Heterogeneous discounting. Time consistency in investment and insurance models

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To my parents

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Introduction

1.1 Introduction

Economic agents often face decisions that imply comparing costs and benefits occurring at different points in time. Decisions made in the present usually have consequences in the future, in the sense that they can restrict the set of available opportunities, extend it or even modify costs and benefits associated with each alternative. For instance, consider someone who is planning how much is he or she going to save for retirement. The amount of money saved today affects not only the individual's current consumption but also his or her future consumption possibilities. Saving more today implies less present consumption, and hence a decline in the current utility. On the other hand, the saving decision enables the agent to consume more in the future, increasing the future utility. These kind of decisions with multi-time consequences are usually referred to as intertemporal choices. From choosing whether to start a diet today or put it off until next week, whether to borrow money in the capital market, whether to build up a pension fund or whether to put effort in our job to improve our position in the future, intertemporal choices are ubiquitous in everyday life.

In order to trade off costs and benefits occurring at different times, agents must make them comparable by discounting future payments at a reference point. The question of how people actually discount delayed payoffs has called the attention not only of economists, but also of psychologists, sociologists and other social science researchers over the last two centuries, and the theory of intertemporal choice has developed dramatically. This literature is doubtless highlighted by Samuelson (1937) who, relying on several assumptions, achieved a simple and elegant formulation that was rapidly adopted as the natural framework for the analysis of intertemporal decisions: the Discounted Utility model.

A central assumption made by Samuelson was that the individual time preference could be characterized by a single parameter: the rate of time preference. Apart from the simplicity of this approach, the appeal of the Discounted Utility model was also that allowed researchers to analyze intertemporal choices by means of the full range of economic and mathematical tools used in other contexts. However, the simplicity of a given model may have a detrimental effect of the model's accuracy, and Samuelson's manifest reservations about the descriptive validity of his model were justified some years after by experimental evidence suggesting that people often behave in ways that are inconsistent with the Discounted Utility model.

The insights derived from empirical research about intertemporal choice led economists and other researchers to develop new models trying to capture the anomalies reported. For instance, hyperbolic discount functions have been used to model the greater impatience in the short term (Laibson (1997), (1998), Angeletos et al. (2001)); habit formation models formalized the idea that the current tastes can depend on past consumption (Duesenberry (1952), Pollack (1970), Heal (1973)); or models considering the anticipation of future pleasures (or pains) as a source of utility (Loewenstein (1987), Caplin and Leahy (2001)). These examples illustrate the ways in which the theory has developed over the last decades. While some models tries to achieve greater realism by modifying the discount function (hyperbolic discounting), others have focused on the utility function by incorporating new arguments (habit formation models; utility of anticipation).

Despite the different theoretical models proposed, nowadays it does not exist, to the best of our knowledge, a clear alternative to the Discounted Utility model. This may well be due to the wide range of different decisions, situations or problems to which the terms "intertemporal choice" could refer, as well as the variety of factors affecting people willingness to trade off present and future satisfactions. In fact, before the formulation made by Samuelson in 1937, discussions about intertemporal choice interpreted these kind of decisions as the joint product of many psychological motives.

The heterogeneous discounting model, which is the central topic of this dissertation, is encompassed in the group of theoretical models that relaxes assumption of a unique constant discount rate of time preference. First proposed in Marín-Solano and Patxot (2012), this model is adequate to model situations in which, in contrast to the hyperbolic and standard discounting, the bias to the present does not remain constant along time. As we discuss in section 5, there are several problems that seem to be good candidates for this description such as human capital formation or retirement and pension models.

Before explain the heterogeneous discount model in more detail, in Section 2 we briefly review the history of intertemporal choice theory until the formulation of the Discounted Utility model. Starting by the ninetieth century, we will try to point out the psychological foundations of the earliest conceptions of intertemporal choice. This will serve to better understand what is encompassed in the rate of time preference as it is used in the Discounted Utility model, the limitations of such formulation, as well as to motivate the heterogeneous preferences. After presenting the Discounted Utility model, in Section 3 we review its corresponding assumptions and implications, paying special attention to those relaxed by the introduction of heterogeneous discounting. Section 4 is devoted to describe the hyperbolic discount function and its main features. Revising hyperbolic models will be useful in order to stress the differences between them and the heterogeneous discounting model, which we present in Section 5. In Section 6 we explain different solution concepts which are usually considered in the non-constant discounting literature. Finally, the structure of the dissertation as well as our contributions are presented in Section 7.

1.2 The early perspectives of intertemporal choice

The evolution of the history of intertemporal choice highlights different stages until the formulation of the Discounted Utility model in 1937. Beginning in the 1830's, early discussions on the topic considered intertemporal choice as an amalgamation of many psychological motives, and although the incorporation of psychological insights were the overall trend during the ninetieth and early twentieth century, it is possible to discern a progressive deviation of the attention towards a more robust mathematical formulation culminating with the Discounted Utility model.

The Sociological Theory of Capital, published in 1834 by the economist John Rae, is considered the starting point for the economic theory of intertemporal choice. Trying to explain why wealth differed among nations, Rae provided the first in-depth discussion of the psychological factors underlying intertemporal choice. Earlier economists, such as Adam Smith, had argued that national wealth differed among countries due to differences in the allocation of the surplus product between production of capital and consumption. Although Rae acknowledged such accounts, he thought they were incomplete since did not reveal on what did that accumulation depend. From his point of view, such differences depended on the "effective desire of accumulation", or the people willingness to defer gratification. In addition, he identified four major determinants that either promoted or limited this desire. The bequest motive and the self restraint, which is defined as "the extend of intellectual powers, and the consequent prevalence of habits of reflection, and prudence, in the minds of the members of society" (p.58), were the two factors that promoted the desire of accumulation. On the other hand, the uncertainty of human life and the excitement for immediate consumption were the factors that limited the desire. According to Rae, the joint product of these factors determined someones' time preference.

Two years after the publication of Rae's book, N.W. Senior expounded a new theory of capital that, like Rae, emphasized the psychological side of the intertemporal choice. Senior (1936) proposed a psychological explanation for interest arguing that the reason why it should be paid on a capital sum was to compensate the holder of the capital for deferring the gratification of immediate consumption. "To abstain from the enjoyment which is in our power, or to seek distant rather than immediate results, are among the most painful exertions of the human will" (p. 60). Hence, Senior's abstinence perspective considered that individuals dispensed equal treatment to present and future, explaining the impatience, or the overweighting of the present, to the pain associated with putting off consumption.

Jevons (1871) took a similar view of intertemporal choice to that of Senior, in the sense that both authors considered decision makers highly influenced by their immediate emotions. However, in Jevons view, the reason that led the agent to defer consumption was the presently felt pleasures he obtained when contemplating future consumption. Thus, the individual defers consumption only if the increase in the anticipated utility more than compensates for the decrease in the utility derived from immediate gratification.

In 1889, Bohm-Bawerk added a new motive to the list of determinants of time preference mentioned by Rae, Senior, and Jevons. His original contribution was a systematic tendency to underestimate future wants, .i.e., humans beings lack the capacity of making a complete picture of their future wants, specially the more distant ones. Although his voluminous work was mostly devoted to the study of the psychological constituent of time preference, Bohm-Bawerk analysis is also highlighted by being the first to consider the allocation of consumption along time as a trade off of satisfactions at different points. In consequence, Bohm-Bawerk's work occupies a pivotal position in the history of intertemporal choice theory, in the sense that his approach paved the way for the subsequent deviation of the focus towards a mathematical formulation of decision making problems to the detriment of the psychological determinants.

Bohm-Bawerk "technical" view of intertemporal choice was formalized by Fisher in 1930 (Fisher (1930)). Fisher (1930) was the first to apply indifference curve tool to study the intertemporal decisions and to formulate Bohm-Bawerk theory in mathematical terms. By plotting consumption in the current year on the abscissa, and consumption in the following year on the ordinate, the indifference diagram showed different levels of current and future consumption between which the individual is indifferent. The slope of the tangent at points intersecting the 45° line departing form the origin, which represent the agent willingness to give up consumption today in exchange for consumption in the next period, can be viewed as the "pure time preference". Fisher writings also included discussions about the psychological constituents of time preference. To the list considered by Rae, Fisher added "foresight" or the ability to imagine future wants (the opposite of Bohm-Bawerk deficit), and "fashion", which he belived to be "of vast importance to a community, in its influence on the rate of interest and on the distribution of wealth itself" (Fisher (1930), p. 88).

1.3 The Discounted Utility model

The Discounted Utility (DU) model was proposed by Samuelson in 1937 on a five page article entitled "A Note on the Measurement of Utility", and was rapidly adopted as the standard model for analyzing problems with a temporal component. Although his predecessors already had described the basic economic relations in intertemporal choice, none of them had proposed a generalized model including any number of outcomes and periods. Samuelson achieved such a formulation with the DU model. However, in his simplified model the individual time preference were characterized by a single parameter, his discount rate, that compressed all the psychological factors discussed by earlier economists.

The first basic assumption made by Samuelson was to consider that "during a specified period of time, the individual behaves so as to maximize the sum of all future utilities, they being reduced to comparable magnitudes by suitable time discounting" (p. 156). Mathematically, the aim of an individual was to maximize

$$J = \int_0^T V(x,t)\,dt\,,$$

where $V(x,t)$ is the utility of income x at time t, and $[0,T]$ is the period of time considered. The second, and perhaps more decisive assumption, was to consider discounting as independent of consumption. Thus, $V(x,t)$ was considered to be composed by two different components: the utility of income $U(x)$ capturing variations in the utility derived from differences in level of consumption, and a discount factor accounting for the weight assigned at every time in the planing horizon $t \in [0,T]$. In fact, Samuelson went further and he considered that such a discount factor was "known to us" (p. 156) and described by an exponential discount function with a constant discount rate of time preference. In consequence, with this second assumption, the individual considered by Samuelson sought to maximize

$$J = \int_0^T e^{-\delta t} U(x)\, dt\,,$$

where δ represents the agent's pure rate of time preference to which all disparate psychological motives were ascribed.[1]

Hence, by means of a simple and general formulation the model was able to capture the basic economic relations between payments distributed over time. However, Samuelson was very cautious in presenting his model, and along his exposition he tried to stress any arbitrariness of the underlaying assumptions: "It is completely arbitrary to assume that the individual behaves so as to maximize an integral of the form envisaged in (the DU model). This involves the assumption that at every instant of time the individual's satisfaction depends only upon the consumption at that time, and that, furthermore, the individual tries to maximize the sum of instantaneous satisfactions reduced to some comparable base by time discount" (p. 159). Despite his manifest reservations, the DU model eventually became the standard framework for the analysis of decisions with a temporal component, at least until the 80's. For this reason, we briefly analyze the implications of the assumptions inherent to Samuelson's model that have special relevance to motivate the heterogeneous discounting.

- **Independence between discounting and consumption**

  Although the DU model was intended to explain only preferences over money income, its subsequents applications has covered a very wide range of problems, from saving and investment behavior to labor supply or even policy issues. In this sense, the model assumes that the discount function is the same for all kind of goods and all categories of intertemporal decisions,

[1] This is why Samuelson's model is also called exponential discounting or constant discounting model. Along this dissertation, we will make use of these terms without distinction.

and this is in contradiction with several empirical evidence. For example, some studies show that people discount gains more than losses (Loewenstein (1987), Thaler (1981)), or that small amounts are discounted more than large amounts (Thaler (1981)). The literature has usually dealt with this questions by incorporating new arguments in the utility function. To cite an instance, Loewenstein and Prelec (1992) used the so called reference-point dependent utilities, which evaluates outcomes considering departures from a reference point, to account for these kind of anomalies.

However, it is important to remark that the rate of time preference, just as presented by Samuelson, condensed all the disparate factors considered in the earliest discussions about intertemporal choice. Consequently, within the framework of the DU model, the diversity of factors that could affect one's willingness to trade off between current and future satisfaction, such as patience or impatience, imagination of future, anticipation or bequest motive, are supposed to play the same role in any kind of decision, ruling out any sort of heterogeneity in the degree to which people discount different sources of utility.

- **Constant discounting and time consistency**

The assumption of an exponential discount function with a constant rate of time preference implies a neutral attitude toward time delay, i.e., shifting a given outcome in time has the same impact on the individual preferences regardless of when it occurs. This assumption permits to summarize the individual time preference in his single discount rate. In addition, preferences described by such a discount function are time-consistent. It means that what is optimal from today perspective will be still optimal for tomorrow point of view and, consequently, plans made for the future will be carried out.

In fact, although Samuelson proposed exponential discounting as an arbitrary assumption, in 1956 Robert Strotz showed that the constant exponential discounting is the only discount function that guarantees dynamic consistency. Departing from the DU model with a general discount function, Strotz (1956) sought to determine under which circumstances an individual who is allowed to reconsider his plans continuously would confirm his previous choices. His answer was that an agent would stick with her previous plans if, and only if, the logarithmic rate of change in the discount function is a constant, i.e., the discount function must be of the form proposed by

Samuelson. The intuition behind the constant logarithmic rate of change is that the marginal rate of substitution between two future outcomes depends only on the length of time between them, thus "the relative importance of 1957 and 1958 is the same in 1957 as in 1956. Consequently, when in 1956 one decides how to apportion consumption between 1957 and 1958, this is the same decision one would make in 1957" (Strotz (1956), p.172).

Although these kind considerations about the independence between consumption and discounting, as well as about the constant discounting and time consistency, are at the core of the heterogeneous discounting model, there are other assumptions inherent to Samuelson's one. Next, we summarize other features of the DU model as it is commonly used by economists.

- Consumption independence: Consumption independence means that consumption in one period has no effect in consumption in other periods. This means, for example, that someone's preferences about having diner in a Japanese restaurant today is not influenced by the fact of having had a Japanese diner the day before. Besides analytical simplicity, consumption independence does not seen a real assumption. As Koopmans (1960) acknowledged, "one can not claim a high degree of realism for (consumption independence assumption), because there is no clear reason why complementarity of goods could not extend over more than one time period." (p. 292).

- Stationary utility function: The DU model assumes that the instantaneous utility function does not change along time, i.e., that preferences do not change with time. Although this is not a realistic assumption, it is usually maintained for analytical convenience.

- Utility independence: Another underlying assumption of the DU model is that given two outcome streams, its global utility is obtained by adding its discounted values at some reference point, and hence, equal streams have the same overall value. Consequently, this assumption rules out preferences for specific distributions of payments along time.

- Positive time preference: Although is not explicitly assumed in the DU model, it is usual to consider a positive rate of time preference.

To sum up, the formulation made by Samuelson relies on some arbitrary assumptions that provides the model with some important properties. In this sense, dynamic consistency is one of its most appealing features. However, a large body of research has casted serious doubts on whether people really behave as the model predicts or not, and virtually every underlying assumption has been called into question. Of particular interest to this dissertation are the independence between consumption and discounting assumption, and the constant discounting one. In fact, departures from constant discounting have recieved an important part of researchers' efforts to achieve greater descriptive realism and, in this sense, the literature about hyperbolic discounting is vast. The key feature of this models is their capacity to capture qualitative aspects of present biased preferences, a phenomenon widely observed in empirical studies. Nevertheless, like the standard model, hyperbolic discounting does not distinguish between different sources of utility, and this is precisely the assumption relaxed by the heterogeneous discounting model. As a result, as we detail in the next two sections, both approaches give rise to time-inconsistent preferences, although there are several qualitative aspects that set the two models apart.

1.4 Hyperbolic discounting

One of the most challenged assumptions of the DU model is that people discount future payments at a constant rate. Thaler (1981) was the first to check the accuracy of this conjecture. In particular, the hypothesis to be tested was that the discount rate implicit in intertemporal choice vary with the length of time to be waited. By asking university students the amount of money that would be necessary to compensate a delay of a given capital, the results suggested that participants' discount rate did not remain constant but seemed to decline as delay of time increased. The publication of Thaler's work triggered a spate of empirical studies supporting the observation of this phenomenon known as hyperbolic discounting (Benzion et al. (1989), Chapman (1996), Redelmeier and Heller (1993)). In addition, another kind of studies pointed out "preference reversals"(Millar and Navarick (1964), Green et al. (1994b), Kirby and Herrnstein (1995)). Reversion occurs when preferences between two future outcomes switch in favor of the more proximate one when delay is reduced, for instance, one may prefer receive 110 Euros in one year plus one day to receive 100 Euros in one year and, at the same time, she may prefer receive 100 Euros today than 110 Euros tomorrow. In other words, evidence shows that people is often more impatient about choices in the short term compared with those in the long run, and such a dynamically inconsistent behavior is entirely consistent with hyperbolic discounting.

In view of empirical findings, considerable literature has been devoted to the implications of hyperbolic discounting. In fact, yet Strotz (1956) thought that alternatives to standard discounting should be considered, arguing that there is "no reason why an individual should have such special discount function" (p. 172) and, from his point of view, the case of declining discount rates deserved special attention. However, he did not propose an alternative discount function. Phelps and Pollak (1968) introduced the (β, δ) formulation in a intergenerational altruism model

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^t & \text{if } t > 0, \end{cases} \tag{1.1}$$

that has been also called quasi hyperbolic discounting since it captures many of the qualitative aspects of hyperbolic discounting. The (β, δ) formulation has been widely used in the literature. For instance, Laibson (1997) used the formulation in (1.1) to explore the role of illiquid assets (golden eggs) as a commitment mechanism to correct overconsumption. In addition to (1.1), other discount functions have been proposed, for instance the generalized hyperbola $D(t) = (1 + \alpha t)^{\frac{-\gamma}{\alpha}}$, with $\gamma, \alpha > 0$ proposed in Loewenstein and Prelec (1992).

The main characteristic of hyperbolic discount functions is that they decline at faster rate in the short run than in the long term, and hence all induce dynamic inconsistency. The sort of inconsistency arising from hyperbolic discounting is probably one of its more appealing features, since it allows to capture present-biased preferences, whereby people pursue immediate gratification in ways that tend to conflict with their long term well-being. Hyperbolic discount functions have been used to study many situations: addiction and self control (Carrillo (1999)), procrastination, since hyperbolic discounting leads an agent to postpone a tedious activity more than she would like from a previous perspective (O'Donoghue and Rabin (1999a), (2000)), consumption-saving behavior (Angeletos et al. (2001), Laibson (1997)), or retirement planning (O'Donoghue and Rabin (1999b)).

1.5 Heterogeneous discounting

Although hyperbolic discounting relaxes the assumption of using a constant discount rate for all time periods, it does not solve all the DU model anomalies. As discussed above (see also Frederick et al. (2002)) the standard model also assumes that the discount rate is the same for all type of goods and all categories of intertemporal choices, and this is in contradiction with several empirical regularities. Marín-Solano and Patxot (2012) introduced a new approach giving rise (as in the case of hyperbolic discounting) to time-inconsistent preferences: the heterogeneous discounting. In that paper, the authors considered a problem in which the decision maker had to maximize her intertemporal utility over a finite period of time $[0, T]$ considering heterogeneity in the degree to which the individual discounted different sources of utility.

In general, in problems with a bounded planing horizon it is usual to introduce a final utility function accounting for the state of the problem at the final time, i.e., the intertemporal preferences at time $t \in [0, T]$ take the form

$$\int_t^T D(s-t)L(x(s), u(s), s)\, ds + D(T-t)F(x(T), T) \ ,$$

where the discount function $D(\tau - t)$ represents how the agent at time t discounts future utilities enjoyed at some future point $\tau > t$; the function $L(x(s), u(s), s)$ measures the instantaneous utility derived from choosing the control $u(s)$ at time s, when the state is $x(s)$. $L(x(s), u(s), s)$ is usually related to consumption and hence to immediate gratification; and the function $F(x(T), T)$ denotes the terminal value associated with the final state $x(T)$. Depending on the context, $F(x(T), T)$ is called scrap function, terminal value function or bequest function.

Marín-Solano and Patxot (2012) relaxed the assumption of a constant discount rate for all kind of goods by introducing different discount rates for utilities enjoyed along the planning horizon and for the final utility or final function. Hence, the individual sought to maximize

$$\int_t^T e^{-\delta(s-t)}L(x(s), u(s), s)\, ds + e^{-\rho(T-t)}F(x(T), T) \ . \tag{1.2}$$

Impatient agents over-valuing instantaneous utilities $L(x(s), u(s), s)$ in comparison with the final function $F(x(T), T)$ are characterized by $\rho > \delta$. However, when time passes, the final function increases its relative value in comparison with instantaneous utilities in a way that can not be described using the standard or the hyperbolic discount functions. In order to see this effect, consider the case $\rho > \delta$ and rewrite (1.2) as

$$\int_t^T e^{-\delta(s-t)}L(x(s), u(s), s)\, ds + e^{-\delta(T-t)}e^{-(\rho-\delta)(T-t)}F(x(T), T) \ , \tag{1.3}$$

or simply

$$\int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\delta(T-t)} \bar{F}(x(T), t, T) \ , \tag{1.4}$$

with $\bar{F}(x(T), t, T) = e^{-(\rho-\delta)(T-t)} F(x(T), T)$. It is easy to see that

$$\frac{\partial \bar{F}(x(T), t, T)}{\partial t} = (\rho - \delta) e^{-(\rho-\delta)(T-t)} F(x(T), T)$$

is positive or negative depending on the sign of $(\rho - \delta)$. Since for this illustrative example we have assumed $\rho > \delta$, the actual valuation of the final function of the agent, $\bar{F}(x(T), t, T) = e^{-(\rho-\delta)(T-t)} F(x(T), T)$, is an increasing function in t. Hence, as long as the agent approaches to the end of the planning horizon, the current final function increases, i.e., $e^{-(\rho-\delta)(T-s_2)} F(x(T), T) > e^{-(\rho-\delta)(T-s_1)} F(x(T), T)$ for $s_1 < s_2$, $s_i \in (t, T)$.

There are several problems that seem to be good candidates for this description. For example, to motivate their model Marín-Solano and Patxot (2012) focused on a situation in which the agent has to exert herself to consume a particular good, that they labeled as arduous good. In order to stress that effort comes before enjoyment, they considered the extreme case in which the arduous good would be only enjoyed in the final period T, being $F(x(T), T)$ the corresponding utility function. Consequently, this effort affects the corresponding discount rate by increasing it.

Other problems which can be represented by this model include consumption and portfolio rules problems or retirement and pension problems. For instance, consider a decision-maker who is planning on how much to save for her retirement. Typically, individuals are much more concerned with life quality after retirement when retirement age is approaching, in comparison with their concern about their post retirement life when they look at it from a long distance, for instance, when they are young. Alternatively, we could think in an agent solving a consumption-portfolio rules problem where the final function represents a bequest function for her descendants. The individual is much more concerned with life quality of her descendants when she becomes older. Other applications of (1.2) could be cooperative differential games with two (or more) agents, one just concerned about immediate rewards, and the other more conservative and worried by the final state.

1.6 Heterogeneous vs. Hyperbolic discounting

Let us briefly compare the type of time-inconsistency for an impatient agent (say, agent A) with hyperbolic discounting (with a non-increasing discount rate) with the effects of impatience of and agent with heterogeneous discounting with $\rho > \delta$ (agent B). For illustrative purposes, consider the case in which the agents have to decide how much are they going to save for retirement. The saving effort can be viewed as a disutility during the first periods, since the agent does not spend the saved resources in consumption and hence in immediate gratification.

For agent A, the willingness to increase her final year's saving effort in return for a better retirement (and higher subsequent welfare) is higher at the beginning of the planning horizon than at the end of the planning horizon, since she is always more impatient in her short-run decisions than in her long-run decisions. For this reason, this agent would like to commit herself, in the first year, to save harder in the final year, compared to her actual willingness to make the saving effort when the final year arrives. In particular, if this agent is naive (time-inconsistent), when the final year arrives, she actually ends up saving less than she planned in the first year.

Next, we look at the behavior of agent B. For a long time horizon and from the first year perspective, it is natural to assume that the agent can hardly imagine her post-retirement life, so she decides to save an small amount of money. As the prospect of retirement looms, she takes things more seriously and decides in the last year to save harder than she planned at the beginning of her planning horizon. This is the effect that we can capture by using a different instantaneous discount rates for instantaneous utilities and for the final function.

Summarizing, the main difference between agents A and B (or between hyperbolic and heterogeneous discounting) is the time evolution of the bias to the present. An agent taking decisions with hyperbolic preferences has always the same bias to her present, as in the case of standard (exponential) discounting. On the contrary, for agent B (with heterogeneous discounting), there is also a bias to the present, but this bias changes (decreases when $\rho < \delta$) as long as she approaches the end of the planning horizon. If $\rho > \delta$ the agent procrastinates (as in hyperbolic discounting), in the sense of undervaluing the final function, but this procrastination decreases along time. With a similar argument, in case that $\rho < \delta$, the agent will have a decreasing valuation of the final function as long as she reaches the final time T. Recent findings on individuals behavior seems to confirm that the bias to the present is not the same at all ages (Green et al.

(1994a), (1996), (1999)).

1.7 Solution concepts

One of the most relevant effects of using any kind of non standard discounting is that preferences change with time. An agent making decisions at time t has different preferences compared with those at time t' and, consequently, we can consider her at different times as different agents. An agent making a decision at time t is usually called the t-agent. In this sense, dynamic inconsistent preferences forces the decision maker to tussle with her different selves in an interior conflict in which earlier selves wish to force later selves, while later ones do their best to maximize their own interest. Economists have usually modeled the situation as an intrapersonal game among the different t-agents.

In general, a person with time-inconsistent preferences may or may not be aware of her changing preferences. Strotz (1956) and Pollak (1968) discussed two extreme alternatives. On the one hand, an individual could make her decisions considering that her preferences are not going to change in the near future, or as if the future t-agents would act in the interest of the current self. Under this naive belief, the decision maker chooses a sequence of actions maximizing her current preferences, and expecting that future selves will stick to this sequence. However, as time goes on, futures selves conduct their own optimization problem obtaining, in general, a different sequence of actions. As a result, the naive agent ends up by solving a problem at each time and applying the optimal action only when it is obtained. In this sense, the naive solution can be constructed by adapting the standard optimization techniques, such as the Pontryagin's maximum principle or the Hamilton-Jacobi-Bellman (HJB) equation. Specifically, in order to obtain this solution one should solve an optimal control problem for each time t on the planing horizon, and then patch together the "optimal" solutions obtained in each problem.

On the other hand, an individual could be completely sophisticated, and correctly predict how her preferences are going to change. In this case, earlier selves will make decisions taking into account the preferences of the future t-agents. Unfortunately, the sophisticated solutions can not be obtained by means of the standard optimization techniques. In fact, the concept of optimality plays no role here, since what is optimal for the t-agent will not be optimal (in general) for the future t'-agents, $t' > t$. Instead, one should look for the subgame Markov perfect equilibria, which prompts the use of a dynamic programming approach by applying the Bellman optimality principle. To this end, first we need to define what we mean by a Markov equilibrium. A natural approach to the problem consists in considering first the equilibrium of a sequence of planners in discrete time and then passing to the continuous time limit. This is probably the most intuitive and natural approach, and it is in the spirit of the construction of equilibrium concepts in the literature of differential games (see e.g. Friedman (1974)). Following this approach, Karp (2007) defined a Markov perfect equilibrium as the formal continuous time limit (provided that it exists) of a discretized version of the corresponding dynamic game with non constant discounting in a deterministic setting (see Marín-Solano and Navas (2009) for a description of the problem in finite horizon, and free terminal time and Marín-Solano and Navas (2010) for the stochastic case). As a result, the equilibrium rule is obtained as the solution to a dynamic programming equation (DPE) which is a modified HJB equation.

An alternative approach, similar in spirit to the one first suggested in Barro (1999), consists in assuming that the decision maker at time t can precommit his/her future behavior during the period $[t, t+\epsilon]$. In Ekeland and Lazrak (2010) this idea was reformulated by considering that the t-agent is allowed to form a coalition with his/her immediate successors (s-agents, with $s \in [t, t+\epsilon]$), provided that, for $s > t+\epsilon$, the corresponding s-agents choose their equilibrium rule. Then, the equilibrium rule was calculated by taking the limit $\epsilon \to 0$. Marín-Solano and Patxot (2012) used this approach to obtain the Markov perfect equilibrium for the heterogeneous discounting problem in a deterministic environment. It is remarkable that the equilibrium necessary conditions obtained in Karp (2007) and Ekeland and Lazrak (2010) are consistent, although the two approaches are different in nature.

1.8 Structure and contributions of the thesis

This dissertation is organized as follows.

In Chapter 2 we extend the heterogeneous discounting model introduced in Marín-Solano and Patxot (2012) to a stochastic environment. Our main contribution in this chapter is to derive the DPE providing time-consistent solution for both the discrete and continuous time case. For the continuous time problem we derive the DPE following the two different procedures described above: the formal limiting procedure and the variational approach. However, an important limitation of these approaches is that the DPE obtained is a functional equation with a nonlocal term. As a consequence, it becomes very complicated to find solutions, not only analytically, but also numerically. For this reason, we also derive a set of two coupled partial differential equations which allows us to compute (analytically or numerically) the solutions for different economic problems. In particular, we are interested in analyzing how time-inconsistent preferences with heterogeneous discounting modify the classical consumption and portfolio rules (Merton (1971)). The introduction of stochastic terminal time is also discussed.

In Chapter 3, the results of Chapter 2 are extended in several ways. First, we consider that the decision maker is subject to a mortality risk. Within this context, we derive the optimal consumption, investment and life insurance rules for an agent whose concern about both the bequest left to her descendants and her wealth at retirement increases with time. To this end we depart from the model in Pliska and Ye (2007) generalizing the individual time preferences by incorporating heterogeneous discount functions. In addition, following Kraft (2003), we derive the wealth process in terms of the portfolio elasticity with respect to the traded assets. This approach allows us to introduce options in the investment opportunity set as well as to enlarge it by any number of contingent claims while maintaining the analytical tractability of the model. Finally, we analyze how the standard solutions are modified depending on the attitude of the agent towards her changing preferences, showing the differences with some numerical illustrations.

In Chapter 4 we extend the heterogeneous discount framework to the study of differential games with heterogeneous agents, i.e., agents who exhibit different instantaneous utility functions and different (but constant) discount rates of time preference. In fact, although the non-standard models have usually focused on individual agents, the framework has proved to be useful in the study of cooperative solutions for some standard discounting differential games. Our main contribution in this chapter is to provide a set of DPE in discrete and continuous time in order to obtain time-consistent cooperative solutions for N-person differential games with heterogeneous agents. The results are applied to the study of a cake eating problem describing the management of a common property exhaustible natural resource. The extension to a simple common renewable natural resource in infinite horizon is also discussed.

Finally, in Chapter 5, we present a summary of the main results of the thesis.

Chapter 2

Heterogeneous discounting in consumption-investment problems. Time consistent solutions.

2.1 Introduction

In the study of intertemporal choices it is customary in economics to consider the so-called Discounted Utility (DU) Model, introduced in Samuelson (1937). According to the Samuelson's model, time preferences can be characterized by a single parameter, the discount rate. Since the DU model assumes a constant discount rate of time preference, it can be easily shown (due to the properties of the exponential function) that constant discounting implies that agent's time preferences are time-consistent. However, empirical observations seem to show that predictions of the DU model disagree with the actual behavior of decision makers (we refer to Frederick et al (2002) for an analysis on the topic and a review of the literature up to (2002)). These anomalies can be of several types.

The best documented DU anomaly is hyperbolic discounting (or non-constant discounting, in general). Strotz (1956) studied the effects of choosing a variable rate of time preference, illustrating how for a very simple model preferences are time consistent if, and only if, the discount function is an exponential with a constant discount rate. Effects of the so-called quasi-hyperbolic (or quasi-geometric) discount functions introduced by Phelps and Pollak (1968) have been extensively studied in a discrete time context, within the field of behavioral economics. The most relevant effect of non-constant discounting is that preferences change along time. In this sense, an agent making a decision at time t has different time preferences compared with those at the initial time t_0. In a continuous time setting, a dynamic programming equation (DPE) providing a time-consistent solution was introduced in Karp (2007) in a deterministic framework. This DPE was extended to the case where the evolution of the state variables is governed by a set of stochastic differential equations in Ekeland and Pirvu (2008) and Marín-Solano and Navas (2010).

Although hyperbolic discounting relaxes the assumption of using a constant discount rate for all time periods, it does not solve all the anomalies of the DU model. As pointed out in Frederick et al (2002), the DU model assumes also that the discount rate should be the same for all types of goods and all categories of intertemporal decisions, and this is in contradiction with several empirical regularities.

In this chapter we study a simple approach (giving rise, as in the case of hyperbolic discounting, to time-inconsistent preferences) which can provide a model for certain behaviors that can not be explained by the DU model or more general hyperbolic preferences. More precisely, we are interested in preferences representing a situation in which the agent discounts in a different way the utilities enjoyed along the planning horizon and that of the bequest or final function. Hence, the intertemporal utility function takes the form

$$U_t = \int_t^T d(s,t)u(x,c,s)\,ds + d(T,t)F(x(T),T)\ .$$

with $d(s,t) = e^{-\rho(s-t)}$ for $s < T$, and $d(T,t) = e^{-\bar{\rho}(T-t)}$, for $\rho \neq \bar{\rho}$, in general.

Impatient agents over-valuing the utilities $u(x(s), c(s), s)$ in comparison with the final function $F(x(T), T)$ are characterized by $\bar{\rho} > \rho$. However, when time passes, the final function increases its relative value in comparison with the instantaneous utilities $u(x(s), c(s), s)$ (usually due to consumption and hence to an immediate benefit). This asymmetric valuation cannot be described by a standard discount function or in general with non-constant discounting. There are several problems that seem to be good candidates for this description: human capital formation, where the the final function represents the utility obtained after a period of continuous effort; consumption and portfolio rule problems, where the final function represents a bequest function (the individual is more concerned with the welfare of her descendants when life is arriving to the end); or, along the same lines, retirement and pension problems. Since preferences are time-inconsistent, no optimal solutions exist, and the standard techniques in optimal control theory (the Pontryagin's Maximum Principle or the Hamilton-Jacobi-Bellman equation) give rise to time-inconsistent solutions. By reproducing the literature of non-constant discounting, we can say that an agent is naive if she does not take into account that her preferences will change in the future, so she is time-inconsistent. In order to obtain time-consistent solutions (agents are sophisticated, using the standard terminology in non-constant discounting), Markov perfect equilibria must be calculated.

This problem with heterogeneous discounting was introduced in Marín-Solano and Patxot (2012) in a deterministic setting. In that paper, a DPE providing a time-consistent solution was derived by using a variational approach, and an economic motivation was given. Such DPE is rather similar to the one first derived by Karp (2007) for the problem with non-constant discounting. An important limit in the approach introduced in that paper is that the DPE is a functional equation with a nonlocal term. As a consequence, it becomes very complicated to find solutions, not only analytically, but also numerically. In this chapter we extend the results in the deterministic setting to a stochastic environment, by deriving a set of two coupled partial differential equations which are equivalent (in the deterministic setting) to the DPE derived in Marín-Solano and Patxot (2012). This approach allows us to compute (analytically or numerically) the solutions for different economic problems. In particular, we are interested in analyzing how time-inconsistent preferences with heterogeneous discounting modify the classical consumption and portfolio rules (Merton (1971)). We show that, similar to the problem with non-constant discounting, within the HARA (hyperbolic absolute risk aversion) utility functions, if the relative risk aversion is constant (logarithmic and power utility functions), the equilibrium portfolio rule does not depend on the rate of time preference. This nice property is not satisfied for more general utility functions, such as the (constant absolute risk aversion) exponential function. With respect to the consumption rules, for the case of heterogeneous discounting, they are different, not only quantitatively, but mainly qualitatively, to the equilibria derived for the case of non-constant discounting in continuous time in Marín-Solano and Navas (2010). The effects on the consumption rule of introducing heterogeneous discounting are illustrated numerically for the case of power and exponential utility functions. As a final contribution we show that, if the final time is a random variable, our problem with heterogeneous discounting transforms into a problem which is equivalent to a model introduced (in a deterministic setting) in Marín-Solano and Shevkoplyas (2011). In this case, we must search for a time-consistent equilibrium in a cooperative differential game with heterogeneous agents.

The chapter is organized as follows. In Section 2 we introduce the model. In Section 3 we first derive the DPE in a discrete time setting and then, we find the formal continuous time limit. As a result, we recover the DPE in the deterministic setting as a particular case. This provides a justification to the mathematically rigorous but less intuitive procedure used in Marín-Solano and Patxot (2012). Next, we define the notion of equilibrium rule as in Marín-Solano and Patxot (2012) (which is based on the one in Ekeland and Pirvu (2008)), and the DPE is obtained by using a variational approach. In Section 4, this equation is solved for the consumption and portfolio rules problem for some particular utility functions. Section 5 analyzes the problem for the case of random time horizon. Finally, Section 6 contains the main conclusions of the chapter.

2.2 The Model

We introduce the problem in a discrete time and deterministic setting. For each period s, $s = 0, 1, 2, \ldots, T-1$, let $x_s = (x_s^1, \ldots, x_s^n)$ be the vector of state variables and $c_s = (c_s^1, \ldots, c_s^m)$ the vector of control (or decision) variables. If $u_s(x_s, c_s, s)$ is the utility function at period s and $F(x_T, T)$ is the final (or bequest) function, in the conventional model, the intertemporal utility function of an agent taking decisions at period t takes the form

$$U_t = \sum_{s=t}^{T-1} \delta^{s-t} u_s(x_s, c_s, s) + \delta^{T-t} F(x_T, T) \ ,$$

where the state variables evolve according to the state equation

$$x_{s+1} = f(x_s, c_s, s) \ ,$$

for $s = t, \ldots, T-1$. In order to maximize U_t we must solve an optimal control problem and, since the discount factor $\delta \in (0, 1]$ is always the same, the solution becomes time consistent. In general, if we consider an arbitrary discount $d(s, t)$ representing how the agent at time t discounts future utilities enjoyed at time $s \geq t$, the intertemporal utility function at period t is given by

$$U_t = \sum_{s=t}^{T-1} d(s, t) u_s(x_s, c_s, s) + d(T, t) F(x_T, T) \ .$$

In the standard case,

$$d(s,t) = \delta^{s-t} .$$

If time preferences are quasi-hyperbolic,

$$d(s,t) = \beta\delta^{s-t} \quad \text{for} \quad s > t , \quad \text{and} \quad d(t,t) = 1 .$$

In this chapter we are interested in preferences representing a situation in which the agent discounts in a different way the utilities enjoyed along the planning horizon, and the final function. In particular, we assume that the discount rate takes the form

$$d(s,t) = \delta^{s-t} \quad \text{for} \quad s < T , \quad \text{and} \quad d(T,t) = \bar{\delta}^{T-t} .$$

The intertemporal utility function becomes

$$U_t = \sum_{s=t}^{T-1} \delta^{s-t} u_s(x_s, c_s, s) + \bar{\delta}^{T-t} F(x_T, T) .$$

Following Marín-Solano and Patxot (2012), we call these time preferences *heterogeneous discounting*.

Next, we extend the model to a continuous time setting. Let $x = (x^1, \ldots, x^n) \in X \subseteq \mathbf{R}^n$ be the vector of state variables, $c = (c^1, \ldots, c^m) \in U \subseteq \mathbf{R}^m$ the vector of control (or decision) variables, $u(x(s), c(s), s)$ the instantaneous utility function at time s, T the planning horizon (terminal time) and $F(x(T), T)$ the final or bequest function. Then the corresponding intertemporal utility function is

$$U_t = \int_t^T e^{-\rho(s-t)} u(x, c, s)\, ds + e^{-\bar{\rho}(T-t)} F(x(T), T) . \tag{2.1}$$

As we present in the Introduction, impatient agents over-valuing utilities $u(x, c, s)$ in comparison with the final function $F(x(T), T)$ are characterized by $\bar{\rho} > \rho$ (or $\delta > \bar{\delta}$ in the discrete time setting). However, with these time preferences, when time passes, the final function increases its value in comparison with the utilities $u(x, c, s)$. This asymmetric valuation cannot be described by using a standard geometric discounting or, in general, with hyperbolic preferences (with a unique non-constant discount rate). Note that with (non)constant discounting the bias to the present (to *their* present) does not change from the viewpoint of the different t-agents (in the hyperbolic discounting literature, an agent taking decisions at time t is called the t-agent). With heterogeneous discounting, the bias to the present changes along time. We refer to Marín-Solano and Patxot (2012) for a discussion of this effect (in that paper heterogeneous discounting were used as an attempt to describe, e.g., the behavior of an undergraduate student who is planning on how hard to work in each of the years of her program).

Problems which can be represented by this model include consumption and portfolio rule problems or retirement and pension problems. For instance, consider a decision-maker who is planning on how much to save for her retirement. Typically, individuals are much more concerned with life quality after retirement when retirement age is approaching[1], in comparison with their concern about their post retirement life when they look at it from a long distance, for instance, when they are young. This saving effort can be viewed as a disutility during the first periods, since the agent does not spend the saved resources in consumption and hence in immediate gratification. Within this setting, let us briefly compare the type of time-inconsistency for an impatient agent (say, agent A) with hyperbolic discounting (with a non-increasing discount rate) with the effects of impatience of and agent with heterogeneous discounting with $\bar{\rho} > \rho$ (agent B).

For agent A, the willingness to increase her final year's saving effort in return for a better retirement (and higher subsequent welfare) is higher at the beginning of the planning horizon than at the end of the planning horizon, since she is always more impatient in her short-run decisions than in her long-run decisions. For this reason, this agent would like to commit herself, in the first year, to save harder in the final year, compared to her actual willingness to make the saving effort when the final year arrives. In particular, if this agent is naive (time-inconsistent), when the final year arrives, she actually ends up saving less than she planned in the first year.

Next, we look at the behavior of agent B. For a long time horizon and from the first year perspective, it is natural to assume that the agent can hardly imagine her post-retirement life, so she decides to save an small amount of money. As the prospect of retirement looms, she takes things more seriously and decides in the last year to save harder than she planned at the beginning of her planning horizon. This is the effect that we can capture by using a different instantaneous discount rates for instantaneous utilities and for the final function. In order to see this effect, consider the case $\bar{\rho} > \rho$ and rewrite the final function in (2.1) as

$$e^{-\rho(T-t)} e^{-(\bar{\rho}-\rho)(T-t)} F(x(T), T) \, .$$

[1]Alternatively, we could think in an agent solving a consumption-portfolio rules problem where the final function represents a bequest function for her descendants. The individual is much more concerned with life quality of her descendants when she becomes older.

In this way, the actual valuation of the final function of the agent is given by

$$e^{-(\bar{\rho}-\rho)(T-t)}F(x(T),T)\,,$$

which is an increasing function in t. Hence, as long as the agent approaches to the end of the planning horizon, the current final function increases, i.e.,

$$e^{-(\bar{\rho}-\rho)(T-s_2)}F(x(T),T) > e^{-(\bar{\rho}-\rho)(T-s_1)}F(x(T),T)\,.$$

for $s_1 < s_2$, $s_i \in (t,T)$.

Summarizing, the main difference between agents A and B (or between hyperbolic and heterogeneous discounting) is the time evolution of the bias to the present. An agent taking decisions with hyperbolic preferences has always the same bias to her present, as in the case of standard (exponential) discounting. On the contrary, for agent B (with heterogeneous discounting), there is also a bias to the present, but this bias changes (decreases when $\rho < \bar{\rho}$) as long as she approaches the end of the planning horizon. If $\bar{\rho} > \rho$ the agent procrastinates (as in hyperbolic discounting), in the sense of undervaluing the final function, but this procrastination decreases along time. With a similar argument, in case that $\rho > \bar{\rho}$, the agent will have a decreasing valuation of the final function as long as she reaches the final time T.

We finish this section by introducing the problem in a stochastic setting. In the discrete time case, the difference equation is now subject to random disturbances and the state equation becomes

$$X_{t+1} = f(X_t, c_t, t, V_{t+1})\,, \qquad X_0 = x_0\,, \qquad V_0 = v_0\,.$$

We restrict our attention to the case when V_{t+1} is a random variable taking values in a finite set $\mathcal{V}$. Each t-agent will look for maximizing in c_t the expected intertemporal utility function

$$E\left[\sum_{s=t}^{T-1} \delta^{s-t} u_s(X_s, c_s, s|x_t, v_t) + \bar{\delta}^{T-t} F(X_T, T|x_t, v_t)\right] \tag{2.2}$$

subject to

$$X_{s+1} = f(X_s, c_s, s, V_{s+1})\,,\;\; X_s = x_s\,,\;\; V_s = v_s\,,\;\; s = t,\ldots,T-1\,. \tag{2.3}$$

In continuous time, the problem becomes

$$\max E\left[\int_t^T e^{-\rho(s-t)}u(X(s),c(s),s)+e^{-\bar{\rho}(T-t)}F(X(T),T)\mid x_t\right] \tag{2.4}$$

subject to

$$dX(s)=f(X(s),c(s),s)ds+\sigma(X(s),c(s),s)\cdot dW(s),\quad X(t)=x_t \quad \text{given}\ . \tag{2.5}$$

2.3 Dynamic programming equation

The solution provided by the use of standard optimal control techniques is time-inconsistent if the intertemporal utility function takes the form (2.2) or (2.4). In Marín-Solano and Patxot (2012) a DPE for sophisticated (time-consistent) agents in a deterministic framework was derived by following a variational approach. In this section we derive first a Dynamic Programming Equation (DPE) for the stochastic problem in a discrete time setting. Next, we obtain the DPE in continuous time by discretizing first the problem and defining then the DPE as the (formal) continuous time limit. This derivation is similar to that in Karp (2007) and Marín-Solano and Navas (2010) for the case of non-constant discounting in deterministic and stochastic environments, respectively. Finally, we provide an alternative derivation of the DPE by using a variational approach.

2.3.1 Dynamic Programming Equation in discrete time

First, let us assume that the probability that $V_{t+1}=v\in\mathcal{V}$, $P_t(v|v_t)$, may depend on the outcome v_t at time t, as well as explicitly on time t, but it is independent on the state and control variables x_t and c_t. In addition, functions u and f are assumed to be continuous in (x,c). We search for an equilibrium rule $c_t^*=\phi_t(x_t,v_t)$, characterized by the property that no decision-maker in the sequence of decision-makers wants to deviate from it. Let T be finite. The value function for the t-agent is given by

$$W(x_t,t,v_t)=\sup_{\{c_t\}} E\left[\sum_{s=t}^{T-1}\delta^{s-t}u_s(X_s,c_s,s|x_t,v_t)+\bar{\delta}^{T-t}F(X_T,T|x_t,v_t)\right] \tag{2.6}$$

where $c_s = \phi_s(x_s, v_s)$, for $s = t+1, \ldots, n$. The computation of the expectation in (2.6) is based on conditional probabilities of the form

$$p^*(v_{t+1}, \ldots, v_s) = P_t(v_{t+1} | \, v_t) \cdot P_{t+1}(v_{t+2} | \, v_{t+1}) \cdots P_{s-1}(v_s | \, v_{s-1}) \, .$$

We adapt the derivation of the DPE in the classical case $\delta = \bar{\delta}$ (see e.g. Seierstad (2009)) as follows. In the final period T we define

$$W(x_T, T, v_T) = F(x_T, T)$$

as usual. At period $T-1$,

$$W(x_{T-1}, T-1, v_{T-1}) = \sup_{\{c_{T-1}\}} \{E\left[u_{T-1}(x_{T-1}, c_{T-1}, T-1) + \right.$$

$$\left. + \bar{\delta} F(X_T, T) \, | \, x_{T-1}, v_{T-1}\right]\} \, ,$$

where the expectation is calculated over V_T given v_{T-1}. Since $F(X_T, T)$ depends on V_T via

$$X_T = f(x_{T-1}, c_{T-1}, T-1, V_T) \, ,$$

we can write

$$W(x_{T-1}, T-1, v_{T-1}) = u_{T-1}(x_{T-1}, \phi_{T-1}(x_{T-1}, v_{T-1}), T-1) +$$

$$+ \bar{\delta} E\left[F(X_T, T) \, | \, x_{T-1}, v_{T-1}\right] = u_{T-1}(x_{T-1}, \phi_{T-1}(x_{T-1}, v_{T-1}), T-1) + \bar{\delta} L_T^{T-1} \, ,$$

where we define

$$L_T^{T-1} = E\left[F(X_T, T) \, | \, x_{T-1}, v_{T-1}\right] \, .$$

In general, if

$$L_\tau^s = E[\cdots [E[E[u(X_\tau, \phi_\tau(X_\tau, \tau), \tau) \, | \, X_{\tau-1}, V_{\tau-1}] \, | \, X_{\tau-2}, V_{\tau-2}] \cdots] \, | \, x_s, v_s] \, ,$$

it is clear that

$$W(x_t, t, v_t) = \sup_{\{c_t\}} \left\{ u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} L_s^t + \bar{\delta}^{T-t} L_T^t \right\} \, . \qquad (2.7)$$

In a similar way,

$$W(x_{t+1},t+1,v_{t+1}) = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^{t+1} + \bar{\delta}^{T-t-1} L_T^{t+1} \, ,$$

and therefore

$$E\left[W(X_{t+1},t+1,V_{t+1}\,|\,x_t,v_t)\right] = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^{t} + \bar{\delta}^{T-t-1} L_T^{t} \, . \tag{2.8}$$

By solving L_T^t in (2.8) and substituting in (2.7) we obtain the Dynamic Programing Equation, which proceeds backward in time:

$$W(x_T,T,v_T) = F(x_T,T) \ ,$$

$$\bar{\delta}^{T-t-1} W(x_t,t,v_t) = \sup_{\{c_t\}} \Bigg\{ \bar{\delta}^{T-t-1} u_t(x_t,c_t,t) + \sum_{s=t+1}^{T-1} \left[\delta^{s-t}\bar{\delta}^{T-t-1} - \delta^{s-t-1}\bar{\delta}^{T-t}\right] L_s^t + $$
$$+ \bar{\delta}^{T-t} E\left[W(X_{t+1},t+1,V_{t+1}\,|\,x_t,v_t)\right]\Bigg\} \ , \tag{2.9}$$
$$X_{s+1} = f(X_s,c_s,s,V_{s+1}) \, , \ \ X_s = x_s, \ \ V_s = v_s \ .$$

The decision rules solving the right hand term in equation (2.9) are the Markov Perfect Equilibria.

Remark 1 *Note that, if the discount rates coincide, $\delta = \bar{\delta}$, the term in the sum in (2.9) vanishes and we recover the standard Bellman equation.*

We can easily extend our previous results to the case when

$$P_t[V_{t+1} = v] = P_t(v|\, x_t, c_t, v_t)$$

depends, not only on time t and the previous outcome v_t, but also on the state and control variables x_t and c_t. We present the details in the Appendix.

2.3.2 The continuous time case: a formal limiting procedure

Now, let us extend the DPE (2.9) to a continuous time setting, by following a formal limiting procedure as in Karp (2007) and Marín-Solano and Navas (2010). In the continuous time setting, the agent at time t (the t-agent) aims to solve

Problem 2.4-2.5. Let us discretize the problem by following the classical Euler (or Euler-Mayurama) method. If we divide the interval $[0, T]$ into N periods of constant length ϵ, in such a way that we identify $T = N\epsilon$, and $s = j\epsilon$, for $j = 0, 1, \ldots, N$, then Equation (2.5) becomes

$$X(t+1) = X(t) + f(X(t), c(t), t) + \sigma(X(t), c(t), t)(w(t+1) - w(t)),$$

where $w(t)$ is a Wiener process. Denoting $X(j\epsilon) = X_j$ and $c(j\epsilon) = c_j$, for $j = 0, \ldots, N-1$, the objective of the agent in period $t = j\epsilon$ is to maximize

$$E\left[\sum_{s=j}^{N-1} e^{-\rho(s-j)\epsilon} u(X_s, c_s, s) + e^{-\bar{\rho}(N-j)\epsilon} F(X_T, T)\right] \tag{2.10}$$

subject to

$$X_{i+1} = X_i + f(X_i, c_i, i) + \sigma(X_i, c_i, i)(w_{i+1} - w_i) , \tag{2.11}$$

for $i = j, \ldots, T-1$, x_j given. Note that Problem 2.10-2.11 is equivalent to Problem 2.2-2.3.

Remark 2 *For a given decision rule $c(x, s)$, a condition assuring the uniform convergence (in the mean square sense) of the solution of the discretized equation (2.11) to the true solution to (2.5) is that functions f and σ satisfy uniform growth and Lipschitz conditions in x, and are Hölder continuous of order $1/2$ in the second variable.*

Definition 1 *We define the value function $V(x, t)$ for Problem (2.4-2.5) as the solution to the DPE obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the DPE (2.9) obtained for the discrete approximation (2.10-2.11) to the problem, assuming that such a limit exists and that the solution is of class $C^{2,1}$.*

Next, let us derive the DPE for the problem with heterogeneous discounting in the spirit of the previous definition. Let $V(x, t)$ be the value function of the t-agent, with initial condition $x(t) = x_t$. Since $s = j\epsilon$ and

$$X(t+\epsilon) = x(t) + f(x(t), c(t), t)\epsilon + \sigma(x(t), c(t), t)(w(t+\epsilon) - w(t)) ,$$

then $W(x_j, j\epsilon, v_j) = V(x_t, t)$ and

$$V(x_{t+\epsilon}, t+\epsilon) = V(x_t, t) +$$

$$+\nabla_{x_t} V(x_t,t) f(x_t,c(t),t)\epsilon + \nabla_{x_t} V(x_t,t)\sigma(x,c(t),t)\cdot(w_{t+\epsilon}-w_t)+$$

$$+\nabla_t V(x_t,t)\epsilon + \frac{1}{2} tr\left(\sigma(x_t,c(t),t)\cdot\sigma'(x_t,c(t),t)\cdot\nabla_{x_t x_t} V(x_t,t)\right)\epsilon + o(\epsilon)$$

where $\lim_{\epsilon\to 0}\frac{o(\epsilon)}{\epsilon}=0$. In addition,

$$e^{-\bar{\rho}(n-j)\epsilon} = e^{-\bar{\rho}(n-j-1)\epsilon}\left[1-\bar{\rho}\epsilon + o(\epsilon)\right],$$

and

$$e^{-\rho k\epsilon} = e^{-\rho(k-1)\epsilon}\left[1-\rho\epsilon + o(\epsilon)\right].$$

By substituting in (2.9) we obtain

$$V(x_t,t) = \sup_{\{c_t\}}\{u(x_t,c_t,t)\epsilon + \sum_{k=j+1}^{n-1}\left[e^{-\rho(k-j-1)\epsilon}(\bar{\rho}-\rho)\epsilon\right] L_k^j\,\epsilon + V(x_t,t)+$$

$$+\nabla_{x_t} V(x_t,t) f(x_t,c_t,t)\epsilon + E\left[\nabla_{x_t} V(x_t,t)\sigma(x_t,c_t,t)\cdot(w_{t+\epsilon}-w_t)\right]+$$

$$\nabla_t V(x_t,t)\epsilon + \frac{1}{2} tr\left(\sigma(x_t,c_t,t)\cdot\sigma'(x_t,c_t,t)\cdot\nabla_{x_t x_t} V(x_t,t)\right)\epsilon -$$

$$-\rho\epsilon V(x_t,t) - \rho\epsilon E\left[\nabla_{x_t} V(x_t,t)\sigma(x_t,v,t)(w_{t+\epsilon}-w_t)\right] + o(\epsilon)\}.$$

Therefore,

$$0 = \sup_{\{c_t\}}\{u(x_t,c_t,t)\epsilon + \sum_{k=j+1}^{n-1}\left[e^{-\rho(k-j-1)\epsilon}(\bar{\rho}-\rho)\epsilon\right] L_k^j\,\epsilon +$$

$$+\nabla_{x_t} V(x_t,t) f(x_t,c_t,t)\epsilon + \nabla_t V(x_t,t)\epsilon - \tag{2.12}$$

$$\left. -\rho\epsilon V(x_t,t) + \frac{1}{2} tr\left(\sigma(x_t,c_t,t)\cdot\sigma'(x_t,c_t,t)\cdot\nabla_{x_t x_t} V(x_t,t)\right)\epsilon + o(\epsilon)\right\}.$$

Dividing equation (2.12) by ϵ and taking the limit $\epsilon\to 0$ we obtain:

Proposition 1 *Let $V(x,t)$ be a function of class $C^{2,1}$ in (x,t) satisfying the DPE*

$$\bar{\rho}V(x,t) - \nabla_t V(x,t) - K(x,t) = \tag{2.13}$$

$$= \sup_{\{c\}}\left\{u(x,c,t) + \nabla_x V(x,t) f(x,c,t) + \frac{1}{2} tr\left(\sigma(x,c,t)\cdot\sigma'(x,c,t)\cdot\nabla_{xx} V(x,t)\right)\right\},$$

with

$$V(x,T) = F(x,T), \tag{2.14}$$

and

$$K(x,t) = (\bar{\rho} - \rho)E\left[\int_t^T e^{-\rho(s-t)} u(X_s, \phi(X_s, s), s)\, ds\right] . \tag{2.15}$$

Then $V(x,t)$ is the value function for Problem 2.4-2.5. If, for each pair (x,t), there exists a decision rule $c^* = \phi(x,t)$, with corresponding state trajectory $X^*(t)$, such that c^* maximizes the right hand side term of (2.13), then $c^* = \phi(x,t)$ is called a Markov equilibrium rule for the problem with heterogeneous discounting.

Remark 3 *Again, if $\rho = \bar{\rho}$, the term $K(x,t)$ vanishes and we recover the standard Hamilton-Jacobi-Bellman equation.*

In the proof of the previous proposition the pass to the limit is "formal" and needs to be mathematically justified. With respect to the classical DPE, in Fleming and Soner (2006) the convergence of finite difference approximations to Hamilton-Jacobi-Bellman equations is discussed. We refer also to Kushner and Dupuis (2001) for a study of the convergence of numerical methods to the value function in the standard case.

Finally, note that we can write

$$K(x,t) = (\bar{\rho} - \rho)E\left[\int_t^T e^{-\rho(s-t)} u(X(s), \phi(X(s), s), s)\, ds\right] \tag{2.16}$$

and, by differentiating K in (2.16) with respect to t we obtain the "auxiliary dynamic programming equation"

$$\rho K(x,t) - \nabla_t K(x,t) = (\bar{\rho} - \rho)u(x, \phi(x,t), t) + \nabla_x K(x,t) \cdot f(x, \phi(x,t), t) +$$
$$+\frac{1}{2} tr\left(\sigma(x, \phi(x,t), t) \cdot \sigma'(x, \phi(x,t), t) \cdot \nabla_{xx} K(x,t)\right) . \tag{2.17}$$

Hence we have:

Corollary 1 *Let $V(x,t)$ and $K(x,t)$ be two functions of class $C^{2,1}$ in (x,t) such that $V(x,t)$, $K(x,t)$ and the strategy $c^* = \phi(x,t)$ satisfy the set of two DPEs (2.13) and (2.17) with boundary conditions $V(x,T) = F(x,T)$, $K(x,T) = 0$. Then $V(x,t)$ is the value function for Problem (2.4-2.5), and the strategy $c^* = \phi(x,t)$ maximizing the right hand side term of Equation (2.13) is a Markov equilibrium rule for the problem with heterogeneous discounting.*

2.3.3 Dynamic programming equation in continuous time: a variational approach

Next we provide an alternative derivation of the DPE (2.13-2.15), by using a variational approach similar to that introduced, for the case of non-constant discounting, in Ekeland and Pirvu (2008). In particular, we extend to a stochastic setting the derivation of a DPE in the deterministic problem with heterogeneous discounting first derived in Marín-Solano and Patxot (2012). To do that we assume that decision rules are progressively measurable processes such that the stochastic differential equation (2.5) admits a unique strong solution (see e.g. Theorem 6.3 in Yong and Zhou (1999) for conditions for the existence of strong solutions). For the problem analyzed in Section 4, described by a linear SDE, the existence of strong unique solutions is guaranteed.

Equilibrium policies are defined as follows. If $c^*(s) = \phi(X(s), s)$ is the equilibrium rule, for $\epsilon > 0$ let us consider the variations

$$c_\epsilon(s) = \begin{cases} v(s) & \text{if} \quad s \in [t, t+\epsilon] \ , \\ \phi(X, s) & \text{if} \quad s > t+\epsilon \ . \end{cases}$$

If the t-agent can precommit her behavior during the period $[t, t+\epsilon]$, the value function for the perturbed control path c_ϵ is given by

$$V_\epsilon(x,t) = \max_{\{v(s),\, s\in[t,t+\epsilon]\}} E\left[\int_t^{t+\epsilon} e^{-\rho(s-t)} u(X(s), v(s), s)\, ds + \right.$$

$$\left. + \int_{t+\epsilon}^{T} e^{-\rho(s-t)} u(X(s), \phi(X(s), s), s)\, ds + e^{-\bar{\rho}(T-t)} F(X(T), T)\right] \ .$$

Definition 2 *Let $V_\epsilon(x,t)$ be differentiable in ϵ in a neighbourhood of $\epsilon = 0$. Then $c^*(s) = \phi(x(s), s)$ is called an equilibrium rule if*

$$\lim_{\epsilon\to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} \geq 0 \ .$$

The definition above can be interpreted as follows. For ϵ sufficiently small, from the continuity of V_ϵ with respect to ϵ, the maximum of V_ϵ in the limit when $\epsilon = 0$ is $V(x,t)$.

Proposition 2 *If the value function is of class $C^{2,1}$, then the solution $c = \phi(X,t)$ to the right hand term of the DPE (2.13-2.15) is an equilibrium rule, in the sense that it satisfies Definition 2.*

Proof: See the Appendix.

Remark 4 *In Marín-Solano and Shevkoplyas (2011) a DPE characterizing time-consistent solutions was derived for the general problem of mazimizing*

$$\int_t^T d(s,t)u(x(s),c(s),s)\,ds + d(T,t)F(x(T),T)$$

in a deterministic setting, where $d(s,t)$ *is an arbitrary discount function. For this problem, the following DPE for time-consistent equilibria was obtained:*

$$\frac{\partial d(T,t)}{\partial t}V(x,t) + \int_t^T \left[d(T,t)\frac{\partial d(s,t)}{\partial t} - d(s,t)\frac{\partial d(T,t)}{\partial t} \right] u(x(s),\sigma(x(s),s),s)ds -$$

$$-d(T,t)\frac{\partial V(x,t)}{\partial t} = d(T,t)\max_{\{c\}} \left[u(x,c,t) + \frac{\partial V(x,t)}{\partial x} \cdot f(x,c,t) \right] .$$

If we extend the proof in Marín-Solano and Shevkoplyas (2011) to the stochastic case, we have just to add the expectation operator in the integral term in the equation above, and the standard second order term $\frac{1}{2}tr(\sigma(x,c,t)\cdot\sigma'(x,c,t)\cdot\nabla_{xx}V(x,t))$ *in the right hand term.*

2.4 An investment-consumption model with heterogeneous discounting

In this section, we apply the results in the previous section in order to analyze which are the effects of introducing different discount rates for utilities obtained, in an investment-consumption problem, from consumption enjoyed along time and from bequest. We obtain the equilibrium consumption and portfolio rules for this modified version of the classical Merton's model (Merton (1971)).

The financial market consists of 2 securities. One of them is risk-free (a cash account, for instance), and the price $P_0(t)$ of one unit is assumed to evolve according to the ordinary differential equation

$$\frac{dP_0(t)}{P_0(t)} = \mu_0\,dt\,,$$

where $P_0(0) = p_0 > 0$ and $\mu_0 > 0$ accounts for the return on the sure asset. There is also a risky security whose price $P_1(t)$ evolves according to

$$\frac{dP_1(t)}{P_1(t)} = \mu_1\, dt + \sigma\, dz\,,$$

where $P_1(0) = p_1 > 0$, μ_1 is the expected percentage change in price per unit time and $z(t)$ is a standard Brownian motion process. The agent can invest a proportion $w(t)$ of her wealth at time t, $W(t)$, in the risky asset and a proportion $(1 - w(t))$ in the risk free asset. In addition the agent can allocate an amount of $c(t)$ to consumption. The consumer's wealth process evolves according to

$$dW(t) = [w(t)(\mu_1 - \mu_0)W(t) + (\mu_0 W(t) - c(t))]\, dt + w(t)\sigma W(t) dz(t)\ , \quad (2.18)$$

with $W(0) = W_0$. The objective of the agent at time t is to choose the consumption and investment strategies, $c(s)$, $w(s)$, $s \in [t, T]$, in order to maximize

$$E\left[\int_t^T e^{-\rho(s-t)} u(c(s)) ds + e^{-\bar{\rho}(T-t)} F(W(T))\right] \quad (2.19)$$

subject to (2.18), given $W(t) = W_t$. Both the utility function $u(\cdot)$ and the bequest function $F(\cdot)$ are assumed to be strictly concave functions on their arguments[2].

If the agent can commit herself to follow in the future the "optimal" solution obtained from the viewpoint of her preferences at time $t = 0$, she will solve the classical Hamilton-Jacobi-Bellman (HJB) equation

$$\rho V^0 - \frac{\partial V^0}{\partial s} =$$

$$= \max_{\{c,w\}} \left\{ u(c) + [w(\mu_1 - \mu_0)W + (\mu_0 W - c)] \frac{\partial V^0}{\partial W} + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2 V^0}{\partial W^2} \right\}\ , \quad (2.20)$$

where $V^0(W, s)$ denotes the current value function. The "optimal" controls are the solution to

$$u'(c(s)) = \frac{\partial V^0}{\partial W}\ , \quad w(s) = -\frac{(\mu_1 - \mu_0)}{\sigma^2} \left[\frac{\frac{\partial V^0}{\partial W}}{W \frac{\partial^2 V^0}{\partial W^2}} \right]\ . \quad (2.21)$$

Both the HJB equation (2.20) and the decision rules (2.21) do not depend explicitly on the new discount rate $\bar{\rho}$. The difference with the standard problem with a unique discount rate appears via the final condition. Note that we can write

[2]The extension to the problem with an arbitrary number of risky assets is straightforward.

the bequest function as

$$e^{-\bar{\rho}T}F(W(T),T) = e^{-\rho T}e^{-(\bar{\rho}-\rho)T}F(W(T),T)\,.$$

Hence, in the current value formulation, the terminal condition to be imposed in (2.20) is now

$$V^0(W,T) = e^{-(\bar{\rho}-\rho)T}F(W)\,. \tag{2.22}$$

If $\rho = \bar{\rho}$ we recover the classical solution, which is time consistent. Otherwise, if the agent can not precommit her future actions, she will be time-inconsistent. Note that, if $V^t(W,s)$, $s \in [t,T]$, denotes the current value function at time t according to the time-preferences of the t-agent, she will look for the solution to the classical HJB equation

$$\rho V^t - \frac{\partial V^t}{\partial s} =$$
$$\max_{\{c,w\}}\left\{u(c) + [w(\mu_1-\mu_0)W + (\mu_0 W - c)]\frac{\partial V^t}{\partial W} + \frac{1}{2}w^2\sigma^2W^2\frac{\partial^2 V^t}{\partial W^2}\right\} \tag{2.23}$$

with the boundary condition

$$V^t(W,T) = e^{-(\bar{\rho}-\rho)(T-t)}F(W)\,. \tag{2.24}$$

At different initial times $t \in [0,T]$ the agent has to solve the same HJB equation (2.23) but she applies a different terminal condition (2.24). In general, if the agent does not commit her decision rule at any time t, and does not take into account that her time preferences will change in the future, she will be continuously modifying her choices. This kind of extremely time-inconsistent behavior is usually referred to as the naive behavior or the naive solution in the non-constant discounting literature. In order to obtain time consistent solutions we must solve the DPE (2.13-2.15). We will do it for the family of CRRA (power and logarithmic) and CARA (exponential) utility functions.

2.4.1 Power utility function

Let us study the problem for the case of power utilities

$$u(c) = \frac{c^\gamma}{\gamma}\,, \qquad\qquad F(W(T)) = \frac{W(T)^\gamma}{\gamma},$$

with $\gamma < 1$, $\gamma \neq 0$.

First we briefly derive the time-inconsistent (naive) solution. The "optimal solution" according to the time preferences of the t-agent can be obtained by solving the HJB equation (2.23) with the boundary condition (2.24). It is easy to prove that, in this case, the value function is given by

$$V^t(W,s) = \alpha^t(s)\frac{W(s)^\gamma}{\gamma}\,,$$

where

$$\alpha^t(s) = \left[\frac{1-\gamma}{\varsigma^t} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)(T-t)} - \frac{1-\gamma}{\varsigma^t}\right)e^{\frac{\varsigma^t}{\gamma-1}(T-s)}\right]^{1-\gamma}$$

with

$$\varsigma^t = \rho - \mu_0\gamma + \frac{1}{2}\frac{\gamma(\mu_1-\mu_0)^2}{\sigma^2(\gamma-1)}\,.$$

The corresponding consumption and investment rules are

$$c^t(s) = (\alpha^t(s))^{\frac{1}{\gamma-1}}W\,, \qquad w^t(s) = \frac{-(\mu_1-\mu_0)}{\sigma^2(\gamma-1)}\,.$$

In particular, if the agent can precommit her decision rule at time $t=0$, we obtain the precommitment solution, characterized by

$$\alpha^P(s) = \left[\frac{1-\gamma}{\varsigma} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)T} - \frac{1-\gamma}{\varsigma}\right)e^{\frac{\varsigma}{\gamma-1}(T-s)}\right]^{1-\gamma}. \tag{2.25}$$

Otherwise, if the agent is naive, since the naive t-agent follows her decision rule just at time $s=t$, her actual consumption rule can be obtained by taking $s=t$, so

$$\alpha^N(t) = \left[\frac{1-\gamma}{\varsigma^N} + \left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)(T-t)} - \frac{1-\gamma}{\varsigma^N}\right)e^{\frac{\varsigma^N}{\gamma-1}(T-t)}\right]^{1-\gamma}.$$

In order to obtain a time-consistent solution, according to Proposition 1, Markov equilibria can be obtained by solving the DPE

$$\bar{\rho}V^S(W,t) - K(W,t) - V^S_t(W,t) = \max_{\{c,w\}}\{u(c)+$$

$$+\left[w(\mu_1-\mu_0)W + (\mu_0 W - c)\right]V^S_W(W,t) + \frac{1}{2}w^2\sigma^2W^2V^S_{WW}(W,t)\}\,, \tag{2.26}$$

with $K(W,t)$ given by

$$K(W,t) = E\left[\int_t^T e^{-\rho(s-t)}(\bar{\rho}-\rho)u(\phi(W,s))ds\right]\,, \tag{2.27}$$

where $c^* = \phi(W,s)$ is the equilibrium consumption rule obtained by solving the right hand term in (2.26). In particular, if we apply Corollary 1, we obtain the set of two coupled partial differential equations

$$\bar{\rho}V^S(W,t) - K(W,t) - V_t^S(W,t) = \max_{\{c,w\}}\left\{\frac{c^\gamma}{\gamma} + \right.$$

$$\left. + \left[w(\mu_1-\mu_0)W + (\mu_0 W - c)\right]V_W^S(W,t) + \frac{1}{2}w^2\sigma^2 W^2 V_{WW}^S(W,t)\right\}, \qquad (2.28)$$

$$\rho K(W,t) - K_t(W,t) = (\bar{\rho}-\rho)\frac{c^{*\gamma}}{\gamma} +$$

$$+ \left[w(\mu_1-\mu_0)W + (\mu_0 W - c^*)\right]K_W(W,t) + \frac{1}{2}w^2\sigma^2 W^2 K_{WW}(W,t). \qquad (2.29)$$

As a candidate to the value function and to the function $K(W,t)$ we guess

$$V^S(W,t) = \alpha^S(t)\frac{W(t)^\gamma}{\gamma}, \qquad K(W,t) = A(t)\frac{W(t)^\gamma}{\gamma}.$$

From the maximization problem in (2.28) we easily obtain

$$c^* = (\alpha(t)^S)^{\frac{1}{\gamma-1}}W, \qquad w^* = \frac{-(\mu_1-\mu_0)}{\sigma^2(\gamma-1)}.$$

Then by substituting in (2.28-2.29) and collecting terms in $W(t)^\gamma$, we obtain that functions $A(t)$ and $\alpha^S(t)$ are the solution to the following system of ordinary differential equations:

$$\rho\frac{1}{\gamma}A(t) - \frac{1}{\gamma}\dot{A}(t) =$$

$$= (\bar{\rho}-\rho)\frac{1}{\gamma}(\alpha^S(t))^{\frac{\gamma}{\gamma-1}} - A(t)\frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2(\gamma-1)} + A(t)\mu_0 - (\alpha^S(t))^{\frac{1}{\gamma-1}}A(t), \qquad (2.30)$$

$$\bar{\rho}\frac{1}{\gamma}\alpha^S(t) - \frac{1}{\gamma}\dot{\alpha}^S(t) - \frac{1}{\gamma}A(t) =$$

$$= \frac{1}{\gamma}(\alpha^S(t))^{\frac{\gamma}{\gamma-1}} - \alpha^S(t)\frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2(\gamma-1)} + \alpha^S(t)\mu_0 - (\alpha^S(t))^{\frac{\gamma}{\gamma-1}}. \qquad (2.31)$$

Table 1 summarizes the results obtained for the power utility for the different behaviors of the agent: precommitment, naive or time-consistent. The results for the particular case in the limit $\gamma = 0$ (logarithmic utility) are presented in Table 2.

It is interesting to observe that, in the case of logarithmic utility functions

Consumption rule	Portfolio rule
$c^P(t) = \frac{W_t}{\frac{1-\gamma}{\varsigma}+\left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)T}-\frac{1-\gamma}{\varsigma}\right)e^{\frac{\varsigma}{\gamma-1}(T-s)}}$	$w^P = \frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$
$c^N(t) = \frac{W_t}{\frac{1-\gamma}{\varsigma^N}+\left(e^{\frac{1}{\gamma-1}(\bar{\rho}-\rho)(T-t)}-\frac{1-\gamma}{\varsigma^N}\right)e^{\frac{\varsigma^N}{\gamma-1}(T-t)}}$	$w^N = \frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$
$c^S(t) = \left(\alpha(t)^S\right)^{\frac{1}{1-\gamma}} W_t$, $\alpha^S(t)$ given by (2.30-2.31)	$w^S = \frac{\mu_1-\mu_0}{(1-\gamma)\sigma^2}$

Table 2.1: Power utility function.

Consumption rule	Portfolio rule
$c^P(t) = \frac{W_t}{e^{-\bar{\rho}T+\rho t}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^P = \frac{\mu_1-\mu_0}{\sigma^2}$
$c^N(t) = \frac{W_t}{e^{-\bar{\rho}(T-t)}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^N = \frac{\mu_1-\mu_0}{\sigma^2}$
$c^S(t) = \frac{W_t}{e^{-\bar{\rho}(T-t)}+\frac{1}{\rho}\left[1-e^{-\rho(T-t)}\right]}$	$w^S = \frac{\mu_1-\mu_0}{\sigma^2}$

Table 2.2: Logarithmic utility function.

$u(c) = \ln(c)$ and $F(W(T)) = \ln(W(T))$, the naive solution is time-consistent, since it verifies the corresponding DPE. This result is similar to that described in Marín-Solano and Navas (2010) for the case of non-constant discounting (or hyperbolic preferences).

Next we illustrate numerically the above results. In all the figures we consider the following values for the main parameters: $T = 30$, $\gamma = -3$, $W_0 = 1000$, $\mu_0 = 0.03$, $\mu_1 = 0.09$ and $\sigma = 0.3$.

In Figure 1 we compare the consumption rules for the precommitment, naive and time-consistent (sophisticated) solutions. The discount rates are $\rho = 0.03$ (for instantaneous utilities) and $\bar{\rho} = 0.12$ (for the bequest function). Note that, for t small, the three solutions are quite similar. However, when time t approaches to the final time $T = 30$, the three solutions become different. The naive and time-consistent solutions indicate how the time preferences evolve along time, in comparison with the precommitment solution, which does not take into account the changing preferences.

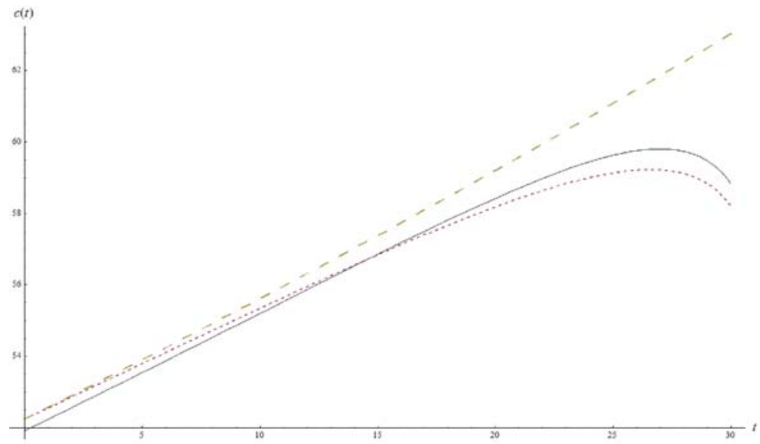

Figure 2.1: Consumption rules for precommitment (Dashed large), naive (Dashed small) and time-consistent (black)

Table 3 represents the values of consumption for several values of time t. The precommitment and naive solutions coincide just at the initial time and, later on, consumption increases faster in the precommitment solution than in the other solution. Time-consistent agents consume less at the beginning and, at the middle of the time horizon, they begin to consume more than naive agents.

t	$c^P(t)$	$c^N(t)$	$c^S(t)$
0	52.2425	52.2425	51.9128
1	52.57	52.5523	52.2354
2	52.8996	52.8625	52.5595
3	53.2313	53.173	52.8851
...	...	...	...
10	55.6118	55.338	55.1966
11	55.9605	55.6426	55.5292
12	56.3113	55.9449	55.8614
13	56.6644	56.2442	56.1928
14	57.0196	56.54	56.5229
15	57.3771	56.8314	56.8509
16	57.7369	57.1176	57.176
17	58.0988	57.3973	57.497
18	58.4631	57.6692	57.8126
19	58.8296	57.9316	58.121
20	59.1985	58.1825	58.4199
...	...	...	...
27	61.8459	59.2191	59.8043
28	62.2336	59.1182	59.736
29	62.6238	58.8375	59.473
30	63.0165	58.2004	58.8342

Table 2.3: Comparison of solutions.

Next, in Figure 2 we analyze the sensitivity of the time-consistent solution for different values of $\bar{\rho}$. For $\rho = 0.03$ we take $\bar{\rho}_1 = 0.03$ (standard case), $\bar{\rho}_2 = 0.10$, $\bar{\rho}_3 = 0.15$ and $\bar{\rho}_4 = 0.20$.

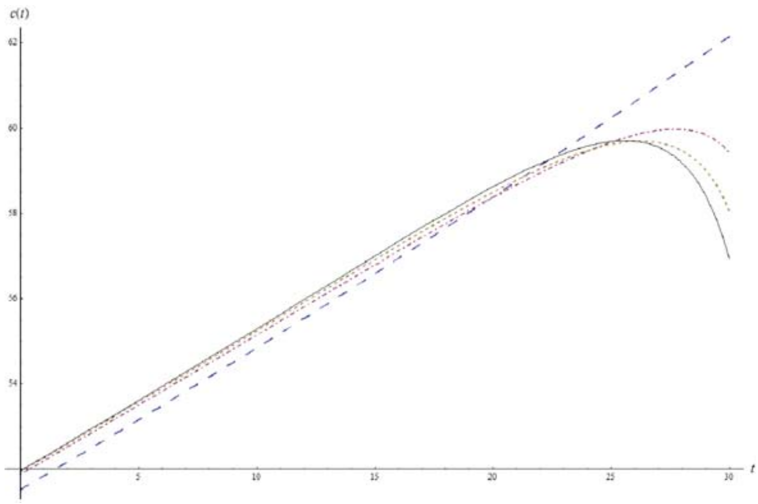

Figure 2.2: Sensitivity of the time-consistent solution for different values of $\bar{\rho}$. Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

Finally, Figure 3 illustrates the sensitivity of consumption in the time-consistent solution for different values of the risk aversion γ.

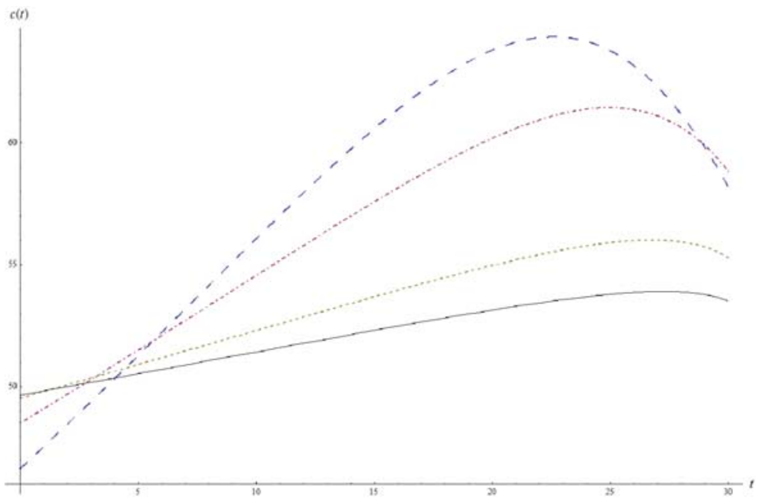

Figure 2.3: Sensitivity of the time-consistent consumption rule for different values of γ. $\gamma = -1$ (Dashing large). $\gamma = -2$ (DotDashed). $\gamma = -5$ (Dashing small). $\gamma = -8$ (Black).

2.4.2 Exponential utility function

Now, let us solve the problem for the (constant absolute risk aversion) exponential utility function

$$u(c) = -\frac{1}{\gamma}e^{-\gamma c}\,, \qquad\qquad F(W(T)) = -ae^{-\gamma W}\,,$$

with $\gamma > 0$. Thus, the decision maker faces the problem

$$\max_{\{c,w\}} E\left[\int_t^T e^{-\rho(s-t)}\frac{-1}{\gamma}e^{-\gamma c}ds + e^{-\bar{\rho}(T-t)}(-ae^{-\gamma W(T)})\right]$$

subject to (2.18) with initial condition $W(t) = W_t$. Once again, we first derive the precommitment and naive solutions. A (time-inconsistent) t-agent looks for the solution to the HJB equation (2.23) with the utility function specified above. By guessing

$$V^t(W,s) = -ae^{-\gamma(\alpha^t(s)+\beta^t(s)W)}\,,$$

the consumption and portfolio rules are given by

$$c^t(s) = \alpha^t(s) + \beta^t(s)W - \frac{\ln(a\gamma\beta^t(s))}{\gamma}\,, \quad w^t(s) = \frac{(\mu_1 - \mu_0)}{\sigma^2\gamma\beta^t(s)W}\,. \tag{2.32}$$

We substitute (2.32) in (2.23) to obtain that $\alpha^t(s)$ and $\beta^t(s)$ must satisfy

$$\dot{\alpha}^t - \alpha^t\beta^t = \frac{\beta^t}{\gamma} - \frac{\rho}{\gamma} - \frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2\gamma} - \frac{\beta^t}{\gamma}\ln(a\gamma\beta^t)\,, \tag{2.33}$$

$$\dot{\beta}^t = (\beta^t)^2 - \mu_0\beta^t\,, \tag{2.34}$$

together with the terminal conditions

$$\alpha^t(T) = \frac{1}{\gamma}(\bar{\rho}-\rho)(T-t)\,, \quad \text{and} \quad \beta^t(T) = 1\,,$$

respectively. The solution to the Bernoulli differential equation (2.34) is

$$\beta^t(s) = \frac{\mu_0}{1 + (\mu_0 - 1)e^{-\mu_0(T-s)}}\,.$$

Note that the function $\beta^t(s)$ does not depend on t. Hence, the value of $\beta(s)$ for both, the 0-agent under commitment and the naive t-agent, coincides for all

$s \in [0, T]$, i.e.

$$\beta(s) = \beta^0(s) = \beta^N(s)\,.$$

By substituting the value of $\beta^t(s)$ in equation (2.33) we find that

$$\alpha^t(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)(T - t) - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right] ,$$

where

$$\upsilon^e(\tau) = \beta(\tau) - \frac{1}{2}\frac{(\mu_1 - \mu_0)^2}{\sigma^2} - \beta(\tau)\ln(a\gamma\beta(\tau)) - \rho\,.$$

Taking $t = 0$ and $s = t$ we obtain the precommitment and naive solutions, respectively,

$$\alpha^P(s) = \alpha^0(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)T - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right] ,$$

$$\alpha^N(s) = \frac{1}{\gamma} e^{-\int_s^T \beta(\tau)d\tau} \left[(\bar{\rho} - \rho)(T - s) - \int_s^T \upsilon^e(\tau) e^{\int_\tau^T \beta(z)dz} d\tau \right] .$$

Finally, let us compute the time consistent equilibrium which, according to Proposition 1, can be obtained by solving the DPE

$$\bar{\rho} V^S(W,t) - K(W,t) - V_t^S(W,t) =$$

$$\max_{\{c,w\}} \left\{ \frac{-1}{\gamma} e^{-\gamma c} + \left[w(\mu_1 - \mu_0)W + (\mu_0 W - c) \right] V_W^S(W,t) + \frac{1}{2} w^2\sigma^2 W^2 V_{WW}^S(W,t) \right\} ,$$

with $K(W,t)$ given by

$$K(W,t) = E\left[\int_t^T e^{-\rho(s-t)} (\bar{\rho} - \rho) \frac{-1}{\gamma} e^{-\gamma c^*} ds \right] .$$

Applying Collorary 1 we obtain the set of two coupled partial differential equations

$$\bar{\rho} V^S(W,t) - K(W,t) - V_t^S(W,t) = \max_{\{c,w\}} \left\{ \frac{-1}{\gamma} e^{-\gamma c} + \right.$$

$$\left. + \left[w(\mu_1 - \mu_0)W + (\mu_0 W - c) \right] V_W^S(W,t) + \frac{1}{2} w^2\sigma^2 W^2 V_{WW}^S(W,t) \right\} , \qquad (2.35)$$

$$\rho K(W,t) - K_t(W,t) = (\bar{\rho} - \rho) \frac{-1}{\gamma} e^{-\gamma c^*} +$$

$$+ \left[w(\mu_1 - \mu_0)W + (\mu_0 W - c) \right] K_W(W,t) + \frac{1}{2} w^2\sigma^2 W^2 K_{WW}(W,t)\,, \qquad (2.36)$$

where c^* is the maximizer of the right hand term in (2.35). As a candidate to the value function and to the function $K(W,t)$ we guess

$$V^S(W,t) = -ae^{-\gamma(\alpha^S(t)+\beta^S(t)W)}\,, \qquad K(W,t) = A(t)e^{-\gamma(\alpha^S(t)+\beta^S(t)W)}.$$

If these choices prove to be consistent the consumption and portfolio rules are

$$c^* = \alpha^S(t) + \beta^S(t)W - \frac{\ln(a\gamma\beta^S(t))}{\gamma}\,, \quad w^* = \frac{(\mu_1 - \mu_0)}{\sigma^2\gamma\beta^S(t)W}\,. \tag{2.37}$$

Next, it is not difficult to check that $\beta^S(t)$ coincides with $\beta(t)$. By substituting (2.37) in (2.35-2.36) we obtain that functions $\alpha^S(t)$ and $A(t)$ are the solution to the following system of ordinary differential equations:

$$\rho A(t) - \dot{A}(t) + \gamma A(t)\dot{\alpha}^S(t) = -a(\bar{\rho} - \rho)\beta^S(t) -$$
$$-\left[\frac{(\mu_1-\mu_0)^2}{\sigma^2\gamma\beta^S(t)} - \alpha^S(t) + \frac{\ln(a\gamma\beta^S(t))}{\gamma}\right]\gamma A(t)\beta^S(t) + \frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2}A(t)\,,$$
$$\bar{\rho}a + a\gamma\dot{\alpha}^S(t) + A(t) =$$
$$a\beta^S(t) - \left[\frac{(\mu_1-\mu_0)^2}{\sigma^2\gamma\beta^S(t)} - \alpha^S(t) + \frac{\ln(a\gamma\beta^S(t))}{\gamma}\right]\gamma a\beta^S(t) + \frac{1}{2}\frac{(\mu_1-\mu_0)^2}{\sigma^2}a\,.$$

In Figure 4 and Figure 5 we analyze the sensitivity of the time-consistent consumption and portfolio rules, respectively, for different values of $\bar{\rho}$. For $\rho = 0.3$ we take $\bar{\rho}_1 = 0.03$ (standard case), $\bar{\rho}_2 = 0.10$, $\bar{\rho}_3 = 0.15$ and $\bar{\rho}_4 = 0.20$. The values for the others parameters are: $T = 30$, $\gamma = 0.1$, $W_0 = 1000$, $\mu_0 = 0.03$, $\mu_1 = 0.09$ and $\sigma = 0.3$.

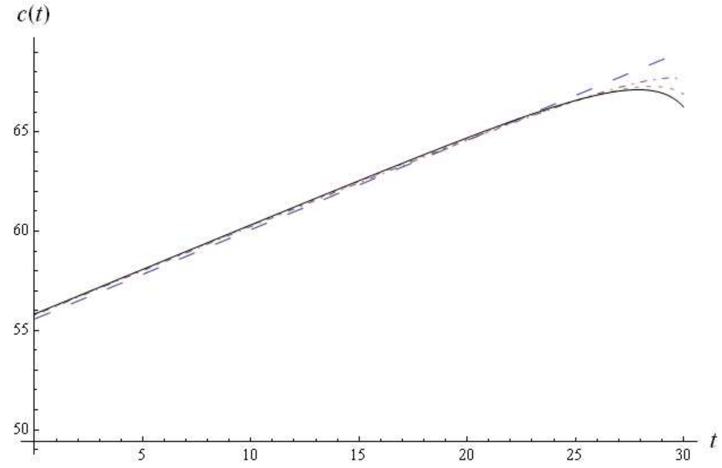

Figure 2.4: Sensitivity of the time-consistent consumption rule for different values of $\bar{\rho}$. Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

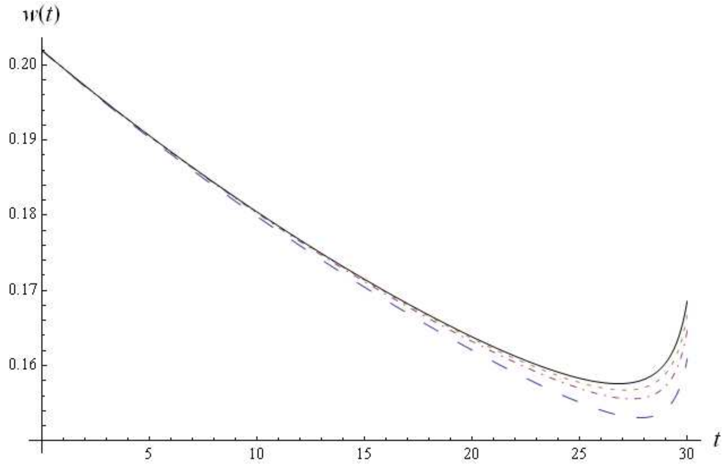

Figure 2.5: Sensitivity of the time-consistent portfolio rule for different values of $\bar{\rho}$. Standard case (Dashing large). $\bar{\rho}_2 = 0.10$ (DotDashed). $\bar{\rho}_3 = 0.15$ (Dashing small). $\bar{\rho}_4 = 0.20$ (Black).

Finally, let us briefly compare the results corresponding to the investment strategy according to the precommitment, naive and time-consistent solutions. In the case of power utilities, we have proved that the portfolio rule is always the same for these three solutions (although the consumption rule differs, as expected). For the case of exponential utilities, the investment rule is calculated according to the same formula

$$w^* = \frac{\mu_1 - \mu_0}{\sigma^2 \gamma \beta(t) W},$$

where $\beta(t)$ coincides for all the solution concepts. However, since w^* depends on W, and W evolves in a different way for precommitment, naive and time-consistent agents, the coincidence of portfolio rules in the power utility case is lost in the case of (CARA) exponential utilities.

2.5 The case of stochastic terminal time

Finally, let us assume that the final time T is a random variable taking values in $[t_0, \bar{T}]$ ($\bar{T}$ can be finite or infinite) with a known (maybe subjective) distribution function $G(\tau)$ and finite expectation. For instance, in the case of uncertain life-time presented by Yaari (1965), the distribution function $G_t(s)$ is the conditional probability that a consumer will die before time s, given that she is alive at time t, for $t < s$. Let us assume that $G(\tau)$ has density function, $G'(\tau) = g(\tau)$. The conditional distribution function satisfies

$$G_t(\tau) = \frac{G(\tau) - G(t)}{1 - G(t)}, \qquad g_t(\tau) = \frac{dG_t(\tau)}{d\tau} = \frac{g(\tau)}{1 - G(t)}.$$

Under heterogeneous discounting and random duration the t-agent will look for maximizing the expected value of (2.4), i.e.,

$$E\left[\int_t^T e^{-\rho(s-t)} u(X(s), c(s), s) + e^{-\bar{\rho}(T-t)} F(X(T), T) \mid x_t\ , t\ ;\ T > t\right] =$$

$$E\left[\int_t^{\bar{T}} dG_t(\tau) \left[\int_t^\tau ds\, e^{-\rho(s-t)} U(X(s), c(s), s)\right] + \right.$$
$$\left. + \int_t^{\bar{T}} dG_t(\tau) e^{-\bar{\rho}(\tau-t)} F(X(\tau), \tau) \mid x_t \right] =$$

$$E\left[\int_t^{\bar{T}} \left[e^{-\rho(s-t)}(1 - G_t(s)) U(X(s), c(s), s) + e^{-\bar{\rho}(s-t)} g_t(s) F(X(s), s)\right] ds \mid x_t\right]. \tag{2.38}$$

For the problem of maximizing (2.38) subject to (2.5), we can easily derive the corresponding dynamic programming equation by reproducing the steps in Section 3. Let $c^*(s) = \phi(x(s), s)$ an equilibrium rule, and assume that functions $V_1(x,t)$, $V_2(x,t)$ given by

$$V_1(x,t) = E\left[\int_t^{\bar{T}} e^{-\rho(s-t)}(1-G(s))U(X(s), \phi(X(s), s)\, ds \mid x_t\right] ,$$

$$V_2(x,t) = E\left[\int_t^{\bar{T}} e^{-\bar{\rho}(s-t)} g(s) F(X(s), s)\, ds \mid x_t\right]$$

are of class $C^{2,1}$ in (x,t). Then the solution to the DPE

$$-\sum_{i=1}^{2} \frac{\partial V_i(x,t)}{\partial t} + \rho V_1(x,t) + \bar{\rho} V_2(x,t) = \max_{\{c\}} \{(1-G(t))U(x,c,t) + g(t)F(x,t) +$$

$$+\sum_{i=1}^{2}\left[\nabla_x V_i(x,t)\cdot f(x,c,t) + \frac{1}{2} tr\left(\sigma(x,c,t)\cdot\sigma'(x,c,t)\cdot\nabla_{xx}V_i(x,t)\right)\right]\Bigg\} \ . \quad (2.39)$$

is an equilibrium policy. Note that, in addition, V_1 and V_2 verify the following partial differential equations system:

$$-\frac{\partial V_1(x,t)}{\partial t} + \rho V_1(x,t) = (1-G(t))U(x,\phi(x,t),t) + \nabla_x V_1(x,t)\cdot f(x,\phi(x,t),t) +$$

$$+\frac{1}{2} tr\left[\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V_1(x,t)\right] , \quad (2.40)$$

$$-\frac{\partial V_2(x,t)}{\partial t} + \bar{\rho} V_2(x,t) = g(t)F(x,t) + \nabla_x V_2(x,t)\cdot f(x,\phi(x,t),t) +$$

$$+\frac{1}{2} tr\left[\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot\nabla_{xx}V_2(x,t)\right] . \quad (2.41)$$

Consider, for instance, the saving-consumption problem of maximizing (2.19), where T is a random variable taking values in $[0,\infty)$, subject to (2.18), with $\mu_1 = \sigma = 0$ (there is just one risk-free asset). In the log-utility case, $U(c) = \ln c$, $F(W) = a \ln W$, by maximizing the right hand term in (2.39) we obtain the consumption rule

$$c^*(x,t) = \frac{1-G(t)}{\nabla_W V_1(W,t) + \nabla_W V_2(W,t)} \ .$$

By substituting in (2.40) and (2.41) and by guessing

$$V_1(W,t) = \alpha(t)\ln W + \beta(t)\,,$$

$$V_2(W,t) = \gamma(t)\ln W + \delta(t)\,,$$

we obtain that $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are the solution to the system of coupled nonlinear differential equations

$$\rho\alpha(t) - \dot{\alpha}(t) = \ln\frac{1-G(t)}{\alpha(t)+\gamma(t)}\,,$$

$$\rho\beta(t) - \dot{\beta}(t) = (1-G(t))\ln\frac{1-G(t)}{\alpha(t)+\gamma(t)} + \alpha(t)\left(\mu_0 - \frac{1-G(t)}{\alpha(t)+\gamma(t)}\right)\,,$$

$$\bar{\rho}\gamma(t) - \dot{\gamma}(t) = ag(t)\,,$$

$$\bar{\rho}\delta(t) - \dot{\delta}(t) = \gamma(t)\left(\mu_0 - \frac{1-G(t)}{\alpha(t)+\gamma(t)}\right)\,.$$

2.6 Conclusions

In this chapter we analyze the problem of deriving the optimal/equilibrium rules for an agent with heterogeneous preferences in a stochastic framework. We consider both the discrete and the continuous time case deriving the SDPE whose solutions are time consistent. In the discrete time setting, we adapt the derivation of the SDPE for the standard case optimizing backwards (Seierstad (2009)). For the continuous time case we follow two different approaches. First, we define the SDPE as the formal continuous time limit of the SDPE corresponding to a discretized version of the model (see Karp(2007) and Marín-Solano and Navas (2010)). Second, by using a variational approach as in Marín-Solano ans Patxot (2012). The equation obtained present a more complicated functional form than the standard one. For this reason, we derive an ”auxiliary dynamic programming equation” that can provide additional information when solving the problem.

The SDPE derived is used to solve the classical consumption and portfolio rules model for some utility functions (CRRA and CARA) and considering the heterogeneous discount function, illustrating the results numerically. As in the standard case, the portfolio rule coincide for the time inconsistent and for the time consistent agent if the utility considered is of the CRRA family. This coincidence is lost for the case of CARA utilities.

Finally, we briefly analyze the problem when an uncertain final time is introduced, adapting the SDPE to this case and solving a simple consumption-saving problem with logarithmic utility functions.

2.7 Appendix

DPE in discrete time: general case. Let us assume that the probabilities

$$P_t[V_{t+1} = v] = P_t(v|\, x_t, c_t, v_t)$$

depend, not only on time t and the previous outcome v_t, but also on the state and control variables x_t and c_t. Given the policies $c_0(x_0, v_0), \ldots, c_T(x_T, v_T)$, the state X_s depends on the outcomes $V_1, \ldots, V_s$, i.e.,

$$X_s = X_s(V_1, \ldots, V_s)\,.$$

If $p^*(v_1, \ldots, v_t)$ denotes the probability of the joint event $V_1 = v_1, \ldots, V_t = v_t$, then the expectation in (2.2) becomes

$$\sum_{s=t}^{n-1} \sum_{v_t,\ldots,v_s} \delta^{s-t} u_s(X_s, c_s(X_s, V_s), s) p^*(v_t, \ldots, v_s) + \sum_{v_t,\ldots,v_T} \bar{\delta}^{T-t} F(X_T, T) p^*(v_t, \ldots, v_T)\,.$$

Since the probabilities $p^*(v_t, \ldots, v_s)$, and hence the expected value, depend on the policies chosen, we can denote the above expectation as $E_{c_t,\ldots,c_T}$. We define the value function

$$W(x_t, t, v_t) = \sup_{\{c_t\}} E_{c_t,\ldots,c_T} \left[\sum_{s=t}^{T-1} \delta^{s-t} u_s(X_s, c_s, s) + \bar{\delta}^{T-t} F(X_T, T) \,|\, x_t, v_t \right] \,,$$

where the supremum is taken over the policy $c_t = c_t(x_t, v_t)$, provided that future s-agents follow the equilibrium rule $c^*_s = \phi_s(x_s, v_s)$, for $s = t+1, \ldots, n$. In the final period T, the value function is

$$W(x_T, T, v_T) = F(X_T, T)\,.$$

For $s \geq \tau$, we define

$$L_s^\tau = \sum_{v_{\tau+1}} \cdots \sum_{v_s} P_\tau(v_{\tau+1}|\, x_\tau, c_\tau^*, v_\tau) P_{\tau+1}(v_{\tau+2}|\, x_{\tau+1}, c_{\tau+1}^*, v_{\tau+1}) \cdots$$
$$\cdots P_{s-1}(v_s|\, x_{s-1}, c_{s-1}^*, v_{s-1}) u_s(X_s, \phi_s(X_s, V_s), s)\,. \quad (2.42)$$

For $s = T-1$ we have

$$W(x_{T-1}, T-1, v_{T-1})$$
$$= \sup_{\{c_{T-1}\}} \{u_{T-1}(x_{T-1}, c_{T-1}, T-1) + E_{c_{T-1}} \bar{\delta} F(X_T, T) | x_{T-1}, v_{T-1}]\}.$$

Let $c_{T-1}^* = \phi(x_{T-1}, v_{T-1})$ be the solution to this equation. Since

$$E_{c_{T-1}}[F(X_T)|x_{T-1}, v_{T-1}] = \sum_{v_T \in \mathcal{V}} P_{T-1}(v_T | x_{T-1}, c_{T-1}^*, v_{T-1}) F(X_T, T)\,,$$

then

$$W(x_{T-1}, T-1, v_{T-1}) = u_{T-1}(x_{T-1}, \phi(x_{T-1}, v_{T-1}), T-1) +$$
$$+ \bar{\delta} \sum_{v_T \in \mathcal{V}} P_{T-1}(v_T | x_{T-1}, c_{T-1}^*, v_{T-1}) F(X_T, T) =$$
$$= u_{T-1}(x_{T-1}, \phi(x_{T-1}, v_{T-1}), T-1) + \bar{\delta} L_T^{T-1}\,.$$

In general

$$W(x_{t+1}, t+1, v_{t+1}) = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^{t+1} + \bar{\delta}^{T-t-1} L_T^{t+1}\,,$$

and

$$W(x_t, t, v_t) = \sup_{\{c_t\}} \left\{ u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} E_{c_t}\left[L_s^{t+1} \,|\, x_t, v_t\right] + \right.$$
$$\left. + \bar{\delta}^{T-t} E_{c_t}\left[L_n^{t+1} \,|\, x_t, v_t\right] \right\} = \sup_{\{c_t\}} \{u_t(x_t, c_t, t) + \sum_{s=t+1}^{T-1} \delta^{s-t} L_s^t + \bar{\delta}^{T-t} L_T^t\}\,. \quad (2.43)$$

Taking the expectation of $W(x_{t+1}, t+1, V_{t+1})$ conditioned to x_t and v_t we have

$$E_{c_t}\left[W(X_{t+1}, t+1, V_{t+1}) \,|\, x_t, v_t\right] = \sum_{s=t+1}^{T-1} \delta^{s-t-1} L_s^t + \bar{\delta}^{T-t-1} L_T^t\,. \quad (2.44)$$

Finally, solving L_T^t in (2.44) and substituting in (2.43) we obtain the DPE (2.9).

□

Proof of Proposition 2. It is a rather straightforward extension of the proof in the deterministic case (see Marín-Solano and Patxot (2012)). We include a sketch of the proof.

First note that, if $\bar{x}(s)$ is the state trajectory corresponding to the decision rule $c_\epsilon(s)$, then

$$V(x,t)-V_\epsilon(x,t) = E\left[\int_t^{t+\epsilon} e^{-\rho(s-t)}\left[u(X(s),\phi(X(s),s),s) - u(\bar{X}(s),v(s),s)\right] ds + \right.$$

$$+\int_{t+\epsilon}^{T} e^{-\rho(s-t)}\left[u(X(s),\phi(X(s),s),s) - u(\bar{X}(s),\phi(\bar{X}(s),s),s)\right] ds +$$

$$\left. + e^{-\bar{\rho}(T-t)}(F(X(T),T) - F(\bar{X}(T),T))\right].$$

Next, we can write

$$E\left[\int_{t+\epsilon}^{T} e^{-\rho(s-t)}u(X(s),\phi(X(s),s),s)\,ds + e^{-\rho(T-t)}F(X(T),T)\right] =$$

$$V(x(t+\epsilon),t+\epsilon) - E\left[\int_{t+\epsilon}^{T}\left(e^{-\rho(s-t-\epsilon)} - e^{-\rho(s-t)}\right)u(X(s),\phi(X(s),s),s)\,ds + \right.$$

$$\left. + \left(e^{-\bar{\rho}(T-t-\epsilon)} - e^{-\bar{\rho}(T-t)}\right)F(X(T),T)\right],$$

and

$$E\left[\int_{t+\epsilon}^{T} e^{-\rho(s-t)}u(\bar{X}(s),\phi(\bar{X}(s),s),s)\,ds + e^{-\bar{\rho}(T-t)}F(\bar{X}(T),T)\right] =$$

$$V(\bar{x}(t+\epsilon),t+\epsilon) - E\left[\int_{t+\epsilon}^{T}\left(e^{-\rho(s-t-\epsilon)} - e^{-\rho(s-t)}\right)u(\bar{X}(s),\phi(\bar{X}(s),s),s)\,ds + \right.$$

$$\left. \left(e^{-\bar{\rho}(T-t-\epsilon)} - e^{-\bar{\rho}(T-t)}\right)F(\bar{X}(T),T)\right].$$

Third, note that

$$\lim_{\epsilon\to 0^+}\frac{V(x,t)-V_\epsilon(x,t)}{\epsilon} = A + B + C$$

where

$$A = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} E \left[\int_t^{t+\epsilon} e^{-\rho(s-t)} \left(u(X(s), \phi(X(s), s), s) - u(\bar{X}(s), v(s), s) \right) ds \right] =$$

$$= u(x(t), \phi(x(t), t), t) - u(x(t), v(t), t) \,,$$

$$B = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} E \left[\int_{t+\epsilon}^{T} \left(e^{-\rho(s-t)} - e^{-\rho(s-t-\epsilon)} \right) (u(X(s), \phi(X(s), s), s) - \right.$$

$$-u(\bar{X}(s), \phi(\bar{X}(s), s), s)\big) \, ds + \left(e^{-\bar{\rho}(T-t)} - e^{-\bar{\rho}(T-t-\epsilon)} \right) (F(x(T), T)$$

$$-F(\bar{x}(T), T))] = 0,$$

and

$$C = \lim_{\epsilon \to 0^+} \frac{V(x(t+\epsilon), t+\epsilon) - V(x(t), t)}{\epsilon} - \lim_{\epsilon \to 0^+} \frac{V(\bar{x}(t+\epsilon), t+\epsilon) - V(x(t), t)}{\epsilon} =$$

$$= \left(\nabla_x V(x,t) f(x, \phi(x,t), t) + \frac{1}{2} tr(\sigma(x, \phi(x,t), t) \sigma'(x, \phi(x,t), t) \nabla_{xx} V(x,t)) \right) -$$

$$- \left(\nabla_x V(x,t) f(x, v(t), t) + \frac{1}{2} tr(\sigma(x, v, t) \sigma'(x, v, t) \nabla_{xx} V(x,t)) \right) .$$

Therefore,

$$\lim_{\epsilon \to 0^+} \frac{V(x,t) - V_\epsilon(x,t)}{\epsilon} = [u(x, \phi(x,t), t) + \nabla_x V(x,t) f(x, \phi(x,t), t) +$$

$$\left. + \frac{1}{2} tr(\sigma(x, \phi(x,t), t) \sigma'(x, \phi(x,t), t) \nabla_{xx} V(x,t)) \right] - [u(x, v(t), t) +$$

$$\left. + \nabla_x V(x,t) f(x, v(t), t) + \frac{1}{2} tr(\sigma(x, v, t) \sigma'(x, v, t) \nabla_{xx} V(x,t)) \right] \geq 0 \,,$$

since $c^* = \phi(x,t)$ is the maximizer of the right hand term in (2.13).

□

Chapter 3

Consumption, investment and life insurance strategies with heterogeneous discounting

3.1 Introduction

The introduction of an uncertain lifetime in portfolio optimization models has proved to be useful in the study of the demand for life insurance, which has usually been derived from a bequest function. The starting point for modern research on the subject dates back to Yaari (1965) who studied the problem of life insurance in a deterministic financial environment with the stochastic time of death as the only source of uncertainty. Later on, Richard (1975) combined the portfolio optimization model in Merton (1969, 1971) with the model in Yaari (1965) to deal with a life-cycle consumption/investment problem in the presence of life insurance and random terminal time. However, the model introduced by Richard (1975) had several unsatisfactory aspects. First, the value function was not well-defined at the final time because the random variable used to model the lifetime was assumed to be bounded. This is a very important point in view of the fact that the problem was analyzed using a dynamic programming approach, which proceeds backward in time. Second, as Leung (1994) pointed out, there is a problem with the existence of interior solutions. In order to overcome these difficulties, Pliska and Ye (2007) incorporated the randomness of the planning horizon by means of the uncertain life model found in reliability theory. In contrast to Richard (1975), in which the random lifetime took values on a bounded interval, in that paper the authors considered an intertemporal model and allowed the random lifetime to take values on $[0, \infty)$. In addition, the authors refined the theory in the following ways. First, the planning horizon was considered to be some fixed point in the future T (the retirement time for the decision maker) in contrast with the model in Richard (1975) in which the planning horizon was interpreted as the finite upper bound on the lifetime. Second, at T a utility was introduced accounting for the agent wealth at the final time. After setting up the HJB equation and deriving the optimal feedback control law, Pliska and Ye (2007) obtained explicit solutions for the family of discounted CRRA utilities. As it is customary in the analysis of intertemporal decision problems, the decision maker considered was characterized by a constant discount rate of time preference, i.e., she discounted the stream of utilities of any category using an exponential discount function with a constant discount rate of time preference according to the Discounted Utility (DU) model introduced in Samuelson (1937). Within this framework, the marginal rate of substitution between payments at different times depends only on the length of the time interval contemplated, being this fact probably the main limitation of the DU model with regard to its capacity to describe the actual time preference patterns.

In fact, the empirical findings on individual behavior seem to challenge some of the predictions of the standard discounting model (see Frederick et al. (2002) for a review of the literature until then). For this reason, variable rates of time preference have received an increasing attention in recent years, in attempts to capture the reported anomalies. In this sense, for instance, several papers focused on the greater impatience of decision makers about the choices in the short run compared with those in the long term using the hyperbolic discount function introduced by Phelps and Pollak (1968). Along the same lines, Karp (2007) and Marín-Solano and Navas (2010) dealt with the problem with non-constant discounting. Also, in a recent paper by Ekeland et al. (2012), the model of Pliska and Ye (2007) was extended with the introduction of non-constant discount rates.

The choice of the discount function will depend, in general, on the problem under consideration. For instance, in intertemporal problems with a bequest motive, like those studying the demand for life insurance, it is useful to account for the fact that the agent concern about the bequest left to her descendants is not the same when she is young than when she is an adult. A similar effect could be considered in retirement and pension models, in which the willingness to save for a better retirement is likely to be greater at the end of the working life than at the beginning. In addition, for such a long planning horizon the greater impatience in the short run may still play a role, although this bias should evolve according to the different valuations over time of the bequest and the pension plan. In order to capture this asymmetric valuation Marín-Solano and Patxot (2012) introduced the heterogeneous discounting model. According to these authors, the individual preferences at time t take the form

$$\int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T), T)\ , \tag{3.1}$$

i.e., the agent uses a constant discount rate of time preference, but this rate is different for the instantaneous utilities $L(x(s), u(s), s)$ and for the final function $F(x(T), T)$ which, in the previous examples, would account for the bequest or the agent wealth at retirement. The most relevant effect of using any non-constant discount function is that preferences change with time. Impatient agents over-valuing instantaneous utilities in comparison with the final function are characterized by $\rho > \delta$ in equation (3.1). However, as we approach the end of the planning horizon T the relative value of the final function increases compared with the instantaneous utilities and consequently, the bias to the present decreases with time (see Marín-Solano and Patxot (2012) and de-Paz et al. (2011) for a detailed discussion of this effect).

The aim of this chapter is to derive the optimal consumption, investment and life insurance rules for an agent whose concern about both the bequest left to her descendants and her wealth at retirement increases with time. To this end we depart from the model in Pliska and Ye (2007) generalizing the individual time preferences by incorporating heterogeneous discount functions. In contrast to the extension of Pliska and Ye's (2007) model in Ekeland et al. (2012), where an intergenerational problem is introduced by assigning different discount functions to different generations, our setting of heterogeneous discounting focuses on the time preference dynamics of the decision maker, i.e., our setting faces an intra-generational problem. In addition, following Kraft (2003), we derive the wealth process in terms of the portfolio elasticity with respect to the traded assets. This approach allows us to introduce options in the investment opportunity set as well as to enlarge it by any number of contingent claims while maintaining the analytical tractability of the model. Finally, we analyze how the standard solutions are modified depending on the attitude of the agent towards her changing preferences, showing the differences with some numerical illustrations.

In effect, the individual facing the problem of maximizing (3.1) can act in two different ways. On the one hand, she could solve the problem by ignoring the fact that her preferences are going to change in the near future, and applying the classical HJB equation. In this case, the strategies obtained will be only optimal from the point of view of her preferences at time t and, in general, will be only obeyed at that time; therefore they are time-inconsistent. On the other hand, she could take into account her changing preferences and obtain the time-consistent strategies by calculating Markov Perfect Equilibria (MPE). These different solutions are usually referred to as naive (in general time-inconsistent) and sophisticated (time-consistent) in the non-constant discounting literature. In order to obtain the MPE, Marín-Solano and Patxot (2012) derived the Dynamic Programming Equation (DPE) in a deterministic framework following a variational approach. The extension to the stochastic case, in which the state dynamics is described by a set of diffusion equations of the form

$$dx(t) = f(x(t), u(t), t)\, dt + \sigma(x(t), u(t), t) dz(t)\,,$$

where $z(t)$ is a standard Wiener process, was studied in de-Paz et al. (2011). In that paper the DPE providing time-consistent solutions was derived following two different approaches. The first one consisted in obtaining the DPE for the heterogeneous discounting problem in discrete time and then taking the formal continuous time limit, following Karp (2007) for the non-constant discounting problem in a deterministic setting (see Marín-Solano and Navas (2010) for the stochastic case). The second one was the variational approach, as in Marín-Solano and Patxot (2012) (which is based on Ekeland and Lazrak (2010)). It is important to remark that, despite the fact that the two approaches are different in nature, the equilibrium conditions coincide.

According to de-Paz et al. (2011), if $V(x,t)$ is the value function of the time-consistent (sophisticated) agent for the problem of maximizing (3.1) subject to the corresponding state equation, and assuming that it is of class $C^{2,1}$, then $V(x,t)$ satisfies the following DPE

$$\rho V(x,t) - V_t(x,t) - K(x,t) = \tag{3.2}$$

$$= \sup_{\{u\}} \left\{ L(x,u,t) + V_x(x,t) f(x,u,t) + \frac{1}{2} tr \left(\sigma(x,u,t) \cdot \sigma'(x,u,t) \cdot V_{xx}(x,t) \right) \right\}\,,$$

where

$$K(x,t) = (\rho - \delta) E \left[\int_t^T e^{-\delta(s-t)} L(x_s, \phi(x_s,s), s)\, ds \right] \tag{3.3}$$

with $V(x,T) = F(x,T)$, and being $\phi(x_s,s)$ the equilibrium rule . The subscripts denote the partial derivative. If, for each pair (x,t), there exists a decision rule

$u^* = \phi(x,t)$, with corresponding state trajectory $x^*(t)$, such that u^* maximizes the right hand side term of (3.2), then $u^* = \phi(x,t)$ is the Markov equilibrium rule for the problem with heterogeneous discounting.

It is worth mentioning that, unlike the standard DPE, a new term $K(x,t)$ appears in (3.2). Checking equation (3.3) it is obvious that $K(x,t) = 0$ in the standard discounting case ($\delta = \rho$). By differentiating (3.3) with respect to t we obtain an "auxiliary dynamic programming equation"

$$\delta K(x,t) - K_t(x,t) = (\delta - \rho)L(x,\phi(x,t),t) + K_x(x,t)\cdot f(x,\phi(x,t),t) + \\ + \frac{1}{2} tr\left(\sigma(x,\phi(x,t),t)\cdot\sigma'(x,\phi(x,t),t)\cdot K_{xx}(x,t)\right) \,, \tag{3.4}$$

so that instead of solving (3.2) and (3.3), the solution can be characterized by solving the system of partial differential equations (3.2) and (3.4) with the corresponding boundary conditions $V(x,T) = F(x,T)$, and $K(x,T) = 0$.

The rest of the chapter is organized as follows. In Section 2, we present the model we want to address describing the underlying financial and insurance market as well as the optimal control problem to be solved. In Section 3, we consider the case of CRRA and CARA utility functions and we discuss the time-consistency of the solutions obtained. In Section 4, we provide some numerical illustrations of the main results, comparing our solutions with the standard ones. Finally, Section 5 concludes.

3.2 The model

Consider a decision maker with a working life that extends from t_0 to T years who is subject to a mortality risk. Let $\tau \in [0,\infty)$ be a random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing the agent time of death. We assume that τ has a known distribution function $F(\tau)$ and density function $F'(\tau) = f(\tau)$. At each time $t \in [t_0, \min\{T,\tau\}]$, the agent has to decide how to allocate her personal wealth $W(t)$ between consumption, investment, and life insurance purchase.

The consumption process rate is denoted by $c(t)$. Obviously, the agent enjoys consumption as long as she is alive, i.e., for all $t \le \min\{T,\tau\}$. The life insurance contract can be purchased by the agent by paying premiums per euro of coverage for age t at a rate denoted by $p(t)$. We assume that contracts of this kind are offered continuously in the insurance market. If $Q(t)$ denotes the total amount of life insurance purchased, the total premium paid at time t is $p(t)Q(t)$. In addition to consumption and purchase of a life insurance policy, we assume that the agent invests the full amount of her savings in a financial market. Let us briefly derive the wealth process when the market comprises two securities, one risk-free and the other risky. The risk-free asset price $M(t)$ is assumed to evolve according to

$$dM(t) = M(t)rdt\,,$$

where $r > 0$ and $M(t_0) = m > 0$, while the risky asset follows a geometric Brownian motion described by

$$dP(t) = P(t)\mu dt + P(t)\sigma dz(t)\,,$$

where $P(t_0) = p > 0$ and $z(t)$ is a standard Brownian motion process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the chapter we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space and that its filtration $\{\mathcal{F}_t, t \in [t_0, T]\}$ is the $\mathbb{P}$-augmentation of the filtration generated by $z(t)$. Besides the return on her investment, the agent receives her income at a rate $i(t)$ until her retirement time or until her death time, whichever happens first. Denoting by w the proportion of savings invested in the risky asset, the wealth process is described by the stochastic differential equation

$$dW = [(r + w(\mu - r))\,W + i(t) - c(t) - p(t)Q(t)]\,dt + w\sigma W dz(t)\,, \tag{3.5}$$

defined on $[t_0, \min\{T, \tau\}]$, with $W(t_0) = W_0$.

Assume now that the opportunity set for investments is not only composed by the two securities described above but that an option $C(t, P(t))$ on the stock is also available in the market. The introduction of options and other derivatives is a natural generalization of the standard portfolio problem due to their wide use as investment opportunities. However, the extension of the stochastic optimal control approach leads to a much more complicated form of the HJB equation, since the option price $C(t, P(t))$ is a non-linear function of the underlying stock price. Kraft (2003) proposed a kind of two step procedure that greatly simplifies the problem. By introducing the elasticity of the portfolio with respect to the stock price, it is shown that this elasticity can be used as the control variable instead of the share of wealth invested in each asset. Thus, in the first step, investment problems with contingent claims of the form $C(t, P(t))$ can be solved as if the portfolio only contained a risky security and a risk-free security (the reduced portfolio problem). Once the optimal elasticity is obtained, the second step consists in calculating a portfolio tracking this elasticity.

Therefore, according to the elasticity approach, the optimal wealth process can be determined by the optimal elasticity of the portfolio with respect to the stock price. We first define the elasticity of the option price with respect to the price of the underlying

$$\epsilon_C = \frac{dC/C}{dP/P},$$

where dC is obtained using Ito's lemma

$$dC = \left(C_t + C_P P\mu + \frac{1}{2}C_{PP}P^2\sigma^2\right)dt + C_P P\sigma dz(t). \tag{3.6}$$

An application of the Black-Scholes partial differential equation,

$$C_t + C_P Pr + \frac{1}{2}C_{PP}P^2\sigma^2 - Cr = 0$$

leads to the option dynamics

$$dC = (rC + (\mu - r)C_P P)\,dt + C_P P\sigma dz(t).$$

Hence, we have

$$\epsilon_C = \frac{dC/C}{dP/P} = \frac{C_P P}{C}$$

and equation (3.6) becomes

$$dC = C\left[(r + \epsilon_C(\mu - r))dt + \epsilon_C \sigma dz(t)\right].$$

Let w_P and w_C denote the proportion of the wealth invested in the risky asset and in the call option, respectively. The remainder $1 - w_P - w_C$ is the proportion invested in the risk-free security. In this case, the wealth process is described by

$$dW = \left[(r + (w_P + w_C\epsilon_C)(\mu - r))\,W + i(t) - c(t) - p(t)Q(t)\right]dt+$$

$$+(w_P + w_C\epsilon_C)\sigma W dz(t). \tag{3.7}$$

In addition, the portfolio's elasticity with respect to the stock price is defined as the weighted sum of the elasticities of the portfolio components

$$\epsilon = (1 - w_P - w_C)\epsilon_M + w_P\epsilon_P + w_C\epsilon_C.$$

Since $\epsilon_P = 1$ and $\epsilon_M = 0$ respectively, we have

$$\epsilon = w_P + w_C \epsilon_C \,,$$

and the stochastic differential equation describing the wealth process can be written in terms of the elasticity of the investor's portfolio provided that $w = (w_P, w_C)$ is hold constant (static elasticity), i.e.,

$$dW = \left[(r + \epsilon(\mu - r))\, W + i(t) - c(t) - p(t)Q(t) \right] dt + \epsilon \sigma W dz(t) \,, \tag{3.8}$$

for $t \in [t_0, \min\{T, \tau\}]$, with $W(t_0) = W_0$.

Note that the only difference between equations (3.8) and (3.5) is that the control variable w in (3.5) is replaced by the static elasticity ϵ in (3.8). In addition, since ϵ is independent of a particular asset, the opportunity set for investment can be enlarged by any number of contingent claims.

The problem for the wage earner is then to choose the consumption, portfolio elasticity and life insurance rules so as to maximize

$$\max_{\{c,\epsilon,Q\}} E \left[\int_t^{T\wedge\tau} e^{-\delta(s-t)} U(c_s) ds + e^{-\rho(\tau-t)} B(Z(\tau), \tau) \mathbb{1}_{\tau \leq T} + \right.$$
$$\left. + e^{-\rho(T-t)} L(W(T)) \mathbb{1}_{\tau > T} \mid \tau > t, \mathcal{F}_t \right] \,, \tag{3.9}$$

where $T \wedge \tau \equiv \min\{T, \tau\}$; $\mathbb{1}_A$ is the indicator function of event A; $U(c)$ is the utility derived from consumption; $L(W(T))$ denotes the utility derived from the wealth available for retirement in case of being alive at T; and $B(Z(\tau), \tau)$ is the utility from the legacy left to her descendants in case of dying before retirement, with $Z(\tau) = W(\tau) + Q(\tau)$. Functions $U(\cdot)$, $B(\cdot)$ and $L(\cdot)$ are assumed to be strictly concave functions on their arguments.

Note that the discount function is the same for $B(Z(\tau), \tau)$ and $L(W(T))$, which are the final functions depending on dying before retirement or not, and it is different from the discount function for the utility derived form consumption, with $\rho > \delta$. In contrast to intergenerational models, in which different generations can be modeled by introducing different discount functions (as in the case of hyperbolic discounting), we are interested in modeling the individual's increasing concern about his/her bequest and his/her wealth available for retirement, i.e., from an intragenerational point of view. As discussed in de-Paz et al. (2011), this asymmetric valuation can not be described by standard exponential discounting or hyperbolic discount functions.

Finally, if the mortality risk is independent of the financial risk, equation (3.9) with random terminal time transforms into

$$\max_{\{c,\epsilon,Q\}} E\left[\int_t^T \left(S(s,t)e^{-\delta(s-t)}U(c_s) + f(s,t)e^{-\rho(s-t)}B(Z(s),s)\right) ds + \right.$$

$$\left. + S(T,t)e^{-\rho(T-t)}L(W(T)) \mid \mathcal{F}_t\right] =$$

$$= \max_{\{c,\epsilon,Q\}} \frac{1}{S(t)} E\left[\int_t^T \left(S(s)e^{-\delta(s-t)}U(c_s) + f(s)e^{-\rho(s-t)}B(Z(s),s)\right) ds + \right.$$

$$\left. + S(T)e^{-\rho(T-t)}L(W(T)) \mid \mathcal{F}_t\right] , \qquad (3.10)$$

where $S(t)$ denotes the survivor function, i.e., the probability that the decision maker survives to some time beyond t

$$S(t) = \mathbb{P}(\tau \geq t) = 1 - F(t) ,$$

the function $f(s,t)$ is the conditional probability density for death at time s, given that the agent is alive at time $t \leq s$

$$f(s,t) = \mathbb{P}(\tau = s \mid \tau \geq t) = \frac{f(s)}{S(t)} ,$$

and the function $S(s,t)$ denotes the conditional probability that the decision maker survives to some time beyond s, given that he or she is alive at time $t \leq s$

$$S(s,t) = \mathbb{P}(\tau \geq s \mid \tau \geq t) = \frac{S(s)}{S(t)} .$$

3.3 The case of CRRA and CARA utility functions

In this section we derive explicit solutions for the problem (3.10) and (3.8) considering first, utility functions with a constant relative risk aversion, and second, utility functions with a constant absolute risk aversion. We then compare the standard solutions with the time-inconsistent and with the time-consistent solutions for the problem with heterogeneous discounting.

Let c^*,ϵ^* and Q^* denote the optimal consumption, portfolio elasticity and life insurance purchase. Then the current value function at time t is

$$\bar{V}(W,t) = \frac{1}{S(t)} E\left[\int_t^T \left(S(s)e^{-\delta(s-t)}U(c_s^*) + f(s)e^{-\rho(s-t)}B(Z^*(s),s)\right)ds + \right.$$
$$\left. + S(T)e^{-\rho(T-t)}L(W^*(T)) \mid \mathcal{F}_t\right] .$$

Throughout the chapter we will work with the value function multiplied by the survivor probability function

$$V(W,t) = S(t)\bar{V}(W,t) .$$

Note that, when transforming the functional (3.9) in (3.10), the utility function $B(Z(s),s)$ enters in the integral term and it can therefore be viewed as an instantaneous utility. In order to have a constant discount rate for the instantaneous utilities different to that for the final function, we rewrite the objective function (3.10) as follows. For each $\omega \in \Omega$, we define a new state variable $y_\omega^t(u)$ as

$$y_\omega^t(u) = \int_t^u f(s)e^{-\rho(s-u)}B(Z_\omega(s),s)ds .$$

For simplicity, in the following we will omit the subindex ω. Then, maximizing (3.10) subject to (3.8) is equivalent to

$$\max_{\{c,\epsilon,Q\}} \frac{1}{S(t)} E\left[\int_t^T S(s)e^{-\delta(s-t)}U(c_s)ds + e^{-\rho(T-t)}\left[y^t(T) + S(T)L(W(T))\right] \mid \mathcal{F}_t\right] \tag{3.11}$$

subject to (3.8) and to

$$\dot{y}^t(s) = \rho y^t(s) + f(s)B(Z(s),s) . \tag{3.12}$$

As mentioned in the introduction, the wage earner solving problem (3.11) subject to (3.8) and (3.12) can act in two different ways. The time-consistent agent must solve the DPE (3.2). Otherwise, the naive agent making decisions at time t without taking into account that her preferences change with time will maximize (3.11) subject to (3.8) and (3.12) by solving the standard HJB equation

$$\delta V(y^t,W,s) - V_s(y^t,W,s) = \max_{\{c,\epsilon,Q\}} \{S(s)U(c)+$$
$$+ \left[\rho y^t(s) + f(s)B(Z(s),s)\right] V_{y^t}(y^t,W,s) + \left[(r+\epsilon(\mu-r))W + i(t) - c(t)\right.$$

$$-p(t)Q(t)]\, V_W(y^t,W,s) + \frac{1}{2}\epsilon^2\sigma^2W^2V_{WW}(y^t,W,s)\Big\} \,. \qquad (3.13)$$

The difference between this solution and the solution in the standard case ($\rho = \delta$) comes from the boundary condition used in each problem. While in the standard case the boundary condition is

$$V(y^t,W,T) = y^t(T) + S(T)L(W(T))\,,$$

the value function at T for the time-inconsistent agent, in its current value form, is

$$V(y^t,W,T) = e^{-(\rho-\delta)(T-t)}(y^t(T) + S(T)L(W(T))\,.$$

This boundary condition changes depending on the moment t at which the solution is calculated. In fact, an agent acting in this way constructs her solution by solving the HJB equation (3.13) together with the family of boundary conditions

$$V(y^t,W,T) = e^{-(\rho-\delta)(T-t)}(y^t(T) + S(T)L(W(T))$$

for $t \in [t_0,T]$ and patching together the solutions obtained. In order to highlight the moment at which the problem is solved, in the following we will denote the value function for the time-inconsistent (naive) agent by $V^t(y^t,W,s)$. In addition, we will omit the superscript t in $y^t(s)$.[1]

3.3.1 Logarithmic utility function

Consider first the case of logarithmic utility functions

$$U(c_s) = \ln c_s\,, \qquad B(Z(s),s) = a\ln Z(s)\,, \qquad \text{and} \qquad L(W(T)) = b\ln W(T), \qquad (3.14)$$

where a and b are positive real parameters. Let us briefly derive the time-inconsistent strategy solving equation (3.13) at some particular time $t \in [t_0,T]$, i.e., with the boundary condition

$$V^t(y,W,T) = e^{-(\rho-\delta)(T-t)}(y(T) + bS(T)\ln W(T))\,. \qquad (3.15)$$

[1] In the optimal solution from the viewpoint of the t_0-agent, $t = t_0$ in equation (3.11) (the so called precommitment solution in the literature of hyperbolic discounting), one should add the initial condition $y^{t_0}(t_0) = 0$ in (3.12). The same initial condition is considered in the time-consistent solution. On the contrary, in the naive solution, the initial condition in the problem for each t-agent is $y^t(t) = 0$, for every t.

From the maximization problem in (3.13) one easily obtains

$$c^t(s) = \frac{S(s)}{V_W^t}\,, \qquad \epsilon^t(s) = \frac{-(\mu - r)}{\sigma^2 W}\frac{V_W^t}{V_{WW}^t}\,, \qquad Q^t(s) = a\frac{f(s)}{p(s)}\frac{V_y^t}{V_W^t} - W\,. \quad (3.16)$$

As a candidate to the value function, we guess

$$V^t(y, W, s) = \alpha^t(s)\ln(W + \beta^t(s)) + \varphi^t(s)y\,.$$

By substituting this guessing into (3.13), we obtain that the rules given by (3.16) become

$$c^t(s) = \frac{S(s)}{\alpha^t(s)}(W + \beta^t(s))\,, \qquad \epsilon^t(s) = \frac{(\mu - r)}{\sigma^2 W}(W + \beta^t(s))\,,$$

and

$$Q(s)^t = a\frac{f(s)\varphi^t(s)}{p(s)\alpha^t(s)}(W + \beta^t(s)) - W\,,$$

where the functions appearing in the guessed value function are given by

$$\alpha^t(s) = bS(T)e^{-\rho(T-t)+\delta(s-t)} + \int_s^T \left(e^{-\delta(\tau-s)}S(\tau) + e^{-\rho(\tau-t)+\delta(s-t)}f(\tau)\right)d\tau$$

and

$$\beta^t(s) = \int_s^T e^{-\int_s^u (r+p(v))dv} i(u)du\,, \qquad \varphi^t(s) = e^{-(\rho-\delta)(s-t)}\,.$$

From the above expressions for $\alpha^t(s)$, $\beta^t(s)$, and $\varphi^t(s)$, it becomes clear that our guessing for the value function is consistent.

Therefore, either the agent is able to commit herself to following the decisions initially taken at t_0 or the rules above will be only obeyed at the time at which they have been calculated, i.e., $s = t$. Thus, either

$$\alpha^{t_0}(s) = bS(T)e^{-\rho(T-t_0)+\delta(s-t_0)} + \int_s^T \left(e^{-\delta(\tau-s)}S(\tau) + e^{-\rho(\tau-t_0)+\delta(s-t_0)}f(\tau)\right)d\tau$$

and

$$\varphi^{t_0}(s) = e^{-(\rho-\delta)(s-t_0)}\,,$$

or

$$\alpha^t(t) = bS(T)e^{-\rho(T-t)} + \int_t^T \left(e^{-\delta(\tau-t)}S(\tau) + e^{-\rho(\tau-t)}f(\tau)\right)d\tau\,, \qquad \varphi^t(t) = 1\,.$$

respectively. With respect to function $\beta^t(s)$, since it does not depend on the moment t, it coincides whether or not the agent can commit herself

$$\beta^t(s) = \beta^{t_0}(s) = \beta^s(s)\,.$$

Now we turn the attention to the time-consistent strategy.

Proposition 3 *Assume that* $U(c_s)$, $B(Z(s), s)$, *and* $L(W(T))$ *are given by (3.14). Then* $V(y, W, t) = \alpha(t)\ln(W + \beta(t)) + \varphi(t)y$, *and the optimal controls are given by*

$$c^*(t) = \frac{S(t)}{\alpha(t)}(W + \beta(t))\,, \qquad \epsilon^*(t) = \frac{(\mu - r)}{\sigma^2 W}(W + \beta(t))\,,$$

$$Q^*(t) = a\frac{f(t)}{p(t)}\frac{\varphi(t)}{\alpha(t)}(W + \beta(t)) - W\,, \tag{3.17}$$

where

$$\alpha(t) = bS(T)e^{-\rho(T-t)} + \int_t^T \left(e^{-\delta(s-t)}S(s) + e^{-\rho(s-t)}f(s)\right) ds\,, \qquad \varphi(t) = 1\,,$$

$$\beta(t) = \int_t^T e^{-\int_t^s (r+p(v))dv} i(s)ds\,. \tag{3.18}$$

Proof: According to Proposition 1 in de-Paz et al. (2011), a time-consistent solution can be obtained by solving the DPE (3.2), which in this specific case becomes

$$\rho V(y, W, t) - K(W, t) - V_t(y, W, t) = \max_{\{c,\epsilon,Q\}} \{S(t)\ln c(t) + \left[(r + \epsilon(t)(\mu - r))\,W + \right.$$

$$\left. + i(t) - c(t) - p(t)Q(t)\right] V_W(y, W, t) + \left[\rho y + af(t)\ln Z(t)\right] V_y(y, W, t) +$$

$$+\frac{1}{2}\epsilon(t)^2\sigma^2 W^2 V_{WW}(y, W, t)\}\,, \tag{3.19}$$

with $K(W, t)$ given by

$$K(W, t) = (\rho - \delta)E\left[\int_t^T e^{-\delta(s-t)}S(s)\ln c^*(s)ds\right]\,, \tag{3.20}$$

where $c^*(s)$ is the equilibrium consumption rule obtained by solving the right hand side in (3.19). In particular, by applying Corollary 1 in de-Paz et al. (2011), we obtain the system of two coupled partial differential equations

$$\rho V(y, W, t) - K(W, t) - V_t(y, W, t) = \max_{\{c,\epsilon,Q\}} \{S(t)\ln c + \left[(r + \epsilon(\mu - r))\,W + \right.$$

$$+i(t) - c(t) - p(t)Q(t)]\, V_W(y,W,t) + [\rho y + af(t)\ln Z(t)]\, V_y(y,W,t)$$
$$+\frac{1}{2}\epsilon^2\sigma^2 W^2 V_{WW}(y,W,t)\} \,, \tag{3.21}$$

and

$$\delta K(W,t) - K_t(W,t) = (\rho-\delta)S(t)\ln c + [(r+\epsilon(\mu-r))\,W +$$
$$+i(t) - c(t) - p(t)Q(t)]\, K_W(W,t) + \frac{1}{2}\epsilon^2\sigma^2 W^2 K_{WW}(W,t) \,. \tag{3.22}$$

We guess a solution of the form

$$V(y,W,t) = \alpha(t)\ln(W+\beta(t)) + \varphi(t)y \,,$$

with

$$V(y,W,T) = bS(T)\ln(W(T)) + y(T) \,,$$

for the value function. With respect to the function $K(W,t)$, we guess

$$K(W,t) = A(t)\ln(W+\beta(t)) \qquad \text{with} \qquad K(W,T) = 0 \,.$$

If theses choices prove to be consistent, from the maximization problem in (3.21) we have that the guessed optimal policies are given by (3.17). Substituting into (3.21) and (3.22), we obtain that the functions $\alpha(t)$, $\beta(t)$, $\varphi(t)$ must satisfy

$$\dot\alpha(t) - \rho\alpha(t) = A(t) - S(t) - af(t)\varphi(t) \,, \qquad \dot\beta(t) - (\mu + p(t))\beta(t) = i(t) \,,$$

$$\dot\varphi(t) = 0 \,,$$

together with the boundary conditions

$$\alpha(T) = bS(T) \,, \qquad \beta(T) = 0 \,, \qquad \varphi(T) = 1.$$

Solving these equations we get (3.18). □

Finally, with respect to the function $A(t)$, we find that must satisfy

$$\dot A(t) - \delta A(t) = -(\rho-\delta)S(t) \qquad \text{with} \qquad A(T) = 0 \,.$$

Thus,

$$A(t) = \int_t^T e^{-\delta(s-t)}(\rho-\delta)S(s)ds \,.$$

Note that this solution coincides with the *a priori* time-inconsistent solution.

This feature, also arising in non-constant discounting problems (see Marín-Solano and Navas (2010)), is a property of the logarithmic utilities and it is not preserved for more general utility functions such as the power utilities, as we analyze in the next subsection.

3.3.2 Power utility function

Next, let us study the problem for the case of power utilities

$$U(c_s) = \frac{c_s^\gamma}{\gamma}\,, \qquad B(Z(s),s) = a\frac{Z(s)^\gamma}{\gamma}\,, \qquad \text{and} \qquad L(W(T)) = b\frac{W(T)^\gamma}{\gamma}, \tag{3.23}$$

with $\gamma < 1$, $\gamma \neq 0$. As in the previous subsection, we first solve equation (3.13) to obtain the "optimal" solution from the point of view of the agent making decisions at time t and then we distinguish between the case of acting under commitment and acting without commitment.

We guess a value function of the form

$$V^t(y,W,s) = \alpha^t(s)\frac{(W+\beta^t(s))^\gamma}{\gamma} + \varphi^t(s)y\,,$$

with

$$V^t(y,W,T) = e^{-(\rho-\delta)(T-t)}\left(bS(T)\frac{W(T)^\gamma}{\gamma} + y(T)\right)\,.$$

Then, by maximizing the right hand side of equation (3.13) we obtain that the "optimal" control rules satisfy

$$c^t(s) = \left(\frac{\alpha^t(s)}{S(s)}\right)^{\frac{1}{\gamma-1}}(W+\beta^t(s))\,, \qquad \epsilon^t(s) = \frac{(\mu-r)}{\sigma^2W(1-\gamma)}(W+\beta^t(s))\,,$$

$$Q^t(s) = \left(\frac{p(s)}{af(s)}\frac{\alpha^t(s)}{\varphi^t(s)}\right)^{\frac{1}{\gamma-1}}(W+\beta^t(s)) - W\,,$$

where $\alpha^t(s)$, $\beta^t(s)$ and $\varphi^t(s)$ are obtained by substituting the guessed value function together with the corresponding guessed controls into (3.13), and hence given by

$$\beta^t(s) = \int_s^T e^{-\int_s^u (r+p(v))dv} i(u)du\,, \qquad \varphi^t(s) = e^{-(\rho-\delta)(s-t)}\,,$$

$$\alpha^t(s) = \upsilon(s)^{\gamma-1}\left[\int_s^T \upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}} + \left(\frac{ae^{-(\rho-\delta)(u-t)}f(u)}{p(u)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)du + \right.$$

$$+\left(e^{-(\rho-\delta)(T-t)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\Big]^{1-\gamma},$$

with

$$\upsilon(s)=exp\left\{-\frac{1}{1-\gamma}\int_0^s\left(\delta-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\gamma-\gamma r-\gamma p(u)\right)du\right\},$$

so that our guessing for the value function is consistent.

Once again, the function $\beta^t(s)$ does not depend on t (the moment at which the decision is made) and therefore there is no difference between the committed and the time-inconsistent agent. However, both $\alpha^t(s)$ and $\varphi^t(s)$ show the deviation between these two different behaviors. While the agent who is able to commit herself will compute her decision rule according to

$$\alpha^{t_0}(s)=\upsilon(s)^{\gamma-1}\left[\int_s^T\upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}}+\left(\frac{ae^{-(\rho-\delta)(u-t_0)}f(u)}{p(u)^{\gamma}}\right)^{\frac{1}{1-\gamma}}\right)du+\right.$$

$$\left.+\left(e^{-(\rho-\delta)(T-t_0)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\right]^{1-\gamma}$$

and

$$\varphi^{t_0}(s)=e^{-(\rho-\delta)(s-t_0)},$$

the time-inconsistent agent will follow the decisions taken only when they are calculated; so at $s=t$

$$\alpha^{t}(t)=\upsilon(t)^{\gamma-1}\left[\int_t^T\upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}}+\left(\frac{ae^{-(\rho-\delta)(u-t)}f(u)}{p(u)^{\gamma}}\right)^{\frac{1}{1-\gamma}}\right)du+\right.$$

$$\left.+\left(e^{-(\rho-\delta)(T-t)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\right]^{1-\gamma},$$

and $\varphi(t)=1$.

With respect to the time-consistent solution, we have:

Proposition 4 *Assume that $U(c_s)$, $B(Z(s),s)$, and $L(W(T))$ are given by (3.23). Then*

$$V(y,W,t)=\alpha(t)\frac{(W+\beta(t))^{\gamma}}{\gamma}+\varphi(t)y\,,\qquad K(W,t)=A(t)\frac{(W+\beta(t))^{\gamma}}{\gamma},$$

and the optimal controls are given by

$$c^*(t) = \left(\frac{\alpha(t)}{S(t)}\right)^{\frac{1}{\gamma-1}} (W + \beta(t)) , \qquad \epsilon^*(t) = \frac{(\mu - r)}{\sigma^2 W(1-\gamma)}(W + \beta(t)) ,$$

$$Q^*(t) = \left(\frac{p(t)}{af(t)}\frac{\alpha(t)}{\varphi(t)}\right)^{\frac{1}{\gamma-1}} (W + \beta(t)) - W , \tag{3.24}$$

where

$$\beta(t) = \int_t^T e^{-\int_t^s (r+p(v))dv} i(s)ds , \qquad \varphi(t) = 1 , \tag{3.25}$$

while functions $\alpha(t)$ and $A(t)$ are the solution to the following system of differential equations

$$\rho\alpha(t) - A(t) - \dot{\alpha}(t) =$$

$$\alpha(t)^{\frac{\gamma}{\gamma-1}} \left[(1-\gamma)\left(S(t)^{\frac{1}{1-\gamma}} + \left(\frac{af(t)}{p(t)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)\right] + \gamma\alpha(t)\left[\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)} + r + p(t)\right] , \tag{3.26}$$

$$\delta A(t) - \dot{A}(t) = (S(t))^{\frac{1}{1-\gamma}} \left[(\rho-\delta)\alpha(t)^{\frac{\gamma}{\gamma-1}} - \gamma\alpha(t)^{\frac{1}{\gamma-1}}A(t)\right] +$$

$$+\gamma A(t)\left[\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)} + r + p(t) - p(t)\left(\frac{p(t)}{af(t)}\alpha(t)\right)^{\frac{1}{\gamma-1}}\right] , \tag{3.27}$$

with $\alpha(T) = bS(T)$, and $A(T) = 0$.

Proof: To obtain the time-consistent solution we must solve the DPE (3.2). Specifically, according to Corollary 1 in de-Paz et al. (2011) we must solve the set of DPE

$$\rho V(y, W, t) - K(W, t) - V_t(y, W, t) =$$

$$= \max_{\{c,\epsilon,Q\}} \left\{ S(t)\frac{c^\gamma}{\gamma} + \left[(r + \epsilon(\mu - r))W + i(t) - c(t) - p(t)Q(t)\right] V_W(y, W, t) + \right.$$

$$\left. + \left(\rho y + af(t)\frac{c^\gamma}{\gamma}\right) V_y(y, W, t) + \frac{1}{2}\epsilon^2\sigma^2W^2V_{WW}(y, W, t)\right\} , \tag{3.28}$$

$$\delta K(W, t) - K_t(W, t) = (\rho - \delta)S(t)\frac{c^\gamma}{\gamma} + \left[(r + \epsilon(\mu - r))W + \right.$$

$$\left. + i(t) - c(t) - p(t)Q(t)\right] K_W(W, t) + \frac{1}{2}\epsilon^2\sigma^2W^2K_{WW}(W, t) . \tag{3.29}$$

We try as a candidates for the value function and to the function $K(W,t)$

$$V(y,W,t)=\alpha(t)\frac{(W+\beta(t))^{\gamma}}{\gamma}+\varphi(t)y\,,\qquad \text{and}\qquad K(W,t)=A(t)\frac{(W+\beta(t))^{\gamma}}{\gamma}\,,$$

respectively. Maximizing the right hand side of (3.28) we obtain (3.24). Finally, substituting the guessed functions and the corresponding optimal controls in (3.28-3.29), together with the terminal conditions

$$V(y,W,T)=y(T)+bS(T)\frac{W(T)^{\gamma}}{\gamma}\,,\qquad \text{and}\qquad K(W,T)=0\,,$$

we obtain that functions $\beta(t)$ and $\varphi(t)$ are given by (3.25), while functions $\alpha(t)$ and $A(t)$ are the solution to the system of differential equations (3.26-3.27). □

3.3.3 Exponential utility function

Finally, let us solve the problem for the case of (constant absolute risk aversion) exponential utility functions

$$U(c_s)=\frac{-1}{\gamma}e^{-\gamma c_s}\,,\qquad B(Z(s),s)=\frac{-a}{\gamma}e^{-\gamma Z(s)}\,,\quad \text{and}\quad L(W(T))=-be^{-\gamma W(T)},\tag{3.30}$$

with $\gamma>0$. Once again, we first derive the "optimal" rules for the point of view of an agent deciding at $t\in[t_0,T]$ by means of equation (3.13). Then we differentiate between the agent who is able to commit herself and the time-inconsistent agent.

By guessing

$$V^t(y,W,s)=-ae^{-\gamma(\alpha^t(s)+\beta^t(s)W)}+\varphi^t(s)y\,,$$

the maximization problem in (3.13) gives

$$c^t(s)=\alpha^t(s)+\beta^t(s)W-\frac{1}{\gamma}\ln\left(\frac{a\gamma\beta^t(s)}{S(s)}\right)\,,\quad \epsilon^t(s)=\frac{(\mu-r)}{\sigma^2\gamma\beta^t(s)W}\,,\tag{3.31}$$

$$Q^t(s)=\alpha^t(s)+\beta^t(s)W-\frac{1}{\gamma}\ln\left(\frac{p(s)}{f(s)}\frac{\gamma\beta^t(s)}{\varphi^t(s)}\right)-W\,.\tag{3.32}$$

We substitute (3.31) and (3.32) in (3.13), and after several calculations, we obtain that the functions $\alpha^t(s)$, $\beta^t(s)$, and $\varphi^t(s)$ satisfy

$$\alpha^t(s) = \frac{-1}{\gamma}\left[\left(\ln\left(\frac{b}{a}S(T)\right) - (\rho-\delta)(T-t)\right)e^{-\int_s^T (1+p(v))\beta^t(v)dv} + \right.$$

$$\left. + \int_s^T \left(\vartheta^t(u) e^{-\int_s^u (1+p(v))\beta^t(v)dv}\right) du\right],$$

$$\beta^t(s) = \frac{1}{e^{-\int_s^T (r+p(v))dv} + \int_s^T \left((1+p(u))e^{-\int_s^u (r+p(v))dv}\right) du}, \tag{3.33}$$

and

$$\varphi^t(s) = e^{-(\rho-\delta)(s-t)}$$

where

$$\vartheta^t(u) = \beta^t(u)\left[1 + p(u) + \ln\left(\frac{a\gamma\beta^t(u)}{S(u)}\right) - p(u)\ln\left(\frac{p(u)}{f(u)}\frac{\gamma\beta^t(u)}{\varphi^t(u)}\right) - i(u)\gamma\right] -$$

$$-\delta - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}.$$

Although the function $\beta^t(s)$ does not depend on t, and hence

$$\beta^t(s) = \beta^{t_0}(s) = \beta^s(s),$$

the "optimal" policies change depending on whether the agent reconsiders her previous decisions or is committed with the initial ones. In the last case, the decision maker will compute her policies according to

$$\alpha^{t_0}(s) = \frac{-1}{\gamma}\left[\left(\ln\left(\frac{b}{a}S(T)\right) - (\rho-\delta)(T-t_0)\right)e^{-\int_s^T (1+p(v))\beta(v)dv} + \right.$$

$$\left. + \int_s^T \left(\vartheta^{t_0}(u) e^{-\int_s^u (1+p(v))\beta(v)dv}\right) du\right],$$

and

$$\varphi^{t_0}(s) = e^{-(\rho-\delta)(s-t_0)}.$$

If she is not committed with the initial decisions, she will be continuously modifying her calculated choices for the future. Consequently $\alpha^t(s)$ and $\varphi^t(s)$ will be only obeyed at $s = t$, i.e.,

$$\alpha(t) = \frac{-1}{\gamma}\left[\left(\ln\left(\frac{b}{a}S(T)\right) - (\rho-\delta)(T-t)\right)e^{-\int_t^T (1+p(v))\beta(v)dv} + \right.$$

$$+\int_t^T \left(\vartheta(u)e^{-\int_t^u (1+p(v))\beta(v)dv}\right) du \Bigg] \,,$$

and $\varphi(t) = 1$.

Next, let us derive the time-consistent solution.

Proposition 5 *Assume that $U(c_s)$, $B(Z(s), s)$, and $L(W(T))$ are given by (3.30). Then*

$$V(y, W, t) = -ae^{-\gamma(\alpha(t)+\beta(t)W)} + \varphi(t)y \,, \qquad K(W, t) = A(t)e^{-\gamma(\alpha(t)+\beta(t)W)} \,,$$

and the optimal controls are given by

$$c^*(t) = \alpha(t) + \beta(t)W - \frac{1}{\gamma}\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right) \,, \quad \epsilon^*(t) = \frac{(\mu - r)}{\sigma^2\gamma\beta(t)W} \,,$$

$$Q^*(t) = \alpha(t) + \beta(t)W - \frac{1}{\gamma}\ln\left(\frac{p(t)}{f(t)}\frac{\gamma\beta(t)}{\varphi(t)}\right) - W \,, \tag{3.34}$$

where

$$\beta^t(s) = \frac{1}{e^{-\int_s^T (r+p(v))dv} + \int_s^T \left((1 + p(u))e^{-\int_s^u (r+p(v))dv}\right) du} \,, \qquad \varphi(t) = 1 \,, \tag{3.35}$$

while functions $\alpha(t)$ and $A(t)$ are the solution to the following system of differential equations

$$a\gamma\dot{\alpha}(t) + a\rho + A(t) = a\beta(t)(1 - \gamma p(t)) - a\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + \tag{3.36}$$

$$+\left[\alpha(t)(1 + p(t)) - \frac{1}{\gamma}\left(\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right) + p(t)\ln\left(\frac{p(t)}{f(t)}\gamma\beta(t)\right)\right) + i(t)\right] a\gamma\beta(t) \,,$$

$$\dot{A}(t) - \delta A(t) - \gamma A(t)\dot{\alpha}(t) = a(\rho - \delta)\beta(t) + \frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2}A(t) - \tag{3.37}$$

$$-\left[\alpha(t)(1 + p(t)) - \frac{1}{\gamma}\left(\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right) + p(t)\ln\left(\frac{p(t)}{f(t)}\gamma\beta(t)\right)\right) - i(t)\right] \gamma A(t)\beta(t) \,,$$

with $\alpha(T) = \dfrac{-1}{\gamma}\ln\left(\dfrac{b}{a}S(T)\right)$, and $A(T) = 0$.

Proof: According to Corollary 1 in de-Paz et al.(2011), Markov Perfect Equilibria can be obtained by solving the set of two coupled PDE

$$\rho V(y,W,t) - K(W,t) - V_t(y,W,t) = \tag{3.38}$$

$$\max_{\{c,\epsilon,Q\}} \left\{ \frac{-1}{\gamma} e^{-\gamma c} S(t) + \left[\left(r + \epsilon(\mu - r) \right) W + i(t) - c(t) - p(t)Q(t) \right] V_W(y,W,t) + \right.$$

$$\left. + \left(\rho y + f(t) \frac{-a}{\gamma} e^{-\gamma(W+Q)} \right) V_y(y,W,t) + \frac{1}{2}\epsilon^2\sigma^2 W^2 V_{WW}(y,W,t) \right\},$$

$$\delta K(W,t) - K_t(W,t) = (\rho - \delta)\frac{-1}{\gamma} e^{-\gamma c} S(t) + \tag{3.39}$$

$$+ \left[\left(r + \epsilon(\mu - r) \right) W + i(t) - c(t) - p(t)Q(t) \right] K_W(W,t) + \frac{1}{2}\epsilon^2\sigma^2 W^2 K_{WW}(W,t).$$

We guess as a candidate to the value function

$$V(y,W,t) = -a e^{-\gamma(\alpha(t)+\beta(t)W)} + \varphi(t) y,$$

and with respect to $K(W,t)$ we try

$$K(W,t) = A(t) e^{-\gamma(\alpha(t)+\beta(t)W)}.$$

If these choices proves to be consistent, then from (3.38) we get (3.34). By substituting in (3.38-3.39) and collecting terms in W, on the one hand, and collecting terms in x, on the other hand, we get that $\beta(t)$ and $\varphi(t)$ are given by (3.35). With respect to the functions $\alpha(t)$ and $A(t)$, we obtain that they must be the solution to the system of differential equations (3.36-3.37). □

3.4 Numerical illustrations

In this section we provide some numerical examples to illustrate the results for the case of power utility functions. As a baseline case, we consider a 25 years old agent endowed with an initial wealth of 1000 euros and with an initial wage of 25000 euros which grows at 3% every year until $T = 65$, when the agent retires. The agent exhibits a risk aversion parameter of $\gamma = -3$ and her heterogeneous preferences are characterized by $\delta = 0.03$ and $\rho = 0.1$. We assume that the individual is subject to an instantaneous force of mortality or hazard rate given by the Gompert law of mortality

$$\lambda(t) = \frac{1}{h} e^{\frac{(t-\eta)}{h}} ,$$

with $t \geq 0$. Following Milevsky (2006), we take $\eta = 82.3$ and $h = 11.4$. Due to the well-known relationship between the hazard rate and the density and survivor probability functions we have

$$f(t) = \lambda(t) e^{-\int_0^t \lambda(s) ds} , \qquad S(t) = e^{-\int_0^t \lambda(s) ds} .$$

Regarding the life insurance market, we assume that the insurance company sets the premium in order to make a profit. In general, the insurance is said to be actuarially fair when the expected profit rate equals 0, which in this case means $p(t) = \lambda(t)$. Consequently, in order to be profitable the insurance company must charge a loading factor θ accounting for the percentage markup from the fair value of insurance, i.e.,

$$p(t) = (1 + \theta)\lambda(t) .$$

For this particular example we consider $\theta = 10\%$ so that the premium per euro of coverage at age t is

$$p(t) = (1 + 0.1)\lambda(t) .$$

Finally, we assume that the risk-free asset yields a return of $r = 0.03$ while the risky security has an expected return of $\mu = 0.09$ and volatility $\sigma = 0.3$.

Before comparing our solutions with the standard solutions, note that the agent makes all her decisions according to her total available wealth (her current wealth $W(t)$ plus the present value of her future earnings $\beta(t)$). Although the present value of future earnings has a positive effect in all the control variables, Figure 1 shows that in this case the current wealth has a negative effect on the total amount of insurance purchased, i.e., the more wealthy the agent is, the less life insurance she purchases. However, since the wage earner has a a small current wealth relative to her future earnings, she depends on her wages to make her decisions. Figure 2 shows the present value of future earnings, two possible trajectories of the total available wealth together with the corresponding time-consistent life insurance rule, and their expected values.

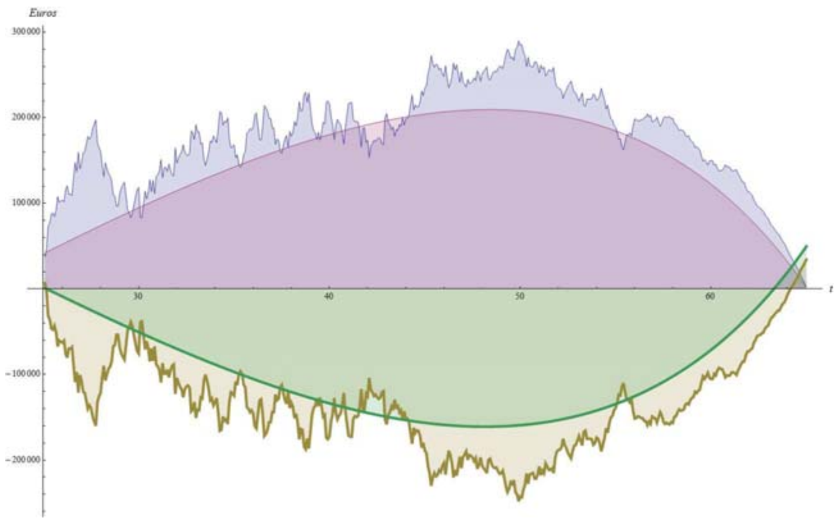

Figure 3.1: Simulated $W(t)$ (thick), expected $W(t)$ (thick), simulated time-consistent life insurance rule (thin) and expected time-consistent life insurance rule (thin).

Note that in spite of the negative current wealth, the amount of life insurance purchased is enough to leave a positive bequest if premature death occurs.

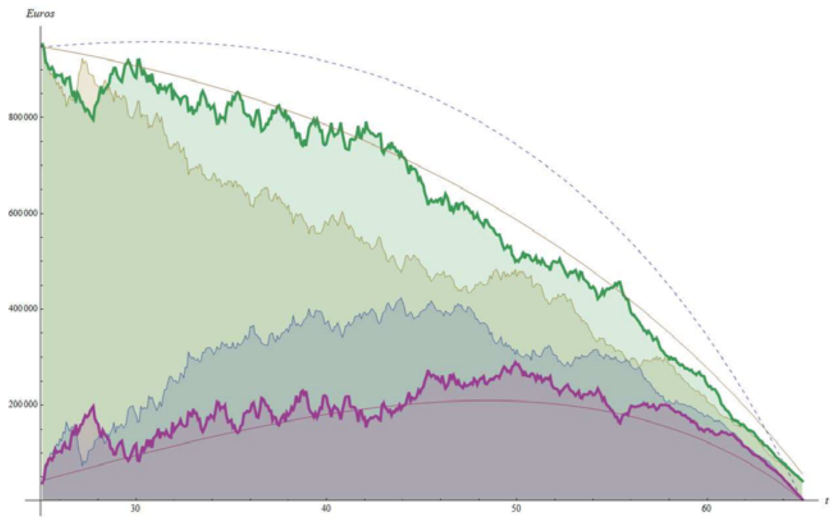

Figure 3.2: Present value of future income (dashed) and two possible trajectories of the total available wealth and the corresponding time-consistent life insurance rule (the thickness reflects the correspondence). Smooth lines represents the expected values.

For this reason the following comparisons are made for different values of initial wage. Figures 3 to 7 show the differences between standard and heterogeneous behaviors. At the beginning of the planning horizon, the wage earner with heterogeneous discounting is more impatient than the agent with standard discounting, since we have assumed $\rho > \delta$. However, as time goes on the bias to the present decreases as her concern about her bequest and her retirement increases. In order to highlight how the heterogeneous preferences evolve over time, we focus first on the differences with the standard case from the point of view of a 25 years old agent who is able to commit herself with the decisions initially taken. Then we look at how these differences change if the agent reconsiders her choices at any time (time-inconsistency), and we end by analyzing the differences from the point of view of the time-consistent agent.

On the one hand, if the wage earner does not modify the decisions made at the age of 25, when she underestimates the bequest left to her descendants and her wealth at retirement, one should expect her to purchase less life insurance and to consume more than the standard agent. On the other hand, if the agent does not commit herself, her policies should change according to her preferences at different ages. Therefore, she should purchase more life insurance and consume less than the committed agent. Finally, although the time-consistent agent also overvalues the instantaneous utilities at the beginning of the planing horizon, she knows that her preferences are going to change in the near future. In this case, her policies should reflect the equilibrium between her preferences at different times.

Figure 3 shows the difference of the life insurance purchased by the committed 25 years old agent and the standard discounting case. Departing from a similar level of life insurance purchased, the difference is negative from that moment until the ages close to the retirement date, when it becomes positive. This means that the individual using the heterogeneous discount function postpones the purchase of life insurance when she is 25 years old to the later adulthood. Note that the deviation attains the maximum length around the age of 50 and decreases from that point onwards. In addition, for a given age, an increasing initial wage leads the agent to buy more life insurance under the standard preferences than under the heterogeneous ones, except at ages closer to 65 years. In Figure 4, we compare the life insurance purchased by the time-inconsistent agent and the standard solution. In this case, the difference is positive since the agent reconsiders her choice at each time point according to her increasing concern about the bequest. The comparison of the time-consistent and standard behaviors is shown in Figure 5. The difference is also positive although it is larger than the difference in Figure 4, i.e., time-consistent planning leads the agent to buy more life insurance than the time-inconsistent one. Note that, in contrast to the committed agent, the agent with heterogeneous discounting (both the time-inconsistent and the time-consistent) reacts to an increase in her salary by buying more life insurance than the standard agent.

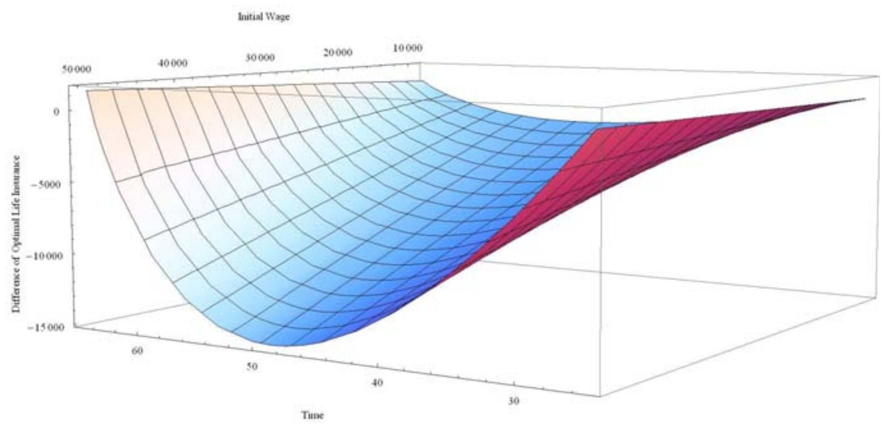

Figure 3.3: Difference of optimal life insurance between the heterogeneous agent committed with her preferences at the age of 25 and the standard agent for different values of initial wage.

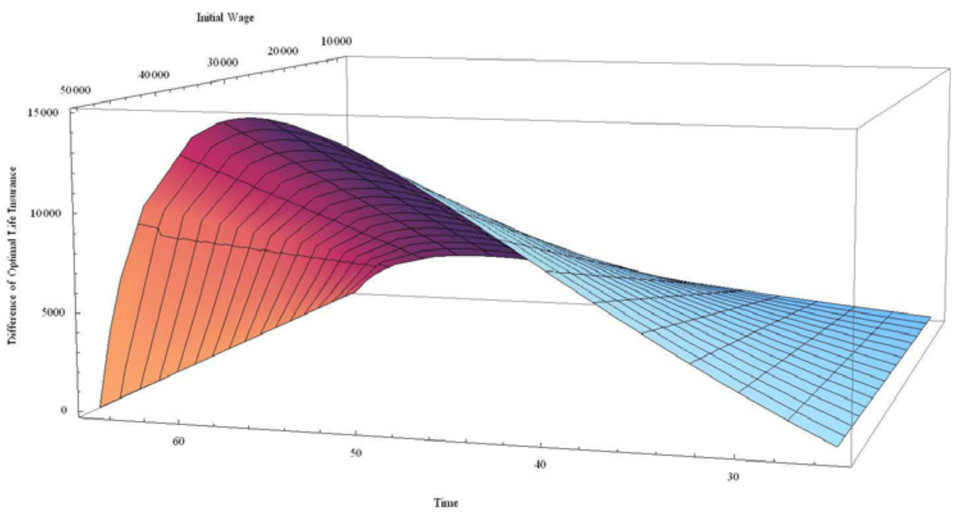

Figure 3.4: Difference of optimal life insurance between the time-inconsistent heterogeneous agent and the standard agent for different values of initial wage.

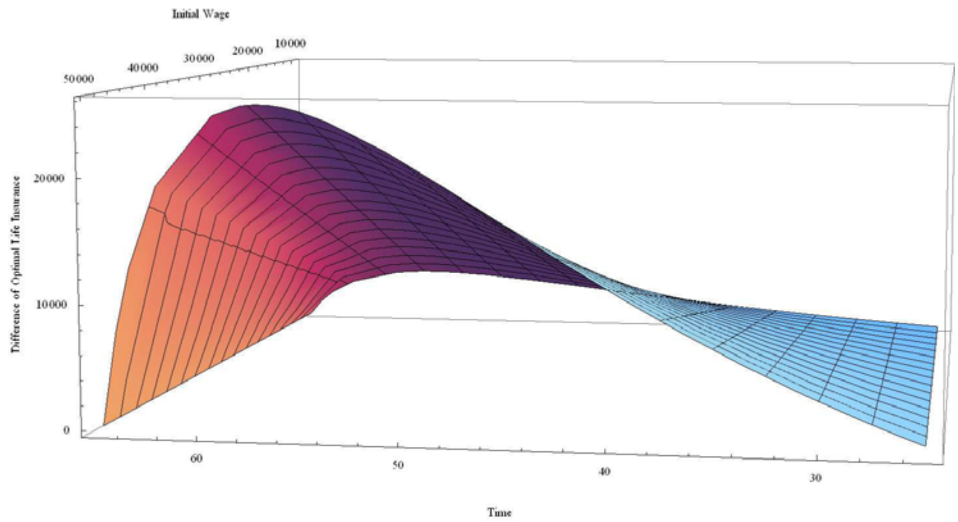

Figure 3.5: Difference of optimal life insurance between the time-consistent heterogeneous agent and the standard agent for different values of initial wage.

In figure 6 we show the life insurance paths (simulated and expected values) for the committed, the time-inconsistent and the time-consistent agent and for the baseline initial wage (25000 euros).

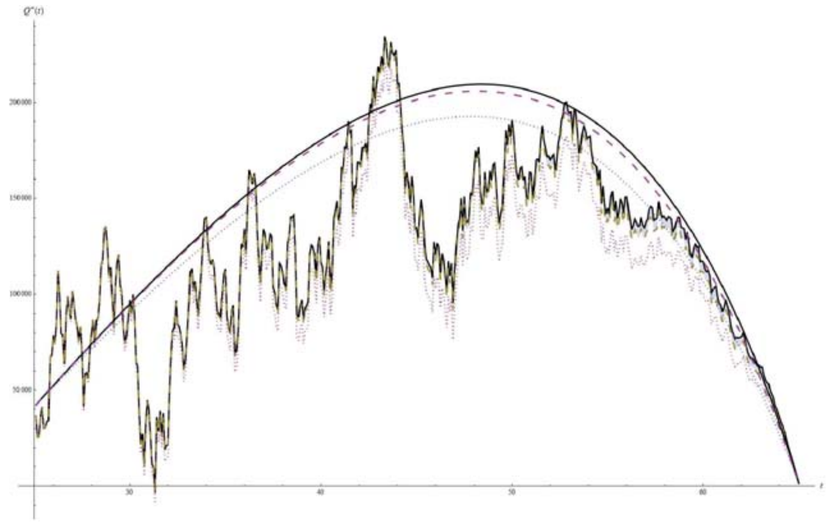

Figure 3.6: Comparison of the optimal life insurance purchase for time-consistent (solid), time-inconsistent (large dashing) and committed 25-years old (small dashing) agents.

Figure 7 highlights the deviation of consumption patterns for different initial wages. Consumption brings immediate benefit so the heterogeneous agent, who is more impatient, decides to allocate larger amounts to consumption at least in the first periods. The committed agent ends up allocating larger amounts to consumption than the standard agent at all ages, since her path reflects the preferences from the perspective of the 25 years old. The time-inconsistent wage earner starts consuming more than the standard. However, as time goes on she modifies (reduces) her previous choices according to her decreasing bias to the present. As a result, her consumption path intersects the standard one between the ages of 45 and 50, and ends in a lower level. Finally, the time-consistent trajectory starts above the other three solutions and ends below them. This is so because this agent makes her plan knowing how her preferences are going to evolve and she decides to take advantage of the different levels of impatience at each time point. Thus, her consumption is greater while she more impatient, since she knows that in the future her preferences will lead her to consume less. Observe that an increase in the initial wage shifts the curves upwards though, unlike the life insurance purchase, it hardly modifies the differences between the four consumption paths.

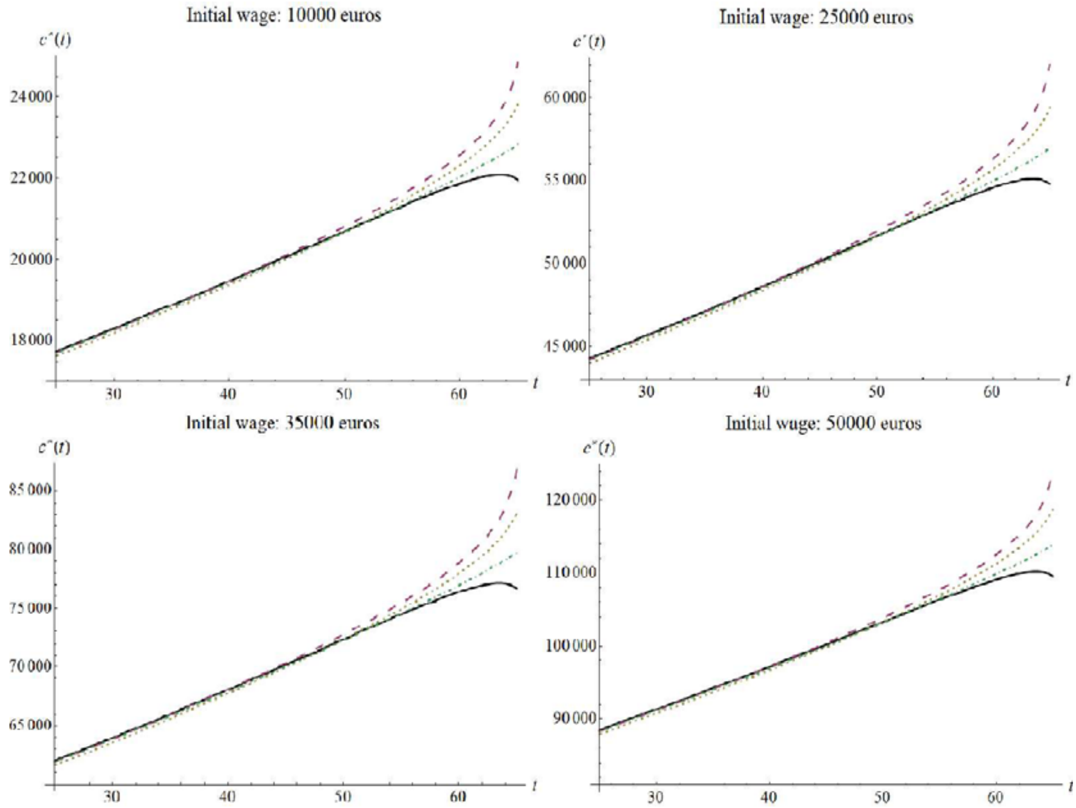

Figure 3.7: Consumption paths for the standard case (dotted), the committed agent (large dashing), the time-inconsistent agent (dot-dahsed) and the time-consistent agent (solid).

To conclude this section, we analyze how the time-consistent life insurance and consumption rules are modified when we vary the heterogeneous preferences. Figures 8 and 9 show that the previous results are intensified if the discount rate for the final function ρ is increased. For $\delta = 0.03$ we plot the different paths taking $\rho = 0.06$, $\rho = 0.15$ and $\rho = 0.2$. In particular, Figure 8 shows that the life insurance purchase increases with ρ, while Figure 9 shows that the consumption path rotates as ρ increases, i.e., the agent consumes more when she is young and less when she is older.

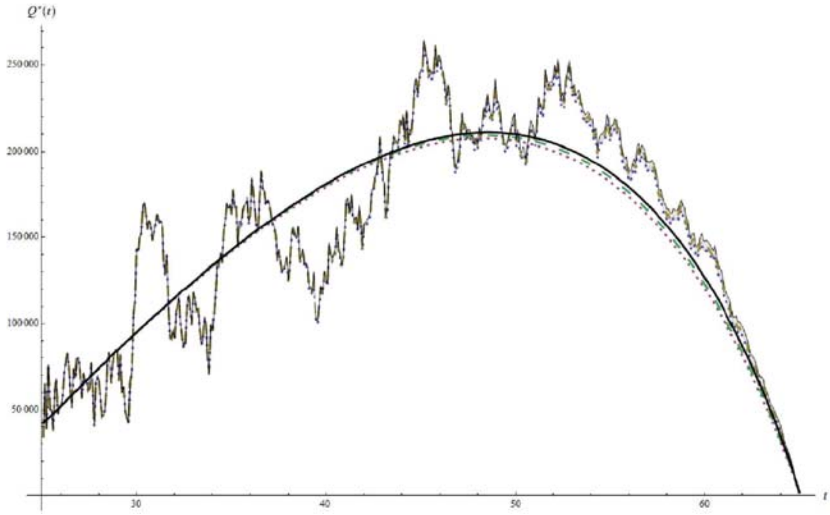

Figure 3.8: Sensitivity of the time-consistent life insurance for different values of ρ, $\rho = 0.06$ (small dashing), $\rho = 0.15$ (large dashing), $\rho = 0.2$ (solid).

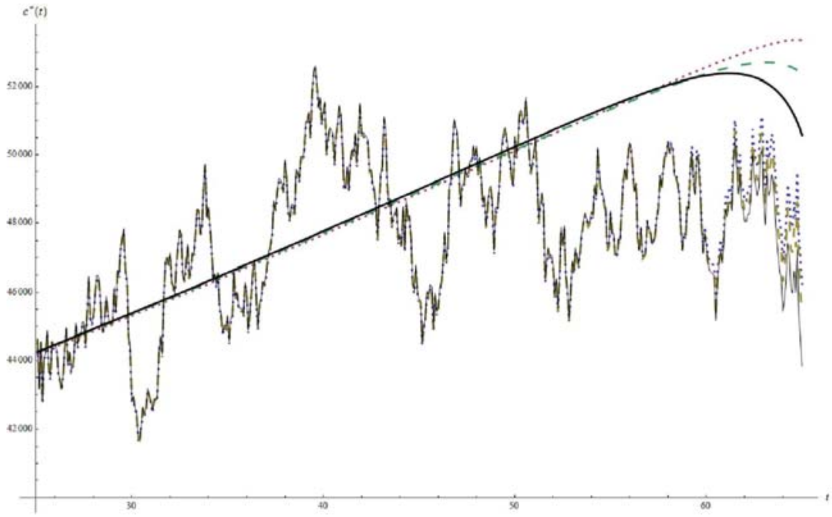

Figure 3.9: Sensitivity of the time-consistent consumption for different values of ρ, $\rho = 0.06$ (small dashing), $\rho = 0.15$ (large dashing), $\rho = 0.2$ (solid).

3.5 Conclusions

In this chapter, we have studied the effects of introducing heterogeneous discounting into a stochastic continuous time model with random lifetime in which the wage earner decides between three different strategies: consumption, investment and life insurance purchase. In contrast with the standard case, heterogeneous preferences capture the different valuations that the individual gives to the bequest left to her descendants and to her wealth at retirement along the planning horizon. Consequently, the optimal policies for an agent using the heterogeneous discount function differ from those for an agent with standard discounting. In order to illustrate these effects, we have departed from the model in Pliska and Ye (2007) generalizing the individual time preferences with the heterogeneous discount function introduced by Marín-Solano and Patxot (2012). In addition, we have derived the wealth dynamics in terms of the portfolio elasticity (Kraft (2003)). This procedure allows us to generalize the investment problem by introducing contingent claims in the opportunity set while maintaining the analytical tractability of the model. Explicit solutions have been obtained for the case of CRRA and CARA utility functions for both the time-inconsistent and the time-consistent agent. The implications of the use of the heterogeneous discount function have been illustrated, showing the differences between our results and the standard ones.

Chapter 4

Time-Consistent Equilibria in a Cooperative Differential Game with Heterogeneous Agents

4.1 Introduction

In the analysis of intertemporal decision problems with several agents, when players can communicate and coordinate their strategies in order to optimize their collective pay-off, cooperative solutions are introduced. Although the natural framework for most economic problems is to assume that the agents compete among each other, in some models -for instance, those related to the analysis of international trade agreements, topics in environmental economics concerning climate change policies, or the exploitation of common property natural resources; see Jorgensen et al (2010) and Long (2011) for two recent surveys on dynamic games in these topics- it is natural to look for mechanisms inducing cooperation between economic agents (see e.g. Breton and Keoula (2011) and references therein for a recent study of coalition formation and stability of coalitions in resource economics).

Although it is customary to assume that all economic agents have the same rate of time preference, there is no reason to believe that consumers, firms or countries have identical time preferences for utility streams (see e.g. Jouini et al (2010) and references therein). For instance, in a non-cooperative setting, for the problem of extraction of exhaustible resources under common access, feedback Nash equilibria have been studied in the case of equal (Clemhout and Wan (1985)) and different (Long et al (1999)) discount rates. With respect to the

Pareto optimum in the cooperative framework, if there is a unique (constant) discount rate for all agents, it is easily obtained by solving a standard optimal control problem. However, in the case of different discount rates, when looking for time-consistent cooperative solutions, standard dynamic optimization techniques fail. The reason is that time preferences become time-inconsistent, as in the case of hyperbolic preferences. In Gollier and Zeckhauser (2005) effects of aggregation of heterogeneous time preferences were studied by assuming that there is a representative agent and that agents can commit to their future consumption plan at date $t = 0$ (this is the so called precommitment solution according to the literature of non-constant discounting). Li and Löfgren (2000) characterized long-run steady states for a renewable resource model with two agents under similar assumptions. If we remove the commitment assumption, time-consistent policies can be computed by solving the dynamic programming equation (DPE) first derived in Karp (2007). This chapter aims to fill the gap in the search for time-consistent solutions in a cooperative continuous time setting if agents are heterogeneous, in the sense that their preferences are represented by different utility functions (there is not a representative agent) and they also use different discount rates. It is important to realize that when agents lack commitment power, they act at different times t as sequences of independent coalitions (the t-coalitions). The solution we compute assumes cooperation among players at every time t, but is a non-cooperative equilibrium for the non-cooperative sequential game defined by these infinitely many t-coalitions.

In recent years, papers departing from standard discounting have received increasing attention. Strotz (1956) called attention to the problem of time inconsistency arising when non-constant discount rates of time preference are introduced. We refer to Frederick et al (2002) for a review of the literature up to 2002. Time-inconsistency also arises in problems where the decision-maker discounts instantaneous utilities and final gains in a different way. Equilibrium conditions for time-consistent solutions have recently been obtained for both kind of problems in a continuous time setting (see Karp (2007) for the case of non-constant discounting, and Marín-Solano and Patxot (2011) for the problem with heterogeneous discounting).

As we have mentioned above, despite the fact that non-standard discounting models have focused on individual agents, this framework has proved to be useful in the study of multi-agent problems if decision-makers cooperate among them (although the different t-coalitions act in a non-cooperative way). If players share the same joint instantaneous utility function (there is a representative agent) but have different rates of time preference, say $r_1 \neq \cdots \neq r_N$, the cooperative problem can be rewritten as a non-constant discounting problem and previous results in the literature can be applied in order to obtain a time-consistent (subgame perfect) solution (see Remark 2 in Karp (2007)) as follows. Let us consider an N-player cooperative differential game where, as usual, the joint coalition maximizes the weighted sum of their respective pay-offs,

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \lambda_m J^m \,,$$

where

$$J^m = \int_t^T e^{-r_m s} U^m \left(x(s), c(s), s\right) ds$$

represents the individual pay-off of player m, $\lambda_m \geq 0$ characterizes the weight of player m in the coalition, and $x(t)$ and $c(t)$ are the vectors of state and control variables. Thus, the joint payoff is

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \lambda_m \int_t^T e^{-r_m(s-t)} U^m \left(x(s), c(s), s\right)\, ds \,.$$

If there is a representative agent we can write the (joint) utility function as $U(x, c, s)$, and the payoff for the group can be rewritten as

$$J\left(c(\cdot)\right) = \int_t^T \theta(s-t) U\left(x(s), c(s), s\right) ds \,,$$

where

$$\theta(s-t) = \sum_{m=1}^{N} \lambda_m e^{-r_m(s-t)}$$

is the discount function, which can be also rewritten as

$$\theta(s-t) = e^{-\int_t^s \bar{r}(\tau - t) d\tau} = e^{-\int_0^{s-t} \bar{r}(\tau) d\tau}$$

where the time preference rate $\bar{r}(\tau)$ is a non-constant function of its argument,

$$\bar{r}(\tau) = -\frac{\theta'(\tau)}{\theta(\tau)} = \frac{\sum_{m=1}^{N} \lambda_m r_m e^{-r_m \tau}}{\sum_{m=1}^{N} \lambda_m e^{-r_m \tau}} \,.$$

For $N = 2$, this non-constant discounting model has been applied to study a model of catastrophic climate-related damages in Karp and Tsur (2011).

In this chapter, we tackle the more general problem that consists in maximizing

$$J\left(c(\cdot)\right)=\sum_{m=1}^{N}\lambda_m\int_t^T e^{-r_m(s-t)}U^m\left(x(s),c(s),s\right)ds \tag{4.1}$$

subject to

$$\dot{x}(s)=f(x(s),c(s),s),\qquad x(t)=x_t\,. \tag{4.2}$$

Hence, we focus on the case when agents exhibit different instantaneous pay-off functions and different (but constant) rates of time preference. This problem cannot be transformed into a problem with non-constant discounting.

There are two sources of time-inconsistency in Problem (4.1-4.2). First, there is the time-consistency problem related to the changing time preferences of the different t-coalitions, as we have discussed in the previous paragraphs. In addition, if players are not committed themselves to cooperate at every instant of time t, a problem of dynamic inconsistency or time-inconsistency (both words are synonymous and are used indistinctly in the literature of cooperative differential games) can arise, independently of the form of the discount function: it is possible that players initially agree on a cooperative solution that generates incentives for them, but it is profitable for some of them to deviate from the cooperative behavior at later periods. Haurie (1976) proved that the extension of the Nash bargaining solution to differential games is typically not dynamically consistent. We refer to Zaccour (2008) for a recent review on the topic. For the case of transferable utilities, if the agents can redistribute the joint payoffs of players in any period, Petrosyan proposed in a series of papers a payoff distribution procedure in order to solve this problem of dynamic inconsistency (see e.g. Yeung and Petrosyan (2006) or Petrosyan and Zaccour (2003) and references therein). If transferable utilities or payoff distributions are not allowed, Sorger (2006) proposed, for the problem with heterogeneous agents in multiperiod (discrete time) problems, the concept of recursive Nash bargaining solution, which is a dynamically consistent equilibrium. This solution assigns different weights $\lambda_m(x)$ to the different players. Since weights are non constant but depend on the state variable x, they evolve along time. We do not consider this issue of dynamic consistency (related to the stability of the whole coalition) in this work. Throughout the chapter we assume that the agents commit themselves to cooperate at every instant of time t. However, whereas the approach in Sorger (2006) is different to ours, if utilities are transferable, payoff (imputation) distribution procedures can be introduced, extending in a rather easy way this method to our problem with heterogeneous agents, as in the case of differential games with non-constant discounting (see Marín-Solano and Shevkoplyas (2011)).

Our main contributions are the following. First, for a finite horizon two person cooperative differential game, we introduce a computationally tractable approach based in transforming the problem into a one-agent problem with heterogeneous discounting (see Marín-Solano and Patxot (2011)). As a result, we must solve two coupled DPEs. A second approach enables us to study problems with an arbitrary number of players. In the derivation of the DPE we adopt the procedure given in Karp (2007) for the non-constant discounting problem. And third, we apply the approach in Marín-Solano and Shevkoplyas (2011) for the analysis of the problem in an infinite horizon setting.

While our contributions are mainly methodological, we illustrate the effects of using different discount rates by solving an exhaustible resource extraction model with common access (see e.g. Dockner et al (2000)), and a basic common property renewable natural resource model (see e.g. Clark (1990)). We prove that, for these problems, if all the agents have the same parameter σ in their utility functions

$$U^i(c_i) = \frac{c_i^{1-\sigma_i} - 1}{1-\sigma_i}\,,$$

the extraction rates of all agents in the time-consistent solution coincide. A similar result has recently been obtained in a discrete time setting in a fisheries model in the limit $\sigma = 1$ for a logarithmic utility function (see Breton and Keoula (2010)).

The chapter is organized as follows. In Section 2, we study a general cooperative problem for the two-player case in finite horizon. We transform the problem into a heterogeneous discounting model and derive the corresponding DPEs. Next we discuss the issue of time (in)consistency through a nonrenewable resource extraction problem. In Section 3, we extend the two-player case to the N-player case, and obtain the corresponding equilibrium conditions. We illustrate the main result by solving an exhaustible resource extraction model. The extension to the infinite time horizon setting is studied in Section 4, and we extend our previous results to a common access renewable resource model. Finally, Section 5 presents a summary of the main results of the chapter.

4.2 The case of two heterogeneous agents

Heterogeneous discounting problems were introduced in Marín-Solano and Patxot (2011) in order to study problems where the agent discounts in a different way the utilities enjoyed along the planning horizon (typically due to consumption) and the final function (which has normally a different nature), i.e., the decision-maker faces the problem of maximizing

$$\int_t^T e^{-r_1(s-t)} U(x(s), c(s), s)\, ds + e^{-r_2(T-t)} F(x(T), T) \tag{4.3}$$

subject to

$$\dot{x}(s) = f(x(s), c(s), s), \qquad x(t) = x_t\ . \tag{4.4}$$

We refer to Marín-Solano and Patxot (2011) for an economic motivation of Problem (4.3-4.4), as well as a discussion on the time-inconsistency of these time preferences.

Next, for the two-player case, $N = 2$, we connect our cooperative problem with an heterogeneous discounting problem. In order to do this, we rewrite the functional objective for one of players in the Mayer form, in such a way that Problem (4.1-4.2) for the t-coalition becomes equivalent to the problem of maximizing

$$\lambda_1 \int_t^T e^{-r_1(s-t)} U^1\left(x(s), c_1(s), c_2(s), s\right) ds + \lambda_2 e^{-r_2(T-t)} y(T)$$

subject to:

$$\dot{x}(s) = f(x(s), c_1(s), c_2(s), s)\ , \quad \dot{y}(s) = r_2 y(s) + U^2\left(x(s), c_1(s), c_2(s), s\right)\ .$$

With the addition of a new state variable y, we transform the cooperative problem with asymmetric players into a Bolza problem for just one agent with integral and terminal value terms, but with different time preferences rates.

Although time-consistent equilibrium conditions for problems with heterogeneous discounting were already derived in Marín-Solano and Patxot (2011) following a variational approach, we provide here an alternative derivation to their main theorem in the spirit of Karp (2007) for non-constant discounting models, by first obtaining the DPE for a discretized version of Problem (4.3-4.4), and passing next to the continuous time limit. In addition, we propose a very simple method for transforming the functional DPE into a system of two partial differential equations, thereby facilitating the solution to the problem.

4.2.1 A set of dynamic programming equations

For Problem (4.3-4.4), let us assume the usual regularity conditions, i.e., functions U, F and f^i are continuously differentiable in all their arguments. In addition, $x \in \mathbf{R}^n$, $c \in \mathbf{R}^m$. Next, let us divide the interval $[0, T]$ into n periods of constant length $\epsilon = T/n$, in such a way that we identify $ds = \epsilon$, and $s = j\epsilon$, for $j = 0, 1, \ldots, n$. Then equation (4.4) becomes

$$x(s+\epsilon) - x(s) = f(x(s), c(s), s)\epsilon\,.$$

By denoting by $x(j\epsilon) = x_j$ and $c(k\epsilon) = c_k$ $(j, k = 0, \ldots, n-1)$, the objective of the agent in period $t = j\epsilon$ will be to maximize

$$V_j = \sum_{i=0}^{n-j-1} e^{-r_1(i\epsilon)} U(x_{(i+j)}, c_{(i+j)}, (i+j)\epsilon)\epsilon + e^{-r_2(n-j)\epsilon} F(x_n, n\epsilon) \tag{4.5}$$

subject to

$$x_{i+1} = x_i + f(x_i, c_i, i\epsilon)\epsilon\ ,\ \ i = j, \ldots, n-1\ ,\ \ x_j \text{ given}\ , \tag{4.6}$$

provided that future j' agents choose their best response actions. Let us state the dynamic programming algorithm for the discrete Problem (4.5-4.6). In the final period, we define

$$V_n^* = F(x(T), T)\,,$$

as usual. For $j = n-1$, the equilibrium value for (4.5) will be given by the solution to the problem

$$V^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = \max_{\{c_{(n-1)}\}} \left\{ U(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon + e^{-r_2\epsilon} V_n^*(x_n, n\epsilon) \right\}$$

with

$$x_n = x_{(n-1)} + f(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon\,.$$

If $c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon)$ is the maximizer of the right hand term of the above equation, let us denote

$$\bar{U}_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = U(x_{(n-1)}, c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon), (n-1)\epsilon)\ .$$

In general, for $j=1,\ldots,n-1$, the value $V_j^*(x_j, j\epsilon)$ in (4.5) can be written as

$$V_j^* = \max_{\{c_j\}} \left\{ U(x_j, c_j, j\epsilon)\epsilon + \sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon} \bar{U}_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon + \right.$$

$$\left. + e^{-r_2(n-j)\epsilon} V_n^* \right\} \tag{4.7}$$

with

$$x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon\,.$$

Since

$$V_{(j+1)}^*(x_{(j+1)}, (j+1)\epsilon) = \sum_{i=0}^{n-j-2} e^{-r_1 i\epsilon} \bar{U}_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon$$

$$+ e^{-r_2(n-j-1)\epsilon} V_n^*\,, \tag{4.8}$$

then solving for $V_n^*(x_n, n)$ in (4.8) and substituting in (4.7), we obtain

$$V_j^*(x_j, j\epsilon) = \max_{\{c_j\}} \left\{ U(x_j, c_j, j\epsilon)\epsilon + \right.$$

$$+ \sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon} \left(1 - e^{-(r_2 - r_1)\epsilon}\right) \bar{U}_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon +$$

$$\left. + e^{-r_2\epsilon} V_{(j+1)}^*(x_{(j+1)}, (j+1)\epsilon) \right\}, \tag{4.9}$$

with

$$x_{(j+1)} = x_j + f(x_j, u_j, j\epsilon)\epsilon\,, \qquad j = 0, \ldots, n-1\,,$$

and

$$V_n^* = F(x_n, n\epsilon)\,.$$

For the continuous time case, we take the following definition.

Definition 3 *We define the value function for Problem (4.3-4.4) as the solution to the DPE obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the DPE (4.9) obtained for the discrete time approximation to the problem, assuming that such a limit exists and that the solution is of class C^1 in all their arguments.*

In order to obtain the DPE for the problem with heterogeneous discounting, let $W(x,t)$ represent the value function of the t-agent, with initial condition $x(t) =$

x. We assume that $W(x,t)$ is continuously differentiable in all its arguments. Since $s = j\epsilon$ and

$$x(s+\epsilon) - x(s) = f(x(s), c(s), s)\epsilon$$

then

$$W(x(t), t) = V_j(x_j, j\epsilon)$$

and

$$W(x(t+\epsilon), t+\epsilon) = W(x(t), t) + \nabla_x W(x(t), t) \cdot f(x(t), c(t), t)\epsilon + \nabla_t W(x(t), t)\epsilon + o(\epsilon)\,,$$

where

$$\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0\,.$$

Substituting in (4.9), since

$$e^{-r_2\epsilon} = 1 - r_2\epsilon + o(\epsilon)\,,$$

we obtain

$$W(x(t), t) = \max_{\{c(t)\}} \{U(x(t), c(t), t)\epsilon + W(x(t), t) + \nabla_x W(x(t), t) \cdot f(x(t), c(t), t)\epsilon$$
$$+\nabla_t W(x(t), t)\epsilon - r_2\epsilon W(x(t), t) - K(x(t), t) + o(\epsilon)\}\,, \tag{4.10}$$

where

$$K(x(t), t) = -\sum_{k=1}^{n-j-1} e^{-r_1 k\epsilon}\left(1 - e^{-(r_2 - r_1)\epsilon}\right)\bar{U}_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon. \tag{4.11}$$

Finally, since

$$\left(1 - e^{-(r_2 - r_1)\epsilon}\right) = (r_2 - r_1)\epsilon + o(\epsilon)$$

by dividing (4.10) and (4.11) by ϵ and taking the limit $\epsilon \to 0$, we obtain the following result:

Proposition 6 *Let $W(x,t)$ be a continuously differentiable function in (x,t) satisfying the dynamic programming equation*

$$r_2 W(x,t) + K(x,t) - \nabla_t W(x,t) = \max_{\{c\}} \{U(x,c,t) + \nabla_x W(x,t) \cdot f(x,c,t)\}\;, \tag{4.12}$$

with $W(x,T) = F(x,T)$, *and*

$$K(x,t) = (r_1 - r_2)\int_t^T e^{-r_1(s-t)}\bar{U}(x,s)ds \ .$$

Then $W(x,t)$ *is the value function for Problem (4.3-4.4). If, for each pair* (x,t), *there exists* $c^* = \phi(x,t)$, *with corresponding state trajectory, such that* c^* *maximizes the right hand side term of (4.12), then* $c^* = \phi(x,t)$ *is called a Markov equilibrium rule for the problem with heterogeneous discounting.*

Next, note that

$$\bar{U}(x,s) = U(x(s),\phi(x(s),s),s)$$

where $x(s)$ is the solution to

$$\dot{x}(s) = f(x,\phi(x,s),s)$$

with $x(t) = x$. Hence,

$$K(x,t) = (r_1 - r_2)\int_t^T e^{-r_1(s-t)}U(x(s),\phi(x(s),s),s)\,ds \tag{4.13}$$

and, by differentiating K in (4.13) with respect to t, we obtain the "auxiliary DPE"

$$r_1K(x,t) - \nabla_t K(x,t) = (r_1 - r_2)U(x,\phi(x,t),t) + \nabla_x K(x,t)\cdot f(x,\phi(x,t),t)\ . \tag{4.14}$$

Corollary 2 *Let* $W(x,t)$, $K(x,t)$ *be two continuously differentiable functions in* (x,t) *such that* $W(x,t)$, $K(x,t)$ *and the strategy* $c^* = \phi(x,t)$ *satisfy the set of two DPEs (4.12) and (4.14) with boundary conditions*

$$W(x,T) = F(x,T)\,, \qquad\qquad K(x,T) = 0\,.$$

Then $W(x,t)$ *is the value function for Problem (4.3-4.4), and the strategy* $c^* = \phi(x,t)$ *maximizing the right hand side term of (4.12) is a Markov equilibrium rule.*

4.2.2 An exhaustible resource model under common access

Let us analyze a simple model of a common-property nonrenewable resource with two agents, $N = 2$, with equal weights $\lambda_1 = \lambda_2$, in a finite time horizon T. Let $x(t)$ and $c_m(t)$, $m = 1, 2$, denote the stock of the resource and player m's rate extraction at time t, respectively, while the evolution of the system follows

$$\dot{x}(t) = -c_1(t) - c_2(t), \quad x(0) = x_0, \quad x(T) = 0. \tag{4.15}$$

Each player m has an increasing and concave utility function $U^m(c_m)$. Let us assume that the utility functions are logarithmic,i.e.

$$U^m(c_m) = \ln(c_m)\,,$$

and are discounted at constant time preference rates $r_m > 0$, with $r_1 \neq r_2$. If the agents at time $t = 0$ decide to cooperate throughout all the planning horizon $[0, T]$, the objective for the coalition is to maximize

$$\int_0^T \ln\left(c_1(s)\right) e^{-r_1 s} ds + \int_0^T \ln\left(c_2(s)\right) e^{-r_2 s} ds \tag{4.16}$$

subject to (4.15). If we solve problem (4.16) subject to (4.15) by means of Pontryagin's Maximum Principle we obtain

$$c_m^0(s) = \frac{e^{-r_m s}}{\sum_{i=1}^2 \frac{1-e^{-r_i T}}{r_i}} x_0 = \frac{e^{-r_m s}}{\sum_{i=1}^2 \frac{e^{-r_i s}-e^{-r_i T}}{r_i}} x_s\,, \tag{4.17}$$

where the superscript 0 in c_m^0 accounts for the moment at which the decision has been made. This is the so called (in the hyperbolic discounting literature) precommitment solution, which is optimal from the viewpoint of the 0-coalition, $c^P(s) = c^0(s)$, and can be associated with the existence of some binding agreement between players at the beginning of the game, in the sense that both agents will follow the decision rule taken at time $t = 0$, despite having incentives to deviate in the future from the previously calculated decision rule. However, if such an agreement does not exist, players in the coalition can re-calculate the cooperative solution at some instant $t \in (0, T]$. The maximum of

$$\int_t^T \ln\left(c_1(s)\right) e^{-r_1(s-t)} ds + \int_t^T \ln\left(c_2(s)\right) e^{-r_2(s-t)} ds, \tag{4.18}$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s), \quad x(t) = x_t, \quad x(T) = 0 \tag{4.19}$$

is given by

$$c^t_m(s) = \frac{e^{-r_m(s-t)}}{\sum_{i=1}^{2} \frac{1-e^{-r_i(T-t)}}{r_i}} x_t \ , \quad s \in [t, T] \ . \tag{4.20}$$

Note that this solution differs from that calculated in (4.17). For instance,

$$c^t_1(t) = c^t_2(t) \, ,$$

whereas

$$c^0_1(t) \neq c^0_2(t) \, ,$$

for every $t > 0$. Thus, the joint solution becomes time inconsistent as long as the coalition has the possibility of re-optimizing at any instant after $t = 0$.

In general, if players in the coalition can continuously re-calculate the "co-operative" solution, they will follow what we call the (time inconsistent) naive decision rule $c^N_m(t)$. Note that a coalition making a decision at time t will choose the decision rule (4.20). However, at time $t' > t$ the coalition will re-compute the decision rule. Hence, $c^t_m(s)$ in (4.20) is followed only at the time $s = t$ at which the agents of the t-coalition have calculated the extraction rate, so that the actual extraction rate becomes

$$c^N_m(t) = c^t_m(t) = \frac{1}{\sum_{i=1}^{2} \frac{1-e^{-r_i(T-t)}}{r_i}} x_t . \tag{4.21}$$

Note that the precommitment and naive solutions do not coincide unless $r_1 = r_2$. In fact,

$$c^P_1(t) \neq c^P_2(t) \, ,$$

for every $t \in (0, T]$, whereas

$$c^N_1(t) = c^N_2(t) \, ,$$

for every $t \in [0, T]$. If the agents can split the resource at time $t = 0$ in an irreversible way so that

$$x_0 = x^1_0 + x^2_0 \, ,$$

where

$$x^m_0 = \int_0^T c_m(s) \, ds \, ,$$

$i = 1, 2$, then the precommitment solution becomes time-consistent.

In order to determine a time-consistent equilibrium, we first reformulate Problem (4.18-4.19) by rewriting the payoff of player 2 in the Mayer form. The objective functional becomes

$$\int_t^T e^{-r_1(s-t)} \ln(c_1(s))\, ds + e^{-r_2(T-t)} y(T)$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s)\,, \quad \dot{y}(s) = r_2 y(s) + \ln(c_2(s)) \tag{4.22}$$

with $x(T) = 0$. Although we have derived Proposition 6 and Corollary 1 for the case of free terminal states $x(T)$ and $y(T)$, it is easy to check that the corresponding DPEs are preserved if a terminal condition on $x(T)$ is imposed. According to Proposition 6, we look for the solution to the DPE (4.12), i.e.,

$$r_2 W(x,y,t) + K(x,y,t) - W_t(x,y,t) =$$

$$= \max_{\{c_1,c_2\}} \{\ln c_1 + W_x(x,y,t)(-c_1 - c_2) + W_y(x,y,t)\,(r_2 y + \ln(c_2))\}\,, \tag{4.23}$$

where

$$K(x,y,t) = (r_1 - r_2)\int_t^T e^{-r_1(s-t)} \ln(c_1^*, s) ds\,.$$

We guess for a value function of the form

$$W(x,y,t) = A(t)\ln(x) + B(t)y + C(t)\,.$$

If this choice proves to be consistent, the extraction rates for both agents are given by

$$c_1(t) = \frac{1}{W_x} = \frac{x}{A(t)} \qquad\qquad c_2(t) = \frac{W_y}{W_x} = \frac{B(t)x}{A(t)}\,.$$

In order to solve (4.23) we calculate the expression for $K(x,t)$. To do that, we substitute our "guessed" controls in (4.22). Hence,

$$x(s) = x_t \exp(\Lambda_t(s))\,, \qquad \text{with} \qquad \Lambda_t(s) = -\int_t^s \frac{1 + B(\tau)}{A(\tau)} d\tau\,.$$

Therefore,

$$K(x,y,t) = (r_1 - r_2)\int_t^T e^{-r_1(s-t)} \ln\left(\frac{x_t e^{\Lambda_t(s)}}{A(s)}\right) ds =$$

$$= \frac{r_1 - r_2}{r_1} \left(1 - e^{-r_1(T-t)}\right) \ln(x_t) + (r_1 - r_2) \int_t^T e^{-r_1(s-t)} \ln\left(\frac{e^{\Lambda_t(s)}}{A(s)}\right) ds \, .$$

By substituting in (4.23) and simplifying, we obtain

$$r_2 \left[A(t)\ln(x) + B(t)y + C(t)\right] - \left[A'(t)\ln(x) + B'(t)y + C'(t)\right] +$$

$$+\frac{r_1 - r_2}{r_1} \left(1 - e^{-r_1(T-t)}\right) \ln(x) + (r_1 - r_2) \int_t^T e^{-r_1(s-t)} \ln\left(\frac{e^{\Lambda_t(s)}}{A(s)}\right) ds =$$

$$= \ln(x) - \ln\left(A(t)\right) - 1 - B(t) + B(t)\left(r_2 y + \ln(x) + \ln\left(\frac{B(t)}{A(t)}\right)\right) .$$

Since the above equation must be satisfied for every x and y, then

$$r_2 A(t) - A'(t) + \frac{r_1 - r_2}{r_1} \left(1 - e^{-r_1(T-t)}\right) = 1 + B(t) \, , \quad B'(t) = 0 \, . \tag{4.24}$$

Using the terminal condition $B(T) = 1$, we obtain

$$B(t) = 1 \, ,$$

and

$$c_1(t) = c_2(t) = \frac{x}{A(t)} \, ,$$

for every $t \in [0, T]$. With respect to $A(t)$, note that, if

$$A(t) = \sum_{i=1}^{2} \frac{1 - e^{-r_i(T-t)}}{r_i} \, ,$$

which describes the solution for a naive coalition (see (4.21)), then equation (4.24) is satisfied and, in addition, the solution to the state equation

$$\dot{x}(t) = \frac{-2x(t)}{A(t)}$$

verifies the terminal condition

$$\lim_{t \to T} x(t) = 0 \, .$$

Therefore, the solution obtained for the naive coalition is a time-consistent policy. This feature, also arising in non-constant discounting models (see Pollak (1968) and Marín-Solano and Navas (2009)), is a consequence of using logarithmic utility functions, and it no longer holds when more general utility functions are considered.

Next, we solve the model for a general isoelastic utility function. Consider the problem of maximizing

$$\int_t^T \left[e^{-r_1 s} \frac{c_1^{1-\sigma}-1}{1-\sigma} + e^{-r_2 s} \frac{c_2^{1-\sigma}-1}{1-\sigma} \right] ds \tag{4.25}$$

subject to (4.15). If $\gamma_i = \frac{r_i}{\sigma}$, the precommitment and naive solutions are given by

$$c_i^P(t) = \frac{e^{-\gamma_i t}}{\sum_{j=1}^2 \frac{1}{\gamma_j}(e^{-\gamma_j t} - e^{-\gamma_j T})} x(t) ,$$

and

$$c_i^N(t) = \frac{1}{\sum_{j=1}^2 \frac{1}{\gamma_j}(1 - e^{-\gamma_j (T-t)})} x(t) ,$$

respectively. The precommitment and naive solutions coincide if, and only if, $r_1 = r_2$.

For the calculation of the time-consistent solution, we transform Problem (4.25) subject to (4.15) into the equivalent one-player problem of maximizing

$$\int_t^T e^{-r_1(s-t)} \frac{c_1^{1-\sigma}-1}{1-\sigma} ds + e^{-r_2(T-t)} y(T)$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s) , \quad \dot{y}(s) = r_2 y(s) + \frac{c_2^{1-\sigma}-1}{1-\sigma} .$$

From Corollary 1, we have to solve the set of two partial differential equations

$$r_2 W(x,y,t) + K(x,y,t) - W_t(x,y,t) = \tag{4.26}$$

$$= \max_{\{c_1,c_2\}} \left\{ \frac{c_1^{1-\sigma}-1}{1-\sigma} + W_x(x,y,t)(-c_1 - c_2) + W_y(x,y,t) \left(r_2 y + \frac{c_2^{1-\sigma}-1}{1-\sigma} \right) \right\} ,$$

$$r_1 K(x,y,t) - K_t(x,y,t) = \tag{4.27}$$

$$= (r_1 - r_2) \frac{(c_1^*)^{1-\sigma}-1}{1-\sigma} + K_x(x,y,t)(-c_1^* - c_2^*) + K_y(x,y,t) \left(r_2 y + \frac{(c_2^*)^{1-\sigma}-1}{1-\sigma} \right) ,$$

where c_1^*, c_2^* in (4.27) are the maximizers to the right hand term in (4.26). Note that

$$(c_1^*)^{-\sigma} = W_x\,, \qquad \text{and} \qquad (c_2^*)^{-\sigma} = \frac{W_x}{W_y}\,.$$

Hence $c_1^* = c_2^*$ if, and only if, $W_y = 1$. It is easy to prove that a solution exists with $W_y = 1$ (and $K_y = 0$), so that the extraction rules for the two agents coincide for every σ. It can be shown that, unless $\sigma = 1$ (the log-utility case), the naive solution is time inconsistent in general. We study this model in more detail in the following section, for a general N-player cooperative differential game.

4.3 The case of N heterogeneous agents

In this section we extend the two-player case analyzed above. Let us consider the case of N players who decide to form a coalition seeking for a time-consistent solution maximizing

$$J\,(c(\cdot)) = \sum_{m=1}^{N} \lambda_m \int_t^T e^{-r_m(s-t)} U^m\,(x(s), c(s), s)\,ds \tag{4.28}$$

subject to

$$\dot{x}(s) = f(x(s), c(s), s), \qquad x(t) = x_t\,. \tag{4.29}$$

4.3.1 Dynamic programming equation

By proceeding in a similar way to that in Section 3, we discretize (4.28-4.29). The corresponding problem in discrete time is

$$\max_{\{c_1,\ldots,c_n\}} V_j = \sum_{m=1}^{N} V_j^m = \sum_{i=0}^{n-j-1} \sum_{m=1}^{N} \lambda_m e^{-r_m(i\epsilon)} U^m(x_{(i+j)}, c_{(i+j)}, (i+j)\epsilon)\epsilon \tag{4.30}$$

subject to

$$x_{i+1} = x_i + f(x_i, c_i, i\epsilon)\epsilon\;,\; i = j, \ldots, n-1\;,\; x_j \text{ given}\,. \tag{4.31}$$

Let us state the dynamic programming algorithm for the discrete time Problem (4.30-4.31). In the final period, we define

$$V_n^* = 0\,,$$

as usual. For $j = n-1$, the optimal value for (4.30) will be given by the solution to the problem

$$V^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = \max_{\{c_{(n-1)}\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x_{(n-1)}, c_{(n-1)}, (n-1)\epsilon)\epsilon \right\} ,$$

with

$$x_n = x_{(n-1)} + f(x_{(n-1)}, u_{(n-1)}, (n-1)\epsilon)\epsilon .$$

If $c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon)$ is the maximizer of the right hand term of the above equation, let us denote

$$\bar{U}^m_{(n-1)}(x_{(n-1)}, (n-1)\epsilon) = U^m(x_{(n-1)}, c^*_{(n-1)}(x_{(n-1)}, (n-1)\epsilon), (n-1)\epsilon) .$$

In general, for $j = 1, \ldots, n-1$, the value $V^*_j(x_j, j\epsilon)$ in (4.30) can be written as

$$V^*_j = \max_{\{c_j\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x_j, c_j, j\epsilon)\epsilon + \sum_{k=1}^{n-j-1} \sum_{m=1}^{N} \lambda_m e^{-r_m k\epsilon} \bar{U}^m_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon \right\} \tag{4.32}$$

with

$$x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon .$$

Since

$$V^*_{(j+1)}(x_{(j+1)}, (j+1)\epsilon) = \sum_{i=0}^{n-j-2} \sum_{m=1}^{N} \lambda_m e^{-r_m i\epsilon} \bar{U}^m_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon,$$

then we can write

$$V^*_{(j+1)}(x_{(j+1)}, (j+1)\epsilon) - \sum_{i=0}^{n-j-2} \sum_{m=1}^{N} \lambda_m e^{-r_m i\epsilon} \bar{U}^m_{(j+i+1)}(x_{(j+i+1)}, (j+i+1)\epsilon)\epsilon = 0 .$$

Adding the former expression to (4.32), we obtain

$$V^*_j(x_j, j\epsilon) = \max_{\{c_j\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x_j, c_j, j\epsilon)\epsilon + \right. \tag{4.33}$$

$$\left. + \sum_{k=1}^{n-j-1} \sum_{m=1}^{N} \lambda_m \left(1 - e^{r_m \epsilon}\right) e^{-r_m k\epsilon} \bar{U}^m_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon + V^*_{(j+1)}(x_{(j+1)}, (j+1)\epsilon) \right\} ,$$

with

$$x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon\,, \qquad j = 0, \ldots, n-1\,, \qquad\qquad \text{and} \qquad V_n^* = 0\,.$$

Next, as in the previous section, we obtain a DPE for the problem with heterogeneous discounting in continuous time by taking the limit $\epsilon \to 0$ in (4.33).

Definition 4 *We define the value function for Problem (4.28-4.29) as the solution to the DPE obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the DPE (4.33) obtained from the discrete approximation to the problem, assuming that the limit exists and that the solution is of class C^1 in all their arguments.*

Let $W^m(x,t)$ be a continuously differentiable function representing the value function of player m in the t-coalition, and let

$$W(x,t) = \sum_{m=1}^{N} W^m(x,t)$$

be the value function for the t-coalition, with initial condition $x(t) = x$. Since $s = j\epsilon$ and

$$x(s+\epsilon) - x(s) = f(x(s), c(s), s)\epsilon\,,$$

then $W(x(t),t) = V_j(x_j, j\epsilon)$ and

$$W(x(t+\epsilon), t+\epsilon) = W(x(t),t) + \nabla_x W(x(t),t)\cdot f(x(t),c(t),t)\epsilon + \nabla_t W(x(t),t)\epsilon + o(\epsilon)\,.$$

Substituting in (4.33) we obtain

$$\begin{aligned} W(x(t),t) = \max_{\{c_t\}} \Bigg\{ & \sum_{m=1}^{N} \lambda_m U^m(x(t),c(t),t)\epsilon + \nabla_x W(x(t),t)\cdot f(x(t),c(t),t)\epsilon \\ & + W(x(t),t) + \nabla_t W(x(t),t)\epsilon - \sum_{m=1}^{N} \left(1 - e^{r_m\epsilon}\right) W^m(x(t),t) + o(\epsilon) \Bigg\}, \end{aligned} \tag{4.34}$$

where

$$W^m(x(t),t) = -\sum_{k=1}^{n-j-1} \lambda_m e^{-r_m k\epsilon} \bar{U}^m_{j+k}(x_{(j+k)}, (j+k)\epsilon)\epsilon. \tag{4.35}$$

Finally, by dividing (4.34) and (4.35) by ϵ and taking the limit $\epsilon \to 0$, we obtain:

Proposition 7 *Let $W^m(x,t)$, $m = 1, \ldots, N$, be a set of continuously differentiable functions in (x,t), satisfying the dynamic programming equation*

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \nabla_t W^m(x,t) =$$

$$= \max_{\{c\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x,c,t) + \sum_{m=1}^{N} \nabla_x W^m(x,t) \cdot f(x,c,t) \right\} \tag{4.36}$$

with $W^m(x,T) = 0$, for every $m = 1, \ldots, N$, and

$$W^m(x,t) = \lambda_m \int_t^T e^{-r_m(s-t)} U(x(s), \phi(x(s),s), s)\, ds \;, \tag{4.37}$$

where, $c^*(t) = \phi(x(t),t)$ is the maximizer of the right hand term in Equation (4.36). Then

$$W(x,t) = \sum_{m=1}^{N} W^m(x,t)$$

is the value function of the whole coalition, the decision rule $c^* = \phi(x,t)$ is the (time-consistent) Markov Perfect Equilibrium, and $W^m(x,t)$, for $m = 1, \ldots, N$, is the value function of player m in the cooperative problem (4.28-4.29).

Remark 5 *Note that, throughout the equilibrium rule $c^* = \phi(x,t)$, for every player m, $W^m(x,t)$ in Equation (4.37) is a solution to the partial differential equation*

$$r_m W^m(x,t) - \nabla_t W^m(x,t) = \lambda_m U^m(x, \phi(x,t), t) + \nabla_x W^m(x,t) \cdot f(x, \phi(x,t), t) \;, \tag{4.38}$$

for $m = 1, \ldots, N$, with $W^m(x,T) = 0$. Hence, we can compute the value function by first determining the decision rule solving the right hand term in Eq. (4.36) as a function of $\nabla_x W^m(x,t)$, $m = 1, \ldots, N$, and then substituting the decision rule into the system of N partial differential equations (4.38).

4.3.2 An exhaustible resource model under common access: the case of N-asymmetric players

Let us extend the results for the nonrenewable resource model in Section 2.2 to the general case of N asymmetric players. If $\lambda_1 = \cdots = \lambda_N = 1$, we must solve

$$\max_{\{c_1,\ldots,c_n\}} \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m}-1}{1-\sigma_m}\, ds \tag{4.39}$$

subject to

$$\dot{x}(s) = -\sum_{m=1}^{N} c_m(s), \qquad x(t) = x_t, \quad x(T) = 0\,. \tag{4.40}$$

For $m = 1, \ldots, N$, the precommitment and naive solutions for Problem (4.39-4.40) are given by

$$c_m^P(t) = \frac{e^{-\gamma_m t}}{\sum_{i=1}^{N} \frac{1}{\gamma_i}\left(e^{-\gamma_i t} - e^{-\gamma_i T}\right)} x_t \quad \text{and} \quad c_m^N(t) = \frac{1}{\sum_{i=1}^{N} \frac{1}{\gamma_i}\left(1 - e^{-\gamma_i (T-t)}\right)} x_t\,, \tag{4.41}$$

respectively, where $\gamma_m = \frac{r_m}{\sigma_m}$. In the naive case the extraction rates of all agents coincide.

In order to look for a time-consistent equilibrium, we apply the results in Proposition 7 and Remark 5 to Problem (4.39-4.40)[1]. From equation (4.36) we have to solve

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial t} = \tag{4.42}$$

$$= \max_{c_1,\ldots,c_N} \left\{ \sum_{m=1}^{N} \frac{c_m(s)^{1-\sigma_m}-1}{1-\sigma_m} + \left(\sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial x}\right)\left(-\sum_{m=1}^{n} c_m(s)\right) \right\}.$$

The maximizer of the right side term in (4.42) is

$$c_m^S(t) = \left(\sum_{j=1}^{N} \frac{\partial W^j(x,t)}{\partial x}\right)^{-\frac{1}{\sigma_m}},$$

for $m = 1, \ldots, N$. Therefore, the extraction rates of agents m and m' coincide ($c_m^S = c_{m'}^S$) if, and only if, $\sigma_m = \sigma_{m'}$. Thus, if there are two players m and m' such that $\sigma_m \neq \sigma_{m'}$ (hence $c_m^S \neq c_{m'}^S$), the naive solution is always time-inconsistent.

In order to compute the actual decision rule we can solve the family of N

[1] As in the standard case, the same DPE in Proposition 7 is obtained if $x(T)$ is fixed

coupled partial differential equations (4.38), which in our particular case becomes

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} =$$

$$= \frac{1}{1-\sigma_m}\left[\left(\sum_{j=1}^{N}\frac{\partial W^j(x,t)}{\partial x}\right)^{\frac{\sigma_m-1}{\sigma_m}} - 1\right] - \frac{\partial W^m(x,t)}{\partial x}\sum_{j=1}^{N}\left(\sum_{i=1}^{N}\frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma_j}},$$

for $m = 1, \ldots, N$. If

$$\sigma_1 = \cdots = \sigma_N = \sigma\,,$$

the above system simplifies to

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} =$$

$$= \frac{1}{1-\sigma}\left[\left(\sum_{j=1}^{N}\frac{\partial W^j(x,t)}{\partial x}\right)^{1-\frac{1}{\sigma}} - 1\right] - N\frac{\partial W^m(x,t)}{\partial x}\left(\sum_{i=1}^{N}\frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma}},$$

for $m = 1, \ldots, N$. We guess

$$W^m(x,t) = A^m(t)\frac{x^{1-\sigma}-1}{1-\sigma} + B^m(t)\,,$$

for $m = 1, \ldots, N$, with $A^m(t) > 0$ for every $t \in [0, T)$. By substituting in the above DPE, we find that the functions $A^m(t)$ are the solution to the system of ordinary differential equations

$$\dot{A}^m - r_m A^m = N(1-\sigma)A^m\left(\sum_{j=1}^{N}A^j\right)^{-\frac{1}{\sigma}} - \left(\sum_{j=1}^{N}A^j\right)^{1-\frac{1}{\sigma}}, \quad m = 1, \ldots, N\,. \tag{4.43}$$

For instance, in the limit case $\sigma = 1$ (which corresponds to a logarithmic utility function), the above system simplifies to

$$\dot{A}^m - r_m A^m + 1 = 0\,,$$

for $m = 1, \ldots, N$. Note that

$$A^m(t) = \frac{1}{r_m}[1 - e^{-r_m(T-t)}]\,,$$

which is the naive solution, satisfies this set of differential equations. Hence, the naive solution also becomes time-consistent in the case of N asymmetric players, extending in this way the result obtained in Section 2.2. Summarizing, we have proved:

Proposition 8 *In Problem (4.39-4.40), in the time-consistent solution, the extraction rates of two agents coincide if, and only if, they have the same marginal elasticity σ. In particular, if $\sigma_1 = \cdots = \sigma_N = 1$, then the naive solution (4.41) is time-consistent.*

If $\sigma \neq 1$ note that

$$c_m^S(t) = \left(\sum_{j=1}^{N} A^j(t) \right)^{-\frac{1}{\sigma}} ,$$

and the solution to the state equation is

$$x(t) = x_0 e^{-\int_0^t \frac{N}{\left(\sum_{j=1}^{N} A^j(s)\right)^{1/\sigma}} ds} .$$

In order to achieve the terminal condition $x(T) = 0$, from the positivity of $A^m(t)$ for $t < T$ we obtain that

$$\lim_{t \to T} \sum_{j=1}^{N} A^j(t) = 0 ,$$

therefore, $A^m(T) = 0$, for every $m = 1, \ldots, N$. It can be shown that the naive solution is time-inconsistent, in general, for $\sigma \neq 1$, as we illustrate numerically in Section 3.3.

Remark 6 *If $U^m(c_m) = U(c_m)$, i.e., all the agents have the same utility function (in the isoelastic case, $\sigma_1 = \cdots = \sigma_N = \sigma$), along the equilibrium rule all players extract the resource at the same rate and Problem (4.39-4.40) becomes equivalent to the problem of a representative agent using the discount function*

$$\sum_{m=1}^{N} e^{-r_m(s-t)} .$$

This result is not preserved for the precommitment solution. The time-inconsistency of the naive solution if $\sigma \neq 1$ for the corresponding cake-eating problem with nonconstant discounting was already proved in Marín-Solano and Navas (2009). On the contrary, if there two agents m and m' with different marginal utilities ($\sigma_m \neq \sigma_{m'}$), the problem cannot be simplified to a non-constant discounting problem.

4.3.3 Numerical illustrations

Next we illustrate numerically the above results. We consider as a baseline case the problem for three players, $N = 3$, exhibiting as time preference rates $r_1 = 0.03$, $r_2 = 0.06$ and $r_3 = 0.09$, respectively, i.e., agent 1 being the most patient and agent 3 the most impatient. Agents face the "optimal" exploitation of a common property exhaustible resource with an initial stock of $S_0 = 100$ during a time interval that extends from $t_0 = 0$ to $T = 50$ periods. Utilities from consumption are assumed to be of the iso-elastic type with equal intertemporal elasticity of substitution $(1/\sigma)$ for all three players in the coalition.

Figures 1 and 2 show the individual extraction rate for every agent in the coalition under the assumption of cooperation for the naive (dot dashed line) and the sophisticated solutions (dashed line), with $\sigma = 0.6$ (Figure 1) and $\sigma = 2$ (Figure 2). In both graphs, the solid line shows the extraction rate for logarithmic utilities.

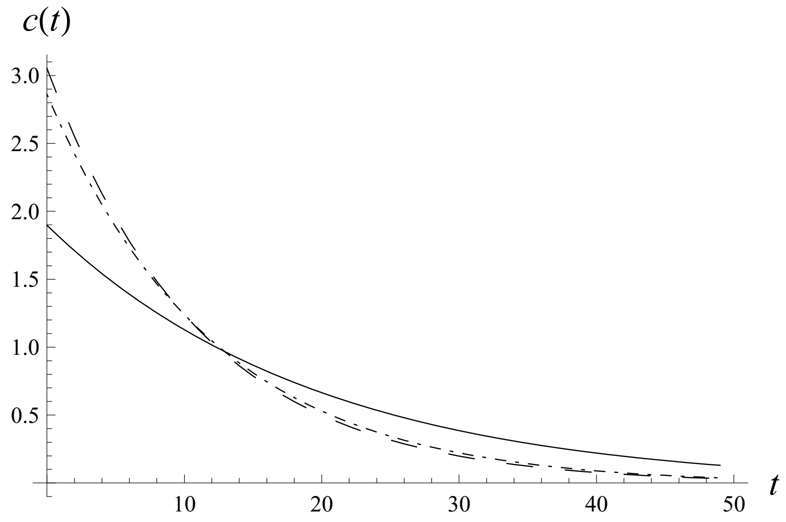

Figure 4.1: Extraction rates for naive and sophisticated agents ($\sigma = 0.6$) and logarithmic case.

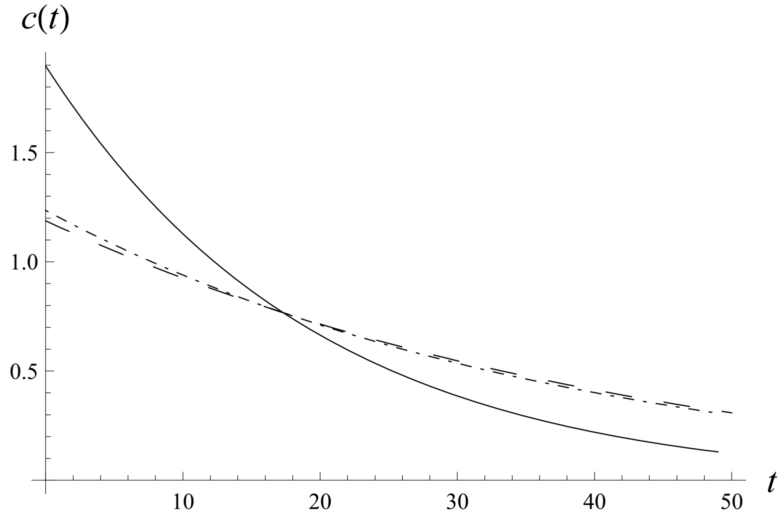

Figure 4.2: Extraction rates for naive and sophisticated agents ($\sigma = 2$) and logarithmic case.

Unless $\sigma = 1$ (logarithmic utilities), the time-consistent and naive solutions do not coincide, as expected. For $\sigma = 0.6$, the time-consistent agents' extraction rate is higher at initial periods compared with naive agents, this behavior being reversed for $\sigma = 2$. It is noteworthy to observe that the equilibrium appears to be more sensitive to the value of σ than to the behavior (naive or time-consistent) of the t-coalitions. In addition, higher values of σ lead agents to smooth their extraction rate path along the time horizon.

Finally, in Figure 3 we compare the precommitment solutions $(c_m^0(s), s \in [0, 50], m = 1, 2, 3)$ with the time-consistent solution assuming now that utilities are of the logarithmic type:

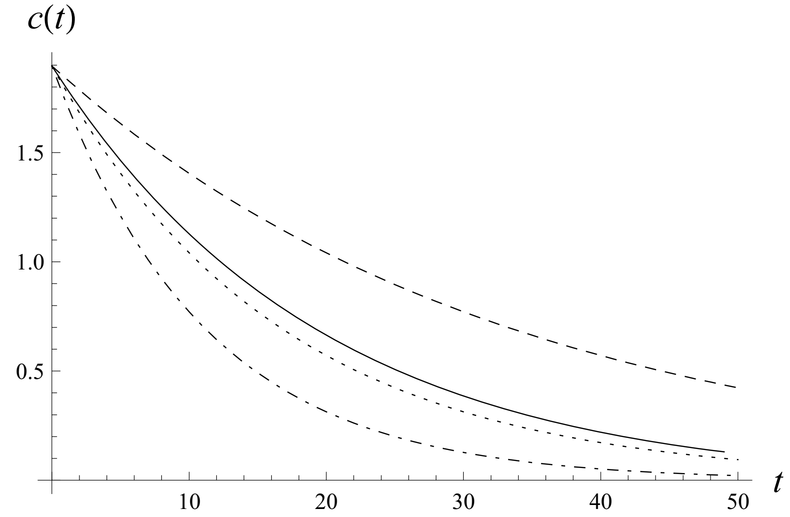

Figure 4.3: Extraction rates for sophisticated agents in the coalition (solid line) and individual extraction rates under precommitment at $t = 0$ (dashed, dotted and dot dashed lines correspond to players 1, 2 and 3, respectively). Logarithmic utility.

We observe that in the precommitment solution, each player's extraction rate in the coalition is different, (patient) player 1 being the agent in the coalition with higher aggregate extraction (and hence exploitation) of the resource (patient agents have a higher weight in the joint functional pay-off than impatient agents). In the time-consistent solution, extraction rates are equal for all three players in the coalition, as shown indicated by the solid line.

4.4 An extension: infinite planning horizon

In most economic models and, in particular, in the economic modeling of natural resources, it is customary to work in an infinite horizon setting. For instance, an important issue in the management of natural resources (such as forests, aquifers or fish species) is the existence of positive steady state levels. In this section we briefly extend the previous results for the nonrenewable resource model to a simple model of management of a common-property renewable resource. If preferences of agent m, for $m = 1, \ldots, N$, are characterized by the utility function

$$U^m(c_m) = \frac{c_m^{1-\sigma_m} - 1}{1 - \sigma_m}$$

and the discount rate of time preference r_m, then, at time t, we must solve

$$\max_{\{c_1,\dots,c_n\}} \sum_{m=1}^{N} \int_t^\infty e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m} - 1}{1-\sigma_m}\, ds \,, \tag{4.44}$$

subject to

$$\dot{x}(s) = g(x) - \sum_{m=1}^{N} c_m(s) \,, \qquad x(t) = x_t \,, \tag{4.45}$$

where $c_m(t)$ is the harvest rate of agent m, for $m = 1, \dots, N$, and $g(x)$ is the natural growth function of the resource stock x. In the case of a representative agent applying a unique utility function, this problem was already studied in Barro (1999) for the neoclassical growth model.

In general, consider the problem of looking for the decision rule "maximizing"

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \int_t^\infty e^{-r_m(s-t)} U^m\left(x(s), c(s), s\right) ds, \tag{4.46}$$

subject to (4.29). From Proposition 2, a natural candidate for a DPE is given by (4.36-4.37) with $T = \infty$. However, in our derivation we assumed that T is finite. Next we provide a mathematical justification of this DPE by using a different procedure. Following the approach in Marín-Solano and Shevkoplyas (2011) (which is based on the one by Ekeland and Lazrak (2010)), if $c^*(s) = \phi(s, x(s))$ is the equilibrium rule, then the value function is

$$W(x,t) = \sum_{m=1}^{N} \int_t^\infty e^{-r_m(s-t)} U^m(x(s), \phi(x(s),s), s)\, ds \tag{4.47}$$

where

$$\dot{x}(s) = f(x(s), \phi(x(s),s), s) \,, \qquad\qquad x(t) = x_t \,.$$

Next, for $\epsilon > 0$ let us consider the variations

$$c_\epsilon(s) = \begin{cases} v(s) & \text{if} \quad s \in [t, t+\epsilon] \,, \\ \phi(x,s) & \text{if} \quad s > t+\epsilon \,. \end{cases}$$

If the t-agent can precommit her behavior during the period $[t, t+\epsilon]$, the value function for the perturbed control path c_ϵ is given by

$$W_\epsilon(x,t) = \max_{\{v(s),\, s\in[t,t+\epsilon]\}} \left\{ \sum_{m=1}^{N} \int_t^{t+\epsilon} e^{-r_m(s-t)} U^m\left(x(s), v(s), s\right) ds \right.$$

$$\left. + \sum_{m=1}^{N} \int_{t+\epsilon}^{\infty} e^{-r_m(s-t)} U^m(x(s), \phi(x(s), s), s)\, ds \right\} . \tag{4.48}$$

Definition 5 *Let W_ϵ be differentiable in ϵ in a neighborhood of $\epsilon = 0$. Then $c^*(s) = \phi(s, x(s))$ is called an equilibrium rule if*

$$\lim_{\epsilon\to 0^+} \frac{W(x,t) - W_\epsilon(x,t)}{\epsilon} \geq 0 \;.$$

The above definition can be interpreted as follows: for ϵ sufficiently small, the maximum of W_ϵ in the limit when $\epsilon = 0$ is precisely $W(x,t)$. Although this notion of equilibrium is not as natural as in the approach described in the previous sections, it allows us to provide a mathematical justification to the DPE (4.36-4.37) with $T = \infty$.

Proposition 9 *If the value function (4.47) is of class C^1, then the solution $c = \phi(x,t)$ to the right hand term of the DPE*

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \nabla_t W^m(x,t) =$$

$$= \max_{\{c\}} \left\{ \sum_{m=1}^{N} U^m(x,c,t) + \sum_{m=1}^{N} \nabla_x W^m(x,t) \cdot f(x,c,t) \right\} \tag{4.49}$$

with

$$W^m(x,t) = \int_t^{\infty} e^{-r_m(s-t)} U(x(s), \phi(x(s), s), s)\, ds \tag{4.50}$$

is an equilibrium rule, in the sense that it satisfies Definition 5.

Proof: In order to prove that $c^*(t) = \phi(x,t)$ solving the right hand term in (4.49) is an equilibrium rule, we have to check Definition 5. We do it in several steps:

If $\bar{x}(s)$ denotes the state trajectory corresponding to the decision rule $c_\epsilon(s)$, then

$$W(x,t) - W_\epsilon(x,t) =$$

$$= \sum_{m=1}^{N} \int_t^{t+\epsilon} e^{-r_m(s-t)} \left[U^m(x(s), \phi(x(s), s), s) - U^m(\bar{x}(s), v(s), s) \right] ds +$$

$$+\sum_{m=1}^{N}\int_{t+\epsilon}^{\infty} e^{-r_m(s-t)}\left[U^m(x(s),\phi(x(s),s),s) - U^m(\bar{x}(s),\phi(\bar{x}(s),s),s)\right]\,ds\ .$$

Note that

$$\sum_{m=1}^{N}\int_{t+\epsilon}^{\infty} e^{-r_m(s-t)}U^m(x(s),\phi(x(s),s),s)\,ds\ = W(x(t+\epsilon),t+\epsilon)-$$

$$-\sum_{m=1}^{N}\int_{t+\epsilon}^{\infty}[e^{-r_m(s-t-\epsilon)} - e^{-r_m(s-t)}]U^m(x(s),\phi(x(s),s),s)\,ds\ .$$

In a similar way,

$$\sum_{m=1}^{N}\int_{t+\epsilon}^{\infty} e^{-r_m(s-t)}U^m(\bar{x}(s),\phi(\bar{x}(s),s),s)\,ds\ = W(\bar{x}(t+\epsilon),t+\epsilon)-$$

$$-\sum_{m=1}^{N}\int_{t+\epsilon}^{\infty}\left[e^{-r_m(s-t-\epsilon)} - e^{-r_m(s-t)}\right]U^m(\bar{x}(s),\phi(\bar{x}(s),s),s)\,ds\ .$$

Therefore,

$$\lim_{\epsilon\to 0^+}\frac{W(x,t)-W_\epsilon(x,t)}{\epsilon} =$$

$$\lim_{\epsilon\to 0^+}\frac{\sum_{m=1}^{N}\int_{t}^{t+\epsilon} e^{-r_m(s-t)}\left[U^m(x(s),\phi(x(s),s),s) - U^m(\bar{x}(s),v(s),s)\right]ds}{\epsilon}+$$

$$+\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\sum_{m=1}^{N}\left[\int_{t+\epsilon}^{\infty}\left[e^{-r_m(s-t)} - e^{-r_m(s-t-\epsilon)}\right][U^m(x(s),\phi(x(s),s),s)-\right.$$

$$\left.-U^m(\bar{x}(s),\phi(\bar{x}(s),s),s)]\,ds\right] + \lim_{\epsilon\to 0^+}\frac{W(x(t+\epsilon),t+\epsilon)-W(\bar{x}(t+\epsilon),t+\epsilon)}{\epsilon} =$$

$$=\sum_{m=1}^{N}\left[U^m(x(t),\phi(x(t),t),t) - U^m(x(t),v(t),t)\right] + 0 +$$

$$+\lim_{\epsilon\to 0^+}\frac{W(x(t+\epsilon),t+\epsilon)-W(x(t),t)}{\epsilon} - \lim_{\epsilon\to 0^+}\frac{W(\bar{x}(t+\epsilon),t+\epsilon)-W(x(t),t)}{\epsilon} =$$

$$=\sum_{m=1}^{N}\left[U^m(x(t),\phi(x(t),t),t) - U^m(x(t),v(t),t)\right] +$$

$$+\left[\frac{\partial W(x,t)}{\partial t} + \nabla_x W(x,t) f(x,\phi(x,t),t)\right] - \left[\frac{\partial W(x,t)}{\partial t} + \nabla_x W(x,t) f(x,v(t),t)\right] =$$

$$= \sum_{m=1}^{N} [U^m(x, \phi(x,t), t) + \nabla_x W^m(x,t) \cdot f(x, \phi(x,t), t)] -$$

$$- \sum_{m=1}^{N} [U^m(x, v(t), t) + \nabla_x W^m(x,t) \cdot f(x, v(t), t)] \geq 0 \,,$$

since $c^* = \phi(x,t)$ is the maximizer of the right hand term in (4.49). □

Remark 7 *Proposition 9 can easily be generalized to the general problem where agents' time preferences are represented by arbitrary discount functions* $d_m(s,t)$, $m = 1, \ldots, N$. *In this case,* $W^m(x,t)$ *in (4.50) becomes*

$$W^m(x,t) = \int_t^{\infty} d(s,t) U(x(s), \phi(x(s), s), s) \, ds \,,$$

and the DPE in (4.49) transforms into

$$\sum_{m=1}^{N} \int_t^{\infty} \frac{\partial d_m(s,t)}{\partial t} U(x(s), \phi(x(s), s), s) \, ds - \sum_{m=1}^{N} \nabla_t W^m(x,t) =$$

$$= \max_{\{c\}} \left\{ \sum_{m=1}^{N} U^m(x, c, t) + \sum_{m=1}^{N} \nabla_x W^m(x,t) \cdot f(x, c, t) \right\} \,.$$

For instance, if $d_m(s,t) = \theta_m(s-t)$ *we obtain the problem of* N*-hyperbolic heterogeneous agents using different non-constant discount rates of time preference.*

Now, we analyze Problem (4.44-4.45). Since both the utility functions and the state equation are autonomous, it seems natural to restrict our attention to time-independent value functions $W^m(x)$, for $m = 1, \ldots, N$. From Proposition 9 we have to solve

$$\sum_{m=1}^{N} r_m W^m = \max_{c_1, \ldots, c_N} \left\{ \sum_{m=1}^{N} \frac{c_m^{1-\sigma_m} - 1}{1 - \sigma_m} + \left(\sum_{j=1}^{N} W_x^j \right) \left(g(x) - \sum_{m=1}^{N} c_m \right) \right\} \,, \quad (4.51)$$

hence

$$c_m^* = \phi_m(x) = \left(\sum_{j=1}^{N} W_x^j \right)^{-\frac{1}{\sigma_m}} \,. \quad (4.52)$$

Therefore, $c_m^* = c_{m'}^*$ if, and only if, $\sigma_m = \sigma_{m'}$. In general, along the equilibrium rule, $U'(c_m^*) = U'(c_{m'}^*)$, for all m, m'. In addition, we have the set of DPEs

$$r_m W^m = \frac{(\phi_m(x))^{1-\sigma_m} - 1}{1-\sigma_m} + W_x^m \left(g(x) - \sum_{j=1}^{N} (\phi_j(x))^* \right) , \tag{4.53}$$

for all $m = 1, \ldots, N$, where $\phi_m(x)$ are given by (4.52).

Next, let us restrict our attention to the case of linear decision rules. Since $(c_i^*)^{-\sigma_i} = (c_j^*)^{-\sigma_j}$, for all $i, j = 1, \ldots, N$, if

$$c_m^* = \phi_m(x) = \alpha_m x ,$$

then

$$(\alpha_i x)^{-\sigma_i} = (\alpha_j x)^{-\sigma_j} .$$

Therefore, no linear decision rules exist unless $\sigma_i = \sigma_j$, for all i, j. For $\sigma_i = \sigma_j = \sigma$, then $\alpha_i = \alpha_j$ and the DPE (4.51) becomes

$$\sum_{m=1}^{N} r_m W^m = \frac{N}{1-\sigma} \left(\alpha^{1-\sigma} x^{1-\sigma} - 1 \right) + \alpha^{-\sigma} x^{-\sigma} \left(g(x) - N\alpha x \right) .$$

This equation has a solution if $g(x) = ax$. In this case, we obtain

$$\sum_{m=1}^{N} r_m W^m(x) = \left[\frac{N\sigma}{1-\sigma} \alpha^{1-\sigma} + a\alpha^{-\sigma} \right] x^{1-\sigma} - \frac{N}{1-\sigma} ,$$

together with

$$\sum_{m=1}^{N} W_x^m(x) = \alpha^{-\sigma} x^{-\sigma}$$

and (4.53). If we try

$$W^m(x) = A^m \frac{x^{1-\sigma} - 1}{1-\sigma} + B^m ,$$

by simplifying we obtain that A^m, B^m and α are obtained by solving the equation system

$$\left[r_m - (1-\sigma)(a - N\alpha) \right] A^m = \alpha^{1-\sigma} , \tag{4.54}$$

$$r_m A^m - (1-\sigma) r_m B^m = 1 , \tag{4.55}$$

$$\sum_{m=1}^{N} A^m = \alpha^{-\sigma} . \tag{4.56}$$

For instance, if $\sigma = 1$ (logarithmic utility) we have

$$A^m = \frac{1}{r_m}, \qquad \text{and} \qquad \alpha = \frac{1}{\sum_{m=1}^N \frac{1}{r_m}}.$$

If $r_1 = \cdots = r_N = r$ then

$$\alpha = \frac{r - (1-\sigma)a}{N\sigma}.$$

The following proposition summarizes the main results for this simple model.

Proposition 10 *In Problem (4.44-4.45), along the equilibrium rule, the extraction rates of two agents are equal if, and only if, they have the same marginal elasticity. If there are two players with different marginal elasticities, no linear decision rules exist. If the natural growth function is linear and all the agents have the same marginal elasticity σ, then the decision rules $c_m = \alpha x$ and the value functions*

$$W^m(x) = A^m \frac{x^{1-\sigma} - 1}{1-\sigma} + B^m,$$

$m = 1, \ldots, N$ solve Problem (4.44-4.45), where the coefficients α, A^m and B^m are the solutions to (4.54-4.56).

Remark 8 *It is easy to show that the qualitative properties of the problem given in Proposition 10 (coincidence of extraction rates, existence of linear decision rules) are preserved if time preferences of agent m, $m = 1, \ldots, N$, are given by $d_m(s,t)$ (see Remark 7).*

4.5 Conclusions

In this chapter, we address the problem of searching time-consistent solutions for cooperative differential games with heterogeneous agents (in the sense that they exhibit different instantaneous pay-off functions and different discount rates of time preference). We analyze the time-consistency problem related to the changing preferences of the different t-coalitions. In order to avoid the possible time-consistency problem associated to the stability of the grand coalition, we assume that agents commit themselves to cooperate at every instant of time t (although we don't assume that the different t-coalitions cooperate among them). First we restrict our attention to problems in a finite horizon setting. For this case we introduce two alternative approaches in order to find time-consistent equilibria. In the first approach, we transform a two-player cooperative differential game into a one-agent problem with heterogeneous discounting. Whereas non-constant discounting models are typically very difficult to solve since the DPE is not a standard (partial) differential equation, our problem with two heterogeneous agents (and the problem with heterogeneous discounting) is more computationally tractable. The second approach allows us to study problems with an arbitrary number of players. A time-consistent solution can be obtained by solving, first, an algebraic equation (for the determination of the decision rule as a function of the value functions of all the agents) and, next, a system of coupled dynamic programming equations. Finally, we extend our results to an infinite horizon setting. We illustrate the results by studying a simple common access natural resource model.

Chapter 5

Concluding Remarks

In this dissertation we depart from the heterogeneous discounting model first proposed in Marín-Solano and Patxot (2012), extending and applying it to the stochastic case and to the differential games framework. The key feature of heterogeneous preferences is that an agent discounts future payments using a constant rate of time preference, but this rate is different for the instantaneous utilities and for the final function. By means of this simple modification of the standard theory, the approach can provide a model for certain behaviors that can not be explained by the DU model or more general hyperbolic discounting. In particular, heterogeneous discounting is appropriate to model situations in which exist a bias to the present that is not constant along time, but evolve with it. Although the model can account for decreasing and increasing levels of impatience, most of the applications in this dissertation focus on the former case. For instance, in Chapter 2 we analyze the effects of introducing this kind of preferences in consumption-investment models, illustrating how the evolution of people concerns about their retirement well-being or bequest, which are supposed to increase with the agent's age, influence the way in which their wealth is allocated between such activities.

The heterogeneous discounting model is hence encompassed within the group of theoretical ones that, like hyperbolic discounting, try to achieve a greater descriptive realism by relaxing the assumption of a constant discount rate of time preference. In the last decades, literature considering deviations from standard theory has developed dramatically challenged by empirical evidence suggesting that people often behaves in ways that are inconsistent with the DU model. In fact, since the publication of Thaler's work in 1981 (Thaler (1981)), a spate of empirical studies has been published, and virtually every assumption of the DU model has been called into question.

However, it is well-known that the introduction of non-constant discount functions give rise to time-inconsistent preferences (Strotz (1956)). Within the framework of utility maximization, for instance, this implies that the application of the standard optimization techniques, such as the Pontryagin's Maximum Principle or the HJB equation, fails in providing time-consistent solutions. Consequently, in order to obtain time-consistent solutions it is necessary to derive an algorithm for each model. In this sense, although some departures from the standard model achieved a greater level of descriptive realism, this had come often at expense of simplicity.

For the particular case of the heterogeneous discounting model, Marín-Solano and Patxot (2012) derived a DPE providing time-consistent solution in a deterministic framework. Nevertheless, some of the natural candidates to be described with their model, such as retirement planning or demand for life insurance, are better fitted in a stochastic environment in which one could consider, for example, financial markets with risky assets.

One of the contributions of this dissertation is precisely the derivation of a DPE providing time-consistent solutions for the heterogeneous discounting problem in a stochastic framework. This is done in Chapter 2, where the problem is analyzed in both discrete and continuous time. For the discrete time case, the derivation of the DPE is based on the principle of optimality, which enables us to optimize backwards. For the second case, the DPE is derived following two different procedures. The first one consists in a formal limiting procedure, departing from a discretized version of the problem. The second is the variational approach. However, both the formal limiting procedure and the variational approach have an important limitation: the resultant DPE is a functional equation with a nonlocal term, and consequently, it becomes very complicated to find solutions, not only analytically, but also numerically. In order to overcome this difficulty, we also derive in Chapter 2 a set of two coupled partial differential equations which allows us to compute (analytically or numerically) the solutions for different economic problems. To illustrate how the equations obtained can be used, we study the classical consumption-investment problem (Merton (1971)), generalizing the individual time preference with the introduction of the heterogeneous discount function, and considering utility functions of the CRRA and CARA type. Numerical illustrations of the solutions obtained in each case help us to highlight the differences with the standard solutions. In addition, the introduction of a stochastic terminal time is also briefly discussed, although this extension is fully developed in the next chapter.

In Chapter 3 the DPE obtained in Chapter 2 is used to analyze a more general problem. Specifically, it is assumed that the decision maker may die before achieving the retirement time, and hence two different sources of uncertainty are incorporated in the model: financial risk and mortality risk. The introduction of uncertain lifetime in the model enables us to study the optimal demand for life insurance, together with the optimal consumption and investment decisions. The amount of life insurance purchased is derived from a bequest function. Hence, by discounting this utility at a different rate of time preference compared with the instantaneous utilities, we are able to model not only an increasing concern about the well-being after retirement, but also an increasing concern about the agent's family protection in case of premature death occurrence, and consequently his or her life insurance strategy differs from that predicted by the standard model. The study is done by generalizing the model proposed in Pliska and Ye (2007) with the introduction of heterogeneous preferences. In addition, Chapter 3 also extends Chapter 2 with regard to the financial market. In particular, we consider that the agent can invest in options apart from riskless and risky assets. The introduction of options and other derivatives is a natural generalization of the standard portfolio problem due to their wide use as investment opportunities. However, in order to maintain the analytical tractability of the problem, we derive the wealth process in terms of the portfolio elasticity with respect to the risky assets (Kraft (2003)). Numerical examples illustrate the results and differences with the standard model for the case of CRRA and CARA utility functions.

Finally, in Chapter 4 the heterogeneous discounting framework is applied to the study of differential games. Although most of non-standard discounting models have focused on individual decision maker problems, this framework has proved to be useful in the study of cooperative differential games. In this chapter, the agents forming the coalition are supposed to differ in both the utility function and the discount rate of time preference. Consequently, one can not assume that there is a representative agent and therefore, the problem cannot be solved by rewriting the functional as a non-constant discounting problem. Our main contribution in this chapter is to provide a way in order to obtain time-consistent cooperative solutions for N-person differential games with asymmetric (heterogeneous) agents. With this goal, we propose two alternative approaches. In the first one, we transform a two-person cooperative differential game into a one-agent problem with heterogeneous discounting. Then, we derive the DPE for this one-agent problem. The second approach allows us to study problems with an arbitrary number of players. For this case, we also obtain the equilibrium for the cooperative time-consistent solutions. The results are illustrated by solving a general exhaustible resource extraction model. Finally, we also study the extension to the infinite time horizon, and we extend previous results to a common access renewable resource model.

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