L}{2}\log\frac{1}{1-|z|^2}.$$

There is one particular case that is well-understood, when we choose $L = 1$. Then the function we are studying takes the simple form

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

In this case Peres and Virág gave a complete description of the random variable $n_1(r)$ [PV05, Theorem 2], this description followed from showing that the corresponding zero set is a so-called determinantal process, but this holds for no other value of L. In fact we know of no other (non-trivial) GAF whose zero set is a determinantal process. Since we are interested in the full range of L, our techniques are accordingly quite different. We are interested in two particular results [PV05, Corollary 3]; the variance,

$$\mathbb{V}[n_1(r)] = \frac{r^2}{1-r^4},$$

and the hole probability,

$$\mathbb{P}[n_1(r) = 0] = \exp\left(-\frac{\pi^2}{6(1-r^2)}(1+o(1)\right)$$

as $r \to 1^-$.

For large values of L, Sodin and Tsirelson [ST04] computed the asymptotic behaviour of the variance of the 'smooth linear statistics'. Let ψ be a smooth real-valued function (which is not identically zero) supported in a compact subset of $\mathbb{D}$ and define

$$n(\psi, L) = \frac{1}{L}\int \psi \, dn_L = \frac{1}{L}\sum_{a\in\mathcal{Z}(f_L)} \psi(a).$$

Then

$$\mathbb{V}[n(\psi, L)] = \frac{\zeta(3)}{16\pi}\frac{1}{L^3}\|\Delta\psi\|_{L^2}(1+o(1)) \text{ as } L \to \infty.$$

There are two sections in this chapter. In the first section we compute the asymptotic behaviour of $\mathbb{V}[n_L(r)]$ as $r \to 1^-$, for the full range of L. In the second section we compute the asymptotic decay of the hole probability for large values of L, and give some large deviation estimates for $n(\psi, L)$.

5.1 The variance

We are interested in computing $\mathbb{V}[n_L(r)]$ as r approaches 1. Our first result is the following, which computes the asymptotic behaviour for the full range of L. Throughout this section we write $o(1)$ to denote a quantity (that may depend on L) that can be made arbitrarily small as r approaches 1 for each fixed L.

Theorem 5.1. *(a) For each fixed $L > 1/2$, as $r \to 1^-$,*

$$\mathbb{V}[n_L(r)] = c_L \frac{1}{1-r^2}(1+o(1)),$$

where

$$c_L = \frac{L^2}{\pi}\int_0^\infty \frac{1}{(1+x^2)^L - 1}\frac{x^2}{1+x^2}\,dx = \frac{L^2}{4\sqrt{\pi}}\sum_{n=1}^\infty \frac{\Gamma(Ln-\frac{1}{2})}{\Gamma(Ln+1)}.$$

Moreover the quantity $o(1)$ can be taken to be uniform in L for all $L \geq 1$.

(b) We have, as $r \to 1^-$,

$$\mathbb{V}[n_{1/2}(r)] = \frac{1}{4\pi}\frac{1}{1-r^2}\log\frac{1}{1-r^2}(1+o(1)).$$

(c) For each fixed $L < 1/2$, as $r \to 1^-$,

$$\mathbb{V}[n_L(r)] = c_L \frac{1}{(1-r^2)^{2-2L}}(1+o(1)),$$

where

$$c_L = \frac{L^2\Gamma(\frac{1}{2}-L)}{\sqrt{\pi}4^L\Gamma(1-L)}.$$

Corollary 5.2. *Given $\epsilon > 0$ there exist $r_0(\epsilon)$ and $L_0(\epsilon)$ such that for all $r > r_0$ and $L > L_0$ we have*

$$\left|\frac{1-r^2}{\sqrt{L}}\mathbb{V}[n_L(r)] - \frac{1}{4\sqrt{\pi}}\zeta\left(\frac{3}{2}\right)\right| < \epsilon.$$

In other words

$$\lim_{\substack{L\to\infty \\ r\to 1^-}} \frac{1-r^2}{\sqrt{L}}\mathbb{V}[n_L(r)] = \frac{1}{4\sqrt{\pi}}\zeta\left(\frac{3}{2}\right)$$

independent of the manner in which $L \to \infty$ and $r \to 1^-$.

In the particular cases $L = 1, 2$ we can show the following more precise result.

Theorem 5.3. *For any $0 < r < 1$*

$$\mathbb{V}[n_1(r)] = \frac{r^2}{1-r^4}$$

and

$$\mathbb{V}[n_2(r)] = \frac{4r^2}{1-r^2}\left(\frac{1}{1+r^2} - \frac{1}{2\sqrt{1+r^4}}\right)$$

In particular we recover [PV05, Corollary 3 (iii)].

We begin with the following.

Lemma 5.4. *For any* $0 < r < 1$

$$\mathbb{V}[n_L(r)] = \frac{L^2 r^4}{2\pi(1-r^2)^2} I_L(r),$$

where

$$I_L(r) = \int_{-\pi}^{\pi} \frac{(1-r^2)^{2L}}{|1-r^2e^{i\theta}|^{2L} - (1-r^2)^{2L}} \frac{2(1-\cos\theta)}{|1-r^2e^{i\theta}|^2} \, d\theta.$$

Proof. We define

$$J_L(z,w) = \frac{|K_L(z,w)|^2}{K_L(z,z)K_L(w,w)} = J_1(z,w)^L$$

where

$$J_1(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2} = 1 - \left|\frac{z-w}{1-z\overline{w}}\right|^2.$$

We recall that the dilogarithm

$$\mathrm{Li}_2(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n^2}$$

satisfies

$$\frac{d}{d\zeta}\mathrm{Li}_2(\zeta) = \frac{1}{\zeta}\log\frac{1}{1-\zeta}.$$

Now for any $D \subset \mathbb{D}$ with piecewise smooth boundary we have (see [SZ08, Theorem 3.1] or [NS11, Lemma 2.3]; we have indicated a proof in Chapter 4)

$$\begin{aligned}\mathbb{V}[n_L(D)] &= \int_D\int_D \Delta_z\Delta_w \frac{1}{4}\mathrm{Li}_2(J_L(z,w))\frac{dm(z)}{2\pi}\frac{dm(w)}{2\pi} \\ &= -\frac{1}{4\pi^2}\int_{\partial D}\int_{\partial D} \frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\bar{w}}\mathrm{Li}_2(J_L(z,w))\, d\bar{z}d\bar{w}\end{aligned}$$

where we have applied Stokes' Theorem. Now

$$\frac{\partial}{\partial\bar{w}}\mathrm{Li}_2(J_L(z,w)) = \frac{1}{J_L}\log\frac{1}{1-J_L}\frac{\partial J_L}{\partial\bar{w}} = \frac{L}{J_1}\log\frac{1}{1-J_L}\frac{\partial J_1}{\partial\bar{w}}$$

and so

$$\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial\bar{w}}\mathrm{Li}_2(J_L(z,w)) = \frac{L^2}{J_1^2}\frac{J_L}{1-J_L}\frac{\partial J_1}{\partial\bar{z}}\frac{\partial J_1}{\partial\bar{w}} + L\log\frac{1}{1-J_L}\left(\frac{1}{J_1}\frac{\partial^2 J_1}{\partial\bar{z}\partial\bar{w}} - \frac{1}{J_1^2}\frac{\partial J_1}{\partial\bar{z}}\frac{\partial J_1}{\partial\bar{w}}\right)$$

Routine but tedious calculations yield

$$\frac{\partial J_1}{\partial\bar{z}} = \frac{1-|w|^2}{|1-z\overline{w}|^2}\frac{w-z}{1-\overline{z}w}$$

$$\frac{\partial J_1}{\partial\bar{w}} = \frac{1-|z|^2}{|1-z\overline{w}|^2}\frac{z-w}{1-z\overline{w}}$$

and

$$\frac{\partial^2 J_1}{\partial \bar{z} \partial \bar{w}} = -\frac{(z-w)^2}{|1-z\overline{w}|^4},$$

so that

$$\frac{1}{J_1}\frac{\partial^2 J_1}{\partial \bar{z} \partial \bar{w}} - \frac{1}{J_1^2}\frac{\partial J_1}{\partial \bar{z}}\frac{\partial J_1}{\partial \bar{w}} = 0.$$

We conclude that

$$\mathbb{V}[n_L(D)] = \frac{L^2}{4\pi^2}\int_{\partial D}\int_{\partial D} \frac{J_L}{1-J_L}\frac{1}{(1-|z|^2)(1-|w|^2)}\frac{(z-w)^2}{|1-z\overline{w}|^2}\, d\bar{z} d\bar{w}.$$

We now suppose that $D = D(0,r)$ for $r < 1$. Then, writing $z = re^{i\theta}$ and $w = re^{i\phi}$, after some simplifications we have

$$\mathbb{V}[n_L(D)] = \frac{L^2 r^4}{4\pi^2(1-r^2)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)^{2L}}{|1-r^2e^{i(\theta-\phi)}|^{2L} - (1-r^2)^{2L}}\frac{2(1-\cos(\theta-\phi))}{|1-r^2e^{i(\theta-\phi)}|^2}\, d\theta d\phi$$

We note that the integrand depends on the difference $\theta - \phi$, so one of the integrals immediately evaluates to 2π. We are left with

$$\mathbb{V}[n_L(D)] = \frac{L^2 r^4}{2\pi(1-r^2)^2}\int_{-\pi}^{\pi} \frac{(1-r^2)^{2L}}{|1-r^2e^{i\theta}|^{2L} - (1-r^2)^{2L}}\frac{2(1-\cos\theta)}{|1-r^2e^{i\theta}|^2}\, d\theta$$

as claimed. ■

It remains only to compute $I_L(r)$. We begin with the values of L where we can compute this exactly.

Proposition 5.5. *For any* $0 < r < 1$

$$I_1(r) = \frac{2\pi(1-r^2)}{r^2(1+r^2)}$$

and

$$I_2(r) = \frac{2\pi(1-r^2)}{r^2}\left(\frac{1}{1+r^2} - \frac{1}{2\sqrt{1+r^4}}\right).$$

Proof. We first suppose that L is an integer. Then

$$\begin{aligned}
I_L(r) &= \int_{-\pi}^{\pi} \frac{(1-r^2)^{2L}}{|1-r^2e^{i\theta}|^{2L} - (1-r^2)^{2L}}\frac{2(1-\cos\theta)}{|1-r^2e^{i\theta}|^2}\, d\theta \\
&= \int_{\partial\mathbb{D}} \frac{(1-r^2)^{2L}}{(1-r^2z)^L(1-r^2/z)^L - (1-r^2)^{2L}}\frac{(1-z)(1-1/z)}{(1-r^2z)(1-r^2/z)}\frac{dz}{iz} \\
&= \int_{\partial\mathbb{D}} -\frac{1}{i}\frac{(1-r^2)^{2L}z^{L-1}}{(1-r^2z)^L(z-r^2)^L - z^L(1-r^2)^{2L}}\frac{(1-z)^2}{(1-r^2z)(z-r^2)}\, dz.
\end{aligned}$$

We note that the integrand has simple poles at r^{-2}, which lies outside the disc, and at r^2 with residue

$$\frac{1}{ir^2}\frac{1-r^2}{1+r^2}.$$

Finally there are poles at the zeroes of the polynomial

$$(1-r^2z)^L(z-r^2)^L - z^L(1-r^2)^{2L}.$$

This is equivalent to finding the zeroes of

$$(1-r^2z)(z-r^2) - \omega z(1-r^2)^2. \tag{5.1}$$

for each Lth root of unity ω satisfying $\omega^L = 1$, which is in turn equivalent to finding the zeroes of

$$(z-1)^2 - \frac{1-\omega}{r^2}(1-r^2)z. \tag{5.2}$$

Now if $L = 1$ we have only $\omega = 1$, and there is a double zero at 1. This pole is removable, since there is a factor $(1-z)^2$ in the numerator of the integrand. We conclude that

$$I_1(r) = \frac{2\pi(1-r^2)}{r^2(1+r^2)}.$$

If $L = 2$ we have $\omega = 1, -1$. Once more if $\omega = 1$ there is a double zero at 1 which gives a removable pole. If $\omega = -1$ we can solve (5.2) easily and get two distinct zeroes

$$z^{(i)} = 1 + \frac{(1-r^2)^2}{r^2} - \frac{1-r^2}{r^2}\sqrt{1+r^4}$$

which is inside the unit disc and

$$z^{(o)} = 1 + \frac{(1-r^2)^2}{r^2} + \frac{1-r^2}{r^2}\sqrt{1+r^4}$$

outside. This yields

$$(1-r^2z)^2(z-r^2)^2 - z^2(1-r^2)^4 = r^4(z-1)^2(z-z^{(i)})(z-z^{(o)})$$

and so the integrand simplifies to

$$-\frac{1}{i}\frac{(1-r^2)^4 z}{r^4(1-r^2z)(z-r^2)}\frac{1}{(z-z^{(i)})(z-z^{(o)})}.$$

Noting that $z^{(i)}$ is a zero of (5.1), we compute the residue at $z^{(i)}$ to be

$$-\frac{1}{i}\frac{(1-r^2)^4}{-r^4(1-r^2)^2}\frac{1}{(z^{(i)}-z^{(o)})} = -\frac{1-r^2}{2ir^2\sqrt{1+r^4}}$$

which gives

$$I_2(r) = 2\pi\left(\frac{1}{r^2}\frac{1-r^2}{1+r^2} - \frac{1-r^2}{2r^2\sqrt{1+r^4}}\right) = \frac{2\pi(1-r^2)}{r^2}\left(\frac{1}{1+r^2} - \frac{1}{2\sqrt{1+r^4}}\right). \qquad \blacksquare$$

Remarks. 1. If we are only interested in the case $L = 1$, we may compute $I_1(r)$ without recourse to residue calculus. First note that the integrand simplifies to $|1 - r^2 e^{i\theta}|^{-2}$ (and some factors that depend on r). From the geometric series we have

$$|1 - r^2 e^{i\theta}|^{-2} = \sum_{n,m=0}^{\infty} r^{2(n+m)} e^{i\theta(n-m)}.$$

Integrating this expression term by term yields the result.

2. In principle we can compute $I_L(r)$ in this manner for any integer L. We need to compute the zeroes of (5.2). We always have a double zero at 1 if $\omega = 1$, which gives a removable pole. For $\omega \neq 1$, since the product of the zeroes is 1 and the sum of the zeroes is $2 + \frac{1-\omega}{r^2}(1 - r^2)$ which has real part strictly greater than 2, we see that there are two distinct zeroes, one inside the disc and one outside, which we label $z_\omega^{(i)}$ and $z_\omega^{(o)}$ respectively. We therefore have

$$(1 - r^2 z)^L (z - r^2)^L - z^L (1 - r^2)^{2L} = (-r^2)^L (z - 1)^2 \prod_\omega (z - z_\omega^{(i)})(z - z_\omega^{(o)})$$

where the product ranges over the $L - 1$ non-trivial roots of unity. Thus the integrand simplifies to

$$-\frac{1}{i} \frac{(1 - r^2)^{2L} z^{L-1}}{(-r^2)^L (1 - r^2 z)(z - r^2)} \frac{1}{\prod_\omega (z - z_\omega^{(i)})(z - z_\omega^{(o)})}.$$

Noting that $z_\omega^{(i)}$ is a zero of (5.1), we compute the residue at $z_\omega^{(i)}$ to be

$$\begin{aligned} -\frac{1}{i} & \frac{(1 - r^2)^{2L} (z_\omega^{(i)})^{L-1}}{(-r^2)^L (1 - r^2 z_\omega^{(i)})(z_\omega^{(i)} - r^2)} \frac{1}{(z_\omega^{(i)} - z_\omega^{(o)}) \prod_{\tilde{\omega} \neq \omega} (z_\omega^{(i)} - z_{\tilde{\omega}}^{(i)})(z_\omega^{(i)} - z_{\tilde{\omega}}^{(o)})} \\ & = -\frac{1}{i} \frac{(1 - r^2)^{2L-2} (z_\omega^{(i)})^{L-2}}{(-r^2)^L \omega} \frac{1}{(z_\omega^{(i)} - z_\omega^{(o)}) \prod_{\tilde{\omega} \neq \omega} (z_\omega^{(i)} - z_{\tilde{\omega}}^{(i)})(z_\omega^{(i)} - z_{\tilde{\omega}}^{(o)})} \end{aligned}$$

and so we conclude that

$$I_L(r) = 2\pi \left(\frac{1}{r^2} \frac{1 - r^2}{1 + r^2} - \sum_\omega \frac{(1 - r^2)^{2L-2} (z_\omega^{(i)})^{L-2}}{(-r^2)^L \omega (z_\omega^{(i)} - z_\omega^{(o)}) \prod_{\tilde{\omega} \neq \omega} (z_\omega^{(i)} - z_{\tilde{\omega}}^{(i)})(z_\omega^{(i)} - z_{\tilde{\omega}}^{(o)})} \right).$$

From here the algebra seems intractable and we have contented ourselves with considering only the values $L = 1, 2$. Mathematica yields an explicit expression for $L = 4$, however we have not been persistent enough to establish its veracity.

3. Mathematica also yields a closed expression if $L = 1/2$ in terms of some special function, that is not terribly enlightening.

We now compute the asymptotic behaviour of $I_L(r)$ for all values of L. By examining the integrand it is clear that for θ smaller than $1 - r^2$ the integrand is approximately constant, so we get a contribution of size $(1 - r^2)$. However if $|\theta|$ is close to π the integrand is approximately $(1 - r^2)^{2L}$. The important region of integration therefore depends on whether or not $L > 1/2$. The next proposition makes this reasoning precise.

Proposition 5.6. *(a) For each fixed $L > 1/2$, as $r \to 1^-$,*

$$\begin{aligned} I_L(r) &= 2(1-r^2)\int_0^\infty \frac{1}{(1+x^2)^L - 1}\frac{x^2}{1+x^2}\,dx(1+o(1)) \\ &= \frac{1-r^2}{2\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma(Ln-\frac{1}{2})}{\Gamma(Ln+1)}(1+o(1)). \end{aligned}$$

Moreover the quantity $o(1)$ can be taken to be uniform in L for all $L \geq 1$.

(b) We have, as $r \to 1^-$,

$$I_{1/2}(r) = 2(1-r^2)\log\frac{1}{1-r^2}(1+o(1)).$$

(c) For each fixed $L < 1/2$, as $r \to 1^-$,

$$I_L(r) = \frac{2\sqrt{\pi}\Gamma(\frac{1}{2}-L)}{4^L\Gamma(1-L)}(1-r^2)^{2L}(1+o(1)).$$

Proof. We first note that

$$|1-r^2e^{i\theta}|^2 = (1-r^2)^2 + 2r^2(1-\cos\theta)$$

and so, from Lemma 5.4,

$$\begin{aligned} I_L(r) &= \int_{-\pi}^{\pi}\frac{(1-r^2)^{2L}}{|1-r^2e^{i\theta}|^{2L}-(1-r^2)^{2L}}\frac{2(1-\cos\theta)}{|1-r^2e^{i\theta}|^2}\,d\theta \\ &= \frac{2}{r^2}\int_0^{\pi}\left(\left(1+2r^2\frac{1-\cos\theta}{(1-r^2)^2}\right)^L - 1\right)^{-1}\left(1+\frac{(1-r^2)^2}{2r^2(1-\cos\theta)}\right)^{-1}\,d\theta. \end{aligned}$$

Making the change of variables $x = \frac{2r^2}{(1-r^2)^2}(1-\cos\theta)$ we see that

$$I_L(r) = \frac{1-r^2}{r^3}\int_0^{\frac{4r^2}{(1-r^2)^2}}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} \tag{5.3}$$

(a) We first assume that $L > 1/2$. Bearing in mind the remarks preceding the statement of this lemma, we expect the main contribution to come from the 'small' values of x. Now

$$\int_0^{\frac{1}{1-r^2}}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} = \int_0^{\frac{1}{1-r^2}}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx(1+o(1))$$

where the term $o(1)$ is uniform in L. Also

$$\int_0^{\frac{1}{1-r^2}}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx = \int_0^{\infty}\frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx(1+o(1))$$

which clearly converges in this range of L. Moreover since

$$\frac{1}{(1+x)^L-1} \leq \frac{1}{(1+x)^M-1}$$

for $L \geq M$ and $x > 0$ the term $o(1)$ may be taken to be uniform in L for all $L \geq 1$ (say). The change of variables $t = \sqrt{x}$ yields

$$\int_0^\infty \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx = 2\int_0^\infty \frac{1}{(1+t^2)^L-1}\frac{t^2}{1+t^2}\,dt.$$

The alternative change of variables $s = (1+x)^{-1}$ gives us (B denotes the usual Beta function)

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx &= \int_0^1 \frac{1}{s^{-L}-1}\sqrt{\frac{1}{s}-1}\frac{ds}{s} \\
&= \int_0^1 s^{L-\frac{3}{2}}\frac{1}{1-s^L}\sqrt{1-s}\,ds \\
&= \sum_{n=0}^\infty \int_0^1 s^{L(n+1)-\frac{3}{2}}\sqrt{1-s}\,ds \\
&= \sum_{n=0}^\infty B\left(L(n+1)-\frac{1}{2},\frac{3}{2}\right) \\
&= \sum_{n=0}^\infty \frac{\Gamma(L(n+1)-\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(L(n+1)+1)} \\
&= \frac{\sqrt{\pi}}{2}\sum_{n=1}^\infty \frac{\Gamma(Ln-\frac{1}{2})}{\Gamma(Ln+1)}.
\end{aligned}$$

We now show that the remaining contributions to (5.3) are $o(1)$. We have

$$\int_{\frac{1}{1-r^2}}^{\frac{2r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} \lesssim \int_{\frac{1}{1-r^2}}^{\frac{2r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\,dx = o(1)$$

and, making the change of variables $y = \frac{(1-r^2)^2}{4r^2}x$, we see that

$$\begin{aligned}
\int_{\frac{2r^2}{(1-r^2)^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L-1}\frac{\sqrt{x}}{1+x}\frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} &\lesssim \int_{\frac{2r^2}{(1-r^2)^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{x^{L+1/2}}\frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} \\
&= (1-r^2)^{2L-1}\int_{1/2}^1 \frac{1}{y^{L+1/2}}\frac{dy}{\sqrt{1-y}} = o(1).
\end{aligned}$$

Again noting that

$$\frac{1}{(1+x)^L-1} \leq \frac{1}{x}$$

for $L \geq 1$ and $x > 0$ we see that

$$\int_{\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1 - \frac{(1-r^2)^2}{4r^2} x}}$$

is uniformly $o(1)$ for all $L \geq 1$.

We conclude that

$$I_L(r) = 2(1-r^2) \int_0^\infty \frac{1}{(1+x^2)^L - 1} \frac{x^2}{1+x^2} \, dx(1 + o(1))$$

for $L > 1/2$, and the term $o(1)$ is uniformly small for all $L \geq 1$.

(c) We now assume that $L < 1/2$. We aim to show that the main contribution to (5.3) comes from the 'big' values of x. Again making the change of variables $y = \frac{(1-r^2)^2}{4r^2} x$ we see that

$$\begin{aligned}
\int_{\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{(1+x)^L - 1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1 - \frac{(1-r^2)^2}{4r^2} x}} &= \int_{\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{x^{L+1/2}} \frac{dx}{\sqrt{1 - \frac{(1-r^2)^2}{4r^2} x}} (1 + o(1)) \\
&= \frac{2(1-r^2)^{2L-1}}{4^L} \int_{\frac{1-r^2}{4r^2}}^1 \frac{1}{y^{L+1/2}} \frac{dy}{\sqrt{1-y}} (1 + o(1)) \\
&= \frac{2(1-r^2)^{2L-1}}{4^L} \int_0^1 \frac{1}{y^{L+1/2}} \frac{dy}{\sqrt{1-y}} (1 + o(1))
\end{aligned}$$

which converges for $L < 1/2$. It remains to show that the remaining parts of (5.3) are small in comparison. Now

$$\int_0^1 \frac{1}{(1+x)^L - 1} \frac{x^{3/2}}{(1+x)^2} \frac{dx}{\sqrt{1 - \frac{(1-r^2)^2}{4r^2} x}} = O(1) = o((1-r^2)^{2L-1})$$

and

$$\begin{aligned}
\int_1^{\frac{1}{1-r^2}} \frac{1}{(1+x)^L - 1} \frac{x^{3/2}}{(1+x)^2} \frac{dx}{\sqrt{1 - \frac{(1-r^2)^2}{4r^2} x}} &\lesssim \int_1^{\frac{1}{1-r^2}} \frac{1}{x^{L+1/2}} \, dx \\
&= O((1-r^2)^{L-1/2}) \\
&= o((1-r^2)^{2L-1}).
\end{aligned}$$

We therefore have

$$I_L(r) = 2(1-r^2)^{2L} 4^{-L} \int_0^1 \frac{1}{y^{L+1/2}} \frac{dy}{\sqrt{1-y}} (1 + o(1))$$

for $L < 1/2$. Now

$$\int_0^1 \frac{1}{y^{L+1/2}} \frac{dy}{\sqrt{1-y}} = B\left(\frac{1}{2} - L, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}-L)\Gamma(\frac{1}{2})}{\Gamma(1-L)} = \frac{\Gamma(\frac{1}{2}-L)\sqrt{\pi}}{\Gamma(1-L)}$$

where, again, B is the Beta function.

(b) We finally consider the critical case $L = 1/2$; the integral we want to estimate is (see (5.3))

$$\int_0^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{\sqrt{1+x}-1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}.$$

It is clear that

$$\int_0^1 \frac{1}{\sqrt{1+x}-1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} = O(1).$$

and that

$$\begin{aligned}\int_1^{\log\frac{1}{1-r^2}} \frac{1}{\sqrt{1+x}-1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} &\lesssim \int_1^{\log\frac{1}{1-r^2}} \frac{1}{x}\,dx \\ &= \log\log\frac{1}{1-r^2} \\ &= o(\log\frac{1}{1-r^2}).\end{aligned}$$

We finally compute that

$$\int_{\log\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{\sqrt{1+x}-1} \frac{\sqrt{x}}{1+x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} = \int_{\log\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}}(1+o(1))$$

Once more making the change of variables $y = \frac{(1-r^2)^2}{4r^2}x$ we see that

$$\begin{aligned}\int_{\log\frac{1}{1-r^2}}^{\frac{4r^2}{(1-r^2)^2}} \frac{1}{x} \frac{dx}{\sqrt{1-\frac{(1-r^2)^2}{4r^2}x}} &= \int_{\frac{(1-r^2)^2}{4r^2}\log\frac{1}{1-r^2}}^{1} \frac{1}{y}\frac{dy}{\sqrt{1-y}} \\ &= \log\frac{1-\sqrt{1-y}}{1+\sqrt{1-y}}\Bigg|_{\frac{(1-r^2)^2}{4r^2}\log\frac{1}{1-r^2}}^{1} \\ &= \log(\frac{4r^2}{(1-r^2)^2\log\frac{1}{1-r^2}}) + O(1) \\ &= 2\log\frac{1}{1-r^2}(1+o(1)).\end{aligned}$$

■

Combining the previous proposition and Lemma 5.4 completes the proof of Theorem 5.1, while Theorem 5.3 follows from Proposition 5.5 and Lemma 5.4. To prove Corollary 5.2 we note that, by Stirling's approximation, we have for large L,

$$\frac{\Gamma(Ln-\frac{1}{2})}{\Gamma(Ln+1)} = \frac{\Gamma(Ln-\frac{1}{2})}{Ln\Gamma(Ln)} = (Ln)^{-3/2}(1+o(1))$$

where the error term $o(1)$ is uniform in n, and so

$$\sum_{n=1}^{\infty} \frac{\Gamma(Ln-\frac{1}{2})}{\Gamma(Ln+1)} = L^{-3/2}\left(\sum_{n=1}^{\infty} n^{-3/2}\right)(1+o(1)).$$

5.2 The hole probability

Recall that n_L denotes the counting measure on the zero set of f_L, the hyperbolic GAF, and that we write $n_L(r) = n_L(D(0,r))$. Recall also that the first intensity is given by $L\nu$ where ν is the hyperbolic area defined by $d\nu(z) = \frac{dm(z)}{\pi(1-|z|^2)^2}$. In particular $\mathbb{E}[n_L(r)] = \frac{Lr^2}{1-r^2}$. We are interested in fluctuations of n_L from the mean, we begin with the following large deviations result.

Theorem 5.7. *For any $\delta > 0$ and any smooth ψ (which is not identically zero) supported in a compact subset of $\mathbb{D}$ there exists $c > 0$ depending on ψ and δ such that*

$$\mathbb{P}\left[\left|\frac{\int \psi\, dn_L}{L\int \psi\, d\nu} - 1\right| > \delta\right] \le e^{-cL^2}$$

as $L \to \infty$.

This theorem has the following corollary.

Corollary 5.8. *(a) Suppose that U is open and contained in a compact subset of $\mathbb{D}$. Let $\delta > 0$. There exists $c > 0$ depending only on δ and U such that for sufficiently large values of L*

$$\mathbb{P}\left[\left|\frac{n_L(U)}{L\nu(U)} - 1\right| > \delta\right] \le e^{-cL^2}.$$

(b) Let $\delta > 0$. There exists $c, \epsilon, M_0 > 0$ depending only on δ such that for all $r < \epsilon$ and $Lr^2 > M_0$

$$\mathbb{P}\left[\left|\frac{n_L(r)}{L\frac{r^2}{1-r^2}} - 1\right| > \delta\right] \le e^{-cL^2r^4}.$$

This corollary may be contrasted with the following 'overcrowding' estimate of Krishnapur.

Theorem 5.9 ([Kri06, Theorem 2]). *For any fixed $L > 0$ and $0 < r < 1$ there are constants β, C_1, C_2 (depending on L and r) such that for every $m \geq 1$*

$$C_1 \exp\left(-\frac{m^2}{|\log r|}\right) \leq \mathbb{P}[n_L(r) \geq m] \leq C_2 \exp(-\beta m^2).$$

We are also interested in the event $n_L(r) = 0$ when $\mathbb{E}[n_L(r)]$ is large. We will not consider the most general regime of $\frac{Lr^2}{1-r^2} \to \infty$, but instead focus on the following cases.

Theorem 5.10. *(a) For each fixed $0 < r < 1$ there exist $c = c(r)$ and $C = C(r)$ such that*

$$e^{-CL^2} \leq \mathbb{P}[n_L(r) = 0] \leq e^{-cL^2}$$

as $L \to \infty$.

(b) If $r \to 0$ but $Lr^2 \to \infty$ then

$$\mathbb{P}[n_L(r) = 0] = e^{-\frac{e^2}{4} L^2 r^4 (1+o(1))}.$$

The upper bound in part (a) of this theorem follows from Corollary 5.8 (a). Part (b) of Corollary 5.8 gives an upper bound of the form $e^{-cL^2r^4}$ in part (b) of this theorem. However to arrive at the constant $\frac{e^2}{4}$ we shall require a different proof.

The lower bounds in Theorem 5.10

We begin by showing the lower bounds in Theorem 5.10.

Proof of the lower bounds in Theorem 5.10. To give a lower bound for the hole probability we will define some events that force f_L to have no zeroes in the disc $D(0, r)$, and compute the probability of these events. Trivially we have

$$|f_L(z)| \geq |a_0| - \left| \sum_{n=1}^{N} a_n \binom{L+n-1}{n}^{1/2} z^n \right| - \left| \sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n \right|.$$

We will choose an appropriate N, and three independent events defined in terms of the coefficients a_n, that force $|a_0|$ to be 'large' and the remaining two terms to be 'small' when $z \in D(0, r)$. This will imply that $|f_L(z)| > 0$ and so we can bound the hole probability below by the probability of these events.

(a) Define $N = [c_0 L]$ where c_0 is a constant to be chosen. Stirling's approximation yields, for $n > N$,

$$\frac{\Gamma(L+n)}{\Gamma(n+1)\Gamma(L)} \simeq \frac{1}{\sqrt{n}} \left(1 + \frac{n}{L}\right)^L \left(1 + \frac{L}{n}\right)^n.$$

Thus, for $|z| \le r$, we have

$$\left|\sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n\right| \lesssim \sum_{n>N} \frac{|a_n|}{n^{1/4}} \left(\left(1+\frac{n}{L}\right)^{L/n}\left(1+\frac{L}{n}\right) r^2\right)^{n/2}.$$

Now, for $n > N$

$$\left(1+\frac{n}{L}\right)^{L/n}\left(1+\frac{L}{n}\right) \le (1+c_0)^{1/c_0}\left(1+\frac{1}{c_0}\right)$$

which decreases to 1 as $c_0 \to \infty$, and so we can choose c_0 depending on r such that

$$(1+c_0)^{1/c_0}\left(1+\frac{1}{c_0}\right) r^2 \le (1-\delta)^2$$

for some $\delta > 0$. We define the event

$$\mathcal{E}_1 = \{|a_n| \le n^{1/4} : n > N\}$$

and we see that $\mathcal{E}_1$ implies that for $|z| \le r$

$$\left|\sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n\right| \lesssim \sum_{n>N} (1-\delta)^n \le \frac{1}{2}$$

for N sufficiently large.

We next define the event

$$\mathcal{E}_2 = \left\{|a_n|^2 \le (1-r^2)^L \frac{1}{4N} : 1 \le n \le N\right\}.$$

Now $\mathcal{E}_2$ implies that for $|z| \le r$

$$\left|\sum_{n=1}^{N} a_n \binom{L+n-1}{n}^{1/2} z^n\right|^2 \le \sum_{n=1}^{N} |a_n|^2 \sum_{n=0}^{\infty} \binom{L+n-1}{n} r^{2n} \le \frac{1}{4}(1-r^2)^L (1-r^2)^{-L} = \frac{1}{4}$$

We finally define the event

$$\mathcal{E}_3 = \{|a_0| > 1\}.$$

Thus the intersection of the (independent) events $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ implies that $|f_L(z)| > 0$ for $|z| \le r$ and so

$$\mathbb{P}[n_L(r) = 0] \ge \mathbb{P}[\mathcal{E}_1]\mathbb{P}[\mathcal{E}_2]\mathbb{P}[\mathcal{E}_3].$$

Now

$$\mathbb{P}[\mathcal{E}_3] = \frac{1}{e}$$

and

$$\mathbb{P}[\mathcal{E}_1] = \prod_{n>N} 1 - e^{-\sqrt{n}} \ge \frac{1}{2}$$

for N sufficiently large. Finally, applying the elementary estimate $1 - e^{-x} \geq x/2$ for $x \leq 1$, we have

$$\begin{aligned}\mathbb{P}[\mathcal{E}_2] &= \left(1 - \exp\left(-(1-r^2)^L \frac{1}{4N}\right)\right)^N \\ &\geq \left((1-r^2)^L \frac{1}{8N}\right)^N = e^{NL\log(1-r^2) - N\log(8N)} \geq e^{-CL^2}\end{aligned}$$

for L sufficiently large, since $N = [c_0 L]$.

(b) Define $N = [c_0 L r^2]$ where $c_0 > e$. Once more applying Stirling's approximation we see that for $|z| \leq r$ we have

$$\left|\sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n\right| \lesssim \sum_{n>N} \frac{|a_n|}{n^{1/4}} \left(\left(1+\frac{n}{L}\right)^{L/n}\left(1+\frac{L}{n}\right) r^2\right)^{n/2}.$$

Now, for $n > N$

$$\left(1+\frac{n}{L}\right)^{L/n}\left(1+\frac{L}{n}\right) r^2 \leq (1+c_0 r^2)^{1/c_0 r^2}\left(1+\frac{1}{c_0 r^2}\right) r^2$$

and for r sufficiently close to zero we have

$$(1+c_0 r^2)^{1/c_0 r^2}\left(1+\frac{1}{c_0 r^2}\right) r^2 \leq e\left(r^2 + \frac{1}{c_0}\right) \leq (1-\delta)^2$$

for some $\delta > 0$. We define the event

$$\mathcal{A}_1 = \{|a_n| \leq n^{1/4} : n > N\}$$

and we see that $\mathcal{A}_1$ implies that for $|z| \leq r$

$$\left|\sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n\right| \lesssim \sum_{n>N} (1-\delta)^n \leq \frac{1}{2}$$

for N sufficiently large.

We next define the event

$$\mathcal{A}_2 = \left\{|a_n| \leq \frac{1}{3\sqrt{L}r}\binom{L+n-1}{n}^{-1/2} r^{-n} : 1 \leq n \leq N\right\}.$$

Now $\mathcal{A}_2$ implies that for $|z| \leq r$

$$\left|\sum_{n=1}^{N} a_n \binom{L+n-1}{n}^{1/2} z^n\right| \leq \frac{N}{3\sqrt{L}r} \leq \sqrt{L}r$$

We finally define the event

$$\mathcal{A}_3 = \{|a_0| \geq 2\sqrt{L}r\}.$$

Thus the intersection of the (independent) events $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ implies that $|f_L(z)| > 0$ for $|z| \leq r$ and so

$$\mathbb{P}[n_L(r) = 0] \geq \mathbb{P}[\mathcal{A}_1]\mathbb{P}[\mathcal{A}_2]\mathbb{P}[\mathcal{A}_3].$$

Now

$$\mathbb{P}[\mathcal{A}_3] = e^{-4Lr^2}$$

and

$$\mathbb{P}[\mathcal{A}_1] = \prod_{n>N} 1 - e^{-\sqrt{n}} \geq \frac{1}{2}$$

for N sufficiently large. Finally

$$\mathbb{P}[\mathcal{A}_2] = \prod_{n=1}^{N} \left(1 - \exp\left(-\frac{1}{9Lr^2}\binom{L+n-1}{n}^{-1} r^{-2n}\right)\right)$$

A final application of Stirling's formula gives

$$\frac{1}{9Lr^2}\binom{L+n-1}{n}^{-1} r^{-2n} \leq \frac{1}{9Lr^2}\frac{n!}{(Lr^2)^n} e^{-(c_1 n^2/L)} \leq 1$$

for $1 \leq n \leq N$ and some $c_1 > 0$, where the last estimate follows from [Nis10, Lemma 2.2]. Applying once more the estimate $1 - e^{-x} \geq x/2$ for $x \leq 1$ we have

$$\begin{aligned}\mathbb{P}[\mathcal{A}_3] &\geq \prod_{n=1}^{N}\left(\frac{1}{18Lr^2}\frac{n!}{(Lr^2)^n} e^{-(c_1 n^2/L)}\right) \\ &= \exp\left(-\sum_{n=1}^{N} \log\frac{(Lr^2)^n}{n!} + O(N\log Lr^2) + O(N^3/L)\right)\end{aligned}$$

As in [Nis10, Lemma 2.3] we compute

$$\sum_{n=1}^{N} \log\frac{(Lr^2)^n}{n!} = e^2(\frac{3}{4} - \frac{1}{2}\log c_0)L^2 r^4 + O(N\log N).$$

Now $N^3/L \simeq r^2 L^2 r^4 = o(1) L^2 r^4$ for small r. Choosing c_0 sufficiently close to e and then Lr^2 large enough completes the proof. ∎

Large deviations

We now turn to the large deviations. We first recall that we write

$$\hat{f}_L(z) = \frac{f_L(z)}{\sqrt{K_L(z,z)}} = f_L(z)(1 - |z|^2)^{L/2},$$

and that the distribution of $\log|\hat{f}_L|$ is invariant under automorphisms of the disc.

We define, for $z, w \in \mathbb{D}$, the hyperbolic distance by $d(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|$ and the hyperbolic discs by $B(z,r) = \{w \in \mathbb{D} : d(z,w) < r\}$ for $0 < r < 1$.

The key ingredient in the proof of Theorem 5.7 is the following lemma.

Lemma 5.11. *(a) Given $0 < R < 1$ and $\delta > 0$ there exist $c = c(\delta)$ and $L_0 = L_0(R,\delta)$ such that, for all $L > L_0$,*

$$\frac{1}{\nu(D(0,R))}\int_{D(0,R)} \left|\log|\hat{f}_L(z)|\right| \, d\nu(z) \le \delta L$$

outside an exceptional set of probability at most e^{-cL^2}.

(b) For any $c_1 > c_0 > 0$ there exist $\epsilon = \epsilon(c_0, c_1)$, $M_0 = M_0(c_0,c_1)$ and $c = c(c_0,c_1)$ such that for all $R < \epsilon$, $LR^2 > M_0$ and $c_0R^2 < \delta < c_1R^2$

$$\frac{1}{\nu(D(0,R))}\int_{D(0,R)} \left|\log|\hat{f}_L(z)|\right| \, d\nu(z) \le \delta L$$

outside an exceptional set of probability at most $e^{-cL^2R^4}$.

We begin with the following.

Lemma 5.12. *(a) Given $\delta > 0$ and $0 < r < 1$ there exists $L_0 = L_0(r,\delta)$ such that for all for $L \ge L_0$ and $z_0 \in \mathbb{D}$*

$$\max_{z\in\overline{B(z_0,r)}} \log|\hat{f}_L(z)| \le \delta L \log\frac{1}{1-r^2}$$

outside an exceptional set of probability at most $\exp(-(1-r^2)^{-\delta L})$.

(b) Given $\delta > 0$ there exists $\epsilon = \epsilon(\delta)$ and $M_0 = M_0(\delta)$ such that for all $r < \epsilon$ and $Lr^2 \ge M_0$

$$\max_{z\in\overline{B(z_0,r)}} \log|\hat{f}_L(z)| \le \delta L \log\frac{1}{1-r^2}$$

outside an exceptional set of probability at most $\exp(-e^{-\delta Lr^2})$.

Proof. We first note that, by invariance, we have

$$\begin{aligned}
\mathbb{P}\left[\max_{z\in\overline{B(z_0,r)}} \log|\hat{f}_L(z)| > \delta L\log\frac{1}{1-r^2}\right] &= \mathbb{P}\left[\max_{z\in\overline{D(0,r)}} \log|\hat{f}_L(z)| > \delta L\log\frac{1}{1-r^2}\right]\\
&\le \mathbb{P}\left[\max_{|z|=r} \log|\hat{f}_L(z)| > \delta L\log\frac{1}{1-r^2}\right]\\
&= \mathbb{P}\left[\max_{|z|=r} \log|f_L(z)| > \left(\frac{1}{2}+\delta\right) L\log\frac{1}{1-r^2}\right].
\end{aligned}$$

It therefore suffices to consider this last event.

(a) Chose $\alpha > 0$ such that $r < e^{-\alpha} < 1$ and define $N = [c_0 \delta L]$. As in the proof of the upper bound in Theorem 5.10, by choosing c_0 sufficiently large (depending on r and δ), we have

$$\binom{L+n-1}{n}^{1/2} r^n \leq e^{-\alpha n}$$

for all $n \geq N$. We further suppose that $c_0 > \frac{1}{\alpha} \log \frac{1}{1-r^2}$ and define the event

$$\mathcal{E}_1 = \{|a_n| \leq e^{\alpha n/2} : n > N\}.$$

Thus $\mathcal{E}_1$ implies that for $|z| = r$ we have

$$\left| \sum_{n>N} a_n \binom{L+n-1}{n}^{1/2} z^n \right| \lesssim \sum_{n>N} e^{-\alpha n/2} \simeq e^{-\alpha N/2} \leq (1-r^2)^{-L/2}$$

by our choice of c_0.

We now define the event

$$\mathcal{E}_2 = \{|a_n| \leq (1-r^2)^{-2\delta L/3} : 0 \leq n \leq N\}$$

and see that $\mathcal{E}_2$ implies that for $|z| = r$ we have

$$\begin{aligned} \left| \sum_{n=0}^{N} a_n \binom{L+n-1}{n}^{1/2} z^n \right| &\leq \left(\sum_{n=0}^{N} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \binom{L+n-1}{n} r^{2n} \right)^{1/2} \\ &\leq (1-r^2)^{-2\delta L/3} \sqrt{N} (1-r^2)^{-L/2}. \end{aligned}$$

Thus the intersection of the (independent) events $\mathcal{E}_1$ and $\mathcal{E}_2$ implies that for $|z| = r$ and some constant c_1

$$|f_L(z)| \leq (1-r^2)^{-2\delta L/3} \sqrt{N} (1-r^2)^{-L/2} + c_1 (1-r^2)^{-L/2} \leq (1-r^2)^{-L(\frac{1}{2}+\delta)}$$

for L sufficiently large.

It remains only to compute the corresponding probabilities; for large L

$$\log \mathbb{P}[\mathcal{E}_2] \geq -\frac{N}{2} \exp(-(1-r^2)^{-4\delta L/3}) \geq -\frac{1}{2} \exp(-(1-r^2)^{-\delta L})$$

and

$$\log \mathbb{P}[\mathcal{E}_1] \geq -\frac{1}{2} \sum_{n>N} \exp(-e^{\alpha n}) \gtrsim -\exp(-e^{\alpha N}) \geq -\frac{1}{2} \exp\left(-(1-r^2)^{-\delta L}\right)$$

by our choice of c_0. Thus

$$\log \mathbb{P}[\mathcal{E}_1] + \log \mathbb{P}[\mathcal{E}_2] \geq -\exp(-(1-r^2)^{-\delta L} \geq \log\left(1 - \exp(-(1-r^2)^{-\delta L})\right)$$

(b) Define $N = [c_0 \delta L r^2]$ where $c_0 > e/\delta$. As in the proof of the upper bound in Theorem 5.10 we have

$$\binom{L+n-1}{n}^{1/2} r^n \leq e^{-\alpha n}$$

for all $n \geq N$ and for some $\alpha > 0$, for r sufficiently small. Furthermore, we may choose $c_0 > 1/\alpha$. We define the events

$$\mathcal{A}_1 = \{|a_n| \leq e^{\alpha n/2} : n > N\}.$$

and

$$\mathcal{A}_2 = \{|a_n| \leq (1-r^2)^{-2\delta L/3} : 0 \leq n \leq N\}$$

which by identical calculations to in part (a) imply that

$$|f_L(z)| \leq (1-r^2)^{-L(\frac{1}{2}+\delta)} \leq e^{Lr^2(\frac{1}{2}+\delta)}$$

for $|z| = r$ and Lr^2 sufficiently large. Identical calculations to part (a) show that we have the claimed lower bound on the probability of these events. ■

Lemma 5.13. *(a) Given $\delta > 0$ and $0 < r < 1$ there exist $L_0 = L_0(r, \delta)$ and $c = c(r, \delta)$ such that for all for $L \geq L_0$ and any $z_0 \in \mathbb{D}$*

$$\max_{z \in B(z_0, r)} \log |\hat{f}_L(z)| \geq -\delta L \log \frac{1}{1-r^2}$$

outside an exceptional set of probability at most e^{-cL^2}.

(b) Given $\delta > 0$ there exists $\epsilon = \epsilon(\delta)$ and $M_0 = M_0(\delta)$ such that for all $r < \epsilon$ and $Lr^2 \geq M_0$, and any $z_0 \in \mathbb{D}$

$$\max_{z \in B(z_0, r)} \log |\hat{f}_L(z)| \geq -\delta L \log \frac{1}{1-r^2}$$

outside an exceptional set of probability at most $e^{-\delta L^2 r^4}$.

Proof. By invariance, it is enough to consider $z_0 = 0$. Define $z_j = re^{2\pi i j/N}$ for $0 \leq j \leq N-1$, where N is an integer to be chosen (large). We have

$$\begin{aligned}\mathbb{P}\left[\max_{|z|\leq r} \log |\hat{f}_L(z)| \leq -\delta L \log \frac{1}{1-r^2}\right] &= \mathbb{P}\left[\max_{|z|\leq r} |\hat{f}_L(z)| \leq (1-r^2)^{\delta L}\right] \\ &\leq \mathbb{P}\left[|\hat{f}_L(z_j)| \leq (1-r^2)^{\delta L} \text{ for all } 0 \leq j \leq N-1\right]\end{aligned}$$

and we now estimate the probability of this event. Consider the vector

$$\xi = \begin{pmatrix} \hat{f}_L(z_1) \\ \vdots \\ \hat{f}_L(z_N) \end{pmatrix}$$

which is a mean-zero N-dimensional complex normal with covariance matrix $\sigma = (\sigma_{jk})_{j,k=1}^N$ given by

$$\sigma_{jk} = \frac{(1-r^2)^L}{(1-r^2 e^{2\pi i(j-k)/N})^L}.$$

Note that σ is a circulant matrix, that is, each row of σ is a cyclic permutation of the first row. Writing the Nth roots of unity $\omega_m = e^{2\pi i m/N}$ for $0 \le m \le N-1$, it is easy to see that the vectors

$$v_m = \begin{pmatrix} 1 \\ \omega_m \\ \omega_m^2 \\ \vdots \\ \omega_m^{N-1} \end{pmatrix}$$

are eigenvectors of σ with corresponding eigenvalues

$$\lambda_m(\sigma) = \sum_{j=0}^{N-1} \sigma_{0j}\omega_m^j$$

for $0 \le m \le N-1$. We compute

$$\begin{aligned}
\lambda_m(\sigma) &= \sum_{j=0}^{N-1} \sigma_{0j}\omega_m^j \\
&= (1-r^2)^L \sum_{j=0}^{N-1} (1-r^2 e^{2\pi i(-j)/N})^{-L}\omega_m^j \\
&= (1-r^2)^L \sum_{j=0}^{N-1}\sum_{n=0}^{\infty} \binom{L+n-1}{n} r^{2n} e^{-2\pi i j n/N}\omega_m^j \\
&= (1-r^2)^L \sum_{n=0}^{\infty} \binom{L+n-1}{n} r^{2n} \sum_{j=0}^{N-1} e^{2\pi i(m-n)j/N} \\
&= N(1-r^2)^L \sum_{n\equiv m \bmod N} \binom{L+n-1}{n} r^{2n}.
\end{aligned}$$

Now since $\binom{L+n}{n+1} = \frac{L+n}{n+1}\binom{L+n-1}{n}$ we see that $\binom{L+n-1}{n} r^{2n}$ has a maximum at $n = [\frac{(L-1)r^2}{1-r^2}]$ (which gets arbitrarily large under our hypotheses) and so we choose $N = [\epsilon\frac{(L-1)r^2}{1-r^2}]$, where ϵ is to be chosen small. Then, for each m, there is at least one $n \equiv m \bmod N$ satisfying

$$(1-\epsilon)\frac{(L-1)r^2}{1-r^2} < n < \frac{(L-1)r^2}{1-r^2}. \tag{5.4}$$

For such n we have, applying Stirling's approximation again,

$$\binom{L+n-1}{n} r^{2n} \simeq n^{-1/2}\sqrt{\frac{L}{L+n}}\left(1+\frac{n}{L}\right)^L\left(1+\frac{L}{n}\right)^n r^{2n}.$$

Now

$$\frac{n}{L} \leq (1 - L^{-1})\frac{r^2}{1-r^2} \leq \frac{r^2}{1-r^2}$$

which implies that

$$\left(1 + \frac{L}{n}\right)^n r^{2n} \geq 1.$$

Moreover

$$\begin{aligned}
\left(1 + \frac{n}{L}\right)^L &\geq \left(1 + (1-\epsilon)(1-L^{-1})\frac{r^2}{1-r^2}\right)^L \\
&= \frac{(1 - \epsilon r^2 - L^{-1}r^2 + \epsilon L^{-1} r^2)^L}{(1-r^2)^L} \\
&\geq \frac{(1-2\epsilon r^2)^L}{(1-r^2)^L}
\end{aligned}$$

for sufficiently large L. Finally it is easy to see that for n satisfying (5.4),

$$\frac{L}{L+n} \geq 1 - r^2.$$

Combining all of these estimates we have

$$\lambda_m(\sigma) \gtrsim \sqrt{\epsilon N(1-r^2)}(1-2\epsilon r^2)^L$$

and so for any B such that $BB^* = \sigma$ then

$$\|B^{-1}\|_2 \leq (\epsilon N(1-r^2))^{-1/4}(1-2\epsilon r^2)^{-L/2}.$$

Now the components of the vector $\zeta = B^{-1}\xi$ are iid $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables, which we denote ζ_j, and moreover

$$\|\zeta\|_\infty \leq \|B^{-1}\xi\|_2 \leq \|B^{-1}\|_2\|\xi\|_2 \leq \|B^{-1}\|_2\sqrt{N}\|\xi\|_\infty \leq \left(\frac{N}{\epsilon(1-r^2)}\right)^{1/4}(1-2\epsilon r^2)^{-L/2}\|\xi\|_\infty.$$

This means that

$$\begin{aligned}
&\mathbb{P}\left[|\hat{f}_L(z_j)| \leq (1-r^2)^{\delta L} \text{ for all } 0 \leq j \leq N-1\right] \\
&\quad\leq \mathbb{P}\left[|\zeta_j| \leq \left(\frac{N}{\epsilon(1-r^2)}\right)^{1/4}(1-2\epsilon r^2)^{-L/2}(1-r^2)^{\delta L} \text{ for all } 1 \leq j \leq N\right] \\
&\quad= \left(1 - \exp\Big(-\left(\frac{N}{\epsilon(1-r^2)}\right)^{1/2}(1-2\epsilon r^2)^{-L}(1-r^2)^{2\delta L}\Big)\right)^N.
\end{aligned}$$

If r is fixed we choose ϵ so small that $1 - 2\epsilon r^2 > (1-r^2)^{2\delta}$. This yields (a). If r is small enough we choose $\epsilon < \delta/2$. This yields (b). ■

Lemma 5.14. *There exist $\epsilon, M_0 > 0$ such that, for any $z_0 \in \mathbb{D}$ and some absolute constant c, we have, for all $r < \epsilon$ and $Lr^2 \geq M_0$,*

$$\frac{1}{\nu(B(z_0,r))}\int_{B(z_0,r)} \left|\log|\hat{f}_L|\right| \, d\nu \leq cLr^2$$

outside an exceptional set of probability at most $e^{-L^2r^4/2}$.

Proof. By invariance, we may assume that $z_0 = 0$. We first note that for any $\zeta \in \mathbb{D}$ and $0 < \rho < 1$

$$\int_{B(\zeta,\rho)} \log\frac{1}{1-|z|^2}\, d\nu(z) = \int_{D(0,\rho)} \log\frac{|1+\overline{\zeta}w|^2}{(1-|\zeta|^2)(1-|w|^2)}\, d\nu(w).$$

Now since $w \mapsto \log|1+\overline{\zeta}w|^2$ is harmonic on $\mathbb{D}$ we have

$$\int_{D(0,\rho)} \log|1+\overline{\zeta}w|^2\, d\nu(w) = 0$$

and so

$$\begin{aligned}\int_{B(\zeta,\rho)} \log\frac{1}{1-|z|^2}\, d\nu(z) &= \int_{D(0,\rho)} \log\frac{1}{(1-|\zeta|^2)(1-|w|^2)}\, d\nu(w)\\ &= \nu(B(\zeta,\rho))\left(\log\frac{1}{1-|\zeta|^2} + \frac{1}{\rho^2}\log\frac{1}{1-\rho^2} - 1\right).\end{aligned}$$

Equivalently

$$\begin{aligned}\log\frac{1}{1-|\zeta|^2} &= \frac{1}{\nu(B(\zeta,\rho))}\int_{B(\zeta,\rho)} \log\frac{1}{1-|z|^2}\, d\nu(z) + 1 - \frac{1}{\rho^2}\log\frac{1}{1-\rho^2}\\ &\geq \frac{1}{\nu(B(\zeta,\rho))}\int_{B(\zeta,\rho)} \log\frac{1}{1-|z|^2}\, d\nu(z) - \frac{\rho^2}{2}. \qquad (5.5)\end{aligned}$$

Now Lemma 5.13 implies that, outside of an exceptional set, there exists $\zeta \in B(0, r/2)$ such that

$$\log|\hat{f}_L(\zeta)| \geq -L\log\frac{1}{1-r^2/4} \geq -Lr^2/4.$$

The subharmonicity of $\log|f_L|$ implies that

$$\log|f_L(\zeta)| \leq \frac{1}{\nu(B(\zeta,\rho))}\int_{B(\zeta,\rho)} \log|f_L(z)|\, d\nu(z)$$

which combined with (5.5) yields

$$\log|\hat{f}_L(\zeta)| \leq \frac{1}{\nu(B(\zeta,\rho))}\int_{B(\zeta,\rho)} \log|\hat{f}_L(z)|\, d\nu(z) + L\rho^2/4.$$

We therefore have

$$0 \le \frac{1}{\nu(B(\zeta,\rho))} \int_{B(\zeta,\rho)} \log|\hat{f}_L(z)|\, d\nu(z) + \frac{L}{4}\left(r^2+\rho^2\right)$$

and so

$$\frac{1}{\nu(B(\zeta,\rho))} \int_{B(\zeta,\rho)} \log^-|\hat{f}_L(z)|\, d\nu(z) \le \frac{1}{\nu(B(\zeta,\rho))} \int_{B(\zeta,\rho)} \log^+|\hat{f}_L(z)|\, d\nu(z) + \frac{L}{4}\left(r^2+\rho^2\right)$$

which means that

$$\frac{1}{\nu(B(\zeta,\rho))} \int_{B(\zeta,\rho)} \Big|\log|\hat{f}_L(z)|\Big|\, d\nu(z) \le \frac{2}{\nu(B(\zeta,\rho))} \int_{B(\zeta,\rho)} \log^+|\hat{f}_L(z)|\, d\nu(z) + \frac{L}{4}\left(r^2+\rho^2\right).$$

Finally we see by Lemma 5.12 that, for $z \in B(\zeta,\rho)$ we have $\log^+|\hat{f}_L(z)| \le L\log\frac{1}{1-\rho^2} \le L\rho^2$ outside of another exceptional set. Thus, choosing $\rho = 3r/2$,

$$\int_{B(\zeta,\rho)} \Big|\log|\hat{f}_L(z)|\Big|\, d\nu(z) \le L\nu(B(\zeta,\rho))\left(2\rho^2 + \frac{r^2}{4} + \frac{\rho^2}{4}\right) \lesssim Lr^2\nu(D(0,r)).$$

Since $B(0,r) \subseteq B(\zeta,\rho)$ we are done. ■

Proof of Lemma 5.11. (a) It suffices to only consider small values of δ. We cover $D(0,R)$ with discs $(B(z_j,r))_{j=1}^N$ such that $z_j \in D(0,R)$ and $\nu(B(z_j,r)) = \frac{r^2}{1-r^2} = \delta$. The Vitali covering lemma implies that we may assume that

$$N \lesssim \nu\left(D\left(0, \frac{R+r}{1+rR}\right)\right)/\delta \lesssim \frac{R^2}{\delta(1-R^2)} = \nu(D(0,R))/\delta.$$

Now, applying Lemma 5.14, we see that outside of an exceptional set of probability at most $Ne^{-L^2r^4/2} \le e^{-cL^2}$ we have

$$\int_{D(0,R)} \Big|\log|\hat{f}_L|\Big|\, d\nu \le \sum_{j=1}^N \int_{B(z_j,r)} \Big|\log|\hat{f}_L|\Big|\, d\nu \lesssim NLr^2\nu(B(z_j,r)) \lesssim \delta L\nu(D(0,R)).$$

Appropriately changing the value of δ completes the proof.

(b) We may use the same proof as in part (a), and note that $Ne^{-L^2r^4/2} \le e^{-cL^2R^4}$ so that the exceptional set has the claimed behaviour. ■

Proof of Theorem 5.7. We first note that, by the Edelman-Kostlan formula,

$$\begin{aligned}\Big|\int \psi\, dn_L - L\int \psi\, d\nu\Big| &= \frac{1}{2\pi}\Big|\int \Delta\psi \log|\hat{f}_L|\, dm\Big| \\ &\le \frac{1}{2\pi}\max_{z\in\mathbb{D}}|\Delta\psi(z)| \int_{\operatorname{supp}\psi} \Big|\log|\hat{f}_L|\Big|\, dm.\end{aligned}$$

We choose $R < 1$ such that $\operatorname{supp}\psi \subset D(0,R)$, and note that Lemma 5.11 implies that for any $\delta' > 0$

$$\int_{\operatorname{supp}\psi} \left|\log|\hat{f}_L|\right| \, dm \le \int_{D(0,R)} \left|\log|\hat{f}_L|\right| \, d\nu \le \delta' L\nu(D(0,R))$$

outside an exceptional set of probability at most $e^{-c(\delta')L^2}$. Choosing

$$\delta' = 2\pi\delta \left|\int \psi \, d\nu\right| \|\Delta\psi\|_\infty^{-1} \nu(D(0,R))^{-1}$$

we see that

$$\left|\int \psi \, dn_L - L\int \psi \, d\nu\right| \le \delta L \left|\int \psi \, d\nu\right|$$

outside an exceptional set, as claimed. ■

Proof of Corollary 5.8. (a) It suffices to show this for sufficiently small δ. Let $0 < \delta < 1$ and choose smooth, compactly supported ψ_1 and ψ_2 satisfying

$$0 \le \psi_1 \le \chi_U \le \psi_2 \le 1,$$

$$\int \psi_1 \, d\nu \ge \nu(U)(1-\delta)$$

and

$$\int \psi_2 \, d\nu \le \nu(U)(1+\delta).$$

Outside an exceptional set of probability e^{-cL^2} we have, by Theorem 5.7,

$$\int \psi_2 \, dn_L \le (1+\delta)L\int \psi_2 \, d\nu.$$

We see that

$$n_L(U) \le \int \psi_2 \, dn_L \le (1+\delta)L\int \psi_2 \, d\nu \le (1+\delta)^2 L\nu(U)$$

whence

$$\frac{n_L(U)}{L\nu(U)} - 1 \lesssim \delta.$$

Similarly, using ψ_1, we see that

$$\frac{n_L(U)}{L\nu(U)} - 1 \gtrsim -\delta.$$

outside another exceptional set of probability e^{-cL^2}, which after appropriately changing the value of δ completes the proof of (a).

(b) It is clear that we may use the same proof, we merely have to take care of the constants in the exceptional sets. We may choose ψ_1 such that $\psi_1 \equiv 1$ on $D(0,r_1)$ where

$$\nu(D(0,r_1)) = (1-\delta)\nu(D(0,r))$$

and moreover $\|\Delta\psi_1\|_\infty \simeq (r-r_1)^{-2} \simeq (\delta r)^{-2}$. Thus, in the proof of Theorem 5.7, we have

$$\delta' = 2\pi\delta\frac{1-r^2}{r^2}\left|\int \psi_1\, d\nu\right| / \|\Delta\psi_1\|_\infty \simeq \delta^3 r^2,$$

that is, we have

$$c_0(\delta)r^2 \leq \delta' \leq c_1(\delta)r^2.$$

We thus see that the exceptional set corresponding to ψ_1 has probability at most $e^{-cL^2r^4}$, where we have used Lemma 5.11 (b). Similarly we may choose ψ_2 to be supported in the disc $D(0, r_2)$ where $\nu(D(0,r_2)) = (1+\delta)\nu(D(0,r)$ and $\|\Delta\psi_2\|_\infty \simeq (r-r_2)^{-2} \simeq (\delta r)^{-2}$. This completes the proof. ■

The upper bound in Theorem 5.10 (b)

We finally turn to the upper bound in Theorem 5.10 (b). We are going to follow the ideas from [Nis10, Section 4] and so we introduce the notation

$$S_L(r) = \sum_{n=1}^{N} \log\left(\binom{L+n-1}{n} r^{2n}\right)$$

where $N = [eLr^2]$. We have already seen that

$$S_L(r) = \frac{e^2}{4}L^2r^4(1+o(1))$$

as $r \to 0$ and $Lr^2 \to \infty$. We begin with the following elementary observation.

Lemma 5.15. *We have*

$$\mathbb{P}[n_L(r) = 0] \leq \mathbb{P}\left[\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} \leq 2\log(Lr^2)\right] + e^{-L^4r^8}.$$

Proof. By Jensen's formula, if there are no zeroes in the disc of radius r, then

$$\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} = \log|f_L(0)| = \log|a_0|.$$

Now

$$\mathbb{P}[\log|a_0| > 2\log(Lr^2)] = \mathbb{P}[|a_0| > L^2r^4] = e^{-L^4r^8}.$$ ■

We can therefore restrict ourselves to estimating $\mathbb{P}[\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} \leq 2\log(Lr^2)]$. We begin with the following lemma.

Lemma 5.16. *We have*

$$\max_{|z|\le r} |f_L(z)| \ge 1$$

outside an exceptional set of probability at most $e^{-S_L(r)}$.

Remark. This lemma holds for all posible values of L and r, not just the regime $r \to 0$ and $Lr^2 \to \infty$.

Proof. Suppose that $|f_L(z)| \le 1$ for $|z| \le r$. Then Cauchy's estimate yields

$$|a_n| \binom{L+n-1}{n}^{1/2} r^n \le 1.$$

We may therefore bound the probability of this event by

$$\begin{aligned}\prod_{n=0}^{N} \mathbb{P}\left[|a_n| \le \binom{L+n-1}{n}^{-1/2} r^{-n}\right] &= \prod_{n=0}^{N} \left(1 - \exp\left(\left(\binom{L+n-1}{n}^{-1} r^{-2n}\right)\right)\right) \\ &\le \prod_{n=0}^{N} \binom{L+n-1}{n}^{-1} r^{-2n} \\ &= e^{-S_L(r)}. \end{aligned}$$

∎

We next approximate $\int_0^{2\pi} \log|f_L(re^{i\theta})| \frac{d\theta}{2\pi}$ by a sum (with high probability). This discretisation will allow us to estimate the probability we are interested in by considering a multivariate complex normal vector. We fix $0 < \delta < 1$, write $\kappa = 1 - \delta^{1/2}$ and

$$z_j = \kappa r \exp\left(\frac{2\pi i j}{N}\right)$$

for $1 \le j \le N$.

Lemma 5.17. *There exists an absolute constant C such that*

$$\frac{1}{N}\sum_{j=0}^{N} \log|f_L(z_j)| \le \int_0^{2\pi} \log|f_L(re^{i\theta})| \frac{d\theta}{2\pi} + \frac{C}{\delta^2}$$

outside an exceptional set of probability at most $2e^{-S_L(\kappa r)}$.

The proof of this lemma is omitted, it is identical to the proof of [Nis10, Lemma 4.3], replacing the estimates from [Nis10, Lemma 4.2] and [Nis10, Lemma 4.1] by corresponding estimates from Lemma 5.16 and Lemma 5.12 (b) respectively.

We define

$$\zeta = \begin{pmatrix} f_L(z_1) \\ \vdots \\ f_L(z_N) \end{pmatrix}$$

which is a mean-zero N-dimensional complex normal with covariance matrix $\sigma = (\sigma_{jk})_{j,k=1}^N$ given by

$$\sigma_{jk} = \frac{1}{(1-\zeta_j\overline{\zeta_k})^L}.$$

We consider the event

$$\mathcal{A}' = \left\{\zeta : \prod_{j=1}^N |\zeta_j| \le \exp\left(2N\log(Lr^2) + Ce\frac{Lr^2}{\delta^2}\right)\right\}$$

where C is the constant appearing in Lemma 5.17 and ζ_j are the entries in the vector ζ. We have the following estimate for the event that we are interested in.

Lemma 5.18. *We have*

$$\mathbb{P}\left[\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} \le 2\log(Lr^2)\right] \le \mathbb{P}[\mathcal{A}'] + 2e^{-S_L(\kappa r)}$$

Proof. Let $\mathcal{E}$ be the event whose probability we are estimating, and let $\mathcal{F}$ be the exceptional set of probability at most $2e^{-S_L(\kappa r)}$ from Lemma 5.17. We have, outside of $\mathcal{F}$,

$$\sum_{j=0}^N \log|f_L(z_j)| \le N\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} + \frac{CN}{\delta^2}$$

or equivalently

$$\prod_{j=0}^N |\zeta_j| \le \exp\left(N\int_0^{2\pi} \log|f_L(re^{i\theta})|\frac{d\theta}{2\pi} + \frac{CeLr^2}{\delta^2}\right).$$

Thus $\mathcal{E}\setminus\mathcal{F}$ is contained in $\mathcal{A}'$. This gives the result. ∎

We now estimate $\mathbb{P}[\mathcal{A}']$ by considering the event

$$\mathcal{A} = \left\{\zeta \in \mathcal{A}' : |\zeta_j| \le \exp(2Lr^2) \text{ for all } 1 \le j \le N\right\}.$$

and separately estimating the probability of the events $\mathcal{A}$ and $\mathcal{A}'\setminus\mathcal{A}$. We begin with the easier of the two.

Lemma 5.19. *We have*

$$\mathbb{P}[\mathcal{A}'\setminus\mathcal{A}] \le \exp(-e^{Lr^2}).$$

Proof. Applying Lemma 5.12 (b) we see that

$$1-\mathbb{P}[\mathcal{A}] = \mathbb{P}[|\zeta_j| > \exp(2Lr^2) \text{ for some } 1\le j\le N] \le N\exp(-e^{3Lr^2}) \le \exp(-e^{Lr^2})$$

and since

$$\mathbb{P}[\mathcal{A}'\setminus\mathcal{A}] \le 1-\mathbb{P}[\mathcal{A}]$$

we are done. ∎

To estimate $\mathbb{P}[\mathcal{A}]$ we require the following two results.

Lemma 5.20 ([Nis10, Corollary 4.7])**.** *For all $0 < \alpha < 1$ there exists $C_1 > 0$ such that for all $\delta \geq (Lr^2)^{-\alpha}$ and Lr^2 sufficiently large we have*

$$\pi^{-N} \operatorname{vol}_{\mathbb{C}^N}(\mathcal{A}) \leq \exp\left\{C_1 Lr^2(\log(Lr^2) + \delta^{-2})\right\}.$$

Remarks. 1. We are abusing notation slightly here by using $\mathcal{A}$ to refer to an event, and the set it defines in $\mathbb{C}^N$.

2. The large parameter in [Nis10, Corollary 4.7] is r^2, which should be replaced by Lr^2 throughout to get the result we have just stated.

Lemma 5.21. *If σ is the covariance matrix defined earlier then*

$$\det\sigma \geq e^{S_L(\kappa r)}.$$

Proof. We follow the proof of [Nis10, Lemma 4.5]. We first see that

$$\sigma = VV^*$$

where

$$V = \begin{pmatrix} b_0 & b_1 z_1 & \cdots & b_N z_1^N & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ b_0 & b_1 z_N & \cdots & b_N z_N^N & \cdots \end{pmatrix}$$

and $b_j = \binom{L+j-1}{j}^{1/2}$. With this notation

$$e^{S_L(\kappa r)} = \prod_{j=1}^{N} b_j^2 (\kappa r)^{2j}.$$

The Cauchy-Binet formula gives

$$\det\sigma = \sum_t |\det m_t(V)|^2$$

where the sum is over all minors $m_t(V)$ of size $N \times N$ of the matrix V. In particular

$$\det\sigma \geq \left|\det\begin{pmatrix} b_1 z_1 & b_2 z_1^2 & \cdots & b_N z_1^N \\ \vdots & \vdots & & \vdots \\ b_1 z_N & b_2 z_N^2 & \cdots & b_N z_N^N \end{pmatrix}\right|^2 = \prod_{j=1}^{N} b_j^2 \prod_{j=1}^{N} |z_j|^2 \prod_{1\leq j\neq k\leq N} |z_j - z_k|.$$

Now

$$\prod_{1\leq j\neq k\leq N} |z_j - z_k| = \left(\prod_{j=1}^{N-1} |z_N - z_j|\right)^N$$

and since z_j are the zeroes of $z^N - (\kappa r)^N$ we have

$$\prod_{j=1}^{N-1}(z_N - z_j) = N(\kappa r)^{N-1}.$$

Thus

$$\prod_{1\le j\neq k\le N} |z_j - z_k| = N^N(\kappa r)^{N(N-1)}$$

which yields

$$\prod_{j=1}^{N}|z_j|^2 \prod_{1\le j\neq k\le N} |z_j - z_k| = (\kappa r)^{2N}N^N(\kappa r)^{N(N-1)} = N^N(\kappa r)^{N(N+1)} = N^N\prod_{j=1}^{N}(\kappa r)^{2j}.$$

We conclude that

$$\det\sigma \ge N^N\prod_{j=1}^{N} b_j^2(\kappa r)^{2j} = N^N e^{S_L(\kappa r)} \ge e^{S_L(\kappa r)}.$$ ■

We can now prove the following estimate.

Lemma 5.22. *For all $0 < \alpha < 1$ there exists $C_1 > 0$ such that for all $\delta \ge (Lr^2)^{-\alpha}$ and Lr^2 sufficiently large we have*

$$\mathbb{P}[\mathcal{A}] \le \exp(-S_L(\kappa r) + C_1 Lr^2\big(\log(Lr^2) + \delta^{-2}\big)).$$

Proof. By definition

$$\mathbb{P}[\mathcal{A}] = \frac{1}{\pi^N\det\sigma}\int_{\mathcal{A}} e^{-\zeta^*\sigma^{-1}\zeta}\, d\operatorname{vol}_{\mathbb{C}^N}(\zeta).$$

Estimating crudely and using the previous two lemmas we get

$$\mathbb{P}[\mathcal{A}] \le \frac{1}{\pi^N\det\sigma}\operatorname{vol}_{\mathbb{C}^N}(\mathcal{A}) \le e^{-S_L(\kappa r)+C_1Lr^2(\log(\sqrt{L}r)+\delta^{-2})}$$

as claimed. ■

Proof of the upper bound in Theorem 5.10 (b). Gathering Lemmas 5.15, 5.18, 5.19 and 5.22 we see that

$$\mathbb{P}[n_L(r) = 0] \le e^{-S_L(\kappa r)+C_1(\log(Lr^2)+\delta^{-2})Lr^2} + \exp(-e^{Lr^2}) + 2e^{-S_L(\kappa r)} + e^{-L^4r^8}$$

for any $\delta \ge (Lr^2)^{-\alpha}$ with $0 < \alpha < 1$. Choosing $\delta = (Lr^2)^{-1/4}$, say, we have

$$S_L(\kappa r) = \frac{e^2}{4}L^2r^4(1 + o(1))$$

and

$$(\log(Lr^2) + \delta^{-2})Lr^2 \lesssim (Lr^2)^{3/2}$$

which completes the proof. ■

Chapter 6

The Paley-Wiener process

In this chapter we shall consider the zeroes of the random function

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(z-n),$$

where a_n are iid random variables with zero mean and unit variance and $\operatorname{sinc} z = \frac{\sin \pi z}{\pi z}$ is the cardinal sine. Since

$$\sum_{n\in\mathbb{Z}} |\operatorname{sinc}(z-n)|^2$$

converges uniformly on compact subsets of the plane, this series almost surely defines an entire function [HKPV09, Lemma 2.2.3]. The covariance kernel is given by

$$K(z,w) = \mathbb{E}[f(z)\overline{f(w)}] = \operatorname{sinc}(z-\overline{w}),$$

this follows from the fact that $(\sin \pi(z-n)/\pi(z-n))_{n\in\mathbb{Z}}$ constitute an orthonormal basis for the Paley-Wiener space

$$PW = \left\{ f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)e^{izt}\,dt \ : \ \phi \in L^2[-\pi,\pi] \right\},$$

of all entire functions of exponential type at most π that are square integrable on the real line. The covariance kernel is thus the reproducing kernel for the Paley-Wiener space, which is known to be given by the cardinal sine.

We will be chiefly concerned with the functions given by taking a_n to be *real* Gaussian random variables, the resulting function is then an example of a stationary symmetric GAF, and we call this the Paley-Wiener process. Here stationary means that for any $z_1,\ldots,z_n \in \mathbb{C}$ and any real t

$$(f(z_1),\ldots,f(z_n)) \stackrel{d}{=} (f(z_1+t),\ldots,f(z_n+t)),$$

which follows in our case by simply observing that $K(z+t,w+t) = K(z,w)$. Symmetric means that $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$, which is immediate from the definition. Symmetry indicates that the zeroes of this function are quite different in nature to those of the functions considered in previous chapters, in particular the non-real zeroes come in conjugate pairs. We once more denote by n_f the counting measure on the set of zeros of f.

Feldheim [Fel10] has computed a counterpart of the Edelman-Kostlan formula for general stationary symmetric GAFs, (c.f. the Kac-Rice formula [Kac43]). In our case Feldheim's result reduces to

$$\mathbb{E}[n_f(z)] = S(y)m(z) + \frac{1}{2\sqrt{3}}\mu(x), \tag{6.1}$$

where $z = x+iy$, m is again the planar Lebesgue measure, μ is the singular measure with respect to m supported on $\mathbb{R}$ and identical to Lebesgue measure there, and

$$S\Big(\frac{y}{2\pi}\Big) = \pi \left| \frac{d}{dy}\left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}} \right) \right|.$$

(Here S is defined only for $y \neq 0$, in fact the singular part of (6.1) is the distributional derivative at 0.) We observe that the first intensity is symmetric with respect to the real line, and invariant with respect to horizontal shifts, which is of course a general property of stationary symmetric GAFs. Indeed since $S(y) = O(|y|)$ as y approaches zero there are almost surely zeros on the real line, but they are sparse close to the real line. Moreover the real zeroes are on average uniformly distributed on the real line.

We are interested in the 'gap probability', that is, the probability that there are no zeros in a large interval on the real line. Our result is the following asymptotic estimate.

Theorem 6.1. *Let n_f be the counting measure on the zero set of the Paley-Wiener process. Then there exist constants $c, C > 0$ such that for all $r \geq 1$,*

$$e^{-cr} \leq \mathbb{P}[n_f((-r,r)) = 0] \leq e^{-Cr}.$$

Remarks. 1. By stationarity, the same result holds for any interval of length $2r$.

2. If instead of considering intervals we consider the rectangle $D_r = (-r,r) \times (-a,a)$ for some fixed $a > 0$, then we obtain a similar exponential decay for $\mathbb{P}[n_f(D_r) = 0]$

3. It seems that oscillations of the kernel $K(x,0)$ are somehow playing a rôle here, though this is not entirely obvious from the proof. The results of Newell and Rosenblatt [NR62] and Slepian [Sle62] suggest that it is not enough to consider merely the decay.

4. If we consider instead the case when the a_n are iid Rademacher random variables, that is, each a_n is equal to either -1 or 1 with equal probability, then we obtain a similar decay of the gap probability. Since $f(n) = a_n$ for $n \in \mathbb{N}$, it follows that if not all a_n for $|n| \leq N$ are of equal sign, then f has to have a zero in $(-N, N)$, by the mean value theorem. A simple modification of Lemma 6.2 shows that the remaining two choices of the a_n for $|n| \leq N$ each yield an f without zeroes in $(-N, N)$, whence the desired probability is exactly $e^{-2N\log 2}$ for $r = N$.

5. Whereas the Rademacher distribution is in some sense a simplified Gaussian, the Cauchy distribution, given by the density

$$p(x) = \frac{1}{\pi}\frac{1}{x^2+1},$$

is in some sense its opposite: It has neither an expectation, nor a standard deviation. If we suppose that the a_n are iid Cauchy random variables, it is not difficult to see that with probability one the sum $\sum a_n/n$ diverges, whence the related random function diverges everywhere almost surely. For a study of random zeros in the polynomial case see [LS68].

6.1 Proof of Theorem 6.1

Upper bound

We want to compute the probability of an event that contains the event of not having any zeroes on $(-N, N)$, for $N \in \mathbb{N}$. One such event is that the values $f(n)$ have the same sign for $|n| \leq N$. The probability of this event is

$$\mathbb{P}\Big[a_n > 0 \text{ for all } |n| \leq N \quad \text{or} \quad a_n < 0 \text{ for all } |n| \leq N\Big] = 2(1/2)^{2N+1} = e^{-CN},$$

for some constant $C > 0$.

Remark. The same upper bound holds when a_n are iid real random variables with $0 < \mathbb{P}[a_n > 0] < 1$ or $0 < \mathbb{P}[a_n > 0] < 1$ for which the random function $\sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(x-n)$ converges.

Lower bound

To compute a lower bound for the hole probability, we use the following scheme. First, we introduce the deterministic function

$$f_0(x) = \sum_{n=-2N}^{2N} \operatorname{sinc}(x-n).$$

We show in Lemma 6.2 that it has no zeroes on $(-N, N)$, and we find an explicit lower bound on $(-N, N)$ for it. This lower bound does not depend on N. Then we consider the functions

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1) \operatorname{sinc}(x-n) \qquad \text{and} \qquad f_2(x) = \sum_{|n|>2N} a_n \operatorname{sinc}(x-n),$$

which induce the splitting

$$f = f_0 + f_1 + f_2.$$

We show that for all $x \in [-N, N]$ we have $|f_1(x)| \leq \epsilon$ with probability at least e^{-cN} for large N and some constant $c > 0$. Moreover, we show that

$$\mathbb{P}\left[\sup_{x\in[-N,N]} |f_2(x)| \leq \epsilon\right]$$

is larger than, say, $1/2$ for N sufficiently large. As the events on f_1 and f_2 are clearly independent, the lower bound now follows by choosing ϵ small.

A zero free function

Lemma 6.2. *Given $N \in \mathbb{N}$ define*

$$f_0(x) = \sum_{n=-2N}^{2N} \operatorname{sinc}(x-n) = \sin \pi x \sum_{n=-2N}^{2N} \frac{(-1)^n}{\pi(x-n)}.$$

Then, there exists a constant $C > 0$ such that, for N sufficiently large,

$$1 - \frac{C}{N} \le \inf_{|x| \le N} f_0(x) \le \sup_{|x| \le N} f_0(x) \le 1 + \frac{C}{N}.$$

Proof. Let $Q = Q(N)$ be the boundary of the square with sides of length $4N+1$, centred at the origin. By the residue theorem, for any non-integer $x \in (-N, N)$, we have

$$\frac{1}{2\pi i} \oint_Q \frac{d\zeta}{(\zeta - x) \sin \pi\zeta} = \frac{1}{\pi} \sum_{n=-2N}^{2N} \frac{(-1)^n}{n-x} + \frac{1}{\sin \pi x},$$

or equivalently

$$\frac{\sin \pi x}{\pi} \sum_{n=-2N}^{2N} \frac{(-1)^n}{x-n} = 1 + \frac{\sin \pi x}{2\pi i} \oint_Q \frac{d\zeta}{(x-\zeta) \sin \pi\zeta}.$$

It is now easy to bound this last integral by C/N. ■

Remark. The same bound holds for all points z in a rectangular strip $[-N, N] \times [-c, c]$ for some fixed $c > 0$ and N large enough (depending on c).

The middle terms

Let $\epsilon > 0$ be given, and consider a fixed $N \in \mathbb{N}$. We look at the function

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1) \operatorname{sinc}(x-n) = \frac{\sin \pi x}{\pi} \sum_{n=-2N}^{2N} (a_n - 1) \frac{(-1)^n}{x-n}.$$

To simplify the expression, we set $b_n = (a_n - 1)(-1)^n$. We want to compute a lower bound for the probability that, for $x \in [-N, N]$,

$$|f_1(x)| \lesssim \epsilon.$$

Define $B_n = b_{-2N} + \cdots + b_n$ for $|n| \le 2N$, and suppose that $x \notin \mathbb{Z}$. With this, summation by parts yields

$$\sum_{-2N}^{2N} \frac{b_n}{x-n} = -\sum_{-2N}^{2N-1} \frac{B_n}{(x-n)(x-n-1)} + \frac{B_{2N}}{x-2N}. \tag{6.2}$$

We now claim that under the event

$$\mathcal{E} = \Big\{ |B_n| \leq \epsilon \quad \text{for} \quad |n| \leq 2N \Big\},$$

we have $|f_1(x)| \leq C\epsilon$ for $|x| \leq N$, with the constant C independent of N. Indeed E implies that the second term on the right hand side of (6.2) converges to zero uniformly for $|x| \leq N$, because

$$\left| \frac{B_{2N}}{x - 2N} \right| \leq \frac{\epsilon}{N}.$$

Suppose that $x \in (k, k+1)$ and split the sum on the right hand side of (6.2) into

$$\sum_{\substack{n=-2N \\ n \neq k-1,k,k+1}}^{2N-1} \frac{B_n}{(x-n)(x-n-1)} + \sum_{n=k-1}^{k+1} \frac{B_n}{(x-n)(x-n-1)}.$$

Then

$$\left| \sum_{\substack{n=-2N \\ n \neq k-1,k,k+1}}^{2N-1} \frac{B_n}{(x-n)(x-n-1)} \right| \leq \sum_{n \geq k+2} \frac{\epsilon}{(k+1-n)^2} + \sum_{n \leq k-2} \frac{\epsilon}{(k-1-n)^2} \lesssim \epsilon.$$

For the remaining terms, the function $\sin \pi x$ comes into play. For example, if $k < x \leq k + 1/2$ then

$$\left| \sin \pi x \frac{B_k}{(x-k)(x-k-1)} \right| \lesssim \frac{\epsilon}{|x-k-1|} \left| \frac{\sin \pi(x-k)}{\pi(x-k)} \right| \lesssim \epsilon.$$

The remaining terms are treated in exactly the same manner. We have shown that the event $\mathcal{E}$ implies that $|f(x)| \lesssim \epsilon$ for all non-integer $x \in (-N, N)$. By continuity this bound also holds for $x \in \mathbb{Z} \cap (-N, N)$.

It remains to compute the probability of the event $\mathcal{E}$. We recall that the b_n were all defined in terms of the real and independent Gaussian variables a_n. So the event $\mathcal{E}$ defines a set

$$V = \left\{ (t_{-2N}, \dots t_{2N}) \in \mathbb{R}^{4N+1} : \left| \sum_{-2N}^{n} t_n \right| \leq \epsilon, \ |n| \leq 2N \right\}.$$

Hence

$$\mathbb{P}[E] = c^{4N+1} \int_V e^{-((t_{-2N}-1)^2 + \cdots (t_{2N}-1)^2)/2} \, dt_{-2N} \cdots dt_{2N}.$$

where c is the normalising constant of the one dimensional Gaussian. It follows that

$$\mathbb{P}[E] \geq c^{4N+1} e^{-(4N+1)(1+C\epsilon)^2/2} \int_V dt_{-2N} \cdots dt_{2N} = c^{4N+1} e^{-(4N+1)(1+C\epsilon)^2/2} \operatorname{vol}(V).$$

We now seek a lower bound for this Euclidean $(4N+1)$-volume. To simplify the notation, we pose this problem as follows. For real variables $x_1, \ldots, x_N$, we wish to compute the Euclidean volume of the solid V_N defined by

$$\begin{aligned} &|x_1| \leq \epsilon, \\ &|x_1 + x_2| \leq \epsilon, \\ &\qquad\vdots \\ &|x_1 + x_2 + \cdots + x_N| \leq \epsilon. \end{aligned}$$

One way to do this is as follows. Write $y_N = x_1 + \cdots + x_{N-1}$, then

$$\mathrm{vol}(V_N) = \int_{V_{N-1}} \left(\int_{-\epsilon-y_N}^{\epsilon-y_N} dx_N \right) dx_1 \cdots dx_{N-1}.$$

This is illustrated in Figure 6.1. Clearly, if $y_N < 0$, $\epsilon - y_N \geq \epsilon$ and $-\epsilon - y_N \leq 0$, while if $y_N > 0$ then $\epsilon - y_N \geq 0$ and $-\epsilon - y_N \leq -\epsilon$.

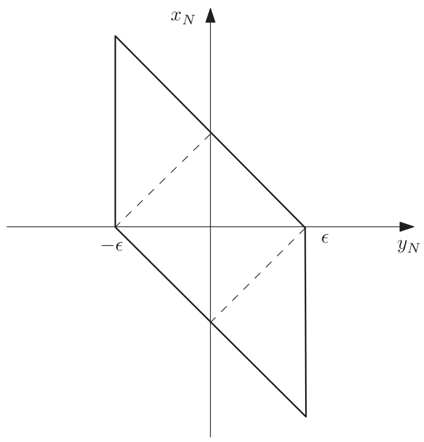

Figure 6.1: Illustration of the solid V_N.

Therefore

$$\begin{aligned} \mathrm{vol}(V_N) \geq &\int_{V_{N-1}\cap\{y_N<0\}} \left(\int_0^{\epsilon} dx_N \right) dx_1 \cdots dx_{N-1} \\ &+ \int_{V_{N-1}\cap\{y_N>0\}} \left(\int_{-\epsilon}^{0} dx_N \right) dx_1 \cdots dx_{N-1} \\ = &\ \epsilon \, \mathrm{vol}(V_{N-1}). \end{aligned}$$

Iterating this, we get

$$\mathrm{vol}(V_N) \geq \epsilon^N.$$

In conclusion,

$$\mathbb{P}[E] \geq e^{-cN},$$

which concludes this part of the proof.

The tail

We now turn to the tail term

$$f_2(x) = \sin \pi x \sum_{|n|>2N} \frac{a_n(-1)^n}{\pi(x-n)}.$$

Clearly, we need only consider the terms for which n is positive. Set $c_n = (-1)^n a_n$. We apply summation by parts to get

$$\sum_{n>2N}^{L} \frac{c_n}{x-n} = -\sum_{2N+1}^{L-1} C_n \frac{1}{(x-n)(x-n-1)} + \frac{C_L}{x-L}. \tag{6.3}$$

where

$$C_n = c_{2N+1} + \cdots + c_n.$$

We want to take the limit as $L \to \infty$ on the right hand side of (6.3). It is easy to see that the second term almost surely converges to zero, uniformly for $|x| \leq N$. We first note that for all $|x| \leq N$

$$\left|\frac{C_L}{x-L}\right| \leq \left|\frac{C_L}{L-2N}\right|.$$

Now C_L is a sum of $L-2N$ independent Gaussian variables with mean 0 and variance 1, so the strong law of large numbers shows that $\frac{C_L}{L-2N}$ converges to 0 almost surely, whence we are allowed to let $L \to \infty$ in (6.3).

We prove the following. With a positive probability, we have for $|x| \leq N$

$$\left|\sum_{2N+1}^{\infty} C_n \frac{1}{(x-n)(x-n-1)}\right| \leq \epsilon.$$

As $n^2 \simeq |(x-n)(x-n-1)|$ for $|x| \leq N$ and $n > 2N$, it is enough to consider the expression

$$\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2}.$$

The absolute value of a Gaussian random variable has the folded-Gaussian distribution. In particular, if $X \sim N(0, \sigma^2)$, then

$$\mathbb{E}[|X|] = \sigma\sqrt{\frac{2}{\pi}}.$$

Since in our case $\sigma^2 = n - 2N$, this yields

$$\mathbb{E}\left[\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2}\right] \lesssim \sum_{2N+1}^{\infty} \frac{\sqrt{n-2N}}{n^2} \lesssim \sum_{1}^{\infty} \frac{1}{(n+2N)^{3/2}} \lesssim \frac{1}{\sqrt{N}}.$$

Finally, by Markov's inequality,

$$\mathbb{P}\left[\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2} \leq \epsilon\right] \geq 1 - \frac{1}{\epsilon}\mathbb{E}\left[\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2}\right] \geq 1 - \frac{C}{\epsilon\sqrt{N}}.$$

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