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Pasquale del Pezzo, Duke of Caianello, Neapolitan mathematician

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Abstract This article is dedicated to a reconstruction of some events and achievements, both personal and scientific, in the life of the Neapolitan mathematician Pasquale del Pezzo, Duke of Caianello.

Contents

1 Introduction

Francesco Tricomi (1897–1978), in his collection of short biographies of Italian mathematicians, said of Del Pezzo[1]

[1] The preposition *del* in a noble surname, such as that of Pasquale del Pezzo, is written in lower-case letters when preceded by the given name. There are different schools of thought on the orthography when the surname is not preceded by the given name: in this case we write the first letter in upper-case, as Benedetto Croce (1866–1952) used to do, e.g., see Croce (1981). However, in citations, the original orthography is maintained.

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> Pasquale Del Pezzo, Duke of Caianello, the most Neapolitan of the Neapolitan mathematicians …. He received a law degree at the University of Naples in 1880, and another in Mathematics in 1882, and soon obtained the professorship in projective geometry at that university after success in the contest for that position; he remained at the University of Naples his entire career, becoming rector, dean of the faculty, etc. He was also mayor of the city of Naples (1914–16) and (from 1919 on) senator.
> Del Pezzo's scientific production is quite meager, but reveals an acute and penetrating ingenuity; his name is now remembered primarily for the surfaces that bear it—these are the surfaces having elliptic curves as plane sections. He was one of the most notable and influential professors at the University of Naples, and, potentially, one of the greatest mathematicians of his time, but he was too distracted by politics and other matters. Innumerable anecdotes, generally salacious, and not all baseless, circulated about him, finding substance as well in his characteristic faunlike figure. As a politician, he had only local importance (Tricomi 1962).[2]

Colorful and allusive words. However, it is certainly not true that Del Pezzo's scientific production was "quite meager", as we will later see.

This paper consists of two parts. The first is dedicated to aspects of Del Pezzo's biography with the aim of putting his intellectual world, his multiple interests, and ultimately his way of doing mathematics in a more accurate perspective. In the second we concentrate on a rather detailed analysis of his more notable scientific results in algebraic geometry. We present this reconstruction also in the light of later developments.

One novelty of this paper consists in describing, also in the light of new archival sources and private correspondence, Del Pezzo's versatile character, as embedded in his time and his cultural and political environment. Although Del Pezzo's name has been attached to some fundamental objects in algebraic geometry, a detailed analysis of his original papers and new ideas contained therein was still missing, with the only exception of an account of the harsh polemic with Corrado Segre (Gario 1988, 1989). The present paper is devoted to fill up this gap, and, in doing this, we give also some new contribution to the understanding and outcomes of the aforementioned polemic.

2 Del Pezzo's life

2.1 The first years

Pasquale del Pezzo[3] was born in Berlin on May 2, 1859 to Gaetano (1833–1890), Duke of Caianello, and Angelica Caracciolo, of the nobility of Torello. Gaetano was in Berlin as ambassador from the court of Francesco II, King of the Two Sicilies, to the King of Prussia.

[2] All quotations have been translated; the original texts have been reproduced only for those which have not been published.

[3] For further biographical information, see Rossi (1990), Gallucci (1938), Palladino and Palladino (2006) and Gatto (2000).

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Del Pezzo's family, originally from Cilento, was of very old nobility from Amalfi and Salerno.

With the fall of the Bourbons and the end of the Kingdom of the Two Sicilies, the family returned to Naples, the city in which Del Pezzo finished his studies. In 1880, he received his law degree, and two years later, in 1882, he completed his degree in mathematics.

2.2 Scholarly activity

The academic career of Del Pezzo unfolded rapidly and intensely. He became "professore pareggiato" in 1885 and the holder of the professorship in Higher Geometry, first by temporary appointment beginning in 1886/87, then as "Professore straordinario" in 1889, and later as "Professore ordinario" (full professor) beginning in 1894. Previous holders of this professorship were Achille Sannia (1823–1892) and Ettore Caporali (1855–1886) from 1878/79 until 1885/86. Del Pezzo held the professorship until 1904/05. From 1905/06, he was successor to the professorship in Projective Geometry previously held by Domenico Montesano (1863–1930). Del Pezzo held this professorship until 1932/33, when he retired, having reached the age limit for the position. He was then named Emeritus Professor of the University of Naples in 1936.

In the course of his career, Del Pezzo had many other responsibilities: from 1897/98 until 1889/99, he was docent and director of the Institute of Geodesy; in 1913/14, and again from 1917/18 until 1918/19, he was in charge of the course of Higher Mathematics; from 1911/12 until 1932/33 he was head of the Institute of Projective Geometry.[4]

Del Pezzo was dean of the faculty in 1902/03 and 1913/14, and rector of the University of Naples for two two-year terms, in 1909–1911 and 1919–1921. From 1905 until 1908 he was a member of the "Consiglio Superiore della Pubblica Istruzione" (a government advisory board for public education).

He was a member of many academic societies, both Italian and international, such as the "Società reale di Napoli" (of which he was also president), the "Accademia delle Scienze", the "Accademia Pontaniana", the "Istituto di Incoraggiamento di Napoli", the "Pontificia Accademia Romana dei Nuovi Lincei", the "Société Mathématique de France", and the "Circolo Matematico di Palermo". Honors awarded include being named as "Commendatore dell'Ordine Mauriziano", "Grande Ufficiale della Corona d'Italia", and Knight of the French Légion d'Honneur.

In the Italian mathematical community, Del Pezzo was a well-known figure of his time. In 1893, he was a protagonist in a lively quarrel with Corrado Segre (1863–1924) caused by the denials of promotion to Full Professor of Del Pezzo himself, Giovan

[4] In the twenties, Maria Del Re (1894-1970) was an assistant in that Institute; she had received her mathematics degree in Naples in 1922 with highest honors. From 1926 on, Del Re was Assistant Professor of Projective Geometry and later "Libero docente" in the same discipline; then she was for a long time in charge of the course of Descriptive Geometry with Projective Aspects in the architecture faculty at the University of Naples. In the Jahrbuch Database are found 16 of her works published in the period 1923–1932, some of these presented by Del Pezzo at the Academy of Sciences of Naples. These articles, perhaps in part inspired by Del Pezzo, really should be given a more thorough analysis.

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Battista Guccia (1855–1914), and Francesco Gerbaldi (1858–1934). We will discuss this in more detail in the second part of this paper (Sect. 3.2.5).

Del Pezzo's activities were not limited to the national level. For example, in October of 1890, he wrote to his friend Federico Amodeo (1859–1946) from Stockholm:

> Now I'm thinking about Abelian–Fuchsian functions, etc., beautiful things that have very close ties with geometry, and it is necessary to study them so as not to find oneself behind the times and grown old. But without the living voice of a teacher it would be impossible for me to ~~masimagester~~ these topics. I then repay these Swedes, for what I take, with the *involutions*. In the next lecture, I will cover up to par. 7 of Sannia (Palladino and Palladino 2006, pp. 353–354).

This text is indicative of the scientific contacts Del Pezzo had with his brother-in-law Gösta Mittag–Leffler (1846–1927).[5]

Pasquale del Pezzo died in Naples on June 20, 1936.

2.3 Del Pezzo's vision of science, society, and university

In the academic year 1895/96, Del Pezzo was in charge of the inaugural lecture at the University of Naples, titled *The Rebellions of Science*. A group of students prevented him from giving his speech:

> In the Great Hall of our University, on the 16th, the solemn inauguration of the new academic year should have taken place.
> Prof. Del Pezzo, Duke of Cajaniello, should have read the address entitled "The Rebellions of Science"; however, the ceremony, which should have been noble and elevated, was instead transformed into a ruckus absolutely unworthy of the Neapolitan student body.[6]

The newspaper LA VANGUARDIA of Barcelona[7] has a lively account of this episode and does not spare any witticisms regarding the turbulence that dominated various Italian universities of the time. Of course, Barcelona too had plenty of experience with student demonstrations in those days. Beyond his scientific prestige, Del Pezzo, according to the newspaper, had been chosen to speak based on his reputation of being *ultraliberal, a declared radical, and, scientifically, a complete revolutionary*. And, in fact, he says:

> The true upholders of a doctrine are those who deny it, the true heirs of the great founders of schools are those that rebel against their authority. (Del Pezzo 1897d, p. 4).

Del Pezzo then ventures forth on an analysis of a historical and epistemological nature of various fields of science, in particular mathematics:

[5] Del Pezzo had married the sister of Mittag–Leffler, Anne Charlotte Leffler, in May of that year (1890).

[6] F. Colonna, "Vita Napoletana" in *La vita Italiana*, Anno II, Roma, December 1, 1895, N. 2, p. 176.

[7] LA VANGUARDIA, December 7, 1895, p. 4.

> […] the development of modern mathematics is largely due to the criticism of fundamental notions (Del Pezzo 1897d, p. 6).

and

> It is not appropriate to ask of a Mathematician: is this theorem true or not? It would be more useful to ask: up to what point is this theorem true? How much truth and how much falsity does it contain? (Del Pezzo 1897d, p. 20).

This last sentence illuminates Del Pezzo's point of view regarding scientific truth in his discipline. The viewpoint on science that emerges from this essay can be illuminated by the following sentence:

> Man resigns himself with difficulty to his inability to understand the true nature of things. He does not want to persuade himself that the mind can only comprehend some relations between things. The things themselves escape him (Del Pezzo 1897d, p. 18).

Del Pezzo recognizes the validity of scientific knowledge, including that of Mathematics, only insofar as it is derived from and tied to experience:

> The fundamental concepts of Mathematics, whether pure or applied, are given to us by experience …(Del Pezzo 1897d, p. 13).
> Mathematics develops under the impulse of perception, but constructions that are logical in origin are hidden beneath (Del Pezzo 1897d, p. 14).

The conclusion of this work is a series of questions and exhortations:

> If Mathematics, Analysis, Geometry, Mechanics, Physics are limited and provisional, if they do not have validity except in an extremely restricted part of space and under conditions imposed by our current means of observation, shall one then find in Ethics and Law, History and Economics those laws worthy to be called absolute and eternal? […]
> And is it then true that the relations among men will always be such: on one hand, a group of outcasts and disinherited struggling with hunger, misery and disease, and on the other, a handful of pleasure-seeking little despots who oppress and confiscate the production of common labor to secure their own advantage? Are these the economic laws of humanity, or are they rather the laws of the dominant class, boasted to be natural and eternal, and imposed on the weak and ignorant? (Del Pezzo 1897d, p. 21).
> […] The atheneum should be the center from which waves of light stream forth, it should incessantly rejuvenate the thought of the masses, which are by nature lazy and conservative. But do not hope for this, you young people, don't expect that the movement comes down from on high, do not rely on the old in spirit. The rebel of yesterday is the tyrant of today …Instead, count on yourselves …Observe, read, learn, but reflect and criticize: and do not have too much faith in dogmas and theories, without having first inspected them, do not accept the inheritance of antiquity without reservation (Del Pezzo 1897d, p. 22).

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In short, it is true, Del Pezzo was an *ultraliberal*, even if he was an aristocrat, even if he belonged, as he was fully entitled to do, to that *handful of pleasure-seekers* and of that *dominant class* that he himself criticized. Populist influences swayed him, but he could not hide a noble's disdain for the *lazy and conservative masses*, profound contradictions for a restless spirit. This passage seems most definitely to us quite an illumination of some facets of Del Pezzo's character and way of thinking, and of the scientific and cultural environment in which he lived.

Finally, we quote a few lines which indicate what Del Pezzo's model for the Italian University should be:

> […] perhaps an institution where young people are trained in the practice of the so-called liberal professions? Or, shall it be a purely scientific institution, where doctrines are expounded only for their abstract value? (Del Pezzo 1897d, p. 3).

He answers his question saying that the Italian University should represent a "middle ground between a scientific and professional institute"; it should, therefore, form qualified professionals, but also train scholars capable "of contradicting and denying the doctrines of the masters".[8]

2.4 Political activity

Pasquale del Pezzo was a politically engaged citizen. Even as a young man, though a member of one of the most noble southern Italian families, with strong ties to the Bourbon monarchy, he openly declared himself as a supporter of the new Italian state and of liberal ideas, on which he often discoursed in the salons he frequented. These ideas are re-echoed in Del Pezzo (1911), a speech given in occasion of the fiftieth anniversary of the proclamation of Rome as the capital of Italy.

In later years, Del Pezzo aligned himself with the liberal-democratic coalition, and was a backer in 1906 of the "Fascio Liberale" that reunited the opposition to the moderate party of Ferdinando del Carretto (1865–1937). In July 1914, he was a candidate in the municipal elections as a member of the "Blocco popolare", which united the constitutional democratic party, the radicals, the republicans, and the socialist reformers, in opposition to the "Fascio dell'Ordine" of a conservative ideology. Other Neapolitan intellectuals were also members of the "Blocco popolare" (the "bloccardi")—for example, the famous poet Salvatore di Giacomo (1860–1934)—while the "Fascio dell'Ordine" could count on the support of the philosopher B. Croce.

The electoral battle was fierce and unsparing in its attacks Alosco et al. (1992, pp. 128–129). The results of the elections were favorable, though only by a little, to the "Blocco". Del Pezzo was thus called to take on the responsibilities of the mayorship. The new city government was successful in realizing some reforms, the first of which was the introduction of lay public instruction. But the outbreak of the world war and the subsequent Italian participation in the conflict caused new, grave problems for the city of Naples—the greatest being providing basic necessities and controlling the

[8] For further discussions on the contribution of other Neapolitan mathematicians to subjects like the dualism between science and philosophy, and the model of university, see Gatto (2000, pp. 121–142).

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rise of prices. In this situation, Del Pezzo's coalition was not successful in realizing the principal aims of its program and was forced to make compromises with the old powers. This caused bitter divisions in the majority. After having tried to avoid a crisis with various reshufflings, Del Pezzo resigned in May 1917 (Rossi 1990).

A hint of the difficulties encountered by Del Pezzo is found in the correspondence between B. Croce and the philosopher Giovanni Gentile (1875–1944), which we will take into consideration in a moment. Del Pezzo, in any case, did not abandon politics: after the end of the war, he was, in fact, nominated senator on October 6, 1919.

Del Pezzo also distinguished himself in different humanitarian activities. For example, in 1915, he was awarded a gold medal for his efforts in organizing aid after the earthquake in the Abruzzi.

2.4.1 Del Pezzo's relationship with Benedetto Croce

Pasquale del Pezzo made regular appearances at the salon of Benedetto Croce, of whom he was an old friend; Mario Vinciguerra recalls how Croce held regular Sunday afternoon gatherings at his house:

> […] these [gatherings] were crowded and almost fashionable then. […] There were some representatives of highest strata of Neapolitan aristocracy, some of these old schoolmates, others known since early childhood, like Riccardo Carafa d'Andria, who in a single day transformed from an adversary in a duel into a fast friend; or, the Duke of Caianello, Pasquale del Pezzo, with that faunlike face and astute and allusive intelligence. Scion of a family so devoted to the deposed Bourbon monarchy, he had jumped the fence, even joining the freemasons, becoming a dignitary there: a strange character, ambitious, and skeptical at the same time, he made a point of telling Croce the secrets of the closed-door lodge meetings, mixed with personal petty gossip about common acquaintances. Del Pezzo was a professor of mathematics at the University; but seemingly took meticulous care to hide this side of his life from the public eye. In this scene, the representation from the university world was quite limited, indeed hostility towards that world was open, and lasted all of Croce's life.

In the correspondence between Croce and Gentile (Croce 1981), various references to Del Pezzo appear concerning different topics.[9] A letter regarding the crisis in the Neapolitan Committee for Civic Organization and Social Assistance is of particular interest; Croce was a member of this committee in 1915, during the time Del Pezzo was mayor of Naples. This letter gives evidence of moments of tension between Croce and Del Pezzo due to political reasons:

> Dearest Giovanni, I've calmed down now, but I have endured a lot of distress concerning this Neapolitan committee over which I presided. […] The majority

[9] The letters of Del Pezzo to Croce are conserved in the Croce Library Foundation in Naples, in the Institute of Philosophical Studies. These consist of about thirty letters spanning the period from 1892 until 1926. This correspondence is currently being studied by Prof. L. Carbone of the University of Naples and Dr. Talamo.

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> of the Community Board, “bloccarda”, or, rather, camorristic, did not take into consideration that the means to achieve its electoral aims might be snatched from its hands. It demanded that the mayor oppose every one of our initiatives and that he should seek to disband the Committee. And the mayor, Pasqualino del Pezzo, he who named me president in a grand popular assembly in front of the entire city […] has obtained our resignations […] Del Pezzo does not have much moral clarity.[10]

2.5 Aspects of private life

2.5.1 Anne Charlotte Leffler

Pasquale del Pezzo was married for the first time to the Swedish writer Anne Charlotte Leffler (1849–1892) in Rome on May 7, 1890.

Anne Charlotte Leffler, the sister of the mathematician Gösta Mittag–Leffler,[11] had been first married to Gustaf Edgren. She met Del Pezzo in 1888, during a voyage to Naples with her brother.[12] She had to face difficult challenges for her love of Pasquale. A free and modern woman, often frequenting the salons of the grand European capitals, she had to endure the hostility of Del Pezzo’s family. She was forced to ask for and obtain the annulment of her first marriage and obliged to convert to Catholicism.

Anne Charlotte was a friend of Sonya Kowalevsky (1850–1891). On the advice of Mittag–Leffler, Kowalevsky was appointed to a professorship at the Stockholm College, where Mittag–Leffler himself was one of the first professors. When Sonya died, Anne Charlotte completed Kowalevsky’s memoirs of childhood (Kovalevsky 1895). An Italian version of this work, translated by Del Pezzo, was published in the *Annali di Matematica* (Leffler 1891). Leffler and Kowalevsky co-authored the drama *Kampen för lyckan* (*The Struggle for Happiness*) in 1888, that achieved some success in theatrical performances.

Hallegren reports on a letter of Anne Charlotte’s to her brother G. Mittag–Leffler, Capri, June 2, 1888, in which she points out the parallels between her friend’s personality and that of Pasquale del Pezzo:

> In him I see little features that remind me of Sonja. He has her same talent; the exactly similar versatility, vivacity, intensity of expression; the equal lack of logic and compliance, the same quickness of spirit, the identical mixture of satire and skepticism towards romanticism and enthusiasm, the same perception of love seen as an essential element of life, the same dreams of a complete compatibility with a companion, for whom one could perform heroics. He continually speaks words that Sofya herself could have spoken. You have always said that only a woman can have her vision of the world, but in this case I find in front of me a man who represents her perfect counterpart. I often think that surely they were

[10] B. Croce to G. Gentile, June 8, 1915 in Croce (1981), p. 495.

[11] For a general reference on Mittag–Leffler and his family see Stubhaug (2010).

[12] Hallegren (2001) gives an account of the life of Anne Charlotte, first at Capri and then in Naples, until her premature death due to peritonitis in 1892, some months after the birth of her son Gaetano.

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> made for each other; she would always be fascinated by recognizing in a man her own thoughts and dreams, and moreover, in a mathematician! He understands her need for collaboration. At the moment, Pasquale hopes to become a writer in order to collaborate with me, just as she did earlier! (Hallegren 2001, pp. 63–64).

This text sheds some light on the figure of Del Pezzo, in his suspension between impulsiveness, fantasy, dedication and logic.

Leffler must have been also attracted by Del Pezzo's antiaristocratic attitude. He appeared to her to possess an "incredible liberalism and a freedom from prejudice, that astonishes on every point […] The only title that is dear to him is that which he obtained with his own work".[13]

Leffler wrote dramas, novels, and short stories in which women, victims of social convention, were protagonists. Her last novel, *Kvimlighet och erotik*, translated in Italian as *Femminilità ed amore* (*Femininity and love*), 1890, is quite autobiographical. It describes the love story of a Swedish woman and a noble Italian poet, Andrea Serra, the counterpart of Pasquale del Pezzo.

Benedetto Croce, who was also an important literary critic, more than once in his writings, praised Anne Charlotte Leffler. In particular in *Conversazioni critiche* he describes Anne Charlotte as a fervid admirer of Henrik Ibsen (1828–1906) and advises reading her "*In lotta con la società*" ("*In battle with society*") translated in Italian by Del Pezzo and published by him in 1913 (Croce 1918, pp. 344–347).

> Many of those finding themselves holding the novel *In lotta con la società*, will be somewhat disoriented by its external appearance as well as by its frontispiece. The author's name is foreign, and conjoined with a quite Neapolitan title of nobility: "Duchess of Cainello". The volume is printed more in the form of a little schoolbook rather than in the manner usual for an artistic work; and, along with the publication date, bears the name of a bookstore and handbook repository, as if it was distributed by one's professors, for use on exams: not to mention certain bibliographical references that pop out in the first pages, constructed of numbers, letters, square parentheses, resembling algebraic formulas! …. And the strangeness of the impression left by this jumble of exotic and scholastic is magnified when it is seen that the preface is signed by a poet, whose spiritual aspect is as far from and discordant with exoticism as it is from and with academicism: Salvatore di Giacomo. In the present case, I am, I would say, already an initiate, none of this can astonish me, because I hold in my soul the image of Anne Charlotte Leffler, the wife of my friend Pasquale del Pezzo, Duke of Caianello, professor of higher geometry, and now of projective geometry, at our university. She died after a few years of marriage, in Naples in 1892; and I remember that indeed it was I and Di Giacomo who numbered among the few who in that brief time had the pleasure of her company (Croce 1918, p. 341).

The echo of Anne Charlotte Leffler's passing from this world did not end with the praises of Croce and Di Giacomo. Leffler is still mentioned today as a part of Swedish literature. And, indeed, 20 years after her death her fame still endured in Italy;

[13] Letter of May 17, 1888 (Hallegren 2001, p. 28).

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among the letters of the Volterra archive, conserved in the Library of the Accademia dei Lincei, there is one, dated 1911, addressed by the young Gaetano Gösta Leffler del Pezzo to Vito Volterra (1860–1940) in which he accepts an invitation to give a lecture in remembrance of his mother.

Gaetano del Pezzo (1892–1971), the only child of the Del Pezzo-Leffler couple, was quite devoted to the memory of his mother and to the Swedish side of his family and kept up an enduring contact with his uncle Gösta, whose name he bore as his middle name. Gaetano became an instructor of analytic geometry in the years from 1917/18 until 1920/21 at the University of Naples (Gatto 2000, p. 492).

Del Pezzo remarried in 1905, to another Swedish woman, Elin Maria Carlsson, the governess of his son Gaetano.

2.5.2 Del Pezzo's relationship with Gösta Mittag-Leffler

Del Pezzo met Gösta Mittag-Leffler and had personal and scientific contacts with him before knowing his sister. A relationship which lasted well beyond the short period of marriage of Del Pezzo with Anne Charlotte, extending till Mittag–Leffler died in 1927. Their relationship is witnessed by an intense correspondence between the two: the letters of Del Pezzo to Mittag-Leffler and the drafts of the letters of the latter to the former are now at the Kungliga Bibioteket Stokholm. For a great part, this correspondence deals with family issues mainly related to the young Gaetano Gösta, whose relationship with his uncle was quite strong: he used to spend vacation periods visiting his Swedish relatives, and his father sometimes joined him.

Occasionally this correspondence touches on mathematical matters. For example, Mittag–Leffler invited Del Pezzo to join him in a scientific meeting with Karl Weirstrass (1815–1897) and Sonya Kowalevski at Werningerode (Germany). Vito Volterra also attended this meeting. The relationship of Volterra with Del Pezzo and his family probably grew out of the one of Volterra with Mittag–Leffler.

A very interesting aspect, which we want to touch upon here, concerns the involvement of Del Pezzo and Mittag–Leffler in various financial initiatives, among which one, at a very high level, with the aim of getting resources for the development of agriculture in the South of Italy. To this purpose, they tried to create a bank and obtain the issuing of state bonds. This aspect cannot be treated here in more detail. We mention it here to show how complex and varied were the interests of Del Pezzo.

3 Written works

Pasquale del Pezzo wrote more than fifty papers. Most of these concern algebraic geometry. They can be subdivided according to their subject matter as follows:

(i) Algebraic curves: Del Pezzo (1883, 1884, 1889a, 1892b);
(ii) Algebraic surfaces: Del Pezzo (1885c, 1886a,b, 1887a,c,d, 1888b, 1897c);
(iii) Singularities of algebraic curves and surfaces: Del Pezzo (1888a, 1889b, 1892a, 1893c,b);
(iv) Projective geometry: Del Pezzo (1885b,a, 1887b); Del Pezzo and Caporali (1888); Del Pezzo (1893a, 1908, 1933, 1934b, 1935);

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(v) Cremona transformations: Del Pezzo (1895a, 1896a,b, 1897b, 1904, 1932, 1934a);
(vi) Other mathematical papers: Del Pezzo (1881, 1893d);[14]
(vii) Polemical writings (the polemic with C. Segre): Del Pezzo (1894, 1897e,f,a);
(viii) Various papers (speeches, commemorations, etc.): Del Pezzo (1895b, 1897d, 1906, 1911, 1912).

3.1 A general overview

Del Pezzo dealt with various topics, concerning the study of algebraic varieties, and above all, surfaces in projective space of any dimension. His techniques are mainly those of a projective nature, based for the most part on synthetic considerations. In general Del Pezzo avoided calculations even if at times he resorted to doing so to treat some particular aspect of the problems he confronts. Del Pezzo thus seems completely a part of the *Italian School* of algebraic geometry founded by Luigi Cremona (1830–1903).[15]

The characteristic feature of the School, of discovering *often without exertion, hidden properties* (Castelnuovo 1930, p. 613), seems to have engaged Pasquale del Pezzo and guided his lines of inquiry. He was directed by one of his mentors, Ettore Caporali, who was not much older than Del Pezzo.

Caporali had been appointed Assistant Professor of Higher Geometry at the University of Naples in 1878 at the age of twenty three, and became Full Professor in 1884. To the great consternation of his colleagues, Caporali committed suicide when he was only thirty one on July 2, 1884, obsessed by the idea that his intellectual capacity was declining. His research area was projective geometry, whose study he undertook using Cremona's synthetic point of view; he was considered to be one of Cremona's most brilliant students. He published 12 memoirs, but others were left still unedited when he died, and were submitted for publication posthumously due to the efforts of his colleagues, including Del Pezzo (Caporali 1888).

Besides Caporali and Sannia, among researchers in geometry in Naples perhaps the most illustrious was Giuseppe Battaglini (1826–1894) (Castellana and Palladino 1996). Battaglini was the mentor of the algebraist Alfredo Capelli (1855–1910), who also taught at Naples. Battaglini, who had been appointed Professor of Higher Geometry in 1860, founded the *Giornale di Matematiche* with Nicola Trudi (1811–1894) and Vincenzo Janni (1819–1891) in 1863. This journal published research and teaching

[14] The paper (Del Pezzo 1881) is the first mathematical contribution by Del Pezzo. At the time he was still a student in mathematics, but he had already graduated in law and he was interested in mathematical aspects of political economy. This article contains the exposition of a talk that Del Pezzo gave at the "Circolo universitario Antonio Genovesi" in Naples in which he presented a mathematical restatement of ~~Lèon~~ Walras' (1834–1910) theories of exchange and money. This exposition was praised by Walras himself (Jaffe 1965, Letter no. 488, p. 673, vol. 2). In the years preceding his professorship, Del Pezzo's was quite oriented towards applications of mathematics to social sciences as witnessed by his correspondence with Walras (Jaffe 1965, Letter no. 675, p. 71, vol. 2). This is a further sign of his multiple interests, which would be worth going deeper into.

[15] For specific considerations about various aspects of this school see, for example, Brigaglia and Ciliberto (1995, 1998).

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articles, as well as expository papers: Del Pezzo (1893a) appeared there. Battaglini moved to Rome in 1871, but returned to Naples in 1885. Certainly Del Pezzo had scientific connections to the active mathematicians in Naples in his youth, in particular with Battaglini, who appears as one of the presenters of some of Del Pezzo's first papers at the Academy of Sciences of Naples, along with another main character of the Neapolitan school, Emanuele Fergola (1830–1915).

Del Pezzo's guiding star, upon which he entrusted his work almost completely, was geometric intuition, a gift with which he was certainly amply endowed. This is clear even from a superficial reading of his work. However, in the opinion of the mathematicians of the time and in their working practices, intuition was not a gift of nature. It came, according to Cremona, from the acquisition of a refined technique consisting in mastering a series of propositions and methods, founded on the extension to projective spaces of higher dimension of properties and concepts holding in plane and three-dimensional projective geometry. These extensions to higher dimensions were not purely intellectual exercises, but they were motivated by natural developments of the discipline. For example, this happened in the study of curves and surfaces, even those considered to be the most simple, such as *rational* curves and surfaces.

Del Pezzo's work proceeds in this direction, along the lines drawn by Cremona and his master Caporali. However, even along these new tracks, one could remain in a *routine* line of inquiry. This is not Del Pezzo's case. Indeed, he ventured forth on unexplored and very fertile terrain. In fact, next to various more standard works—groups (iv) and (v)—Del Pezzo attacked some of the most interesting open problems of the time as the ones in (ii) and (iii).

Del Pezzo, in his most daring research, furnished with only his acumen and a few higher-dimensional projective techniques, ventured on a terrain at his time little explored after the pioneering work of Bernhard Riemann (1826–1866), Alfred Clebsch (1833–1872), Cremona and Max Noether (1844–1922): the study of surfaces in projective space of any dimension, their projective and birational classification, and the resolution of singularities. On these subjects, Del Pezzo indicated some of the main directions of research and accomplished some key results that formed the base of future developments. However, the lack of adequate tools, developed only later, prevented him from presenting complete proofs.

To the modernity and audacity of Del Pezzo's research, one should add a feature which limited that research, according to his contemporaries, and which was at the heart of a heated polemic that opposed him to Corrado Segre (cfr. the following Sect. 3.2.5). Del Pezzo in fact often trusted too much in his intuitive capacity, and did not ~~not~~ subject some immature ideas, however brilliant and exciting, to the scrutiny of an attentive and necessary criticism. It seems that sometimes Del Pezzo convinced himself of the validity of some *plausible assumptions* that were clear to him, and deduced consequences *as if* they had already been proved or even had no need at all of a proof. By contrast, not all such assumptions turned out to be true. This left his works, even his important ones, spangled with gaps, imprecisions, and even unfixable and glaring errors.

Accompanying this attitude was a writing style that was too terse that left much tacitly understood, and required the reader to be already an expert. Del Pezzo did not stop to explain details, giving instead, in a rapid chain of ideas, the elements he considered essential for the reader to reconstruct the reasoning himself. The same aristocratic trait shows itself in a neglectful attitude towards citations: a specific example of this is the preamble to Del Pezzo (1889a), where no care is taken to cite the articles in which the results he mentions and uses are found. As another example, one may examine the introduction of Del Pezzo (1892a), as regards an article by Eugenio Bertini (1846–1933).[16]

3.2 Principal contributions

Del Pezzo's principal contributions concern surfaces, some of their projective-differential properties and their singularities. They belong to the groups (ii) and (iii) listed above, and were made, for the most part, between 1885 and 1893. We will concentrate our attention on these, not necessarily following chronological order, giving the rest of his work only a rapid glance later.

3.2.1 Algebraic surfaces and their hyperplane sections

We begin with Del Pezzo (1885c). This is a brief note, whose importance should not be underestimated. In fact, as noted by two of today's eminent algebraic geometers (Eisenbud and Harris 1987), this note is the basis of later important developments taking place over the course of a century. In it surfaces of degree n in a projective space $\mathbb{P}^{n+1}$ of dimension $n+1$ are classified. The degree of such surfaces is the minimum possible for a surface in $\mathbb{P}^{n+1}$ that is *nondegenerate*, i.e., not contained in any hyperplane. The hyperplane sections of these surfaces are *rational normal curves*.

Del Pezzo proved that such a surface is either one of those that are today called *rational ruled surfaces*, or is the *Veronese surface* of degree 4 in $\mathbb{P}^5$, and that they are all rational. As pointed out in the introduction of Del Pezzo (1885c), these surfaces had already been studied, the first group by Segre (1883–1884) and the last surface by Veronese (1882, 1883–1884). The interest of Del Pezzo's result lies in the proof that these are the *only* surfaces of such minimal degree. From this result, one deduces, with simple enough arguments, the classification of varieties of *minimum degree*, that is, of nondegenerate varieties of dimension m in $\mathbb{P}^r$ of degree $r-m+1$ (Eisenbud and Harris 1987)—Del Pezzo speaks very briefly of this in (1886b).

Del Pezzo's proof is simple and elegant. It is discussed in the classic texts of Bertini (1907) and Fabio Conforto (1909–1954) (Conforto 1939). This last text collects the lectures given by Enriques in Rome in the 1930s which were not allowed to appear under his name because of the racial laws against Jews. The proof also appears in more recent texts like that of Griffiths and Harris (1978, p. 525). Del Pezzo observed that if S is one of these minimal degree surfaces with $n>2$ (the case $n=2$ is clear), after projecting the surface to $\mathbb{P}^3$ from $n-4$ general points on it, one obtains a quadric; the projection is birational, i.e., invertible on an open set. This proves the rationality of S, since the quadric itself is rational. One then obtains the theorem with an accurate study of the birational inverse of the projection.

[16] Del Pezzo probably refers to Bertini (1891) (see also Bertini 1894).

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As noted in Conforto (1939, p. 278), this theorem implies a later result of Charles Émile Picard (1854–1941)[17] which asserts that the surfaces whose hyperplane sections are rational are those described by Del Pezzo, or their projections. This is equivalent to the classification, at least up to plane birational transformations, of linear systems of rational curves of dimension at least three, by way of their models of minimum degree. Such a classification for all linear systems of rational curves of positive dimension (that is including those of dimension one and two) is most delicate. It is related to another classical problem, which we will discuss soon, that of the generation of the group of birational transformations of the plane by projectivities and quadratic transformations.[18]

The paper (Del Pezzo 1887c) deals with this same cluster of ideas; this may be perhaps considered as Del Pezzo's most important work. In any case, it is that for which he is most famed. In this article, nondegenerate surfaces S of degree n in $\mathbb{P}^n$ are studied and classified. This paper studies surfaces having degree one more than the minimum possible. Their general hyperplane sections are either rational or *elliptic*, that is, of genus one. Del Pezzo came to the following conclusions: if S has rational curves as hyperplane sections, then it is the projection to $\mathbb{P}^n$ of a surface of minimum degree in $\mathbb{P}^{n+1}$. If, instead, S has elliptic curves as sections, then either S is a cone, and this is the only case possible if $n > 9$, or it is a rational surface. Del Pezzo concentrated his attention on these last surfaces, studying them with his projection method invented in Del Pezzo (1885c). In fact, such a surface, projected to $\mathbb{P}^3$ from $n - 3$ general points lying on it, has a non-ruled surface of degree 3 as a birational image. These last surfaces, in turn, had been studied in detail by various authors, among them Cremona in his famous memoir for which he was awarded the Steiner Prize of the Berlin Academy of Sciences in 1866 (Cremona 1867a,b). Profiting from Cremona's results, Del Pezzo succeeded in subdividing the surfaces under consideration into two types: the first type appears for every value of n between 3 and 9, and the second only if $n = 8$. For the surfaces of the first type, Del Pezzo explicitly identified its *plane representation*, or, the linear system of plane curves of genus one and minimal degree corresponding to the hyperplane sections of S: this is the linear system of plane cubics passing through $9 - n$ sufficiently general base points. Del Pezzo postponed to a later exposition the plane representation of the surfaces of the second type, which appear only for $n = 8$, but no trace of such a work appears in his bibliography.[19] However, from his analysis, one may easily deduce that this representation is given by the system of plane curves of degree four passing with multiplicity two through two base points. All such surfaces are today called *Del Pezzo surfaces*. The later note (Del Pezzo 1897c) concerns the study of an interesting particular surface of this type with $n = 6$, whose projection to $\mathbb{P}^3$ presents a singular curve formed by nine double lines, while, in general, it is given by a double irreducible curve of degree nine. The

[17] See Picard and Simart (1897, 1906, Tome II, pp. 59–63).

[18] For more details on this subject, see the historical note on Conforto (1939, p. 3[illegible]r more recent results and a bibliography, both classic and modern, see Calabri and Ciliberto ~~(2009)~~.

[19] The plane representation of these specific surfaces is given by a linear system of curves of degree 4 with two base points of multiplicity 2, see Guccia (1887); Martinetti (1887). More details will be given in a moment.

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Del Pezzo surfaces are ubiquitous in the classification of varieties, as we try to explain now.

The whole of chapter III in the second part of Conforto (1939) is dedicated to the classification of surfaces whose hyperplane sections are elliptic curves. As shown in the first section of this chapter, such surfaces are either ruled (and thus are part of the classification of Segre 1885–1886a), or are Del Pezzo surfaces or their projections, and are therefore rational. Almost contemporaneously to Del Pezzo's studies, various other authors (Bertini 1877; Guccia 1887; Martinetti 1887) were conducting research of their own on the reduction to minimal order of linear systems of positive dimension of plane elliptic curves, as well as of linear systems of curves of larger genus (Conforto 1939, p. 329). A good number of these last papers are affected by an objection made by Segre (1900–1901) to an argument used therein. This argument went back to M. Noether in his erroneous proof of the fact that the group of birational transformations of the plane, called the *Cremona group*, is generated by projective and quadratic transformations. This theorem was later proved by Castelnuovo and is therefore called the *Noether–Castelnuovo* theorem.[20] The link between the studies on the reduction to minimal order of systems of rational and elliptic curves with Del Pezzo's research was explained explicitly in Segre (1887), in which the essential identity of the two points of view was elucidated.

But what is the real importance of the classification of Del Pezzo surfaces, or more generally, of linear systems of elliptic curves of positive dimension? In order to appreciate this, one needs to jump roughly 10/, years forward in time and consider the fundamental work of Castelnuovo and Enriques on the classification of algebraic surfaces. One of the cornerstones of this classification is the *rationality criterion* of Castelnuovo (1893, 1894). This states that a surface is rational if and only if its bigenus and its irregularity are both zero. The method used by Castelnuovo in his proof is quite modern: it is not substantially dissimilar from what today is called an application of the *minimal model* program, invented by S. Mori for the classification of varieties of any dimension, for which Mori was awarded the Fields Medal in 1990. Castelnuovo's proof begins with the consideration of a *very ample* linear system on a surface S, that is, a system obtained by the intersection of hyperplanes with a smooth birational model of S embedded in a projective space $\mathbb{P}^r$. Next, the *successive adjoints* of L are considered; these are the systems of type $L+nK_S$, where n is any nonnegative integer and K_S is the *canonical system* of S. Castelnuovo observes that, under the hypotheses of the criterion, the *adjunction vanishes*, which means that there is an integer $n \geq 0$ such that $D = L + nK_S$ is nonempty, while $D + K_S = L + (n+1)K_S$ is empty. This implies that the curves in D are rational. If the dimension of D is at least one, then by Noether's criterion recalled above, S is rational. If instead D has dimension 0, one considers $D' = L + (n-1)K_S$ and observes that this system consists of elliptic curves. Reiterating this argument, one can suppose that D' has positive dimension. We then have a surface with a positive dimensional system of elliptic curves, and here Del Pezzo's work plays a crucial role, allowing the conclusion that, also in this case,

[20] Cfr. Noether (1875–1876, 1870); Castelnuovo (1901); for historical notes on this subject, cfr. Calabri (2006), where a proof of the Noether–Castelnuovo theorem, inspired by the one in Alexander (1916), is given.

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S is rational. Certainly, if it is true that Castelnuovo's criterion is the cornerstone of the classification of surfaces, then it is also true that Del Pezzo's theorem forms its indispensable base.

In Enriques (1893, 1896), the role played by the multiples of the canonical linear system $|K_S|$, whose dimensions give, in essence, the plurigenera, is displayed in its full fundamental importance. Enriques' classification of surfaces is based on the behavior of the multiples of the canonical system and hence of the plurigenera. From this point of view, the Del Pezzo surfaces occupy a very special and important position. They are the only surfaces in a projective space for which the opposite of the canonical system $|-K_S|$ is cut out on the surface by the hyperplanes of the ambient space. In today's language, these are the only surfaces S such that the *anticanonical linear system* $|-K_S|$ is *big* and *nef*—meaning that $K_S^2 > 0$ and for each curve C on S one has $K_S \cdot C \leq 0$. The analogues of these surfaces in higher dimensions are the so-called *Fano varieties*.[21] These varieties were classically studied by Gino Fano (1871–1952) in a long series of papers from 1936 on.[22] Fano varieties are, in a sense that can be made precise, some of the basic building blocks in the classification of varieties. For this reason, they have been extensively studied, both classically and recently. In particular, *Del Pezzo varieties*, those in which the spatial surface sections are Del Pezzo surfaces, arise in these studies and come up in problems of classification, even today, more than a century after the publication of the research we reviewed here. Classically, Enriques dedicated two important notes to Del Pezzo varieties (Enriques 1894a,b), while in Enriques (1897), he touches on a problem that is still of great interest, that is, the study of rationality for families of Del Pezzo surfaces in relation to rationality problems for varieties of higher dimension.

3.2.2 The beginnings of projective differential geometry in Italy

Del Pezzo's article (1886a) played a foundational role in the development of the so-called *school of projective differential geometry* and its flowering in Italy in the first half of the last century.

Projective differential geometry studies properties of locally closed differentiable or analytic subvarieties of real or complex projective space. Some of the notions introduced in Del Pezzo (1886a) are typical concepts used in the discipline.

The Italian school of projective differential geometry was born at the beginning of the twentieth century in some of C. Segre's work. These papers of Segre's relate the classic results of G. Darboux (1842–1917) to those of E. J. Wilczynski (1876–1932) on the projective-differential study of curves and surfaces, but also refer explicitly to the geometric approach inaugurated by Del Pezzo. Segre discusses, in a series of articles from 1897 on, various results and problems that will form the basis of later developments, and which will come to involve a huge number of colleagues and students. The

[21] These are varieties such that the anticanonical system is *ample*, that is, such that a multiple is very ample.

[22] Cfr. the bibliography in Brigaglia et al. (2010).

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principal names to mention here are, in alphabetical order: E. Bompiani (1889–1975), G. Fubini (1879–1943), B. Segre (1903–1977), A. Terracini (1889–1968).[23]

Coming back to Del Pezzo's contributions, he made use in (1885c) of the technique of projection of a surface S in $\mathbb{P}^r$ from a *sufficiently general* subspace of dimension $r - 4$; he used this in later works as well. He was aware, however, that at times it might be necessary to effect *special projections*: those projections from subspaces not in general position with respect to S. For example, it can be useful to project S from a subspace that is *tangent* or *osculating* to S. This concept would be applied by Del Pezzo in later papers (1886b; 1887d). These ideas are crucial and used today routinely in the area of classification of projective varieties. However, at the time of Del Pezzo, not only the notion of an osculating space, but also that of tangent space to a projective variety had not yet been formalized. One of the purposes of Del Pezzo (1886a) is precisely that of introducing these concepts, that, in themselves, have not only a projective character, but also a differential one. Del Pezzo, however, did not limit himself to this alone. He also investigated how the osculating spaces to curves that are hyperplane section passing through a smooth point p of the surface S are distributed. He observed that these osculating spaces, in general, fill out a quadric cone of dimension 4 and rank 3, having as vertex the tangent plane to S at p. This cone is a notable *projective-differential* invariant later called the *Del Pezzo cone* by Alessandro Terracini in his introduction to the second volume of Segre's works (Segre 1957–1958–1961–1963). These concepts were briefly extended by Del Pezzo to the case of higher dimensional varieties. Moreover, this brief but extremely pithy note also contains two results that Del Pezzo just tossed at the reader, with proofs that are barely sketched. These proofs are even approximative and somewhat insufficient, as if they were of a minor relevance. By contrast, these are important results. The first is a basic *technique*, the second is a theorem that was fully appreciated only several years later, a true and proper cornerstone in the geometry of projective varieties.

The first result asserts that the general tangent plane to a surface intersects it in a curve if and only if the surface is ruled or lies in $\mathbb{P}^3$. It is not difficult to deduce from this an analogous result for varieties of higher dimension, see Ciliberto et al. (2004, Proposition 5.2).

The second result affirms that the Veronese surface of degree 4 in $\mathbb{P}^5$ is the only surface (besides cones) in any $\mathbb{P}^r$, with $r \geq 5$, such that any general pair of its tangent planes have non–empty intersection. The profound significance of this theorem was not fully appreciated until 1911 when the paper by Terracini (1911) appeared: this work was Terracini's thesis, with C. Segre as advisor. In this fundamental work, what is today known as *Terracini's lemma* was proved; namely, given a variety X of dimension n in $\mathbb{P}^r$, the lemma determines the tangent space at a general point of the variety $\mathrm{Sec}_h(X)$ described by the spaces $\mathbb{P}^h$ generated by $h+1$ independent points of X, with $h \leq r$. The general point of this variety depends on $(h+1)n+h$ parameters, and thus this number is the *expected dimension* of $\mathrm{Sec}_h(X)$, unless $(h+1)n+h \geq r$, in which case one expects that $\mathrm{Sec}_h(X)$ is all of $\mathbb{P}^r$. Now, it can well happen that the parameters in question are dependent. In such a case, the dimension of $\mathrm{Sec}_h(X)$ is less

[23] Some historical references can be found in Terracini (1927, 1949–1950), in the introduction to the second volume of Segre (1957–1958–1961–1963), and in Bompiani (1935, 1966).

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than the expected, that is less than $\min\{(h+1)n+h, r\}$. If this happens, X is called *h-defective*. Examples of defective varieties are cones. Since the dimension of a variety coincides with that of its tangent space at a smooth point, to understand whether X is h-defective or not, it is enough to determine the tangent space to $\mathrm{Sec}_h(X)$ at a general point x. Terracini's lemma affirms that if x belongs to the subspace generated by $x_0, \ldots, x_h \in X$, then the tangent space to $\mathrm{Sec}_h(X)$ is generated by the tangent spaces to X at $x_0, \ldots, x_h$. It follows that the dimension of $\mathrm{Sec}_h(X)$ is the expected dimension if and only if the tangent spaces to X at $h+1$ independent points on X are in *general position*, that is, these points generate a subspace of $\mathbb{P}^r$ of maximum possible dimension, this maximum being exactly $\min\{(h+1)n+h, r\}$. From here, it is not difficult to deduce that a curve is never defective. Passing to the case of surfaces, one verifies that a surface in $\mathbb{P}^r$, with $r \leq 4$, is never 1-defective. For a surface in $\mathbb{P}^r$, with $r \leq 4$, the expected dimension of the *variety of secant lines* $\mathrm{Sec}(X)$ (we omit here the subscript 1) is 5. Terracini's lemma tells us that $\mathrm{Sec}(X)$ has dimension 4, less than that expected, if and only if two general pairs of tangent planes to X intersect in a point and therefore, in accord with Del Pezzo's theorem, if and only if X is a cone or the Veronese surface.

But, why be concerned with knowing the dimension of $\mathrm{Sec}(X)$? The projection of a smooth variety $X \subset \mathbb{P}^r$ to $\mathbb{P}^s$ from a general *center of projection* $\mathbb{P}^{r-s-1}$ has as its image a variety X' *isomorphic* to X if and only if the center of projection does not intersect $\mathrm{Sec}(X)$. Therefore, after a series of such projections, one succeeds in embedding X in $\mathbb{P}^s$, with $s = \dim(\mathrm{Sec}(X))$. Furthermore, the smaller the dimension of $\mathrm{Sec}(X)$, the smaller also the dimension of the space in which one can embed X, and, thus, the easier it will be to describe X. In fact, the smaller the codimension of a variety, the smaller one expects to be the number of equations necessary to define it (for example, hypersurfaces, having codimension one, are described by only one equation). Del Pezzo's theorem is thus equivalent to the following one, proved in 1901 by F. Severi in his memoir (Severi 1901): the only smooth nondegenerate surface S in $\mathbb{P}^r$, $r \geq 5$, that can be projected in $\mathbb{P}^4$ yielding an isomorphism onto its image, is the Veronese surface in $\mathbb{P}^5$.

Classically, Gaetano Scorza (1876–1939) made important contributions to the study of defective varieties; his papers (Scorza 1908, 1909b) precede Terracini's work, and take Del Pezzo's point of view.[24]

[24] Since the 1970s the classification of defective varieties progressed tremendously, with starting point exactly the theorems of Del Pezzo, Terracini, and Severi mentioned above. To give a brief sketch of these developments, we first recall a fundamental theorem of Barth and Larsen (1972), which shows that the lower the codimension of a smooth variety X in $\mathbb{P}^r$, the stronger the topological constraints on X become: the cohomology of X *resembles* that of the ambient space $\mathbb{P}^r$ more closely as its codimension lessens. This fact led R. Hartshorne to formulate two important conjectures (Hartshorne 1974). The first affirms that if $X \subset \mathbb{P}^r$ is smooth, irreducible and nondegenerate of dimension n, and if $3n > 2r$ then X is a *complete intersection*, in other words, it is the zero set of $r-n$ homogeneous polynomials in r variables, and these $r-n$ polynomials generate the ideal of polynomials which vanish on X. This is true, as we have said, if $n = r-1$, but the conjecture is still open for $n < r-1$ (for recent results and bibliographic information on this subject, cfr. Ionescu and Russo 2009). The second of Hartshorne's conjectures affirms that if X is as above, and if $3n > 2(r-1)$ then X is *linearly normal*, that is, X is not isomorphic via a projection to a nondegenerate variety X' in $\mathbb{P}^s$ with $s > r$. This is equivalent to saying that if X is a smooth variety of dimension n, then $\dim(\mathrm{Sec}(X)) \geq \frac{3}{2}n + 1$. This second conjecture was proven in 1979 by F. Zak whose

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Before concluding the discussion on Del Pezzo (1886a), we should make some remarks on the exposition therein, clarifying some general comments made previously in Sect. 3.1. As pointed out there, various of Del Pezzo's arguments leave something to be desired. For example, in the calculation of the dimension of osculating spaces, he implicitly makes assumptions of generality that he never explicitly states, and without which the results are invalid. The imprecision of the beginning of §8 is ever more serious. Here, he affirms that a family of planes, not lying in a $\mathbb{P}^4$, such that any two intersect in a point, *in general* lie in a $\mathbb{P}^5$. Exactly what *in general* means is not explained. The fact is that there are other possibilities that Del Pezzo does not contemplate. To be precise, the planes may also pass through one single point, or all intersect a fixed plane in a line. The missing consideration of these cases is a gap in his argument. This gap is also present in §12 of Del Pezzo (1887c) and in §12 of the memoir (Del Pezzo 1893a), which is a partial collection of notes for a course on projective hyperspace geometry.[25] These deficiencies in Del Pezzo's proofs were well known to his contemporaries. For example, Scorza points them out elegantly in this passage:

> One of the most notable characteristic properties of Veronese surfaces is that stated by Prof. *Del Pezzo* in his memoir on V_2^n in S_n and proved rigorously for the first time by Prof. *Bertini* in his recent works on the projective geometry of hyperspaces.

3.2.3 General results on the classification of surfaces according to degree and genus of their hyperplane sections

Del Pezzo's articles (1886b; 1887a; 1887d; 1888b) are all related, and address a very interesting question. In the course of his research into surfaces with rational or elliptic curves as sections, Del Pezzo became aware of the validity of a general result, which he had proved in those initial cases. The result, expounded in Del Pezzo (1886b), is as follows: *there exists a function $\phi(g)$, $g \in \mathbb{N}$, such that if S is a surface of degree d having general hyperplane sections of genus g (having sectional genus g), and if $d > \phi(g)$ then S is a ruled surface.* To this is added the following: *there exists a function $\psi(r) > r - 1$, $r \in \mathbb{N}$, such that if $S \subset \mathbb{P}^r$ is a nondegenerate surface of degree d and $r - 1 \leq d < \psi(r)$ then S is a ruled surface.* Del Pezzo made some extensions to varieties of higher dimension as well, and then dedicated the articles (Del Pezzo 1887a,d) to an attempt to determine the functions ϕ and ψ.

Footnote 24 continued
work is exposed in the monograph (Zak 1993). Zak does not limit himself to discussing the proof of this conjecture. He considers smooth defective *extremal* varieties X—those satisfying $r > \dim(\mathrm{Sec}(X)) = \frac{3}{2}n + 1$—and calls them *Severi varieties*. The reason to name them so is that the first example of such a variety arises for $n = 2$, and according to Severi's theorem, is the Veronese surface in $\mathbb{P}^5$. It would be justified to ask whether a more appropriate name, given the priority of contributions, might not be *Del Pezzo varieties*. In any case, one of the major accomplishments of Zak is the classification of these varieties. Recent extensions of the results of Del Pezzo, Severi, Terracini and Scorza, other than the cited memoir of Zak, are also found in Chiantini and Ciliberto (2008).

[25] The general classification of these families of planes, with extensions to families of subspaces of higher dimension, is owed to U. Morin (1901–1968) in (1941; 1941–1942).

In order to understand the value of these results, it is enough to notice that investigations of the same type were presented a few years later in the fundamental works (Castelnuovo 1890; Enriques 1894c).[26] The theorem of Castelnuovo and Enriques, which are more precise than Del Pezzo's, states that *if* $S \subset \mathbb{P}^r$ *is a nondegenerate surface of degree d and sectional genus g, then S is a ruled surface if* $d > 4g+4+\epsilon$ *or if* $r > 3g+5+\epsilon$*, where* $\epsilon = 1$ *if* $g = 1$ *and* $\epsilon = 0$ *if* $g \neq 1$.[27] The classical approach of Castelnuovo and Enriques is not dissimilar to that proposed in Del Pezzo (1886b): Del Pezzo in fact analyzed the projection of the surface in $\mathbb{P}^3$ from $r-3$ of its general points, while Castelnuovo and Enriques considered projections from tangent spaces (see Ciliberto et al. 2008). Del Pezzo's proof applies only to the case of a surface $S \subset \mathbb{P}^r$ of degree d and sectional genus g such that $r = d - g + 1$; in particular, his argument applies to regular surfaces. As usual, Del Pezzo did not take care to make this restriction explicit, but it should be noted that this sort of subtle restriction was not used at the time of his research—the differences in behavior between regular and irregular surfaces, one of the crucial points in the theory of surfaces, were unknown then (see Brigaglia et al. 2004). Del Pezzo's proof consists of the observation that the degree of the image of the projection $S' \subseteq \mathbb{P}^3$ is $g+2$, but that S' must contain $r-3$ skew lines, the images of the points from which S is projected. For d very large, r is also very large, while the number of lines in a surface of fixed degree, if finite, is bounded. This implies that, for d very large, S' is ruled, from which Del Pezzo deduces that S is ruled as well. The second theorem is proved in an analogous way.[28] Del Pezzo's argument is very elegant and even today may be further exploited. It has not received the attention it is due; Castelnuovo and Enriques themselves seemed to ignore Del Pezzo and did not cite him; indeed he was not cited in their works coming after those mentioned here.[29]

Another theorem in Del Pezzo (1886b, §13) is *for a nondegenerate ruled surface* $S \subset \mathbb{P}^r$ *of degree d and sectional genus g, that is not a cone, then* $r \leq d - g$, a result also proved in Segre (1885–1886b).[30]

Unfortunately also Del Pezzo (1886b) cannot escape from the sort of criticisms discussed previously. We point out a couple of points where Del Pezzo paid too little attention to details that would be fully understood only later, and with much effort. Apart from the usual hypotheses of generality that were never made precise and some glaring oversights (cfr. the clearly erroneous assertion at the end of the first part of

[26] See also Jung (1887–1888, 1888–1889); related work in recent times include (Hartshorne 1969; Dicks 1987; Ciliberto and Russo 2006): the reader is referred to the latter paper for its ample bibliography and more up-to-date results.

[27] From a modern viewpoint, this result follows from a property of the adjoint system to the system of hyperplane sections—namely, that the adjoint system is nef if the surface is not ruled—a result proved in its maximal generality in Ionescu (1986).

[28] For a modern proof, see Harris 1981.

[29] It is difficult to explain this strange reaction, especially on Castelnuovo's side, since he was very careful with citation. Either they simply were not aware of Del Pezzo's work, or they considered it a minor, partial result. Castenuovo–Enriques correspondence (Bottazzini et al. 1996) starts in 1892 and it does not shed any light on this matter.

[30] For a modern version and a snapshot of recent bibliographical references on rulings and vector bundles on curves, cfr. Ghione (1981), Calabri et al. (2008).

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§ 9), we point out two assertions that, even if not proved correctly, are in themselves interesting.

The first is a basic classical result, continually used in projective algebraic geometry. This result is today known as the *trisecant lemma* or the *general position lemma*, which Del Pezzo tried to prove with a tortuous and incomplete argument at the beginning of the paper. The result is as follows: *if $S \subset \mathbb{P}^r$ is a nondegenerate surface, with $r > 3$, then its projection in $\mathbb{P}^3$ from $r-3$ of its general points is birational to its image*. This is equivalent to the statement that, if $r > 3$, the space $\mathbb{P}^{r-3}$ generated by $r-2$ general points of S intersects the surface only in those $r-2$ points.[31]

The second assertion is found in §14 of Del Pezzo (1886b): *a nondegenerate three-dimensional variety in $\mathbb{P}^6$ having the Veronese surface of degree 4 in $\mathbb{P}^5$ as a general hyperplane section is a cone*. In modern terminology, this means that the Veronese surface is not *extendible*: an extendible variety is one that is a hyperplane section of another variety that is not a cone. It is worth noting that every variety is a hyperplane section of a cone with vertex a single point. The argument proposed by Del Pezzo is incomplete: he bases it on the faulty reasoning we have already noticed when given in Del Pezzo (1886a, 1887c, 1893a), regarding families of pairwise incident linear spaces. This proposition was also stated in Segre (1885–1886b). A proof appears in the book by Bertini (1907, Chap. 15, §10). Scorza refers to this text, and to C. Segre, but not to Del Pezzo in his short, very elegant note (Scorza 1909a) in which he generalized the theorem, proving the inextendibility of all Veronese varieties.[32]

In (1887a; 1887d) Del Pezzo attempts to determine the functions ϕ and ψ mentioned earlier.[33] Also here Del Pezzo makes errors that lead him to state results that in general are not true. The principal is the following: he asserts that *every linearly normal surface S of degree d^2 in $\mathbb{P}^{\frac{d(d+3)}{2}}$ is a Veronese surface, that is, the immersion of the plane in $\mathbb{P}^{\frac{d(d+3)}{2}}$ determined by the complete linear system of curves of degree d* (cfr. §5). This assertion is false already for $d = 2$ and $\mathbb{P}^5$—other than the Veronese surface of degree 4, there are also the normal ruled rational surfaces, as Del Pezzo knew quite well. In general the existence of ruled surfaces, for example cones, contradicts Del Pezzo's assertion. But these are not the only counterexamples; one can

[31] For modern versions, cfr. for example Griffiths and Harris (1978, p. 249), Laudal (1978) and Chiantini and Ciliberto (1993).

[32] Scorza also proved the analogous theorem concerning the inextendibility of *Segre varieties*, that is, product varieties of two or more projective spaces. A different proof of the inextendibility of Veronese varieties, which uses techniques from differential geometry, was given in Terracini (1913–1914, note I, §6), which cites in order Segre, Scorza, Bertini, A. Tanturri (1877–1924) (Tanturri 1907), but not Del Pezzo. A proof of the inextendibility of *Grassmann varieties* other than $\mathbb{G}(1, 3)$, inspired by the arguments of Scorza, is found in Di Fiore and Freni (1981). For an elegant recent approach to these questions, see GR08. In the past 20 years, problems of extendibility have seen a renaissance, beginning with the papers (Wahl 1987; Beauville and Merindol 1987) that point out a fundamental cohomological invariant of a canonical curve that controls extendibility. Following these papers, various contributions have been made, for example see Bădescu (1989); Ballico and Ciliberto (1993); L'vovski (1989); Zak (1991) for more information, and for a glance at the principal results in this line of inquiry.

[33] For a modern exposition and extensions of these results, see Ciliberto (2006); Ciliberto et al. (2008).

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construct many others.[34] Del Pezzo's error in his proof of this proposition lies in a mistaken use of projections from osculating spaces. He assumes implicitly that the generic d-osculating space intersects the surface in a finite number of points, while this is not always so: a surprising error, seeing that Del Pezzo himself was the first, as we have seen, to characterize surfaces for which the general tangent plane intersects it in a curve. This error invalidates all other results in Del Pezzo (1887d), which, even so, remains interesting: it leaves open the problem of characterizing those surfaces for which the general osculating space to the surface intersects it in a curve, as well as the problem of finding a characterization of the Veronese surface in the spirit suggested by Del Pezzo.

Finally we point out the strange note (Del Pezzo 1888b), merely an announcement of results and only a few lines in length. In this the author stated that he has found the following result: *every non-ruled surface of degree d and sectional genus $g \leq d-2$ is rational*—a inescapably flawed result. The first counterexamples are surfaces of degree $d = 6$ and sectional genus 4: one, a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ (a *K3 surface*, that is, a regular surface with trivial canonical system), the other is the famous *Enriques surface* in $\mathbb{P}^3$ whose curves of double points form the edges of a tetrahedron. Putting this note in its correct context, we notice that it precedes the famous Castelnuovo criterion for rationality by some years. Thus, at the time, to recognize the rationality of a surface was not an easy task, and, of the two counterexamples listed above, the first was perhaps known, but its irrationality was not clear, and the second was not yet known: it was first pointed out by Enriques to Castelnuovo in a famous letter dated July 22, 1894, Bottazzini et al. (1996, p. 125, letter no. 111), and was decisive in suggesting to Castelnuovo the correct hypotheses for his rationality criterion. Indeed, at the time, researchers in this area still walked on quicksand, and the note (Del Pezzo 1888b) confirms this, making us appreciate even more the giant step forward made by the contributions of Castelnuovo and Enriques. On the other hand, the fact that Del Pezzo (1888b) was not followed by a publication with the proof of the announced result, suggests that Del Pezzo himself had become aware of his error.

3.2.4 Singularities of curves and surfaces

Del Pezzo's works on this subject are those in group (iii). Apart from Del Pezzo (1893b,c), which are in sequence and concern singularities of plane curves, the remaining papers deal with the problem of resolution of singularities for surfaces. These papers constitute a focal point for the lively polemic between Del Pezzo and Corrado

[34] As shown in Castelnuovo (1890) and in Ciliberto et al. (2008, Theorem 7.3), for every $g \geq 2$, there exist rational, nondegenerate and linearly normal surfaces $S \subset \mathbb{P}^{3g+5}$ of degree $4g+4$ and sectional genus g that possess a linear pencil of conics and thus have general hyperplane sections that are *hyperelliptic*, that is, double covers of $\mathbb{P}^1$. Fixing $d \geq 5$, let $g = \binom{d-1}{2}$, and consider such a surface, projecting it from $d^2 - 6d + 8 > 0$ of its general points. The image is a linearly normal surface of degree d^2 in $\mathbb{P}^{\frac{d(d+3)}{2}}$. It too has a linear pencil of conics and thus is not a Veronese surface of degree d^2, since all curves on this last surface have degree multiple of d.

Segre—several of the writings in group (vii) also concern this quarrel. The papers (Del Pezzo 1888a, 1889b, 1892a, 1893b), as well as the polemical notes listed in (vii), and the contributions of Segre (1897, 1896–1897, 1897–1898) have been analyzed and commented on critically, with many bibliographic references and with a glance at later developments as well, in Gario (1988, 1989, 1991, 1994) and Palladino and Palladino (2006). The interested reader should consult these references for more insight into the conflict.

The resolution of singularities of algebraic varieties is a fundamental problem, posited at the very beginnings of algebraic geometry. The problem is that of assigning a smooth birational model to any projective irreducible variety. The interest in doing this lies in the fact that, for smooth varieties, basic techniques such as intersection theory for subvarieties or linear equivalence, work without problems, while for singular varieties things are complicated, at times in an inextricable way, rendering the classification problematic.

For curves, the resolution of singularities was realized by Noether (1871), Leopold Kronecker (1823–1891) (Kroneker 1881) and George Halphen (1844–1889) (Halphen 1874, 1875, 1876). At the time Del Pezzo's contributions appeared, that is, between 1888 and 1893, the analogous problem for surfaces was one of the most important open questions considered by geometers. Del Pezzo, without question, deserves the recognition for having first tackled this problem, which would remain open until 1935 when it was solved by R. Walker (1909–1992) in Walker (1935), followed by the work Zariski (1939) of O. Zariski (1899 –1986), in which a different proof was given for the resolution of singularities for a surface embedded in a smooth three-dimensional variety by way of successive *blowups*. The papers of Walker and Zariski followed a long series of partial and incomplete contributions of various authors, including Del Pezzo and Segre. Among these we mention the following: B. Levi (1875–1961), who was a student of C. Segre and had been directed by Segre towards this topic;—Levi's first work Levi (1897) consisted of an attempt to correct and complete some gaps in Segre's approach; O. Chisini (1889–1967), who in (1917) confronted the problem of the *immersed resolution* of surfaces in $\mathbb{P}^3$; F. Severi in (1914), of which we will speak more shortly; G. Albanese (1890–1947), who in (1924a) furnished an ingenious proof of the resolution of singularities of curves with a method of iterated projections and then attempted an extension to the case of surfaces in (1924b), a method that was later to be extended to higher dimensional varieties by G. Dantoni (1909–2005) in 1951; 1953 (cfr. Lipman (1975) for general considerations on this subject and the introduction in Ciliberto et al. (1996) to the collected works of G. Albanese).

As Zariski observes, commenting on contributions to the resolution of singularities (cfr. the book Zariski 1935, Chapter I, §6, p. 16)

> The proofs of these theorems are very elaborate and involve a mass of details which it would be impossible to reproduce in a condensed form. It is important, however, to bear in mind that in the theory of singularities the details of the proofs acquire a special importance and make all the difference between theorems which are rigorously proved and those which are only rendered highly plausible.

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This sentence suggests that, in Zariski's view, all works cited above, and first of all those of Del Pezzo, contain only plausibility arguments for the resolution of singularities, but no proof.[35]

Returning to Del Pezzo, the first article in this line of inquiry, Del Pezzo (1888a) is only five pages long. In it, rather than offering proofs he suggested a method for resolving singularities. Given an irreducible surface S in $\mathbb{P}^3$, Del Pezzo considered a linear system $\mathcal{L}$ of surfaces of very large degree, with general element having the *same singularities* as S. Letting r be the dimension of this linear system, it determines a rational map $\phi_{\mathcal{L}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^r$ which, restricted to S, induces a birational map from S onto its image, which, according to Del Pezzo, should be a smooth surface. This procedure would thus realize the resolution of singularities of S. We remark that this idea is not at all a mistaken one. It reappears in a more articulated form, in the attempt of Severi (1914) as well. To be precise, Del Pezzo's assertion is completely equivalent to the resolution of singularities. The only problem is that of *proving the existence of the system* $\mathcal{L}$ and requires first a precise definition of what it means for the general surface in the system to have the *same singularities* as S. This is not only is not clarified, but also not even considered in Del Pezzo (1888a).

Del Pezzo must have soon been well aware of this shortcoming, or it must have been pointed out to him by some critic, since he returns to this question in Del Pezzo (1889b), in which he attempts to elucidate his assertions. One sees the echo of these objections in the polemical note Del Pezzo (1897e):

> Some voices have been raised against the value of my writings, hinting at grave errors threaded throughout, and I have had to often confront this in private conversations, striking down some observations, refuting some mistaken claims about the validity of the theorems I have stated, and every single time that I have had the opportunity to sit down at my desk calmly with one of my critics and examine my papers, I have always had the fortune of convincing them of their soundness and of converting them to my side (Del Pezzo 1897e, p. 3).

Del Pezzo proposes the following definition:

> We will say that two surfaces have the same singularity ω or λ at the point O or along the curve L, when any plane π cuts them in two curves, having at O or at all the points of L, the same singularity (Del Pezzo 1889b, p. 238).

Obviously Del Pezzo assumed that the reader knows the analogous notion for curves, which he reviews tersely in the first part of the note. The problem is that the definition cited above is clearly lacking something. In fact, if by *any plane* Del Pezzo really meant, as it would seem, *each plane*, then the definition is too restrictive. In this case, in fact even two surfaces having a simple point at O and tangent there may not have the same singularity at O. Here it is enough to consider two quadrics, one smooth and one a cone, tangent at a point O where both are smooth. The tangent plane cuts the first quadric along two lines through O, and the second in a double

[35] The resolution of singularities for any variety over the complex numbers, was proved by Hironaka (1964), who was awarded the Fields Medal for this accomplishment in 1970.

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line through O, and the singularities of these two curves are not the same. If instead Del Pezzo meant by *any plane*, *a general plane*, then the definition is too weak. Here one may consider the surfaces having, near the origin O, defining equations of the form $x^2 + y^2 + z^2 + \cdots = 0, \quad x^2 + y^2 + \cdots = 0$, where $\cdots$ stands for terms of degree at least three in x, y, z. These are intersected by a general plane through O in a curve with a node, and the two curves have the *same singularity* at O. However, one certainly should not consider that the singularities of the two surfaces are *equal* at O: one has as tangent cone an irreducible quadric (O is a *conic* double point) and the other a pair of planes (O is a *biplanar* double point).[36]

Del Pezzo then unsuccessfully proposed in (1889b) the construction of a linear system $\mathcal{L}$ with the properties he required. If S has homogeneous defining equation $F = 0$ of degree m, it is enough to take $\mathcal{L}$ to be the system of surfaces defined by equations $FG + H = 0$, where H has degree $d >> 0$ and the surface defined by $H = 0$ passes through each singular point of S with multiplicity greater than that of S at the point, and where G is any homogeneous polynomial of degree $d - m$. Obviously this creates a circular argument, since it is not clear what is meant by saying that $H = 0$ passes through each singular point of S with multiplicity greater than that of S at that point.

On the other hand, also Del Pezzo considered an analogous questions, also in Del Pezzo (1893c, §I), in which he examines the case of plane curves, with the aim of giving a new proof of the desingularization of such curves. Given a plane curve C with homogeneous equation $f(x_0, x_1, x_2) = 0$, the problem is to construct a linear system $\mathcal{L}$ of plane curves passing through *all of the singular points* of C. According to Del Pezzo, taking the image of C under the corresponding rational map, one then has a birational map from C onto its image, that would then be a smooth model. Again, the problem with this reasoning, a priori correct, is that of constructing $\mathcal{L}$. Del Pezzo proposed to define $\mathcal{L}$ using the system of curves with equations

$$\sum_{i=0}^{2} G_i \frac{\partial f}{\partial x_i} = 0, \tag{1}$$

where G_i, $i = 0, 1, 2$, are homogeneous polynomials of degree $d >> 0$. Thus, this is the system of curves of degree $d >> 0$ *generated* by the *polars* of the curve, with equations

$$\frac{\partial f}{\partial x_i} = 0, \quad i = 0, 1, 2. \tag{2}$$

The system of equations (2) defines, as is well known, the locus of singular points of the curve. Thus it is natural to claim that the general curve with equation of type (1) contains all the singular points of the curve. However, for Del Pezzo's argument to work, it is necessary that each such curve not only passes through the actual, *proper*,

[36] The problem of reducing the concept of *equal singularities* for surfaces at isolated double points to that of their plane curve sections was resolved many years later in Franchetta (1946): he correctly interpreted the notion of *having the same singularity* as the existence of an analytic isomorphism in a neighborhood of the singular point that maps one surface to the other in that neighborhood.

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singular points of C, but also through the *infinitely near* singular points, obtained by iteratively blowing up the plane at the singular points of C, and then at the singular points of its subsequent transformed curves. However, this does not always happen. The first who showed that it is not true that the curves in system (2) pass through all the singular points of C, even those infinitely near, with the expected multiplicity, was Segre (1952).[37]

Finally, Del Pezzo (1892a) deals with the embedded resolution of the singularities of a surface in $\mathbb{P}^3$. One can make the same objections noted above to this paper as well.

3.2.5 The polemic with C. Segre: scientific controversy or academic quarrel?

The polemic with C. Segre unfolded in two quite distinct phases, of which only the second, taking place in 1897, is explicit and violent. Given the landscape of personalities and the importance of the material, the polemic expands to involve, at least emotionally, other illustrious mathematicians such as Castelnuovo and Enriques, as seen from the letters of May 19 and 20, 1897 from Enriques to Castelnuovo in Bottazzini et al. (1996, pp. 334–335).

The polemic began with some objections made by Segre (1897, §27) to Del Pezzo's reasoning: objections not dissimilar to those we discussed above. Segre's remarks, even though their tone appears neither polemical nor particularly aggressive, were made point by point in a very detailed manner; in short, he offered a true *account* in which Del Pezzo's errors were exposed completely. Del Pezzo's reaction was extremely animated and, in no time, the polemic escalated to a level that was scarcely scientific in nature. To the point that the editors of Segre's Selected Works (Segre 1957–1958–1961–1963), i.e., B. Segre, F. Severi, A. Terracini, and Eugenio G. Togliatti (1890–1977), decided to omit these notes (Segre 1896–1897, 1897–1898) from the volumes.[38] Due to the slight scientific content of the quarrel in the last phases, and given that, as we said, others have already written about it, we will not further dwell on this here. Instead, we would like to shed some light on the first phase of the polemic, which took place around 1893. This was mostly underneath the surface and therefore less evident. However, we think it constitutes a precedent to the later polemic and in part explains the violence of that second phase and its departure from scientific motivations.

Del Pezzo and Segre seemed to have had a cordial relationship before 1893, apparently imbued with mutual esteem and consideration. This is underscored by various reciprocal citations, in which each gives ample credit to the other for results they use. It is worth pointing out an already cited note of Segre (1887), which highly praises Del Pezzo's results, defining them "very important", and which gives evidence of a rather regular correspondence between the two in the course of the second half of the 1880s. This correspondence was not really a true and proper collaboration, though it did resemble one. Moreover, the results of Del Pezzo that were praised are those of Del Pezzo (1887a,d); though open to a fair amount of criticism, as we have remarked

[37] Cfr. also Vesentini (1953) and for later developments, Ciliberto et al. (2008).

[38] In this regard, also see the comments in Palladino and Palladino (2006, pp. 51–52).

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already, apparently this escaped the attention of the hypercritical Segre. Segre's friendship, and that of other mathematicians, with Del Pezzo is witnessed in F. Amodeo's correspondence (Palladino and Palladino 2006). For example, Segre writes to Amodeo in a letter dated February 19, 1892 as following:

> And, what is Del Pezzo up to? What sort of research is he doing? What is the subject of his course? Tell him to write me, to write me, that I am sorry that he never gives me any news about himself – I have so much in common with him as regards outlook and ideals!

For his part, Del Pezzo regarded Segre with equal esteem and friendliness. For example, in regards to another famous polemic opposing Segre to Giuseppe Peano (1858–1932), Del Pezzo writes to Amodeo from Naples on May 18, 1891 as follows:

> I do like Segre's article, and find it interesting. Peano's response seems a *play on words*. Peano has thousands of reasons, if one is limited to speak of the definitive exposition of a subject, but the inexactnesses and outright errors in very new research areas are very freq., and do not detract an often superior merit to those investigations.

Irony of a sort, in the polemic with Peano, which flared up after Segre (1891), Segre, who was usually the one to give lessons on rigor to others, was attacked exactly on logical grounds as regards the principles of his discipline. In his defense, he pointed out that the researcher who found himself exploring new terrain must have a certain audacity not hampered by too many scruples regarding rigor—an argument that one would expect from Del Pezzo more than Segre.[39]

Notwithstanding this relationship of mutual esteem, a committee, with members Ferdinando Aschieri (1844–1907), E. Bertini, Enrico D'Ovidio (1843–1933), C. Segre and Giuseppe Veronese (1854–1917), rejected the applications of the candidates F. Gerbaldi, G. B. Guccia—founder of the *Circolo Matematico di Palermo*—and Del Pezzo himself, to promotion to Full Professor. Segre was perhaps the most active member of that committee, and he was the one who wrote up the final report on the competition. These negative judgements were annulled only a few days later by the "Consiglio Superiore della Pubblica Istruzione" (Higher Commission on Public Instruction) because of a minor quibble regarding a faulty formulation of the evaluations by the members of the committee. The first committee was then dissolved and a new one formed, with members Valentino Cerruti (1850–1909), Francesco Chizzoni (1848–1904), L. Cremona, Nicola Salvatore Dino (1843–1919) and Salvatore Pincherle (1853–1936). The new committee pronounced a judgement in favor of promoting the candidates. In particular, in the part of this second committee's report concerning the final decision about Del Pezzo, one reads:

> The committee, even if admitting that Prof. Del Pezzo's works contain errors due to negligence in writing and a disregard for details which the A[uthor] leaves to the reader's comprehension, recognizes a notable scientific value in them. In proposing difficult problems, as well as in the undertaking of their solutions,

[39] For the Peano-Segre polemic, see also, the discussion in Borga et al. (1980).

> he has shown himself to be in possession of the most delicate instruments of Geometry and Analysis. The memoir on singular points of surfaces is small in length and could have been – should have been – much longer in order to benefit the reader more, but, even so, as it is, it offers the complete solution to a very important question.[40]

Not only influential academics but also politicians tied to the failed candidates put pressure on Minister Ferdinando Martini (1841–1928) to annul the first committee and form a more accommodating new one. Giustino Fortunato (1848–1932) intervened weightily on Del Pezzo's behalf, writing to Martini on October 28, 1893, immediately after the conclusion of the first committee's deliberations the following letter. In its few lines, one may find various interesting key points. First, one notices a hint of the aversion that Francesco Brioschi (1824–1897), teacher and friend of Cremona and the *grand old man* of Italian mathematics at the time, held for the conclusions of the committee. Later, Fortunato, as an advocate of the cause of south Italy, complained of an attack on Neapolitan culture launched, in his opinion, by northern academics. This point of view was also, in part, taken by Palladino and Palladino (2006).

> Dear Ferdinando,
> more on the promotion of the Duke of Cajanello, Prof. Del Pezzo, to Full Professor of Higher Geometry here in Naples.
> Be that as it may; but the Higher Commission has, as you know, rejected the report of the committee to the Minister. Thus, justice is done. Brioschi was right to call the committee's verdict insane.
> Now what do I complain of? Well …
> As regards a professorship at the University of Naples, it was not right to trust the judgement of two Turinese, two Pavians, and a Paduan; furthermore it was not right to exclude faculty members from Naples.
> Bertina [sic], because of old scientific quarrels, was always, as is well known, hostile to Cajanello. Why marvel, then, that the verdict was pronounced with such passionate words? But, by the grace of God, the Higher Commission was not passionate in passing a summary judgement on that verdict.
> I hope that the [new] Committee, when reconsidering the desired promotion, will be formed a bit more humanely. Just so.
> I remain yours, dear Ferdinando, Giustino Fortunato.[41]

[40] La commissione, pure ammettendo che i lavori del prof. del Pezzo contengono mende dovute a negligenza di redazione e quasi a disprezzo di particolari che l'A. lascia all'intelligenza del lettore, riconosce in esso un notevole valore scientifico. Così nel proporsi ardui problemi, come nell'intraprenderne la soluzione, egli mostra di possedere i più delicati stromenti della Geometria e dell'Analisi. La memoria su' punti singolari delle superficie è piccola di mole ed avrebbe potuto e dovuto essere molto più ampia con grande beneficio del lettore, ma, anche così com'è, offre la completa soluzione di una importantissima questione. Cfr. "Del Pezzo, Pasquale", Archivio Centrale dello Stato (ACS), Roma.

[41] Caro Ferdinando, ancora della promozione a ordinario nella cattedra di Geometria Superiore qui in Napoli del duca di Cajanello prof. Del Pezzo. Sarà quel che sara [sic]; ma il Consiglio Superiore ha, come sai, respinto al Ministero la relazione della Commissione. Così, giustizia è fatta. Il Brioschi aveva ragione a dare del matto al verdetto della Commissione. Or di che mi dolgo? Ecco. Trattandosi di una cattedra della Università di Napoli, non fu equo affidare il giudizio a due torinesi, a due pavesi e a un padovano; non fu

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The letter was accompanied by an urgent telegram whose date we have not been able to discern:

> Telegram to the Ministry of Instruction, Rome.
> Evidently Professor Del Pezzo had to be sacrificed given the way that promotion committee higher geometry university Naples was composed – Do you want to promote him despite this? You would be acting justly. Giustino Fortunato.[42]

Francesco Siacci (1839–1907), Senator and member of the Accademia dei Lincei, intervened on behalf of Del Pezzo, from the academic side. Siacci wrote to G. Ferrando, General Director of the Ministry of Public Instruction, the following letter, dated September 19, 1894:

> Prof. Del Pezzo writes me from Stockholm: "The time for nominating the committee of Higher Geometry for my promotion is drawing near. You recall that when we spoke with Comm. Ferrando he agreed with us on the appropriateness of naming another committee, exactly as the Higher Commission has ruled."
> Then, he requested that I write to you, in order to kindly request, also on behalf of Guccia and Gerbaldi, that this new [underlined twice] committee be named, all three declaring that in case any member of the old committee would be named, they would withdraw their application.
> Thus, I do request all this of you, and quite willingly, because I know all of three professors and I hold them in much esteem, as does everyone certainly.
> Believe me, esteemed Comm., your v. devoted,
> Francesco Siacci[43]

At this point it is worthwhile noting the highly authoritative and influential intervention of Cremona in the dispute: Cremona at the time had been a Senator since 1877 and a member of the Central Office of the Senate—he would also be Minister of Public Instruction himself, for a month, some years later, in 1898. To this end, we

Footnote 41 continued
equo, cioè, escludere un membro della Facoltà di Napoli, ha v'ha di più [sic]. Il Bertina [sic], per antiche dispute scientifiche, fu sempre, ed è notorio, ostile al Cajanello. Quale maraviglia, che il verdetto sia stato emesso in quei termini passionati? Ma non passionato, per grazia di Dio, è stato il Consiglio Superiore, che di quel verdetto ha fatto giustizia sommaria. Io spero, che ripresentandosi la proposta di promozione voglia la Commissione essere composta un po' più umanamente. Propio così [sic]. Tu caro Ferdinando riarma [sic] il tuo, Giustino Fortunato. Cfr. "Del Pezzo, Pasquale" (ACS), Roma.

42 Telegramma al Ministro Istruzione Roma.
Dal modo come fu composta commissione promozione geometria superiore università Napoli evidentemente professore del Pezzo doveva essere sagrificato – Vuoi promuoverlo malgrado accaduto? Faresti opera equa. Giustino Fortunato. Cfr. "Del Pezzo, Pasquale" (ACS), Roma.

43 Il Prof. Del Pezzo mi scrive da Stoccolma: "Si approssima l'epoca in cui dovrà nominarsi la commissione di Geom. Superiore per la mia promozione. Ella ricorda che quando parlammo col Comm. Ferrando egli convenne con noi della opportunità di nominare un'altra commissione, giusta il deliberato del Consiglio Sup.^{e}." In seguito mi prega di scriverle perché io la preghi, anche a nome di Guccia e Gerbaldi, a far nominare codesta nuova [doppia sottolineatura] commissione dichiarando tutti e tre che qualora fosse nominata la vecchia commissione essi ritirerebbero i loro titoli. Dunque io la prego di tutto ciò, e ben volentieri perchè conosco tutti e tre i professori e li stimo assai, come tutti certamente li stimano. Mi creda, egregio Comm. suo Dev.mo Francesco Siacci. Cfr. "Del Pezzo, Pasquale" (ACS), Roma.

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reproduce the following letter from Del Pezzo to Cremona on December 3, 1894, after the conclusion of the second committee's deliberations:

> Most esteemed Professor,
> Permit me to thank you for all that you did for me in this difficult battle I have had to undergo regarding my promotion. You have been like a father to me, and I confess to you that it was my greatest joy to see your support and defense of me and to hear the benevolent words you spoke about me at the committee deliberations, words that encouraged me and compensated me for the damaging effects of the evil that others have tried to do to me. It is superfluous to add that you have my lifelong unalterable devotion, because I have already wholly dedicated that to you in my heart; I only desire now to have the opportunity to be able to actively show you my gratitude.
> Guccia told me that you would like to read my wife's biography of Kovalevsky. I will send that to you as soon as it appears in German, French or English. The translation rights have been given to three publishers for these three languages, but the volumes have not yet come out.
> Sonja Kovalevsky's 'Souvenirs d'enfance' have been published in the July and August issues of the *Review de France*, a work to which the biography written by my wife is a sequel. I do not have another copy of it; if I had one, I would send it to you.
> Permit me to thank you again, to present my respects to your wife and to declare my lifelong devotion to you, my dear and venerated master.
> Pasquale del Pezzo.[44]

As one sees in Del Pezzo (1894), a polemical note self-published in Stockholm, the works Del Pezzo presented for the promotion were (Del Pezzo, 1892a,b, 1893a,c,d). In Del Pezzo (1894), besides defending himself passionately, Del Pezzo vigorously criticizes the author—i.e., Segre—of the evaluatory report, without however, directly attacking any particular member of the committee. It is worth noting that the report had not been made public for confidentiality reasons, a negative judgement having been passed on the competitors. However, Del Pezzo had been able to get a copy of it

[44] Chiarissimo Professore, Mi permetta di ringraziarla di tutto quanto ella ha fatto per me in questa dura battaglia che ho dovuto sostenere per la mia promozione. Ella è stata per me un padre, e le confesso che la mia gioia maggiore è stata di vedermi sostenuto e difeso da lei e di udire le benevoli parole che ella ha detto per me in seno alla commissione, parole che mi incoraggiano e mi compensano ad usura del male che da altri si è tentato di farmi. È inutile che aggiunga che la mia inalterabile devozione le è acquistata per la vita, perché già prima di ora glie l'avevo interamente dedicata in cuor mio; solamente desidero di avere occasione di poterle mostrare coi fatti la mia gratitudine. Guccia mi ha detto che ella desidera leggere la biografia della Kovalevsky scritta da mia moglie. Io gliela manderò appena sarà comparsa in tedesco, o francese, o inglese. I diritti di traduzione sono stati ceduti a tre editori per queste tre lingue, ma i volumi non sono ancora usciti. Nei fascicoli di Luglio e Agosto della *Revue de France* sono stati pubblicati i 'Souvenirs d'enfance' di Sonja Kovalevsky opera a cui fa seguito la biografia scritta da mia moglie. Io non ne posseggo alcuna copia, se no gliela manderei. Mi permetta di ringraziarla di nuovo, di presentare i miei omaggi alla sua signora, e di professarmi di lui, mio amato e venerato maestro, devoto per la vita. Pasquale del Pezzo. This letter, kindly brought to our attention by Prof. Aldo Brigaglia, whom we thank here, is available among Cremona's correspondence held at the Mazzini Institute of Genoa (letter no. 053–12451).

and reproduces some passages from it. Del Pezzo complains of "an excessively critical spirit" present therein as well as

> […] the impression of not having found myself in front of impartial and benevolent judges—as older, esteemed, well-established scientists ought to be, able to discern how much new, good and praiseworthy has been done in youthful works and not to focus on the inevitable errors when making their evaluations—but instead, confronted by people resolute on a merciless demolition. Given their behavior, they did not deserve to be called judges, but public accusers. The unpublished delivery of the committee should not be called a report, but rather a prosecutor's speech (Del Pezzo 1894, pp. 1–2).

Del Pezzo did admit some responsibility:

> Naturally it is a fault to make errors, or use ambiguous terminology in writing up papers, and more care in this would be desirable (Del Pezzo 1894, p. 5).

But, at the same time, he laments the vagueness of the main points in the report:

> When they hint at proofs that are invalid, to restrictions that they believe are necessary, etc., in place of using an precise language, indicating exactly the incriminating propositions, where the holes are, or the sophisms, which restrictions they, with their elevated wisdom and foresight, would have introduced, they only make vague allusions with flowery expressions, worthy of the lawyer's art but not of the serene good sense of a mathematician. And thus they make it impossible, not only for a mere reader, but even for the author himself, to give point by point the appropriate clarifications (Del Pezzo 1894, p. 2).

By way of example, as regards the main points of (1892a), Del Pezzo reports the following sentence from the report, relative to the paragraphs §I and II, that

> […] seem to indicate that I do not have a clear conception of singularities and of the various ways in which a Cremona transformation can change them (Del Pezzo 1894, p. 6).

> And, he adds

> A severe judgement, severely expressed. But here I cannot do more than repeat what I have said at the beginning about this report. It is not scientific and it is not serious to be critical with vague words. If the author of this incredible judgement had taken the care to point out in what way and how I lacked a clear conception of singularities, maybe he would have been able to convince me of the correctness of his assertion; or, he would have come to see that, regarding singularities and transformations, his conceptions are not less clear, but different than mine, which happens many times among mathematicians who argue about the way of posing a problem; or, maybe, he would have convinced the public that he is the one lacking that clear conception (Del Pezzo 1894, p. 6).

The point that Del Pezzo made is a serious one: the report of a committee must be precise and clearly reasoned, especially when a negative judgement has been made. It

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is thus not strange that Segre, years later, returns to the question, and in Segre (1897) takes the opportunity to write the detailed and reasoned report that Del Pezzo had accused him of not having taken the time to write previously.

Finally, Del Pezzo complained about the committee having

> […] on one hand an excessive and obstinate pedantry, and on the other an immoderate ambition to rise to dictatorship, when yesterday marching in the infantry. Certain newcomers mean to assign tasks to others, to sketch out paths, and to oppose themselves even to eminent men, fathers and forebears to generations of mathematicians, have already tightly linked their name to the most ingenious and fertile scientific theories, thus immortalizing it (Del Pezzo 1894, p. 13).

Here we clearly see the allusion to true intellectual confrontation between the old professor Cremona and the brilliant young men of whom Segre was perhaps the coryphеus.[45] And here one notices Del Pezzo's annoyance, so much more acute for an aristocrat like him, in confronting the final judgement of the committee, made in a certainly very severe and paternalistic tone, not lacking in a sort of haughtiness of those who want to "rise to dictatorship, when yesterday marching in the infantry" (Del Pezzo 1894, p. 13):

> Prof. Del Pezzo has a lively and original ingenuity: however, he must restrain and direct it better, considering much more carefully his assertions and his line of reasoning, and making more accurate criticisms and revisions of his works before publishing them. On this point, as in all its preceding judgements, the committee was unanimous.[46]

Such a heavy judgement, that we hear its echo a good 70 years later, in Terracini's memoirs:

> In the committees for promotion to Full Professor, Segre was not what one would call an easy-going member. Perhaps it would be worthwhile to remember this, now that promotion to Full Professor has generally become a ordinary bureaucratic process (as a friend of mine once said, it is not denied to anyone, unless maybe to someone who has murdered his father and mother: both of them, because it seems that only one death would not suffice). Del Pezzo's denied promotion did cause a certain ruckus in his time (Terracini 1968, p. 20).

What was Segre's reason for changing his evaluation of Del Pezzo so unexpectedly, from an excellent one, to a less than mediocre judgement, to the point of denying him the promotion? We have already alluded to one reason: the not-so-secret academic quarrel with Cremona, who was a well-known mentor of Guccia, and was probably involved in the annulment of the first committee and in the chairmanship of the

[45] Concerning Segre and his school, see Giacardi (2001).

[46] Il Prof. Del Pezzo ha un ingegno vivace ed originale: ma deve frenarlo ed indirizzarlo meglio, pesando molto di più le sue asserzioni ed i suoi ragionamenti, e facendo una più accurata critica e lima dei suoi lavori prima di pubblicarli. Su questo, come in tutti i precedenti giudizi, la commissione fu unanime. Cfr. "Del Pezzo, Pasquale" (ACS), Roma.

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new one. Another reason is related to the fact that C. Segre was working quite hard on establishing the resolution of singularities for surfaces in the years of which we are speaking (Gario 1994). He perhaps felt that this ought to have been his indelible contribution to the construction of a theory that he saw realized in Castelnuovo and Enriques' works. Segre's efforts in this direction were intense, to the point that he dedicated his course on Higher Geometry in the academic years 1894–95 and 1896–97 to the study of singularities.[47] Segre might have regarded Del Pezzo's intrusion on this territory with annoyance. Finally, the main reason might be found in Segre's character: hypercritical even regarding himself, and obsessed with rigor, he could not help attacking those who did not aspire to the levels of precision he held so dear. Even Enriques, at the beginning of his career, was not exempt from his criticisms, as witnessed by a famous letter from Segre to Castelnuovo, dated May 27, 1893 (Gario 2008; Giacardi 2001), in which Segre, criticizing a preliminary draft of the famous paper (Enriques 1893) submitted for publication in the *Memorie dell'Accademia delle Scienze di Torino*, writes:

> I fervently advise rigor, rigor, rigor.

An ingenious, messy thinker like Del Pezzo must have been, on one hand, attractive to Segre because of his intuitive capacity, but on the other hand, antipodal to him as regards precision and care with details. In any case, Segre's obsession with rigor was well known, as even Castelnuovo, in his commemorative address at the Accademia dei Lincei for his colleague and lifelong friend, hinted at it, implicitly lamenting how this obsession limited Segre:

> It is really worth observing that, while he aspired to open new roads to geometric investigations, he did not make an effort then to fully explore these paths up to where they appeared fruitful. The search for simplicity and elegance that made his papers so attractive, the aversion for complicated, strained arguments and for daring endeavors which one must make in the discovery phase, perhaps kept him from fully entering into the regions that he had begun to explore. It almost seems as if a desire for artistic perfection had sometimes dulled the researcher's curiosity.[48]

We also refer the reader to a letter cited by Babbitt and Goodstein (2009, p. 803), written by Severi to B. Segre on January 2, 1932, in which Severi pronounced a cutting and ungenerous judgement on his old mentor C. Segre.

On the other hand, the existence of an academic conflict which ended up with a temporary defeat of the emergent group of which Segre was the leading exponent, is witnessed by the battle for the control of the Circolo Matematico di Palermo, which took place at around the same time as the promotion ~~context.~~ Hints of this can be found in a letter written by Gerbaldi to Amodeo, on December 28, 1892:

[47] Cfr. the notebook Gario and Segre (1995) edited by S. Di Sieno e P. Gario, with an introduction by D. Cerutti and P. Gario, and Giacardi and Segre (2002, notebooks 6 and 8) edited by P. Gario.

[48] Cfr. Castelnuovo (1924). Also in: G. Castelnuovo, *Opere Matematiche, Memorie e Note*, published under the auspices of the National Academy of hte Lincei, vol. 3, 1907–1930, Roma, Accademia Nazionale dei Lincei, 2004, p. 375.

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> Next January 21st, as you must know, the elections of the Board of Directors of the Circolo Matematico di Palermo will take place.
> From what we hear, someone (perhaps Segre) is agitating to remove Del Pezzo's name, substituting him with Veronese. If things turn out that way, we will have as Board of Directors the entire committee (D'Ovidio, Segre, Bertini, Veronese) which for some years has lorded it over and bullied everyone taking part in the ~~contexts~~ and promotions; then you know what I am talking about!
> Del Pezzo, Guccia and I have now sworn to fight this committee to the death (Palladino and Palladino 2006, p. 491).

However, it is worthwhile to hear what Segre himself said about all this. Writing in the heat of the moment to Castelnuovo on October 16, 1883, immediately after the end of the ~~context~~, he said

> All three promotions were denied (with five votes against them). The reports on Del Pezzo and Guccia, written by me, outlined all of their errors and the insufficiency of the presented documents. The papers of Gerbaldi seemed insufficient as well, especially on the geometric side, as Veronese reported.
> We were tormented by the presence of Gerbaldi, Del Re, Amodeo, Del Pezzo! Does it seem to you that we were harsh? We made all of our deliberations in full agreement, convinced that we were doing the right thing by introducing a greater seriousness in regards to ~~contexts~~ and promotions. Young people can now see that one cannot get by with sloppy little mishmashes just thrown together at the last minute. I think that the reports against promotion will not be published; if they were, you would see exactly what kind of blunders I pointed out in Guccia's stuff![49]

Another three letters to Castelnuovo followed only a few days later, on October 21 and 27, and November 5, 1892;[50] here are some excerpts:

> I just received another very bitter letter from our friend D.P. He denies that his two papers on singularities are incorrect: he says that we have not understood them! And he says some other things to me – that I will not repeat – and for which I must forgive him since they were written by an unfortunate. I begin to feel the consequences of our courage.

[49] Le promozioni furono tutte e tre respinte (con cinque no). Nelle relazioni su Del Pezzo e Guccia, fatte da me, furono rilevati tutti i loro errori e l'insufficienza dei titoli presentati. Insufficienti pure parvero i titoli di Gerbaldi, specialmente dal lato geometrico i [sic] relatore fu Veronese. Fummo afflitti dalla presenza di Gerbaldi, Del Re, Amodeo, Del Pezzo! Ti pare che siamo stati severi? Noi abbiamo preso tutte le nostre deliberazioni in pieno accordo, convinti di far bene e d'introdurre maggior serietà nei concorsi e promozioni. I giovani possono vedere ora che non si va avanti coi pasticcetti tirati fuori al momento di concorrere. Non si pubblicheranno, credo, le relazioni contrarie alle promozioni; altrimenti vedresti che razza di spropositi io ho rilevato nelle cose di Guccia!

[50] These letters, like the preceding one, are in Gario (2008).

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> Besides to D.P., I had also written to G^{a} [Guccia] but I have not yet had an answer from him. We will see.[51]
>
> Read the three letters that have cheered me so in the past few days, and then send me your thoughts on them.
>
> In explanation of G^{a}'s letter I will tell you that when writing to him I had only cited *as an example* an incorrect argument, suggesting to him a way of changing it: that besides, the report (to which I repeatedly referred him) contained a lot of criticisms. I had said that (parenthetically, I believe) I thought that the reports would not be published because it seems that reports contrary to promotions are never published. But I regret having written that if he interprets it …his way. It would be my most ardent desire that it be published!
>
> I will not write again, neither to him nor to Del Pezzo. I confess to you that I was not expecting letters so …how to describe them?
>
> The best part is that the Consiglio Superiore (spurred by Guccia?) has annulled all of our decisions relative to the promotions (so at least Cossa writes)! We gave our judgements saying (and signing) that they were all unanimous; we voted with five votes against the promotion …it was not enough! The requirement was that the secretary should have recorded in the minutes the same judgement five different times, attributing each in succession to the five individual committee members!![52]
>
> G^{a} was in Pisa tormenting the excellent b^{i} [Bertini] for two days. Then he went to Genoa with L^{a} [Loria]. I hope that they would not be seen in Turin!
>
> I am quite disgusted by the way that Cr^{a} [Cremona] has taken his protege's defeat. It is really disheartening! So much more so to think that a Cons. Sup. would stoop to such things![53]

It is of note that, after the outcome of the concorso, Segre felt it his duty to write to Del Pezzo and Guccia, probably to let them know the negative results and give an explanation. That he expected a different reaction from the actual one of open contestation, is quite singular and perhaps illuminates the *professorial* character of

[51] Un'altra lettera, molto amara, ho ricevuto or ora dall'amico D.P. Egli nega che i 2 lavori sulle singolarit à siano sbagliati: dice che noi non li abbiamo capiti! E mi dice qualche altra frase – che non trascrivo – che debbo perdonargli perché scritta da un infelice. Comincio a sentir le conseguenze del nostro coraggio. Oltre che a D. P. avevo scritto al G^{a} [Guccia] ma di lui non ho ancora la risposta. Vedremo.

[52] Leggi le tre lettere che m'han rallegrato nei giorni scorsi, e poi rinviamele raccomandate. A spiegazione di quella di G^{a}. ti dirò che scrivendogli gli avevo solo citato *come esempio* un ragionamento sbagliato, accennandogli un modo di sostituirlo: ché del resto la relazione (a cui ripetutamente l'avevo rimandato) conteneva un gran numero di critiche. Della relazione avevo detto, (credo fra parentesi), che credevo non si pubblicasse perché pare che le relazioni contrarie alle promozioni non si pubblichino. Ma mi rammarico di aver scritto ciò se egli lo interpreta …a modo suo. Sarebbe mio desiderio vivissimo che si pubblicasse! Nè a lui, nè a Del Pezzo scrivo altro. Ti confesso che non m'aspettavo due lettere così …, come chiamarle? Il bello è che il Consiglio Superiore (mosso da Guccia?) ha annullato tutti i nostri atti relativi alle promozioni (almeno così scrive Cossa)! Noi avevamo dato dei giudizi dicendo (e firmando) che erano tutti unanimi; avevamo votato cinque no …Non basta! Bisognava che il segretario trascrivesse nei verbali cinque volte lo stesso giudizio attribuendolo successivamente ai cinque commissari!!

[53] G^{a} è stato a Pisa ad affliggere per due giorni l'ottimo B^{i} [Bertini]. Poi fu a Genova con L^{a} [Loria]. Spero che non si farà vedere a Torino! Sono molto disgustato dal modo come Cr^{a} [Cremona] ha presa la sconfitta del suo protetto. Davvero è sconfortante! Tanto più a pensare che un Cons. sup. s'inchina a tali cose!

his personality, even in regards to older, though inferior in rank, colleagues. Segre himself then hinted at Guccia's pressure on the Consiglio Superiore and emphasizes Cremona's defensive shielding of his *protégé*. The use of the word *sconfitta* (*defeat*) concerning the failures seems interesting to us.

But the story does not end here; a striking final scene awaits. In fact we find, in the Volterra archive at the Accademia dei Lincei, a little postcard addressed to Del Pezzo from Volterra, dated April 16, 1899 from Turin (where Volterra taught at that time):

> Esteemed Professor, I wholeheartedly thank you for directing me to the memoir of Prof. Mittag-Leffler, excellently translated, that I presented this very day at the Accademia, which is so grateful to you for the task that you undertook. I communicated what you told me to Prof. Segre, who conveys those same sentiments to you with equal affection and feeling.
> I hope to see you in Turin when you pass through. Meanwhile …I remember with lively pleasure the days spent in Perugia, …with the greatest esteem, your most devoted and affectionate Vito Volterra.[54]

Since it would not be right to assert that Volterra's words on "same sentiments" and "equal affection and feelings" were ironic, we must think that, without fanfare, the two—Del Pezzo and Segre—had made peace with each other, less than 2 years from the outbreak of the polemic. Whether the reconciliation happened because of the intervention of third parties, or through the initiative of the two participants themselves, we do not know now. This correspondence witnesses the mutual respect between Del Pezzo and Volterra.[55]

3.3 Other writings on algebraic geometry

Del Pezzo's writings which have not yet been discussed are definitely worth considering *minor*. However, it is more worthwhile to point out some in particular.

Among the papers in (i), Del Pezzo (1889a) is a little gem. This paper treats the problem of determining the maximum number of cusps that one can impose on an irreducible plane curve of degree d. The problem is trivial if $d \leq 4$. On the other hand, no example of a rational curve with nodes and cusps, and with more than 4 cusps is yet known, and the problem of determining the maximum number of cusps on such a curve is still open. It has been conjectured that this maximum number is 4, independent of the degree of the curve. In Fernández de Bobadilla et al. (2006) this problem was attributed to F. Sakai, while evidently the question had already been considered by Del Pezzo. It is notable that Del Pezzo affirms, at the beginning of Del

[54] Egregio Signor Professore, La ringrazio sentitamente dell'invio della memoria del prof. Mittag–Leffler, ottimamente tradotta, che ho presentata oggi stesso all'Accademia, che Le è ben grata dell'incarico che Ella si è preso. Ho comunicato quanto Ella mi disse al Prof. Segre, che Le ricambia gli eguali sentimenti con altrettanta affezione ed affetto. Spero di vederLa a Torino quando Ella vi passerà. Intanto …ricordo con vivo piacere i giorni passati a Perugia, …con la massima stima, suo dev.mo aff.mo Vito Volterra.

[55] In another message with no date from Del Pezzo to Volterra, former introduces to the latter the young Oscar Veblen (1880–1960) from Chicago. This shows the presence of international contacts that Del Pezzo maintained.

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Pezzo (1889a), and without giving references, that there are no existing rational curves of degree 5 with more than 4 cusps. In Del Pezzo (1889a), with an elegant argument that makes use of quadratic transformations, Del Pezzo exhibits the equation of a curve of degree 5 having the maximum possible number of cusps, namely 5, and otherwise nonsingular.[56]

The papers in (iv) deal with classical questions of projective geometry. Among these, we cite the memoir by Del Pezzo and Caporali (1888), dedicated to a synthetic study of Grassmanians and line complexes, which though incomplete, was published after Caporali's death. The works Del Pezzo (1885b,a) are dedicated to the study of certain interesting configurations of quadrics.

The articles in (v) are for the most part dedicated to the study of quadratic transformations in $\mathbb{P}^4$. At Del Pezzo's time, the classification of quadratic transformations of $\mathbb{P}^2$ and $\mathbb{P}^3$ was assumed to be known to the experts.[57] Little was known at the time about the analogous classification of quadratic transformations of $\mathbb{P}^r$, with $r \geq 4$.[58] These works of Del Pezzo are cited and analyzed, and placed in context with later developments, in Chapter VIII, due to A. B. Coble (1878–1966), of the invaluable book AAVV (1928), which collects a large part of the classical bibliography with algebro-geometric content. In this group of papers we also point out the note Del Pezzo (1896a) in which the birational transformations of $\mathbb{P}^r$ defined by linear systems of cones are studied.

4 Conclusions

The aim of this paper has been twofold. On one side we made an analysis, gave an account of, and put in perspective, the scientific production of Pasquale del Pezzo, which was mostly devoted to projective algebraic geometry in the framework of the so-called Italian school founded by Luigi Cremona. In doing this, it has been important for us to put the accent on his way of conceiving and doing mathematics. In particular, we have tried to illustrate the role payed by these aspects in the case of the harsh polemic in which Del Pezzo confronted Corrado Segre. We have also tried to elucidate the scientific, cultural, and social context in which Del Pezzo was embedded, because we think that this is important to understand his scientific character. In this perspective we have given a suitable space to the biographical initial part of this paper.

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[56] The problem that Del Pezzo considers in this paper became quite important, for example, in the study of the fundamental group of the complement of a curve in the projective plane, cfr. Zariski (1935, Chapt. VIII). For other aspects of the question and for an extensive bibliography on classical and recent results, cfr. the already cited Fernández de Bobadilla et al. (2006).

[57] A summary of the classical results on quadratic transformations of $\mathbb{P}^3$ (the case of $\mathbb{P}^2$ is easy), can be found in Conforto (1939, Libro I, Cap. 1). Notwithstanding the many classical studies on this topic, the classification of quadratic transformations of $\mathbb{P}^3$ up to projectivities, is recent (Pan et al., 2001).

[58] This is still an open problem. Del Pezzo's works should be unquestionably useful to one who would like to undertake research here.

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