Spaces of bandlimited functions on compact manifolds

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Bharti Pridhnani Pridhnani

Certifico que la present memòria ha estat
realitzada per Bharti Pridhnani Pridhnani i
dirigida per mi,

Joaquim Ortega-Cerdà.

For Kumar, Hema and my parents

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Resum

En aquesta tesi, estudiem les famílies d'interpolació i *sampling* en espais de funcions de banda limitada en varietats compactes. Les nocions de sampling i interpolació juguen un rol fonamental en problemes com ara recuperar un senyal continu a través de les mostres discretes. Aquestes dues nocions són, en part, de caràcter oposat: un conjunt de sampling ha de ser suficientment dens per tal de poder recuperar la informació i, en un conjunt d'interpolació, els punts han de ser suficientment separats per tal de poder trobar una funció que interpola certs valors.

Considerem l'espai de Paley-Wiener, $PW^2_{[-\pi,\pi]}$, que consisteix en funcions de banda limitada amb banda π, és a dir, funcions de quadrat integrable tal que la transformada de Fourier té suport en l'interval $[-\pi,\pi]$. Pels espais de Paley-Wiener, diem que una successió $\Lambda = \{\lambda_n\}$ és de sampling si existeixen constants $0 < A \leq B < \infty$ tal que per tota $f \in PW^2_{[-\pi,\pi]}$,

$$A\|f\|_2^2 \leq \sum_{\lambda_n \in \Lambda} |f(\lambda_n)|^2 \leq B\|f\|_2^2.$$

Diem que Λ és d'interpolació per $PW^2_{[-\pi,\pi]}$ si el problema d'interpolació

$$f(\lambda_n) = c_n, \quad \text{per tot } n,$$

admet alguna solució amb $f \in PW^2[-\pi,\pi]$, on $\{c_n\}_n$ és qualsevol successió de quadrat sumable. Un resultat fonamental en la història és el teorema de sampling de Whittaker-Shannon-Kotelnikov que conclou que qualsevol funció de l'espai de Paley-Wiener es pot recuperar a través dels valors que pren la funció en els enters, és a dir,

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)}, \quad \forall f \in PW^2_{[-\pi,\pi]}.$$

Per tant, els enters formen una successió tant d'interpolació com de sampling per $PW^2_{[-\pi,\pi]}$.

Si en lloc de prendre els enters, considerem successions $\Lambda = \{\lambda_n\}_n$ tals que

$$\sup_n |\lambda_n - n| < \delta,$$

amb δ suficientment petit, encara obtenim una successió de sampling i interpolació per l'espai de Paley-Wiener. En 1932, aquest fet va ser provat per Paley-Wiener amb $\delta < 1/\pi^2$.

El seu treball va donar lloc al començament de l'estudi de les sèries de Fourier no armòniques i la geometria de les successions d'interpolació i sampling.

El cas de les funcions amb transformada de Fourier amb suport en un interval està ben estudiat. Per un cas més general, com ara les funcions amb banda limitada en un conjunt acotat $E \subset \mathbb{R}^m$, H.J. Landau va trobar condicions necessàries per les successions de sampling i interpolació en termes de les densitats superiors i inferiors de $\Lambda = \{\lambda_n\}_n$ que es defineixen com:

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^m} \#(\Lambda \cap Q_r(x))}{r^m},$$

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{\inf_{x\in\mathbb{R}^m} \#(\Lambda \cap Q_r(x))}{r^m},$$

on $Q_r(x)$ és el cub centrat a x amb llargària r del costat. Aquestes nocions s'anomenen les densitats de Beurling-Landau ja que Beurling les va introduir en una dimensió i Landau les va generalitzar a dimensions superiors. Quan un conjunt Λ és separat (és a dir, $|\lambda_n - \lambda_m| \geq \delta > 0$ per tot $n \neq m$), Landau va provar que una condició necessària per a què Λ sigui d'interpolació (o sampling) és que la densitat sigui inferior (o superior) que un valor crític, la raó de Nyquist.

Un problema semblant però diferent és caracteritzar les successions d'interpolació i sampling per l'espai de polinomis restringits a $\mathbb{S}^1$. Considerem per tot $n \in \mathbb{N}$, l'espai de polinomis de grau menor o igual que n:

$$\mathcal{P}_n = \left\{ q(z) = \sum_{k=0}^{n} a_k z^k, \ z \in \mathbb{S}^1 \right\},$$

dotat de la norma L^2:

$$\|q\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \right)^{1/2}.$$

Els polinomis juguen el paper de les funcions de banda limitada de banda donada pel grau del polinomi. Els espais $\mathcal{P}_n$ tenen propietats semblants a les de l'espai de Paley-Wiener però no els hem de confondre ja que hem passat d'una situació no compacta a una de compacta. Un problema natural és estudiar la interpolació i sampling per l'espai de polinomis $\mathcal{P}_n$. En aquest nou context, no podem prendre successions de punts donat que una successió separada en $\mathbb{S}^1$ seria finita i, per tant, el problema de sampling per polinomis de qualsevol grau no tindria sentit. Doncs, sembla natural reemplaçar les successions de punts per famílies de punts en $\mathbb{S}^1$ de manera que cada nivell n de la família recupera la norma de qualsevol polinomi $p \in \mathcal{P}_n$ (amb constants independents de n). El concepte de família separada s'entén com que la distància entre dos punts de la generació n és almenys $\delta/n > 0$. Per tant, el rol d'una successió de sampling el fa una família triangular anomenada família de Marcinkiewicz -Zygmund (M-Z). La definició formal és la següent.

Definició. *Donada una família triangular de punts* $\mathcal{Z} = \{\mathcal{Z}(n)\}_{n\geq 0} \subset \mathbb{S}^1$ *amb* $Z(n) =$

$\{z_{nj}\}_{j=0,\dots,m_n}$, *diem que* $\mathcal{Z}$ *és* M-Z *per* L^p $(1 \le p < +\infty)$ *si per tot* $n \in \mathbb{N}$ *i* $q \in \mathcal{P}_n$ *se satisfà:*

$$\frac{C_p^{-1}}{n}\sum_{j=0}^{m_n}|q(z_{nj})|^p \le \int_0^{2\pi}|q(e^{i\theta})|^p d\theta \le \frac{C_p}{n}\sum_{j=0}^{m_n}|q(z_{nj})|^p,$$

amb C_p *independent de* n.

Per exemple, les arrels $(n+1)$-èssimes satisfan la desigualtat anterior. Aquest tipus de desigualtats són similars a les de ser successions de sampling en el context de Paley-Wiener. De fet, aquesta similitud és més que superficial. La primera desigualtat en la definició de M-Z s'anomena desigualtat del tipus Plancherel-Pólya i és conseqüència d'una caracterització de les mesures de Carleson. La segona desigualtat és la més complicada de caracteritzar. En [OCS07], J. Ortega-Cerdà i J. Saludes van estudiar amb detall el cas de $\mathbb{S}^1$. Van provar condicions necessàries i suficients per a què una família sigui de M-Z en termes de les densitats de Beurling-Landau.

Teorema (J.Ortega-Cerdà, J. Saludes). *Sigui* $\mathcal{Z}$ *una família separada, és a dir,*

$$d(z_{nj}, z_{nk}) \ge C/n, \quad \forall j \ne k, n \in \mathbb{N}.$$

Si

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty}\liminf_{n\to\infty}\frac{\min_{x\in[0,2\pi]}\#(\mathcal{Z}(n)\cap(x, x+R/n))}{R} > \frac{1}{2\pi},$$

on (x,y) *simbolitza l'arc en la* $\mathbb{S}^1$ *amb extrems* e^{ix} *i* e^{iy}, *aleshores* $\mathcal{Z}$ *és una família* M-Z *per qualsevol* $p \in [1,\infty]$. *Si* $\mathcal{Z}$ *és* M-Z *per algun* $p \in [1,\infty]$, *aleshores* $D^-(\mathcal{Z}) \ge 1/2\pi$.

Amb aquest treball i les idees de Landau, J. Marzo va provar, en la seva tesi [Mar08], condicions necessàries per a què una família sigui d'interpolació o sampling en termes de les densitats de Beurling-Landau. J. Marzo va considerar espais de funcions que consisteixen en combinacions lineals d'armònics esfèrics de $\mathbb{S}^m$ de grau més petit que L. Aquestes funcions són de banda limitada i la banda ve donada pel grau L.

L'objectiu d'aquest treball és generalitzar els resultats obtinguts en la tesi d'en J. Marzo al cas d'una varietat compacta de dimensió $m \ge 2$ arbitrària (el cas 1-dimensional està completament estudiat en [OCS07]). Els armònics esfèrics es caracteritzen com els vectors propis de l'operador de Laplace-Beltrami de la $\mathbb{S}^m$. Doncs, la generalització natural dels armònics esfèrics són els vectors propis de l'operador de Laplace-Beltrami associat a la varietat. Ens concentrarem en la norma L^2. La dificultat i diferència principal amb el cas de $\mathbb{S}^m$ és el fet que un no pot fer el producte de dues funcions de banda limitada i obtenir una altra funció dels espais considerats. També, en una varietat compacta arbitrària, no tenim una expressió explícita del nucli reproductor dels espais que considerem. Per tant, una de les dificultats més importants és la construcció de funcions de banda limitada amb un control del seu decaïment fora d'una bola geodèsica fixada. En moltes ocasions, el nucli reproductor sol ser una funció test per provar condicions necessàries en alguna caracterització. Aquesta funció, també anomenada com la funció espectral associada al

Laplacià, ha sigut tema de recerca de molts autors, dels quals destaca L. Hörmander que va provar algunes estimacions de la funció espectral associada a qualsevol operador el·líptic en varietats compactes (vegeu [Hör68] per més detalls).

Aquest treball s'estructura en quatre capítols.

En el primer capítol, introduïm el context del nostre problema i els resultats principals provats al llarg d'aquesta tesi. També descrivim el comportament asimptòtic del nucli reproductor i la construcció de nous nuclis associats als nostres espais amb un decaïment fora de la diagonal. A més a més, explicarem algunes eines que jugaran un paper fonamental en les proves dels nostres resultats.

En el segon capítol, estudiem el problema del *sampling continu*. El rol d'una família discreta de sampling el realitza una successió de conjunts en la varietat anomenada successió de Logvinenko-Sereda. Un problema més dèbil és trobar una caracterització de les mesures de Carleson. Aquesta qüestió també s'ha resolt en termes d'una condició geomètrica.

En el tercer capítol, provem algunes condicions (qualitatives) necessàries i suficients per a la interpolació i sampling. Definim l'anàleg a la densitat de Beurling-Landau i provem, seguint les idees de Landau en el context dels espais de Paley-Wiener, condicions quantitatives necessàries per a què una família sigui de sampling o d'interpolació.

En el quart capítol, donem una aplicació dels resultats de densitat obtinguts en el Capítol 3. Estudiem les famílies de punts de Fekete en varietats compactes amb certa propietat (vegeu la Definició 4.1 per a més informació). Els punts de Fekete són punts que maximitzen un determinant del tipus Vandermond que apareix en la fórmula d'interpolació del polinomi de Lagrange. Són punts adients per les fórmules d'interpolació i la integració numèrica. Els punts de Fekete tenen la propietat que són casi d'interpolació i M-Z. Per tant, aquest tipus de punts estan ben distribuïts en la varietat ja que contenen informació suficient per recuperar la norma L^2 d'una funció de banda limitada i, són suficientment separats per tal d'interpolar alguns valors fixats.

Els resultats d'aquest treball són part dels següents articles:

- J. Ortega-Cerdà, B. Pridhnani. Carleson measures and Logvinenko-Sereda sets on compact manifolds. Forum Mathematicum, to appear ([OCP11b]).
- J. Ortega-Cerdà, B. Pridhnani. Beurling-Landau's density on compact manifolds. Preprint ([OCP11a]).

Introduction

In this monograph we study the interpolating and sampling families for the spaces of bandlimited functions on compact manifolds. Sampling and interpolation play a fundamental role in problems such as recovering a continuous signal from discrete samples or assessing the information lost in the sampling process. Somehow, sampling and interpolating sequences are opposite in nature. A set of sampling should be *dense* enough in order to recover the information and, in an interpolating set, the points should be far enough so that one can find a function that interpolates any given data.

A bandlimited function is a function such that its Fourier transform has compact support. We denote by $PW^2_{[-\pi,\pi]}$ the space of bandlimited functions with bandwidth π (i.e. square integrable functions with Fourier transform vanishing outside $[-\pi,\pi]$). For the Paley-Wiener space, $PW^2_{[-\pi,\pi]}$, a set $\Lambda = \{\lambda_n\}_n$ is a set of sampling if there are constants $0 < A \leq B < \infty$ such that for any $f \in PW^2_{[-\pi,\pi]}$,

$$A\|f\|_2^2 \leq \sum_{\lambda_n \in \Lambda} |f(\lambda_n)|^2 \leq B\|f\|_2^2.$$

We say that Λ is an interpolating set for $PW^2_{[-\pi,\pi]}$ if the interpolation problem

$$f(\lambda_n) = c_n \quad \text{for all } n,$$

has a solution $f \in PW^2_{[-\pi,\pi]}$ for every square-summable sequence $\{c_n\}_n$. The well known Whittaker-Shannon-Kotelnikov sampling theorem states that any function belonging to the Paley-Wiener space can be recovered from its values at the integers:

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)}, \quad \forall f \in PW^2_{[-\pi,\pi]}.$$

Thus, the integers are both sampling and interpolating for the Paley-Wiener spaces $PW^2_{[-\pi,\pi]}$.

If, instead of the integers, we take a set $\Lambda = \{\lambda_n\}_n$ such that

$$\sup_n |\lambda_n - n| < \delta,$$

with δ small enough, we still get a sampling and interpolating set for the Paley-Wiener space. This fact was proved by Paley-Wiener in 1934. They proved the result for $\delta < 1/\pi^2$.

Their work gave birth to the study of non-harmonic Fourier series and the geometry of interpolating and sampling sequences.
In 1964, M.I. Kadec proved that one can have this property for $\delta < 1/4$ (this is called the Kadec's 1/4-theorem) and this bound is sharp.

Thus, the case of functions with Fourier transform supported in an interval is well explored. For a more general case, like functions bandlimited to some general bounded set $E \subset \mathbb{R}^m$, H.J. Landau found necessary conditions for sampling and interpolation for functions in PW_E^2 in terms of the upper and lower density of the set $\Lambda = \{\lambda_n\}_n$ given by

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^m} \#(\Lambda \cap Q_r(x))}{r^m},$$

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{\inf_{x\in\mathbb{R}^m} \#(\Lambda \cap Q_r(x))}{r^m},$$

where $Q_r(x)$ is the cube centered at x of sidelength r. These are called the Beurling-Landau densities, because Beurling introduced them in one dimension and Landau generalized them for higher dimensions. For a separated set Λ (i.e. $|\lambda_n - \lambda_m| \geq \delta > 0$ for all $n \neq m$), Landau proved that a necessary condition for Λ to be interpolating (or sampling) is that the density is smaller (or bigger) than a critical value, the Nyquist rate.

A similar problem but yet very different is characterizing the interpolating and sampling sequences for the spaces of polynomials restricted to $\mathbb{S}^1$. Consider for all $n \in \mathbb{N}$, the space of polynomials of degree less than n:

$$\mathcal{P}_n = \left\{ q(z) = \sum_{k=0}^{n} a_k z^k,\ z \in \mathbb{S}^1 \right\},$$

endowed with the L^2-norm:

$$\|q\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \right)^{1/2}.$$

The polynomials play the role of bandlimited functions with band given by the degree of the polynomial. We remark that these spaces share many properties with the Paley-Wiener spaces but they are not to be confused with them, because we have moved from a non-compact setting to a compact one. A natural question is to ask for interpolation and sampling. In this context, we cannot take sequences because a separated sequence in $\mathbb{S}^1$ would be finite and thus the sampling problem for polynomials of any degree will not make sense. It seems very natural to replace the sequences by families of points in $\mathbb{S}^1$ so that each level n of the family recovers the norm of any polynomial $p \in \mathcal{P}_n$ (with constants independent of n). The notion of a separated family is now translated into the fact that the distance between two points of the n-th generation is at least $\delta/n > 0$. Thus, the role of a sampling sequence is developed by a triangular family called Marcinkiewicz-Zygmund (M-Z) family. The precise definition is the following.

Definition. *Given a triangular family of points $\mathcal{Z} = \{\mathcal{Z}(n)\}_{n\geq 0} \subset \mathbb{S}^1$ with $Z(n) = \{z_{nj}\}_{j=0,\dots,m_n}$, we say that $\mathcal{Z}$ is* M-Z *for L^p $(1 \leq p < +\infty)$ if the following inequality holds for all $n \in \mathbb{N}$ and $q \in \mathcal{P}_n$,*

$$\frac{C_p^{-1}}{n}\sum_{j=0}^{m_n}|q(z_{nj})|^p \leq \int_0^{2\pi}|q(e^{i\theta})|^p d\theta \leq \frac{C_p}{n}\sum_{j=0}^{m_n}|q(z_{nj})|^p, \tag{1}$$

with C_p independent of n.

For instance, the $(n+1)$-roots of unity satisfy (1). These sort of inequalities are similar to the sampling sequences estimates in the Paley-Wiener setting. In fact, this similarity is more than superficial. The first inequality in (1) is called Plancherel-Pólya type inequality and follows from a characterization of the Carleson measures. The second inequality in (1) is the harder one to characterize (the so-called reverse Carleson inequality). In [OCS07], J. Ortega-Cerdà and J. Saludes studied in detail the case of the $\mathbb{S}^1$. In this direction, they proved necessary and sufficient conditions for a family to be M-Z in terms of the Beurling-Landau densities.

Theorem (J.Ortega-Cerdà, J. Saludes)**.** *Let $\mathcal{Z}$ be a separated family, i.e.*

$$d(z_{nj}, z_{nk}) \geq C/n, \quad \forall j \neq k, n \in \mathbb{N}.$$

If

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty}\liminf_{n\to\infty}\frac{\min_{x\in[0,2\pi]}\#(\mathcal{Z}(n)\cap(x, x+R/n))}{R} > \frac{1}{2\pi},$$

where (x, y) denotes the arc in $\mathbb{S}^1$ with endpoints e^{ix} and e^{iy}, then $\mathcal{Z}$ is a M-Z *family for any $p \in [1,\infty]$. Conversely, if $\mathcal{Z}$ is* M-Z *family for some $p \in [1,\infty]$, then $D^-(\mathcal{Z}) \geq 1/2\pi$.*

Taking into account the case of the circle and following the ideas of Landau, J. Marzo proved, in his thesis (see [Mar08]), the necessary condition for interpolation and sampling in terms of density in a more general space. He considered the spaces of linear combinations of spherical harmonics in $\mathbb{S}^m$ of degree less than L. These functions are again bandlimited and the band is given by the degree L.

The goal of this monograph is to extend the results in the thesis of J. Marzo to a general compact manifold of dimension $m \geq 2$ (the one-dimensional case is completely studied in [OCS07]). The spherical harmonics have an intrinsic characterization as the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^m$. Thus, the natural generalization of the spherical harmonics, on an arbitrary compact manifold, are the eigenfunctions of the Laplacian associated to the manifold. We will focus our attention on the case of the L^2-norm. The major difficulty and difference from the case of $\mathbb{S}^m$ is that one cannot multiply two bandlimited functions and obtain another function in our spaces. Furthermore, in a general compact manifold, we lack of an explicit expression of the reproducing kernel of the spaces under consideration. Hence, one of the main difficulties is to construct bandlimited functions with a control of their decay outside a fixed geodesic ball. Usually, a test function to prove necessary conditions of a characterization is the reproducing kernel. This function, also called the spectral function associated to the Laplacian, has been subject of research for many authors, specially L. Hörmander in his paper [Hör68]. He proved some estimates for the spectral function associated to any elliptic operator on compact manifolds. All the information about the reproducing kernel is useful in our context because one can express the notion of being interpolating or M-Z in terms of the reproducing kernel (see Chapter 3, Section 3.1 for more details).

Till now, we have been discussing about estimating the L^2-norm of a bandlimited function by a discrete norm of its evaluation on some sequence of points. One can think about controlling the L^2-norm of a function in our space by the L^2-norm restricted to some sets, i.e. can we study a *continuous* sampling? This turns out to be an easier problem than discrete sampling. Thus, a natural question is to find for which sequences of sets $\{A_L\}_L$ in the manifold,

$$\|f\|_2 \simeq \|\chi_{A_L} f\|_2,$$

for any function f in our spaces, with constants independent of L and f. These comparison of norms are called the Logvinenko-Sereda sets inequality. In the case of the $\mathbb{S}^m$, in [MOC08], a geometric characterization has been obtained. Intuitively, Logvinenko-Sereda sets should be *relatively dense* in order to recover the L^2-norm of any bandlimited function.

Beyond interpolating and sampling, there are other families of points, called the Fekete points, that have their own interest. Fekete points are the points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. They are well suited points for interpolation formulas and numerical integration. The geometric properties of the distribution of the Fekete points on the sphere has been studied by many authors like in [Rei90], [BLW08] or [SW04]. In the circle, the roots of unity are simultaneously interpolating and M-Z arrays. On higher dimension, i.e. in the $\mathbb{S}^m$, it has been proved (see [Mar07, Theorem 1.7]) that there are no arrays which are simultaneously interpolating and M-Z for the L^p-norm ($p \neq 2$) when $m \geq 2$. But, in [MOC10], it was showed that the Fekete points are a very reasonable substitute of the roots of unity. In the beginning, we said that sampling and interpolating sequences are of opposite nature. Fekete points have the property that they are almost interpolating and M-Z. Thus, these kind of points are well distributed in the manifold because they have enough information in order to recover the L^2-norm of a bandlimited function and they are far enough in order to interpolate some given data. Furthermore, understanding the densities of M-Z and interpolating arrays will help to get some geometric information about the Fekete families. The main difficulty here is that in our proofs, we need functions, in the spaces under consideration, with a desired decay and such that they preserve certain values. The technique we use, is to multiply two bandlimited functions and still obtain a function in our spaces. Thus, we will analyze the Fekete points on compact manifolds with some restriction.

This monograph is structured in four chapters.

In Chapter 1, we present the context of our problem and the main results proved in this work. We describe the asymptotic behaviour of the reproducing kernel and the construction of new kernels associated to our spaces with a decay away from the diagonal. We shall also explain some tools that will play a fundamental role in the proof of our results.

In Chapter 2, we study the problem of a *continuous* sampling. The role of a discrete family of sampling is played now by a sequence of sets in the manifold called Logvinenko-Sereda sets. We give a complete geometric characterization. A weaker problem is to find a characterization of the Carleson's measures. This question has been also answered in terms of a geometric condition.

In Chapter 3, we provide some (qualitative) necessary and sufficient conditions for interpolation and sampling. We define an analog of the Beurling-Landau's density and prove a quantitative necessary condition for sampling and interpolation following the scheme of Landau in the context of the Paley-Wiener spaces.

In Chapter 4, we give an application of the density results obtained in Chapter 3 and study the Fekete arrays on compact manifolds with some restriction. Furthermore, we prove from the results of Chapter 3, the equidistribution of the Fekete families on compact manifolds that have a product property (see Definition 4.1 for more details).

The results of this monograph are part of the following articles:

- J. Ortega-Cerdà, B. Pridhnani. Carleson measures and Logvinenko-Sereda sets on compact manifolds. Forum Mathematicum, to appear ([OCP11b]).
- J. Ortega-Cerdà, B. Pridhnani. Beurling-Landau's density on compact manifolds. Preprint ([OCP11a]).

Chapter 1

Main results and Preliminaries

In this chapter, we present the context and notation of our problem. In the first section, we explain the motivation of our work and the main results obtained. In the second section, we define the reproducing kernel of the spaces under consideration and state its asymptotics proved by L. Hörmander. Furthermore, we consider Bochner-Riesz type kernels associated to our spaces with a certain decay off the diagonal. In the last section, we illustrate a technical tool in order to obtain gradient estimates of functions in the spaces under consideration.

1.1 Notation and statement of the results

Let (M, g) be a smooth, connected, compact Riemannian manifold without boundary, of dimension $m \geq 2$. Let dV and Δ_M be the volume element and the Laplacian on M associated to the metric g, respectively. The Laplacian is given in local coordinates by

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

where $|g| = |\det(g_{ij})|$ and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$. Since M is compact, g_{ij} and all its derivatives are bounded and we assume that the metric g is non-singular at each point of M.

By the compactness of M, the spectrum of the Laplacian is discrete and there is a sequence of eigenvalues

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \ldots \to \infty$$

and an orthonormal basis ϕ_i of smooth real eigenfunctions of the Laplacian i.e. $\Delta_M \phi_i = -\lambda_i^2 \phi_i$ (without loss of generality, we assume that $\lambda_i \geq 0$). Thus, $L^2(M)$ decomposes into an orthogonal direct sum of eigenfunctions of the Laplacian.

We consider the following subspaces of $L^2(M)$.

$$E_L = \left\{ f \in L^2(M) \ : \ f = \sum_{i=1}^{k_L} \beta_i \phi_i, \ \Delta_M \phi_i = -\lambda_i^2 \phi_i, \ \lambda_{k_L} \leq L \right\},$$

where $L \geq 1$ and $k_L = \dim E_L$. We see that E_L consists of functions in $L^2(M)$ with a restriction on the support of its Fourier transform. It is, in a sense, the Paley-Wiener space on M with bandwidth L.

The motivation of this work is to show that the spaces E_L behave like the spaces defined in $\mathbb{S}^m$ ($m > 1$) of linear combinations of spherical harmonics of degree not exceeding L. In fact, the space E_L is a generalization of the spherical harmonics and the role of them are played by the eigenfunctions. The cases $M = \mathbb{S}^1$ and $M = \mathbb{S}^d$ ($d > 1$) have been studied in [OCS07] and [Mar07], respectively.

This similarity between eigenfunctions of the Laplacian and polynomials is not new; for instance, Donnelly and Fefferman showed in [DF90, Theorem 1] that on a compact manifold, an eigenfunction of eigenvalue λ^2 behaves essentially like a polynomial of degree λ. In this direction they proved the following stated below local L^∞-Bernstein inequality.

Theorem (Donnelly-Fefferman)**.** *Let M be as above with $m = \dim M$. If u is an eigenfunction of the Laplacian $\Delta_M u = -\lambda^2 u$, then there exists $r_0 = r_0(M)$ such that for all $r < r_0$ we have*

$$\max_{B(x,r)} |\nabla u| \leq \frac{C\lambda^{(m+2)/2}}{r} \max_{B(x,r)} |u|.$$

The proof of the above estimate is rather delicate. Donnelly and Fefferman conjectured that it is possible to replace $\lambda^{(m+2)/2}$ by λ in the inequality. If the conjecture holds, we have in particular, a global Bernstein type inequality:

$$\|\nabla u\|_\infty \lesssim \lambda \|u\|_\infty . \tag{1.1}$$

In fact, this weaker estimate holds and a proof will be given later. This fact suggests that the right metric to study the space E_L should be rescaled by a factor $1/L$ because in balls of radius $1/\lambda$, a bounded eigenfunction of eigenvalue λ^2 oscillates very little.

In Chapter 2, we study for which measures $\mu = \{\mu_L\}_L$ one has

$$\int_M |f|^2 d\mu_L \simeq \int_M |f|^2 \, dV, \quad \forall f \in E_L, \tag{1.2}$$

with constants independent of f and L.
We also consider a weaker inequality

$$\int_M |f|^2 d\mu_L \lesssim \int_M |f|^2 dV,$$

that defines the Carleson measures and we present a geometric characterization of them. Inequality (1.2) will be studied only for the special case $d\mu_L = \chi_{A_L} dV$, where $\mathcal{A} = \{A_L\}_L$ is a sequence of sets in the manifold. In case (1.2) holds, we say that $\mathcal{A}$ is a sequence of Logvinenko-Sereda sets. Our two main results are the following:

Theorem 1.1. *The sequence of sets $\mathcal{A} = \{A_L\}_L$ is Logvinenko-Sereda if and only if there is an $r > 0$ such that*

$$\inf_L \inf_{z \in M} \frac{\mathrm{vol}(A_L \cap B(z, r/L))}{\mathrm{vol}(B(z, r/L))} > 0.$$

Theorem 1.2. *Let $\mu = \{\mu_L\}_L$ be a sequence of measures on M. Then μ is L^2-Carleson for M if and only if there exists a $C > 0$ such that for all $L \geq 1$,*

$$\sup_{\xi \in M} \frac{\mu_L(B(\xi, 1/L))}{\mathrm{vol}(B(\xi, 1/L))} \leq C.$$

In Chapter 3 we consider the interpolating and M-Z families associated to the spaces $\{E_L\}_{L \geq 1}$. In this context, we say that a triangular family $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 1}$ with $\mathcal{Z}(L) = \{z_{Lj}\}_{j=1}^{m_L}$, is M-Z (or interpolating) if for each level L, the sequence $\mathcal{Z}(L)$ is sampling (or interpolating) for the space E_L with constants independent of L (see Definitions 3.1 and 3.2).
As explained in the introduction, intuitively an interpolating family is sparse and a M-Z family is dense. We formalize these notions in this chapter. More precisely, we prove:

- A necessary condition for a family $\mathcal{Z}$ to be interpolating is that $\mathcal{Z}$ should be *uniformly separated* (see Definition 2.4 and Proposition 3.4 for a precise statement).
- If a family $\mathcal{Z}$ is enough separated then it is interpolating (check Proposition 3.6 for a proof).
- A M-Z family contains a subfamily that is uniformly separated and M-Z (see Theorem 3.7).
- If a family $\mathcal{Z}$ is dense enough then it is M-Z (see Theorem 3.9 for a precise formulation and proof).

The results mentioned above are qualitative. We lack of a precise quantity for measuring how much sparse or dense a family should be in order to be interpolating or M-Z. The second goal of Chapter 3 is to extend the theory of Beurling-Landau on the discretization of functions in the Paley-Wiener space on $\mathbb{R}^n$ to functions in M. This should be possible because there is already a literature on the subject in the case $M = \mathbb{S}^m$ (see [Mar07] for more details). In terms of the Beurling-Landau's density, we prove a quantitative result that is a necessary condition for interpolating and M-Z families. More precisely, our main result in this direction is:

Theorem 1.3. *Let $\mathcal{Z}$ be a triangular family in M. If $\mathcal{Z}$ is an L^2-*M-Z *family then there exists a uniformly separated L^2-*M-Z *family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that*

$$D^-(\tilde{\mathcal{Z}}) \geq 1.$$

If $\mathcal{Z}$ is an L^2-interpolating family then it is uniformly separated and

$$D^+(\mathcal{Z}) \leq 1,$$

where D^+ and D^- are the upper and lower Beurling-Landau's density (see Definition 3.10 for more details), respectively.

The proof of this result relies on the scheme used by Landau for the Paley-Wiener spaces. We define a concentration operator over a set $A \subset M$ that maps E_L onto E_L. There is a relation between the number of *big* eigenvalues of this operator with the Beurling-Landau density. To this end, the number of *big* eigenvalues of the concentration operator can be controlled by the trace of this operator. Thus, estimating the density is reduced to estimating the trace of the concentration operator.

In Chapter 4, we consider the Fekete arrays on compact manifolds with a certain product property. The Fekete points are the points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. There is a huge literature for the case $M = \mathbb{S}^m$. As explained before, in the circle the roots of unity are simultaneously interpolating and M-Z. On higher dimensions, $\mathbb{S}^m$ $(m > 1)$ there is no model of points like the roots of unity. In [MOC10], it has been showed that the Fekete points are a very reasonable substitute of the roots of unity. Thus, they are well distributed points in $\mathbb{S}^m$. In [MOC10], J. Marzo and J. Ortega-Cerdà proved that as $L \to \infty$, the number of Fekete points in a spherical cap $B(z, R)$ gets closer to $k_L \tilde{\sigma}(B(z, R))$, where $\tilde{\sigma}$ is the normalized Lebesgue measure on $\mathbb{S}^m$. The key idea in proving this result is a connection between the Fekete points and the M-Z and interpolating arrays. Following their approach, we define the Fekete points on M and show their connection with the interpolating and M-Z families and prove the asymptotic equidistribution of the Fekete points on the manifold. Intuitively, the Fekete families are almost interpolating and M-Z. Another main tool for the proof of the equidistribution of the Fekete families is the necessary condition for the interpolating and Marcinkiewicz-Zygmund arrays in terms of the Beurling-Landau densities. More precisely, we prove:

Theorem 1.4. *Let M be an admissible manifold and $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 1}$ be any array such that $\mathcal{Z}(L)$ is a set of Fekete points of degree L. Consider the measures $\mu_L = \frac{1}{k_L} \sum_{j=1}^{k_L} \delta_{z_{Lj}}$. Then μ_L converges in the weak-$*$ topology to the normalized volume measure on M.*

By admissibility we mean that there exists a constant $C > 0$ such that for all $0 < \epsilon < 1$ and $L \geq 1$:

$$E_L \cdot E_{\epsilon L} \subset E_{L(1+C\epsilon)}.$$

We need to restrict to these kind of manifolds because our proof requires to construct a bandlimited function with a certain decay that preserves some given values. This has been done by taking the product of two bandlimited functions. Examples of such manifolds are the two-point compact homogeneous spaces, the torus, the Klein bottle and many more can be constructed by taking products of admissible manifolds (check Chapter 4, Section 4.3 for a complete description).

In what follows, when we write $A \lesssim B$, $A \gtrsim B$ or $A \simeq B$, we mean that there are constants depending only on the manifold such that $A \leq CB$, $A \geq CB$ or $C_1 B \leq A \leq C_2 B$, respectively. Also, the value of the constants appearing during a proof may change, but they will be still denoted by the same letter. A geodesic ball in M and an Euclidean ball in $\mathbb{R}^m$ are represented by $B(\xi, r)$ and $\mathbb{B}(z, r)$, respectively.

1.2 Kernels associated to E_L

Let

$$K_L(z,w) := \sum_{i=1}^{k_L} \phi_i(z)\phi_i(w) = \sum_{\lambda_i \le L} \phi_i(z)\phi_i(w).$$

This function is the reproducing kernel of the space E_L, i.e.

$$f(z) = \langle f, K_L(z,\cdot)\rangle, \quad \forall f \in E_L.$$

Note that $\dim(E_L) = k_L = \#\{\lambda_i \le L\}$. The function K_L is also called the spectral function associated to the Laplacian. Hörmander proved in [Hör68] the following estimates:

1. $K_L(z,z) = \frac{\sigma_m}{(2\pi)^m} L^m + O(L^{m-1})$ (uniformly in $z \in M$), where $\sigma_m = \frac{2\pi^{m/2}}{m\Gamma(m/2)}$.

2. $k_L = \frac{\mathrm{vol}(M)\sigma_m}{(2\pi)^m} L^m + O(L^{m-1})$.

In fact, in [Hör68] there are estimates for the spectral function associated to any elliptic operator of order $n \ge 1$ with constants depending only on the manifold.

So, for L big enough we have $k_L \simeq L^m$ and

$$\|K_L(z,\cdot)\|_2^2 = K_L(z,z) \simeq L^m \simeq k_L$$

with constants independent of L and z.

We will also make use of the Bochner-Riesz kernel associated to the Laplacian that is defined as

$$S_L^N(z,w) := \sum_{i=1}^{k_L} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(w).$$

Here $N \in \mathbb{N}$ is the order of the kernel. Using the definition, one has that for all $g \in L^2(M)$, the Bochner-Riesz transform of g is

$$S_L^N(g)(z) = \int_M S_L^N(z,w)g(w)dV(w) = \sum_{i=1}^{k_L} \left(1 - \frac{\lambda_i}{L}\right)^N c_i\phi_i(z) \in E_L,$$

where $c_i = \langle g, \phi_i\rangle$. Observe that $\|S_L^N(g)\|_2 \le \|g\|_2$.

Note that $S_L^0(z,w) = K_L(z,w)$. The Bochner-Riesz kernel satisfies the following estimate.

$$|S_L^N(z,w)| \le C_N L^m \left(1 + Ld_M(z,w)\right)^{-N-1}, \tag{1.3}$$

where C_N is a constant depending on the manifold and the order N. This estimate has its origins in Hörmander's article [Hör69, Theorem 5.3]. Estimate (1.3) can be found also in [Sog87, Lemma 2.1].

Note that on the diagonal, $S_L^N(z,z) \simeq C_N L^m$. The upper bound is trivial by the definition and the lower bound follows from

$$S_L^N(z,z) \geq \sum_{\lambda_i \leq L/2} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(z) \geq 2^{-N} K_{L/2}(z,z) \simeq C_N L^m.$$

Similarly we observe that $\left\|S_L^N(\cdot,\xi)\right\|_2^2 \simeq C_N L^m$.

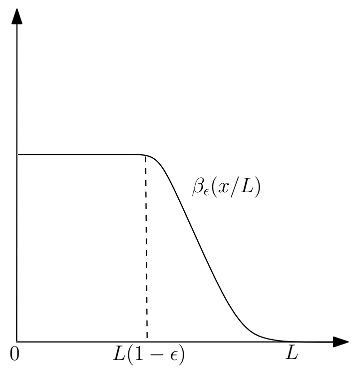
We can consider other Bochner-Riesz type kernels. From now on, we fix an $\epsilon > 0$ and B_L^ϵ will denote a transform from $L^2(M)$ to E_L with kernel

$$B_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w), \tag{1.4}$$

i.e. for any $f \in L^2(M)$,

$$B_L^\epsilon(f)(z) = \int_M B_L^\epsilon(z,w)f(w)dV(w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\langle f,\phi_i\rangle\phi_i(z),$$

where $\beta_\epsilon : [0,+\infty) \to [0,1]$ is a function of class $\mathcal{C}^\infty$ supported in $[0,1]$ such that $\beta_\epsilon(x) = 1$ for $x \in [0,1-\epsilon]$ and $\beta_\epsilon(x) = 0$ if $x \notin [0,1)$.



Observe that for $\epsilon = 0$, the transform B_L^0 is just the orthogonal projection of the space E_L i.e. the kernel $B_L^0(z,w) = K_L(z,w)$.

We recall now an estimate for the kernel $B_L^\epsilon(z,w)$ that is similar to the Bochner-Riesz kernel estimate (1.3).

Lemma 1.5. *Let $H : [0,+\infty) \to [0,1]$ be a function with continuous derivatives up to order $N > m$ with compact support in $[0,1]$. Then there exists a constant C_N independent of L such that*

$$\left|\sum_{i=1}^{k_L} H(\lambda_i/L)\phi_i(z)\phi_i(w)\right| \leq C_N L^m \frac{1}{(1 + L d_M(z,w))^N}, \quad \forall z,w \in M. \tag{1.5}$$

For a proof see [FM10b, Theorem 2.1] and some ideas can be traced back from [Sog87].

1.3 Harmonic extension

Now we will explain a technical tool in order to have gradient estimates for functions of the spaces E_L. Consider the product manifold $N = M \times \mathbb{R}$ endowed with the product metric denoted by $\tilde{g} = (\tilde{g}_{ij})_{ij}$. Thus, for $i = 1, \dots, m+1$,

$$(\tilde{g}_{ij})_{i,j=1,\dots,m+1} = \begin{pmatrix} (g_{ij})_{i,j=1}^m & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this matrix, we can calculate the gradient and the Laplacian for N. If $h(z,t)$ is a function defined on N then

$$|\nabla_N h(z,t)|^2 = |\nabla_M h(z,t)|^2 + \left(\frac{\partial h}{\partial t}(z,t)\right)^2$$

and

$$\Delta_N h(z,t) = \Delta_M h(z,t) + \frac{\partial^2 h}{\partial t^2}(z,t).$$

Note that $|\nabla_M h(z,t)| \leq |\nabla_N h(z,t)|$.

Let $f \in E_L$, we know that

$$f = \sum_{i=1}^{k_L} \beta_i \phi_i, \qquad \Delta_M \phi_i = -\lambda_i^2 \phi_i, \qquad 0 \leq \lambda_i \leq L.$$

Define for $(z,t) \in N$,

$$h(z,t) = \sum_{i=1}^{k_L} \beta_i \phi_i(z) e^{\lambda_i t}.$$

Observe that $h(z,0) = f(z)$. Moreover $|\nabla_M f(z)| \leq |\nabla_N h(z,0)|$.

The function h is harmonic in N because

$$\Delta_N h(z,t) = \sum_{i=1}^{k_L} \left[\beta_i e^{\lambda_i t} \Delta_M \phi_i(z) + \beta_i \phi_i(z) \Delta_{\mathbb{R}}(e^{\lambda_i t})\right] = 0.$$

In what follows, given $f \in E_L$, we will denote by h the defined harmonic extension in N of f. Observe that there are other possible ways of defining an harmonic extension of functions of E_L.
We don't have the mean-value property for an harmonic function, because it is not true for all manifolds (only for the harmonic manifolds, see [Wil50] for a complete characterization of them). What is always true is a "submean-value property" (with a uniform constant) for positive subharmonic functions, see for example [SY94, Chapter II, Section 6]).

Observe that since h is harmonic on N, $|h|^2$ is a positive subharmonic function on N. In fact, $|h|^p$ is subharmonic for all $p \geq 1$ (for a proof see [GW74, Proposition 1]). Therefore, we know that for all $r < R_0(M)$,

$$|h(z,t)|^2 \lesssim \fint_{B(z,r/L)\times I_r(t)} |h(w,s)|^2 dV(w) ds,$$

where $R_0(M) > 0$ denotes the injectivity radius of the manifold M and $I_r(t) = (t - r/L, t + r/L)$. In particular,

$$|f(z)|^2 \leq C_r L^{m+1} \int_{B(z,r/L)\times I_r} |h(w,s)|^2 dV(w) ds, \tag{1.6}$$

where $I_r = I_r(0)$.

Remark 1.1. The injectivity radius on M is defined as:

$$R_0(M) = \inf_{\xi \in M} R_0(\xi),$$

where

$$R_0(\xi) = \sup \left\{ r > 0 : \exp_\xi \text{ is defined on } \mathbb{B}(0,r) \subset T_\xi M \text{ and is injective} \right\}.$$

By the compactness of M, $R_0(M) > 0$ and the coordinate map $\exp_\xi$ maps $\mathbb{B}(0,r)$ to $B(\xi,r)$ diffeomorphically for all $r \leq R_0(M)$. Thus,

$$\operatorname{vol}(B(\xi,r)) = \int_{\mathbb{B}(0,r)} \sqrt{|g|(\exp_\xi(w))} dm(w) \simeq |\mathbb{B}(0,r)| \simeq r^m,$$

where $|\cdot|$ denotes the euclidean volume and the constants are uniform in ξ and r. Hence, if $r \leq R_0(M)$, then

$$C_1 r^m \leq \operatorname{vol}(B(\xi,r)) \leq C_2 r^m,$$

with constants depending only on the manifold. This last estimate is enough for the major part of this work. But, much more is known. For instance, in [Blü90, Lemma 2], it is proved that for all $\xi \in M$,

$$\left| \frac{\frac{\operatorname{vol}(B(\xi,r))}{\operatorname{vol}(M)}}{|\mathbb{B}(0,cr)|} - 1 \right| \leq C r^2,$$

with c and C independent of ξ and r. This estimate will be used in the last part of this work.

The following result relates the L^2-norm of f and h.

Proposition 1.6. *Let $r > 0$ be fixed. If $f \in E_L$, then*

$$2re^{-2r} \|f\|_2^2 \leq L \|h\|_{L^2(M\times I_r)}^2 \leq 2re^{2r} \|f\|_2^2. \tag{1.7}$$

Therefore, for $r < R_0(M)$,

$$\frac{L}{2r} \|h\|_{L^2(M\times I_r)}^2 \simeq \|f\|_2^2,$$

with constants depending only on the manifold M.

Proof. Using the orthogonality of $\{\phi_i\}_i$ we have

$$\begin{aligned}\|h\|^2_{L^2(M\times I_r)} &= \int_{I_r}\int_M \left|\sum_{i=1}^{k_L}\beta_i\phi_i(z)e^{\lambda_i t}\right|^2 dV(z)dt \\ &= \int_{I_r}\sum_{i=1}^{k_L}\int_M |\beta_i|^2|\phi_i(z)|^2 dV(z)e^{2\lambda_i t}dt \le \int_{I_r} e^{2Lt}dt\,\|f\|_2^2 .\end{aligned}$$

Similarly, one can prove the left hand side inequality of (1.7). □

We have a relation between the L^2-norm and L^∞-norm for functions of our spaces. It is clear that $\|f\|_2 \lesssim \|f\|_\infty$, for all $f\in E_L$. The reverse inequality is true with a dependance on L. More precisely,

Proposition 1.7. *Given $f\in E_L$,*

$$k_L^{-1}\|f\|_\infty^2 \lesssim \|f\|_2^2 .$$

Proof. By the submean-value inequality for $|h|^2$, we know that for $0<r<R_0(M)$,

$$\begin{aligned}|f(z)|^2 &\lesssim \fint_{B(z,r/L)\times I_r} |h(w,s)|^2 dV(w)ds \lesssim e^{2r}\|f\|_2^2\frac{L^m}{r^m} \\ &\simeq \frac{e^{2r}}{r^m}k_L\|f\|_2^2 = C_r k_L\|f\|_2^2 ,\end{aligned}$$

where we have used Remark 1.1.
Taking $r=R_0(M)/2$ we get $\|f\|_\infty^2 \le Ck_L\|f\|_2^2$ with C a constant depending only on M.

Alternatively, we can prove this estimate using the reproducing property. Indeed, for any $z\in M$,

$$f(z)=\int_M f(w)K_L(z,w)dV(w).$$

Thus, using Hölder inequality,

$$|f(z)|^2 \le \|f\|_2^2\int_M |K_L(z,w)|^2 dV(w) = \|f\|_2^2 K_L(z,z)\simeq \|f\|^2 k_L, \quad \forall z\in M.$$

□

We recall now a result proved by Schoen and Yau that estimates the gradient of harmonic functions.

Theorem (Schoen-Yau)**.** *Let N be a complete Riemannian manifold with Ricci curvature bounded below by -(n-1)K (n is the dimension of N and K a positive constant). Suppose B_a is a geodesic ball in N with radius a and h is an harmonic function on B_a. Then*

$$\sup_{B_{a/2}}|\nabla h| \le C_n\left(\frac{1+a\sqrt{K}}{a}\right)\sup_{B_a}|h|, \tag{1.8}$$

where C_n is a constant depending only on the dimension of N.

For a proof see [SY94, Corollary 3.2., page 21].

Remark 1.2. We will use Schoen and Yau's estimate in the following context. Take $N = M \times \mathbb{R}$. Observe that $\mathrm{Ricc}(N) = \mathrm{Ricc}(M)$ which is bounded from below because M is compact. Note that N is complete because it is a product of two complete manifolds. We put $a = r/L$ ($r < R_0(M)$) and $B_a = B(z, r/L) \times I_r$ (this is not the ball of center $(z, 0) \in N$ and radius r/L, but it contains and it is contained in such ball of comparable radius).

Using Schoen and Yau's theorem, we deduce the global Bernstein inequality for a single eigenfunction.

Corollary 1.8 (Bernstein inequality)**.** *If u is an eigenfunction of eigenvalue λ^2, then*

$$\|\nabla u\|_\infty \lesssim \lambda \|u\|_\infty . \tag{1.9}$$

Proof. The harmonic extension of u is $h(z,t) = u(z)e^{\lambda t}$. Applying inequality (1.8) to h (taking $a = R_0(M)/(2\lambda)$),

$$|\nabla u(z)| \lesssim \lambda \|h\|_{L^\infty(M \times I_{R_0/2})} \simeq \lambda \|u\|_\infty .$$

□

We conjectured that in inequality (1.9), one can replace u by any function $f \in E_L$, i.e.

$$\|\nabla f\|_\infty \lesssim L \|f\|_\infty . \tag{1.10}$$

This estimate has been proved recently in a work of F. Filbir and H.N. Mhaskar (see [FM10b, Theorem 2.2] for more details). The proof is rather delicate and it requires an estimate like (1.5) of the gradient of the kernels $B_L^\epsilon(z,w)$.
For instance, as a direct consequence of Green's formula, we have the L^2-Bernstein inequality for the space E_L:

$$\|\nabla f\|_2 \lesssim L \|f\|_2 \quad \forall f \in E_L.$$

For our purpose, it is sufficient to have a weaker Bernstein type inequality that compares the L^∞-norm of the gradient with the L^2-norm of the function.

Proposition 1.9. *Let $f \in E_L$. Then there exists a universal constant C such that*

$$\|\nabla f\|_\infty \leq C\sqrt{k_L} L \|f\|_2 .$$

For the proof, we need the following lemma.

Lemma 1.10. *For all $f \in E_L$ and $0 < r < R_0(M)/2$,*

$$|\nabla f(z)|^2 \leq C_r L^{m+2+1} \int_{B(z,r/L) \times I_r} |h(w,s)|^2 dV(w) ds.$$

Proof. Using inequality (1.8) and the submean-value inequality for $|h|^2$, we have

$$|\nabla f(z)|^2 \leq |\nabla h(z,0)|^2 \lesssim \frac{L^2}{r^2} \sup_{B(z,r/L)\times I_r} |h(w,t)|^2$$
$$\lesssim \frac{L^{m+1+2}}{\tilde{r}^{m+2+1}} \int_{B(z,\tilde{r}/L)\times I_{\tilde{r}}} |h(\xi,s)|^2 dV(\xi) ds,$$

where $\tilde{r} = 2r$. □

Proof of Proposition 1.9. By Lemma 1.10, given $0 < r < R_0(M)/2$, there exists a constant C_r such that

$$|\nabla f(z)|^2 \leq C_r k_L L^2 L \int_{M\times I_r} |h(w,s)|^2 dV(w) ds \simeq C_r k_L L^2 \|f\|_2^2,$$

where we have used Proposition 1.6. Taking $r = R_0(M)/4$, we get $|\nabla f(z)|^2 \leq C k_L L^2 \|f\|_2^2$ for all $z \in M$.

Alternatively, we can prove this result using the Bernstein inequality (1.10) for E_L. Indeed, given $f \in E_L$ we have:

$$\|\nabla f\|_\infty \lesssim L\|f\|_\infty \lesssim L\sqrt{k_L}\|f\|_2,$$

where we have used Proposition 1.7. □

Chapter 2

Carleson Measures and Logvinenko-Sereda sets

In this chapter, we want to give a complete geometric characterization for the Carleson measures on M. A straight forward application is the Plancherel-Pólya inequality that says that a family is a finite union of uniformly separated families if and only if the normalized reproducing kernels form a Bessel sequence. This will be used later on, to prove that a M-Z family contains a separated family which is also M-Z.

We will focus our attention on the reverse Carleson's inequalities that are also called Logvinenko-Sereda (L-S) sets inequality when the measures are the characteristic function of a set with the usual Lebesgue measure. Thus, M-Z inequalities can be seen as the "discrete" versions of the L-S sets. Moreover, one can obtain a sequence of L-S sets starting by a M-Z family, just by taking unions of balls centered at the points of the family of radius smaller than their separation. Consequently, these comparison results seem easier than the M-Z inequality.

The outline of the chapter is the following. In the first section, we define the Carleson measures and prove a geometric characterization of them. In the second section, we give the precise definition of the L-S sets and prove their characterization. In this later case, for an easy read, the proof is divided in two propositions: the necessity and the sufficiency.

2.1 Characterization of Carleson measures

Carleson measures are of its own interest. Let us recall a bit of history and the intrinsic relation of the Carleson measures with the interpolating sequences. The theory of Carleson measures and interpolating sequences has its roots in L. Carleson's paper of 1958 (see [Car58]). Carleson characterized completely the interpolating sequences $\Lambda = \{\lambda_n\}$ for the space of bounded analytic functions $H^\infty(\mathbb{D})$. He showed that Λ is interpolating for $H^\infty(\mathbb{D})$ if and only if

- A separation condition on the sequences of points holds:

$$\beta(\lambda_j, \lambda_k) \geq c > 0, \quad k \neq j,$$

  where β is the hyperbolic distance.

- The measure

$$\mu := \sum_j (1 - |\lambda_j|^2)\delta_{\lambda_j}$$

  is a Carleson measure.

In this context, we say that μ is a Carleson measure for $H^p(\mathbb{D})$ if

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \lesssim \|f\|^p_{H^p(\mathbb{D})}, \quad \forall f \in H^p(\mathbb{D}).$$

There is a geometric characterization for the Carleson measures, sometimes easier to check (for more details, see [Car62, Theorem 1]):

$$\mu(S) \leq Cl,$$

for all sets S of the form

$$S = \left\{ re^{i\theta}; \quad r \geq 1 - l, \theta_0 \leq \theta \leq \theta_0 + l \right\}, \quad l \leq 1.$$

Now, the Hardy space $H^2(\mathbb{D})$ is a Hilbert space with reproducing kernel, i.e. for any $z \in \mathbb{D}$, there is a function $k_z \in H^2(\mathbb{D})$, the reproducing kernel for z, which is characterized by the fact that for any $f \in H^2(\mathbb{D})$, $f(z) = \langle f, k_z \rangle$. A result of Shapiro and Shields yields that interpolating sequences for $H^2(\mathbb{D})$ are characterized by the following two conditions:

- Λ is uniformly separated.
- The measure

$$\mu := \sum_j \frac{1}{\|k_{\lambda_j}\|^2} \delta_{\lambda_j}$$

  is a Carleson measure for $H^2(\mathbb{D})$.

Thus, the Carleson measures are strongly related with the interpolating sequences.

Motivated by this intrinsic relation, we study the Carleson measures on M. The precise definition is the following.

Definition 2.1. *Let $\mu = \{\mu_L\}_{L\geq 0}$ be a sequence of measures on M. We say that μ is an L^2-**Carleson sequence** for M if there exists a positive constant C such that for all $L \geq 1$ and $f_L \in E_L$,*

$$\int_M |f_L|^2 d\mu_L \leq C \int_M |f_L|^2 dV.$$

It is easy to check that if a triangular family of points $\mathcal{Z}$ is interpolating (see Definition 3.2 for a precise description), then the measures

$$\mu_L = \sum_{j=1}^{m_L} \frac{1}{k_L} \delta_{z_{Lj}}$$

form a sequence of Carleson measures (this follows from the definition of an interpolating family in terms of the reproducing kernel, see Remark 3.4). We want to give a geometric characterization that should be easy to check. Our main result is:

Theorem 2.2. *Let μ be a sequence of measures on M. Then, μ is L^2-Carleson for M if and only if there exists a $C > 0$ such that for all $L \geq 1$,*

$$\sup_{\xi \in M} \mu_L(B(\xi, 1/L)) \leq \frac{C}{k_L} \simeq \frac{1}{L^m}. \tag{2.1}$$

This characterization is known in the case of the $M = \mathbb{S}^m$ (a detailed discussion can be found in [Mar07, Theorem 4.5]). To the best of our knowledge, for the general case, Theorem 2.2 is new.

Remark 2.1. Condition (2.1) can be viewed as

$$\sup_{\xi \in M} \frac{\mu_L(B(\xi, 1/L))}{\operatorname{vol}(B(\xi, 1/L))} \lesssim 1.$$

First, we prove a technical result that allows us to modify slightly the condition (2.1).

Lemma 2.3. *Let μ be a sequence of measures on M. Then, the following conditions are equivalent.*

1. *There exists a constant $C = C(M) > 0$ such that for each $L \geq 1$,*

$$\sup_{\xi \in M} \mu_L(B(\xi, 1/L)) \leq \frac{C}{k_L}.$$

2. *There exist $c = c(M) > 0$ ($c < 1$ small) and $C = C(M) > 0$ such that for all $L \geq 1$,*

$$\sup_{\xi \in M} \mu_L(B(\xi, c/L)) \leq \frac{C}{k_L}.$$

Proof. Obviously, the first condition implies the second one since

$$B(\xi, c/L) \subset B(\xi, 1/L).$$

Let's see the converse. The manifold M is covered by the union of balls of center $\xi \in M$ and radius c/L. Taking into account the 5-r covering lemma (see [Mat95, Chapter 2, page 23] for more details), we get a finite family of disjoint balls, denoted by $B_i = B(\xi_i, c/L)$, such that M is covered by the union of $5B_i$. Let $\xi \in M$ and consider $B := B(\xi, 1/L)$.

Denote by n the number of balls $\bar{B}_i$ such that $\bar{B} \cap 5\bar{B}_i \neq \emptyset$. Since $\bar{B}$ is compact, we have a finite number of these balls so that

$$\bar{B} \subset \bigcup_{i=1}^{n} \overline{5B_i}.$$

We claim that n is independent of L. In this case, we get

$$\mu_L(B) \leq \sum_{i=1}^{n} \mu_L(B(\xi_i, 5c/L)) \lesssim \frac{n}{k_L}$$

and thus our statement is proved. Indeed, using the triangle inequality, for all $i = 1, \ldots, n$,

$$B(\xi_i, c/L) \subset B(\xi, 10/L).$$

Therefore,

$$\bigcup_{i=1}^{n} B(\xi_i, c/L) \subset B(\xi, 10/L),$$

where the union is a disjoint union of balls. Now,

$$\frac{10^m}{L^m} \simeq \mathrm{vol}(B(\xi, 10/L)) \geq \sum_{i=1}^{n} \mathrm{vol}(B_i) \simeq n\frac{c^m}{L^m},$$

where we have used Remark 1.1.
Hence, $n \lesssim (10/c)^m$ and we can choose it independently of L. $\square$

Now we can prove the characterization of the Carleson measures.

Proof of Theorem 2.2. Assume condition (2.1) holds. We need to prove the existence of a constant $C > 0$ (independent of L) such that for each $f \in E_L$,

$$\int_M |f|^2 d\mu_L \leq C \int_M |f|^2 dV.$$

Let $f \in E_L$ with L and $r > 0$ (small) fixed. Using (1.6) and Proposition 1.6, we have:

$$\begin{aligned}
\int_M |f(z)|^2 d\mu_L &\leq C_r L^{m+1} \int_M \int_{B(z,r/L)\times I_r} |h(w,s)|^2 dV(w) ds d\mu_L(z) \\
&= C_r L^{m+1} \int_{M\times I_r} |h(w,s)|^2 \mu_L(B(w,r/L)) dV(w) ds \\
&\leq C_r L^{m+1} \frac{1}{k_L} \int_{M\times I_r} |h(w,s)|^2 dV(w) ds \simeq \|f\|_2^2
\end{aligned}$$

with constants independent of L. Therefore, $\mu = \{\mu_L\}_L$ is L^2-Carleson for M.

For the converse, assume that μ is L^2-Carleson for M. We have to show the existence of a constant C such that for all $L \geq 1$ and $\xi \in M$, $\mu_L(B(\xi, c/L)) \leq C/k_L$ (for some small constant $c > 0$). We will argue by contradiction, i.e. assume that for all $n \in \mathbb{N}$ there exists L_n and a ball B_n of radius c/L_n such that $\mu_{L_n}(B_n) > n/k_{L_n} \simeq n/L_n^m$ (c will be chosen later). Let b_n be the center of the ball B_n. Define $F_n(w) = K_{L_n}(b_n, w)$. Note that the function $L_n^{-m/2} F_n \in E_{L_n}$ and $\|F_n\|_2^2 = K_{L_n}(b_n, b_n) \simeq L_n^m$. Therefore,

$$\begin{aligned} C &\simeq \int_M |L_n^{-m/2} F_n|^2 dV \gtrsim \int_M |L_n^{-m/2} F_n|^2 d\mu_{L_n} \gtrsim \int_{B_n} |L_n^{-m/2} F_n|^2 d\mu_{L_n} \\ &\geq \inf_{w \in B_n} |L_n^{-m/2} F_n(w)|^2 \mu_{L_n}(B_n) \gtrsim \inf_{w \in B_n} |F_n(w)|^2 \frac{n}{L_n^{2m}}. \end{aligned}$$

Now we will study this infimum. Let $w \in B_n = B(b_n, c/L_n)$. Then, by Proposition 1.9,

$$\begin{aligned} ||F_n(b_n)| - |F_n(w)|| \leq |F_n(b_n) - F_n(w)| &\leq \frac{c}{L_n} \|\nabla F_n\|_\infty \\ &\leq \frac{c}{L_n} C_1 \sqrt{k_{L_n}} L_n \|F_n\|_2 \simeq c C_1 k_{L_n}. \end{aligned}$$

We pick c small enough so that

$$\inf_{B_n} |F_n(w)|^2 \geq C L_n^{2m}.$$

Finally, we have shown that $C \gtrsim n$ for all $n \in \mathbb{N}$. This leads to a contradiction. $\square$

Using the characterization of the Carleson measures, we can prove a Plancherel-Pólya type inequality for the spaces E_L. We recall that in the context of the Paley-Wiener spaces $PW^2_{(-\pi,\pi)}$, the Plancherel-Pólya inequality bounds the discrete norm given by the evaluation at the integers in terms of the L^2-norm:

$$\sum_{j=-\infty}^{\infty} |f(j)|^2 \leq C \|f\|_2^2, \quad \forall f \in PW^2_{(-\pi,\pi)}.$$

Moreover, there exists a constant C such that

$$\sum_j |f(z_j)|^2 \leq C \|f\|_2^2, \quad \forall f \in PW^2_{(-\pi,\pi)},$$

if and only if the sequence $\{z_j\}_j$ can be expressed as a finite union of separated sequences. In our setting, a similar result can be proved. Before we give the precise statement, we shall introduce the concept of a separated family of points.

Definition 2.4. *Let $\mathcal{Z} = \{z_{Lj}\}_{j \in \{1,\dots,m_L\}, L \geq 1} \subset M$ be a triangular family of points, where $m_L \to \infty$ as $L \to \infty$. We say that $\mathcal{Z}$ is uniformly separated if there exists $\epsilon > 0$ such that*

$$d_M(z_{Lj}, z_{Lk}) \geq \frac{\epsilon}{L}, \quad \forall j \neq k, \quad \forall L \geq 1,$$

where ϵ is called the separation constant of $\mathcal{Z}$.

The following result is a Plancherel-Pólya type theorem but in the context of the Paley-Wiener spaces E_L. To the best of our knowledge, this result is new.

Theorem 2.5 (Plancherel-Pólya Theorem)**.** *Let $\mathcal{Z}$ be a triangular family of points in M, i.e. $\mathcal{Z} = \{z_{Lj}\}_{j\in\{1,\dots,m_L\},L\geq 1} \subset M$. Then $\mathcal{Z}$ is a finite union of uniformly separated families, if and only if there exists a constant $C > 0$ such that for all $L \geq 1$ and $f_L \in E_L$,*

$$\frac{1}{k_L}\sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq C \int_M |f_L(\xi)|^2 dV(\xi). \tag{2.2}$$

Remark 2.2. The above result is interesting because the inequality (2.2) means that the sequence of normalized reproducing kernels is a Bessel sequence for E_L, i.e.

$$\sum_{j=1}^{m_L} |\langle f, \tilde{K}_L(\cdot, z_{Lj})\rangle|^2 \lesssim \|f\|_2^2 \quad \forall f \in E_L,$$

where $\left\{\tilde{K}_L(\cdot, z_{Lj})\right\}_j$ are the normalized reproducing kernels. Note that $|\tilde{K}_L(\cdot, z_{Lj})|^2 \simeq |K_L(\cdot, z_{Lj})|^2 k_L^{-1}$. That is the reason why the quantity k_L appears in the inequality (2.2).

Proof. This is a consequence of Theorem 2.2 applied to the measures

$$\mu_L = \frac{1}{k_L}\sum_{j=1}^{m_L} \delta_{z_{Lj}}, \;\; L \geq 1.$$

□

2.2 Characterization of Logvinenko-Sereda Sets

Before we state the characterization, we would like to recall some history of these kind of inequalities. The classical Logvinenko-Sereda (L-S) theorem describes some equivalent norms for functions in the Paley-Wiener space PW_Ω^p. The precise statement is the following:

Theorem 2.6 (L-S)**.** *Let Ω be a bounded set and $1 \leq p < +\infty$. A set $E \subset \mathbb{R}^d$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C_p \int_E |f(x)|^p dx, \;\; \forall f \in PW_\Omega^p,$$

if and only if there is a cube $K \subset \mathbb{R}^d$ such that

$$\inf_{x\in\mathbb{R}^d} |(K + x) \cap E| > 0.$$

One can find the original proof in [LS74] and another proof can be found in [HJ94, p. 112-116].

Luecking studied in [Lue83] this notion for the Bergman spaces. Following his ideas, in [MOC08], the following result has been proved.

Theorem 2.7. *Let $1 \le p < +\infty$, μ be a doubling measure and let $\mathcal{A} = \{A_L\}_{L\ge 0}$ be a sequence of sets in $\mathbb{S}^d$. Then $\mathcal{A}$ is $L^p(\mu)$-L-S if and only if $\mathcal{A}$ is μ-relatively dense.*

For the precise definitions and notations see [MOC08]. Using the same ideas, we prove the above theorem for the case of an arbitrary compact manifold M and the measure given by the volume element ($p = 2$).

In what follows, $\mathcal{A} = \{A_L\}_L$ will be a sequence of sets in M.

Definition 2.8. *We say that $\mathcal{A}$ is* L-S *if there exists a constant $C > 0$ such that for any $L \ge 1$ and $f_L \in E_L$,*

$$\int_M |f_L|^2 dV \le C \int_{A_L} |f_L|^2 dV.$$

Definition 2.9. *The sequence of sets $\mathcal{A} \subset M$ is relatively dense if there exists $r > 0$ and $\rho > 0$ such that for all $L \ge 1$,*

$$\inf_{z\in M} \frac{\mathrm{vol}(A_L \cap B(z, r/L))}{\mathrm{vol}(B(z, r/L))} \ge \rho > 0.$$

Remark 2.3. It is equivalent to have this property for all $L \ge L_0$ for some L_0 fixed.

A natural example of relatively dense sets is the following. Consider a separated family in M, $\mathcal{Z} = \{z_{Lj}\}_{j=1,\dots,m_L; L\ge 1}$ with separation constant s. Let us denote by $A_L = M \setminus \cup_{j=1}^{m_L} B(z_{Lj}, \frac{s}{10L})$. It is easy to check that the family $\mathcal{A} = \{A_L\}_L$ is relatively dense.

Our main statement is the following:

Theorem 2.10. *$\mathcal{A}$ is* L-S *if and only if $\mathcal{A}$ is relatively dense.*

A straight forward consequence is the stated below uncertainty principle for functions in $L^2(M)$ modelled after the uncertainty principle proved by Havin and Joricke in the context of the Paley-Wiener spaces (see [HJ94]).

Corollary 2.11. *The sequence of sets $\mathcal{A}$ is relatively dense if and only if there exists a constant $C > 0$ such that for all $f \in L^2(M)$ and $L \ge 1$,*

$$\int_M |f|^2 dV \le C \left(\int_{A_L} |f|^2 + \sum_{\lambda > L} \|f_\lambda\|_2^2 \right), \tag{2.3}$$

where f_λ is the orthogonal projection of f into the eigenspace of eigenvalue λ^2.

Proof. For any $f \in L^2(M)$, the decomposition $f = \sum_\lambda f_\lambda$ holds with $f_\lambda = \langle f, \phi_\lambda \rangle \phi_\lambda$, where $\Delta_M \phi_\lambda = -\lambda^2 \phi_\lambda$. Assume $\mathcal{A}$ is relatively dense. Then, using Theorem 2.10, estimate

(2.3) is satisfied because

$$\begin{aligned}\|f\|_2^2 &= \sum_{\lambda\leq L}\|f_\lambda\|_2^2+\sum_{\lambda>L}\|f_\lambda\|_2^2=\int_M |P_{E_L}(f)|^2dV+\sum_{\lambda>L}\|f_\lambda\|_2^2\\ &\leq C\left(\int_{A_L}|P_{E_L}(f)|^2dV+\int_{A_L}|P_{E_L}^{\perp}(f)|^2dV+\sum_{\lambda>L}\|f_\lambda\|_2^2\right)\\ &= C\left(\int_{A_L}|f|^2dV+\sum_{\lambda>L}\|f_\lambda\|_2^2\right).\end{aligned}$$

Conversely, if (2.3) holds then it is trivial that $\mathcal{A}$ is relatively dense. Indeed, for any $L \geq 1$ and $f_L \in E_L$, $\langle f_L, \phi_\lambda\rangle = 0$ if $\lambda > L$. Thus, condition (2.3) applied to functions of the space E_L is reduced to the fact that $\mathcal{A}$ is a sequence of L-S sets. Hence, by Theorem 2.10 $\mathcal{A}$ is relatively dense. □

We shall prove the two implications in the statement of Theorem 2.10 separately. First we will show that this condition is necessary. Before proceeding, we construct functions in E_L with a desired decay of its L^2-integral outside a ball.

Proposition 2.12. *Given $\xi \in M$ and $\epsilon > 0$, there exist functions $f_L = f_{L,\xi} \in E_L$ and $R_0 = R_0(\epsilon, M) > 0$ such that*

1. $\|f_L\|_2 = 1$.
2. *For all $L \geq 1$,*
$$\int_{M\setminus B(\xi,R_0/L)}|f_L|^2dV<\epsilon.$$
3. *For all $L \geq 1$ and any subset $A \subset M$,*
$$\int_A |f_L|^2dV\leq C_1\frac{\mathrm{vol}(A\cap B(\xi,R_0/L))}{\mathrm{vol}(B(\xi,R_0/L))}+\epsilon,$$
where C_1 is a constant independent of L, ξ and f_L.

Remark. In the above Proposition, the R_0 does not depend on the point ξ.

Proof. Given $z, \xi \in M$ and $L \geq 1$, let $S_L^N(z,\xi)$ denote the Bochner-Riesz kernel of index $N \in \mathbb{N}$ associated to the Laplacian, i.e

$$S_L^N(z,\xi)=\sum_{i=1}^{k_L}\left(1-\frac{\lambda_i}{L}\right)^N\phi_i(z)\phi_i(\xi).$$

Note that $S_L^0(z,\xi) = K_L(z,\xi)$. Recall that the Bochner-Riesz kernel satisfies the following inequality.

$$|S_L^N(z,\xi)|\leq CL^m(1+Ld_M(z,\xi))^{-N-1}. \tag{2.4}$$

Also, on the diagonal, $S_L^N(z,z) \simeq C_N L^m$ and $\left\|S_L^N(\cdot,\xi)\right\|_2^2 \simeq C_N L^m$.

Given $\xi \in M$, define for all $L \geq 1$,

$$f_{L,\xi}(z) := f_L(z) = \frac{S_L^N(z,\xi)}{\|S_L^N(\cdot,\xi)\|_2}.$$

We will choose the order N later. Each f_L belongs to the space E_L and has unit L^2-norm. Let us verify the second property claimed in Proposition 2.12. Fix a radius R. Using the estimate (2.4),

$$\int_{M\setminus B(\xi,R/L)} |f_L|^2 dV \leq C_N L^m \int_{M\setminus B(\xi,R/L)} \frac{dV}{(L d_M(z,\xi))^{2(N+1)}} = (\star)$$

For any $t \geq 0$, consider the following set.

$$A_t := \left\{ z \in M : \quad d_M(z,\xi) \geq \frac{R}{L}, \quad d_M(z,\xi) < \frac{t^{-1/(2(N+1))}}{L} \right\}.$$

Note that for $t > R^{-2(N+1)}$ we have $A_t = \emptyset$, and for $t < R^{-2(N+1)}$ we obtain $A_t \subset B(\xi, t^{-1/(2(N+1))}/L)$. Using the distribution function and Remark 1.1, we have:

$$(\star) = C_N L^m \int_0^{R^{-2(N+1)}} \mathrm{vol}(A_t) dt \leq C_N \frac{1}{R^{2(N+1)-m}},$$

provided $N+1 > m/2$. Thus if we pick R_0 big enough we get

$$\int_{M\setminus B(\xi,R_0/L)} |f_L|^2 dV < \epsilon. \tag{2.5}$$

Now the third property claimed in Proposition 2.12 follows from (2.5). Indeed, given any subset A in the manifold M,

$$\int_A |f_L|^2 dV \leq \int_{A\cap B(\xi,R_0/L)} |f_L|^2 dV + \epsilon.$$

Observe that

$$\begin{aligned}\int_{A\cap B(\xi,R_0/L)} |f_L|^2 dV &\lesssim C_N L^m \int_{A\cap B(\xi,R_0/L)} \frac{dV(z)}{(1+L d_M(z,\xi))^{2(N+1)}} \\ &\lesssim C_N R_0^m \frac{\mathrm{vol}(A\cap B(\xi,R_0/L))}{\mathrm{vol}(B(\xi,R_0/L))}.\end{aligned}$$

□

Now we are ready to prove one of the implications in the characterization of the L-S sets.

Proposition 2.13. *Assume $\mathcal{A}$ is* L-S. *Then $\mathcal{A}$ is relatively dense.*

Proof. Assume $\mathcal{A}$ is L-S, i.e.

$$\int_M |f_L|^2 dV \leq C \int_{A_L} |f_L|^2 dV.$$

Let $\xi \in M$ be an arbitrary point. Fix $\epsilon > 0$ and consider the R_0 and the functions $f_L \in E_L$ given by Proposition 2.12. Using the third property of Proposition 2.12 for the sets A_L, we get for all $L \geq 1$,

$$1 = \|f_L\|_2^2 \leq C \int_{A_L} |f_L|^2 \leq CC_1 \frac{\mathrm{vol}(A_L \cap B(\xi, R_0/L))}{\mathrm{vol}(B(\xi, R_0/L))} + C\epsilon,$$

where C_1 is a constant independent of L, ξ and f_L. Therefore, we have proved that there exist constants c_1 and c_2 such that

$$\frac{\mathrm{vol}(A_L \cap B(\xi, R_0/L))}{\mathrm{vol}(B(\xi, R_0/L))} \geq c_1 - c_2\epsilon.$$

Hence, $\mathcal{A}$ is relatively dense provided $\epsilon > 0$ is small enough. □

Now, we shall prove the sufficient condition of the main result. Before we continue, we will prove a fact concerning the uniform limit of harmonic functions with respect to some metric.

Lemma 2.14. *Let $\{H_n\}_n$ be a family of uniformly bounded real functions defined on the ball $\mathbb{B}(0,\rho) \subset \mathbb{R}^d$ for some $\rho > 0$. Let g be a non-singular $\mathcal{C}^\infty$ metric such that g and all its derivatives are uniformly bounded and $g_{ij}(0) = \delta_{ij}$. Define $g_n(z) = g(z/L_n)$ (the rescaled metrics) where L_n is a sequence of values tending to ∞ as n increases. Assume the family $\{H_n\}_n$ converges uniformly on compact subsets of $\mathbb{B}(0,\rho)$ to a limit function $H : \mathbb{B}(0,\rho) \to \mathbb{R}$ and H_n is harmonic with respect to the metric g_n (i.e. $\Delta_{g_n} H_n = 0$). Then, the limit function H is harmonic in the Euclidean sense.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{B}(0,\rho))$. We have

$$\int_{\mathbb{B}(0,\rho)} \Delta_g f \varphi dV = \int_{\mathbb{B}(0,\rho)} f \Delta_g \varphi dV.$$

Let Δ be the Euclidean Laplacian. We claim that $\Delta_{g_n}\varphi \to \Delta\varphi$ uniformly and $\Delta_{g_n}\varphi$ is uniformly bounded on $\mathbb{B}(0,\rho)$. Then

$$0 = \int_{\mathbb{B}(0,\rho)} H_n \Delta_{g_n} \varphi dV_{g_n} \to \int_{\mathbb{B}(0,\rho)} H \Delta\varphi dm(z) = \int_{\mathbb{B}(0,\rho)} \Delta H \varphi dm(z).$$

Therefore, the limit function H is harmonic in the weak sense. Applying Weyl's lemma, we conclude that H is harmonic in the Euclidean sense.
In order to finish the statement, we shall prove the claim. The uniform boundedness of $\Delta_{g_n}\varphi$ is clear since g_n is the rescaled metric of g that has derivatives uniformly bounded. It only remains to check that $\Delta_{g_n}\varphi$ tends to the Euclidean Laplacian of φ. Observe the following facts.

1. Since g_n is the rescaled metric of g, we have $g_{n,ij}(z) = g_{ij}(z/L_n) \to g_{ij}(0) = \delta_{ij}$ (also $g_n^{ij}(z) \to \delta_{ij}$). Therefore, g_n is converging (uniformly) to the Euclidean metric.
2. All the derivatives of $g_{n,ij}$ and g_n^{ij} are going to 0 uniformly since
$$\left|\frac{\partial}{\partial x_i} g_{n,ij}(z)\right| = \left|\frac{\partial g_{ij}}{\partial x_i}(z/L_n)\right| \frac{1}{L_n} \le C\frac{1}{L_n} \to 0,\ n \to \infty.$$
3. The determinant of the metric g_n tends to 1 since $|g_n|(z) = |g|(z/L_n)$ and $|g|$ is a combination of $g_{ij}(z/L_n) \to \delta_{ij}$.
4. $\partial_{x_i}|g_n|(z) \to 0$ (as the derivative of $|g_n|(z)$ is a combination of products of the form $(\partial_{x_i} g_{n,jk})g_{n,lm}$ which tends to 0). Therefore we have
$$\frac{1}{\sqrt{|g_n|}}\frac{\partial}{\partial x_i}\sqrt{|g_n|} = \frac{1}{2}\frac{1}{|g_n|}\frac{\partial}{\partial x_i}|g_n| \to \frac{1}{2}\cdot 0 = 0.$$

Now using the above observations,
$$\Delta_{g_n}\varphi = \sum_{i,j}\left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right) g_n^{ij} + \sum_{i,j}\left(\frac{1}{\sqrt{|g_n|}} g_n^{ij}\frac{\partial}{\partial x_i}\sqrt{|g_n|}\right)\frac{\partial\varphi}{\partial x_j} + \sum_{i,j}\frac{\partial g_n^{ij}}{\partial x_i}\frac{\partial\varphi}{\partial x_j} \to \Delta\varphi,$$
that proves the claim. □

Remark 2.4. The above argument also holds if we have a sequence of metrics g_n converging uniformly to g whose derivatives also converge uniformly to the derivatives of g. In this case, the conclusion would be that the limit is harmonic with respect to the limit metric g.

Now, we have all the tools to prove the sufficient condition of the main result.

Proposition 2.15. *If $\mathcal{A}$ is relatively dense then it is* L-S.

Proof. Fix $\epsilon > 0$ and $r > 0$. Let $D := D_{\epsilon,r,f_L}$ be
$$D = \left\{z \in M :\ |f_L(z)|^2 = |h_L(z,0)|^2 \ge \epsilon \fint_{B(z,\frac{r}{L})\times I_r} |h_L(\xi,t)|^2 dV(\xi)dt\right\},$$
where h_L is the harmonic extension of f_L defined as
$$h_L(z,t) = \sum_{i=1}^{k_L}\beta_i\phi_i(z)e^{\lambda_i t},\quad f_L(z) = \sum_{i=1}^{k_L}\beta_i\phi_i(z).$$
The norm of f_L is almost concentrated on D because
$$\begin{aligned}\int_{M\setminus D}|f_L(z)|^2 dV(z) &\lesssim \\ &\lesssim \epsilon\frac{1}{l(I_r)}\int_{M\times I_r}|h_L(\xi,t)|^2\frac{L^m}{r^m}\int_{(M\setminus D)\cap B(\xi,r/L)} dV(z)dV(\xi)dt \\ &\lesssim \epsilon\frac{1}{l(I_r)}\int_{M\times I_r}|h_L(\xi,t)|^2 dV(\xi)dt \lesssim e^{2r}\epsilon\int_M |f_L|^2 dV,\end{aligned}$$

where we have used Proposition 1.6.
It is enough to prove
$$\int_D |f_L|^2 dV \lesssim \int_{A_L} |f_L|^2 dV \tag{2.6}$$
with constants independent of L and for this, it is sufficient to show that there exists a constant $C > 0$ such that for all $w \in D$
$$|f_L(w)|^2 \le \frac{C}{\mathrm{vol}(B(w, r/L))} \int_{A_L \cap B(w,r/L)} |f_L(\xi)|^2 dV(\xi). \tag{2.7}$$
Because then, (2.6) follows by integrating (2.7) over D. So we need to prove (2.7).

This is the outline of the proof: we assume that (2.7) is not true in order to construct functions that satisfy the opposite inequality. Then we parametrize these functions and prove that their limit is harmonic with unit norm and is zero in a subset of positive measure. This will lead to a contradiction. Now we proceed with the details.

Step 1. *Parametrization and rescalement of the functions.*

If (2.7) is not true, then for all $n \in \mathbb{N}$ there exists L_n, functions $f_n \in E_{L_n}$ and $w_n \in D_n = D_{\epsilon,r,f_n}$ such that
$$|f_n(w_n)|^2 > \frac{n}{\mathrm{vol}(B(w_n, r/L_n))} \int_{A_{L_n} \cap B(w_n,r/L_n)} |f_n|^2 dV.$$
By the compactness of the manifold M, there exists $\rho_0 = \rho_0(M) > 0$ such that for all $w \in M$, the exponential map, $\exp_w : \mathbb{B}(0, \rho_0) \to B(w, \rho_0)$, is a diffeomorphism and $(B(w, \rho_0), \exp_w^{-1})$ is a normal coordinate chart, where w is mapped to 0 and the metric g verifies $g_{ij}(0) = \delta_{ij}$.
For all $n \in \mathbb{N}$, take $\exp_n(z) := \exp_{w_n}(rz/L_n)$ which is defined in $\mathbb{B}(0, 1)$ and act as follows:
$$\begin{array}{rcccccc} \exp_n : & \mathbb{B}(0,1) & \longrightarrow & \mathbb{B}(0, r/L_n) & \longrightarrow & B(w_n, r/L_n) \\ & z & \longmapsto & \frac{rz}{L_n} & \longmapsto & \exp_{w_n}\left(\frac{rz}{L_n}\right) =: w \end{array}$$
Consider $F_n(z) := c_n f_n(\exp_n(z)) : \mathbb{B}(0, 1) \to B(w_n, r/L_n) \overset{c_n f_n}{\to} \mathbb{R}$ and the corresponding harmonic extension h_n of f_n. Set
$$H_n(z, t) := c_n h_n(\exp_n(z), rt/L_n),$$
defined on $\mathbb{B}(0, 1) \times J_1$ (where $J_1 = (-1, 1)$), where c_n is a normalization constant such that
$$\int_{\mathbb{B}(0,1)\times J_1} |H_n(z, s)|^2 d\mu_n(z) ds = 1$$
and μ_n is the measure such that
$$d\mu_n(z) = \sqrt{|g|(\exp_{w_n}(rz/L_n))} dm(z).$$

Step 2. *The functions H_n are uniformly bounded.*

Note that
$$\int_{B(w_n,\frac{r}{L_n})} |f_n|^2 dV = \frac{r^m}{L_n^m}\frac{1}{|c_n|^2}\int_{\mathbb{B}(0,1)} |F_n(z)|^2 d\mu_n(z).$$
Therefore, we have
$$⨍_{B(w_n,r/L_n)} |f_n|^2 dV \simeq \frac{1}{|c_n|^2}\int_{\mathbb{B}(0,1)} |F_n|^2 d\mu_n.$$
As $w_n \in D_n$, we obtain
$$|F_n(0)|^2 = |c_n|^2|f_n(w_n)|^2 \geq |c_n|^2\epsilon ⨍_{B(w_n,r/L_n)\times I_r} |h_n(w,t)|^2 dV dt$$
$$\simeq \epsilon \int_{\mathbb{B}(0,1)\times J_1} |H_n(w,s)|^2 d\mu_n(w) ds = \epsilon.$$
Since $|h_n|^2$ is subharmonic,
$$|F_n(0)|^2 = |c_n|^2|h_n(w_n,0)|^2 \lesssim \int_{\mathbb{B}(0,1)\times J_1} |H_n(w,s)|^2 d\mu_n(w) ds = 1.$$
Hence, we have $0 < \epsilon \lesssim |F_n(0)|^2 \lesssim 1$ for all $n \in \mathbb{N}$.

Using the assumption,
$$\frac{1}{n} \gtrsim \frac{|c_n|^2}{\mathrm{vol}(B(w_n,r/L_n))}\int_{A_{L_n}\cap B(w_n,\frac{r}{L_n})} |f_n|^2 dV \simeq \int_{B_n\cap\mathbb{B}(0,1)} |F_n|^2 d\mu_n,$$
where B_n is such that $\exp_n(B_n \cap \mathbb{B}(0,1)) = A_{L_n} \cap B(w_n, r/L_n)$. So we have that
$$\begin{cases} \forall n \; 0 < \epsilon \lesssim |F_n(0)|^2 \lesssim 1 \\ \forall n \; \int_{\mathbb{B}(0,1)\cap B_n} |F_n|^2 d\mu_n \lesssim \frac{1}{n} \end{cases}.$$
In fact, $0 < \epsilon \lesssim |H_n(0,0)|^2 \lesssim 1$ (by definition) and one can prove that $|H_n|^2 \lesssim 1$. Indeed, if $(z,s) \in \mathbb{B}(0,1/2) \times J_{1/2}$, let $w = \exp_n(z) \in B(w_n, r/(2L_n))$ and $t = rs/L_n \in I_{r/2}$. Then
$$|H_n(z,s)|^2 = |c_n|^2|h_n(w,t)|^2 \lesssim |c_n|^2 ⨍_{B(w,r/(2L_n))\times I_{r/2}(t)} |h_n|^2$$
$$\lesssim |c_n|^2 ⨍_{B(w_n,r/L_n)\times I_r} |h_n|^2 dV dt \simeq 1.$$
Therefore, working with $1/2$ instead of 1 we have $|H_n|^2 \lesssim 1$ for all n.

Step 3. *The family $\{H_n\}_n$ is equicontinuous in $\mathbb{B}(0,1) \times J_1$.*

Consider $(w,t) \in B(w_n, r/(4L_n)) \times I_{r/4}$ and $(\tilde{w},\tilde{t}) \in B(w, \tilde{r}r/L_n) \times I_{\tilde{r}r}(t)$, then there exists some small $\delta > 0$ such that

$$|c_n||h_n(w,t) - h_n(\tilde{w},\tilde{t})| \leq |c_n| \frac{\tilde{r}}{L_n} r \sup_{B(w,\delta/L_n)\times I_\delta(t)} |\nabla h_n| \leq (\star)$$

Taking $\tilde{r}$ small enough so that $\delta \leq r/4$ and using Schoen and Yau's estimate (1.8), we have

$$(\star) \leq |c_n| \frac{\tilde{r}r}{L_n} \sup_{B(w_n, r/(2L_n))\times I_{r/2}} |\nabla h_n| \lesssim \frac{\tilde{r}r}{L_n} \frac{1}{\frac{r}{L_n}} \sup_{B(w_n,r/L_n)\times I_r} |c_n||h_n| \lesssim \tilde{r}.$$

So we have proved that $|c_n||h_n(w,t) - h_n(\tilde{w},\tilde{t})| \leq C\tilde{r}$. Take $\tilde{r}$ small enough so that $C\tilde{r} < \epsilon$. Let $(z,s) \in \mathbb{B}(0,1/4) \times J_{1/4}$ and $(\tilde{z},\tilde{s}) \in \mathbb{B}(z,\tilde{r}) \times (s-\tilde{r}, s+\tilde{r})$. Consider $w = \exp_n(z)$, $t = rs/L_n$, $\tilde{w} = \exp_n(\tilde{z})$ and $\tilde{t} = r\tilde{s}/L_n$. Then, we have proved that for all $\epsilon > 0$ there exists $\tilde{r} > 0$ (small) such that for all $(z,s) \in \mathbb{B}(0,1/4) \times J_{1/4}$:

$$|H_n(z,s) - H_n(\tilde{z},\tilde{s})| < \epsilon \ \text{ if } \ |z-\tilde{z}| < \tilde{r}, \ |s-\tilde{s}| < \tilde{r} \quad \forall n \in \mathbb{N}.$$

Change $1/4$ to 1. So the sequence H_n is equicontinuous.

Step 4. *There exists a limit function of H_n that is real analytic.*

The family $\{H_n\}_n$ is equicontinuous and uniformly bounded on $\mathbb{B}(0,1) \times J_1$. Therefore, by Ascoli-Arzela's theorem there exists a partial sequence (denoted as the sequence itself) such that $H_n \to H$ uniformly on compact subsets of $\mathbb{B}(0,1) \times J_1$. Since $F_n(z) = H_n(z,0)$, we get a function $F(z) := H(z,0) : \mathbb{B}(0,1) \to \mathbb{R}$ which is the limit of F_n (uniformly on compact subsets of $\mathbb{B}(0,1)$).

Now we will prove that H is real analytic. In fact, we will show that H is harmonic. We have the following properties:

1. The family of measures $d\mu_n$ converges uniformly to the ordinary Euclidean measure because $g_{ij}(\exp_{w_n}(rz/L_n)) \to g_{ij}(\exp_{w_0}(0)) = \delta_{ij}$, where w_0 is the limit point of some subsequence of w_n (recall that we are taking normal coordinate charts).

2. If $g_n(z) := g(rz/L_n)$ (i.e. g_n is the rescaled metric), then $\Delta_{(g_n,Id)} H_n(z,s) = 0$ for all $(z,s) \in \mathbb{B}(0,1) \times J_1$, by construction.

3. The functions H_n are uniformly bounded and converge uniformly on compact subsets of $\mathbb{B}(0,1) \times J_1$.

We are in the hypothesis of Lemma 2.14 that guarantees the harmonicity of H in the Euclidean sense.

Step 5. *Using the hypothesis, we will construct a measure τ such that $|F| = 0$ τ-a.e. and $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$. These two properties and the real analyticity of*

F will lead to a contradiction.

By hypothesis, the sequence $\{A_L\}_L$ is relatively dense. Taking into account that

$$\mathrm{vol}(B(w_n, r/L_n)) = \frac{r^m}{L_n^m}\mu_n(\mathbb{B}(0,1)),$$

we get that

$$\inf_n \mu_n(B_n) \geq \rho > 0, \tag{2.8}$$

where we have denoted $B_n \cap \mathbb{B}(0,1)$ by B_n.
Let τ_n be such that $d\tau_n = \chi_{B_n} d\mu_n$. From a standard argument (τ_n are supported in a ball), we know the existence of a weak-$*$ limit of a subsequence of τ_n, denoted by τ. This subsequence will be noted as the sequence itself. Using (2.8), we know that τ is not identically 0. Now we have that

$$\int_{\mathbb{B}(0,1)} |F|^2 d\tau = 0.$$

Therefore, $F = 0$ τ-a.e. in $\mathbb{B}(0,1)$. Now for all $K \subset \mathbb{B}(0,1)$ compact

$$\int_K |F|^2 d\tau = 0,$$

therefore $F = 0$ in $\mathrm{supp}\tau$. Let $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$ satisfy $\overline{\mathbb{B}(a,s)} \cap \mathrm{supp}\tau \neq \emptyset$. Then using the fact $B_n \subset \mathbb{B}(0,1)$, we obtain

$$\tau_n(\overline{\mathbb{B}(a,s)}) \leq \int_{\overline{\mathbb{B}(a,s)}} d\mu_n \simeq \frac{L_n^m}{r^m}\mathrm{vol}(B(\exp_n(a), sr/L_n)) \simeq s^m.$$

Therefore $\tau_n(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all n. Hence, in the limit case $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$. In short,

1. We have sets $B_n \subset \mathbb{B}(0,1)$ such that
$$\rho \leq \mu_n(B_n) \leq \mu_n(\mathbb{B}(0,1)) \simeq 1.$$
2. We have measures τ_n weakly-$*$ converging to τ (not identically 0).
3. $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ for all $\overline{\mathbb{B}(a,s)} \subset \mathbb{B}(0,1)$.
4. $|F| = 0$ τ-a.e. in $\mathbb{B}(0,1)$.
5. $|F(0)| > 0$ and $|F| \lesssim 1$.

We know that H is real analytic, then $F(z)$ is real analytic. Federer ([Fed69, Theorem 3.4.8]) proved that the $(m-1)$-Hausdorff measure $\mathcal{H}^{m-1}(F^{-1}(0)) < \infty$. Hence $\mathcal{H}^{m-1}(\mathrm{supp}\tau) \leq \mathcal{H}^{m-1}(F^{-1}(0)) < \infty$. As a consequence, we get an upper bound for the Hausdorff dimension of $\mathrm{supp}\tau$: $\dim_{\mathcal{H}}(\mathrm{supp}\tau) \leq m-1$. On the other hand, using Frostman's Lemma, since $\tau(\overline{\mathbb{B}(a,s)}) \lesssim s^m$ and $\tau(\mathrm{supp}\tau) > 0$, we have

$$0 < \mathcal{H}^m(\mathrm{supp}\tau).$$

Thus, $\dim_{\mathcal{H}}(\mathrm{supp}\tau) \geq m$. So we reached to a contradiction and the proof is complete. This concludes the proof of the proposition. □

The following remark shows us the interest for studying the relatively dense sets.

Remark 2.5. A natural question is if one can replace the condition of being L-S, i.e.

$$\int_M |f|^2 dV \leq C \int_{A_L} |f|^2 dV, \quad \forall f \in E_L, \tag{2.9}$$

by a weaker condition like

$$\int_M |f|^2 dV \leq C \int_{A_L} |f|^2 dV, \quad \forall f \in W_L, \tag{2.10}$$

and still obtains the fact that $\{A_L\}_L$ are relatively dense, where W_L is the eigenspace of Δ_M with eigenvalue $-L^2$ endowed with the L^2-norm. If this was achieved, one could try to use this fact together with the recent work of Colding-Minicozzi (see [CM11]) in order to make some progress towards a proof of the lower bound in Yau's conjecture on the size of nodal sets. More precisely, the nodal set of an eigenfunction u_L of eigenvalue $-L^2$ is its set of zeros. The conjecture of Yau claims that

$$\mathcal{H}^{m-1}(\{u_L = 0\}) \geq CL, \quad \Delta_M u_L = -L^2 u_L.$$

The authors in [CM11] proved that

$$\mathcal{H}^{m-1}(\{u_L = 0\}) \geq CL^{\frac{3-m}{2}}. \tag{2.11}$$

We sketch the idea of their proof (see [CM11] for a detailed discussion). Let B_i be balls of radius a/L so that $M \subset \cup_i B_i$, where a is a fixed constant given by [CM11, Lemma 1]. Given a constant $d > 1$, B_i is d-good if

$$\int_{2B_i} |u_L|^2 \leq 2^d \int_{B_i} |u_L|^2.$$

Let G_d be the union of d-good balls:

$$G_d = G_d(L) := \cup \{B_i; \quad B_i \text{ is a } d\text{-good ball}\}.$$

In [CM11, Lemma 3] it is proved that most of the L^2-norm of u_L comes from the d_M-good balls, where d_M is a large constant that depends only on M and not on L. Thus, if we could replace the condition of being L-S by (2.10), we would obtain that $\{G_d(L)\}_L$ is a sequence of relatively dense sets. Let N be the number of d_M-good balls. By [CM11, Lemma 4], $N \geq CL^{(m+1)/2}$. Now, the idea in proving (2.11) relies on the following estimate (check [CM11] for a proof):

$$\mathcal{H}^{m-1}(\{u_L = 0\}) \geq CL^{1-m} N. \tag{2.12}$$

Thus, obtaining better estimates of N will improve the lower bound for the $m-1$ Hausdorff measure of the nodal set of u_L. Observe that the number of balls $B_\xi = B(\xi, R/L)$ (pairwise disjoint) so that $5B_\xi$ cover M is of order L^m/R^m. If $G_{d_M}(L)$ is relatively dense, then at least in each B_ξ we have a d_M-good ball. Therefore, $N \geq CL^m$ with C depending on the manifold. As a consequence, replacing this lower bound in (2.12), we would get a proof

for Yau's conjecture.
Unfortunately, a simple example shows that condition (2.9) cannot be replaced by (2.10). Indeed, take $M = \mathbb{S}^1$. Thus, we are considering the space of polynomials of the form $p_n(z) = az^n + b\overline{z}^n$. Note that $|p_n(z)| = |az^{2n} + b|$ for all $z \in \mathbb{S}^1$. Now consider the sets

$$A_n = \left\{ z \in \mathbb{S}^1; \quad \mathrm{Im}(z) < 0 \right\}.$$

Trivially,

$$\int_{\mathbb{S}^1} |p_n| dV \leq 2 \int_{A_n} |p_n| dV \quad \forall n \in \mathbb{N},$$

but the sets A_n are not relatively dense.
Of course an interesting question which is left open is a geometric/metric description of the L-S sets for W_L.

Chapter 3

Interpolating and Marcinkiewicz-Zygmund families.

In the previous chapter, we studied inequalities that represents a *continuous* sampling. Now we pay attention to a more delicate question of a discrete sampling.

This chapter is devoted to prove the intuitive facts about the interpolating and M-Z arrays in the Paley-Wiener spaces E_L mentioned in the Introduction. We prove necessary and sufficient conditions for interpolation and M-Z. These results do not provide quantitative conditions. For this reason, in this chapter we focus our attention on the Beurling-Landau densities and the conditions that measure the density of interpolating and M-Z families. In the special case of the $\mathbb{S}^m$ ($m \geq 2$), J. Marzo in [Mar07], found necessary conditions in terms of the Beurling-Landau densities for M-Z and interpolating families, adapting H.J. Landau's approach developed for the Paley-Wiener case. His result provides a critical density (the Nyquist rate) necessary for being a M-Z and interpolating family. Following the ideas in [Mar07] and [Lan67a], we are able to find necessary density conditions for interpolation and sampling in the general case of compact manifolds.

Outline of the chapter: in the first section, we state the main definitions of M-Z and interpolating families. In the second section, we provide qualitative results about the interpolating and M-Z arrays. The goal of the third section is to study the Beurling-Landau densities and provide necessary conditions for families in order to be interpolating or M-Z. In the last section, we discuss the main difference from the case of $\mathbb{S}^m$ (the reader may want to skip, in a first lecture, the proof of the main result for a general compact manifold and read this last section, where we explain the scheme used for $M = \mathbb{S}^m$ that works for some compact manifolds).

3.1 Definitions and Notations

Given $L \geq 1$ and $m_L \in \mathbb{N}$, we consider a triangular family of points in M, $\mathcal{Z} = \{\mathcal{Z}(L)\}_L$, denoted as

$$\mathcal{Z}(L) = \{z_{Lj} \in M \ : \ 1 \leq j \leq m_L\}, L \geq 1,$$

and we assume that $m_L \to \infty$ as L increases.

Recall that we say that a family $\mathcal{Z}$ is uniformly separated if there exists a positive ϵ such that for all $L \geq 1$,

$$d_M(z_{Lj}, z_{Lk}) \geq \frac{\epsilon}{L}, \quad j \neq k,$$

and ϵ is called the separation constant of $\mathcal{Z}$.

Remark 3.1. The natural separation is of order $1/L$ in view of Proposition 3.4 (see below) that shows that a necessary condition for interpolation is that the family should be uniformly separated with this order of separation. The key idea is Bernstein's inequality:

$$\|\nabla f_L\|_\infty \lesssim L \|f_L\|_\infty, \quad \forall f_L \in E_L.$$

This estimate has been proved recently in [FM10b, Theorem 2.2]. Thus, on balls of radius $1/L$, a bounded function of E_L oscillates little.

Definition 3.1. *Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ be a triangular family in M with $m_L \geq k_L$ for all $L \geq 1$. Then $\mathcal{Z}$ is a L^2-**Marcinkiewicz-Zygmund*** (M-Z) *family, if there exists a constant $C > 0$ such that for all $L \geq 1$ and $f_L \in E_L$,*

$$\frac{C^{-1}}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq \int_M |f_L|^2 dV \leq \frac{C}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2.$$

Remark 3.2. The condition of being M-Z can be expressed in terms of the reproducing kernel of E_L: a family $\mathcal{Z}$ is M-Z if and only if the normalized reproducing kernels form a frame with uniform bounds in L, i.e.

$$\sum_{j=1}^{m_L} |\langle f_L, \tilde{K}_L(z_{Lj}, \cdot)\rangle|^2 \simeq \|f_L\|_2^2,$$

with constants independent of L, where $\tilde{K}_L(z,w) = \frac{K_L(z,w)}{\|K_L(z,\cdot)\|_2}$.

Definition 3.2. *Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ be a triangular family in M with $m_L \leq k_L$ for all L. Then $\mathcal{Z}$ is an L^2-**interpolating family** if for all family of values $c = \{c(L)\}_{L\geq 1}$, $c(L) = \{c_{Lj}\}_{1\leq j\leq m_L}$ such that*

$$\sup_{L\geq 1} \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty,$$

there exists a sequence of functions $f_L \in E_L$ with

$$\sup_{L\geq 1} \|f_L\|_2 < \infty$$

and $f_L(z_{Lj}) = c_{Lj}$ $(1 \leq j \leq m_L)$. That is, $f_L(\mathcal{Z}(L)) = c(L)$ for all $L \geq 1$.

Remark 3.3. Equivalently, a family is interpolating if the normalized reproducing kernels form a Riesz sequence, i.e.

$$\frac{1}{k_L}\sum_j |c_{Lj}|^2 \simeq \|\sum_j c_{Lj}\tilde{K}_L(z_{Lj},\cdot)\|_2^2,$$

with constants independent of L, whenever $c = \{c_{Lj}\}_{j,L}$ is a family satisfying

$$\sup_L \frac{1}{k_L}\sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty.$$

Remark 3.4. The interpolating families are strongly related to the Carleson measures (see Definition 2.1). If $\mathcal{Z}$ is interpolating then the measures

$$\mu_L = \frac{1}{k_L}\sum_{j=1}^{m_L} \delta_{z_{Lj}}$$

are L^2-Carleson measures.

Proof. Let G be the Grammian matrix associated to a family $\mathcal{Z}$, i.e. $G = (G_{ij})_{ij}$ with

$$G_{ij} = \frac{\langle K_L(z_{Lj},\cdot), K_L(z_{Li},\cdot)\rangle}{\|K_L(z_{Lj},\cdot)\|\|K_L(z_{Li},\cdot)\|}.$$

A family $\mathcal{Z}$ is interpolating if the normalized reproducing kernels form a Riesz sequence. This is equivalent to the fact that the Grammian associated to $\mathcal{Z}$, G, is bounded above and below. In particular, G is bounded above. This last property is equivalent to the fact that μ_L are Carleson measures (see [AM02, Proposition 9.5] for a proof).

Alternatively, if a family $\mathcal{Z}$ is interpolating then it is uniformly separated (see Proposition 3.4 below). Thus, μ_L are Carleson measures by the Plancherel-Pólya inequality (Theorem 2.5). □

Intuitively, a M-Z family should be *dense* in order to recover the L^2-norm of functions of the space E_L and an interpolating family should be *sparse*.

3.2 Interpolating and M-Z families

In this section we present some qualitative results about the interpolating and M-Z families.

3.2.1 Interpolating families

In this section, we consider the following Banach spaces.

- $E := \{E_L\}_L$ endowed with the norm

$$\|f\|_E^2 := \sup_L \|f_L\|_2^2 .$$

-

$$\ell_L^2 := \left\{ v_L = \{v_{Lk}\}_{k=1,\dots,m_L} \; ; \; \|v_L\|_{\ell_L^2}^2 := \frac{1}{k_L} \sum_{k=1}^{m_L} |v_{Lk}|^2 (< \infty) \right\} .$$

- $\mathcal{A} = \{v = \{v_L\}_L \; ; \; v_L \in \ell_L^2, \; \|v\|_{\mathcal{A}} < \infty\}$, where

$$\|v\|_{\mathcal{A}}^2 = \sup_L \|v_L\|_{\ell_L^2}^2 = \sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}|^2 .$$

The result stated below shows that the interpolation can be done in a stable way.

Lemma 3.3. *Let $\mathcal{Z}$ be a triangular family in M. Assume $\mathcal{Z}$ is interpolating. Then the interpolation can be done by functions $f_L \in E_L$ such that*

$$\|f_L\|_2^2 \leq \frac{C}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 ,$$

where C is independent of L.

Proof. For all $L \geq 1$, define the operator $T_L : \ell_L^2 \to E_L/I_L$ as $T_L(v_L) = [f_L]$ such that $f_L(z_{Lj}) = v_{Lj}$, where

$$I_L = [K_L(z_{L1}, \cdot), \dots, K_L(z_{Lm_L}, \cdot)]^{\perp} .$$

Thus, $[f_L] = [g_L]$ if $f_L(z_{Lj}) = g_L(z_{Lj})$. The operator T_L is well defined because $\mathcal{Z}$ is an interpolating sequence.

Let us consider the following Banach space.

$$D := \{[f] = ([f_L])_L , \quad [f_L] \in E_L/I_L\} ,$$

endowed with supremum norm, i.e.

$$\|[f]\|_D = \sup_L \|[f_L]\|_{E_L/I_L} = \sup_L \min_{g_L \in [f_L]} \|g_L\|_2 .$$

Now define the operator $T : \mathcal{A} \to D$ as $T(v) := (T_L(v_L))_L = ([f_L])_L = [f]$, where $[f_L] = T_L(v_L)$, i.e $f_L(z_{Lj}) = v_{Lj}$ with minimal norm. T is well defined by the definition of an interpolating sequence.

We claim that T is bounded. Thus, the operators T_L are bounded with uniform constant in L. This means that there exists $f_L \in E_L$ such that $f_L(z_{Lj}) = v_{Lj}$ and

$$\|f_L\|_2^2 \lesssim \frac{1}{k_L}\sum_{j=1}^{m_L} |v_{Lj}|^2,$$

i.e. f_L is the solution of the interpolation problem with minimal norm and the proof is finished. Now we proceed in order to prove the claim. We will use the Closed Graph's theorem to prove that T is bounded. Obviously T is a linear operator between two Banach spaces. We claim that the graph of T is closed. Indeed, let $v^n = \{v_L^n\}_L \subset \mathcal{A}$ be such that $v^n \xrightarrow{\|\cdot\|_{\mathcal{A}}} v = (v_L)_L \in \mathcal{A}$. Let $[f^n] = ([f_L^n])_L = (T_L(v_L^n))_L \xrightarrow{\|\cdot\|_D} [f] = ([f_L])_L \in D$. We have to prove that for all $L \geq 1$, $[f_L] = T_L(v_L)$. Let $L \geq 1$ be fixed. By hypothesis, we know that $\|v^n - v\|_{\mathcal{A}} \to 0$ as $n \to \infty$ and

$$0 \leftarrow \|[f^n] - [f]\|_D = \sup_L \min_{g_L^n \in [f_L^n], g_L \in [f_L]} \|g_L^n - g_L\|_2 =: \sup_L \|h_L^n - h_L\|_2,$$

where $h_L^n(z_{Lj}) = f_L^n(z_{Lj})$ and $h_L(z_{Lj}) = f_L(z_{Lj})$. Using Proposition 1.7, we get

$$\begin{aligned}
0 &\leq |h_L(z_{Lj}) - v_{Lj}| \leq |h_L(z_{Lj}) - h_L^n(z_{Lj})| + |h_L^n(z_{Lj}) - v_{Lj}| \\
&\leq \|h_L - h_L^n\|_\infty + |v_{Lj}^n - v_{Lj}| \\
&\leq C\sqrt{k_L}(\|h_L - h_L^n\|_2 + \|v^n - v\|_{\mathcal{A}}) \to 0,\ n \to \infty.
\end{aligned}$$

Hence, $T_L([v_L]) = [h_L] = [f_L]$ for all $L \geq 1$ and the graph of T is closed. Now the Closed Graph's theorem guarantees the boundedness of T. $\square$

Now, we provide a necessary condition for an interpolating family.

Proposition 3.4. *Let $\mathcal{Z}$ be an L^2-interpolating triangular family in M. Then $\mathcal{Z}$ is uniformly separated.*

Proof. Fix $L_0 \geq 1$ and $1 \leq j_0 \leq m_{L_0}$. Using Lemma 3.3, there exist functions $f_{L_0} \in E_{L_0}$ such that $f_{L_0}(z_{L_0 j}) = \delta_{j j_0}$ and $\|f_{L_0}\|_2^2 \leq C/k_{L_0}$ (C independent of L). Applying Proposition 1.9, we get the following.

$$\begin{aligned}
1 &= |f_{L_0}(z_{L_0 j_0}) - f_{L_0}(z_{L_0 j})| \leq \|\nabla f_{L_0}\|_\infty\, d_M(z_{L_0 j_0}, z_{L_0 j}) \\
&\lesssim \sqrt{k_{L_0}} L_0 \,\|f_{L_0}\|_2\, d_M(z_{L_0 j_0}, z_{L_0 j}) \lesssim L_0 \sqrt{k_{L_0}} \frac{1}{\sqrt{k_{L_0}}} d_M(z_{L_0 j_0}, z_{L_0 j}) \\
&\simeq L_0 d_M(z_{L_0 j_0}, z_{L_0 j}).
\end{aligned}$$

Thus,

$$d_M(z_{L_0 j_0}, z_{L_0 j}) \gtrsim \frac{1}{L_0}, \quad \forall L_0 \geq 1,\ j \neq j_0,$$

where the constant does not depend on L_0 and j_0. $\square$

Theorem 3.5. *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two triangular families in M. Assume that $\mathcal{Z}$ is an L^2-interpolating family. Then there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, Z' is also L^2-interpolating provided*

$$d_M(z'_{Lj}, z_{Lj}) < \delta/L, \quad \forall j = 1, \dots, m_L; L \geq 1.$$

Proof. Let $v = \{v_L\}_{L\geq 1}$, $v_L = \{v_{Lk}\}_{k=1,\dots,m_L}$ be such that

$$\sup_{L\geq 1} \frac{1}{k_L} \sum_{k=1}^{m_L} |v_{Lk}|^2 \leq 1.$$

Define $v^0 = v$ and $v_L^0 = v_L$. Since $\mathcal{Z}$ is interpolating, we know that there exists $f_0 = \left\{f_0^L\right\}_L \in E$ such that $f_0^L(\mathcal{Z}(L)) = v_L^0$ i.e. $f_0^L(z_{Lj}) = v_{Lj}^0$ for all $L \geq 1$ and $j = 1, ..., m_L$. We also know, by Lemma 3.3, that we can take f_0^L such that

$$\left\|f_0^L\right\|_2^2 \leq C \left\|v_L^0\right\|_{\ell_L^2}^2 \leq C.$$

Now consider $v_L^1 = v_L^0 - f_0^L(\mathcal{Z}'(L)) = f_0^L(\mathcal{Z}(L)) - f_0^L(\mathcal{Z}'(L))$. We need to check that $\{v_L^1\}_L \in \mathcal{A}$. Observe that $\mathcal{Z}$ is uniformly separated in view of Proposition 3.4. The family $\mathcal{Z}'$ is also uniformly separated because it is close to $\mathcal{Z}$. Indeed,

$$\begin{aligned}\frac{\epsilon}{L} &< d_M(z_{Lj}, z_{Lk}) \leq d_M(z_{Lj}, z'_{Lj}) + d_M(z'_{Lj}, z'_{Lk}) + d_M(z'_{Lk}, z_{Lk}) \\ &\leq \frac{2\delta}{L} + d_M(z'_{Lj}, z'_{Lk}).\end{aligned}$$

Hence, if δ is small enough,

$$d_M(z'_{Lj}, z'_{Lk}) \geq \frac{\epsilon - 2\delta}{L} > 0.$$

Thus, the family $\mathcal{Z}'$ is uniformly separated. Applying the Plancherel-Pólya estimate (Theorem 2.5) to $\mathcal{Z}'$, we get

$$\frac{1}{k_L} \sum_{j=1}^{m_L} |f_0^L(z'_{Lj})|^2 \lesssim \left\|f_0^L\right\|_2^2 \leq \|f_0\|_E^2 < \infty,$$

with constants independent of L and f_0. Using this fact, we get that $v^1 \in \mathcal{A}$. Indeed,

$$\begin{aligned}\frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}^1|^2 &\lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}^0|^2 + \frac{1}{k_L} \sum_{j=1}^{m_L} |f_0^L(z'_{Lj})|^2 \\ &\leq \|v^0\|_{\mathcal{A}}^2 + \frac{1}{k_L} \sum_{j=1}^{m_L} |f_0^L(z'_{Lj})|^2 \lesssim 1 + \|f_0\|_E^2 < \infty.\end{aligned}$$

Therefore, $v^1 \in \mathcal{A}$.

We claim that $\|v_L^1\|_{\ell^2_L} \leq \gamma \|v_L^0\|_{\ell^2_L}$, for some $\gamma < 1$ (γ independent of L). In such case, consider $f_1^L \in E_L$ such that $f_1^L(\mathcal{Z}(L)) = v_L^1$ and $\left\|f_1^L\right\|_2 \leq C \|v_L^1\|_{\ell^2_L} \leq C\gamma \|v_L^0\|_{\ell^2_L} \leq C\gamma$. Now take $\{v_L^2\}_L := \left\{v_L^1 - f_1^L(\mathcal{Z}'(L))\right\}_L \in \mathcal{A}$. Then $\|v_L^2\|_{\ell^2_L} \leq \gamma \|v_L^1\|_{\ell^2_L} \leq \gamma^2 \|v_L^0\|_{\ell^2_L}$. Iterating this process, we get for all j a function $f_j^L \in E_L$ such that $f_j^L(\mathcal{Z}(L)) = v_L^j = v_L^{j-1} - f_{j-1}^L(\mathcal{Z}'(L))$, $\left\|f_j^L\right\|_2 \leq C\gamma^j \|v_L^0\|_{\ell^2_L} \leq C\gamma^j$. Let $f^L = \sum_j f_j^L$. This sum is well defined (the series converges absolutely) and $f^L \in E_L$. Furthermore,

$$\begin{aligned} f^L(\mathcal{Z}'(L)) &= \sum_j f_j^L(\mathcal{Z}'(L)) = \sum_{j\geq 0}(v_L^j - v_L^{j+1}) \\ &= v_L^0 - v_L^1 + v_L^1 - v_L^2 + \ldots = v_L^0. \end{aligned}$$

This shows that $\mathcal{Z}'$ is interpolating. The only thing left is to check the claim: $\|v_L^1\|_{\ell^2_L} \leq \gamma \|v_L^0\|_{\ell^2_L}$, for some $\gamma < 1$.

$$\begin{aligned} \left\|v_L^1\right\|_{\ell^2_L}^2 &= \frac{1}{k_L}\sum_{k=1}^{m_L} |v_{Lk}^1|^2 = \frac{1}{k_L}\sum_{k=1}^{m_L} |f_0^L(z_{Lk}) - f_0^L(z'_{Lk})|^2 \\ &\leq \frac{1}{k_L}\sum_{k=1}^{m_L} |\nabla f_0^L(\xi_{Lk})|^2 d_M(z_{Lk}, z'_{Lk})^2 \\ &\leq \frac{\delta^2}{L^2}\frac{1}{k_L}\sum_{k=1}^{m_L} |\nabla f_0^L(\xi_{Lk})|^2 \leq (\star) \end{aligned}$$

Let h_0^L be the harmonic extension of $f_0^L \in E_L$. Using Lemma 1.10 we know that

$$|\nabla f_0^L(\xi_{Lk})|^2 \leq C\frac{L^{m+2}}{r^{(m+2)}}\frac{L}{r}\int_{B(\xi_{Lk}, r/L)\times I_r} |h_0^L|^2.$$

Let r be small enough (independent of L) such that $B(\xi_{Lk}, r/L)$ are pairwise disjoint (such r exists because $\{\xi_{Lk}\}_{L,k}$ is uniformly separated since $\xi_{Lk} \in B(z_{Lk}, \delta/L)$ and $\mathcal{Z}$ is uniformly separated). Hence

$$\begin{aligned} \sum_{k=1}^{m_L}|\nabla f_0^L(\xi_{Lk})|^2 &\lesssim \frac{L^{m+2}}{r^{(m+2)}}\frac{L}{r}\int_{\cup_{k=1}^{m_L} B(\xi_{Lk}, r/L)\times I_r} |h_0^L|^2 \\ &\leq \frac{L^{m+2}}{r^{(m+2)}}\frac{L}{r}\int_{M\times I_r} |h_0^L|^2 \lesssim \frac{k_L L^2}{r^{m+2}} e^{2r} \left\|f_0^L\right\|_2^2. \end{aligned}$$

Using this last estimate, we get

$$(\star) \lesssim \frac{\delta^2 e^{2r}}{r^{m+2}} \left\|f_0^L\right\|_2^2 \lesssim \frac{\delta^2 e^{2r}}{r^{m+2}} \left\|v_L^0\right\|_{\ell^2_L}^2.$$

This shows that

$$\left\|v_L^1\right\|_{\ell^2_L}^2 \leq C\frac{\delta^2 e^{2r}}{r^{m+2}} \left\|v_L^0\right\|_{\ell^2_L}^2.$$

Let δ be small enough so that $\gamma := \delta\frac{\sqrt{C}e^r}{r^{(m+2)/2}} < 1$ (γ is independent of L). $\square$

Proposition 3.6. *Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_L = \{z_{Lj}\}_{L\geq 1, j=1,\dots,m_L} \subset M$ be a triangular family of points with $m_L \leq k_L$. Assume $\mathcal{Z}$ is separated enough, i.e. there exists $R > 0$ (big enough) such that*

$$d_M(z_{Lj}, z_{Lk}) \geq \frac{R}{L}, \quad \forall j \neq k, \quad \forall L.$$

Then $\mathcal{Z}$ is an interpolating family.

Proof. Let $\mathcal{R}: \ E \to \mathcal{A}$ be the evaluating operator, i.e. if $v := \mathcal{R}(f)$ for some $f \in E$, then $v_{Lj} = f_L(z_{Lj})$. This operator is linear and continuous by the Plancherel-Pólya type inequality (Theorem 2.5). Now, consider the operator $\mathcal{S} : \mathcal{A} \to E$ defined as follows: if $v \in \mathcal{A}$, then $\mathcal{S}(v) =: f$ with

$$f_L(z) := \sum_{j=1}^{m_L} v_{Lj} \frac{S_L^N(z_{Lj}, z)}{S_L^N(z_{Lj}, z_{Lj})},$$

where $S_L^N(z,w)$ is the Bochner-Riesz Kernel of order N associated to the Laplacian (see Chapter 1, Section 1.2 for the definition). The order N will be chosen later. Note that the functions f_L belong to E_L and

$$f_L(z_{Lk}) = v_{Lk} + \sum_{j\neq k} v_{Lj} \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})}.$$

The operator $\mathcal{S}$ is well defined. Indeed, let $v \in \mathcal{A}$ and $f := \mathcal{S}(v)$. We need to prove that $f \in E$. Using Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} \|f_L\|_2 = \sup_{\|g\|_2=1} |\langle f_L, g\rangle| &= \sup_{\|g\|_2=1} \left| \sum_{j=1}^{m_L} v_{Lj} \frac{\langle S_L^N(z_{Lj}, \cdot), g\rangle}{S_L^N(z_{Lj}, z_{Lj})} \right| \\ &\lesssim \|v\|_{\mathcal{A}} \sup_{\|g\|_2=1} \|S_L^N g\|_2 \leq \|v\|_{\mathcal{A}}, \end{aligned}$$

where we have applied Theorem 2.5 to $S_L^N(g)$. Therefore, $\|f\|_E \lesssim \|v\|_{\mathcal{A}} < \infty$. This proves that $\mathcal{S}$ is well defined and continuous. Obviously this operator is linear.

If $\|\mathcal{R} \circ \mathcal{S} - Id\| < 1$, then $\mathcal{R}$ is invertible. Furthermore, $\mathcal{R}$ is exhaustive and as a consequence the family $\mathcal{Z}$ is interpolating. We only need to check that $\|\mathcal{R} \circ \mathcal{S} - Id\| < 1$. We claim that

$$\sum_{j\neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| << 1, \tag{3.1}$$

uniformly in L for R big enough, provided $N + 1 > m$. Thus,

$$\|\mathcal{R} \circ \mathcal{S} - Id\|^2 = \sup_{v\in\mathcal{A};\|v\|_{\mathcal{A}}=1} \|\mathcal{R}(\mathcal{S}(v)) - v\|_{\mathcal{A}}^2 = \sup_{v\in\mathcal{A};\|v\|_{\mathcal{A}}=1} \|w\|_{\mathcal{A}}^2,$$

where $w = \{w_{Lk}\}_{k;L}$ with

$$w_{Lk} = \sum_{j\neq k} v_{Lj} \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})}.$$

Using the claim (3.1), we get a control of the L^∞-norm of w:

$$\sup_L |w_{Lk}| \leq \sup_L \sum_{j\neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| << 1,$$

for all $v = \{v_{Lj}\}_{j;L}$ such that $\sup_L |v_{Lj}| = 1$. Moreover, using again (3.1), we have the same control of the L^1-norm of w:

$$\sup_L \frac{1}{k_L} \sum_{k=1}^{m_L} |w_{Lk}| \lesssim \sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}| \sum_{k\neq j} \left| \frac{S_L^N(z_{Lk}, z_{Lj})}{S_L^N(z_{Lk}, z_{Lk})} \right| << 1,$$

for all $v = \{v_{Lj}\}_{j;L}$ such that $\sup_L \frac{1}{k_L} \sum_j |v_{Lj}| = 1$. Thus, interpolating between the L^1-norm and L^∞-norm, we get the same result for the L^2-norm of w and the proof is complete. Now we will proceed in order to prove the claim (3.1). Let

$$g_k(z) := \frac{1}{(1 + L d_M(z, z_{Lk}))^{N+1}}$$

and $B_j := B(z_{Lj}, 1/L)$. It is easy to check that

$$\inf_{B_j} g_k(z) \geq \frac{1}{2^{N+1}} g_k(z_{Lj}).$$

Using the fact that $\mathcal{Z}$ is separated enough, we know that B_j are pairwise disjoint and $\cup_{j\neq k} B_j \subset M \setminus B(z_{Lk}, (R-1)/L)$. Therefore, applying (1.3),

$$\begin{aligned} \sum_{j\neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| &\leq C_N \sum_{j\neq k} g_k(z_{Lj}) \leq C_N L^m \sum_{j\neq k} \int_{B_j} g_k(z) dV(z) \\ &\leq C_N L^m \int_M \frac{1}{(L d_M(z, z_{Lk}))^{N+1}} \chi_{M\setminus B(z_{Lk},(R-1)/L)}(z) dV. \end{aligned}$$

Consider for any $t \geq 0$, the following set A_t.

$$A_t = \left\{ z \in M : \quad d_M(z, z_{Lk}) \geq \frac{R-1}{L}, \quad d_M(z, z_{Lk}) < \frac{t^{-1/(N+1)}}{L} \right\}.$$

Using the distribution function, one can compute that

$$\begin{aligned} \sum_{j\neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| &\leq C_N L^m \int_0^{(R-1)^{-(N+1)}} \mathrm{vol}(A_t) dt \\ &\lesssim C_N \frac{1}{(R-1)^{(N+1)-m}}, \end{aligned}$$

provided $N + 1 > m$. Taking R big enough we get the desired claim. □

3.2.2 Marcinkiewicz-Zygmund families

In what follows, we will present some qualitative results concerning the M-Z families. The proof of these results follows from standard techniques and the ideas in [Mar07, Theorem 4.7], replacing the corresponding gradient estimates by the ones obtained in Chapter 1, Section 1.3.

The following theorem allows us to assume, without loss of generality, that a M-Z family is uniformly separated.

Theorem 3.7. *Let $\mathcal{Z} \subset M$ be an L^2-*M-Z *family. Then there exists a uniformly separated family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ which is also an L^2-*M-Z *family.*

Proof. By Theorem 2.5, we may assume that $\mathcal{Z}$ is a finite union of N uniformly ϵ-separated families, denoted by $\mathcal{Z}^{(j)}$, $j = 1, ..., N$. Using a standard argument (see for example [Sei95, Page 141]), we can construct for $0 < \delta < \epsilon/4$ a uniformly separated family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that for all $L \geq 1$ and $j = 1, ..., m_L$,

$$d_M(z_{Lj}, \tilde{\mathcal{Z}}(L)) < \frac{\delta}{L}.$$

Let $\tilde{z} \in \tilde{\mathcal{Z}}(L)$ be the closest point to $z \in \mathcal{Z}(L)$. For any $f_L \in E_L$ we have that

$$|f_L(z) - f_L(\tilde{z})| \leq \sup_{B(z,r/L)} |\nabla f_L(\xi)| d_M(z, \tilde{z}) \leq \frac{\delta}{L} \sup_{B(z,r/L)} |\nabla f_L(\xi)|.$$

Using Lemma 1.10, we have

$$\begin{aligned} |f_L(z) - f_L(\tilde{z})|^2 &\leq \frac{\delta^2}{L^2} \sup_{B(z,r/L)} |\nabla f_L(\xi)|^2 \\ &\lesssim \frac{\delta^2}{L^2} \sup_{B(z,r/L)} \frac{L^{m+2+1}}{r^{m+2+1}} \int_{B(\xi,r/L)\times I_r} |h_L|^2 \\ &\lesssim \frac{\delta^2 L^m}{r^{m+2}} \frac{L}{2r} \int_{B(z,2r/L)\times I_{2r}} |h_L|^2. \end{aligned}$$

Using the fact that the family $\mathcal{Z}$ is L^2-M-Z, we get

$$\begin{aligned} \|f_L\|_2^2 &\simeq \frac{1}{k_L} \sum_{z\in\mathcal{Z}(L)} |f_L(z)|^2 \\ &\lesssim \frac{1}{k_L} \sum_{j=1}^{N} \sum_{z\in\mathcal{Z}^{(j)}(L)} (|f_L(z) - f_L(\tilde{z})|^2 + |f_L(\tilde{z})|^2) \\ &\leq \frac{1}{k_L} \sum_{j=1}^{N} \sum_{z\in\mathcal{Z}^{(j)}(L)} \frac{\delta^2}{r^{m+2+1}} L^{m+1} \int_{B(z,r/L)\times I_r} |h_L|^2 \\ &+ \frac{CN}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2 = (\star) \end{aligned}$$

Since $\mathcal{Z}^{(j)}$ is ϵ-separated, we can take $r < \epsilon$ so that the balls $B(z, r/L)$ for $z \in \mathcal{Z}^{(j)}(L)$ are pairwise disjoint. Hence

$$\begin{aligned}
(\star) &= \frac{\delta^2}{r^{m+2}} \frac{L^m}{k_L} \frac{L}{r} \sum_{j=1}^{N} \int_{\cup_{z\in\mathcal{Z}^{(j)}(L)} B(z,r/L)\times I_r} |h_L|^2 + \frac{CN}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2 \\
&\lesssim \frac{N\delta^2}{r^{m+2}} \frac{1}{l(I_r)} \int_{M\times I_r} |h_L|^2 + \frac{CN}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2 \\
&\simeq \frac{N}{r^{m+2}} \delta^2 \|f_L\|_2^2 + \frac{CN}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2.
\end{aligned}$$

Thus,

$$\|f_L\|_2^2 \leq C_{\epsilon,r,M,N} \delta^2 \|f_L\|_2^2 + \frac{CN}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2, \ \forall\delta > 0.$$

Letting $\delta \to 0$ in the last estimate, we get

$$\|f_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{z\in\tilde{\mathcal{Z}}(L)} |f_L(z)|^2.$$

The reverse inequality is obvious using the fact that $\tilde{\mathcal{Z}}$ is uniformly separated and Theorem 2.5. $\square$

The next result shows us that a small perturbation of a M-Z family is still a M-Z family.

Theorem 3.8. *Let $\mathcal{Z}$ be a L^2-M-Z family. There exists $\epsilon_0 > 0$ such that if $\mathcal{Z}'$ is a uniformly separated family with*

$$d_M(z_{Lj}, z'_{Lj}) < \frac{\epsilon}{L},$$

for some $\epsilon \leq \epsilon_0$, then the family of points $\mathcal{Z}'$ is L^2-M-Z.

Proof. Recall that $\mathcal{Z} = \{z_{Lj}\}_{L\geq 1, j=1,\dots,m_L}$ is L^2-M-Z family if and only if for all $L \geq 1$ and for all $f_L \in E_L$,

$$\int_M |f_L|^2 dV \simeq \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2,$$

with constants independent of L and f_L. We will prove that

$$\frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z'_{Lj})|^2 \simeq \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2,$$

with constants not depending on L and f_L. This shows that the family Z' is L^2-M-Z. Using the following inequality

$$\left[\left(\sum_{j=1}^{n} a_j^2 \right)^{1/2} - \left(\sum_{j=1}^{n} b_j^2 \right)^{1/2} \right]^2 \leq \sum_{j=1}^{n} (a_j - b_j)^2, [1]$$

[1] This is equiv. to $\sum_j a_j b_j \leq \sqrt{\sum_j a_j^2 \sum_j b_j^2}$ (i.e. the Cauchy-Schwarz inequality)

we get

$$\left|\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2\right)^{1/2}-\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z'_{Lj})|^2\right)^{1/2}\right|$$
$$\leq\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})-f_L(z'_{Lj})|^2\right)^{1/2}.$$

Now we are going to estimate the right hand side of the above inequality. We know that there exists points $\tilde{z}_{Lj}$ in the segment joining z_{Lj} and z'_{Lj} (therefore, $d_M(\tilde{z}_{Lj},z_{Lj})\leq 2\epsilon/L$) such that

$$\begin{aligned}\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})-f_L(z'_{Lj})|^2 &\leq \frac{1}{k_L}\sum_{j=1}^{m_L}|\nabla f_L(\tilde{z}_{Lj})|^2 d_M(z_{Lj},z'_{Lj})^2\\ &\leq \frac{\epsilon^2}{L^2}\frac{1}{k_L}\sum_{j=1}^{m_L}|\nabla f_L(\tilde{z}_{Lj})|^2\leq(\star)\end{aligned}$$

The separation of $\mathcal{Z}$ implies that

$$\begin{aligned}\frac{s}{L}&\leq d_M(z_{Lj},z_{Lk})\leq d_M(z_{Lj},\tilde{z}_{Lj})+d_M(\tilde{z}_{Lj},\tilde{z}_{Lk})+d_M(\tilde{z}_{Lk},z_{Lk})\\ &\leq\frac{4\epsilon}{L}+d_M(\tilde{z}_{Lj},\tilde{z}_{Lk}).\end{aligned}$$

Hence we have that $d_M(\tilde{z}_{Lj},\tilde{z}_{Lk})\geq(s-4\epsilon)/L$ (take $\epsilon<s/4$). Therefore, the sequence of points $\{\tilde{z}_{Lj}\}$ is uniformly separated. Using this fact and the gradient estimate (see Lemma 1.10) we have for r small enough (so that the balls $B(\tilde{z}_{Lj},r/L)$ are pairwise disjoint)

$$(\star)\lesssim\frac{\epsilon^2}{L^2}\frac{1}{k_L}\frac{L^mL^2}{r^{m+2}}\frac{1}{l(I_r)}\int_{\cup_j B(\tilde{z}_{Lj},r/L)\times I_r}|h_L|^2\lesssim\epsilon^2\int_M|f_L|^2,$$

where we have used Proposition 1.6. Therefore we have obtained that

$$\left|\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2\right)^{1/2}-\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z'_{Lj})|^2\right)^{1/2}\right|\lesssim\epsilon\left(\int_M|f_L|^2\right)^{1/2}$$

$$\overset{\mathcal{Z}\text{ is M-Z}}{\lesssim}\epsilon\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2\right)^{1/2}\leq(\star)$$

Taking ϵ small enough we get

$$(\star)\leq\frac{1}{4}\left(\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2\right)^{1/2}.$$

At this end, the desired result is clear:

$$\frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z'_{Lj})|^2 \simeq \frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2$$

with constants not depending on L and f_L. □

Now we will provide a sufficient condition for a family to be L^2-M-Z. Intuitively, a family should be *dense* in order to be M-Z.

Theorem 3.9. *There exists $\epsilon_0 > 0$ such that if $\mathcal{Z}$ is an ϵ-dense family (not necessarily uniformly separated), i.e. for all $L \geq 1$*

$$\sup_{\xi\in M} d_M(\xi, \mathcal{Z}(L)) < \frac{\epsilon}{L}, \quad (\epsilon \leq \epsilon_0),$$

*then there exists a uniformly separated subfamily which is $\tilde{\epsilon}$-dense and is an L^2-*M-Z *family provided that $\tilde{\epsilon} \leq \epsilon_0$.*

Proof. First, we will prove the result when the family $\mathcal{Z}$ is uniformly separated and then we will generalize it for any family not necessarily separated.

Assume $\mathcal{Z}$ is uniformly separated. We want to prove that there exists a constant $C > 0$ such that for any $L \geq 1$ and $f_L \in E_L$ we have

$$\frac{C^{-1}}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2 \leq \int_M |f_L|^2 dV \leq \frac{C}{k_L}\sum |f_L(z_{Lj})|^2.$$

The left hand side follows easily from the fact that the family of points is uniformly separated and Theorem 2.5.

Thus, we just need to verify the right-hand side inequality, i.e.

$$\int_M |f_L|^2 \lesssim \frac{1}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2.$$

Observe that since the family $\mathcal{Z}$ is ϵ-dense we have that

$$M \subset \bigcup_{j=1}^{m_L} B(z_{Lj}, \epsilon/L).$$

Therefore,

$$\begin{aligned}\int_M |f_L|^2 &\leq \sum_{j=1}^{m_L}\int_{B(z_{Lj},\epsilon/L)} |f_L(w)|^2 dV(w)\\ &\lesssim \sum_{j=1}^{m_L}\int_{B(z_{Lj},\epsilon/L)} |f_L(w) - f_L(z_{Lj})|^2 dV + \frac{\epsilon^m}{k_L}\sum_{j=1}^{m_L}|f_L(z_{Lj})|^2.\end{aligned}$$

Using the gradient estimate for the harmonic extension of f_L (see Lemma 1.10) we have

$$\begin{aligned}
&\sum_{j=1}^{m_L} \int_{B(z_{Lj},\epsilon/L)} |f_L(w) - f_L(z_{Lj})|^2 dV(w) \\
&\quad\leq \frac{\epsilon^2}{L^2} \sum_{j=1}^{m_L} \int_{B(z_{Lj},\epsilon/L)} |\nabla f_L(\xi_{w,z_{Lj}})|^2 dV(w) \\
&\quad\lesssim \frac{\epsilon^2}{L^2} \sum_{j=1}^{m_L} \int_{B(z_{Lj},\epsilon/L)} \frac{L^m L^2}{r^{m+2}} \frac{1}{l(I_r)} \int_{B(\xi_{w,z_{Lj}},r/L)\times I_r} |h_L|^2 \leq (\star)
\end{aligned}$$

Note that $B(\xi_{w,z_{Lj}}, r/L) \subset B(z_{Lj}, C\epsilon/L)$ (for some $C \geq 1$ independent of L). Hence,

$$\begin{aligned}
(\star) &\leq \epsilon^2 \sum_{j=1}^{m_L} \int_{B(z_{Lj},\epsilon/L)} \frac{L^m}{r^{m+2}} \frac{1}{l(I_r)} \int_{B(z_{Lj},C\epsilon/L)\times I_r} |h_L|^2 \\
&= \frac{\epsilon^2 L^m}{r^{m+2}} \sum_{j=1}^{m_L} \mathrm{vol}(B(z_{Lj}, \epsilon/L)) \frac{1}{l(I_r)} \int_{B(z_{Lj},C\epsilon/L)\times I_r} |h_L|^2 \\
&\simeq \frac{\epsilon^{m+2}}{r^{m+2}} \sum_{j=1}^{m_L} \frac{1}{l(I_r)} \int_{B(z_{Lj},C\epsilon/L)\times I_r} |h_L|^2 \\
&= \frac{\epsilon^{m+2}}{r^{m+2}} \frac{1}{l(I_r)} \int_{M\times I_r} \left[\sum_{j=1}^{m_L} \chi_{B(z_{Lj},C\epsilon/L)}(\xi)\right] |h_L(\xi,t)|^2 dV(\xi) dt.
\end{aligned}$$

Observe that there exists a constant $C_1 > 0$ (independent of L) such that for all $\xi \in M$ and L,

$$\sum_{j=1}^{m_L} \chi_{B(z_{Lj},C\epsilon/L)}(\xi) \leq C_1,$$

because the family of points $\mathcal{Z}$ is uniformly separated. Indeed, if $\xi \in M$ then we know that there exists $j \in \{1, ..., m_L\}$ such that $\xi \in B(z_{Lj}, \epsilon/L)$(because these balls covers M). Fix $r > 0$. Let n be the number of balls $\overline{B(z_{Lj}, \epsilon/L)}$ for $j = 1, ..., m_L$ such that $\overline{B(\xi, r/L)} \cap \overline{B(z_{Lj}, C\epsilon/L)}$ is not empty. Since $\overline{B(\xi, r/L)}$ is compact we know that n is finite. We claim that this n does not depend on L. Let us prove it. Observe that by the triangle inequality we know that $B(z_{Lj}, \epsilon/L) \subset B(\xi, ((C+1)\epsilon + r)/L)$ for all $j = 1, ..., n$. Therefore we have

$$\bigcup_{j=1}^{n} B(z_{Lj}, \epsilon/L) \subset B(\xi, ((C+1)\epsilon + r)/L).$$

Let s be the the separation between the family $\mathcal{Z}$. Note that the balls $B(z_{Lj}, s/L)$ are pairwise disjoint.

In the case when $s \geq \epsilon$, the balls $B(z_{Lj}, \epsilon/L)$ are also pairwise disjoint. Thus, in such situation,

$$\frac{((C+1)\epsilon + r)^m}{L^m} \simeq \mathrm{vol}\left(B\left(\xi, \frac{(C+1)\epsilon + r}{L}\right)\right) \geq \mathrm{vol}\left(\bigcup_{j=1}^{n} B(z_{Lj}, \epsilon/L)\right) \simeq n\frac{\epsilon^m}{L^m}.$$

Therefore $n \le C_2(\epsilon) \le C_2$ (taking a proper r). Now if $s < \epsilon$ then we have

$$\begin{aligned}\frac{((C+1)\epsilon + r)^m}{L^m} &\simeq \operatorname{vol}\left(B\left(\xi, \frac{(C+1)\epsilon + r}{L}\right)\right) \ge \operatorname{vol}\left(\bigcup_{j=1}^{n} B(z_{Lj}, \epsilon/L)\right) \\ &\ge \operatorname{vol}\left(\bigcup_{j=1}^{n} B(z_{Lj}, s/L)\right) \simeq n\frac{s^m}{L^m}.\end{aligned}$$

Hence, in this case also $n \le C_2(\epsilon)$. So we can take n independent of L. This implies that each point of M can be at most in n balls $B(z_{Lj}, C\epsilon/L)$, so that

$$\sum_{j=1}^{m_L} \chi_{B(z_{Lj}, C\epsilon/L)}(\xi) = n \le C_1.$$

Using this last estimate we get

$$\begin{aligned}&\sum_{j=1}^{m_L} \int_{B(z_{Lj},\epsilon/L)} |f_L(w) - f_L(z_{Lj})|^2 dV(w) \\ &\qquad \lesssim \epsilon^{m+2} \frac{1}{l(I_r)} \int_{M\times I_r} |h_L(\xi,t)|^2 dV(\xi) dt \simeq \epsilon^{m+2} \int_M |f_L|^2 dV.\end{aligned}$$

Now

$$\begin{aligned}\int_M |f_L|^2 &\le C_3 \epsilon^{m+2} \int_M |f_L|^2 + C_4 \frac{\epsilon^m}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \\ &= \epsilon^m \left(C_3\epsilon^2 \int_M |f_L|^2 + C_4 \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \right) \\ &\le C_3 \epsilon^2 \int_M |f_L|^2 + C_4 \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2.\end{aligned}$$

Therefore we got

$$(1 - \epsilon^2 C_3) \int_M |f_L|^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_L} |f_L(z_{Lj})|^2.$$

Now for ϵ small enough (take for example $\epsilon < 1/(2\sqrt{C_3})$ we have

$$\int_M |f_L|^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2.$$

General case of a family $\mathcal{Z}$ not necessarily separated. Let $B_j := B(z_{Lj}, \epsilon/L)$. Since the family $\mathcal{Z}$ is ϵ-dense we know that

$$M \subset \bigcup_{j=1}^{m_L} B_j.$$

By the 5-r covering Lemma we know that there exists a subsequence $\{B_j\}_{j=1}^{n_L}$ ($n_L \leq m_L$) pairwise disjoint such that

$$M \subset \bigcup_{j=1}^{m_L} B_j \subset \bigcup_{j=1}^{n_L} 5B_j.$$

Take $\tilde{\mathcal{Z}} = \left\{\tilde{\mathcal{Z}}(L)\right\}_L$ to be the family $\tilde{\mathcal{Z}}(L) = \{z_{Lj}\}_{j=1}^{n_L}$. Clearly this is a subfamily of $\mathcal{Z}$ and since B_j are pairwise disjoint for $j = 1, ..., n_L$ this family is uniformly separated. Moreover, $\tilde{\mathcal{Z}}$ is $\tilde{\epsilon}$-dense because $5B_j$ for $j = 1, ..., n_L$ are covering M. Now applying the case of a uniformly separated family, we get that this subfamily $\tilde{\mathcal{Z}}$ is L^2-M-Z if $\tilde{\epsilon}$ is small enough. $\square$

Remark 3.5. Theorem 3.9 has been also proved by F. Fibir and H.N. Mhaskar using other techniques (see [FM10a, Theorem 5.1]).

3.3 Beurling-Landau density

In this section, we prove necessary conditions for a family to be interpolating or sampling in terms of the following Beurling-Landau type densities.

Definition 3.10. *Let $\mathcal{Z}$ be a triangular family of points in M. We define the upper and lower Beurling-Landau density, respectively, as*

$$D^+(\mathcal{Z}) = \limsup_{R\to\infty}\left(\limsup_{L\to\infty}\left(\max_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}\right)\right)\right),$$

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty}\left(\liminf_{L\to\infty}\left(\min_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}\right)\right)\right).$$

Remark 3.6. Let μ_L be the normalized counting measure, i.e.

$$\mu_L = \frac{1}{k_L}\sum_{j=1}^{m_L} \delta_{z_{Lj}}$$

and σ the normalized volume measure, i.e. $d\sigma = dV/\mathrm{vol}(M)$. Then the densities defined above can be viewed as the asymptotic behaviour of the quantity

$$\frac{\mu_L(B(\xi,R/L))}{\sigma(B(\xi,R/L))}.$$

Our main result is:

Theorem 3.11. *Let M be an arbitrary smooth compact Riemannian manifold without boundary of dimension $m \geq 2$ and $\mathcal{Z}$ a triangular family in M. If $\mathcal{Z}$ is an L^2-M-Z family then there exists a uniformly separated L^2-M-Z family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that*

$$D^-(\tilde{\mathcal{Z}}) \geq 1.$$

If $\mathcal{Z}$ is an L^2-interpolating family then it is uniformly separated and

$$D^+(\mathcal{Z}) \leq 1.$$

This result was proved in the particular case when $M = \mathbb{S}^m$ in [Mar07]. Following the ideas in [Mar07], we prove Theorem 3.11 in the general case of a manifold. In [Mar07], the key idea to prove this result was the comparison of the trace of the concentration operator and its square with an estimate of the eigenvalues of this operator. In general, the main difference from the case of the Sphere is that we lack of an explicit expression of the reproducing kernel. Thus, in the general setting, we need to work with a "modified" concentration operator. In Section 3.4, we show that Theorem 3.11 can be proved, using the same procedure as in [Mar07] with the classical concentration operator, for manifolds that have a decay of its reproducing kernel off the diagonal. Before we proceed, we shall introduce the concept of the classical and modified concentration operator.

3.3.1 Classical Concentration Operator

Definition 3.12. *The classical concentration operator $\mathcal{K}_A^L$, over a set $A \subset M$, is defined for $f_L \in E_L$ as*

$$\mathcal{K}_A^L f_L(z) = \int_A K_L(z,\xi) f_L(\xi) dV(\xi). \tag{3.2}$$

This operator is the composition of the restriction operator to A with the orthogonal projection of E_L, i.e. $\mathcal{K}_A^L(f_L) = P_{E_L}(\chi_A f_L)$ for all $f_L \in E_L$. The operator $\mathcal{K}_A^L$ is self-adjoint. Indeed, if $f_L, g_L \in E_L$ then:

$$\begin{aligned}\langle \mathcal{K}_A^L f_L, g_L\rangle &= \langle P_{E_L}(\chi_A \cdot f_L), g_L\rangle = \langle \chi_A \cdot f_L, P_{E_L}(g_L)\rangle = \langle \chi_A \cdot f_L, g_L\rangle \\ &= \langle f_L, \chi_A \cdot g_L\rangle = \langle P_{E_L} f_L, \chi_A \cdot g_L\rangle = \langle f_L, \mathcal{K}_A^L(g)\rangle.\end{aligned}$$

Alternatively, we can view the action of the concentration operator as a matrix acting on a sequence $\beta = \{\beta_i\}_{i=1,\dots,k_L}$ that are the Fourier coefficients of a function $f_L \in E_L$ (with respect to the orthonormal basis $\{\phi_i\}$). If we denote by $D_L := (d_{ij})_{i,j=1}^{k_L}$, where

$$d_{ij} = \int_A \phi_i \phi_j,$$

then $\mathcal{K}_A^L(f_L) \equiv D_L(\beta)$.
Using the spectral theorem, we know that the eigenvalues of $\mathcal{K}_A^L$ are all real and E_L has an orthonormal basis of eigenvectors of $\mathcal{K}_A^L$. The trace of $\mathcal{K}_A^L$ is

$$\operatorname{tr}(\mathcal{K}_A^L) = \sum_{i=1}^{k_L} d_{ii} = \int_A K_L(z,z) dV(z).$$

Similarly, we can compute the trace of $\mathcal{K}_A^L \circ \mathcal{K}_A^L$.

$$\operatorname{tr}(\mathcal{K}_A^L \circ \mathcal{K}_A^L) = \sum_{i,j=1}^{k_L} d_{ij} d_{ji} = \int_{A\times A} |K_L(z,w)|^2 dV(w) dV(z).$$

We will choose A as $B(\xi, R/L)$ for some fixed point $\xi \in M$ (note that all the constants in the estimates will not depend on the fixed point $\xi \in M$). Taking into account that

$$\mathrm{vol}(B(\xi, R/L)) \simeq \frac{R^m}{L^m}$$

and using Hörmander's estimates for the reproducing kernel and k_L (see Chapter 1, Section 1.2), we get

$$\mathrm{tr}(\mathcal{K}^L_{B(\xi,R/L)}) = k_L \frac{\mathrm{vol}(B(\xi, R/L))}{\mathrm{vol}(M)} + \frac{o(L^m)}{L^m}. \tag{3.3}$$

3.3.2 Modified Concentration Operator

From now on, we fix an $\epsilon > 0$ and consider the transform B^ϵ_L defined in Chapter 1, Section 1.2 associated with the kernel

$$B^\epsilon_L(z, w) = \sum_{i=1}^{k_L} \beta_\epsilon \left(\frac{\lambda_i}{L}\right) \phi_i(z)\phi_i(w),$$

i.e. for all $f \in L^2(M)$,

$$B^\epsilon_L(f)(z) = \int_M B^\epsilon_L(z, w)f(w)dV(w) = \sum_{i=1}^{k_L} \beta_\epsilon \left(\frac{\lambda_i}{L}\right) \langle f, \phi_i\rangle\phi_i(z).$$

Definition 3.13. *The modified concentration operator $T^\epsilon_{L,A}$, over a set $A \subset M$, is defined for $f_L \in E_L$ as:*

$$T^\epsilon_{L,A} f_L(z) = B^\epsilon_L(\chi_A \cdot B^\epsilon_L(f_L))(z) = \int_M B^\epsilon_L(z, w)\chi_A(w)B^\epsilon_L(f_L)(w)dV(w).$$

Observe that for $\epsilon = 0$, the modified concentration operator is just the classical concentration operator defined previously.

An advantage of $T^\epsilon_{L,A}$ in contrast of $\mathcal{K}^L_A$ is that we have a nice estimate of its kernel: using Lemma 1.5, we know that for any $N > m$, there exists a constant C_N independent of L such that

$$|B^\epsilon_L(z, w)| \le C_N L^m \frac{1}{(1 + Ld_M(z, w))^N}, \quad \forall z, w \in M.$$

It is easy to check (as was done in the case of $\mathcal{K}^L_A$) that the operator $T^\epsilon_{L,A}$ is self-adjoint and by the spectral theorem its eigenvalues are all real and E_L has an orthonormal basis of eigenvectors of $T^\epsilon_{L,A}$. In fact, the main reason to do the first smooth projection in $T^\epsilon_{L,A}$ is to ensure the self-adjointness of the operator (but the calculations work even if we consider only $B^\epsilon_L(\chi_A \cdot)$).

As before, we can compute the trace of $T^{\epsilon}_{L,A}$ and $T^{\epsilon}_{L,A} \circ T^{\epsilon}_{L,A}$ that will be used later on.

$$\operatorname{tr}(T^{\epsilon}_{L,A}) = \sum_{i=1}^{k_L} \beta_{\epsilon}^2\left(\frac{\lambda_i}{L}\right)\int_A \phi_i^2(z)dV(z) =: \int_A \tilde{B}^{\epsilon}_L(z,z)dV(z),$$

where $\tilde{B}^{\epsilon}_L(z,w)$ is a kernel defined as

$$\tilde{B}^{\epsilon}_L(z,w) = \sum_{i=1}^{k_L} \alpha\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w),$$

with $\alpha(x) := \beta_{\epsilon}^2(x)$. Note that the function α has the same properties as β_ϵ and therefore we know that $\tilde{B}^{\epsilon}_L(z,w)$ has the estimate (1.5).

Similarly we can compute the trace of $T^{\epsilon}_{L,A} \circ T^{\epsilon}_{L,A}$.

$$\operatorname{tr}(T^{\epsilon}_{L,A} \circ T^{\epsilon}_{L,A}) = \int_{A\times A} |\tilde{B}^{\epsilon}_L(z,w)|^2 dV(z)dV(w).$$

Since the modified concentration operator is a *small* perturbation of $\mathcal{K}^L_A$, one can estimate $\operatorname{tr}(T^{\epsilon}_{L,A})$ in terms of $\operatorname{tr}(\mathcal{K}^L_A)$. Indeed, using the definition of $\beta_\epsilon(x)$,

$$\operatorname{tr}(\mathcal{K}^{L(1-\epsilon)}_A) \le \operatorname{tr}(T^{\epsilon}_{L,A}) \le \operatorname{tr}(\mathcal{K}^L_A).$$

Applying this computation to $A = A_L := B(\xi, R/L)$ and using (3.3), we get the following.

$$\frac{\operatorname{tr}(T^{\epsilon}_{L,B(\xi,R/L)})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \ge \frac{k_{L(1-\epsilon)}\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} + \frac{o(L^m(1-\epsilon)^m)}{L^m(1-\epsilon)^m}\frac{1}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}.$$

Since $\operatorname{vol}(B(\xi,R/L)) \simeq R^m/L^m$, the second term tends to 0 when $L \to \infty$. Thus, using the expression for k_L (see Chapter 1, Section 1.2), we get:

$$\liminf_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L,A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \ge (1-\epsilon)^m, \quad \forall\epsilon > 0. \tag{3.4}$$

The upper bound for this quantity is trivial since $\operatorname{tr}(T^{\epsilon}_{L,A_L}) \le \operatorname{tr}(\mathcal{K}^L_{A_L})$ and has been computed previously. Hence, using (3.3) we have

$$\limsup_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L,A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \le 1. \tag{3.5}$$

Similarly, if $\rho > 0$ is a fixed number, then

$$\limsup_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L(1+\rho),A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \le (1+\rho)^m. \tag{3.6}$$

3.3.3 Proof of the main result

In the spirit of the original work of Landau, the proof of Theorem 3.11 relies on a trace estimate of $T^\epsilon_{L,A}$ and two technical lemmas (Lemma 3.15 and 3.16 below) that estimate the number of big eigenvalues of the modified concentration operator. First we state these technical results and show the proof of the main result and in Sections 3.3.4 and 3.3.5 we present a proof of them.

The following result is an estimate of the difference of the trace of T^ϵ_{L,A_L} and $T^\epsilon_{L,A_L} \circ T^\epsilon_{L,A_L}$. It will show us, later on, that most of the eigenvalues are either close to 1 or to 0.

Proposition 3.14. *Let $A_L = B(\xi, R/L)$. Then*

$$\limsup_{L\to\infty} \left(\operatorname{tr}(T^\epsilon_{L,A_L}) - \operatorname{tr}(T^\epsilon_{L,A_L} \circ T^\epsilon_{L,A_L})\right) \leq C_1(1-(1-\epsilon)^m)R^m + C_2 R^{m-1},$$

where C_1 (independent of ϵ) and C_2 are constants independent of R.
Similarly, if $\rho > 0$ then

$$\limsup_{L\to\infty} \left(\operatorname{tr}(T^\epsilon_{L(1+\rho),A_L}) - \operatorname{tr}(T^\epsilon_{L(1+\rho),A_L} \circ T^\epsilon_{L(1+\rho),A_L})\right)$$
$$\leq C_1(1+\rho)^m(1-(1-\epsilon)^m)R^m + C_2 R^{m-1},$$

where C_1 (independent of ϵ and ρ) and C_2 are constants independent of R.

Given $L \geq 1$ and $R > 0$, let $A_L, A_L^+ = A_L^+(t)$ and $A_L^- = A_L^-(t)$ be the balls centered at a fixed point $\xi \in M$ and radius R/L, $(R+t)/L$ and $(R-t)/L$, respectively, where t is a parameter such that $s << t << R << L$ and s is the separation constant of the family $\mathcal{Z}$. The value of t will be chosen later on. We denote the eigenvalues of the modified concentration operator T^ϵ_{L,A_L} as

$$1 > \lambda_1^L \geq \ldots \geq \lambda_{k_L}^L > 0.$$

Lemma 3.15. *Let $\mathcal{Z}$ be an s-uniformly separated L^2-MZ family. Then there exist $t_0 = t_0(M, s) > 0$ and a constant $0 < \gamma < 1$ (independent of ϵ, R and L) such that for all $t \geq t_0$,*

$$\lambda^L_{N_L+1} \leq \gamma,$$

where

$$N_L := N_L(t) = \#(\mathcal{Z}(L) \cap A_L^+) = \#(\mathcal{Z}(L) \cap B(\xi, (R+t)/L)).$$

Remark 3.7. In the conditions of Lemma 3.15,

$$\#\left\{\lambda_j^L > \gamma\right\} \leq N_L = \#(\mathcal{Z}(L) \cap A_L^+) \leq \#(\mathcal{Z}(L) \cap A_L) + O(R^{m-1}), R \to \infty,$$

where the constant in $O(R^{m-1})$ does not depend on L.

Proof of Remark 3.7. The first inequality is trivial by Lemma 3.15 and the second inequality follows using the separation of the family $\mathcal{Z}$. Moreover, $N_L \lesssim R^m/s^m$. □

Lemma 3.16. *Let $\mathcal{Z}$ be an L^2-interpolating family with separation constant s and $\rho > 0$. Then there exist $t_1 = t_1(M, s) > 0$ and a constant $0 < \delta < 1$ independent of R and L such that for all $t \geq t_1$,*

$$\lambda_{n_L-1}^{L(1+\rho)} \geq \delta := C\beta_\epsilon^2 \left(\frac{1}{1+\rho}\right),$$

where $\lambda_k^{L(1+\rho)}$ are the eigenvalues associated to $T^\epsilon_{L(1+\rho),A_L}$, C is independent of ρ and ϵ and

$$n_L := n_L(t) = \#(\mathcal{Z}(L) \cap A_L^-) = \#(\mathcal{Z}(L) \cap B(\xi, (R-t)/L)).$$

Remark 3.8. In the conditions of Lemma 3.16 we have

$$\#(\mathcal{Z}(L) \cap A_L) - O(R^{m-1}) \leq n_L = \#(\mathcal{Z}(L) \cap A_L^-) \leq \# \left\{\lambda_j^{L(1+\rho)} \geq \delta\right\} + 1,$$

where the constant in $O(R^{m-1})$ does not depend on L.

Proof of Remark 3.8. The second inequality is trivial by Lemma 3.16 and the first inequality follows using the separation of $\mathcal{Z}$. $\square$

In what follows, we pick the parameter t in the range $\max(t_0, t_1) \leq t << R$, where t_0 and t_1 are the values given by Lemmas 3.15 and 3.16.

Now we have all the tools in order to prove the main result concerning the notion of densities.

Proof of Theorem 3.11. Assume $\mathcal{Z}$ is an L^2-M-Z family. Without loss of generality, we may assume that $\mathcal{Z}$ is uniformly separated (see Theorem 3.7). Consider the following measures:

$$d\mu_L = \sum_{j=1}^{k_L} \delta_{\lambda_j^L}.$$

Note that

$$\mathrm{tr}(T^\epsilon_{L,A_L}) = \int_0^1 x\,d\mu_L(x), \quad \mathrm{tr}(T^\epsilon_{L,A_L} \circ T^\epsilon_{L,A_L}) = \int_0^1 x^2 d\mu_L(x).$$

Let γ be given by Lemma 3.15. We have

$$\begin{aligned}\# \left\{\lambda_j^L > \gamma\right\} &= \int_\gamma^1 d\mu_L(x) \geq \int_0^1 x\,d\mu_L(x) - \frac{1}{1-\gamma}\int_0^1 x(1-x)d\mu_L(x) \\ &= \mathrm{tr}(T^\epsilon_{L,A_L}) - \frac{1}{1-\gamma}(\mathrm{tr}(T^\epsilon_{L,A_L}) - \mathrm{tr}(T^\epsilon_{L,A_L} \circ T^\epsilon_{L,A_L})),\end{aligned}$$

Using the remark following Lemma 3.15 and (3.4), we have

$$\begin{aligned}
&\liminf_{L\to\infty} \frac{\#(\mathcal{Z}(L)\cap A_L)+O(R^{m-1})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \\
&\geq \liminf_{L\to\infty}\left[\frac{\mathrm{tr}(T^{\epsilon}_{L,A_L})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} - \frac{1}{1-\gamma}\frac{\mathrm{tr}(T^{\epsilon}_{L,A_L})-\mathrm{tr}(T^{\epsilon}_{L,A_L}\circ T^{\epsilon}_{L,A_L})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}\right] \\
&\geq (1-\epsilon)^m - \frac{1}{1-\gamma}\limsup_{L\to\infty}\frac{\mathrm{tr}(T^{\epsilon}_{L,A_L})-\mathrm{tr}(T^{\epsilon}_{L,A_L}\circ T^{\epsilon}_{L,A_L})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}
\end{aligned}$$

Observe that

$$k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)} \simeq R^m. \tag{3.7}$$

Applying (3.7) and Proposition 3.14, we have

$$\begin{aligned}
&\liminf_{L\to\infty} \frac{\#(\mathcal{Z}(L)\cap A_L)+O(R^{m-1})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \\
&\geq (1-\epsilon)^m - \frac{C}{1-\gamma}\frac{\limsup_{L\to\infty}(\mathrm{tr}(T^{\epsilon}_{L,A_L})-\mathrm{tr}(T^{\epsilon}_{L,A_L}\circ T^{\epsilon}_{L,A_L}))}{R^m} \\
&\geq (1-\epsilon)^m - \frac{C}{1-\gamma}(1-(1-\epsilon)^m) - \frac{1}{1-\gamma}\frac{O(R^{m-1})}{R^m}.
\end{aligned}$$

Taking inferior limits when $R\to\infty$ in the last estimate, we get that

$$D^-(\mathcal{Z}) \geq (1-\epsilon)^m - \frac{C}{1-\gamma}(1-(1-\epsilon)^m) \quad \forall\epsilon>0,$$

where C and γ are independent of ϵ. Therefore, letting $\epsilon\to 0$ we get the claimed result:

$$D^-(\mathcal{Z}) \geq 1.$$

Assume now that $\mathcal{Z}$ is an L^2-interpolating family, in particular it is uniformly separated by Proposition 3.4. Fix $\rho>0$. Let $\delta>0$ be the value given by Lemma 3.16.

$$\begin{aligned}
&\#\left\{\lambda_j^{L(1+\rho)}\geq\delta\right\} \leq \frac{-1}{\delta}\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L}\circ T^{\epsilon}_{L(1+\rho),A_L}) + \frac{1+\delta}{\delta}\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L}) \\
&= \mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L}) + \frac{1}{\delta}(\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L})-\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L}\circ T^{\epsilon}_{L(1+\rho),A_L})).
\end{aligned}$$

Using the remark following Lemma 3.16, (3.7), (3.6) and Proposition 3.14 we have

$$\begin{aligned}
&\limsup_{L\to\infty}\frac{\#(\mathcal{Z}(L)\cap A_L)-O(R^{m-1})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \leq \limsup_{L\to\infty}\frac{\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \\
&+\frac{1}{\delta}\limsup_{L\to\infty}\frac{\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L})-\mathrm{tr}(T^{\epsilon}_{L(1+\rho),A_L}\circ T^{\epsilon}_{L(1+\rho),A_L})}{k_L\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} + \frac{C_1}{R^m} \\
&\leq (1+\rho)^m + \frac{C(1+\rho)^m}{\delta}(1-(1-\epsilon)^m) + \frac{1}{\delta}\frac{O(R^{m-1})}{R^m} + \frac{C_1}{R^m}.
\end{aligned}$$

Taking superior limits in $R \to \infty$ in the last estimate and using the expression for δ, we get

$$D^+(\mathcal{Z}) \leq (1+\rho)^m + \frac{C(1+\rho)^m}{\beta_\epsilon^2\left(\frac{1}{1+\rho}\right)}(1-(1-\epsilon)^m), \quad \forall \epsilon, \rho > 0,$$

where C is independent of $\epsilon > 0$ and ρ. Thus, taking limits in $\epsilon \to 0$ and then in $\rho \to 0$, we get the claimed result:

$$D^+(\mathcal{Z}) \leq 1.$$

□

3.3.4 Trace estimate

In this section, we will give a proof of Proposition 3.14. For this purpose, we need the following computation.

Lemma 3.17. *Let $H : [0, \infty) \to [0, 1]$ be a $\mathcal{C}^\infty$-function with compact support in $[0, 1]$. Let $B(\xi, R/L)$ be a ball in M. Then*

$$I := \int_{B(\xi,R/L)} \int_{M \setminus B(\xi,R/L)} \left| \sum_{i=1}^{k_L} H(\lambda_i/L)\phi_i(z)\phi_i(w) \right|^2 dV(w)dV(z) \leq CR^{m-1},$$

where C is independent of L and R.

Proof. The proof follows by using Lemma 1.5 and working in local coordinates. Indeed, using Lemma 1.5, we have

$$I \leq CL^{2m} \int_{B(\xi,R/L)} \int_{M \setminus B(\xi,R/L)} \frac{dV(w)}{(1+Ld(z,w))^{2(m+1)}} dV(z) = (\star)$$

Let

$$r_z = L\left(\frac{R}{L} - d(z,\xi)\right)/2.$$

Then, $B(z, r_z/L) \subset B(\xi, R/L)$ and hence

$$(\star) \leq CL^{2m} \int_{B(\xi,R/L)} \int_{M \setminus B(z,r_z/L)} \frac{dV(w)}{(1+Ld(z,w))^{2(m+1)}} dV(z).$$

We claim that

$$\int_{M \setminus B(z,r/L)} \frac{dV(w)}{(1+Ld(z,w))^{2(m+1)}} \leq C\frac{1}{L^m(1+r)^{m+2}}. \tag{3.8}$$

Using this fact and coordinates in $B(\xi, R/L)$, we get

$$\begin{aligned}
I &\lesssim L^m \int_{\mathbb{B}(0,R/L)} \frac{1}{(1+r_{\exp_\xi(w)})^{m+2}} \sqrt{|g|(\exp_\xi(w))}\,dm(w) \\
&\lesssim L^m \int_{\mathbb{B}(0,R/L)} \frac{1}{(1+r_{\exp_\xi(w)})^{m+2}} dm(w) \lesssim L^m \int_0^{R/L} r^{m-1}(2+R-rL)^{-m-2} dr \\
&\lesssim \int_0^R s^{m-1}(2+R-s)^{-m-2} ds = C\int_2^{2+R} (2+R-t)^{m-1} t^{-m-2} dt \\
&\le C(2+R)^{m-1} \int_2^{2+R} t^{-m-2} = \frac{C}{m+1}(2+R)^{m-1}\left[2^{-m-1} - (2+R)^{-m-1}\right] \\
&\le C\frac{2^{-m-1}}{m+1}(2+R)^{m-1} \le CR^{m-1}.
\end{aligned}$$

Now we just need to prove the claim (3.8). Let

$$f(w) = \chi_{M\setminus B(z,r/L)}(w) \frac{1}{(1+Ld(z,w))^{2(m+1)}}.$$

Then,

$$\begin{aligned}
\int_{M\setminus B(z,r/L)} \frac{dV(w)}{(1+Ld(z,w))^{2(m+1)}} &= \int_M f(w)dV(w) \\
&= \int_0^\infty \operatorname{vol}(\{w \in M;\ f(w) > t\})dt =: \int_0^\infty \operatorname{vol}(A_t)dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
A_t &= \left\{ w \in M; \quad d(w,z) \ge \frac{r}{L}, \frac{1}{(1+Ld(z,w))^{2(m+1)}} > t \right\} \\
&= \left\{ w \in M; \quad \frac{r}{L} \le d(w,z) < \frac{t^{-1/(2(m+1))} - 1}{L} \right\}.
\end{aligned}$$

Note that for $t > (1+r)^{-2(m+1)}$ (i.e. $t^{-1/(2(m+1))} - 1 < r$), $A_t = \emptyset$ and for $t \le (1+r)^{-2(m+1)}$, $A_t = B(z, (t^{-1/(2(m+1))} - 1)/L) \setminus B(z, r/L)$. Thus,

$$\operatorname{vol}(A_t) \le C\frac{(t^{-1/(2(m+1))} - 1)^m}{L^m}, \quad \forall t \le \frac{1}{(1+r)^{2(m+1)}}.$$

Therefore,

$$\begin{aligned}
&\int_{M\setminus B(z,r/L)} \frac{dV(w)}{(1+Ld(z,w))^{2(m+1)}} \le \frac{C}{L^m} \int_0^{(1+r)^{-2(m+1)}} (t^{-1/(2(m+1))} - 1)^m dt \\
&\le \frac{C}{L^m} \int_0^{(1+r)^{-2(m+1)}} t^{-m/(2(m+1))} dt = \frac{C}{L^m} \frac{1}{(1+r)^{m+2}}.
\end{aligned}$$

This proves the claim and the desired result. $\square$

Taking into account this last result, we can prove Proposition 3.14.

Proof of Proposition 3.14. Recall the definition of the kernels $B_L^\epsilon(z,w)$ and $\tilde{B}_L^\epsilon(z,w)$:

$$B_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w),$$

$$\tilde{B}_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \alpha\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w) := \sum_{i=1}^{k_L} \beta_\epsilon^2\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w).$$

Let $A = B(\xi, R/L)$. First, we will compute the trace of $T_{L,A}^\epsilon \circ T_{L,A}^\epsilon$.

$$\begin{aligned}\operatorname{tr}(T_{L,A}^\epsilon \circ T_{L,A}^\epsilon) &= \int_{A\times A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z)\\ &= \int_A \sum_{i=1}^{k_L} \alpha^2\left(\frac{\lambda_i}{L}\right)\phi_i^2(z)dV(z) - \int_A\int_{M\setminus A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z).\end{aligned}$$

Thus, we have

$$\begin{aligned}\operatorname{tr}(T_{L,A}^\epsilon) - \operatorname{tr}(T_{L,A}^\epsilon \circ T_{L,A}^\epsilon) &= \int_A \sum_{i=1}^{k_L}\left[\alpha\left(\frac{\lambda_i}{L}\right) - \alpha^2\left(\frac{\lambda_i}{L}\right)\right]\phi_i^2(z)dV(z)\\ &+ \int_A\int_{M\setminus A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z) =: I_1 + I_2.\end{aligned}$$

By Lemma 3.17, $I_2 = O(R^{m-1})$ with constants independent of L (the constant may depend on ϵ). Now we need to estimate I_1. Note that $\alpha(x) \equiv 1$ for $0 \le 0 \le 1-\epsilon$. Hence,

$$\begin{aligned}I_1 &= \int_A \sum_{\lambda_i \in (L(1-\epsilon),L]}\left[\alpha\left(\frac{\lambda_i}{L}\right) - \alpha^2\left(\frac{\lambda_i}{L}\right)\right]\phi_i^2(z)dV(z)\\ &\le \int_A \sum_{\lambda_i \in (L(1-\epsilon),L]} \phi_i^2(z)dV(z) = \int_A (K_L(z,z) - K_{L(1-\epsilon)}(z,z))dV(z).\end{aligned}$$

Using the expression of the reproducing kernel (see Chapter 1, Section 1.2), we obtain:

$$K_L(z,z) - K_{L(1-\epsilon)}(z,z) = c_m L^m(1-(1-\epsilon)^m) + O(L^{m-1})(1-(1-\epsilon)^{m-1}).$$

Thus,

$$\begin{aligned}I_1 &\le c_m(1-(1-\epsilon)^m)L^m \operatorname{vol}(B(\xi,R/L)) + \frac{o(L^m)}{L^m}(1-(1-\epsilon)^{m-1})\\ &\le C(1-(1-\epsilon)^m)R^m + \frac{o(L^m)}{L^m}(1-(1-\epsilon)^{m-1}),\end{aligned}$$

where C is independent of L, R and ϵ. Therefore,

$$\lim_{L\to\infty} I_1 \le C(1-(1-\epsilon)^m)R^m.$$

If $\rho > 0$ then a similar computation, working with $L(1+\rho)$ instead of L, shows the second claim of Proposition 3.14. $\square$

3.3.5 Technical results

In this section, we present a proof of Lemma 3.15 and 3.16. First, we shall prove a localization type property of the functions f_L of the space E_L.

Lemma 3.18. *Let $\mathcal{Z}$ be a s-separated family. Given $f_L \in E_L$ and $\eta > 0$, there exists $t_0 = t_0(\eta)$ such that for all $t \geq t_0$,*

$$\frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+(t)} |f_L(z_{Lj})|^2 \leq C_1 \int_{M \setminus A_L} |f_L|^2 + C_2 \eta \int_{A_L} |f_L|^2,$$

where $A_L^+ = A_L^+(t) = B(\xi, (R+t)/L)$, C_1 and C_2 are constants depending only on the manifold M and the separation constant s of $\mathcal{Z}$.

Proof. Let $f_L \in E_L$. Consider the kernel

$$B_{2L}(z,w) := B_{2L}^{1/2}(z,w),$$

where $B_L^{\epsilon}(z,w)$ is defined in (1.4). Note that the transform $B_{2L}|_{E_L}$ is the identity transform, by construction. Thus,

$$f_L(z) = B_{2L}(f_L)(z) = \int_M B_{2L}(z,w) f_L(w) dV(w), \quad \forall z \in M. \tag{3.9}$$

By Lemma 1.5, for any $N > m$, there exists a constant C_N such that

$$|B_{2L}(z,w)| \leq C_N L^m \frac{1}{(1 + 2L d_M(z,w))^N}. \tag{3.10}$$

We will choose N later on.

In order to prove the claimed result, we will show that

1. Given $\eta > 0$ there exists $t_0 = t_0(\eta)$ such that for all $t \geq t_0$,

$$\frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \leq C_1 \int_{M \setminus A_L} |f_L| + C_2 \eta \int_{A_L} |f_L|, \tag{3.11}$$

   where C_i are uniform constants.

2. Given $\eta > 0$ there exists $t_0 = t_0(\eta)$ such that for all $t \geq t_0$,

$$\max_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \leq C_1 \|f_L\|_{L^\infty(M \setminus A_L)} + C_2 \eta \|f_L\|_{L^\infty(A_L)}, \tag{3.12}$$

   where C_i are uniform constants.

Hence, by interpolating between the L^1-norm and L^∞-norm, we will have the claimed result for the L^2-norm. Let us prove first that this is true in the L^∞-norm.

Observe that the set of points $z_{Lj} \notin A_L^+$ is contained in $M \setminus B(\xi, (R+t)/L)$. Thus,

$$\max_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \leq \|f_L\|_{L^\infty(M \setminus B(\xi,(R+t)/L))} \leq \|f_L\|_{L^\infty(M \setminus B(\xi, R/L))}.$$

Hence, (3.12) is trivially true.

Now we just need to prove (3.11). Let

$$0 \leq h_j(w) := \frac{1}{(1 + 2Ld_M(z_{Lj}, w))^N} \leq 1.$$

Using (3.9) and (3.10), we obtain:

$$\begin{aligned} \frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| &\leq C_N \left\{ \int_{M \setminus B(\xi, R/L)} + \int_{B(\xi, R/L)} \right\} |f_L(w)| \sum_{z_{Lj} \notin A_L^+} h_j(w) \\ &=: I_1 + I_2. \end{aligned}$$

Observe that for all $w \in M$,

$$h_j(w) \lesssim \frac{L^m}{s^m} \int_{B(z_{Lj}, s/L)} \frac{dV(z)}{(1 + 2Ld_M(z,w))^N}.$$

Note that $B(z_{Lj}, s/L)$ are pairwise disjoint and for $w \in B(\xi, R/L)$,

$$\bigcup_{z_{Lj} \notin A_L^+} B\left(z_{Lj}, \frac{s}{L}\right) \subset M \setminus B\left(\xi, \frac{R+t-s}{L}\right) \subset M \setminus B\left(w, \frac{t-s}{L}\right),$$

Therefore, if $w \in B(\xi, R/L)$,

$$\sum_{z_{Lj} \notin A_L^+} h_j(w) \lesssim \frac{L^m}{s^m} \int_{M \setminus B\left(w, \frac{t-s}{L}\right)} \frac{dV(z)}{(1 + 2Ld_M(z,w))^N} \lesssim \frac{C_N}{s^m (t-s)^{N-m}} \leq \eta$$

for all $t \geq t_0(\eta, N)$, provided $N > m$. This implies that

$$I_2 \leq C_2 \eta \int_{B(\xi, R/L)} |f_L|.$$

The only thing left is to bound the integral I_1. Given w, let

$$\#J := \#\left\{ j : B(w, 2s/L) \cap B(z_{Lj}, s/L) \neq \emptyset \right\}.$$

Then there exists a uniform constant $C(s)$ (depending only on s) such that $\#J \leq C(s)$. Hence,

$$\sum_{z_{Lj} \notin A_L^+} h_j(w) = \sum_{\substack{z_{Lj} \notin A_L^+ \\ j \in J}} h_j(w) + \sum_{\substack{z_{Lj} \notin A_L^+ \\ j \notin J}} h_j(w) \leq C(s) + \sum_{j \notin J} h_j(w).$$

Note that for any $w \in M$,

$$\cup_{j \notin J} B(z_{Lj}, s/L) \subset M \setminus B(w, s/L).$$

Hence,

$$\sum_{j \notin J} h_j(w) \lesssim \frac{L^m}{s^m} \int_{M \setminus B(w,s/L)} \frac{dV(z)}{(1 + 2L d_M(z,w))^N} \lesssim C_{s,N},$$

provided $N > m$. So we have that

$$I_1 \leq (C(s) + C_{s,N}) \int_{M \setminus B(\xi, R/L)} |f_L| dV$$

and the claim is proved. □

Lemma 3.19. *Let $\mathcal{Z}$ be a s-separated family. Given $f_L \in E_L$ and $\eta > 0$, there exists $t_1 = t_1(\eta)$ such that for all $t \geq t_1$*

$$\frac{1}{k_L} \sum_{z_{Lj} \in A_L^-(t)} |f_L(z_{Lj})|^2 \leq C_1 \int_{A_L} |f_L|^2 + C_2 \eta \int_{M \setminus A_L} |f_L|^2,$$

where $A_L^- = A_L^-(t) = B(\xi, (R-t)/L)$, C_1 and C_2 are constants depending only on the manifold M and the separation constant s of $\mathcal{Z}$.

The proof of this Lemma is similar to the one of Lemma 3.18.

Now we will prove Lemma 3.15.

Proof of Lemma 3.15. Given $F_L \in E_L$, assume that

$$F_L(z_{Lj}) = 0, \quad \forall z_{Lj} \in A_L^+ = B(\xi, (R+t)/L).$$

Then, using the fact that $\mathcal{Z}$ is L^2-MZ and Lemma 3.18, we have

$$\begin{aligned} \|F_L\|_2^2 &\lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |F_L(z_{Lj})|^2 = \frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+} |F_L(z_{Lj})|^2 \\ &\leq C_1 \int_{M \setminus A_L} |F_L|^2 + C_2 \eta \int_{A_L} |F_L|^2 \leq C_1 \int_{M \setminus A_L} |F_L|^2 + C_2 \eta \|F_L\|_2^2. \end{aligned}$$

Picking $\eta > 0$ small enough (note that it is independent of ϵ, L and R), we get a $t_0(\eta)$ given by Lemma 3.18 so that for all $t \geq t_0$,

$$\|F_L\|_2^2 \leq C_3 \int_{M \setminus A_L} |F_L|^2 dV, \tag{3.13}$$

where $F_L \in E_L$ is any function vanishing at the points z_{Lj} that are contained in A_L^+. Observe that $C_3 > 1$.

Now, we consider an orthonormal basis of eigenvectors G_j^L corresponding to the eigenvalues λ_j^L of the modified concentration operator. Let

$$f_L(z) = \sum_{j=1}^{N_L+1} c_j^L G_j^L \in E_L.$$

Note that $f_L \in E_L$ since $N_L \leq CR^m \leq k_L$ for L big enough, in view of the separation of $\mathcal{Z}$. Consider now $F_L := B_L^\epsilon(f_L) \in E_L$. We will apply inequality (3.13) to F_L. We pick c_j^L such that $F_L(z_{Lj}) = 0$ for all $z_{Lj} \in A_L^+$. Observe that

$$\sum_{j=1}^{N_L+1} \lambda_j^L |c_j^L|^2 = \langle T_{L,A_L}^\epsilon f_L, f_L\rangle = \int_{A_L} |B_L^\epsilon f_L(w)|^2 dV(w).$$

Now, using inequality (3.13),

$$\begin{aligned}\lambda_{N_L+1}^L \sum_{j=1}^{N_L+1} |c_j^L|^2 &\leq \sum_{j=1}^{N_L+1} \lambda_j^L |c_j^L|^2 = \left\{\int_M - \int_{M\setminus A_L}\right\} |B_L^\epsilon f_L(z)|^2 dV \\ &\leq \left(1 - \frac{1}{C_3}\right) \|B_L^\epsilon(f_L)\|_2^2 \leq \left(1 - \frac{1}{C_3}\right) \|f_L\|_2^2 = \left(1 - \frac{1}{C_3}\right) \sum_{j=1}^{N_L+1} |c_j^L|^2.\end{aligned}$$

where the constant C_3 comes from (3.13) (independent of ϵ, L and R). Hence,

$$\lambda_{N_L+1}^L \leq 1 - \frac{1}{C_3} =: \gamma < 1.$$

□

Now we are going to prove the technical lemma corresponding to the interpolating case.

Proof of Lemma 3.16. Let $\mathcal{I} = \{j; \quad z_{Lj} \in A_L^-\}$ and $\rho > 0$ fixed.
Recall that, by Lemma 3.3, if $\mathcal{Z}$ is an interpolating sequence, then for each sequence $\{c_{Lj}\}_{Lj}$ such that

$$\sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty,$$

we can construct functions $f_L \in e(L)$ with $\sup_L \|f_L\|_2 < \infty$ and $f_L(z_{Lj}) = c_{Lj}$, where

$$e(L) := \left\{ f_L \in E_L; \quad \|f_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \right\}.$$

In fact, these functions f_L are the solution of the interpolation problem with minimal norm.

Since we have an interpolating family, we can construct for each $z_{Lj} \in \mathcal{Z}(L)$ a function $f_j \in e(L)$ such that

$$f_j(z_{Lj'}) = \delta_{jj'}.$$

Clearly these functions f_j are linearly independent. Note that, since $B^{\epsilon}_{L(1+\rho)}|_{E_L}$ is bijective, for each j there exists a function $h_j \in E_L$ such that

$$f_j = B^{\epsilon}_{L(1+\rho)} h_j.$$

Let

$$F := \operatorname{span}\left\{h_j; \quad z_{Lj} \in A_L^-\right\}.$$

Note that F has dimension n_L. Let $f_L \in F$ an arbitrary function and $g_L := B^{\epsilon}_{L(1+\rho)} f_L$. Since $f_L \in F$, we know that

$$f_L = \sum_{j \in \mathcal{I}} c_j h_j.$$

Hence,

$$g_L = B^{\epsilon}_{L(1+\rho)} f_L = \sum_{j \in \mathcal{I}} c_j B^{\epsilon}_{L(1+\rho)} h_j = \sum_{j \in \mathcal{I}} c_j f_j \in e(L),$$

where we have used that each $f_j \in e(L)$ and so this g_L is the function of minimal norm that solves the interpolation problem with data $c_j \delta_{jj'}$. Therefore,

$$\|g_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |g_L(z_{Lj})|^2,$$

where the constant do not depend on ϵ and L.

Note that, by construction, f_j vanishes in the points $z_{Lj'}$ with $j \neq j'$. Therefore, for each $j \in \mathcal{I}$ fixed, we have that $f_j(z_{Lk}) = 0$ for all $k \notin \mathcal{I}$. Thus,

$$g_L(z_{Lk}) = \sum_{j \in \mathcal{I}} c_j f_j(z_{Lk}) = 0, \quad \forall k \notin \mathcal{I},$$

This shows that $g_L = 0$ for $z_{Lk} \notin A_L^-$. Hence, applying Lemma 3.19 to $g_L = B^{\epsilon}_{L(1+\rho)} f_L$, we get

$$\begin{aligned}
\|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2 &\lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |g_L(z_{Lj})|^2 = \frac{1}{k_L} \sum_{j \in \mathcal{I}} |g_L(z_{Lj})|^2 \\
&\le C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV + C_2 \eta \int_{M \setminus A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV \\
&\le C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV + C_2 \eta \|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2.
\end{aligned}$$

Picking η small enough (note that it is independent of ρ, ϵ, L and R because all the constants appearing in the above computation are independent of these parameters), we get from Lemma 3.19 a value $t_1 = t_1(\eta)$ such that for all $t \ge t_1$,

$$\|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2 \le C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV. \tag{3.14}$$

Thus, using this last estimate (3.14), we get the following.

$$\beta_\epsilon^2\left(\frac{1}{1+\rho}\right)\|f_L\|_2^2 \le \sum_{i=1}^{k_L}\beta_\epsilon^2\left(\frac{\lambda_i}{L(1+\rho)}\right)|\langle f_L,\phi_i\rangle|^2$$
$$= \sum_{i=1}^{k_{L(1+\rho)}}\beta_\epsilon^2\left(\frac{\lambda_i}{L(1+\rho)}\right)|\langle f_L,\phi_i\rangle|^2 = \|B^\epsilon_{L(1+\rho)}f_L\|_2^2$$
$$\le C_1\int_{A_L}|B^\epsilon_{L(1+\rho)}f_L|^2dV = C_1\langle T^\epsilon_{L(1+\rho),A_L}f_L,f_L\rangle.$$

We have proved that for all $f_L \in F$,

$$\frac{\langle T^\epsilon_{L(1+\rho),A_L}f_L,f_L\rangle}{\langle f_L,f_L\rangle} \ge \delta := C\beta_\epsilon^2\left(\frac{1}{1+\rho}\right), \tag{3.15}$$

where C does not depend on L, ρ, ϵ and f_L. Now, applying Weyl-Courant Lemma (see [DS67, Part 2, p. 908]), we know

$$\lambda_{k-1}^{L(1+\rho)} \ge \inf_{g\in E_{L(1+\rho)}\cap E}\frac{\langle T^\epsilon_{L(1+\rho),A_L}g,g\rangle}{\langle g,g\rangle}$$

for each subspace $E \subset E_{L(1+\rho)}$ with $\dim(E) = k$. Take $E := F \subset E_L \subset E_{L(1+\rho)}$ defined previously. Note that $\dim(E)$ =$\dim(F) = n_L$ and hence, using (3.15)

$$\lambda_{n_L-1}^{L(1+\rho)} \ge \inf_{f_L\in F}\frac{\langle T^\epsilon_{L(1+\rho),A_L}f_L,f_L\rangle}{\langle f_L,f_L\rangle} \ge \delta.$$

Note that $0 < \delta = C\beta_\epsilon^2(1/(1+\rho)) < 1$. □

3.4 Difference from the classical case

In the Paley-Wiener space $PW^2_{[-\pi,\pi]}$ and $M = \mathbb{S}^m$, the study of the classical concentration operator $\mathcal{K}_A^L$ is enough. In the general case, we don't have proper estimates of the reproducing kernel. That is the reason of the replacement of $\mathcal{K}_A^L$ by a smooth version of it. We can still prove Theorem 3.11 using the classical concentration operator in the special case where the reproducing kernel has some L^2-decay away from the diagonal (we call such manifolds permissible, see Definition 3.20 below).

Definition 3.20. *We say that a manifold M is permissible if its normalized reproducing kernel, i.e.*

$$P_L(z,w) := \frac{K_L(z,w)}{L^{m/2}} \quad (\|P_L(z,\cdot)\|_2 \simeq 1),$$

satisfies the following condition for some $\alpha \in (0,1)$,

$$\limsup_{L\to\infty}\int_{M\setminus B(z,r/L)}|P_L(z,w)|^2dV(w) \lesssim \frac{1}{(1+r)^\alpha}, \forall r \ge 1, \tag{3.16}$$

where the constant does not depend on r and z.

We will provide some examples of permissible manifolds in the last part of this section.

Using the same notation as in Section 3.3.3, Proposition 3.14 is easier to prove when M is permissible and can be stated as:

Proposition 3.21. *Let M be a permissible manifold and $A_L = B(\xi, R/L)$. Then*

$$\limsup_{L\to\infty} \left(\operatorname{tr}(\mathcal{K}^L_{A_L}) - \operatorname{tr}(\mathcal{K}^L_{A_L} \circ \mathcal{K}^L_{A_L})\right) = O(R^{m-\alpha}),$$

where α is given as in (3.16)*. In particular,*

$$\liminf_{L\to\infty} \left(\operatorname{tr}(\mathcal{K}^L_{A_L}) - \operatorname{tr}(\mathcal{K}^L_{A_L} \circ \mathcal{K}^L_{A_L})\right) = O(R^{m-\alpha}),$$

Considering the eigenvalues associated to $\mathcal{K}^L_{A_L}$, Lemma 3.15 and 3.16 with the corresponding remarks remain true (in fact, the proofs are more simple). Thus, the proof of Theorem 3.11 follows using the same calculations as in Section 3.3.3. We sketch the proof of Proposition 3.21 for completeness.

Proof of Proposition 3.21. Since M is permissible, we know that for some $\alpha \in (0,1)$, (3.16) holds. Let $P_L(z,w)$ be the normalized kernel for the space E_L, i.e.

$$K_L(z,w) = L^{m/2} P_L(z,w), \quad \|P_L(z,\cdot)\|_2 \simeq 1.$$

Let us compute the trace of $\mathcal{K}^L_{A_L} \circ \mathcal{K}^L_{A_L}$. Using the reproducing property, we have

$$\begin{aligned}
\operatorname{tr}(\mathcal{K}^L_{A_L} \circ \mathcal{K}^L_{A_L}) &= \int_{A_L}\int_{A_L} |K_L(z,w)|^2 dV(w)dV(z) \\
&= \int_{A_L}\int_{M} |K_L(z,w)|^2 dV(w)dV(z) \\
&- \int_{A_L}\int_{M\setminus A_L} |K_L(z,w)|^2 dV(w)dV(z) \\
&= \operatorname{tr}(\mathcal{K}^L_{A_L}) - L^m \int_{A_L}\int_{M\setminus A_L} |P_L(z,w)|^2 dV(w)dV(z).
\end{aligned}$$

Since $A_L = B(\xi, R/L)$, we have

$$\operatorname{tr}(\mathcal{K}^L_{A_L}) - \operatorname{tr}(\mathcal{K}^L_{A_L} \circ \mathcal{K}^L_{A_L}) = L^m \int_{B(\xi,R/L)}\int_{M\setminus B(\xi,R/L)} |P_L(z,w)|^2 dV(w)dV(z).$$

Thus, we need to bound this last integral. Observe that if $z \in B(\xi, R/L)$, then it is easy to check that $B(z, r_z/L) \subset B(\xi, R/L)$, where

$$\frac{r_z}{L} = \frac{1}{2}\left(\frac{R}{L} - r\right), \quad r = d(z,\xi).$$

Therefore,

$$
\begin{aligned}
& L^m \int_{B(\xi,R/L)} \int_{M\setminus B(\xi,R/L)} |P_L(z,w)|^2 dV(w) dV(z) \\
& \leq L^m \int_{B(\xi,R/L)} \int_{M\setminus B(z,r_z/L)} |P_L(z,w)|^2 dV(w) dV(z) \\
& = L^m \int_{B(\xi,R/L)} \int_{M\setminus B(z,r_z/L), r_z\geq 1} |P_L(z,w)|^2 dV(w) dV(z) \\
& + L^m \int_{B(\xi,R/L)} \int_{M\setminus B(z,r_z/L), r_z< 1} |P_L(z,w)|^2 dV(w) dV(z) \\
& =: I_1 + I_2.
\end{aligned}
$$

We will deal first with the second integral. Note that

$$
r_z < 1 \iff r > \frac{R-2}{L}.
$$

Hence,

$$
\begin{aligned}
I_2 & \leq L^m \int_{B(\xi,R/L)} \chi_{\{d(z,\xi)>(R-2)/L\}}(z) \int_M |P_L(z,w)|^2 dV(w) dV(z) \\
& \simeq L^m \int_{B(\xi,R/L)} \chi_{\{d(z,\xi)>(R-2)/L\}}(z) dV(z) \\
& \simeq L^m \int_{(R-2)/L}^{R/L} r^{m-1} dr = O(R^{m-1}).
\end{aligned}
$$

Now, we turn our attention to the first integral. Note that

$$
r_z \geq 1 \iff r \leq \frac{R-2}{L}.
$$

Let

$$
Q_L(z) = \int_{M\setminus B(z,r_z/L)} |P_L(z,w)|^2 dV(w).
$$

Using local coordinates in the ball $B(\xi, R/L)$ we get

$$
\begin{aligned}
I_1 & = L^m \int_{B(\xi,R/L)} \chi_{\{d(z,\xi)\leq(R-2)/L\}}(z) Q_L(z) dV(z) \\
& \lesssim L^m \int_{\mathbb{S}^{m-1}} g(\theta) \int_0^{(R-2)/L} r^{m-1} Q_L(r,\theta) dr d\theta \\
& = \int_{\mathbb{S}^{m-1}} g(\theta) \int_0^{R-2} s^{m-1} Q_L(s/L,\theta) ds d\theta \\
& = \int_{\mathbb{S}^{m-1}} g(\theta) \int_0^{R-2} s^{m-1} \left(\int_{M\setminus B\left(z, \frac{R-s}{2L}\right)} |P_L(z,w)|^2 dV(w) \right) ds d\theta.
\end{aligned}
$$

Using the Reverse Fatou Lemma and estimate (3.16) (note that $(R-s)/2 \geq 1$ for all $s \in [0, R-2]$), we have

$$
\begin{aligned}
&\limsup_{L\to\infty} I_1 \\
&\leq \int_{\mathbb{S}^{m-1}} g(\theta) \int_0^{R-2} s^{m-1} \left(\limsup_{L\to\infty} \int_{M\setminus B\left(z, \frac{R-s}{2L}\right)} |P_L(z,w)|^2 dV(w) \right) ds d\theta \\
&\lesssim \int_{\mathbb{S}^{m-1}} g(\theta) \int_0^{R-2} s^{m-1} \frac{1}{(1+\frac{1}{2}(R-s))^{\alpha}} ds d\theta \\
&\simeq \int_0^{R-2} s^{m-1}(2+R-s)^{-\alpha} ds = \int_4^{2+R} (2+R-r)^{m-1} r^{-\alpha} dr \\
&= O(R^{m-\alpha}).
\end{aligned}
$$

Since $\alpha \in (0,1)$, we have that

$$
\begin{aligned}
\limsup_{L\to\infty} \left(\mathrm{tr}(\mathcal{K}_{A_L}^L) - \mathrm{tr}(\mathcal{K}_{A_L}^L \circ \mathcal{K}_{A_L}^L) \right) &\lesssim I_1 + \limsup_{L\to\infty} I_2 \\
&= O(R^{m-\alpha}) + O(R^{m-1}) = O(R^{m-\alpha}).
\end{aligned}
$$

□

3.4.1 Examples of permissible manifolds

In this subsection, we provide some examples of manifolds that are permissible. The fundamental example is $\mathbb{S}^m$ that has been extensively studied in [Mar07]. For the sake of completeness we will state the calculations that prove that $\mathbb{S}^m$ is permissible.

To proceed with the examples, we focus our attention on the compact two-point homogeneous manifolds (of dimension $m \geq 2$) that coincides with the compact Riemannian symmetric spaces of rank one (see for instance, [Hel62, IX.5] for definitions and basic facts). These have been classified and fall into the following types.

1. The Sphere $\mathbb{S}^m$, $m \geq 2$.
2. The real projective spaces $\mathbb{RP}^m$, $m \geq 2$.
3. The complex projective spaces $\mathbb{CP}^m$, $m \geq 1$.
4. The quaternionic projective spaces $\mathbb{HP}^m$, $m \geq 2$, where $\mathbb{H}$ is the quaternionic field.
5. The Cayley elliptic plane $\mathbb{P}^{16}$(Cayley). This space can be viewed as $\mathbb{OP}^2$, where $\mathbb{O}$ is the octonionic field.

Later we will give the proper definitions of the projective spaces.

The compact two-point homogeneous manifolds share many properties. For instance, all geodesics on these manifolds are closed and have the same length ([Hel62, p. 356]). The main reason to work with these manifolds is that one can compute explicitly the reproducing kernel. Moreover, it comes out that the reproducing kernel depends only on one variable: the geodesic distance. For a general compact manifold, one does not have explicit formulas for the kernel and hence, the condition of permissibility cannot be easily checked. Furthermore, in the compact two-point manifolds we have more advantages than the general case. For instance, one can multiply two polynomials (eigenfunctions) in order to obtain another polynomial. This fact gives us a tool to work with these manifolds as was done in [Mar07] for the specific case of $\mathbb{S}^m$.

The Spheres

On the m-dimensional sphere $\mathbb{S}^m$ ($m \geq 2$), we know that the reproducing kernel has the explicit form (for instance, see [Mar07, Page 565]):

$$K_L(z,w) = C_{m,L} P_L^{(\alpha,\beta)}(\langle z, w\rangle),$$

where $C_{m,L} \asymp L^{m/2}$, $\alpha = 1+\lambda$, $\beta = \lambda$, $\lambda = (m-2)/2$ and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of order n and parameters α and β. So, the normalized kernel is just the Jacobi polynomial. Now we proceed in order to prove that $\mathbb{S}^m$ is permissible.

Let $B(\xi, R/L)$ be a ball in $\mathbb{S}^m$ (i.e. a spherical cup). Let ϕ be a rotation that maps the north pole to ξ (we know that the jacobian of this change of variables is one and $\langle \xi, \phi(z)\rangle = \langle \phi^{-1}(\xi), z\rangle = \langle N, z\rangle$, where N is the north pole). Therefore,

$$\begin{aligned}
&\int_{\mathbb{S}^m\setminus B(\xi,R/L)} |P_L^{(1+\lambda,\lambda)}(\langle \xi, z\rangle)|^2 dV(z)\\
&\quad= \int_{\phi^{-1}(\mathbb{S}^m\setminus B(\xi,R/L))} |P_L^{(1+\lambda,\lambda)}(\langle \xi, \phi(w)\rangle)|^2 |J_\phi(w)| dV(w)\\
&\quad= \int_{\mathbb{S}^m\setminus B(N,R/L)} |P_L^{(1+\lambda,\lambda)}(\langle N, z\rangle)|^2 dV(z)\\
&\quad= \mathrm{vol}(\mathbb{S}^m) \int_{R/L}^{\pi} |P_L^{(1+\lambda,\lambda)}(\cos(\theta))|^2 \sin^{m-1}(\theta) d\theta\\
&\quad\simeq \int_{R/L}^{\pi-R/L} |P_L^{(1+\lambda,\lambda)}(\cos(\theta))|^2 \sin^{m-1}(\theta) d\theta\\
&\quad+ \int_{\pi-R/L}^{\pi} |P_L^{(1+\lambda,\lambda)}(\cos(\theta))|^2 \sin^{m-1}(\theta) d\theta =: I_1 + I_2.
\end{aligned}$$

We always assume that $R >> 1$ and $L >> R$, so that $R/L \simeq 0$. We deal first with the second integral. In what follows, the stated below relations will be used.

1. There exists $\epsilon > 0$ such that for all $\pi - \epsilon \leq \theta \leq \pi$, $\sin\theta \simeq (\pi - \theta)$, and for all $0 \leq \theta < \epsilon$, $\mathrm{cotan}(\theta) = \cos\theta/\sin\theta \simeq (1-\theta^2/2)/\theta \leq 1/\theta$.

2. For all $c/L \leq \theta \leq \pi - \theta/L$, $\sin\theta \geq c/L$.
3. $\sin^2(\theta/2) = (1-\cos\theta)/2$, $\cos^2(\theta/2) = (1+\cos\theta)/2$.

Using Szegö's estimate (see [Sze39, p. 168]), we know that for $-1 \leq x \leq 0$,

$$|P_n^{(1+\lambda,\lambda)}(x)| = O(n^\lambda).$$

Hence,

$$\begin{aligned} I_2 &\lesssim L^{2\lambda} \int_{\pi-R/L}^{\pi} \sin^{m-1}\theta d\theta \simeq L^{m-2} \int_{\pi-R/L}^{\pi} (\pi-\theta)^{m-1} d\theta \\ &= L^{m-2} \int_0^{R/L} x^{m-1} dx \simeq \frac{R^m}{L^2} \to 0, \quad L \to \infty. \end{aligned}$$

Now we turn our attention to the first integral. Szegö (see [Sze39, p. 198]) proved that fixed c, for all $c/n \leq \theta \leq \pi - c/n$, we have

$$P_n^{(1+\lambda,\lambda)}(\cos\theta) = \frac{k(\theta)}{\sqrt{n}} \left[\cos((n+\lambda+1)\theta + \gamma) + \frac{O(1)}{n\sin\theta} \right],$$

where $\gamma = -(\lambda+3/2)\pi/2$ and

$$k(\theta) = \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2} \right)^{-\lambda-3/2} \left(\cos\frac{\theta}{2} \right)^{-\lambda-1/2}.$$

Therefore, applying this estimate to our case, we get for all $\theta \in [R/L, \pi - R/L]$,

$$\begin{aligned} |P_L^{(1+\lambda,\lambda)}(\cos\theta)|^2 \sin^{m-1}\theta &\leq \frac{k(\theta)^2}{L} \left(1 + \frac{O(1)}{L^2\sin^2\theta} \right) \sin^{m-1}\theta \\ &\simeq \frac{1}{L} \frac{1}{\sin^2(\theta/2)} \left(1 + \frac{O(1)}{L^2\sin^2\theta} \right). \end{aligned}$$

Note that $L^2\sin^2\theta \geq R^2 \geq 1$. Hence,

$$|P_L^{(1+\lambda,\lambda)}(\cos\theta)|^2 \sin^{m-1}\theta \lesssim \frac{1}{L} \frac{1}{\sin^2(\theta/2)}.$$

Thus,

$$I_1 \lesssim \frac{1}{L} \int_{R/L}^{\pi-R/L} \frac{1}{\sin^2(\theta/2)} d\theta. \tag{3.17}$$

Computing this last integral, we get

$$\begin{aligned} I_1 &\lesssim \frac{1}{L} \int_{R/L}^{\pi-R/L} \operatorname{cosec}^2(\theta/2) d\theta = \frac{2}{L} \int_{\frac{R}{2L}}^{\frac{\pi}{2}-\frac{R}{2L}} \operatorname{cosec}^2(x) dx \\ &= \frac{2}{L} \left[-\operatorname{cotan}(x) \right]_{\frac{R}{2L}}^{\frac{\pi}{2}-\frac{R}{2L}} = \frac{2}{L} \left[\operatorname{cotan}\frac{R}{2L} - \operatorname{cotan}\left(\frac{\pi}{2} - \frac{R}{2L} \right) \right] \\ &\lesssim \frac{1}{L} \frac{1}{\frac{R}{2L}} \simeq \frac{1}{R} \lesssim \frac{1}{1+R}. \end{aligned}$$

Observe that for any $\alpha \in (0,1)$, $(1+R)^\alpha \le (1+R)$, thus

$$\frac{1}{1+R} \le \frac{1}{(1+R)^\alpha}.$$

Hence, we have that the normalized kernels satisfy

$$\limsup_{L\to\infty} \left(\int_{\mathbb{S}^m \setminus B(\xi,R/L)} |P_L^{(1+\lambda,\lambda)}(\langle \xi, z\rangle)|^2 dV(z) \right) \lesssim \frac{1}{1+R} \le \frac{1}{(1+R)^\alpha}.$$

The Projective Spaces

We will follow the notations and some basic facts used in [Rag71, Section 4].
Let $\mathbb{K}$ be any one of the (skew) fields

$$\begin{aligned}
&\mathbb{O} = \{x = x_0 + x_1 i_1 + \cdots + x_7 i_7 : \quad x_i \in \mathbb{R}\} \quad \text{the octonions,}\\
&\mathbb{H} = \{x \in \mathbb{O} : \quad x_l = 0 \quad \forall l = 4, \cdots, 7\} \quad \text{the quaternions,}\\
&\mathbb{C} = \{x \in \mathbb{H} : \quad x_2 = x_3 = 0\} \quad \text{the complex numbers,}\\
&\mathbb{R} = \{x \in \mathbb{C} : \quad x_1 = x_2 = x_3 = 0\} \quad \text{the reals.}
\end{aligned}$$

Let $d = \dim_{\mathbb{R}} \mathbb{K}$. The projective space over $\mathbb{K}$ of dimension m over $\mathbb{K}$ is defined as a quotient of the unit sphere in $\mathbb{K}^{m+1}$ with an equivalence relation, i.e.

$$\mathbb{KP}^m := \left\{ x \in \mathbb{K}^{m+1}, |x| = \sqrt{\bar{x}x} = 1 \right\} / \sim,$$

where $x \sim y \iff x = \lambda y$ with $\lambda \in \mathbb{K}$, $|\lambda| = 1$. Without loss of generality, we denote the class of an element by the element itself. Note that the real dimension of $\mathbb{KP}^m$ is dm and 16 for $\mathbb{P}^{16}(\text{Cayley}) = \mathbb{OP}^2$ ($d = 8$ and $m = 2$).
The projective space $\mathbb{KP}^m$ can be naturally provided with a Riemannian metric ρ (see [Rag71, Page 166]). We state some properties of ρ (for a proof see [Rag71, Lemma 4.1]).

Lemma 3.22 (Ragozin)**.** *Let $x, y \in \mathbb{KP}^m$. The following properties are satisfied.*

1. $\sqrt{2}\rho(x,y) = \arccos(2|\langle x, y\rangle|^2 - 1)$.
2. $\max(\rho(x,y)) = \pi/\sqrt{2}$.

Now we will present a result of D.L. Ragozin that shows how to compute integrals involving the normalized Riemannian measure on $\mathbb{KP}^m$ denoted by μ. For a proof see [Rag71, Proof of Lemma 4.4]. If f is a radial function, i.e. $f(y) = g(\rho(x,y))$, then choosing polar coordinates about x we have

$$\int_{\mathbb{KP}^m} f(y)d\mu(y) = \int_0^{\pi/\sqrt{2}} g(r)A(r)dr, \tag{3.18}$$

where $A(r)$ is the "area" of the sphere (in $\mathbb{KP}^m$) about x and radius r. The expression of $A(r)$ is

$$A(r) = c' \sin^{md-d}(r/\sqrt{2}) \sin^{d-1}(\sqrt{2}r),$$

with c' a constant depending on the volume of the unit ball in $\mathbb{R}^{md}$. For more details, see [Rag71, Page 168], [AB77, Section 3] and [Hel65, Section 6]. In fact, the Laplace-Beltrami operator on $\mathbb{KP}^m$ of a radial function is the operator

$$\frac{1}{A(r)}\frac{\partial}{\partial r}\left(A(r)\frac{\partial}{\partial r}\right),$$

that is also called as the radial part of $\Delta_{\mathbb{KP}^m}$ (see [Hel65, Section 6] for further details).

In Chapter 4, Section 4.3, we prove that the space E_L on $\mathbb{KP}^m$ is identified with the space of polynomials on $\mathbb{KP}^m$ of degree at most L. Now we recall a result of D.L. Ragozin that gives an explicit expression of the reproducing kernel for the space of real polynomials on $\mathbb{KP}^m$ of degree at most L. For a proof see [Rag72, Theorem 4'] (and [Mea82, Page 112]).

Theorem 3.23 (Ragozin). *Let $\mathbb{KP}^m$ be the dm-dimensional projective space over $\mathbb{K}$ ($dm \geq 2$) and let $K_L(x,y)$ be the reproducing kernel for the space $\mathcal{P}_L$ of real polynomials on $\mathbb{KP}^m$ of degree at most L. Then*

$$K_L(x,y) = \sum_{k=0}^{L}\left\{h_k^{(\alpha,\beta)}\right\}^{-1} P_k^{(\alpha,\beta)}(1)P_k^{(\alpha,\beta)}(\cos(\sqrt{2}\rho(x,y))),$$

where $\alpha = (dm-2)/2$, $\beta = (d-2)/2$, $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial of degree k,

$$h_k^{(\alpha,\beta)} = c\int_{-1}^{1}\left[P_k^{(\alpha,\beta)}(t)\right]^2(1-t)^\alpha(1+t)^\alpha dt$$

and

$$\frac{1}{c} = \int_{-1}^{1}(1-t)^\alpha(1+t)^\alpha dt.$$

One can compute explicitly the reproducing kernel using the Christoffel-Darboux formula (see [Sze39, Page 71]):

$$\sum_{k=0}^{n}\left\{h_k^{(\alpha,\beta)}\right\}^{-1} P_k^{(\alpha,\beta)}(1)P_k^{(\alpha,\beta)}(t) = A_{\alpha,\beta}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)}P_n^{(\alpha+1,\beta)}(t).$$

Note that the normalized reproducing kernel $P_L(x,y)$ is the respective Jacobi polynomial of parameters $\alpha+1$ and β and degree L. Recall that

$$d = \begin{cases} 1, & \mathbb{K}=\mathbb{R}, \\ 2, & \mathbb{K}=\mathbb{C}, \\ 4, & \mathbb{K}=\mathbb{H}, \\ 8, & \mathbb{K}=\mathbb{O} \end{cases}.$$

We summarize these results for $\mathbb{KP}^m$ in Tables 3.1 and 3.2.

	α	β	$A(r)$
$\mathbb{RP}^m$	$\frac{m-2}{2}$	$\frac{-1}{2}$	$c' \sin^{m-1}(r/\sqrt{2})$
$\mathbb{CP}^m$	$m-1$	0	$c' \sin^{2m-2}(r/\sqrt{2}) \sin(\sqrt{2}r)$
$\mathbb{HP}^m$	$2m-1$	1	$c' \sin^{4m-4}(r/\sqrt{2}) \sin^3(\sqrt{2}r)$
$\mathbb{OP}^2$	7	3	$c' \sin^{8}(r/\sqrt{2}) \sin^7(\sqrt{2}r)$

Table 3.1: Results for the Projective Spaces

	$P_L(x,y)$
$\mathbb{RP}^m$	$P_L^{(m/2,-1/2)}(\cos(\sqrt{2}\rho(x,y)))$
$\mathbb{CP}^m$	$P_L^{(m,0)}(\cos(\sqrt{2}\rho(x,y)))$
$\mathbb{HP}^m$	$P_L^{(2m,1)}(\cos(\sqrt{2}\rho(x,y)))$
$\mathbb{OP}^2$	$P_L^{(8,3)}(\cos(\sqrt{2}\rho(x,y)))$

Table 3.2: Normalized kernel for the Projective Spaces

Now we will proceed to prove that the projective spaces are permissible. For this, we will first show the computation of the integrals of the form

$$I := \int_{\mathbb{KP}^m \setminus B(N,R/L)} |P_L^{(\alpha+1,\beta)}(\cos(\sqrt{2}\rho(x,N)))|^2 d\mu(x).$$

In what follows, $R >> 1$ and $L >> R$ so that $R/L \simeq 0$. Without loss of generality, we can work with the north pole N of $\mathbb{KP}^m$ as we did in the case of the sphere.

Observe that using (3.18), we can compute the above integral.

$$\begin{aligned}
I &= \int_{R/L}^{\pi/\sqrt{2}} |P_L^{(\alpha+1,\beta)}(\cos(\sqrt{2}r))|^2 A(r) dr \\
&= c' \int_{R/L}^{\pi/\sqrt{2}} |P_L^{(\alpha+1,\beta)}(\cos(\sqrt{2}r))|^2 \sin^{md-d}\left(\frac{r}{\sqrt{2}}\right) \sin^{d-1}(\sqrt{2}r) dr \\
&= c \int_{\sqrt{2}R/L}^{\pi} |P_L^{(\alpha+1,\beta)}(\cos(\theta))|^2 \sin^{md-d}(\theta/2) \sin^{d-1}(\theta) d\theta \\
&= c \left[\int_{\frac{\sqrt{2}R}{L}}^{\pi - \frac{\sqrt{2}R}{L}} + \int_{\pi - \frac{\sqrt{2}R}{L}}^{\pi} \right] |P_L^{(\alpha+1,\beta)}(\cos(\theta))|^2 \sin^{md-d}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta) d\theta \\
&=: I_1 + I_2.
\end{aligned}$$

We will deal first with the second integral.

Szegö proved that for $-1 \leq x \leq 0$,

$$|P_n^{(a,b)}(x)| = O(n^{\max(b,-1/2)}).$$

Thus,

$$
\begin{aligned}
I_2 &\lesssim L^{2\beta} \int_{\pi-\frac{\sqrt{2}R}{L}}^{\pi} \sin^{md-d}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta) d\theta \leq L^{2\beta} \int_{\pi-\frac{\sqrt{2}R}{L}}^{\pi} \sin^{d-1}(\theta) d\theta \\
&\simeq L^{2\beta} \int_{\pi-\frac{\sqrt{2}R}{L}}^{\pi} (\pi-\theta)^{d-1} d\theta \simeq \frac{R^d}{L^{d-2\beta}}.
\end{aligned}
$$

Note that $d-2\beta = 2$ in any of the projective spaces. Thus,

$$
I_2 \leq C \frac{R^d}{L^2} \to 0, \quad L \to \infty.
$$

Now we will compute the first integral I_1. For this, we will use an estimate proved by Szegö ([Sze39, p. 198]): for $c > 0$ fixed, let $c/n \leq \theta \leq \pi - c/n$. Then

$$
P_n^{(a,b)}(\cos\theta) = \frac{1}{\sqrt{n}} k(\theta) \left\{ \cos(N\theta + \gamma) + \frac{O(1)}{n \sin\theta} \right\},
$$

where $N = n + (a+b+1)/2$, $\gamma = -(a+1/2)\pi/2$ and

$$
k(\theta) = \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2} \right)^{-a-1/2} \left(\cos\frac{\theta}{2} \right)^{-b-1/2},
$$

We will apply this estimate to our normalized kernels $P_L^{(\alpha+1,\beta)}(\cos(\theta))$. Note that

$$
I_1 \lesssim \frac{1}{L} \int_{\frac{\sqrt{2}R}{L}}^{\pi-\frac{\sqrt{2}R}{L}} k(\theta)^2 \sin^{md-d}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta) d\theta.
$$

Now we compute the quantity $k(\theta)^2 \sin^{md-d}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta)$.

$$
\begin{aligned}
&k(\theta)^2 \sin^{md-d}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta) \\
&= \frac{1}{\pi} \sin^{-d-1}\left(\frac{\theta}{2}\right) \cos^{-d+1}\left(\frac{\theta}{2}\right) \sin^{d-1}(\theta) \\
&= \frac{1}{\pi} \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} \left[\frac{1-\cos^2\theta}{4} \right]^{-\frac{d-1}{2}} \sin^{d-1}\theta = \frac{2^{d-1}}{\pi} \frac{1}{\sin^2\left(\frac{\theta}{2}\right)}.
\end{aligned}
$$

Thus, we have:

$$
I_1 \lesssim \frac{1}{L} \int_{\frac{\sqrt{2}R}{L}}^{\pi-\frac{\sqrt{2}R}{L}} \frac{d\theta}{\sin^2(\theta/2)}.
$$

This is just the integral appearing in (3.17). So we have for any $\alpha \in (0,1)$,

$$
I_1 \lesssim \frac{1}{1+R} \leq \frac{1}{(1+R)^\alpha}.
$$

Hence, we have that the normalized kernels for the projective spaces verify the condition of permissibility, i.e.

$$\begin{aligned}&\limsup_{L\to\infty}\left(\int_{\mathbb{KP}^{dm}\setminus B(N,R/L)}|P_L(x,N)|^2 d\mu(x)\right)\\&\lesssim \limsup_{L\to\infty} I_1 + \limsup_{L\to\infty} I_2 \lesssim \frac{1}{(1+R)^\alpha}.\end{aligned}$$

Chapter 4

Fekete arrays on some compact manifolds

In this chapter, we study the Fekete arrays on some compact manifolds. As explained in the introduction, Fekete arrays are well distributed points that are almost sampling and interpolating. Consider the case of $M = \mathbb{S}^m$ ($m > 1$). The spaces E_L for the sphere are identified with the spaces of spherical harmonics of degree at most L, usually denoted by Π_L:

$$\Pi_L = \operatorname{span} \bigcup_{l=0}^{L} \mathcal{H}_l,$$

where $\mathcal{H}_l$ is the space of spherical harmonics of degree l in $\mathbb{S}^m$. These vector spaces have dimensions $\pi_L \simeq L^m$. Let $\left\{Q_1^L, \dots, Q_{\pi_l}^L\right\}$ be any basis in Π_L. The points $\mathcal{Z}(L) = \left\{z_{L1}, \dots, z_{L\pi_L}\right\}$ maximizing the determinant

$$|\Delta(x_1, \dots, x_{\pi_L})| = |\det(Q_i^L(x_j))_{i,j}|$$

are called the Fekete points of degree L for $\mathbb{S}^m$ (these points are sometimes called extremal fundamental systems of points as in [SW04]). They are not to be confused with the elliptic Fekete points that are a system of points that minimize the potential energy. The interest for studying the Fekete points is that they are better suited nodes for cubature formulas and for polynomial interpolation (check [SW04] for a detailed discussion). The interpolatory cubature rule associated with a system of points $x_1, \dots, x_{\pi_L} \in \mathbb{S}^m$ is the rule

$$Q_L(f) := \sum_{j=1}^{\pi_L} w_j f(x_j), \tag{4.1}$$

obtained by integrating exactly the polynomial that interpolates $f \in \mathcal{C}(\mathbb{S}^m)$ at the points $x_1, \dots, x_{\pi_L}$. For $L = 2$ it is proved in [Rei94] that all cubature weights of the rule $Q_L(f)$ are positive. For larger values of L, less is known. A cubature rule that have all the weights positive is of interest for numerical integration.
The condition that the cubature rule (4.1) is exact for all polynomials in Π_L can be written as a linear system $Gw = e$, where w is the vector of cubature weights, e is the vector

of 1's in $\mathbb{R}^{\pi_L}$ and G is the matrix with components $G_{ij} = K_L(z_{Li}, z_{Lj})$. In [SW04, Section 2.2], it is observed that $G = A^T A$, where A is the basis matrix obtained from the spherical harmonic basis. Thus, G is positive semi-definite for any set of points $x_1, \dots, x_{\pi_L}$ and $\det(G) = (\det(A))^2 \geq 0$. Note that a set of Fekete points can be obtained also as the one maximizing the determinant of G. In order to compute the weights w_j, we need to solve a linear system $Gw = e$. This is possible whenever $\det(G) > 0$. For numerical integration it is convenient that the determinant of G should be as big as possible. Thus, a natural candidate of $\{x_1, \dots, x_{\pi_L}\}$ is a set of extremal fundamental system of points. This is a reason why a set of Fekete points is of interest.

A natural problem is to find the limiting distribution of points as $L \to \infty$. In [MOC10], J. Marzo and J. Ortega-Cerdà proved that as $L \to \infty$, the number of Fekete points in a spherical cap $B(z, R)$ gets closer to $\pi_L \tilde{\sigma}(B(z, R))$, where $\tilde{\sigma}$ is the normalized Lebesgue measure on $\mathbb{S}^m$. They emphasize the connection of the Fekete points with the M-Z and interpolating arrays. In [BB08], Berman and Boucksom have found the limiting distribution in the context of line bundles over complex manifolds. The proof is based on a careful study of the weighted transfinite diameter and its differentiability.
Following the approach in [MOC10], we define the Fekete points for an arbitrary compact manifold associated to spaces E_L and study their distribution as $L \to \infty$. The main difficulty in relating the Fekete points with the M-Z and interpolating families is to construct a weighted interpolation formula for E_L where the weight has a fast decay off the diagonal. That is the reason why, we restrict our attention to manifolds that satisfy a product property (see Definition 4.1, below). Under this hypothesis, we are able to prove the equidistribution of the Fekete points.

The outline of the chapter is the following. In the first section we give the precise definition of the Fekete points and the manifolds that are going to be considered. In the second section, we relate the Fekete families with the interpolating and M-Z arrays and prove the main result. In the last section, we provide some examples of manifolds that satisfy the product property.

4.1 Definitions and statement of the results

Definition 4.1. *We say that a manifold is* ***admissible*** *if it satisfies the following product property: there exists a constant $C > 0$ such that for all $0 < \epsilon < 1$ and $L \geq 1$:*

$$E_L \cdot E_{\epsilon L} \subset E_{L(1+C\epsilon)}. \tag{4.2}$$

From now on, M will denote an admissible manifold. Thus we are assuming that we may multiply two functions of our spaces and still obtain a function which is in some space E_L. In the last section, we provide some examples of such manifolds.

Given $L \geq 1$ and $m_L \in \mathbb{N}$, we consider a triangular family of points in M, $\mathcal{Z} = \{\mathcal{Z}(L)\}_L$, denoted as

$$\mathcal{Z}(L) = \{z_{Lj} \in M \ : \ 1 \leq j \leq m_L\}, L \geq 1,$$

and we assume that $m_L \to \infty$ as L increases.

The Fekete points are the points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. We will show their connection with the interpolating and M-Z families and prove the asymptotic equidistribution of the Fekete points on the manifold. But before that, we give the precise definition and notation for the Fekete arrays.

Definition 4.2. *Let $\left\{\phi_1^L, \dots, \phi_{k_L}^L\right\}$ be any basis in E_L. The points*

$$\mathcal{Z}(L) = \{z_{L1}, \dots, z_{Lk_L}\}$$

maximizing the determinant

$$|\Delta(x_1, \dots, x_{k_L})| = |\det(\phi_i^L(x_j))_{i,j}|$$

are called a set of ***Fekete points*** *of degree L for M.*

4.2 Fekete points

Now we proceed to prove the equidistribution of a set of Fekete points (see Theorem 4.6). The scheme is to explore the connection of the Fekete arrays with the interpolating and M-Z families. Then, making use of a density result (see Theorem 4.5 below), known for the interpolating and M-Z arrays, we will be able to prove the equidistribution of the Fekete points.

The following two results give the relation of the Fekete points with the interpolating and M-Z arrays. Intuitively, Fekete families are almost interpolating and M-Z.

Theorem 4.3. *Given $\epsilon > 0$, let $L_\epsilon = [(1+\epsilon)L]$ and*

$$\mathcal{Z}_\epsilon(L) = \mathcal{Z}(L_\epsilon) = [z_{L_\epsilon 1}, \dots, z_{L_\epsilon k_{L_\epsilon}}],$$

where $\mathcal{Z}(L)$ is a set of Fekete points of degree L. Then $\mathcal{Z}_\epsilon = \{\mathcal{Z}_\epsilon(L)\}_L$ is a M-Z *array.*

Proof. Assume that $\mathcal{Z}$ is a Fekete family. We will prove that they are uniformly separated. Consider the Lagrange *polynomial* defined as

$$l_{Li}(z) := \frac{\Delta(z_{L1}, \dots, z_{L(i-1)}, z, z_{L(i+1)}, \dots, z_{Lk_L})}{\Delta(z_{L1}, \dots, z_{Lk_L})}.$$

Note that

- $\|l_{Li}\|_\infty = 1$.
- $l_{Li}(z_{Lj}) = \delta_{ij}$.
- $l_{Li} \in E_L$.

Thus, using the Bernstein inequality for the space E_L (see (1.10)), we have for all $j \neq i$,

$$\begin{aligned}1 = |l_{Li}(z_{Li}) - l_{Li}(z_{Lj})| &\leq \|\nabla l_{Li}\|_\infty d_M(z_{Li}, z_{Lj}) \\ &\lesssim L\|l_{Li}\|_\infty d_M(z_{Li}, z_{Lj}) = Ld_M(z_{Li}, z_{Lj}).\end{aligned}$$

Therefore,

$$d_M(z_{Li}, z_{Lj}) \geq \frac{C}{L},$$

i.e. $\mathcal{Z}$ is uniformly separated. This implies that $\mathcal{Z}_\epsilon$ is also uniformly separated because

$$d_M(z_{L_\epsilon i}, z_{L_\epsilon j}) \geq \frac{C}{L_\epsilon} \overset{L_\epsilon \leq (1+\epsilon)L}{\geq} \frac{C/(1+\epsilon)}{L}.$$

Using Theorem 2.5 we get for any $f_L \in E_L$,

$$\frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2 \lesssim \int_M |f_L|^2 dV.$$

In order to prove that $\mathcal{Z}_\epsilon$ is M-Z, we only need to prove the converse inequality, i.e.

$$\frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2 \gtrsim \|f_L\|_2^2.$$

Consider the Lagrange interpolation operator defined in $\mathcal{C}(M)$ as

$$\Lambda_L(f)(z) := \sum_{j=1}^{k_L} f(z_{Lj}) l_{Lj}(z).$$

Note that

$$\|\Lambda_L(f)\|_\infty \leq k_L \|f\|_\infty.$$

This estimate isn't enough. In order to have better control on the norms, we will make use of a weighted interpolation formula. Fix a point $z \in M$ and let $p(z, \cdot)$ be a function in the space $E_{\frac{\epsilon}{C}L}$ such that $p(z,z) = 1$, where C is the constant appearing in (4.2). Then given $f_L \in E_L$ one has

$$R(w) = f_L(w)p(z,w) \in E_{L_\epsilon}.$$

Note that $R(z) = f_L(z)p(z,z) = f_L(z)$. Thus, we have a weigthed representation formula

$$f_L(z) = \sum_{j=1}^{k_{L_\epsilon}} p(z, z_{L_\epsilon j}) f_L(z_{L_\epsilon j}) l_{L_\epsilon j}(z).$$

We define the operator Q_L from $\mathbb{C}^{k_{L_\epsilon}} \to E_{L_{2\epsilon}}$ [1]as

$$Q_L[v](z) = \sum_{j=1}^{k_{L_\epsilon}} v_j p(z, z_{L_\epsilon j}) l_{L_\epsilon j}(z), \quad \forall v \in \mathbb{C}^{k_{L_\epsilon}}.$$

[1] $E_{L(1+\epsilon)} \cdot E_{L\frac{\epsilon}{C}} \subset E_{L(1+\epsilon)(1+\epsilon/(1+\epsilon))} \subset E_{L(1+2\epsilon)} \subset E_{L_{2\epsilon}}$.

We want to prove that

$$\int_M |Q_L[v](z)|^2 dV(z) \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |v_j|^2, \tag{4.3}$$

with constant independent of L. Once we have proved this estimate, choosing $v_j = f_L(z_{L_\epsilon j})$ we will have

$$\begin{aligned} Q_L[(f_L(z_{L_\epsilon j}))_j](z) &= \sum_{j=1}^{k_{L_\epsilon}} f_L(z_{L_\epsilon j}) p(z, z_{L_\epsilon j}) l_{L_\epsilon j}(z) \\ &= \sum_{j=1}^{k_{L_\epsilon}} R(z_{L_\epsilon j}) l_{L_\epsilon j}(z) = R(z) = f_L(z). \end{aligned}$$

Hence, applying the claimed inequality (4.3) we will obtain

$$\|f_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2,$$

and thus $\mathcal{Z}_\epsilon$ is M-Z.
In order to prove (4.3), we need to choose the weight p with care. We shall construct $p \in E_{L\epsilon/C}$ with a fast decay off the diagonal.

Let $\delta > 0$ and consider the kernels $B_L(z, w) := B_L^\delta(z, w)$ defined in Section 1.2. Let

$$p(z, w) = \frac{B_{L\frac{\epsilon}{C}}(z, w)}{B_{L\frac{\epsilon}{C}}(z, z)} \in E_{L\frac{\epsilon}{C}}.$$

Observe that

- $p(z, z) = 1$.
- $$\begin{aligned} \int_M |p(z, w)| dV(w) &= \frac{1}{B_{L\frac{\epsilon}{C}}(z, z)} \|B_{L\frac{\epsilon}{C}}(z, \cdot)\|_1 \\ &\lesssim \frac{1}{k_L}, \end{aligned}$$

  where we have used $\|B_L(z, \cdot)\|_1 \lesssim 1$ (see [FM10b, Equation (2.11), Theorem 2.1] for a proof).

Now we are ready to prove (4.3). Note that

$$\begin{aligned} \int_M |Q_L[v](z)| dV(z) &\le \int_M \sum_{j=1}^{k_{L_\epsilon}} |v_j| |p(z, z_{L_\epsilon j})| |l_{L_\epsilon j}(z)| dV(z) \\ &\le \sum_{j=1}^{k_{L_\epsilon}} |v_{Lj}| \|p(\cdot, z_{L_\epsilon j})\|_1 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |v_{Lj}|. \end{aligned}$$

On the other hand,

$$|Q_L[v](z)| \leq \sup_j |v_j| \sum_{j=1}^{k_{L_\epsilon}} |p(z, z_{L_\epsilon j})|.$$

Let s be the separation constant of Z_{L_ϵ} and

$$h(z,w) = \frac{1}{(1 + L_\epsilon d_M(z,w))^N} \leq 1.$$

Note that,

$$\inf_{w \in B(z_{L_\epsilon j}, s/L_\epsilon)} h(z,w) \geq C_s h(z, z_{L_\epsilon j}).$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{k_{L_\epsilon}} |p(z, z_{L_\epsilon j})| &= \frac{1}{B_{L\frac{\epsilon}{C}}(z_{L_\epsilon j}, z_{L_\epsilon j})} \sum_{j=1}^{k_{L_\epsilon}} |B_{L\frac{\epsilon}{C}}(z_{L_\epsilon j}, z)| \lesssim \sum_{j=1}^{k_{L_\epsilon}} \frac{1}{(1 + L\frac{\epsilon}{C} d_M(z, z_{L_\epsilon j}))^N} \\
&\lesssim \frac{L_\epsilon^m}{s^m} \int_{\cup_{j=1}^{k_{L_\epsilon}} B(z_{L_\epsilon j}, s/L_\epsilon)} h(z,w) dV(w) \\
&= \frac{L_\epsilon^m}{s^m} \int_{\cup_{j=1}^{k_{L_\epsilon}} B(z_{L_\epsilon j}, s/L_\epsilon) \cap B(z, 2s/L_\epsilon)} h(z,w) dV(w) \\
&+ \frac{L_\epsilon^m}{s^m} \int_{\cup_{j=1}^{k_{L_\epsilon}} B(z_{L_\epsilon j}, s/L_\epsilon) \cap B(z, 2s/L_\epsilon)^c} h(z,w) dV(w) \\
&\leq C_{s,\epsilon} + C_s L_\epsilon^m \int_{M \setminus B(z, 2s/L_\epsilon)} h(z,w) dV(w) \lesssim 1,
\end{aligned}$$

where we have used that

$$\int_{M \setminus B(z, r/L_\epsilon)} h(z,w) dV(w) \lesssim \frac{1}{L_\epsilon^m (1+r)^{N-m}}.$$

This computation follows by integrating $h(z,w)$ using the distribution function. Hence, we have proved that

$$\|Q_L[v]\|_\infty \lesssim \sup_j |v_j|.$$

The claimed estimate (4.3) follows by the Riesz-Thorin interpolation theorem. □

The following result relates the Fekete points with the interpolating families.

Theorem 4.4. *Given $\epsilon > 0$, let $L_{-\epsilon} = [(1-\epsilon)L]$ and let*

$$\mathcal{Z}_{-\epsilon}(L) = \mathcal{Z}(L_{-\epsilon}) = \left\{ z_{L_{-\epsilon}1}, \dots, z_{L_{-\epsilon}k_{L_{-\epsilon}}} \right\},$$

where $\mathcal{Z}(L)$ is a set of Fekete points of degree L. Then the array $\mathcal{Z}_{-\epsilon} = \{\mathcal{Z}_{-\epsilon}(L)\}_L$ is an interpolating family.

Proof. Given any array of values $\{v_{L_{-\epsilon}j}\}_{j=1}^{k_{L_{-\epsilon}}}$, we consider

$$R_L[v](z) = \sum_{j=1}^{k_{L_{-\epsilon}}} v_{L_{-\epsilon}j} p(z, z_{L_{-\epsilon}j}) l_{L_{-\epsilon}j}(z) \in E_L,$$

where $p(\cdot, z) \in E_{L\epsilon/C}$ defined in the proof of the previous Theorem. Note that

$$\begin{aligned} R_L[v](z_{L_{-\epsilon}k}) &= \sum_{j=1}^{k_{L_{-\epsilon}}} v_{L_{-\epsilon}j} p(z_{L_{-\epsilon}k}, z_{L_{-\epsilon}j}) l_{L_{-\epsilon}j}(z_{L_{-\epsilon}k}) \\ &= v_{L_{-\epsilon}k} p(z_{L_{-\epsilon}k}, z_{L_{-\epsilon}k}) = v_{L_{-\epsilon}k}. \end{aligned}$$

Also, as in the proof of the previous theorem we have

$$\sum_{j=1}^{k_{L_{-\epsilon}}} |p(z, z_{L_{-\epsilon}j})| \lesssim 1$$

and

$$\int_M |p(z, z_{L_{-\epsilon}j})| dV(z) \lesssim \frac{1}{k_L}.$$

Thus, as before we have that

$$|R_L[v](z)| \le \sup_j |v_{L_{-\epsilon}j}| \sum_{j=1}^{k_{L_{-\epsilon}}} |p(z, z_{L_{-\epsilon}j})| \lesssim \sup_j |v_{L_{-\epsilon}j}|.$$

Hence

$$\|R_L[v]\|_\infty \lesssim \sup_j |v_{L_{-\epsilon}j}|.$$

Also,

$$\|R_L[v]\|_1 \le \sum_{j=1}^{k_{L_{-\epsilon}}} |v_{L_{-\epsilon}j}| \|p(\cdot, z_{L_{-\epsilon}j})\|_1 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_{-\epsilon}}} |v_{L_{-\epsilon}j}|.$$

By the Riesz-Thorin interpolation theorem we get

$$\|R_L[v]\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_{-\epsilon}}} |v_{L_{-\epsilon}j}|^2.$$

□

Now we are ready to prove the equidistribution of the Fekete points. Since the Fekete families are, essentially, interpolating and M-Z, we will make use of Theorem 3.11, proved in the previous chapter, that gives a necessary condition for interpolation and sampling. In what follows, σ will denote the normalized volume measure, i.e. $d\sigma = dV/\mathrm{vol}(M)$.

Recall that the upper and lower Beurling-Landau density of a uniformly separated family $\mathcal{Z}$ in M are defined as:

$$\mathcal{D}^-(\mathcal{Z}) = \liminf_{R\to\infty}\left(\liminf_{L\to\infty}\left(\min_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\sigma(B(\xi,R/L))}\right)\right)\right),$$

$$\mathcal{D}^+(\mathcal{Z}) = \limsup_{R\to\infty}\left(\limsup_{L\to\infty}\left(\max_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\sigma(B(\xi,R/L))}\right)\right)\right).$$

We proved in the previous chapter the following theorem.

Theorem 4.5. *Let M be an arbitrary compact manifold, without boundary, of dimension $m\geq 2$ and $\mathcal{Z}$ a uniformly separated family. If $\mathcal{Z}$ is M-Z, then $\mathcal{D}^-(\mathcal{Z})\geq 1$. On the other hand, if $\mathcal{Z}$ is an interpolating family, then $\mathcal{D}^+(\mathcal{Z})\leq 1$.*

From this result and the relation of the Fekete points with the M-Z and interpolating families, we will prove the equidistribution of the Fekete points.

Theorem 4.6. *Let $\mathcal{Z}=\{\mathcal{Z}(L)\}_{L\geq 1}$ be any array such that $\mathcal{Z}(L)$ is a set of Fekete points of degree L and $\mu_L = \frac{1}{k_L}\sum_{j=1}^{k_L}\delta_{z_{Lj}}$. Then μ_L converges in the weak-$*$ topology to the normalized volume measure on M.*

Proof. We know that for any $\epsilon>0$ the array $\mathcal{Z}_\epsilon=\{\mathcal{Z}_\epsilon(L)\}_{L\geq 1}$ is M-Z, so if we use the density results (see Theorem 4.5), we get for any $\epsilon>0$, a large $R=R(\epsilon)$ and $L(R(\epsilon))$ such that for all $L\geq L(R(\epsilon))$ and $\xi\in M$,

$$\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\sigma(B(\xi,R/L))}\geq(1-\epsilon). \tag{4.4}$$

Similarly, since $\mathcal{Z}_{-\epsilon}$ is interpolating (because $\mathcal{Z}$ is a family of Fekete) we know that there exist $R=R(\epsilon)$ and $L(R(\epsilon))$ such that for all $L\geq L(R(\epsilon))$ and $\xi\in M$,

$$\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\sigma(B(\xi,R/L))}\leq(1+\epsilon). \tag{4.5}$$

Note that

$$\mu_L(B(\xi,R/L)) = \frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L)).$$

Thus, for any $\epsilon>0$ there is a large R such that for any L big enough and $\xi\in M$,

$$(1-\epsilon)\sigma(B(\xi,r_L))\leq\mu_L(B(\xi,r_L))\leq(1+\epsilon)\sigma(B(\xi,r_L)), \tag{4.6}$$

where $r_L=R/L$. Hence, we have that

$$\lim_{L\to\infty}\frac{\mu_L(B(z,r_L))}{\sigma(B(z,r_L))}=1,\quad r_L\to 0, \tag{4.7}$$

uniformly in $z \in M$. This is enough to prove the equidistribution of the Fekete points. We proceed now with the details. Let $f \in \mathcal{C}(M)$. We will use the notation

$$\nu(f) := \int_M f(z) d\nu(z),$$

where ν is a measure and f_r will denote the mean of f over a ball $B(z,r)$ with respect to the volume measure, i.e.

$$f_r(z) = \frac{1}{\sigma(B(z,r))} \int_{B(z,r)} f(w) d\sigma(w).$$

We want to show that $\mu_L(f) \to \sigma(f)$, when $L \to \infty$, for all $f \in \mathcal{C}(M)$.

$$\begin{aligned} |\mu_L(f) - \sigma(f)| &\leq |(\mu_L - \sigma)(f - f_{r_L})| + |(\mu_L - \sigma)(f_{r_L})| \\ &\leq (\mu_L(M) + \sigma(M)) \|f - f_{r_L}\|_\infty + |(\mu_L - \sigma)(f_{r_L})| \\ &\leq 2\|f - f_{r_L}\|_\infty + |(\mu_L - \sigma)(f_{r_L})|. \end{aligned}$$

We will estimate the second term using [Blü90, Lemma 2] that says

$$\sup_{z \in M} \left| \frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|} - 1 \right| = O(r^2), \tag{4.8}$$

uniformly in $z \in M$, where $|\cdot|$ denotes the Euclidean volume and c is a constant depending only on M. Similarly, one has

$$\sup_{z \in M} \left| \frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))} - 1 \right| = O(r^2), \tag{4.9}$$

because, by the compactness of M (see Remark 1.1),

$$C_1 \leq \frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|} \leq C_2, \tag{4.10}$$

thus,

$$\left| \frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))} - 1 \right| = \left| \frac{1 - \frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|}}{\frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|}} \right| \leq \frac{Cr^2}{C_1} = O(r^2).$$

Similarly,

$$\sup_{w,z \in M} \left| \frac{\sigma(B(w,r))}{\sigma(B(z,r))} - 1 \right| = O(r^2). \tag{4.11}$$

Indeed, using (4.8), (4.9) and (4.10)

$$\begin{aligned} \left| \frac{\sigma(B(w,r))}{\sigma(B(z,r))} - 1 \right| &= \left| \frac{\sigma(B(w,r))}{|\mathbb{B}(0,cr)|} \left(\frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))} - 1 + 1 \right) - 1 \right| \\ &\leq \left| \frac{\sigma(B(w,r))}{|\mathbb{B}(0,cr)|} - 1 \right| + \frac{\sigma(B(w,r))}{|\mathbb{B}(0,cr)|} \left| \frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))} - 1 \right| \leq Cr^2. \end{aligned}$$

Using Fubini, we obtain the following.

$$\begin{aligned}|(\mu_L-\sigma)(f_{r_L})| &= \left|\int_M \fint_{B(z,r_L)} f(w)d\sigma(w)d(\mu_L-\sigma)(z)\right| \\ &= \left|\int_M f(w)\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))}d\sigma(w)\right| \\ &\leq \int_M |f(w)| \left|\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))}\right| d\sigma(w) = (1)\end{aligned}$$

Now we will deal with the second integral.

$$\begin{aligned}&\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))} = \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))}\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))} \\ &= \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))} + \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))}\left(\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right)\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))}\right| \leq \frac{1}{\sigma(B(w,r_L))}\left|\mu_L(B(w,r_L))-\sigma(B(w,r_L))\right| \\ &+ \int_{B(w,r_L)} \frac{1}{\sigma(B(w,r_L))}\left|\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right|(d\mu_L(z)+d\sigma(z)).\end{aligned}$$

Hence, using (4.11),

$$\begin{aligned}&(1) \leq \sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}-1\right| \|f\|_1 \\ &+ \sup_{z,w\in M}\left|\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right| \int_M |f(w)|\left(\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}+1\right)d\sigma(w) \\ &\leq \|f\|_1\left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}-1\right| + Cr_L^2\left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}\right|+1\right)\right).\end{aligned}$$

Briefly, we have obtained

$$\begin{aligned}&|\mu_L(f)-\sigma(f)| \leq 2\|f-f_{r_L}\|_\infty \\ &+ \|f\|_1\left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}-1\right| + Cr_L^2\left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}\right|+1\right)\right)\end{aligned}$$

Letting $L\to\infty$ and using (4.7), we obtain the desired result:

$$\mu_L(f)\to\sigma(f), \quad L\to\infty, \forall f\in\mathcal{C}(M).$$

□

4.3 Examples of admissible manifolds

The basic examples are the compact two-point homogeneous spaces. These spaces, essentially are $\mathbb{S}^m$, the projective spaces over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the Cayley Plane. In these spaces we can multiply two functions of the spaces E_L and obtain another function of some bigger space E_L. Indeed, in the case of the Sphere, E_L represents the spherical harmonics of degree less than L, usually denoted by Π_L. In such spaces, we know that

$$\Pi_{2L} = \text{span}\Pi_L\Pi_L,$$

(see [Mar08, Lemma 4.5]). Moreover, in $\mathbb{S}^m$,

$$\Pi_L \cdot \Pi_{\epsilon L} \subset \Pi_{L(1+\epsilon)}.$$

Thus, the product property holds trivially in $\mathbb{S}^m$.

Projective Spaces.

The case of the Projective spaces is similar to the Sphere. In [Sha01, Sections 3.2 and 3.3], there is a description and an orthogonal decomposition of the harmonic polynomials on the projective spaces.

Let $\mathbb{K}$ be the field of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Consider the sphere $\mathbb{S}^{m-1} \subset \mathbb{K}^m \approx \mathbb{R}^{dm}$, where $d = \dim_{\mathbb{R}}\mathbb{K}$. We define the projective space $\mathbb{KP}^{m-1}$ over the field $\mathbb{K}$ (of dimension $m-1$) as the quotient

$$\mathbb{KP}^{m-1} = \mathbb{S}^{m-1}/\sim,$$

where $x \sim y$ if and only if $y = \gamma x$ with $\gamma \in \mathbb{K}$ and $|\gamma| = 1$. Consider the space of homogeneous polynomials of degree less than L on the projective spaces:

$$\text{Pol}_L = \left\{ p(x)|_{\mathbb{S}^{m-1}};\ x \in \mathbb{R}^{dm}, \deg(p) \le L, p(\gamma x) = |\gamma|^L p(x), \forall \gamma \in \mathbb{K} \right\}.$$

It is immediate that Pol_L verify the product property (4.2). We will show that the spaces E_L associated to $\mathbb{KP}^{m-1}$ are identified with the spaces Pol_L. This proves that the projective spaces are admissible. It is observed in [Sha01, Section 3.2], that Pol_L coincide with its subspace of harmonic polynomials of degree less than L:

$$\text{Pol}_L = \text{Harm}_L = \{p \in \text{Pol}_L;\ \Delta_{\mathbb{R}^{dm}} p \equiv 0\}$$

and an orthogonal decomposition holds:

$$\text{Harm}_L = \text{Harm}(0) \oplus \text{Harm}(2) \oplus \ldots \oplus \text{Harm}(2[L/2]),$$

where $\text{Harm}(2k)$ is the subspace of Pol_L of harmonics of degree $2k$. We claim that the spaces E_L associated to the projective spaces are identified with the spaces Harm_L. Thus, we need to show that $\text{Harm}(2k)$ are the eigenspaces of $\Delta_{\mathbb{KP}^{m-1}}$. For this purpose, it is sufficient to prove that its reproducing kernel, $f(x, y)$, is an eigenfunction because then for any $Y \in \mathrm{Harm}(2k)$,

$$\Delta_{\mathbb{KP}^{m-1}} Y(x) = \Delta_{\mathbb{KP}^{m-1}} \langle Y, f(x, \cdot)\rangle = \langle Y, \Delta_{\mathbb{KP}^{m-1}} f(x, \cdot)\rangle = -\lambda^2 \langle Y, f(x, \cdot)\rangle = -\lambda^2 Y(x).$$

Let h_{2k} be the dimension of $\mathrm{Harm}(2k)$ and $(s_{ki})_{I=1}^{h_{2k}}$ be an orthonormal basis in $\mathrm{Harm}(2k)$. Its kernel, can be expressed as the function

$$f(x, y) = \sum_{i=1}^{h_{2k}} \overline{s_{ki}(x)} s_{ki}(y), \quad x, y \in \mathbb{S}^{m-1}.$$

It is proved, in [Sha01, Section 3.3], that $f(x, y)$ is a function of $|\langle x, y\rangle|^2$,

$$f(x, y) = q_k(|\langle x, y\rangle|^2),$$

where $q_k : [0, 1] \to \mathbb{C}$. Moreover, in [Sha01, Section 3.3], we can find an explicit form of this function:

$$\sum_{i=1}^{h_{2k}} \overline{s_{ki}(x)} s_{ki}(y) = b_k^d P_k^{(\alpha,\beta)}(2|\langle x, y\rangle|^2 - 1) = b_k^d P_k^{(\alpha,\beta)}(\cos(\sqrt{2}\rho(x, y))),$$

where ρ is the geodesic distance, b_k^d is a constant of normalization and

$$\alpha = \frac{dm - d - 2}{2}, \quad \beta = \frac{d - 2}{2}, \quad d = \dim_{\mathbb{R}} \mathbb{K}.$$

Note that, since the reproducing kernel $f(x, y)$ depends only on $|\langle x, y\rangle|^2$, we only need to take account of the radial part of the Laplacian, i.e.

$$\frac{1}{A(r)} \frac{\partial}{\partial r} \left(A(r) \frac{\partial}{\partial r} \right), \tag{4.12}$$

where $A(r) = c' \sin^{d(m-2)}(r/\sqrt{2}) \sin^{d-1}(\sqrt{2}r)$ (see [Rag71, p. 168]). Since we want to calculate the radial part of the Laplacian of functions of the form $f(\cos(\sqrt{2}r))$, we will make a change of variable $t = \cos(\sqrt{2}r)$ in (4.12). We proceed with the details taking into account these basic identities:

$$\sin(\theta/2) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}, \quad \sin(\arccos(x)) = \sqrt{1 - x^2}.$$

$$\begin{aligned} A(r) &= c' \sin^{d(m-2)}(\sqrt{2}r/2) \sin^{d-1}(\sqrt{2}r) \\ &= c'(1 - \cos(\sqrt{2}r))^{\frac{d(m-2)}{2}} \sin^{d-1}(\arccos(t)) \\ &= c'(1 - t)^{\frac{d(m-2)}{2}} (1 - t^2)^{\frac{d-1}{2}} = c'(1 - t)^{\frac{d(m-1)-1}{2}} (1 + t)^{\frac{d-1}{2}}. \end{aligned}$$

Now the radial part of the Laplacian can be written also in the variable t and it turns out to be:

$$\begin{aligned}&\frac{1}{A(r)}\frac{\partial}{\partial r}\left(A(r)\frac{\partial}{\partial r}\right)\\&=c'(1-t)^{-\frac{d(m-1)-2}{2}}(1+t)^{-\frac{d-2}{2}}\frac{\partial}{\partial t}\left((1-t)^{\frac{d(m-1)}{2}}(1+t)^{\frac{d}{2}}\frac{\partial}{\partial t}\right).\end{aligned}$$

Thus, defining

$$\alpha=\frac{d(m-1)-2}{2},\quad \beta=\frac{d-2}{2},$$

we get that the radial part of the Laplacian is of the form

$$c'(1-t)^{-\alpha}(1+t)^{-\beta}\frac{\partial}{\partial t}\left((1-t)^{\alpha+1}(1+t)^{\beta+1}\frac{\partial}{\partial t}\right).$$

It is well known (see [Sze39]) that the precise eigenfunctions of this operator are the Jacobi polynomials $P^{(\alpha,\beta)}(t)$ with eigenvalues $-k(k+\alpha+\beta+1)=-k(k+dm/2-1)$.

Observe that since the polynomials are dense in $L^2(\mathbb{KP}^{m-1})$,

$$L^2(\mathbb{KP}^{m-1})=\bigoplus_{l\geq 0}\mathrm{Harm}(2l),$$

For further details check [Rag72, Page 87]. Therefore, we know that all the eigenvalues of $\Delta_{\mathbb{KP}^{m-1}}$ are of the form $-k(k+dm/2-1)$. A simple calculation shows that the spaces E_L in the projective spaces are identified with the space of spherical harmonics (of the projective spaces) with degree less than L. More precisely,

$$E_L=\mathrm{Harm}_{L^*}=\bigoplus_{l=0}^{[L^*/2]}\mathrm{Harm}(2l)=\mathrm{Pol}_{L^*},$$

where $L^*=\sqrt{(dm/2-1)^2+4L^2}-(dm/2-1)>0$ for $L>0$ (note that $\frac{L^*}{2L}\to 1$, as $L\to\infty$). Therefore, E_L satisfies the product property (4.2) because the spaces Pol_{L^*} verify it. As a consequence, the projective spaces $\mathbb{KP}^{m-1}$ are admissible.

Other examples.

Another example with a different nature is the Torus (Figure 4.1) A Torus can be represented as the unit rectangle $[0,1]\times[0,1]$ with the identification $(x,y)\sim(x,y+1)$ and $(x,y)\sim(x+1,y)$ (as shown in Figure 4.2). The eigenfunctions of the Laplacian are of the form $e^{2\pi i(mx+ny)}$ with $m,n\in\mathbb{N}$. Now we are ready to prove the product property. Let $f_1\in E_L$, i.e. f_1 is a linear combination of eigenvectors of eigenvalues less than L^2, i.e. we are taking pairs (n,m) such that

$$4\pi^2\left(n^2+m^2\right)\leq L^2,$$

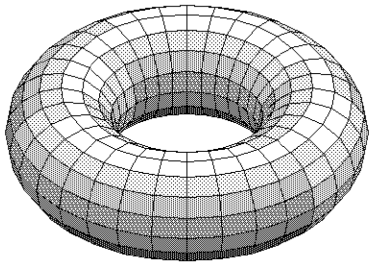


Figure 4.1: Torus

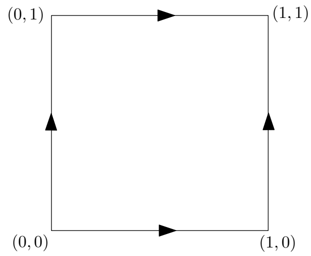


Figure 4.2: Parametrization of a Torus

and let f_2 be a linear combination of eigenvectors of eigenvalue less than $\epsilon^2 L^2$ $(0 < \epsilon < 1)$, i.e. we are taking pairs (k, l) such that

$$4\pi^2 \left(k^2 + l^2\right) \leq \epsilon^2 L^2,$$

We can compute the product of f_1 and f_2:

$$f_1(x, y) f_2(x, y) = \sum_{n,m,k,l} c_{n,m} d_{k,l} e^{i2\pi((n+k)y + x(m+l))}.$$

Thus, we have eigenvalues

$$V^2 := 4\pi^2 \left((n+k)^2 + (m+l)^2\right).$$

We will estimate V by computing $(a+b)^2$ and using the fact that

$$\begin{cases} n, m \leq \frac{L}{2\pi} \\ k, l \leq \frac{\epsilon L}{2\pi} \\ \sqrt{1+x} \leq 1 + x/2, \quad \forall x \geq 0 \end{cases}.$$

Then we get that $V^2 \leq L^2(1+\epsilon^2+4\epsilon) \leq L^2(1+5\epsilon)$. Hence, $V \leq L\sqrt{1+5\epsilon} \leq L(1+5/2\epsilon)$. Therefore, a Torus is admissible.

Another example is the Klein bottle (see Figure 4.3). We take the parametrization of the Klein Bottle as $[-1/2, 1/2] \times [-1/2, 1/2]$, identifying the points $(x, y) \sim (x, y+1)$

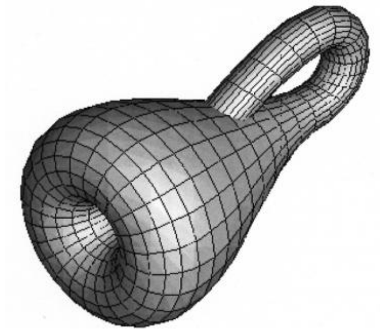


Figure 4.3: Klein Bottle

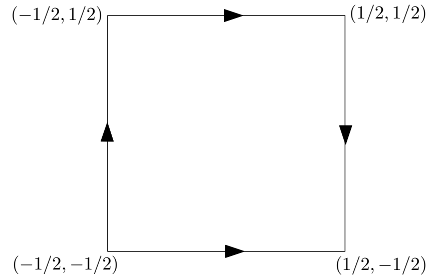


Figure 4.4: Parametrization of the Klein Bottle

and $(x+1,y) \sim (x,-y)$ (as shown in Figure 4.4). We recover the Klein Bottle by a $2 \to 1$ covering, getting a torus defined in $[-1,1] \times [-1/2,1/2]$ with the identification $(x+2,y) \sim (x,y)$ and $(x,y+1) \sim (x,y)$ as shown in Figure 4.5. Thus, we can make use of the results we have in a torus. Each eigenfunction of the Klein Bottle generates an eigenfunction on the torus. We know that the eigenfunctions of the torus are precisely $e^{i2\pi(mx+ny)}$. Now we just need to look which of these functions can reproduce a function in the Klein Bottle and these will be all the eigenfunctions we can have. Since we have identified the points $(x,y+1) \sim (x,y)$ and $(x+2,y) \sim (x,y)$, the only eigenfunctions of the Laplacian in the torus are

$$e^{i2\pi(\frac{m}{2}x+ny)}, \quad \forall m,n \in \mathbb{N}.$$

with eigenvalues $-4\pi^2(n^2+m^2/4)$. We can compute which of these functions make sense in the Klein Bottle, i.e. if

$$g(x,y) = \sum_{m,n} c_{n,m} e^{i2\pi(ny+mx/2)},$$

then imposing that $g(x,y+1) = g(x,y)$ and $g(x+1,y) = g(x,-y)$ we get that

$$\sum_{n,m} (-1)^m c_{n,m} e^{i2\pi(ny+mx/2)} = \sum_{n,m} c_{-n,m} e^{2\pi i(-ny+m/2x)}.$$

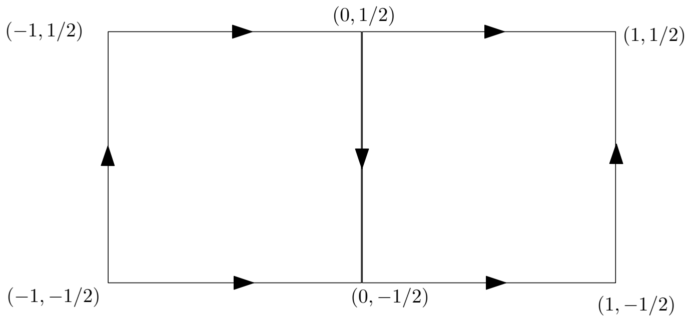


Figure 4.5: Recovering of the Klein Bottle

Solving this, we get that the coefficients $c_{n,m}$ must satisfy $(-1)^m c_{n,m} = c_{-n,m}$. Thus, the functions in the Klein Bottle are of the form

$$f(x,y) = \sum_{n,m} c_{n,m} e^{i2\pi(ny+m/2x)},$$

where

$$\begin{cases} c_{n,2k} = c_{-n,2k}, \\ c_{n,2k+1} = -c_{n,2k+1} \end{cases}.$$

Now we are ready to prove the product property for the Klein Bottle. Let f_1 be a linear combination of eigenvectors of eigenvalues less than L^2, i.e. we are taking pairs (n,m) such that

$$4\pi^2 \left(n^2 + \frac{m^2}{4} \right) \leq L^2,$$

and let f_2 be a linear combination of eigenvectors of eigenvalue less than $\epsilon^2 L^2$ $(0 < \epsilon < 1)$, i.e. we are taking pairs (k,l) such that

$$4\pi^2 \left(k^2 + \frac{l^2}{4} \right) \leq \epsilon^2 L^2,$$

We can compute the product of f_1 and f_2:

$$f_1(x,y) f_2(x,y) = \sum_{n,m,k,l} c_{n,m} d_{k,l} e^{i2\pi((n+k)y + x\frac{m+l}{2})}.$$

Thus, we have eigenvalues $V^2 := 4\pi^2 \left((n+k)^2 + \frac{(m+l)^2}{4} \right)$. Proceeding as in the case of the Torus, one can compute that $V \leq L(1 + 5/2\epsilon)$. Hence, the Klein bottle is admissible.

Product of admissible manifolds.

More examples can be constructed by taking products of manifolds that satisfy the product assumption because if f_1 and f_2 are functions defined on two manifolds M and N, respectively, then

$$\Delta_{M \times N}(f_1 \cdot f_2) = f_2 \Delta_M f_1 + f_1 \Delta_N(f_2).$$

More precisely, let M and N be admissible manifolds, i.e.

$$E_L^M \cdot E_{\epsilon L}^M \subset E_{L(1+C_1\epsilon)}^M,$$

and

$$E_{mL}^N \cdot E_{\epsilon L}^N \subset E_{L(1+C_2\epsilon)}^N,$$

where

$$E_L^M = \langle \{\phi_i; \quad \Delta_M \phi_i = -\lambda_i^2 \phi_i, \lambda_i \leq L\} \rangle,$$

and

$$E_L^N = \langle \{\psi_i; \quad \Delta_N \psi_i = -\mu_i^2 \psi_i, \mu_i \leq L\} \rangle.$$

Thus, if we consider the product manifold $M \times N$, then

$$E_L^{M\times N} = \langle \{\phi_i \psi_j; \quad \lambda_i^2 + \mu_j^2 \leq L^2\} \rangle,$$

Let $f \in E_L^{M\times N}$ and $g \in E_{\epsilon L}^{M\times N}$. Then, the product can be expressed as

$$f \cdot g = \sum_{\substack{\lambda_i^2+\mu_j^2 \leq L^2 \\ \lambda_k^2+\mu_l^2 \leq \epsilon^2 L^2}} c_{ij} d_{kl} \phi_i \psi_j \phi_k \psi_l.$$

We need to show that $\phi_i\psi_j\phi_k\psi_l \in E_{L(1+C\epsilon)}^{M\times N}$, whenever $\lambda_i^2 + \mu_j^2 \leq L^2$ and $\lambda_k^2 + \mu_l^2 \leq \epsilon^2 L^2$. Note that since $\lambda_i \in [0, L]$ and $\lambda_k \in [0, \epsilon L]$, we can put $\lambda_i = r_i L$ and $\lambda_k = s_k \epsilon L$ for some $r_i, s_k \in [0, 1]$. Thus, once we fix the indices i and k, then $\mu_j \leq L\sqrt{1-r_i^2}$ and $\mu_l \leq L\epsilon\sqrt{1-s_k^2}$. Now we compute the product $\phi_i\psi_j\phi_k\psi_l$.

$$\phi_i\phi_k \in E_{r_i L}^M \cdot E_{s_k \epsilon L}^M \subset E_{L(r_i + C_1 \epsilon s_k)}^M,$$

$$\psi_j\psi_l \in E_{L\sqrt{1-r_i^2}}^N \cdot E_{L\epsilon\sqrt{1-s_k^2}}^N \subset E_{L(\sqrt{1-r_i^2}+C_2\epsilon\sqrt{1-s_k^2})}^N.$$

Therefore, we know that

$$\phi_i\psi_j\phi_k\psi_l = \sum_{n,t} c_n d_t \phi_n \psi_t,$$

where the sum runs for all indices n, t with

$$\lambda_n \leq L(r_i + C_1 \epsilon s_k),$$

$$\mu_t \leq L(\sqrt{1-r_i^2} + C_2 \epsilon \sqrt{1-s_k^2}).$$

Now, the functions $\phi_n\psi_t$ are eigenvectors of $\Delta_{M\times N}$ with eigenvalues $-(\lambda_n^2 + \mu_t^2)$. Thus, in order to prove that $M \times N$ is admissible, we need to show that the following estimate holds for some constant $C > 0$:

$$\lambda_n^2 + \mu_t^2 \leq L^2(1 + C\epsilon)^2.$$

Indeed,

$$\frac{\lambda_n^2+\mu_t^2}{L^2} \le [r_i + C_1\epsilon s_k]^2 + \left[\sqrt{1-r_i^2} + C_2\epsilon\sqrt{1-s_k^2}\right]^2 = 1 + \epsilon^2\left[C_1^2 s_k^2 + C_2^2(1-s_k^2)\right]$$
$$+ 2\epsilon\left[C_1 r_i s_k + C_2\sqrt{1-r_i^2}\sqrt{1-s_k^2}\right] =: 1 + \epsilon^2 I_1 + 2\epsilon I_2.$$

Let $C := 2\max(C_1, C_2)$. Using the fact that $r_i, s_k \in [0,1]$, we obtain the following.

$$I_2 \le C_1 + C_2 \le C.$$

$$I_1 \le C_1^2 + C_2^2 \le (C_1 + C_2)^2 \le C^2.$$

Hence,

$$\begin{aligned}\lambda_n^2 + \mu_t^2 &\le L^2\left[1 + C^2\epsilon^2 + 2\epsilon C\right]\\ &= L^2(1+C\epsilon)^2.\end{aligned}$$

Thus, $\phi_i\psi_j\phi_k\psi_l \in E^{M\times N}_{L(1+C\epsilon)}$ with $C = 2\max(C_1, C_2)$ and we conclude that the product manifold $M \times N$ is admissible.

Remark 4.1. Note that the example of the torus can be reduced to this later case because it is the product of two $\mathbb{S}^1$.

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Notation