Workshop I
The $\mathbb{R}^n$ Vector Space. Linear Combinations

Exercise 1. Given the vectors $\vec{u} = (1, 0, 3), \vec{v} = (-2, 1, 2)$, and $\vec{w} = (1, -2, -4)$ compute

a) $\vec{u} + (\vec{v} + \vec{w})$

b) $2\vec{u} - 3\vec{v}$

c) $2\vec{v} - \vec{u} + \vec{w}$

Exercise 2. Draw and describe geometrically (line, plane. . .) all linear combinations of

a) $\{(1, 3)\}$

b) $\{(1, -1), (-3, 3)\}$

c) $\{(0, 1), (1, 0)\}$

Exercise 3. Draw and describe geometrically (line, plane. . .) all linear combinations of

a) $\{(-1, -1, -1), (-4, -4, -4)\}$

b) $\{(2, 0, 0), (1, 2, 2)\}$

c) $\{(2, 2, 2), (1, 0, 2), (3, 2, 3)\}$

d) $\{(2, -1, 3), (1, 4, 1), (5, 2, 7)\}$

Exercise 4. Is the vector $\vec{u} = (2, 1, 4)$ a linear combination of $\{\vec{u}_1 = (2, 2, 1), \vec{u}_2 = (5, 3, 2)\}$?

Exercise 5. Is the vector $\vec{u} = (1, -1, 3)$ a linear combination of $\{\vec{u}_1 = (2, -1, 0), \vec{u}_2 = (3, 1, 1), \vec{u}_3 = (0, -1, 1)\}$?

Exercise 6. Is the vector $\vec{u} = (4, 2, 3)$ a linear combination of $\{\vec{u}_1 = (1, 1, 1), \vec{u}_2 = (3, 2, -1), \vec{u}_3 = (4, 3, 0), \vec{u}_4 = (7, 5, -1)\}$?

Exercise 7. For what values of the parameter k is the vector $\vec{u} = (k, 2, 1)$ a linear combination of $\{\vec{v} = (5, 2, 0), \vec{w} = (3, 0, 1)\}$?

Exercise 8. For what values of the parameter k is the vector $\vec{u} = (4, 2, -5)$ a linear combination of $\{\vec{u}_1 = (1, 0, 1), \vec{u}_2 = (0, 1, k), \vec{u}_3 = (k, 2, k)\}$?

Exercise 9. Let $\vec{u}, \vec{v}$, and $\vec{w}$ be three vectors of $\mathbb{R}^3$. Suppose that the determinant of the matrix build from the three vectors is different from 0. Is $\vec{w}$ a linear combination of $\{\vec{u}, \vec{v}\}$?

Exercise 10. Find the equation of the plane that contains $(2, -1, 3), (1, 4, 1)$, and passes through the origin. Check if the point $(5, 2, 7)$ is contained in the plane. Is the vector $(5, 2, 7)$ a linear combination of $\{(2, -1, 3), (1, 4, 1)\}$?

Workshop 2.
Linear Dependence

Exercise 1. Let $\vec{u}_1 = (1,0,0)$, $\vec{u}_2 = (0,1,0)$, $\vec{u}_3 = (0,0,1)$, and $\vec{u}_4 = (-2,3,1)$. Show that the three vectors are linearly dependent

a) By solving the system $A\vec{\lambda} = \vec{0}$ and finding a solution $\vec{\lambda} \neq \vec{0}$.

b) By showing that one of the vectors is a linear combination of the rest.

Exercise 2. Explain why any set containing the $\vec{0}$ vector is linearly dependent.

Exercise 3. Study the linear dependence/independence of the following sets:

a) $\{(1,-2,3),(3,1,2),(2,-3,1)\}$

b) $\{(-2,-3,3),(3,4,1),(1,2,-7)\}$

c) $\{(5,6,2),(2,3,5),(3,2,-1)\}$

When the set is linearly dependent, find a linear combination of the first two vectors that gives rise to the third vector.

Exercise 4. Let $\vec{u} = (0,2,3)$ and $\vec{v} = (-1,-4,1)$. Find a vector $\vec{w}$ such that $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent

Exercise 5. Find the largest number of linearly independent column vectors of:

$$A = \begin{pmatrix} -2 & 1 & 2 & 3 \\ 0 & 0 & 6 & -1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 & 0 & -6 & 3 \\ 2 & 0 & 4 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 7 \end{pmatrix}$$

Exercise 6. Show that if $a = 0$, $d = 0$, or $f = 0$, the column vectors of the matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

are linearly dependent.

Exercise 7. Let $\vec{u} = (1,3,3)$, $\vec{v} = (1,0,1)$, and $\vec{w} = (6,k,-k)$. For what value of k is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Exercise 8. Let $\vec{u} = (2,2,1)$, $\vec{v} = (k,1,2)$, and $\vec{w} = (-3,k,1)$. For what value of k is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Exercise 9. Study for which value of k is the following set of vectors linearly independent

$$\{\vec{u}_1 = (3,5,1), \vec{u}_2 = (k,4,7), \vec{u}_3 = (2,-k,0), \vec{u}_4 = (k,k,3)\}$$

Exercise 10. Let $\vec{u}_1$ and $\vec{u}_2$ be two linearly independent vectors. Reason out if the following statements are true or false.

a) The vectors $\vec{w}_1$ and $\vec{w}_2$ given by $\vec{w}_1 = \vec{u}_1 + \vec{u}_2$ and $\vec{w}_2 = \vec{u}_1 - \vec{u}_2$ are always linearly independent.

b) The vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ given by $\vec{w}_1 = \vec{u}_1$, $\vec{w}_2 = \vec{u}_1 + \vec{u}_2$, and $\vec{w}_3 = \vec{u}_1 - \vec{u}_2$ are always linearly independent.

Workshop 3.
Linear Span, Basis, and Dimension

Exercise 1. Let $\vec{u}_1 = \left(3, \frac{1}{2}\right)$, $\vec{u}_2 = \left(\frac{9}{2}, \frac{3}{4}\right)$, and $\vec{u}_3 = \left(-\frac{3}{2}, 2\right)$.

a) Is the linear span of $\{\vec{u}_1\}$ a line in $\mathbb{R}^2$?

b) Is $\{\vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

c) Is $\{\vec{u}_1, \vec{u}_2\}$ a spanning set of $\mathbb{R}^2$?

d) Is $\{\vec{u}_1, \vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

e) Is $\{\vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

f) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

g) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

Exercise 2. Let $\vec{u}_1 = (3, 2, -1)$, $\vec{u}_2 = (0, 2, 2)$, $\vec{u}_3 = (3, 0, -3)$, and $\vec{u}_4 = (0, 3, 3)$.

a) Is the linear span of $\{\vec{u}_1, \vec{u}_2\}$ a plane in $\mathbb{R}^3$?

b) Is the linear span of $\{\vec{u}_2, \vec{u}_4\}$ a plane in $\mathbb{R}^3$?

c) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

d) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ a spanning set of $\mathbb{R}^3$? If not, what is the linear span of the set of vectors?

Exercise 3. Is $\{(2, 3, 1), (5, 4, 2), (5, 0, 1)\}$ a spanning set of $\mathbb{R}^3$?

Exercise 4. How many vectors does a spanning set of $\mathbb{R}^6$ have? In other words, how many vectors do we need to span the whole $\mathbb{R}^6$ vector space?

Exercise 5. Let $\vec{u}_1 = (-2, 3, 2)$, $\vec{u}_2 = (a + 2, 0, 3)$, and $\vec{u}_3 = (5, 3, a)$.

a) For what value of the parameter a is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

b) For what value of the parameter a are the three vectors linearly independent?

Exercise 6. Reason out if the following statement is true or false:
"A set of vectors of $\mathbb{R}^n$ is a basis of the $\mathbb{R}^n$ vector space if every vector of $\mathbb{R}^n$ is a linear combination of the vectors of the set and if no vector of the set is a linear combination of the rest".

Exercise 7. Let $\vec{u}_1 = (a, 0, a)$, $\vec{u}_2 = (1, 1, 1)$, and $\vec{u}_3 = (2, 5, a)$. For what value of a is the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a basis of $\mathbb{R}^3$?

Exercise 8. Let $\vec{u}_1 = (-1, 2, 3)$, $\vec{u}_2 = (0, 2, 2)$, and $\vec{u}_3 = (5, -2, -3)$.

a) Check that the three vectors above form a basis of $\mathbb{R}^3$.

b) Find the coordinates of $\vec{u} = (4, 2, 2)$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Exercise 9. Let $\vec{u}_1 = (0, 1, 4)$, $\vec{u}_2 = (2, 1, 0)$, and $\vec{u}_3 = (7, -1, 2)$.

a) Check that the three vectors above form a basis of $\mathbb{R}^3$.

b) Find the coordinates of $\vec{u} = (-9, 2, 6)$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Exercise 10. Is it possible to have a basis of $\mathbb{R}^3$ composed of four vectors? Is it possible to have a spanning set of $\mathbb{R}^3$ composed of four vectors?

Exercise 11. Explain why the columns of every n-by-n invertible (non-singular) matrix form a basis of $\mathbb{R}^n$.

Exercise 12. Reason out if the following statements are true or false.

a) In the vector space $\mathbb{R}^3$, every set of more than three vectors is linearly dependent.

b) In the vector space $\mathbb{R}^4$, every set of four vectors is a basis of $\mathbb{R}^4$.

c) In the vector space $\mathbb{R}^4$, every set composed of less than four vectors is linearly independent.

d) In the vector space $\mathbb{R}^4$, every set of more than four vectors is a spanning set of $\mathbb{R}^4$.

Exercise 13. Show that if $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a spanning set of $\mathbb{R}^n$, then $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{u}_{k+1}\}$ is also a spanning set of $\mathbb{R}^n$.

Exercise 14. Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be a basis of $\mathbb{R}^3$.

a) Show that $\{\vec{w}_1 = \vec{u}_1, \vec{w}_2 = 2\vec{u}_2, \vec{w}_3 = 3\vec{u}_3\}$ is also a basis of $\mathbb{R}^3$.

b) Suppose that the coordinates of a certain vector with respect to $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are $(5, 8, 27)$. Compute the coordinates of that vector with respect to $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

Workshop 4.
Vector Subspaces

Exercise 1. Check that the linear span of

$$\{\vec{u}_1 = (2,3,1), \vec{u}_2 = (1,0,-2)\}$$

is a plane in $\mathbb{R}^3$ and find its analytical expression.

Exercise 2. Describe the vector subspace spanned by $\vec{u} = \left(\frac{1}{2}, 1\right)$ and find its analytic expression.

Exercise 3. Describe the subspace of $\mathbb{R}^3$ spanned by $\vec{u} = (3,-1,2)$ and find its analytic expression.

Exercise 4. Given the following vectors of $\mathbb{R}^2$

$$\vec{u}_1 = (-1,2), \quad \vec{u}_2 = (0,5), \quad \vec{u}_3 = (2,3), \quad \vec{u}_4 = (4,-6)$$

Find a basis and the dimension of the vector subspace $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\})$.

Exercise 5. Find the analytical expression of the vector subspace spanned by the vectors

$$\vec{u}_1 = (-1,0,3), \quad \vec{u}_2 = (2,2,-1).$$

Exercise 6. Find a basis of the vector subspace of $\mathbb{R}^3$ that consists of all vectors of $\mathbb{R}^3$ whose coordinates coincide.

Exercise 7. Find a basis of the vector subspace of $\mathbb{R}^3$ that consists of all vectors of $\mathbb{R}^3$ whose coordinates add up to 1.

Exercise 8. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : x + y = 0, z = 0\}$$

Exercise 9. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : 2x + y = 0, x - z = 0\}$$

Exercise 10. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : 4x - 2y - z = 0\}$$

Exercise 11. Given the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = 0\}$$

a) Show that S is a vector subspace of $\mathbb{R}^3$.

b) Find a basis of S and compute $dim(S)$.

Exercise 12. Check if the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$$

is a vector subspace of $\mathbb{R}^3$.

Exercise 13. Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : \frac{x}{y} = 0\}$$

is not a vector subspace of $\mathbb{R}^2$.

Exercise 14. Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$$

is a vector subspace of $\mathbb{R}^2$.

Workshop 5.
The Euclidean Space

Exercise 1. Find the values of the parameter k so that the inner product of the vectors $\vec{u} = (-1, k, 2)$ and $\vec{v} = (k, 1, k)$ equals 2.

Exercise 2. Study whether the vectors $\vec{u}$ and $\vec{v}$ are orthogonal or not in the following cases:

a) $\vec{u} = (2, 3)$ and $\vec{v} = (3, -2)$.

b) $\vec{u} = (1, 2, 3)$ and $\vec{v} = (-4, -2, 3)$.

c) $\vec{u} = (2, -1, 1)$ and $\vec{v} = (3, 2, -2)$.

d) $\vec{u} = (2, -1, 1, 3)$ and $\vec{v} = (0, 0, 0, 0)$.

Exercise 3. Find the values of the parameter k for which the vectors $\vec{u} = (-1, k, 1)$ and $\vec{v} = (k, k, -6)$ are orthogonal?

Exercise 4. Given $\vec{u} = (1, -2, 3)$ and $\vec{v} = (4, 0, 1)$, compute

a) $\|\vec{u}\|$ and $\|\vec{v}\|$

b) $\|2\vec{u}\|$

c) $\|\vec{u} - \vec{v}\|$

d) $\|2\vec{u} + \vec{v}\|$

Exercise 5. Given $\vec{u} = (1, -2, 3)$ and $\vec{v} = (4, 0, 1)$, check that the following inequalities hold

a) $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$

b) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Exercise 6. Normalize the vectors $\vec{u} = (2, 1, -2)$ and $\vec{v} = (4, -4, -4, 4)$.

Exercise 7. Check if the following set of vectors is an orthogonal basis of $\mathbb{R}^3$:

$$\{(1, 1, 0), (1, -1, 2), (-2, 2, 2)\}.$$

Exercise 8. Check if the following set of vectors is an orthonormal basis of $\mathbb{R}^3$:

$$\left\{ (0, 0, -1), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0 \right) \right\}.$$

Exercise 9. For what value of the parameter k is the norm of the vector $(-4, k, 0)$ equal to 5?

Exercise 10. Find the angle between the vectors $\vec{u} = (1, 0, 1)$ and $\vec{v} = (4, 3, 0)$.

Exercise 11. Find the angle between the vectors $\vec{u} = (2, 0, 2)$ and $\vec{v} = (3, 0, 3)$.

Exercise 12. Find the distance between the vectors $\vec{u} = (3, 1, -2)$ and $\vec{v} = (0, 1, 2)$.

Exercise 13. Find the value of k for which the distance between the following two vectors is 13:

$$\vec{u} = (k, 2, k) \quad \text{and} \quad \vec{v} = (1, 2, -k).$$

Exercise 14. Find the value of k for which the distance between the following two vectors is 25:

$$\vec{u} = (-4, 14, k) \quad \text{and} \quad \vec{v} = (k, -10, k).$$

Workshop 6.
Domain and level curves

Exercise 1. Represent graphically the following real functions of one variable:

a) $f(x) = 3x - 2$

b) $f(x) = -x^2 + 6x - 7$

c) $f(x) = \dfrac{2}{x}$

d) $f(x) = \sqrt{x}$

Exercise 2. Compute the derivatives of the following real functions of one variable.

a) $f(x) = x^4 - 2x^3 + \dfrac{1}{2}x^2 - 4x + 5$

b) $f(x) = (3x - 4)^2$

c) $f(x) = e^x$

d) $f(x) = \ln(x^2 + 1)$

Exercise 3. Given $f(x) = \sqrt{x^2 + 1}$ and $g(x) = e^x$, compute:

a) $f \circ g$

b) $g \circ f$

Exercise 4. Compute the domain of f in the following cases:

a) $f(x) = \dfrac{2x - 1}{x^2 - 1}$

b) $f(x) = \sqrt{-x^2 - 3x + 4}$

c) $f(x) = \ln\left((x - 2)^2 - 4\right)$

d) $f(x) = \dfrac{x}{x^2 - 9}$

Exercise 5. For each of the domains computed in Exercise 4, check if it is a closed, open, bounded, compact, or convex set.

Exercise 6. Study the continuity and differentiability of the following functions:

a) $f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ x + 2 & \text{if } x > 0 \end{cases}$

b) $g(x) = \begin{cases} x^2 + k & \text{if } x \leq 0 \\ kx^2 & \text{if } x > 0 \end{cases}$

The first six Exercises are about real functions of one variable. This topic is a previous requirement that all students are supposed to know.

Exercise 7. Compute and draw the domain of the real function:

$$f(x,y)=\sqrt{1-xy},$$

discuss the properties of the domain (open, closed, bounded, compact, convex).

Exercise 8. Draw the set $A\subseteq\mathbb{R}^2$ in the following cases:

a) $A=\{(x,y)\in\mathbb{R}^2:(x-2)^2+(y+1)^2\leq 1\}$

b) $A=\left\{(x,y)\in\mathbb{R}^2:|y|\geq\dfrac{1}{x}\right\}$

c) $A=\left\{(x,y)\in\mathbb{R}^2:|y|\geq\dfrac{1}{x}\text{ and }x^2+y^2\leq 4\right\}$

d) $A=\left\{(x,y)\in\mathbb{R}^2:|y|\geq\dfrac{1}{x}\text{ or }x^2+y^2\leq 4\right\}$

Exercise 9. For each of the domains computed in Exercise 8, check if it is a closed, open, bounded, compact, or convex set.

Exercise 10. Compute and describe graphically the domain of the following real functions of two variables. For each of the computed domains discuss the properties of the sets.

a) $f(x,y)=\dfrac{3x-1}{x^2-y-1}$

b) $f(x,y)=\sqrt{6x^2+3y-9}$

c) $f(x,y)=\sqrt{x^2+y^2-9}$

d) $f(x,y)=\ln(x+y)$

e) $f(x,y)=\dfrac{\sqrt{xy}}{x}$

Exercise 11. Compute and describe graphically the level curves of the following real functions of two variables.

a) $f(x,y)=\dfrac{x}{y}$

b) $f(x,y)=2x^2+4y^2$

c) $f(x,y)=2x^2+2y^2$

d) $f(x,y)=4xy$

e) $f(x,y)=x^2+y+1$

f) $f(x,y)=\dfrac{\sqrt{x^2-y}}{3x}$

Workshop 7.
Partial and directional derivatives. Gradient vector. Hessian matrix

Exercise 1. Compute all the partial derivatives of the following real functions.

a) $f(x,y) = 5x^3y^2 - 2x$

b) $f(x,y) = \cos\left(x^3 + y^4\right)$

c) $f(x,y) = \dfrac{x^2y - xy^2}{x+y}$

d) $f(x,y) = xe^{x+y^2}$

e) $f(x,y,z) = \sqrt{xe^{x+y} + yz^3}$

f) $f(x,y,z) = \ln\left(\dfrac{xy-6}{x^2-z}\right)$

Exercise 2. Compute the gradient of the real function f and evaluate it at the given point in the following cases:

a) $f(x,y) = \sin(x-y) + \cos(x+y)$ at point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

b) $f(x,y) = x^3e^{x-y} + xy^3$ at point $(2,2)$.

c) $f(x,y) = \ln\left(x^2 + y^4\right)$ at point $(1,3)$.

d) $f(x,y,z) = \dfrac{x+y+z^3}{x^3+y^2+z}$ at point $(1,0,1)$

e) $f(x,y,z) = \sqrt{2yz - xy^2}$ at point $(1,1,1)$.

f) $f(x,y,z,t) = 4x^3 + yz^2 - 6t^5$ at point $(2,3,1,4)$.

Exercise 3. Compute the directional derivative of the real function $f(x,y) = 3x^2y - y$ along the vector $(1,0)$ at point $(2,3)$.

Exercise 4. Compute the directional derivative of the real function $f(x,y,z) = \ln\left(\dfrac{x^2-2yz}{x+y+z}\right)$ along the vector $(1,1,1)$ at point $(1,1,1)$.

Exercise 5. Compute the Hessian matrix of the real functions studied in Exercise 2 and evaluate them at the given point.

Workshop 8.
The tangent plane. Marginality. Elasticity.

Exercise 1. Let $f(x, y, z) = 5x^4 + 3xy^3 - 6yz$ and $p = (0, 1, 2)$.

a) Find the direction of the greatest rate of increase of f at point p.

b) Compute the value of the maximal directional derivative of f at p.

c) Evaluate the Hessian matrix of f at p. Use the Schwartz theorem to ease the computation of second-order derivatives.

Exercise 2. Compute the equation of the tangent plane of f at p in the following cases:

a) $f(x, y) = \dfrac{e^{x^2}}{x + y}$, $p = (4, -3)$

b) $f(x, y) = \dfrac{(y - x^2)(y - 2x^2)}{xy}$, $p = (1, 1)$

Exercise 3. Given $f(x, y) = 3(x + 2)^2(y - 3)$,

a) Compute the equation of the tangent plane of f at point $(6, 6)$.

b) Using the tangent plane, approximate the output of the function at $(7, 5)$. Compute the error of the approximation.

c) Approximate the output at $(8, 4)$ using the tangent plane and compute the error incurred.

Exercise 4. Given a real function of two variables, $f(x, y)$, such that

$$f(100, 100) = 5, \qquad \frac{\partial f}{\partial x}(100, 100) = 2, \quad \text{and} \quad \frac{\partial f}{\partial y}(100, 100) = 4,$$

compute the approximate value of $f(101, 100)$ using the concept of marginality.

Exercise 5. A given company produces three different products, A, B, and C. The overall daily benefit of the company is described by the following function (in €)

$$B(x, y, z) = e^{\frac{x}{100} + \frac{y^2}{1000} + \frac{z^2}{10000}} - 1,$$

where x, y, and z are the quantities of products A, B, and C produced daily.

Nowadays, 100 items of each of the three products are produced daily. Study how would the benefit of the company vary if the production of good C is incremented in one unit.

Exercise 6. Compute the partial elasticity of $f(x, y) = Ax^5y^3$, where A is a fixed parameter, with respect to y.

Exercise 7. The benefits of a given company (in €) are determined by the following real function:

$$B(x, y) = xy^2 - \frac{x^2 y}{4},$$

where x and y are the produced items of two goods. The current production of the two articles is 100 and 50 items, respectively.

a) Estimate the effect (on the benefits) of producing one more item of the second good using the marginal analysis.

b) Compute and interpret the partial elasticity of B with respect to y.

Exercise 8. Suppose that the production cost (in €) of producing x units of good A and y units of good B is given by the following real function

$$C(x, y) = \ln(x + 1) + \ln(y + 1) + 2x + 3y.$$

Compute and interpret the elasticity of the production cost with respect to good A at the production level $(x, y) = (100, 150)$.

Workshop 9.
Composite, implicit, and homogeneous functions

Exercise 1. Given the Cobb-Douglas production function $Q(K,L) = K^{\frac{1}{2}}L^{\frac{1}{2}}$, where Q is the production, K is the capital, and L is the working time. If the capital and working time are functions of the time, $K = K(t) = 100e^{\frac{-t}{2}}$ and $L = L(t) = 50t^{\frac{1}{2}}$, compute dQ/dt and evaluate it at $t = 4$.

Exercise 2. Compute $\partial z/\partial t$ and $\partial z/\partial s$ where $z = \cos(x + y^2) - \sin(x^2 - y)$, $x = \ln(s - t)$, and $y = \ln(s + t)$.

Exercise 3. Let f be a real function of three variables $f(x, y, z)$, where $x = \ln(u+v)$, $y = u-v$, and $z = v - u$. Compute $\dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v}$.

Exercise 4. The equation $(x+y)^2 + \ln(2x-y-2) = 0$ defines y as a function of x ($y = y(x)$) in a neighborhood of the point $(1, -1)$.

a) Find the value of dy/dx at this point.

b) Compute the equation of the tangent line of $y(x)$ at $x = 1$.

Exercise 5. The equation $x^3 + x^2y - 2y^2 - 10y = 0$ defines y as a function of x in a neighborhood of the point $(2, 1)$. Find the equation of the tangent line of the curve $y(x)$ at that point.

Exercise 6. The equation $xyz + x^2\ln(z) + y - 2 = 0$ defines z as a function of x and y in a neighborhood of the point $(1, 1, 1)$. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$ at this point.

Exercise 7. Decide whether the real function f is homogeneous or not and find its degree of homogeneity in case it is homogeneous in the following cases:

a) $f(x, y) = \sqrt{xy}$

b) $f(x, y) = \sqrt{x^2 + y^2}$

c) $f(x, y, z) = x^4 - xy^2 + z^2$

d) $f(x, y, z) = x^4 - xy^2z + y^3z$

e) $f(x, y, z) = xe^{\frac{x^2+y^2}{z^4}}$

f) $f(x, y, z) = \dfrac{x^3y + xy^2z}{x(y - z)}$

Exercise 8. Decide whether the real functions of Exercise 7 are homogeneous or not using Euler's Theorem.

Exercise 9. Let f be a real valued function of two variables where $Dom(f) = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ and such that

$$\frac{\partial f}{\partial x} = \frac{-y}{x} \cdot \frac{\partial f}{\partial y}$$

Reason out if f is homogeneous or not.

Workshop 10.
Quadratic forms

Exercise 1. Find the symmetric matrix associated to the quadratic form

$$q(x,y,z) = 2x^2 + 5y^2 + 4xz + 9yz.$$

Exercise 2. Find the expression of the quadratic form associated to the matrix

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 5 & 3/2 \\ 4 & 3/2 & 0 \end{pmatrix}$$

Exercise 3. Classify the quadratic form associated to the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Exercise 4. Classify the quadratic form associated to the matrix

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

Exercise 5. Classify, depending on the value of the parameter $a \in \mathbb{R}$, the quadratic form

$$q(x,y) = 3x^2 + 2y^2 + axy.$$

Exercise 6. Classify, depending on the value of the parameter $a \in \mathbb{R}$, the quadratic form

$$q(x,y,z) = -x^2 - 4y^2 - z^2 + xy + ayz.$$

Exercise 7. A company produces two goods, A and B. When x units of A and y units of B are produced, the benefits of the company (in €) are described by the following function

$$B(x,y) = 3x^2 + 5y^2 - 16xy$$

a) Check that the company can incur in losses?

b) If the production of the second good is four times the production of the first good, will the company incur in losses?

Workshop 11.
Optimization I

Exercise 1. Given the real function of two variables $f(x,y) = \dfrac{x^2+9}{4-y}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } -6 \leq y \leq 6\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : y \leq 2 \text{ or } y \geq 8\}$

Exercise 2. Given the real function of two variables $f(x,y) = e^{x^3-y^2}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : x^2 - 9 \leq y\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 1, \text{ and } x + y \leq 6\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } -2 \leq y \leq 2\}$

Exercise 3. Reason out if the real function $f(x,y) = \sqrt{(x-1)^2 + (y-1)^2 - 9}$ has global extreme points in its domain using the Extreme Value Theorem.

Exercise 4. Given the real function $f(x,y) = \dfrac{\sin(x+y)}{x-a}$ where $a \in \mathbb{R}$. Study the applicability of the Extreme Value Theorem in the set $A = \{(x,y) \in \mathbb{R}^2 : -6 \leq x \leq 6 \text{ and } -6 \leq y \leq 6\}$ with respect to the values of the parameter a.

Exercise 5. For each of the real functions below, find the stationary points and classify[1] them whenever possible.

a) $f(x,y) = 6x^2 - 3xy^2 + 12xy - 18$

b) $f(x,y) = 2x^4 + y^2$

c) $f(x,y,z) = x^2 - 8x + 6z + y^2 - 4y + 20$

d) $f(x,y) = x^2 - y^2 + xy + 2x + 2y - 2$

e) $f(x,y,z) = x^2 - y^2 + xy + 2x - 2y - z^2 - 2$

[1] Local maximum, local minimum, or saddle point

Workshop 12.
Optimization II

Exercise 1. Study the convexity/concavity of $f(x,y) = \dfrac{-x^3}{6} - \dfrac{y^3}{6} + xy + 9x + 12y - 2$ in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y < 0\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } y > 1\}$

d) $D = \{(x,y) \in \mathbb{R}^2 : x < -1 \text{ and } y < -1\}$

Exercise 2. Study the global extreme points of the real function f in $\mathbb{R}^2$:

a) $f(x,y) = 9x^2 + \sqrt{2}y^2$

b) $f(x,y) = x^4 + 6y^2 - 4x$

c) $f(x,y) = x^2 + y^2 + xy - 8x - 8y$

d) $f(x,y) = x^2 + y^2 - xy - 8x - 8y$

Exercise 3. The inverse demand function of a commodity is described by $p = 650 - q$, where p stands for the price and q for the quantity of product demanded. Suppose that there is a single seller in the market and that the cost function of this monopolist is $C(q) = 4q^2 + 50q + 200$. Find the production level and the retail price that maximizes the profit of the monopolist.

Exercise 4. A given company produces two different commodities. The first one, A, is sold in the domestic market and the second one, B, is sold in the foreign market. The demand function in the domestic market is described by $p_A = 60 - q_A$, where p_A stands for the price and q_A for the quantity of product A demanded. The demand function in the foreign market is described by $p_B = 52 - 2q_B$, where p_B stands for the price and q_B for the quantity of product B demanded. If the cost function of the company is $C(q_A, q_B) = 6q_A^2 + 4q_B^2 - 10q_A - 8q_B + 2$

a) Find the profit function of the company.

b) Compute the production level (of both commodities) that maximizes the profits of the company and the retail prices.

Exercise 5. Study the global extreme points of the real function

$$f(x,y) = 6x^2 - 3xy^2 + 12xy - 18$$

in the set $A = \{(x,y) \in \mathbb{R}^2 : 2x + (y-2)^2 < 0\}$.