Marcinkiewicz-Zygmund inequalities

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Abstract

We study a generalization of the classical Marcinkiewicz-Zygmund inequalities. We relate this problem to the sampling sequences in the Paley-Wiener space and by using this analogy we give sharp necessary and sufficient computable conditions for a family of points to satisfy the Marcinkiewicz-Zygmund inequalities.

Key words: Marcinkiewicz-Zygmund inequalities, sampling sequences, Paley-Wiener spaces

1 Introduction

We recall the classical Marcinkiewicz-Zygmund inequalities (see (MA37) or (Zyg77, Theorem 7.5, chapter X)). Let $\omega_{n,j}$, $j = 0, \dots, n$ be the $(n+1)$-roots of the unity. We denote by $\mathcal{P}_n$ the polynomials of degree smaller or equal than n. Then for any $q \in \mathcal{P}_n$ we have

$$\frac{C_p^{-1}}{n} \sum_{j=0}^{n} |q(w_{n,j})|^p \leq \int_0^{2\pi} |q(e^{i\theta})|^p \, d\theta \leq \frac{C_p}{n} \sum_{j=0}^{n} |q(w_{n,j})|^p, \tag{1}$$

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for any $1 < p < +\infty$. The essential feature is that C_p is independent of the polynomial q and of *the degree* of the polynomial. We aim at generalizing these inequalities to more general families of points.

We will consider a triangular family of points $z_{n,j} \in \mathbb{T}$ of the form

$$\mathcal{Z} = \{z_{n,j}\}_{\substack{n=0,\dots,\infty \\ j=0,\dots,m_n}}.$$

We will denote by $\mathcal{Z}(n)$ the n-th generation of points in the family, i.e. $\mathcal{Z}(n) = \{z_{n,0}, \dots, z_{n,m_n}\}$.

Definition 1 *We say that $\mathcal{Z}$ is a M-Z (Marcinkiewicz-Zygmund) family for L^p $(1 \le p < +\infty)$ if the following inequality holds for all holomorphic polynomials q of degree smaller or equal than n*

$$\frac{C_p^{-1}}{m_n} \sum_{j=0}^{m_n} |q(z_{n,j})|^p \le \int_0^{2\pi} |q(e^{i\theta})|^p \, d\theta \le \frac{C_p}{m_n} \sum_{j=0}^{m_n} |q(z_{n,j})|^p. \tag{2}$$

Of course $m_n \ge n$ for all n. When $p = \infty$ the inequality is replaced by

$$\sup_{|z|=1} |q(z)| \le C \sup_{j=0,\dots,m_n} |q(z_{n,j})|.$$

These sort of inequalities are similar to the sampling sequences in the Paley-Wiener setting. We will show that this similarity is more than superficial and show how the same kind of results are expected. These inequalities had been studied in a Gaussian quadrature setting in (Lub98) and (MT00) and also in (KN94) for Banach spaces. In this work we consider sequences that satisfy inequalities from above and below simultaneously in (2). It is the second one that is harder to characterize (the so called reverse M-Z inequality or reverse Carleson inequality), and it is with these one that we will deal in most of these work.

A minimal M-Z family of points is a M-Z family such that $m_n = n$. These have been studied and described in detail by Chui and Zhong in (CZ99) when $1 < p < \infty$. If $p = 1$ or $p = \infty$ there are no minimal M-Z families (see Theorem 5) but there are plenty of M-Z families. When $1 < p < \infty$ a naive guess suggests that any M-Z family of points minus some points maybe a minimal M-Z family. The following example shows that this is not the case and one cannot reduce the study of M-Z families to the minimal ones.

Example 2 *Consider the triangular family $\mathcal{Z} = \{z_{n,j} = e^{2\pi i j/(n+2)}\}_{\substack{j=0,\dots,(n+1) \\ n=0,\dots,\infty}}$. Clearly $\mathcal{Z}$ is a M-Z family for L^p $(1 < p < \infty)$ but there is not any triangular subfamily $\mathcal{W}$ that is a minimal M-Z family.*

PROOF. We will argue by contradiction. Assume that $\mathcal{W} \subset \mathcal{Z}$ is a minimal subfamily in L^p. In each generation n of $\mathcal{Z}$ there is an excess of one point. Since the problem is invariant under rotations we may assume that the minimal family $\mathcal{W}$ is just $\mathcal{Z}$ minus the point 1 in all generations. Consider the polynomials $p_n(z) = 1 + z + \cdots + z^n$. The norm of p_n can be easily estimated with the classical M-Z inequality,

$$\|p_n\|_{L^p}^p = \int_{|z|=1} |p_n(z)|^p \, |dz| \simeq \frac{|p_n(1)|^p}{n+1} = (n+1)^{p-1},$$

since $p_n(z) = \frac{z^{n+1}-1}{z-1}$, for $z \neq 1$. On the other hand if $\mathcal{W}$ is a M-Z family then

$$\|p_n\|_{L^p}^p \simeq \frac{1}{n+1} \sum_{j=1}^{n+1} |p_n(e^{2\pi i j/(n+2)})|^p,$$

but

$$|p_n(e^{2\pi i j/(n+2)})| = \left| \frac{1 - e^{-2\pi i j/(n+2)}}{1 - e^{2\pi i j/(n+2)}} \right| = 1,$$

which yields a contradiction.

Of the two inequalities in the definition of M-Z families the first one is the easiest to study, it corresponds to the classical Plancherel-Polya theorem in the Paley-Wiener setting. For the sake of completeness we give a characterization in Theorem 3 of the families of points that satisfy only the first inequality. A more delicate problem is the study of such an inequality when restricted to subintervals of the arc. This has been studied in (GLN01). Our main results are Theorems 13 and 15 that provide a near description of the sequences that satisfy both inequalities.

There are several possible motivations for this work. One possible motivation is the approximation of periodic continuous functions by trigonometric polynomials. Consider for instance any triangular family of points $\mathcal{W}$ such that $\mathcal{W}(n)$ has cardinality $2n+1$. There are periodic continuous functions f such that the unique trigonometric polynomial of degree n that interpolates f in $\mathcal{W}(n)$ does not converge (in uniform norm) to f (see (Che98)). To obtain a convergent sequence of trigonometric polynomials p_n to f it is possible to use the following proposition:

Proposition 3 *Let $f \in C(\mathbb{T})$ and let $\mathcal{Z}$ be a M-Z family for L^∞. If p_n is the trigonometric polinomial of degree n that minimizes $\max_{z \in \mathcal{Z}(2n)} |p_n(z) - f(z)|$ then $p_n \to f$ in $L^\infty(\mathbb{T})$.*

PROOF. First observe that if $\mathcal{Z}$ is a M-Z family for the holomorphic polynomials with the norm $L^p(\mathbb{T})$ then $\mathcal{W}$ defined as $\mathcal{W}(n) = \mathcal{Z}(2n)$ is an L^p M-Z family for the harmonic polynomials with the norm $L^p(\mathbb{T})$. The reason is that for any harmonic polynomial π of degree n ($\pi(z) = a_0 + \sum_{1\le i\le n} a_i z^i + b_i \bar{z}^i$), the polynomial $p = z^n \pi$, when restricted to $\mathbb{T}$, coincides with a holomorphic polynomial of degree $2n$. Moreover $|p(e^{ix})| = |\pi(e^{ix})|$ for all $x \in \mathbb{R}$, thus the L^p norm of π and p are the same and the discretized norms are the same too. Therefore the description of M-Z families for harmonic polynomials can be reduced to the study of M-Z families of holomorphic polynomials. We will, as usual, identify any periodic function on $\mathbb{R}$ with a function in $\mathbb{T}$ and the trigonometric polynomials with the harmonic polynomials.

Consider the function $f \in C(\mathbb{T})$. There exists a sequence of harmonic polynomials q_n of degree n that converge to f in the uniform norm by Weierstrass Theorem. Let p_n be the harmonic polynomials of degree n that minimize $\max_{z\in\mathcal{W}(n)} |p_n(z) - f(z)|$. Clearly $\|p_n - f\|_\infty \le \|p_n - q_n\|_\infty + \|q_n - f\|_\infty$ and $\|q_n - f\|_\infty \to 0$. Moreover since $\mathcal{W}$ is a M-Z family $\|p_n - q_n\|_\infty \lesssim \max_{z\in\mathcal{W}(n)} |p_n(z) - q_n(z)|$. Since p_n minimizes the distance to f in $\mathcal{W}(n)$ we have then that $\|p_n - q_n\|_\infty \lesssim \max_{z\in\mathcal{W}(n)} |q_n(z) - f(z)| \to 0$. Thus $p_n \to f$ in the uniform norm.

A full characterization of the M-Z families for $p = \infty$ is given by Theorem 15.

There is also some motivation in the study of M-Z families that comes from a problem in Computerized Tomography. In the setting of the Radon transform in dimension two, one typically knows the integrals of a function supported in the unit disk through a finite number of lines, and one wants to reconstruct the function from the value of these integrals. The lines (in the usual parallel-beam geometry) are grouped in families of parallel lines along a finite number of directions. The number of directions (which can be identified with points in the unit circle) depends on the resolution that we want to achieve. Typically it is required that the set of directions is a uniqueness set for the polynomials of a certain degree. But it has been noted (see (Nat86, p. 70) or (Log75, p. 668)) that uniqueness is not good enough for the numerical stability of the reconstruction. We need certain stability conditions. This is exactly what the M-Z inequalities provide. Thus, in principle, the M-Z family of points provide good sets of directions to sample the Radon transform. A more detailed analysis of the application of M-Z families to the Computerized Tomography deserves a work of its own.

In light of this connection it seems also interesting to study analogous M-Z inequalities in higher dimensions (i.e., replace the circle by the sphere in $\mathbb{R}^3$ and the holomorphic polynomials by harmonic polynomials of a certain degree). Some preliminary work has already been done, (MNW01) but we don't pursue this line further.

We will rather provide metric conditions for $\mathcal{Z}$ to be a M-Z family. Our first main result is Theorem 13 which gives a sharp metric condition for a family $\mathcal{Z}$ to be M-Z. This condition is in terms of a density. When $p = \infty$ the density condition is actually a characterization. This is our other main result (Theorem 15). As mentioned before all this results are parallel to similar results for entire functions in the Paley-Wiener space. A good reference for these is (Sei04). In the next section we prove this metric characterization after some preliminary technical lemmas. Finally in the last section we briefly comment on a full characterization of M-Z families when $p = 2$. This characterization is in terms of the invertibility of certain Toeplitz operators and it is somewhat involved. We have not been able to obtain good computable conditions nor interesting examples from it.

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2 Metric conditions

We state now some preliminary results that we will need for our computation. The following inequality was found by Bernstein and Zygmund, see for instance (Zyg77, Theorem 3.16, p. 11 vol II).

Theorem 4 (Bernstein type inequalities) *For any p, $0 < p \leq \infty$, and any polynomial q_n of degree n:*

$$\|q_n'\|_{L^p(\mathbb{T})} \leq n\|q_n\|_{L^p(\mathbb{T})}$$

There is a good reason that the classical M-Z inequality does not hold in the endpoints cases $p = 1, \infty$. It is not true in this case, but in the irregular setting that we consider this is still the case.

Theorem 5 *There are no minimal Marcinkiewicz-Zygmund families for $p = 1$ and $m_n = n$.*

PROOF. Suppose that there exists such a family $\mathcal{Z}$. We start by proving that in any generation $\mathcal{Z}(n)$ two different points $z_{n,j}$ and $z_{n,k}$, $j \neq k$ are uniformly separated. More precisely there is a constant $C > 0$ such that

$n|z_{n,j} - z_{n,k}| \geq C$ for all $j \neq k$ and all n. To prove this, take the unique polynomial $p \in \mathcal{P}_n$ with values $p(z_{n,j}) = 1$, $p(z_{n,k}) = 0$ for $k = 0, \ldots, n$, $k \neq j$. Since we assume that $\mathcal{Z}$ is M-Z, then $n\|p\|_1 \simeq 1$. The Bernstein inequalities entails that $\|p'\|_1 \leq n\|p\|_1 \simeq 1$. If $\mathcal{Z}$ is a minimal M-Z family for $p = 1$ then for any polynomial q of degree n

$$\sum_{z_{n,j} \in \mathcal{Z}(n)} |q(z_{n,j})| \lesssim n\|q\|_1.$$

If we consider just one point $z_{n,j}$ in each generation and we apply it to any polynomial of the form $q_\lambda(z) = q(\lambda z)$, with $|\lambda| = 1$, we get $\|q\|_\infty \lesssim n\|q\|_1$. Therefore for any polynomial p, $\|p'\|_\infty \lesssim n\|p'\|_1 \simeq n$. Thus

$$1 = |p(z_{n,j}) - p(z_{n,k})| \leq \|p'\|_\infty |z_{n,j} - z_{n,k}| \lesssim n|z_{n,j} - z_{n,k}|.$$

Now we will build a bounded projection from $L^1(\mathbb{T})$ to the Hardy space $H^1(\mathbb{T})$ and this is well known to be impossible, thus we will reach a contradiction. To build such projection, take any function $f \in L^1(\mathbb{T})$ and for any n, consider the values of the Poisson extension $v_j = P[f](z_{n,j}(1 - 1/n))$, $j = 0, \ldots, n$. We denote by $p(z, e^{it})$ the Poisson kernel in the disk. An easy but tedious computation shows that $\sup_t \sum_{j=0}^n p((1 - 1/n)z_{n,j}, e^{it}) \leq Cn$, because the points in $\mathcal{Z}(n)$ satisfy $n|z_{n,j} - z_{n,k}| \geq C'$. Thus, $|v_0| + \cdots + |v_n| \lesssim n\|f\|_1$. Let p_n be the holomorphic polynomial that takes the values v_j at the points $z_{n,j}(1-1/n)$. Its norm $\|p_n\|_1$ is comparable to the norm $\|p_n\|_{L^1(1-1/n)\mathbb{T}}$ evaluated in a concentric circle by Lemma 7 that will be proved later. Since $\mathcal{Z}$ is a M-Z family then $\|p_n\|_{L^1(1-1/n)\mathbb{T}}$ is bounded by $\|f\|_1$. Therefore the operator Q_n that associates to each function f the corresponding interpolating polynomial p_n is a bounded projection from L^1 to the subspace of holomorphic polynomials of degree n. Now take a a partial subsequence of Q_n converging to Q. This is a bounded projection into H^1 (the polynomials are dense and are fixed by Q).

Observe that the same proof shows that there are no minimal M-Z families for $p = \infty$.

It will convenient to evaluate the norm of a polynomial not on the boundary of the unit disk, but on the boundary of some slightly smaller or bigger disk. This can be done without harm as Lemma 7 shows.

We need the following lemma:

Lemma 6 (Hardy) *Let $p > 0$, let f be holomorphic on $D(0, R)$ and define*

$$I(r) = \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

Then $I(r)$ is increasing and log-convex with respect to $\log r$.

This is a classical result of Hardy (Har15) that actually follows from elementary theory of subharmonic functions (see e.g. (HK76, Theorem 2.16)).

Lemma 7 *Let $p \in [1,\infty]$ and let q be any polynomial of degree n. For any $r \in [\frac{n}{n+1}, \frac{n+1}{n}]$ there is a constant C_p (independent of n and q) such that*

$$C_p \|q\|_{L^p} \leq \|q_r\|_{L^p} \leq C_p^{-1} \|q\|_{L^p}, \tag{3}$$

where q_r is the dilation $q_r(z) = q(rz)$.

PROOF. Let $q \in \mathcal{P}_n$. In the case $0 < r < 1$ and $p < \infty$ we obtain $\|q_r\|_p \leq \|q\|_p$ because $\|q_r\|_p^p$ is increasing by Lemma 6. When $p = \infty$, the conclusion follows from the maximum principle since $|q(z)|^p$ is subharmonic.

For $r > 1$, define

$$I_p(r) = \int_0^{2\pi} |q(re^{i\theta})|^p d\theta \quad \text{and} \quad I_\infty(r) = \max_{\theta \in [0,2\pi]} |q(re^{i\theta})|.$$

By using Hadamard's three-circle principle for $p = \infty$ and Lemma 6 for $p < \infty$, we can assume that I_p is log-convex as a function of $\log r$ for $p \in [1,\infty]$. Therefore

$$\log I_p(r) \leq (1-t) \log I_p(1) + t \log I_p(R) \tag{4}$$

where $t = \log r / \log R$. Notice that $I_p(r) = \|q_r\|_p^p$.

Now: $I_p(R) = O(R^{np})$ as $R \to \infty$, thus (4) becomes:

$$\log I_p(r) \leq \log I_p(1) + \varepsilon(R) + np \log r$$

with $\varepsilon(R) \to 0$; so we have

$$\|q_r\|_p \leq \|q\|_p r^n \leq e\|q\|_p, \text{ for } 1 < r < \frac{n+1}{n}. \tag{5}$$

For the left hand side of (3) in the case $1 < r$, we can use again Lemma 6 for $|q(z)|^p$, in the case $p < \infty$ and the subharmonicity of the absolute value of q in the case $p = \infty$.

In the case $1 - 1/n < r < 1$, we consider $\tilde{q}(z) = q(rz) \in \mathcal{P}_n$, now $\tilde{q}_{r^{-1}}(z) = q(z)$ with $1 < r^{-1} < n/(n-1)$. Therefore, using (5) we have

$$\|q\|_p = \|\tilde{q}_{r^{-1}}\|_p \leq C\|\tilde{q}\|_p = C\|q_r\|_p.$$

If we denote by C_n the annulus $\{z \in \mathbb{C} : 1 - 1/n < |z| < 1 + 1/n\}$ and $dm(z)$ the Lebesgue measure, Lemma 7 immediately entails the following corollary

Corollary 8 *For any polynomial q of degree n*

$$\|q\|^p_{L^p(\mathbb{T})} \simeq n\|q\|^p_{L^p(C_n,dm(z))}.$$

Now we are able to prove a Plancherel-Polya type Theorem describing the triangular families that satisfy the first of the M-Z inequalities (the easier one). Other results in a more general setting appear in (EL).

Theorem 9 *Let $p \in [1,\infty)$. If $\mathcal{Z}$ is a triangular family such that*

$$\#(\mathcal{Z}(n) \cap I_n)\frac{n}{m_n} \le C \tag{6}$$

for all $n \in \mathbb{N}$ and all intervals I_n of the unit circle of length $1/n$, then for any polynomial q of degree n

$$\frac{1}{m_n}\sum_{k=0}^{m_n} |q(z_{n,k})|^p \le C_p \int_0^{2\pi} |q(e^{it})|^p\, dt, \tag{7}$$

where the constant C_p is independent of the degree. Conversely if (7) holds, then there is a constant C such that (6) holds for all intervals I_n of length $1/n$.

This result was given earlier by (MT00, thm. 4.2). Nevertheless, we give here another proof for completeness.

PROOF. Take any point $z_{n,k} \in \mathcal{Z}(n)$. By the subharmonicity of $|q|^p$ we have

$$|q(z_{n,k})|^p \le \frac{n^2}{\pi}\int_{D(z_{n,k},1/n)} |q(w)|^p\, dm(w).$$

Now if we add all the points we get

$$\frac{1}{m_n}\sum_{k=0}^{m_n} |q(z_{n,k})|^p \le \frac{n^2}{\pi m_n}\sum_{k=0}^{m_n}\int_{D(z_{n,k},1/n)} |q(w)|^p\, dm(w).$$

and now we replace the sum in the right hand side by the integral over the union of disks. Each point in the annulus C_n is at most in Cm_n/n disks due to the hypothesis (6). Finally the sum is bounded by:

$$\frac{1}{m_n}\sum |q(z_{n,k})|^p \le \frac{n^2}{\pi m_n}\frac{Cm_n}{n}\int_{C_n} |q(w)|^p\, dm(w).$$

Finally we can apply Corollary 8

$$n\int_{C_n} |q(w)|^p\, dm(w) \simeq \|q\|^p_{L^p(\mathbb{T})}.$$

From now on, we will use the following notation for the discrete norm:

$$\|q \mid \mathcal{Z}(n)\|_p^p = \frac{1}{m_n}\sum_{k=0}^{m_n} |q(z_{n,k})|^p, \quad \text{for } q \in \mathcal{P}_n.$$

For the second part, consider the polynomial

$$q_m(z) = \frac{z^m - 1}{m(z-1)} = \frac{1 + z + z^2 + \cdots + z^{m-1}}{m}$$

This polynomial satisfies $\|q_m\|_\infty = 1$ and moreover $q_m(1) = 1$. Let $\mathcal{W} = \{w_{m,j}\}$ be the triangular family of the m-roots of the unity ($w_{m,j} = e^{i2\pi j/m}$, for $j = 0, \ldots, m-1$). We have $q_m(w_{m,j}) = 0$ for $j \neq 0$ and $q_m(w_{m,0}) = 1$. If we fix $p \geq 1$, it is clear that $\|q_m\|_p^p \simeq \|q_m \mid \mathcal{W}(m)\|_p^p = (m-1)^{-1}$ because the roots of unity are the prototypical M-Z family. So

$$\|q_m\|_p^p \leq C_p(\mathcal{W})\|q_m \mid \mathcal{W}(m)\|_p^p = C_p(\mathcal{W})/(m-1) \tag{8}$$

Now assume that (6) is false, but (7) is true for a given constant C_p. Then, by taking $C = 2^{p+2}C_p(\mathcal{W})C_p$ in the reverse of (6), there is $N > 0$ and an arc I of length $1/N$ such that

$$\#(\mathcal{Z}(N) \cap I) > Cm_N/N.$$

Now, divide I in halves; it is clear that there is a half J such that

$$\#(\mathcal{Z}(N) \cap J) \geq Cm_N/(2N). \tag{9}$$

Since $\|q(e^{i\theta}\cdot)\|_p = \|q\|_p$ and $\|q(e^{i\theta}\cdot) \mid \mathcal{Z}(n)\|_p = \|q \mid e^{i\theta}\mathcal{Z}(n)\|_p$, we can assume that J is centered at the point 1, without changing the MZ property of $\mathcal{Z}$. On the other hand, by the Bernstein inequality,

$$\sup_{|z|=1} |q_N'(z)| \leq N \sup_{|z|=1} |q_N(z)| = N;$$

Combined with $|q_N(z) - q_N(1)| \leqslant \sup_{|\xi|=1} |\nabla q_N(\xi)||z-1|$ and the fact that $|J| = (2N)^{-1}$, we obtain a lower bound for q_N on $z \in J$:

$$|q_N(z)| \geq 1 - N|z-1| \geq 1/2.$$

Then, using (9) in the definition of the discrete norm

$$\|q_N \mid \mathcal{Z}(N)\|_p^p \geq \frac{1}{m_N}\inf_{z\in J} |q_N(z)|^p \#(\mathcal{Z}(N) \cap J) \geq$$
$$\frac{1}{m_N}\frac{1}{2^p}C\frac{m_N}{2N} = 2C_pC_p(\mathcal{W})/N \tag{10}$$

But now, by (8) and the assumption that $\mathcal{Z}$ satisfies (7),

$$\|q_N \mid \mathcal{Z}(N)\|_p^p \leq C_p \|q_N\|_p^p \leq C_p C_p(\mathcal{W})/N \tag{11}$$

Inequalities (10) and (11) are incompatible, thus (7) cannot hold for $\mathcal{Z}$.

Definition 10 *Given a triangular family $\mathcal{Z}$ we say that it is separated whenever there is an $\varepsilon > 0$ such that $|z_{n,j} - z_{n,k}| \geq \varepsilon/n$ for all $1 \leq j, k \leq m_n$, $j \neq k$ and all $n \in \mathbb{N}$.*

Theorem 11 *If $\mathcal{Z}$ is a M-Z family then there is a separated subfamily $\mathcal{Z}'$ such that $\mathcal{Z}'$ is also a M-Z family.*

In view of this theorem we will limit ourselves to the study of separated triangular families. Observe that any separated triangular family satisfies $m_n \leq Cn$.

PROOF. The idea of the proof is the following. We take $\varepsilon > 0$ very small (to be determined) and we split the circle $|z| = 1$ into intervals I_n of size ε/n. From the points belonging to $\mathcal{Z}(n)$ we are going to select some to be in $\mathcal{Z}'(n)$. In each interval I_n we only keep at most one point. If the remaining points are still not $\varepsilon/(3n)$-distance one from the other we discard some more points in such a way that all points in $\mathcal{Z}'(n)$ are at least $\varepsilon/(3n)$-distance one from the other and any point in $\mathcal{Z}(n)$ is at most at distance $3\varepsilon/n$ from some of the points in $\mathcal{Z}'(n)$. We need now to prove that $\mathcal{Z}'$ is a M-Z family for a small enough $\varepsilon > 0$.

To begin with need the following stability result.

Lemma 12 *If $\mathcal{Z}$ is a M-Z triangular family then there is an $\epsilon > 0$ (depending only on the constants of the M-Z inequalities for $\mathcal{Z}$) such that for any perturbation $\mathcal{Z}^*$ of the original family with the property $|z_{n,j} - z^*_{n,j}| \leq \varepsilon/n$ is still a M-Z triangular family.*

PROOF. Observe that

$$\left| \left(\frac{1}{m_n} \sum_{j=1}^{m_n} |q_n(z_{n,j})|^p \right)^{1/p} - \left(\frac{1}{m_n} \sum_{j=1}^{m_n} |q_n(z^*_{n,j})|^p \right)^{1/p} \right| \leq \left(\frac{1}{m_n} \sum_{j=1}^{m_n} |q_n(z^*_{n,j}) - q_n(z_{n,j})|^p \right)^{1/p}.$$

There are points $\tilde{z}_{n,j}$ in between $z^*_{n,j}$ and $z_{n,j}$ such that

$$|q_n(z^*_{n,j}) - q_n(z_{n,j})|^p \leq C_p |q'_n(\tilde{z}_{n,j})|^p |z_{n,j} - z^*_{n,j}|^p$$

Now

$$\frac{1}{m_m}\sum_{j=1}^{m_n}|q_n(z_{n,j}^*)-q_n(z_{n,j})|^p\le\frac{C_p\varepsilon^p}{n^pm_n}\sum_{j=1}^{m_n}|q_n'(\tilde z_{n,j})|^p$$

The points in the triangular family $\tilde z_{n,j}$ satisfy (6) because $z_{n,j}$ does and they are very close one to the other. Therefore we can apply Theorem 9 and we get

$$\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j}^*)-q_n(z_{n,j})|^p\le\frac{C_p\varepsilon^p}{n^p}\int_{\mathbb{T}}|q_n'|^p\,dt.$$

Finally we can use Bernstein inequalities (Theorem 4) and the fact that $z_{n,j}$ is a M-Z family and we get

$$\left|\Big(\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j})|^p\Big)^{1/p}-\Big(\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j}^*)|^p\Big)^{1/p}\right|\le\frac14\Big(\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j})|^p\Big)^{1/p}$$

if we pick ε small enough. Therefore

$$\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j})|^p\simeq\frac{1}{m_n}\sum_{j=1}^{m_n}|q_n(z_{n,j}^*)|^p$$

as we wanted to prove.

We finish now the proof of the theorem. Since the family $\mathcal{Z}'$ is $\varepsilon/3$ separated we have automatically the inequality (7). We only have to prove the other inequality.

For any point $z_{n,j}\in\mathcal{Z}_n$ the closest point $z_{n,j}^*$ in $\mathcal{Z}_n$ is at most at distance $3\varepsilon/n$, so we can apply the Lemma. We can't conclude directly that $\mathcal{Z}'$ is a M-Z family because in the discrete norm we may be repeating the same $z_{n,j}^*$ associated to many different $z_{n,j}$. The inequality (6) does the trick: there is a bound of at most $C\frac{m_n}{n}$ different $z_{n,j}$ points in $\mathcal{Z}(n)$ associated to the same point $z_{n,j}^*\in\mathcal{Z}'(n)$.

$$\|q_n\|^p\simeq\frac{1}{m_n}\sum_{i=1}^{m_n}|q_n(z_{n,j}^*)|^p\le\frac{1}{m_n}\sum_{i=1}^{m_n'}\frac{Cm_n}{n}|q_n(z_{n,j}')|^p.$$

Since $\mathcal{Z}'$ is separated then $m_n'\simeq n$ and thus

$$\|q_n\|^p\simeq\frac{1}{m_n'}\sum_{i=1}^{m_n'}|q_n(z_{n,j}')|^p$$

In the statements of Theorem 13 and Theorem 15 we denote by (x,y) to the arc in $\mathbb{T}$ delimited by the endpoints e^{ix} and e^{iy}.

Theorem 13 *Given a separated family $\mathcal{Z}$, if*

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty} \left(\liminf_{n\to\infty} \frac{\min_{x\in[0,2\pi]} \#\mathcal{Z}(n)\cap(x,x+R/n)}{R} \right) > \frac{1}{2\pi},$$

then $\mathcal{Z}$ is a M-Z family (for any $p \in [1,\infty]$). Conversely, if $\mathcal{Z}$ is a M-Z family for some $p \in [1,\infty]$, then

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty} \left(\liminf_{n\to\infty} \frac{\min_{x\in[0,2\pi]} \#\mathcal{Z}(n)\cap(x,x+R/n)}{R} \right) \geq \frac{1}{2\pi}. \quad (12)$$

Remark: In the particular case $p = 1$ this gives a positive answer to the open question (II) in (Lub98), see also (Pel85), where it is asked whether the $n(1+\varepsilon)$-roots of unity are a M-Z sequence for $p = 1$. Moreover Theorem 5 shows that the n-roots of unity are not enough.

PROOF. We start with the sufficiency part for $p = \infty$. We will relate this problem to the similar problem in the Bernstein class which consists of entire functions of exponential type π bounded on the real line. The sampling sequences for such functions were studied and described by Beurling in (Beu89, p. 340).

Let $\mathcal{Z}$ be a separated triangular family. To each generation of points

$$\mathcal{Z}(n) = \{e^{i\theta_{n,1}}, e^{i\theta_{n,2}}, \dots, e^{i\theta_{n,m_n}}\}, \quad \theta_{n,i} \in [-\pi,\pi],$$

we associate a real sequence $\Lambda(n)$ consisting of the points

$$\Lambda(n) = \{n\theta_{n,1}/(2\pi) + nk, n\theta_{n,2}/(2\pi) + nk, \dots, n\theta_{n,m_n}/(2\pi) + nk\}_{k\in\mathbb{Z}}. \quad (13)$$

Since $\mathcal{Z}$ is separated then $\Lambda(n)$ is δ-separated uniformly on n. Moreover the hypothesis on $\mathcal{Z}$ imply that there is an $R > 0$ and $\varepsilon > 0$ such that

$$\frac{\#\Lambda(n)\cap(x,x+R)}{R} > 1+\varepsilon, \quad \forall x \in \mathbb{R}.$$

This means that $\Lambda(n)$ is a sampling sequence for the Bernstein class (see (Beu89, p. 346 Theorem 5). That is there is a constant C which depends only on ε, R and the separation constant δ such that $\sup_{\mathbb{R}} |f(x)| \leq C \sup_{\lambda\in\Lambda(n)} |f(\lambda)|$ for all functions f in the Bernstein class. The constant is independent of n.

Given any polynomial $q \in \mathcal{P}_n$ we have $\sup_{w\in\mathbb{T}} |q(w)| = \sup_{x\in\mathbb{R}} |q(e^{2\pi ix/n})|$. If we define $f \in \mathcal{H}(\mathbb{C})$ as $f(w) = q(e^{2\pi iw/n})e^{-\pi iw}$ then f belongs to the Bernstein class since q is of degree n. Therefore we may apply Beurling's Theorem and we obtain

$$\|q\|_{L^\infty(\mathbb{T})} = \|f\|_{L^\infty(\mathbb{R})} \le C \sup_{\lambda\in\Lambda(n)} |f(\lambda)| = \sup_{z_i\in\mathcal{Z}(n)} |q(z_i)|.$$

Thus we have proved the theorem for $p = \infty$. Now we are going to prove it for $p = 1$ and the others will follow by interpolation. We will use a similar scheme as in (Sei93, p. 36) Indeed, the property that $\mathcal{Z}$ is a M-Z family for p means that the operators $R_n : (\mathcal{P}_n, \|\cdot\|_p) \to (\mathbb{C}^{m_n}, \|\cdot\|_p)$ defined as $R_n(q) = (q(z_{n,1}), \dots, q(z_{n,m_n}))$ are injective and of closed range. Therefore the inverse R_n is defined in the range of R_n and it has bounded norm $\|R_n^{-1}\|_p$. The key point is that the norm of the inverse must be bounded by $Cn^{-1/p}$. We have proved that whenever $D^-(\mathcal{Z}) > 1$ then $\|R_n^{-1}\|_\infty < C$. We will now prove that $\|R_n^{-1}\|_1 < C/n$ is also uniformly bounded, and by interpolation $\|R_n^{-1}\|_p < Cn^{-1/p}$ for any $p \in [1, \infty]$.

We will use that $\mathcal{Z}$ is a M-Z family for $p = \infty$. Let us denote by $(A_n, \|\cdot\|_\infty) \subset \mathbb{C}^{m_n}$ the image of R_n. Any bounded linear functional ϕ on $(\mathcal{P}_n, \|\|_\infty)$ induces a bounded linear functional ϕ on A_n as $\tilde{\phi}(x) = \phi(R^{-1}(x))$, with $\|\tilde{\phi}\| \le K\|\phi\|$. For each $w \in \mathbb{T}$ let ϕ_w denote the point evaluation functional, i.e., $\phi_w(q) = q(w)$ for any $q \in \mathcal{P}_n$. The norm of ϕ is trivially 1. Since the dual space of $(\mathbb{C}^{m_n}, \|\cdot\|_\infty)$ is $(\mathbb{C}^{m_n}, \|\cdot\|_1)$, there is a m_n-tuple of numbers $g_j(w)$ such that $\sum_{j=1}^{m_n} |g_j(w)| \le M$ and moreover

$$q(w) = \sum_{j=0}^{m_n} q(z_{n,j}) g_j(w). \tag{14}$$

Moreover since there is an $\epsilon > 0$, such that $D^-(\mathcal{Z}) > 1 + \varepsilon$ then the MZ-inequality holds not only for polynomials q of degree n but also on polynomials of degree $[(1 + \varepsilon/2)n]$. Thus we have established (14) for all polynomials q of degree $(1+\varepsilon/2)n$. Consider now a collection of auxiliary polynomials $a_n(z)$ of at most degree $[\varepsilon n/2]$ such that $a_n(1) = 1$ and $\|a_n\|_1 \simeq 1/n$. This polynomial can be constructed for instance taking $a_n(z) = b_n^2(z)$ and $b_n(z)$ a polynomial of degree $[\varepsilon n/8]$ which is 1 in 1 and 0 in the other $[\varepsilon n/8]$-roots of unity. Clearly since the roots of unity are a M-Z family $\|b_n\|_2^2 \simeq \frac{1}{n}$. Moreover $\|a_n\|_1 = \|b_n\|_2^2$. Finally, take any polynomial r of degree n and any point $w \in \mathbb{T}$. The polynomial $q(z) = r(z)a_n(\bar{w}z)$ is a polynomial of degree at most $(1 + \varepsilon/2)n$ with the property that $q(w) = r(w)$. We may apply (14) and we get

$$r(w) = \sum_{j=0}^{m_n} r(z_{n,j}) a_n(\bar{w} z_{n,j}) g_j(w).$$

If we now estimate $\|r\|_1$ we get

$$\|r\|_1 \le \sum_{j=0}^{m_n} |r(z_{n,j})| \sup_j \int_{\mathbb{T}} |a_n(\bar{w} z_{n,j}) g_j(w)|\, d|w|.$$

But $|g_j(w)| \leq M$ (even the sum is bounded by M) and $\int_{|w|=1} |a_n(\bar{w} z_{n,j})|\, d|w| = \|a_n\|_1 \simeq 1/n$, therefore

$$\|r\|_1 \lesssim \frac{1}{n} \sum_{j=0}^{m_n} |r(z_{n,j})|,$$

for all polynomials r of degree n which is what we wanted to prove.

To prove the necessity we want to deal only with $p = 2$. The next lemma shows how we can reduce ourselves to this situation.

Lemma 14 *If $\mathcal{Z}$ is a separated L^p M-Z family then for any arbitrary small $\delta > 0$ the family $\mathcal{Z}'$ obtained scaling the indexes (i.e. $\mathcal{Z}'(n) = \mathcal{Z}([n(1+\delta)])$ is an L^2 M-Z family.*

PROOF. We will prove that under the hypothesis $\mathcal{Z}'$ is a M-Z family for L^1 and for L^∞, thus by interpolation it will be a M-Z family for all L^r, $1 \leq r \leq \infty$, in particular for $r = 2$ as in the statement. We start by proving that $\mathcal{Z}'$ is an L^∞ M-Z family. Just as before if $\mathcal{Z}$ is a L^p M-Z family then there are functions $g_j^n : \mathbb{T} \to \mathbb{C}$ such that $\sum_{j=0}^{m_n} |g_{n,j}(z)|^q \leq C$ (where q satisfies $1/p + 1/q = 1$) and for all polynomials of degree n:

$$p(z) = \sum_{j=0}^{m_n} p(z_{n,j}) g_{n,j}(z).$$

If we take polynomials c_n of degree $[\delta n]$ such that $\|c_n\|_p \simeq n^{-1/p}$ and $c_n(1) = 1$, we get that for any $z \in \mathbb{T}$,

$$p(z) = \sum_{j=0}^{m_n} p(z'_{n,j}) c_n(z'_{n,j}\bar{z}) g_{n,j}(z), \tag{15}$$

for all polynomials of degree n and the rescaled sequence $\mathcal{Z}'$. If we use Hölder inequality we obtain

$$|p(z)| \leq (\sup_j |p(z'_{n,j}|) \|c_n(z'_{n,j}\bar{z})\|_{\ell^p} \|g_j^n(z)\|_{\ell^q}.$$

Finally, by Theorem 9 $\|c_n(z'_{n,j}\bar{z})\|_{\ell^p} \lesssim n^{1/p} \|c_n\|_p$ and thus

$$\sup_{\mathbb{T}} |p(z)| \lesssim (\sup_j |p(z'_{n,j})|).$$

That proves that $\mathcal{Z}'$ is an L^∞ M-Z family. To prove that it is an L^1 family we take polynomials b_n of degree $[\delta n]$ such that $\|a_n\|_1 \simeq n^{-1}$ and $a_n(1) = 1$, we get that for any $z \in \mathbb{T}$,

$$p(z) = \sum_{j=0}^{m_n} p(z'_{n,j}) a_n(z'_{n,j}\bar{z}) g_{n,j}(z),$$

if we integrate this

$$\|p\|_1 \leq \sum_{j=0}^{m_n} \int_{\mathbb{T}} |p(z'_{n,j})||a_n(z'_{n,j}\bar{z})g_{n,j}(z)|d|z|.$$

Since $|g_{n,j}(z)| \leq (\sum_j |g_{n,j}(z)|^q)^{1/q} < C$ and $\int |a_n(z'_{n,j}\bar{z})|d|z| \leq n^{-1}$, then

$$\|p\|_1 \lesssim n^{-1} \sum_{j=0}^{m_n} |p(z'_{n,j})|.$$

To prove the inequality (12) we will use the scheme proposed by Ramanathan and Steger in the context of the windowed Fourier transform (see (RS95)). This works well when $p = 2$, for other $p \in [1, \infty]$ we use Lemma 14. Now if we can prove the result for $p = 2$ we obtain the inequality

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty} \left(\liminf_{n\to\infty} \frac{\min_{x\in[0,2\pi]} \#\mathcal{Z}(n) \cap (x, x + R/n)}{R} \right) \geq \frac{1}{2\pi} - \delta.$$

and this proves (12) by taking δ arbitrarily small.

Observe that the polynomial $p_n(z) = (z^n - 1)/(1 - z)$ has the property that

$$\int_{|z-1|>R/n, |z|=1} |p_n(z)|^2 \lesssim \frac{1}{R} \int_{|z|=1} |p_n(z)|^2.$$

That means that for any separated family $\mathcal{Z}$ we have

$$\sum_{|z_{n,i}-1|>R/n} |p_n(z_{n,i})|^2 \lesssim \frac{1}{R} \int_{|z|=1} |p_n(z)|^2. \tag{16}$$

Assume that $\mathcal{Z}$ is a L^2 M-Z family. Consider $\mathcal{P}_n$ the polynomials of degree n as a Hilbert space with reproducing kernel. The corresponding reproducing kernel is $k(z, w) = (1 - (z\bar{w})^{n+1})/(1 - (z\bar{w}))$, that is

$$p(w) = \langle p, k(\cdot, w)\rangle = \frac{1}{2\pi} \int_{|z|=1} p(z)\overline{k(z, w)}\, |dz|, \quad \forall p \in \mathcal{P}_n.$$

Since $\mathcal{Z}$ is a M-Z family that means that the normalized reproducing kernels $\{\frac{1}{\sqrt{n}} k(z, z_{n,i})\}_i$ form a frame in $\mathcal{P}_n$, i.e.

$$\|p\|^2 \simeq \frac{1}{n} \sum_{i=1}^{m_n} |\langle p, k(\cdot, z_{n,i})\rangle|^2, \quad \forall p \in \mathcal{P}_n$$

with constants independent of n. This implies (see (Dau92) for the basic facts on frames), that there are polynomials $\{d_i(z)\}_{i=1}^{m_n}$ (the dual frame) such that for all polynomials p in $\mathcal{P}_n$,

$$p(z) = \frac{1}{\sqrt{n}} \sum_{i=1}^{m_n} \langle p, k(z, z_i)\rangle d_i(z),$$
$$p(z) = \frac{1}{\sqrt{n}} \sum_{i=1}^{m_n} \langle p, d_i(z)\rangle k(z, z_i),$$

and
$$\|p\|^2 \simeq \frac{1}{n} \sum_{i=1}^{m_n} |\langle p, k(\cdot, z_{n,i})\rangle|^2 \simeq \sum_{i=1}^{m_n} |\langle p, d_i\rangle|^2, \quad \forall p \in \mathcal{P}_n$$
Given $x \in \mathbb{T}$ and $t, r > 0$ (t much bigger that r) we denote by $I(\tau)$ the arc-interval in $\mathbb{T}$ with center x and radius τ/n. consider the following two subspaces of $\mathcal{P}_n$:

$$W_S = \langle d_i(z) : z_i \in \mathcal{Z}(n) \cap I(t+r)\rangle$$
$$W_I = \langle \frac{1}{\sqrt{n}} k(z, w_j) : w_j \in I(t), w_j^n = 1\rangle.$$

Let P_S and P_I denote the orthogonal projections of $\mathcal{P}_n$ on W_S and W_I respectively. We estimate the trace of the operator $T = P_I P_S$ in two different ways. To begin with
$$\operatorname{tr}(T) \leq \operatorname{rank} W_S \leq \#(\mathcal{Z}(n) \cap I(t+r)). \tag{17}$$
On the other hand

$$\operatorname{tr}(T) = \sum_{w_i \in I(t)} \langle T(\frac{1}{\sqrt{n}} k(z, w_j)), P_I \kappa_j\rangle,$$

where $\{\kappa_j(z)\}$ is the dual basis of $\frac{1}{\sqrt{n}} k(z, w_j)$ in $\mathcal{P}_n$. Using that P_I and P_S are projections one deduces that

$$\operatorname{tr}(T) \geq \#\{w_j \in I(t)\} \left(1 - \sup_j |\langle P_S(\frac{1}{\sqrt{n}} k(z, w_j)) - \frac{1}{\sqrt{n}} k(z, w_j)), \kappa_j\rangle|\right). \tag{18}$$
Since $\|\frac{1}{\sqrt{n}} k(z, w_j)\| \simeq 1$, also $\|\kappa_j\| \simeq 1$. We now show that $\|P_S(\frac{1}{\sqrt{n}} k(z, w_j)) - \frac{1}{\sqrt{n}} k(z, w_j)\| \leq \varepsilon$ for a suitable r.

We have

$$\left| P_S(\frac{1}{\sqrt{n}} k(z, w_j)) - \frac{1}{\sqrt{n}} k(z, w_j) \right|^2 \lesssim \frac{1}{n} \sum_{z_s \notin I(t+r)} |\langle k(z, w_j), k(z, z_s)\rangle|^2 =$$
$$\frac{1}{n} \sum_{z_s \notin I(t+r)} |k(w_j, z_s)|^2.$$

This last sum is smaller than ε if r is big enough because $|w_j - z_s| \geq r/n$ and we can apply (16). If we put together (17) and (18), we find that for every ε

there is an r such that

$$\#(\mathcal{Z}(n) \cap I(t+r)) \geq (1-\varepsilon)\#\{w_j \in I(t)\} = (1-\varepsilon)t,$$

and this implies (12).

The inequality in (12) can be improved when $p = \infty$ to get a strict inequality, thus providing a description in terms of densities of the M-Z inequalities in this case. For this, we need to adapt part of the arguments of Beurling in (Beu89). We will prove

Theorem 15 *Let $p = \infty$. Given a separated family $\mathcal{Z}$ it is a M-Z family if and only if*

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty}\left(\liminf_{n\to\infty}\frac{\min_{x\in[0,2\pi]}\#\mathcal{Z}(n)\cap(x, x+R/n)}{R}\right) > \frac{1}{2\pi},$$

Definition 16 *The Hausdorff distance between two compact sets K, F in a metric space is defined as the infimum of the $\varepsilon > 0$ such that*

$$K \subset (F + B(0,\varepsilon)) \text{ and } F \subset (K + B(0,\varepsilon)).$$

We denote this distance by $d_H(K, F)$.

A sequence of uniformly separated real sequences Λ_n is said to converge weakly to Λ if for any closed interval I, $d_H((I \cap \Lambda_n) \cup \partial I, (I \cap \Lambda) \cup \partial I) \to 0$.

Definition 17 *Recall that for any triangular family $\mathcal{Z}$ we can associate a sequence of real sequences $\Lambda(n)$ as in (13). We take now an arbitrary family of real numbers τ_n and consider the corresponding translated sequences: $\Sigma(n) = \Lambda(n) - \tau_n$ (this corresponds to making rotations of $\mathcal{Z}(n)$). We say that Λ belongs to a $W(\mathcal{Z})$ if there is a sequence of translates τ_n such that the corresponding $\Sigma(n)$ converges weakly to Λ.*

Definition 18 *We denote by $\mathcal{F}$ the closed subspace of entire functions in the Bernstein class spanned by finite linear combinations of exponentials of the form e^{irz} and $r \in \mathbb{Q} \cap [-\pi, \pi]$. The space $\mathcal{F}$ consists of almost periodic functions when restricted to the real line.*

With the same arguments as in (Beu89) we can prove the following theorem and corollary

Theorem 19 *The triangular family $\mathcal{Z}$ is a L^∞ Marcinkiewicz-Zygmund family if and only if all $\Lambda \in W(\mathcal{Z})$ are uniqueness sets for $\mathcal{F}$.*

Corollary 20 *If $\mathcal{Z}$ is a M-Z triangular family then there is an $\varepsilon > 0$ such that the triangular family $\mathcal{Z}'$ defined as $\mathcal{Z}'(n) = \mathcal{Z}([n(1-\varepsilon)])$ is also a M-Z triangular family.*

Now we apply the necessary condition (12) of Theorem 13 and we obtain that $D^-(\mathcal{Z}) > D^-(\mathcal{Z}') \geq 2\pi$.

3 The model space

Actually it is possible to give a full characterization of M-Z sequences when $p = 2$. It is not easily computable. In this section we present this characterization. We need to introduce the model spaces. Suppose that I is an inner function in the disk. We denote by

$$K_I^2(\mathbb{T}) = H^2(\mathbb{T}) \ominus IH^2(\mathbb{T})$$

If instead of the disk one considers the upper half plane, then $K_I^2(\mathbb{R})$ is the standard L^2-Paley-Wiener space if $I = e^{iz}$. If we return back to the disk and consider the case $I = z^n$ then K_I^2 is the space of holomorphic polynomials of degree smaller or equal than n.

Thus, the setting of the model spaces is common for both the polynomials and the Paley-Wiener space. Therefore any results that can be obtained from general theorems in the model space setting will have the same flavor in both the finite and the infinite-dimensional space.

Let us state the result that is more relevant in our context. A Blaschke sequence $\Gamma \subset \mathbb{D}$ is a sampling sequence for K_I^2 when

$$\|f\|^2 \simeq \sum_{\Gamma} |f(\gamma)|^2 \omega_I(\gamma),$$

for some appropriate weight ω_I. The following theorem was proved by Seip in (Sei04):

Theorem 21 *Denote by B_Γ the Blaschke product with zeros in Γ, If Γ satisfies $\sup_\Gamma |I(\gamma)| < 1$ and it is a Carleson sequence the following are equivalent:*

- *Γ is a sampling sequence for K_I^2.*
- *There is an inner function J such that the Toeplitz operator in H^2 with symbol $JI\bar{B}_\Gamma$ is invertible.*

In our setting we start by a separated triangular family $\mathcal{Z} \subset \mathbb{T}$ and we want a description of whether it is M-Z or not. We can replace this family by the family $\mathcal{W}$ defined as

$$w_{n,j} = z_{n,j}(1 - \epsilon/n) \qquad \forall j = 0, \ldots, m_n,\ n \in \mathbb{N}.$$

If $\epsilon > 0$ is small enough the new triangular family is still separated and by Lemma 7 it will be a M-Z family whenever $\mathcal{Z}$ is a M-Z family. The advantage of $\mathcal{W}$ is that we are uniformly under the hypotheses of Theorem 21. That is if $I_n = z^n$ and Γ_n is the sequence $\mathcal{W}(n) = \{w_{n,0}, \ldots, w_{n,m_n}\}$, then $\sup_n \sup_{\Gamma_n} |I_n(\gamma)| < 1$ and moreover Γ_n is a Carleson measure (uniformly in n). Thus if we define B_n to be the Blaschke product with zeros in Γ_n, then a necessary and sufficient condition so that $\mathcal{Z}$ is M-Z is that there exist inner functions J_n such that the Toeplitz operators T_n in H^2 with symbols $J_n I_n \bar{B}_n$ are invertible with uniform bounds. There are computable criteria for a Toeplitz operator to be invertible (the Widom-Devinatz Theorem). The difficulty of translating Theorem 21 into a computable criteria are the inner functions J_n. If we are given a sequence Γ which we want to check wheter it is sampling or not, we do not have a natural candidate for function J to use the theorem. There are some instances, for example in the Paley-Wiener space (see (OCS02) or (Sei04)) and for certain choices of sequences Γ where this is doable. In the finite dimensional situation that we are dealing with, we do not get any new computable criteria from this more complete theorem.

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