UNIFORM ESTIMATES IN THE POINCARÉ-ARONSZAJN THEOREM ON THE SEPARATION OF SINGULARITIES OF ANALYTIC FUNCTIONS

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ABSTRACT. We study the possibility of splitting any bounded analytic function f with singularities in a closed set $E \cup F$ as a sum of two bounded analytic functions with singularities in E and F respectively. We obtain some results under geometric restrictions on the sets E and F and we provide some examples showing the sharpness of the positive results.

INTRODUCTION

Let $O \subset \mathbb{C}$ be an open set and S_1, S_2 be two relatively closed subsets in O, $S = S_1 \cup S_2$. The following result is due to Aronszajn [Aro35]

Theorem 1. *Any function f analytic in $O \setminus S$ coincides with $(f_1 + f_2)|_{(O \setminus S)}$ where f_j are analytic in $O \setminus S_j$, $j = 1, 2$.*

In modern textbooks this fact is treated (if at all) as a trivial example illustrating general sheaf theoretic and $\bar{\partial}$ approaches and related to the first Cousin problem; [Aro35] is never quoted (see e.g. [BG91, p. 225], [Hör90]) not to mention its famous predecessors Poincaré [Poi92] and Fréchet [Fré30] although their approaches remain interesting even now. The first version of Theorem 1 with an ingenious proof appeared in 1892 [Poi92] ($O = \mathbb{C}$, $S_1 = [-1, 1]$, $S_2 = (-\infty, -1] \cup [1, +\infty)$ which is, of course, equivalent to $S_1 = (-\infty, 0]$, $S_2 = [0, \infty)$). The great author was motivated by his dispute with Borel concerning possible generalizations of the classical notion of analytic continuation. It is not quite clear who was right (see the discussion in [Val54, Chapter IV, section 21-22]), but the explicit (non-linear) construction of f_1, f_2 given in [Poi92] (exposed also in [Val54]) is very elegant (as was shown in [MH71], in the Poincaré situation there is no linear operator $f \to (f_1, f_2)$ (from $Hol(O \setminus S) \to Hol(O \setminus S_1) \times Hol(O \setminus S_2)$ where $Hol(G)$ stands for the space of all functions analytic in an open set G with the usual topology). Various aspects of the separation of singularities in the spirit of the Poincaré-Aronszajn theorem (Theorem 1) are treated in [Val54], [BG91], [Hav58], [Hav84] and [Gau98].

The present article deals with a quantitative aspect of Theorem 1 related to spaces $H^\infty(G)$ of functions *bounded* and analytic in an open set $G \subset \mathbb{C}$. Namely we are interested in the case when f in Theorem 1 is bounded (i.e. $f \in H^\infty(O \setminus S)$) and ask whether f_1, f_2 can be made bounded as well (i.e. $f_j \in H^\infty(O \setminus S_j)$, $j = 1, 2$).

Definition. *Let* O, S_1, S_2, S *be as in Theorem 1. We say* (S_1, S_2) *is a bounded separation pair (bs-pair) in* O *if any* $f \in H^\infty(O \setminus S)$ *is representable by the formula*

$$(1) \qquad\qquad\qquad f = f_1 + f_2 \text{ in } O \setminus S$$

where $f_j \in H^\infty(O \setminus S_j)$.

The problem which is implicit in the definition can be restated as follows (just putting $G_j = O \setminus S_j$, $G = O \setminus S$): *given open sets* $G_1, G_2 \subset \mathbb{C}$ *is it possible to represent an arbitrary* $f \in H^\infty(G)$ *by* (1) *with* $f_j \in H^\infty(G_j)$, $j = 1, 2$?

We want more or less efficient (preferably geometric) conditions imposed on (S_1, S_2), resp (G_1, G_2) and ensuring (1) for any $f \in H^\infty(O \setminus S)$ with $f_j \in H^\infty(O \setminus S_j)$ (resp. $f \in H^\infty(G_1 \cap G_2)$, $f_j \in H^\infty(G_j)$).

One of the first results of this kind is due to Polyakov [Pol83]; it was ancillary in his work on bounded extensions of functions $f \in H^\infty(C)$ to the polydisk $\mathbb{D}^n$ (where $\mathbb{D} = \{|z| < 1\}$) for an analytic curve $C \subset \mathbb{D}^n$; see also [HP84]. For $G_1 = \mathbb{D} \cap \{\Im z > 0\}$, $G_2 = \mathbb{D} \cap \{\Re z > 0\}$ and any $f \in H^\infty(G_1 \cap G_2)$ Polyakov proved (1) in $G_1 \cap G_2$ where f_j is analytic in G_j and bounded in $G_j \cap \frac{1}{2}\mathbb{D}$.

In [HN01] (whose authors were unaware of [Pol83]) and in [Hav04] the bs-pairs were investigated in a systematic way and in much more general setting. As in [Pol83], in [HN01] it was essential that S_1, S_2 meet transversally (a generic situation is presented below on Figure 6 (section 3.2 below); the results of [HN01] apply to more general transversal configurations). In [Hav04] a class of tangent bs-pairs was described. Turning to the present article we start with some remarks on bs-pairs.

If $S_1, S_2 \subset \mathbb{C}$ are compact and disjoint, then (S_1, S_2) is a bs-pair in $\mathbb{C}$: this is an easy consequence of the Cauchy integral formula. Simple examples of pairs which are not bs in $\mathbb{C}$ (in particular the Poincaré pair $S_1 = (-\infty, 0]$, $S_2 = [0, \infty)$) are in [HN01].

The bs-problem is related to the Alice Roth Fusion Lemma (see [Gai87], [Gam69], [Dou66], [HN01]). In particular, the usual localization technique (the Vitushkin operator) gives the following result (see [HN01]): (S_1, S_2) *is a bs-pair in a bounded open set* O *if* $\mathrm{dist}(S_1, S_2) > 0$.

Dealing with the bs-problem for a pair (S_1, S_2) we may always assume that $S_1 \cap S_2 = \emptyset$. (Indeed, (S_1, S_2) is a bs-pair in O iff $(S_1 \setminus s, S_2 \setminus s)$ is a bs-pair in $O \setminus s$ where $s = S_1 \cap S_2$.)

A really subtle and interesting situation arises when $S_1 \cap S_2 = \emptyset$ and $\mathrm{dist}(S_1, S_2) = 0$. Let us denote the closure of a set E by $\mathrm{Clos}\, E$, and put $K_j = \mathrm{Clos}\, S_j$, $j = 1, 2$, $k = K_1 \cap K_2$. It is shown in [HN01] that (S_1, S_2) *is a bs-pair in* O *iff* $(S_1 \cap v, S_2 \cap v)$ *is a bs-pair in* O *for a neighbourhood* v *with respect to* $\mathbb{C}$ *of* k. Thus only "the germs of S_1 and S_2 at k" are responsible for the bs-property of (S_1, S_2) in O.

In the present paper (as in [HN01] and [Hav04]) we concentrate on the simplest case when k is a singleton.

We continue the study of "transversal" pairs (S_1, S_2) meeting at a point and obtain a very general result applicable to *arbitrary* relatively closed parts S_j of the upper half-plane $\mathbb{C}^+$ separated by two rays $\eta = k\zeta$, $\eta = k_1\zeta$, $\zeta > 0$, $0 < \arg(k) < \arg(k_1) < \pi$. In [HN01] the transversally meeting sets S_1 and S_2 were supposed to be parts of certain rectifiable arcs. We believe, however, that the concrete bounded splittings $f \to (f_1, f_2)$ ($f \in H^\infty(O \setminus S)$, $f_j \in H^\infty(O \setminus S_j)$)

constructed in [HN01] are of independent interest; they are also applicable to many *tangent* pairs (S_1, S_2), see [Hav04].

Unlike [HN01], we do not exclude the tangency of S_1 and S_2 and find rather sharp descriptions of tangent bs-pairs (S_1, S_2) in $\mathbb{C}^+$ when $k = \{0\}$; in some cases these descriptions yield necessary and sufficient conditions for a tangent pair of arcs to be bs (Theorem 7).

We mainly put $O = \mathbb{C}^+$; note that (S_1, S_2) is a bs-pair in $\mathbb{C}^+$ iff $(\phi(S_1), \phi(S_2))$ is a bs-pair in $\phi(\mathbb{C}^+)$ for a conformal homeomorphism ϕ of $\mathbb{C}^+$.

The article consists of three sections. In §1 we obtain sufficient conditions for a pair (S_1, S_2) to be bs in $\mathbb{C}^+$ (Theorem 4). Our approach is a reduction to the $\bar{\partial}$-problem

$$\bar{\partial} u = f \bar{\partial} \chi \tag{2}$$

in an open set O where $f \in H^\infty(O \setminus S)$ is a given function to be split as in (1), and χ is a cutting factor which is $\equiv 1$ near S_1 and $\equiv 0$ near S_2 in O; the pair (S_1, S_2) is bs in O if (2) has always (for any f) a bounded solution u.

Theorems 2 and 3 are preparatory; they are corollaries of deep results by Berndtsson, [Ber92] on bounded solutions of the $\bar{\partial}$-problem in a disc. From them we deduce our main result of §1 (Theorem 4), treating transversal and tangent pairs $S_1, S_2 \subset \mathbb{C}^+$ with $k = \{0\}$; concrete examples are discussed in 1.4. In 1.5, we give a direct and explicit solution of (2), not using Berndtsson's results and thus making our §1 essentially self-contained. In 1.6 we give yet another very explicit proof of Theorem 4 for pairs (S_1, S_2) in $\mathbb{C}^+$ where S_1, S_2 are separated by a sector with vertex at the origin. In 1.7 we briefly describe quite explicit bounded splittings $f \to (f_1, f_2)$ applicable to a class of pairs $(S_1, S_2), S_j \subset \mathbb{C}^+$; the splittings of 1.5, 1.7 are linear in $f \in H^\infty(\mathbb{C}^+ \setminus S)$.

The objects of §2 are pairs (S_1, S_2) of smooth arcs in a sector A meeting tangentially at its vertex, which are *not* bs in A (and, in fact, in any domain $O \supset A$).

The construction is a rather involved "condensation of singularities", an accumulation of "badly splittable" pairs of arcs in A based on the Banach theorem on surjective operators. The results are different depending on whether the common tangent of S_1 and S_2 at the vertex is a side of A (Theorem 6) or is strictly inside A (Theorem 5). These theorems show the sharpness of the results of §1.

The results of §§1–2 are combined in §3 to state Theorem 7 which includes, among other things, the tangential case when S_j is the graph of a real $\mathcal{C}^{1+\varepsilon}$-function ϕ_j on $[0, b]$ such that

$$\phi_j(0) = \phi_j'(0) = 0, \; 0 < \phi_1(t) < \phi_2(t) \quad \text{for } t \in (0, b].$$

It turns out that (S_1, S_2) is a bs-pair in $\mathbb{C}^+$ iff

$$\liminf_{x \to 0} \frac{\phi_2(x) - \phi_1(x)}{\phi_1(x)} > 0$$

(i.e. S_1 and S_2 are hyperbollicaly separated in $\mathbb{C}^+$).

In section 3.2 we show that for a Jordan domain $G = G_1 \cap G_2$ where the domains G_j are Jordan as well, a function $f \in H^\infty(G)$ may exist which is not representable as $f_1 + f_2$ with $f_j \in H^\infty(G_j)$. In section 3.3, we construct an example illustrating the difference between the

case of *connected* sets S_1, S_2 (arcs) we dealt with in Theorem 7, and *disconnected* S_j's where, apparently, some new results are needed to grasp the bs-property.

1. SOME CLASSES OF BS-PAIRS IN THE UPPER HALF-PLANE

1.1. **Reduction to a $\bar{\partial}$-problem.** Let S_1, S_2 be disjoint relatively closed subsets of the domain O, and $f \in H^\infty(O \setminus S)$, $S = S_1 \cup S_2$. Let U_1, U_2 be disjoint open neighbourhoods of (resp.) S_1, S_2 in O. Consider a bounded locally Lipschitz function χ in O such that

$$(3) \qquad\qquad \chi|U_1 = 0, \; \chi|U_2 = 1.$$

Extend f to O putting $f|S = 0$. Any (distributional) solution u of the $\bar{\partial}$-problem

$$(4) \qquad\qquad \bar{\partial}u = f\bar{\partial}\chi \quad \text{in } O$$

is a continuous function, since $f\bar{\partial}\chi$ is locally bounded in O. We assume $u \in \mathcal{C}(O)$ and put

$$f_2 = f\chi - u, \quad f_1 = f(1 - \chi) + u \quad \text{in } O,$$

so that $f = f_1 + f_2$ in O. Clearly, f_j is continuous in $O \setminus S_j$ and $\bar{\partial}f_j = 0$ in $O \setminus S_j$, whence it is analytic in $O \setminus S_j$. It is bounded if u is. We arrive at the following conclusion:

if (2) *admits a solution* $u \in L^\infty(O)$ *for any* $f \in H^\infty(O \setminus S)$, *then* (S_1, S_2) *is a bs-pair with respect to* O

1.2. **bs-pairs and a result by Berndtsson.** Bounded solvability of a general $\bar{\partial}$-equation

$$(5) \qquad\qquad \bar{\partial}u = \rho$$

in $\mathbb{D}$ was studied in [Ber92]. The result of [Ber92] we need can be easily adapted to $\mathbb{C}^+ = \{\Im z > 0\}$. By $L^1(E)$, $E \subset \mathbb{C}$, we mean $L^1(E, dA)$ where dA is the area element.

Theorem 2. *Suppose that* $\rho \in L^1(\mathbb{C}^+)$ *vanishes in* $\mathbb{C}^+ \cap \{|z| > R\}$ *for a positive* R. *If*

    (a) $\rho \, dA$ *is a Carleson measure in* $\mathbb{C}^+$, *and*
    (b) $\rho\Im z \in L^\infty(\mathbb{C}^+)$,

then the (distributional) problem (5) *has a solution* $u \in L^\infty(\mathbb{C}^+)$.

Combining this result with section 1.1 we get

Theorem 3. *Suppose* $S_1, S_2 \subset \mathbb{C}^+$ *are bounded, disjoint, and relatively closed. Suppose there exists a locally Lipschitz real function* χ *in* $\mathbb{C}^+$ *satisfying* (3), *vanishing in* $\mathbb{C}^+ \cap \{|z| > R\}$ *and such that*

    (a) $|\nabla\chi| \, dA$ *is a Carleson measure in* $\mathbb{C}^+$,
    (b) $|\nabla\chi(\zeta)| \leq C/\Im\zeta$, $(\zeta \in \mathbb{C}^+)$.

Then (S_1, S_2) *is a bs-pair in any domain* $O \subset \mathbb{C}^+$ *containing* S.

Proof. Note that $2|\bar{\partial}\chi| = |\nabla\chi|$. By Theorem 2 the $\bar{\partial}$-problem 2 has a bounded solution in $\mathbb{C}^+$ (thus in O), and section 1.1 applies. $\qquad\square$

1.3. Sets S_1, S_2 separated by graphs. To get a palpable description of some bs-pairs let us assume that there are a number $\mu > 0$ and a Lipschitz nonnegative function g on $\mathbb{R}$ such that

(6)
$$S_1 \subset U_1 = \{\xi + i\eta \in \mathbb{C}^+ : \ \eta < g(\xi)\},$$
$$S_2 \subset U_2 = \{\xi + i\eta \in \mathbb{C}^+ : \ \eta > (1+\mu)g(\xi)\}.$$

Theorem 4. *Suppose S_1, S_2 are relatively closed in $\mathbb{C}^+$ and bounded. If (6) holds then (S_1, S_2) satisfies the assumptions of Theorem 3 and thus is a bs-pair in any $O \subset \mathbb{C}^+$ containing S.*

Our assumption (6) means that S_1, S_2 are separated by the corridor $\{\xi + i\eta \in \mathbb{C}^+ : \ g(\xi) \le \eta \le (1+\mu)g(\xi)\}$ whose hyperbolic width is positive (since for $g(\xi) > 0$ the hyperbolic distance between $\xi + ig(\xi)$ and $\xi + i(1+\mu)g(\xi)$ exceeds a positive number not depending on $\xi \in \mathbb{R}$). We shall see in §2 that this condition is sharp.

Proof. Put

$$\chi_0(\xi, \eta) = \begin{cases} 0 & \text{if } 0 < \eta < g(\xi), \\ \frac{\eta - g(\xi)}{\mu g(\xi)} & \text{if } \xi \in \mathbb{R}, 0 < g(\xi) \le \eta \le (1+\mu)g(\xi), \\ 1 & \text{if } \eta > (1+\mu)g(\xi). \end{cases}$$

Thus $\chi_0(\xi, \eta) = 1$ if $\xi \in \mathbb{R}$, $g(\xi) = 0$, $\eta > 0$. Clearly χ_0 is locally Lipschitz on the set $\{\xi + i\eta \in \mathbb{C}^+ : \ g(\xi) > 0\}$. If $g(\xi) = 0, \eta > 0$, then $(1+\mu)g(\xi') < \eta'$ for $\xi' \simeq \xi, \eta' \simeq \eta$ whence $\chi_0(\xi', \eta') = 1$. We see that $\chi_0 \equiv 1$ in a neighbourhood of any point $\xi + i\eta \in \mathbb{C}^+$ with $g(\xi) = 0$, and χ_0 is locally Lipschitz in $\mathbb{C}^+$. By definition $\chi_0|U_1 = 0$, $\chi_0|U_2 = 1$. Take a large $R > 0$ (so that $S \subset \{|\zeta| < R\}$) and then a $\mathcal{C}_0^\infty$-function χ_1 which is real and $\equiv 1$ on $\{|\zeta| < R\}$. Then

(7)
$$\chi = \chi_0 \chi_1$$

satisfies (3). We only have to check (a) and (b) of Theorem 3. But $\nabla\chi = 0$ on $U_1 \cup U_2$ and

$$|\nabla\chi(\xi, \eta)| \le \frac{C(\chi_1)}{\mu g(\xi)}\Big(\frac{\eta|g'(\xi)|}{g(\xi)} + 1\Big) \le C(g, \mu, \chi_1)/\eta$$

for almost all $\xi \in \{g > 0\}$ and $\eta \in (g(\xi), (1+\mu)g(\xi))$ (recall that $g' \in L^\infty(\mathbb{R})$). Thus (b), holds. To prove (a) consider a Carleson box $B = (a, b) \times (0, b-a)$ $(a, b \in \mathbb{R}, \ a < b)$. We have

$$\int_B |\nabla\chi|dA = \int_{B \cap \{g(\xi) < \eta < (1+\mu)g(\xi)\}} |\nabla\chi|dA \le C\int_a^b \Big(\int_{g(\xi)}^{(1+\mu)g(\xi)} \frac{d\eta}{\eta}d\xi\Big) = C\log(1+\mu)(b-a).$$

$\square$

1.4. Examples.

1.4.1. Example 1. For $k > 0$ put $g(\xi) = k\xi$ $(\xi \ge 0)$, $g(\xi) = 0$ $(\xi < 0)$. In this case Theorem 4 is illustrated by pairs (S_1, S_2) separated by *a sector* $\{\xi + i\eta : \ \xi > 0, k\xi < \eta < (1+\mu)k\xi\}$. Compared with [HN01] this result is a progress. In [HN01] some transversally meeting bs-pairs were described with an explicit and elementary splitting formula $f = f_1 + f_2$ $(f \in H^\infty(O \setminus S), \ f_j \in H^\infty(O \setminus S_j))$. But in [HN01] certain regularity conditions were imposed on S_1, S_2 (in particular they had to be contained in a union of rectifiable arcs) whereas Theorem 4 allows *arbitrary* relatively closed sets separated by a sector.
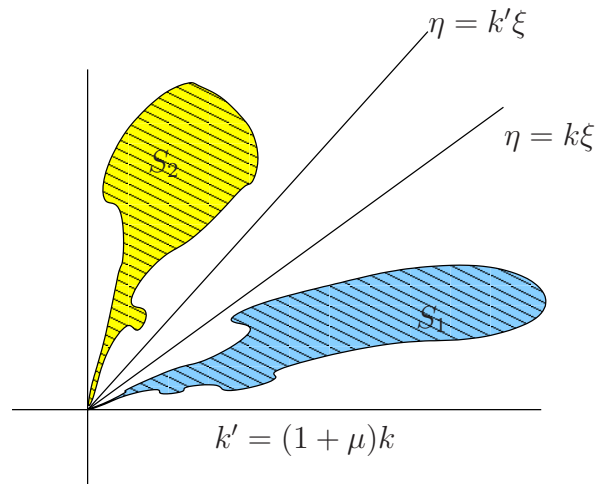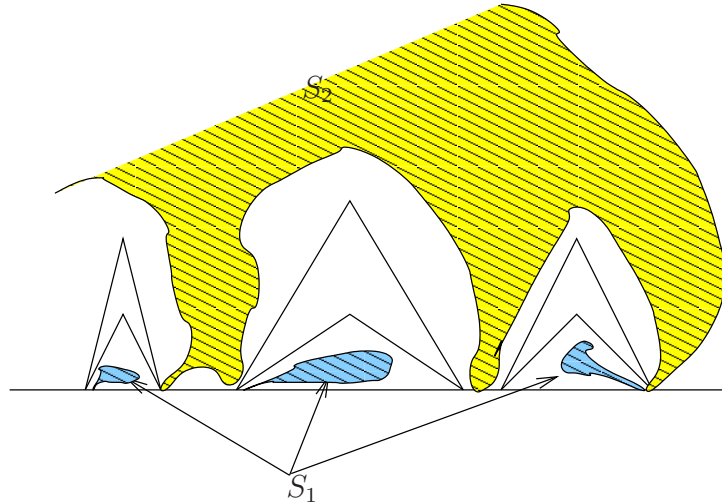
FIGURE 1. Example 1

FIGURE 2. Example 2

1.4.2. *Example 2*. Let $G \subset \mathbb{R}$ be a bounded open set. Theorem 4 is applicable to $g(\xi) = \text{dist}(\xi, \mathbb{R} \setminus G)$ (see figure 2)

1.4.3. *Example 3*. Theorem 4 is also applicable to some *tangentially* meeting bs-pairs (S_1, S_2). Suppose $g \in \mathcal{C}^1(\mathbb{R}), g \geq 0, g(0) = g'(0) = 0$. On figure 3 S_1 and S_2 meet tangentially at the origin, but form a bs-pair in $\mathbb{C}^+$ (and in any subdomain of $\mathbb{C}^+$ containing $S = S_1 \cup S_2$).

1.5. **Theorem 2 revisited: an explicit solution.** The $\bar{\partial}$-estimates that we needed included uniform bounds in the whole set $\mathbb{C}^+$ and not only on the boundary. That is the reason that we could not use Carleson estimates, but had to appeal to the more sophisticated Theorem 2 and we had
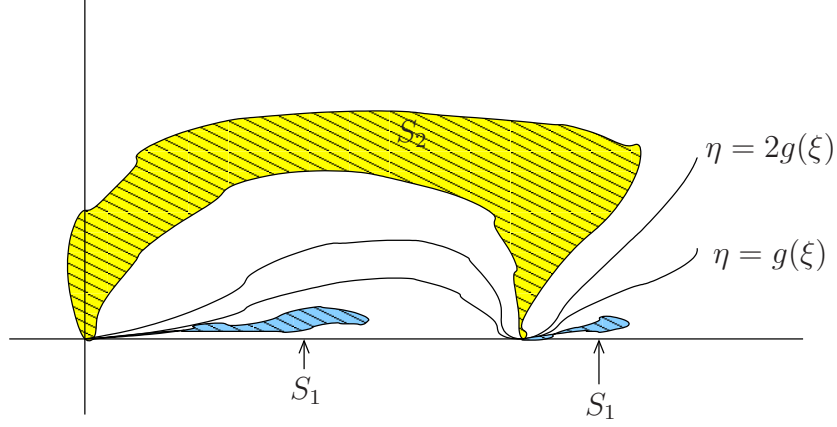
FIGURE 3. Example3

to assume some extra regularity on the data ρ in (5) apart from the Carleson condition. The estimates by Berndtsson rely on some a priori estimates and therefore the solution is not explicit. Nevertheless Jones has found an explicit non-linear formula [Jon83] to solve the $\bar{\partial}$ equation that gives bounded boundary values for the solution when the data is a Carleson measure. Seip has observed (see [Sei04, Chapter1]) that this formula can be adapted to get Theorem 2. For the sake of completeness we present the adapted Jones solution.

Proof. We assume that $\rho \in L^1(\mathbb{C}^+)$ and vanishes in $\mathbb{C}^+ \cap \{|z| > R\}$ for a positive $R > 0$. Furthemore we assume $|\rho|dA$ is a Carleson measure in $\mathbb{C}^+$ and that $\rho \Im z \in L^\infty(\mathbb{C}^+)$. Set

$$u(z) = \frac{2i}{\pi} \int_{\Im\zeta>0} \frac{1}{z-\zeta} \frac{\Im\zeta}{z-\bar\zeta} \exp\Big\{\alpha \int_{\Im w \leq \Im\zeta} \Big(\frac{-i}{z-\bar w} + \frac{i}{\zeta-\bar w}\Big)|\rho(w)|dA(w)\Big\}\rho(\zeta)dA(\zeta).$$

The real number α can be chosen freely. Now, $\bar\partial u = \rho$. Indeed u is of the form

$$u(z) = \frac{1}{\pi}\int \frac{h(z,\zeta)}{z-\zeta}\rho(\zeta)dA(\zeta),$$

where h is continuous in $\mathbb{C}^+ \times \mathbb{C}^+$, and $h(\zeta,\zeta) = 1$ in the diagonal; $h(\cdot,\zeta)$ is holomorphic for any $\zeta \in \mathbb{C}^+$. For any test function $\phi \in \mathcal{C}_0^\infty(\mathbb{C}^+)$, $\zeta \in \mathbb{C}^+$ and small $\varepsilon > 0$,

$$\frac{1}{\pi}\int_{\mathbb{C}^+\setminus B(\zeta,\varepsilon)} \frac{h(z,\zeta)}{z-\zeta}\bar\partial\phi(z)\,dA(z) = \frac{1}{\pi}\int_{\mathbb{C}^+\setminus B(\zeta,\varepsilon)} \bar\partial\left(\frac{h(z,\zeta)}{(z-\zeta)}\phi(z)\right)\,dA(z) =$$

$$-\frac{1}{2\pi i}\int_{|z-\zeta|=\varepsilon} \frac{h(z,\zeta)}{z-\zeta}\phi(z)\,dz \to -\phi(\zeta) \qquad \text{as } \varepsilon \to 0,$$

the countour integral is taken counterclockwise. Thus

$$\int_{\mathbb{C}^+} u(z)\bar\partial\phi(z)dA(z) = \int_{\mathbb{C}^+} \rho(\zeta)\frac{1}{\pi}\int_{\mathbb{C}^+} \frac{h(z,\zeta)}{(z-\zeta)}\bar\partial\phi(z)\,dA(z)dA(\zeta) = -\int_{\mathbb{C}^+}\phi\rho\,dA$$

and $\bar{\partial}u = \rho$ follows. The hard part is to obtain estimates for u. We take

$$\alpha^{-1} = \sup_{\Im\zeta>0} \int_{\Im w \leq \Im\zeta} \frac{2\Im\zeta}{|w - \bar{\zeta}|^2} |\rho(w)| dA(w).$$

This supremum is finite because we assume that $|\rho|dA$ is a Carleson measure. We get

$$|u(z)| \leq$$
$$\frac{2}{\pi\alpha} \int_{\Im\zeta>0} \frac{\alpha}{|z - \zeta|} \frac{\Im\zeta}{|z - \bar{\zeta}|} \exp\left\{\alpha \int_{\Im w \leq \Im\zeta} \left(\frac{-(\Im w + \Im z)}{|z - \bar{w}|^2} + \frac{2\Im\zeta}{|\zeta - \bar{w}|^2}\right)|\rho(w)|dA(w)\right\}|\rho(\zeta)|dA(\zeta).$$

We use the definition of α and we get

$$|u(z)| \leq \frac{2e}{\pi\alpha} \int_{\Im\zeta>0} \frac{\alpha}{|z - \zeta|} \frac{\Im\zeta}{|z - \bar{\zeta}|} \exp\left\{\alpha \int_{\Im w \leq \Im\zeta} \frac{-(\Im(w) + \Im(z))}{|z - \bar{w}|^2}|\rho(w)|dA(w)\right\}|\rho(\zeta)|dA(\zeta).$$

We then write

$$|u(z)| \leq I_1 + I_2,$$

where

$$I_1 = \frac{2e}{\alpha\pi} \int_{2|z-\zeta|\geq\Im z} \frac{\alpha}{|z - \zeta|} \frac{\Im\zeta}{|z - \bar{\zeta}|} \exp\left\{\alpha \int_{\Im w \leq \Im\zeta} \frac{-(\Im(w) + \Im(z))}{|z - \bar{w}|^2}|\rho(w)|dA(w)\right\}|\rho(\zeta)|dA(\zeta)$$

and

$$I_2 = \frac{2e}{\alpha\pi} \int_{2|z-\zeta|<\Im z} \frac{\alpha}{|z - \zeta|} \frac{\Im\zeta}{|z - \bar{\zeta}|} \exp\left\{\alpha \int_{\Im w \leq \Im\zeta} \frac{-(\Im(w) + \Im(z))}{|z - \bar{w}|^2}|\rho(w)|dA(w)\right\}|\rho(\zeta)|dA(\zeta).$$

When $2|z - \zeta| \geq \Im z$, then $|z - \zeta| \geq |z - \bar{\zeta}|/5$ and thus I_1 is bounded by a constant times

$$(8) \qquad \frac{1}{\alpha} \int_{\Im\zeta>0} \frac{\alpha\Im\zeta}{|z - \bar{\zeta}|^2} \exp\left\{\alpha \int_{\Im w \leq \Im\zeta} \frac{-\Im(w)}{|z - \bar{w}|^2}|\rho(w)|dA(w)\right\}|\rho(\zeta)|dA(\zeta).$$

If f is any positive function defined on $t \geq 0$ and $R_0 = \int_0^\infty f(t)dt$ (which may eventually be ∞), then

$$\int_0^\infty f(r) \exp\left(-\int_0^r f(t)\,dt\right) dr = \int_0^{R_0} e^{-t}\,dt \leq 1.$$

We apply this inequality to the function

$$f(t) = \int_{\mathbb{R}} \frac{\alpha t}{|z - (x - it)|^2}|\rho(x + it)|\,dx,$$

and we get that (8) is bounded by α^{-1}.

On the other hand I_2 is easy to estimate if we drop the exponential term and use the fact that $|\rho(z)| \leq C/\Im z$.   □

1.6. An explicit solution of (2) **for transversally meeting sets** $S_1, S_2 \subset \mathbb{C}^+$**.** Here we return to Example 1 of section 1.4 and give another very simple solution of the problem

$$(9) \qquad \qquad \bar{\partial} u = \rho, \ \rho = f \bar{\partial} \chi,$$

where χ (see (7)) corresponds to g which is zero on $(-\infty, 0]$ and is linear on $[0, \infty)$: $g(\xi) = k\xi$, $\xi \geq 0$, for a $k > 0$. The sets S_1, S_2 are separated by the sector

$$A_{k,\mu} = \{(\xi, \eta) : k\xi \leq \eta \leq (1 + \mu)k\xi\},$$

and $\partial \chi$ is supported by a sector $T = A_{k,\mu} \cap \{|z| < R\}$. Put $u = C^\rho - a$, where $C^\rho = -\frac{1}{\pi}\rho \star \frac{1}{z}$ is the standard solution of (9) and

$$a(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}^+} \frac{\bar{z}}{z} \frac{\rho(z)}{\zeta - \bar{z}} dA(z), \qquad \zeta \in \mathbb{C}^+.$$

Note that $\rho \equiv 0$ off T, and $|\rho(z)| \leq C/\Im z \simeq 1/|z|$ for $z \in T$ whence $\rho \in L^1(\mathbb{C}^+, dA)$, and functions C^ρ, a make sense.

Clearly, $\bar{\partial} u = \rho$, since a is analytic in $\mathbb{C}^+$, and we only have to prove that u is bounded in $\mathbb{C}^+$. We have

$$u(\zeta) = \frac{2i\zeta}{\pi} \int_T \frac{\Im z \bar{\partial} \chi(z) f(z) dA(z)}{z(\zeta - z)(\zeta - \bar{z})},$$

whence

$$(10) \qquad |u(\zeta)| \leq c|\zeta| \int_T \frac{dA(z)}{|z||\zeta - z||\zeta - \bar{z}|} = C|\zeta|J(\zeta), \quad \zeta \in \mathbb{C}^+.$$

Fix a small $q > 0$ (to be specified later). Then

$$J(\zeta) \leq \int_{T \cap \{|z-\zeta|<q|\zeta|\}} + \int_{T \cap \{|z-\zeta|>q|\zeta|, \ |z|>q|\zeta|\}} + \int_{T \cap \{|z-\zeta|>q|\zeta|, \ |z|<q|\zeta|\}} = I + II + III.$$

Now, $|z| \simeq \Im z$ for $z \in T$. Estimating I we may write $|z| \geq |\zeta| - |z - \zeta| \geq (1 - q)|\zeta|$,

$$|\bar{z} - \zeta| \geq 2\Im z - |z - \zeta| \geq c|z| - q|\zeta| \geq c'|\zeta|$$

where $c = c(k, \mu) > 0$, $c' = c(1 - q) - q > 0$ if q is small, and

$$(11) \qquad I \leq \frac{c_1}{|\zeta|^2} \int_{|z-\zeta|<q|\zeta|} \frac{dA(z)}{|\zeta - z|} = \frac{c_2}{|\zeta|}.$$

The Hölder inequality (with $p = 3$, $q = 3/2$) and $|\bar{z} - \zeta| \geq |z - \zeta|$ give

$$(12)$$
$$II \leq \int_{|z-\zeta|>q|\zeta|, \ |z|>q|\zeta|} \frac{dA(z)}{|z||z - \zeta|^2} \leq \left(\int_{|z|>q|\zeta|} \frac{dA(z)}{|z|^3} \right)^{1/3} \left(\int_{|z-\zeta|>q|\zeta|} \frac{dA(z)}{|z - \zeta|^3} \right)^{2/3} = \frac{c_3}{|\zeta|}.$$

At last,

$$(13) \qquad III \leq \int_{|z-\zeta|>q|\zeta|, \ |z|<q|\zeta|} \frac{dA(z)}{|z||z - \zeta|^2} \leq \frac{1}{q^2|\zeta|^2} \int_{|z|<q(|\zeta|)} \frac{dA(z)}{|z|} = \frac{c_4}{|\zeta|}.$$

Combining (10) with (11)–(13) we see that u is bounded. $\qquad \qquad \square$

1.7. **Another explicit and bounded solution to equation** (2). Here we again exploit the special form of the right side of (2) (recall that Theorem 2 is aimed at *general* right sides $\rho \in L^1(\mathbb{C}^+)$, but we only deal with $\rho = f\bar{\partial}\chi$, where $f \in H^\infty(\mathbb{C}^+ \setminus S)$ and χ is a smooth cutting factor).

This time we consider a function $g : \mathbb{R} \to [0, +\infty)$ and assume $g \in \mathcal{C}^{1+\varepsilon}(\mathbb{R})$, $\varepsilon > 0$, $g \equiv 0$ on $(-\infty, 0]$, $g(x) > 0$ for $x > 0$ (in Theorem 4 g was a Lipschitz function and it could vanish at some positive points). The functions χ_0, χ_1, χ are the same as in Theorem 4. Fix a small $b > 0$ so that $\chi_1 \equiv 1$ on the corridor

$$G_b = \{\xi + i\eta : \ 0 < \xi < b, \ g(\xi) < \eta < (1+\mu)g(\xi)\},$$

and $\chi_0 \equiv \chi$ on G_b. For $t \in [1, 1+\mu]$ put $\gamma_t(\xi) = \xi + itg(\xi)$, $0 \le \xi \le b$.

Suppose f is as in Theorem 4 ($f \in L^\infty(\mathbb{C}^+)$, $f|S = 0$, f is analytic in $\mathbb{C}^+ \setminus S$). Put

$$u(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}^+} \frac{f(z)\bar{\partial}\chi(z)}{\zeta - z} dA(z) + \frac{\mu}{2\pi i} \int_{\bar{\gamma}_1} \frac{f(\bar{z})dz}{\zeta - z}, \ \zeta \in \mathbb{C}^+.$$

Then u is bounded in $\mathbb{C}^+$ and satisfies (2).

Recall that γ_1 is the path $t \to t + ig(t)$, $t \in [0, b]$, so that the path $\bar{\gamma}_1$ (i.e. the reflected graph of g) does not intersect $\mathbb{C}^+$, and the contour integral is analytic in $\mathbb{C}^+$. At the same time the double integral represents "the standard solution of (2)" (note that $\bar{\partial}\chi$ is summable in G_b, see the last estimate in Section 1.3). Thus our u satisfies (2) in $\mathbb{C}^+$, and the only problem is its boundedness. The standard solution may blow up at the origin, but its growth is counterbalanced by the contour integral. A detailed proof will be given in [HO06].

Our definition of u combined with Section 1.1 generates a *linear* operator $f \to (f_1, f_2)$ splitting $f \in H^\infty(\mathbb{C}^+ \setminus S)$ into the sum of functions $f_j \in H^\infty(\mathbb{C}^+ \setminus S_j)$, $j = 1, 2$.

2. NEGATIVE RESULTS

In this section we describe some pairs (S_1, S_2) of subsets of a sector A which are *not* bs-pairs in A (Theorems 5, 6 in section 2.5.3 and 2.6). Combined with the results of §1 they give some necessary and sufficient geometric conditions for two smooth *graphs* to form a bs-pair in $\mathbb{C}^+$ (see §3). We start with preliminary technical results on "badly splittable" pairs of arcs K_1, K_2 in a domain O which means the existence of $\phi \in H^\infty(O \setminus K)$, $K = K_1 \cup K_2$ with $|\phi| \le 1$ in $O \setminus K$ and such that a representation $\phi = \phi_1 + \phi_2$, $\phi_j \in H^\infty(O \setminus K)$, is only possible with a very big $\sup\{|\phi_1(\zeta)| : \ \zeta \in O \setminus K_1\}$. If $S_1, S_2 \subset O$ contain the elements K_1, K_2 of arbitrarily badly splittable pairs (K_1, K_2) (i.e. $S_j \supset K_j$, $j = 1, 2$), then (S_1, S_2) is not a bs-pair in O. These vague considerations are made precise in sections 2.1–2.4, and then applied to quite concrete Theorems 5, 6.

2.1. **Cells and their rotundities.** A *cell* is by definition a pair (g, A) where g is a Jordan domain with a rectifiable boundary ∂g, and A, "the center" of g, is a point in g. Sometimes we write g instead of (g, A). Put

$$\rho(g)(= \rho((g, A))) = \frac{2\pi \, \mathrm{dist}(A, \partial g)}{l(\partial g)}$$

where $l(\partial g)$ denotes the length of ∂g. Clearly, $\rho(g) \leq 1$, a geometrically obvious fact (an instant proof is given by the formula $1 = (2\pi i)^{-1} \int_{\partial g} \frac{dz}{z-A}$); the equality $\rho(g) = 1$ occurs iff g is a disc centered at A. We call $\rho(g)$ *the rotundity* of the cell $g(= (g, A))$.

2.2. **Functions at a large distance from $H^\infty(g)$ in a cell g.** Let K be a compact subset of g containing A. We assume A to be a boundary point of K (in our applications K will be just a simple arc)

Lemma 1. *Let ϕ be a function analytic in $\mathbb{C} \setminus K$, $\phi(\infty) = 0$. Then any $h \in H^\infty(g)$ satisfies*

$$(14) \qquad \|\phi - h\|_{g\setminus K} \geq \frac{\rho(g)}{2} \limsup_A |\phi|,$$

where $\| \cdot \|_E$ denotes $\| \cdot \|_{\infty, E}$.

In other words, if ϕ is very big near A and (g, A) is sufficiently rotund, then *any $h \in H^\infty(g)$* is far away from ϕ in $H^\infty(g \setminus K)$.

Proof. Any $h \in H^\infty(g)$ has the angular boundary value $h(g)$ at almost every $q \in \partial g$, and

$$h(p) \equiv \frac{1}{2\pi i} \oint_{\partial g} \frac{h(q)}{q - p}\, dq \qquad \text{for } p \in g.$$

Moreover, $\oint_{\partial g} \frac{\phi(q)}{q-p} \equiv 0$ for $p \in g$. Thus putting $\| \cdot \| = \| \cdot \|_{g\setminus K}$, we have

$$|\phi(p)| \leq |h(p)| + \|\phi - h\| =$$
$$\frac{1}{2\pi}\Big|\oint_{\partial g} \frac{h(q) - \phi(q)}{q - p} dq\Big| + \|\phi - h\| \leq$$
$$\Big(\frac{1}{\rho(g)} + 1\Big)\|\phi - h\| \leq \frac{2}{\rho(g)}\|\phi - h\|.$$

Letting $p \to A$ we get (14). $\qquad \square$

2.3. **Pairs not admitting bounded separation in a domain (an abstract scheme).** Lemma 1 suggests a method to construct pairs (S_1, S_2) which are not bs in a domain O.

2.3.1. Suppose S_1, S_2 are relatively closed disjoint and nowhere dense parts of a domain O. Suppose there is a number C such that for any (big) $M > 0$ there exists a cell $g(= (g, A))$, $g \subset O$, $A \in K_1$, and a pair (K_1, K_2) of compact sets such that $K_j \subset S_j$, $j = 1, 2$, and $K_1 \subset g$, $\rho(g) \geq C^{-1}$, and for a pair (ϕ_1, ϕ_2) of functions analytic, respectively, in $\mathbb{C} \setminus K_j$, $j = 1, 2$, we have

$$\phi_1(\infty) = \phi_2(\infty) = 0, \ |\phi_1 + \phi_2| \leq C \text{ in } \mathbb{C} \setminus K, \ K = K_1 \cup K_2,$$

whereas

$$(15) \qquad \limsup_A |\phi_1| > M.$$

Lemma 2. *If S_1, S_2 and O enjoy property 2.3.1, then (S_1, S_2) is not a bs-pair in O.*

Note that $\phi_j \in H^\infty(\mathbb{C} \setminus K_j)$, $j = 1, 2$, since K_1, K_2 are disjoint.

Proof. Put $S = S_1 \cup S_2$. If (S_1, S_2) were a bs-pair in O, then the operator $(f_1, f_2) \to (f_1 + f_2)(O \setminus S)$ from $H^\infty(O \setminus S_1) \times H^\infty(O \setminus S_2))$ to $H^\infty(O \setminus S)$ would be surjective, and, by the Banach open mapping theorem, we could find a number $N > 0$ such that any $f \in H^\infty(O \setminus S)$ would admit a splitting

(16) $$f = f_1 + f_2 \qquad \text{in } O \setminus S,$$

with $f_j \in H^\infty(O \setminus S_j)$, $\|f_j\|_{O \setminus S_j} \le N \|f\|_{O \setminus S}$.

But 2.3.1 forbids the existence of such number N. Indeed, let K_j ϕ_j be as in 2.3.1 where a big $M = M(N)$ will be specified later. Put $f_j^0 = \phi_j|(O \setminus K_j)$, $j = 1, 2$, and $f = f_1^0 + f_2^0$ in $O \setminus K$, $K = K_1 \cup K_2$. Then for any representation (16) of f we get

$$f_1 = f_1^0 - h \text{ in } O \setminus S_1$$
$$f_2 = f_2^0 + h \text{ in } O \setminus S_2$$

where $h \in H^\infty(O)$, since $f_1^0 - f_1$ and $f_2 - f_2^0$ are mutual analytic continuations from $O \setminus S$ across S_2 and S_1. But f_1 is in fact defined and analytic not only on $O \setminus S_1$, but in $O \setminus K_1$, since $f_1^0 \in H^\infty(O \setminus K_1)$, $h \in H^\infty(O)$. Moreover, $\|f_1\|_{O \setminus K_1} = \|f_1\|_{O \setminus S_1}$, since S_1 has no interior points. Now since $h \in H^\infty(g)$ and $\rho(g) \ge C^{-1}$, by Lemma 1 and (15) we have

$$\|f_1\|_{O \setminus S_1} = \|f_1\|_{O \setminus K_1} \ge \frac{M}{2C}.$$

For an arbitrary $N > 0$ take $M > 2NC^2$. We get a contradiction: given $N > 0$, no splitting (16) of f with $f_j \in H^\infty(O \setminus S_j)$ satisfies $\|f_1\|_{O \setminus S_1} \le N \|f\|_{O \setminus S}$ (recall that $\|f\|_{O \setminus S} \le C$). □

2.3.2. Now we are ready to describe a general scheme of constructing triples (O, S_1, S_2) where O is a domain, S_j its relatively closed and disjoint subsets such that (S_1, S_2) is not a bs-pair in O. Namely, suppose we have constructed two families $(K_1^x)_{x \in E}$, $(K_2^x)_{x \in E}$ of compact simple arcs and two families $(\phi_1^x)_{x \in E}$, $(\phi_2^x)_{x \in E}$ of functions in, respectively, $H^\infty(\mathbb{C} \setminus K_j^x)$, $j = 1, 2$, and a family of points $(A^x)_{x \in E}$ such that

  (a) $K_1^x \cap K_2^x = \emptyset$;
  (b) $\phi_j^x(\infty) = 0$, $j = 1, 2$;
  (c) $\sup_{x \in E} \|\phi_1^x + \phi_2^x\|_{\mathbb{C} \setminus K^x} < +\infty$, $K^x := K_1^x \cup K_2^x$;
  (d) $A^x \in K_1^x$ and $\sup_{x \in E} \limsup_{A^x} |\phi_1^x| = +\infty$.

The last element of our construction is a family $(g^x, A^x)_{x \in E}$ of cells such that

  (e) $K_1^x \subset g^x$, $x \in E$, and g^x are uniformly rotund (i.e. $\inf_{x \in E} \rho(g^x) > 0$).

Any family of quintuples $(K_1^x, K_2^x, \phi_1^x, \phi_2^x, g^x)$ enjoying (a)–(e) generates plenty of "bad" triples (O, S_1, S_2). Indeed, for *any* domain O containing $\cup_{x \in E}(g^x \cup K_2^x)$, any pair (S_1, S_2) of its disjoint relatively closed and nowhere dense subsets is not a bs-pair with respect to O provided

$$S_1 \supset \cup_{x \in E} K_1^x, \qquad S_2 \supset \cup_{x \in E} K_2^x.$$

This is an immediate consequence of Lemma 2.

Sections 2.1–2.3 are an abridged version of §1 in [Hav04]. But the technique of section 2.4 is quite different from [Hav04] and results in removing a logarithmic factor from Theorem 4 of [Hav04]; our Theorem 7 is also based on this improvement.
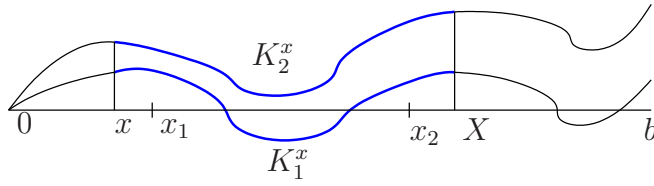
FIGURE 4. $x_1 - x = X - x_2 = h$, $X = X(x)$, $x \to 0$
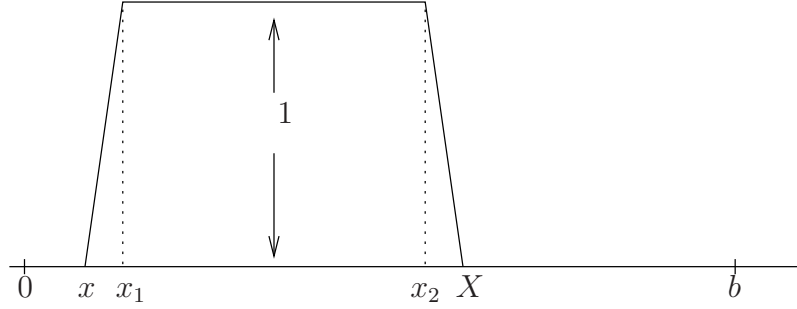
FIGURE 5. $x_1 - x = X - x_2 = h(x)$

2.4. A realization of the general scheme: pairs of graphs.

In what follows the arcs K_j^x will be pieces of the graphs of real functions ϕ_j, $j = 1, 2$ defined on $[0, b]$. We assume

(17) $\qquad \varphi_j \in \mathcal{C}^{1+\varepsilon}([0, b])$, $\varphi_j(0) = \varphi_j'(0) = 0$, $j = 1, 2$, $\varphi_1(t) < \varphi_2(t)$ for $t \in (0, b]$.

It will be also convenient to assume

(18) $\qquad\qquad\qquad\qquad |\varphi_j'(t)| < 1/2$, $t \in [0, b]$

(this assumption doesn't affect the generality).

2.4.1. We put

(19) $\qquad\qquad K_j^x = \{t + i\varphi_j(t) : x \leq t \leq X\}$, $x \in (0, b]$

The choice of $X = X(x)$ will be specified later.

To define the functions ϕ_j^x (see (c) on page 12) we need a piecewise linear function f^x defined on $\mathbb{R}$ as shown on Figure 5, so that $f^x \equiv 0$ on $\mathbb{R} \setminus (x, X)$, $f^x \equiv 1$ on $[x_1, x_2]$, f^x is linear on $[x, x_1]$, $[x_2, X]$; the positive number $h = h(x)$ will be chosen later. Now put

$$\tilde{f}_j^x(\zeta) = f^x(\Re\zeta) \qquad (\zeta \in \mathbb{C}).$$

Define the functions ϕ_j^x by the formulas

(20)
$$\phi_j^x(\zeta) = (-1)^{j-1} \frac{1}{2\pi i} \int_{K_j} \frac{\tilde{f}_j^x(z)}{z - \zeta} dz =$$
$$(-1)^{j-1} \frac{1}{2\pi i} \int_x^X \frac{f^x(t)(1 + i\varphi_j'(t))}{(t - \xi) + i(\varphi_j(t) - \eta)} \, dt, \quad \zeta = \xi + i\eta \in \mathbb{C} \setminus K_j^x.$$

The first (contour) integral in (20) is oriented from the left to the right.

In fact the integral (20) can be easily evaluated, so that ϕ_j^x become explicit linear combinations of some elementary functions of ζ (compositions of $\zeta \log \zeta$ and $\log \zeta$ with some Möbius functions). But we prefer to keep the integral representation of ϕ_j^x, the estimates of their explicit versions not being any shorter. Getting rid of integration makes it possible to apply the scheme of Section 2.3 to general continua S_1, S_2 (not just curves), see [Hav04]; but it also makes the description of bs-pairs of *graphs* not as precise as our Theorem 7.

Put $A^x(= A) = x + i\varphi_1(x)$. Functions ϕ_j^x belong to $H^\infty(\mathbb{C} \setminus K_j^x)$ and vanish at infinity; ϕ_j^x is continuous at both sides of the arc K_j^x. In particular, $\lim_{A_x} \phi_1^x$ exists and coincides with $\frac{1}{2\pi i} \int_{K_1^x} \frac{\tilde{f}^x(z)dz}{z - A^x}$. We denote it by $\phi_1(A)$.

We are going to show that $|\phi_1(A)|$ is big if $h << X - x$. Indeed,

$$\phi_1(A) = \left(\int_x^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^X \right) \frac{f(t)}{2\pi i} \frac{(1 + i\varphi_1'(t))dt}{(t - x) + i(\varphi_1(t) - \varphi_1(x))} = I + II + III,$$

$$|I| \leq \frac{1}{2\pi} \int_x^{x+h} \frac{t - x}{h} \frac{3/2 dt}{t - x} < 1.$$

To estimate III note that any $t \in (x_2, X)$ satisfies $X - t < t - x$, and

$$|III| \leq \frac{1}{2\pi} \int_{X-h}^X \frac{X - t}{h} \frac{3/2 dt}{t - x} < 1.$$

Now,

$$|II| = \frac{1}{2\pi} \left| \int_{x_1}^{x_2} \frac{d(t + i\varphi_1(t))}{(t + i\varphi_1(t)) - A} \right| \geq \frac{1}{2\pi} \left| \log \frac{|(x_2 - x) + i(\varphi_1(x_2) - \varphi_1(x))|}{|(x_1 - x) + i(\varphi_1(x_1) - \varphi_1(x))|} \right|.$$

The numerator of the last fraction is greater than or equal to $x_2 - x$, the denominator is less than or equal to $3/2(x_1 - x)$ whence $|II| \geq \frac{1}{2\pi} \log \frac{x_2 - x_1}{2h}$. But $h/(x_2 - x_1) = (h/(X - x))(1 - 2h/(X - x))^{-1}$, so that

$$(21) \qquad \lim_{x \to 0} |\phi_1^x(A^x)| = +\infty \text{ if } \lim_{x \to 0} \frac{h(x)}{x - X(x)} = 0.$$

2.5. **Upper estimates of $|\phi_1^x + \phi_2^x|$.** Put

$$K^x = K_1^x \cup K_2^x, \ \phi^x = \phi_1^x + \phi_2^x.$$

(Sometimes we omit the index x). We now have to concentrate on the upper estimate of $|\phi|$ in $\mathbb{C} \setminus K$ (see (c) on page 12). We may apply the elementary maximum modulus principle to ϕ, since ϕ is continuous up to the sides of the arcs K_1, K_2. Expressing the boundary values of ϕ on K by the Sohotsky-Plemelj formulas and using the estimate $|f| \leq 1$, we reduce our problem to the upper estimate of $|J(z_0)|$ where

$$J(z_0) = p.v. \int_K \tilde{f}(z) \frac{dz}{z - z_0}, \qquad z_0 \in K,$$

K_1 is oriented from the left to the right, K_2 is oriented in the opposite direction, so that, hopefully, the contributions of K_1 and K_2 will be mutually tempered. Put

$$z_j(t) = t + i\varphi_j(t), \qquad t \in [0, b],$$

and suppose

$$z_0 \in K_1, \; z_0 = z_1(t_0), \; t_0 \in [0, b].$$

(the case $z_0 \in K_2$ is symmetric). We have

$$J(z_0) = p.v. \int_x^X f(t)\mathcal{K}(t, z_0) \, dt,$$

where

$$\mathcal{K}(t, z_0) = \frac{z_1'(t)}{z_1(t) - z_0} - \frac{z_2'(t)}{z_2(t) - z_0}.$$

For a smooth function λ of a real variable t put

$$R_{t_0}(\lambda)(t) = \lambda(t) - \lambda(t_0) - \lambda'(t)(t - t_0).$$

Put $\Delta = \varphi_2 - \varphi_1$. Using this notation the reader can check the following identity:

(22) $$\mathcal{K}(t, z_0) = \mathcal{K}_1(t, z_0) + \mathcal{K}_2(t, z_0)$$

where

(23)
$$\mathcal{K}_1(t, z_0) = \frac{iR(\Delta) + \varphi_2'(t)R(\varphi_1) - \varphi_1'(t)R(\varphi_2)}{(z_1(t) - z_0)(z_2(t) - z_0)},$$

$$\mathcal{K}_2(t, z_0) = \frac{i\Delta(t_0)z_1'(t)}{(z_1(t) - z_0)(z_2(t) - z_0)}, \; R = R_{t_0}.$$

An easy estimate of the kernel $\mathcal{K}_1$ follows from the inequalities

$$|z_j(t) - z_0| \geq |t - t_0|, \; |R(\varphi_j)(t)| \leq c(\varphi_j)|t - t_0|^{1+\varepsilon}, \; j = 1, 2,$$

since φ_j (and Δ) are in $\mathcal{C}^{1+\varepsilon}$; thus, for $t_0 \in [0, b]$,

(24) $$\left| p.v. \int_x^X f(t)\mathcal{K}_1(t, z_0) \, dt \right| \leq c \int_x^X \frac{dt}{|t - t_0|^{1-\varepsilon}} \leq c'$$

where c, c' depend only on φ_1, φ_2 and ε (but not on z_0) for any f satisfying $|f(t)| \leq 1$ on $[0, b]$.

Turn now to

(25) $$J_2 = p.v. \int_x^X \mathcal{K}_2(t, z_0)f(t) \, dt = \int_{[x,X]\backslash a} + \int_a = j_1 + j_2$$

where $a = [x, X] \cap [t_0 - \Delta(t_0), t_0 + \Delta(t_0)]$. The integral j_1 is easy (recall that $|f| \leq 1$, $|\varphi_1'| \leq 1/2$):

(26) $$|j_1| \leq 2\Delta(t_0) \int_{|t-t_0|>\Delta(t_0)} \frac{dt}{(t - t_0)^2} = 4.$$

But j_2, unlike preceding estimates, requires some special properties of f and additional restrictions on the proximity of K_1 and K_2 (i.e. smallness restrictions on Δ). To estimate j_2 put $F(t) = f(t)/(z_2(t) - z_0)$; we get

$$
(27)\qquad
\begin{aligned}
j_2 =& i\Delta(t_0) p.v. \int_a F(t) \frac{z_1'(t)\,dt}{z_1(t) - z_0} = \\
& i\Delta(t_0) \int_a \frac{F(t) - F(t_0)}{z_1(t) - z_1(t_0)} dz_1(t) + i\Delta(t_0) F(t_0) p.v. \int_{K_1^a} \frac{dz}{z - z_0} = I + II,
\end{aligned}
$$

where $K_1^a = \{t + i\varphi_1(t) : t \in a\}$. Now,

$$
(28)\qquad |I| \leq \Delta(t_0) \int_a \left| \frac{F(t) - F(t_0)}{t - t_0} \right| (1 + |\varphi_1'(t)|)\,dt \leq 2\Delta(t_0)\|F'\|_{\infty,a}|a|,
$$

$$
(29)\qquad |F'(t)| \leq |f'(t)|/|z_2(t) - z_1(t)| + |f(t)| \frac{1 + |\varphi_2'(t)|}{|z_2(t) - z_1(t_0)|^2}.
$$

But

$$
|z_2(t) - z_1(t_0)| \geq |\varphi_2(t) - \varphi_1(t_0)| \geq \Delta(t_0) - \|\varphi_2'\|_\infty |t - t_0| \geq \Delta(t_0)/2,
$$

since $\|\varphi_2'\| < 1/2$, $t \in a$. From (28), (29) and the estimates $\|f\|_\infty \leq 1$, $\|f'\|_\infty \leq 1/h$, $|a| \leq 2\Delta(t_0)$ we conclude that

$$
|I| \leq 2\Delta(t_0) \left(\frac{2}{h\Delta(t_0)} + \frac{8}{\Delta(t_0)^2} \right) 2\Delta(t_0) = \frac{8\Delta(t_0)}{h} + 32 \leq 40,
$$

if

$$
(30)\qquad \Delta(t_0) \leq h, \quad \text{for any } t_0 \in [x, X],
$$

an important restriction expressing the proximity of φ_1 and φ_2 (note that $h << X - x$ by (21)).

We are left now with the integral II (see (27)). From $|F(t_0)|\Delta(t_0) = f(t_0)$ we conclude

$$
\begin{aligned}
|II| \leq& f(t_0) \left| p.v. \int_{K_1^a} \frac{dz}{z - z_0} \right|, \\
& a = [x, X] \cap [t_0 - \Delta(t_0), t_0 + \Delta(t_0)], \ K_1^a = K_1 \cap \{\zeta : \Re\zeta \in a\}, \ t_0 \in [x, X].
\end{aligned}
$$

2.5.1. We have to consider two cases:

Case 1: $x \leq t_0 - \Delta(t_0) \leq t_0 + \Delta(t_0) \leq X$;

Case 2: $t_0 - \Delta(t) < x$ or $t_0 + \Delta(t_0) > X$.

In Case 1 $a = [t_0 - \Delta(t_0), t_0 + \Delta(t_0)]$, and

$$
\begin{aligned}
|II| \leq& \left| p.v. \int_{K_1^a} \frac{dz}{z - z_0} \right| = \left| \log \frac{z_1(t_0 + \Delta(t_0)) - z_1(t_0)}{z_1(t_0 - \Delta(t_0)) - z_1(t_0)} \right| \leq \\
\leq& \left| \log \frac{|z_1(t_0 + \Delta(t_0)) - z_1(t_0)|}{|z_1(t_0 - \Delta(t_0)) - z_1(t_0)|} \right| + 2\pi = \\
=& \frac{1}{2} \left| \log \frac{(\Delta(t_0))^2 + (\varphi_1(t_0 + \Delta(t_0)) - \varphi_1(t_0))^2}{(\Delta(t_0))^2 + (\varphi_1(t_0 - \Delta(t_0)) - \varphi_1(t_0))^2} \right| + 2\pi \leq B,
\end{aligned}
$$

B being an absolute constant, since $|\varphi_1(t_0 \pm \Delta(t_0)) - \varphi_1(t_0)| \leq \Delta(t_0)/2$ whence the fraction under the logarithm is in $(4/5, 5/4)$.

In Case 2 we suppose, e.g., that $t_0 - \Delta(t_0) < x$ (the case $X < t_0 + \Delta(t_0)$ is symmetric). Then $0 \leq t_0 - x < \Delta(t_0) \leq h$ by (30), so that $t_0 \in [x, x_1]$ and $f(t_0) = (t_0 - x)/h$. Now $a = [x, t_0 + \Delta(t_0)]$ (note that $t_0 + \Delta(t_0) > X$ is impossible, because this inequality implies $2\Delta(t_0) = (t_0 + \Delta(t_0)) - (t_0 - \Delta(t_0) \geq X - x > 2h$, a contradiction with (30)). Thus,

$$(31) \qquad |II| \leq \frac{t_0 - x}{h} \left[\left| \log \frac{|z_1(t_0 + \Delta(t_0)) - z_1(t_0)|}{|z_1(x) - z_1(t_0)|} \right| + 2\pi \right]$$

The numerator of the last fraction is between $\Delta(t_0)$ and $2\Delta(t_0)$ (since $|\varphi_1'| < 1/2$), its denominator is between $t_0 - x$ and $2(t_0 - x)$ so that, by (30) and (31),

$$|II| \leq \frac{t_0 - x}{\Delta(t_0)} \left[\log \frac{\Delta(t_0)}{t_0 - x} + \log 2 + 2\pi \right] \leq e^{-1} + \log 2 + 2\pi$$

(since $y|\log y| \leq e^{-1}$ for $y \in (0, 1]$).

Summing up, *our functions ϕ_1^x, ϕ_2^x satisfy conditions (a)–(d) of Section 2.3.2, if* (21) *and* (30) *are fulfilled.*

2.5.2. We now have to specify $h(x)$, $X(x)$ to satisfy (21) and (30), and then construct the cells g^x centered at $A^x = x + i\varphi_1(x)$ and satisfying condition (e) in 2.3.2. These definitions will be given twice, one time for the sector $A_k = \{\zeta = \xi + i\eta \in \mathbb{C} : \xi > 0, |\eta| < k\xi\}$, and the second for its upper half $A_k^+ = A_k \cap \mathbb{C}^+$.

For any $R > 0$ we may assume the graphs of φ_1 and φ_2 are in A_k, replacing b by a smaller number if needed. Put

$$\epsilon(x) = \sup\{\frac{|\varphi_1(t)| + |\varphi_2(t)|}{t} : 0 < t \leq 2x\}, \ X(x) = 2x, \ h(x) = 2\varepsilon(x)x.$$

Then $\lim_{x \to 0} \frac{h(x)}{X(x) - x} = \lim_{x \to 0} 2\epsilon(x) = 0$ (see (17) in section 2.4), and we get (21). If $t \in [x, X(x)]$, then $\Delta(t) = \varphi_2(t) - \varphi_1(t) \leq |\varphi_2(t)| + |\varphi_1(t)| \leq \epsilon(x)t \leq h(x)$, so that (30) holds. Put $g^x = A_k \cap \{\zeta \in \mathbb{C} : \Re\zeta < 3x\}$. The inclusion $K_1^x \subset g^x$ is obvious if $x > 0$ is small (again by (17) and (19) in section 2.4) as is the uniform rotundity of g^x with respect to the center A^x (at this point A_k cannot be replaced by A_k^+, since the center A^x may be too close to the boundary of A_k^+ and the rotundity of g^x be very small).

2.5.3. We have arrived at the first result of this section.

Let S_1, S_2 be the graphs of $\varphi_j|(0, b]$, $j = 1, 2$ where φ_j are as in section 2.4, see (17). Without loss of generality (see [HN01] or the Introduction) we may assume $A_k \supset S = S_1 \cup S_2$. Suppose $S \subset \{|\zeta| < R\}(= R\mathbb{D})$.

Theorem 5. *For any $k > 0$ and any domain $O \supset A_k \cap R\mathbb{D}$, (S_1, S_2) is not a bs-pair in O.*

The proof follows from Lemma 2, since the families (ϕ_j^x), (K_j^x), $j = 1, 2$, (g^x) enjoying properties (a)–(e) have been constructed in sections 2.4.1–2.5.2.

2.6. **Two graphs in A_k^+.** We still assume $S_1, S_2, \varphi_1, \varphi_2$ are as in section 2.4 (see (17)), but we also suppose

(32) $$\varphi_1(t) > 0, \ t \in (0, b].$$

Taking $b < b(k)$ we may assume $S \subset A_k^+$. This time we have to impose a special proximity conditions on S_1, S_2 ((17) is not sufficient to apply Lemma 2). Namely, we assume

(33) $$\liminf_{x \to 0} \frac{\Delta(x)}{\varphi_1(x)} = 0.$$

Let $E \subset (0, b]$ be a set with a limit point at the origin and such that

$$\lim_{x \to 0, \, x \in E} \frac{\Delta(x)}{\varphi_1(x)} = 0.$$

We define $h(x), X(x)$ (and thus $K_j^x, \ j = 1, 2$) for $x \in E$ only:

$$X(x) = x + \frac{1}{2}\varphi_1(x), \ h(x) = 2\varphi_1(x)\epsilon(x)$$

where

$$\epsilon(x) = \frac{\Delta(x)}{\varphi_1(x)} + \sup\{|\Delta'(t)| : \ 0 \le t \le X(x)\}, \ x \in E,$$

so that $\lim_{x \to 0, x \in E} h(x)/(X - x) = \lim_{x \to 0, x \in E} \varepsilon(x) = 0$, and we get (21). Now, if $t \in [x, X]$, then, for a $c \in [x, t]$, $\Delta(t) = \Delta(x) + \Delta'(c)(t - x) \le \epsilon(x)\varphi_1(x) + \epsilon(x)\varphi_1(x) = h(x)$ which is (30). The cell g^x will be defined for $x \in E$ as the square $(x - \varphi_1(x), x + \varphi_1(x)) \times (0, 2\varphi_1(x))$ with the center $x + i\varphi_1(x)$. This cell is uniformly rotund. Moreover, $K_1^x \subset g^x (x \in E)$. Indeed, for $t \in [x, x + \varphi_1(x)]$ we have $\varphi_1(t) = \varphi_1(x) + \varphi'(c)(t - x) < 2\varphi_1(x)$, since we may assume $|\varphi_1'(c)| < 1$. Moreover, $g^x \subset A_k^+$ for any small $x \in E$, since $\lim_{x \to 0} \frac{2\varphi_1(x)}{x - \varphi_1(x)} = 0$.

As in section 2.5.2 we get the following result.

Theorem 6. *If $S \subset A_k^+ \cap R\mathbb{D}$, and φ_1, φ_2 satisfy (17) and (33), then, for any domain $O \supset A_k^+; \cap R\mathbb{D}$, (S_1, S_2) is not a bs-pair in O.*

3. CONCLUDING REMARKS

3.1. **A complete description of some bs-pairs of arcs in $\mathbb{C}^+$.** Let $\gamma_j : [0, 1] \to \mathbb{C}$, $j = 1, 2$, be two simple $\mathcal{C}^{1+\varepsilon}$-arcs in $\mathbb{C}$ such that $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1(t) \ne \gamma_2(t)$ for $t \in (0, 1]$. We also assume $\Im\gamma_j(t) > 0$ for $t \in (0, 1]$, $\gamma_j'(t) \ne 0$ for $t \in [0, 1]$.

In this section we define S_j as the trajectory of $\gamma_j \setminus \{0\}$, $j = 1, 2$. Combining Theorem 4 with Theorems 5 and 6 we obtain a complete and very clear description of all bs-pairs of this sort in $\mathbb{C}^+$.

Denote by τ_j the unit tangent vector of γ_j at the origin. Note that $\Im\tau_j \ge 0$, since $S_j \subset \mathbb{C}^+$.

Theorem 7. *I. If $\tau_1 \ne \tau_2$, then (S_1, S_2) is a bs-pair in $\mathbb{C}^+$.*
  II. Suppose $\tau_1 = \tau_2$. Then
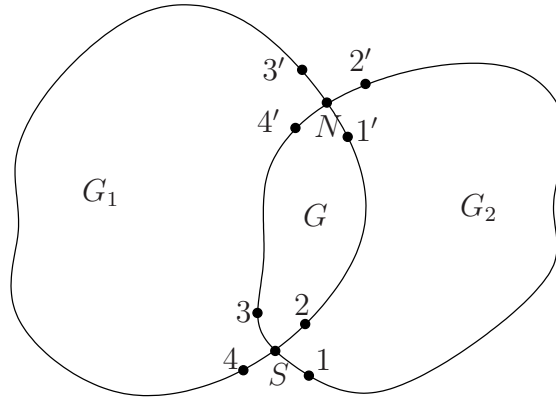   *(a) (S_1, S_2) is not a bs-pair in $\mathbb{C}^+$, if $\Im\tau_1 > 0$.*

FIGURE 6

(b) *If τ_1 is real we may assume that $\tau_1 = 1$, and for a small $b > 0$ the arcs $S_j^b = S_j \cap \{\Re\zeta < b\}$ are graphs (over $(0, b]$) of real functions $\varphi_j \in \mathcal{C}^{1+\varepsilon}([0, b])$ as in Theorem 5 (see (17)), and (S_1, S_2) is a bs-pair in $\mathbb{C}^+$ if, and only if, $\liminf_{x \to 0} \frac{\Delta(x)}{\varphi_1(x)} > 0$, $\Delta = |\varphi_1 - \varphi_2|$.*

Proof. In case I the pieces S_1^b, S_2^b are separated by two rays in $\mathbb{C}^+$ emanating from the origin, and we may apply Theorem 4 of §1 (see also the "transversal" example 1 in section 1.4 and the construction of section 1.6). In case II(a) S_1^b and S_2^b stay in a sector in $\mathbb{C}^+$ whose bisector is parallel to τ_1, and Theorem 5 applies. In case II(b) if (33) holds, then (S_1, S_2) is not a bs-pair in $\mathbb{C}^+$ by Theorem 6, since $S_1^b \cup S_2^b$ is covered by a sector A_k^+; if (33) does not hold, then, taking $g = (\varphi_1 + \varphi_2)/2$ and a small $\mu > 0$ (depending on $\limsup_0 \Delta/\varphi_1$), S_1^b and S_2^b are separated by the graphs of g and $(1 + \mu)g$, and Theorem 4 applies; see example 3 in section 1.4. $\square$

3.2. **Bounded splittings of functions analytic in the intersection of Jordan domains.** Consider two Jordan domains G_1, G_2 as on Figure 6 and their intersection G. Suppose the curves $\partial G_1, \partial G_2$ are piecewise $\mathcal{C}^1$-smooth and intersect *transversally* at S and N (i.e. any pair of arcs $1, 2, 3, 4$ (respectively $1', 2', 3', 4'$) meet under a positive angle at S, respectively at N). It is shown in [HN01, Example 4.1, section 4.6] that for any $f \in H^\infty(G)$ there exist functions $f_j \in H^\infty(G_j)$, $j = 1, 2$, such that

$$(34) \qquad\qquad f = f_1 + f_2 \qquad \text{in } G.$$

Now we are going to show that *the transversality assumption cannot be dropped*. Let G_1, G_2 be the Jordan domains on Figure 7, bounded respectively by the loops $AoDEFA$, $oFDCBo$, so that $\partial G = oDEo$. Suppose $b > 0$ is small, and $\partial G \cap \{0 \le \Im z \le b\} = oM_1 \cup oM_2$ where the compact arcs oM_1, oM_2 are as on Figure 7, so that (oM_1, oM_2) is not a bs-pair with respect to $\mathbb{C}^+$ (see Theorem 7). Then *there exists a function $f \in H^\infty(G)$ which cannot be split as in* (34).

Proof. Take an arbitrary $F \in H^\infty(\mathbb{C}^+ \setminus (oM_1 \cup oM_2))$ and put $f = F|G$. Suppose f can be represented by (34) ($f_j \in H^\infty(G_j)$, $j = 1, 2$); then we prove

$$(35) \qquad\qquad F = F_1 + F_2 \quad \text{in } \mathbb{C}^+ \setminus (oM_1 \cup oM_2)$$
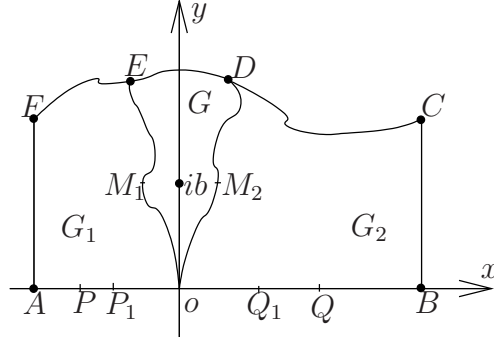
FIGURE 7. $\partial G_1 = AoDEFA, \partial G_2 = oEDCBo$
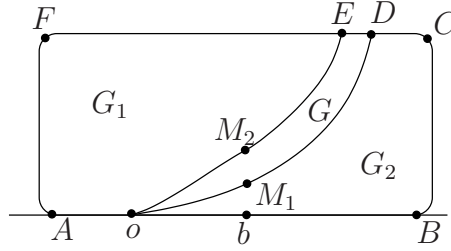
FIGURE 8. $\partial G_1 = AoDEFA, \partial G_2 = oEDCBo$

with $F_j \in H^\infty(\mathbb{C}^+ \setminus oM_j)$, $j = 1, 2$, a contradiction.

To deduce (35) from (34) put $H = G_1 \cup G_2$ (so that $\partial H = AoBCDEFA$) and note that f_j extends analytically from G_j to $H \setminus oM_j$, $j = 1, 2$. Indeed, $f_1 = f - f_2$ in G whence $f - f_2$ is the analytic continuation of f_1 from G to $H \setminus oM_1$; the same argument applies to f_2. Extend f_1, f_2 to $(H \cup \partial H)' = \mathbb{C} \setminus (H \cup \partial H)$ putting

$$f_1(\zeta) = f_s(\zeta) = 0, \ \zeta \in (H \cup \partial H)',$$

thus making $f_j \in H^\infty((\partial H \cup oM_j)')$, $j = 1, 2$. Applying the preseparation Corollary 3.3 from [HN01] we split f_1, f_2 in their domains as follows:

$$f_j = \varphi_j + r_j, \ j = 1, 2$$

where $\varphi_j \in H^\infty([P, Q] \cup oM_j)')$, $r_j \in H^\infty((\partial H \setminus [P_1, Q_1])')$ (see Figure 7). The identity

(36) $$F - \varphi_1 - \varphi_2 = r_1 + r_2 \quad \text{in } H \setminus (oM_1 \cup oM_2)$$

(which is (34)) shows that $r_1 + r_2$ coincides in H with a function $h \in H^\infty(\mathbb{C}^+)$, since the left side of (36) is analytic at any point of $\mathbb{C}^+ \setminus H$, and F, φ_1, φ_2 are bounded in their domains. Therefore (35) holds with $F_1 = \varphi_1$, $F_2 = \varphi_2 + h$.   □

We could get a similar example for the domains G_1, G_2, G as on Figure 8. The arcs oM_2, oM_2 are the graphs of functions $\varphi_1, \varphi_2 \in \mathcal{C}^1([0, b])$ satisfying $\varphi_j(0) = \varphi_j'(0) = 0$, $j = 1, 2$, $0 < \varphi_1(t) < \varphi_2(t)$ ($t \in (0, b]$), $\lim_{t \to 0} \frac{\varphi_2(t) - \varphi_1(t)}{\varphi_1(t)} = 0$ (see Theorem 7). However this kind
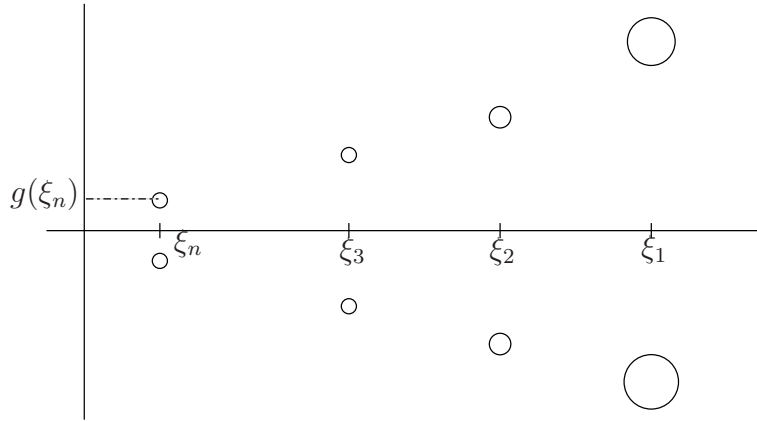
FIGURE 9

of tangency (when oM_1, oM_2 are tangent to ∂H at the origin) is compatible with the splitting formula (34) for *any* $f \in H^\infty(G)$ if $\lim_{t \to 0} \frac{\varphi_2(t) - \varphi_1(t)}{\varphi_1(t)} > 0$. This happens, for example, for

$$G_1 = \{0 < x < 1,\ 0 < y < 2x^2\}, \qquad G_2 = \{0 < x < 1,\ x^2 < y < 2\}.$$

3.3. **bs-pairs of hyperbolically close sets.** Very satisfactory looking Theorem 7 deals with pairs of *graphs* and cannot be applied to disconnected sets. In this section we describe examples of bs-pairs (S_+, S_-) with respect to the *right* half plane Π with S_+ and S_- hyperbolically very close.

Let g be a non-negative function on $[0, b]$ such that $g(\xi) \leq \xi$ ($\xi \in [0, b]$) and $g(\xi) > 0$ for $\xi \in (0, b]$. Consider a *strictly* decreasing sequence $(\xi_n)_{n=1}^\infty$ in $(0, b]$ tending to zero and put $\zeta_n = \xi_n + ig(\xi_n)$. For $r > 0$ denote by $B_n(r)$ the *closed* disc $\{|\zeta - \zeta_n| \leq r\}$. Suppose a sequence $(r_n)_{n=1}^\infty$ of positive numbers satisfies

$$(37) \qquad \sum_{n=1}^\infty \frac{r_n}{g(\xi_n)} < \infty,$$

so that $\sum r_n/\xi_n < \infty$ as well. We assume $r_n < g(\xi_n)$ for all n and $r_n < \frac{1}{2}\min(\xi_n - \xi_{n-1}, \xi_{n+1} - \xi_n)$, so that the discs $B_n = B_n(r_n)$ are disjoint and stay in $\mathbb{C}^+$. Put

$$S_+ = \cup_{n=1}^\infty B_n, \quad S_- = \cup_{n=1}^\infty \overline{B_n}, \qquad S = S_+ \cup S_-.$$

Theorem 8. (S_+, S_-) *is a bs-pair with respect to the right half plane* Π.

Note that the hyperbolic distance between ξ_n and $\bar{\xi}_n$ (and B_n and $\overline{B_n}$) in Π is comparable with $g(\xi_n)/\xi_n$ and tends to zero, if say, $g'(0) = 0$.

Proof. Any $f \in H^\infty(\Pi \setminus S)$ has angular boundary values a.e. on $i\mathbb{R}$ and ∂S, and can be represented as follows:

$$f = f_1 + f_2 = f_1 + f_+ + f_- \qquad \text{in } \Pi$$

where

$$(38) \quad f_1 = -(1+z)C_{i\mathbb{R}}^{f/(1+z)}, \qquad f_+ = (1+z)\sum_{n=1}^{\infty} C_{\partial B_n}^{f/(1+z)}, \qquad f_- = (1+z)\sum_{n=1}^{\infty} C_{\partial \overline{B}_n}^{f/(1+z)}.$$

(C_A^k is the Cauchy type integral $\frac{1}{2\pi i}\int_A \frac{k(z)}{z-\zeta}dz$, $\zeta \notin A$). The imaginary axis is oriented "upwards", and the circles $\partial B_n, \partial \overline{B}_n$ are oriented clockwise. The proof of (38) is standard ($f(iy)/(1+y^2) \in L^1(\mathbb{R}, dy)$, $f(z)/(1+z) = O(1/z)$, $|z| \to \infty$). Now, $f_\pm = (1+z)C^{\mu_\pm}$ where $\mu_\pm$ are *finite* complex charges on $S_\pm$, so that f_2 is defined and analytic in $\mathbb{C} \setminus (S \cup \{0\})$ and $f_2(z) = O(1)$ ($|z| \to \infty$). Clearly $f_1 = f - f_2$ is analytic in Π, and $f_1(z) = O(1)$ ($|z| \to \infty$).

We are going to prove that

$$(39) \qquad\qquad\qquad f_1 \in H^\infty(\Pi)$$

whence $f_2 \in H^\infty(\Pi \setminus S)$. At last we prove $f_\pm$ are bounded in $\Pi \setminus S_\pm$ and get the final splitting of f into the sum of elements of $H^\infty(\Pi \setminus S_\pm)$:

$$f = (f_1 + f_+) - f_-.$$

(a) f_+ is bounded on any set $\mathbb{C} \setminus A_k^+$, $k > 2$, where $A_k^+ = \{\xi + i\eta : \xi > 0, 0 < \eta < k\xi\}$ (according to our assumptions $S_+ \subset A_2$). Indeed,

$$(40) \qquad\qquad |C_{\partial B_n}^f(\zeta)| \leq \|f\|_\infty r_n/\operatorname{dist}(\zeta, B_n), \qquad \zeta \notin S_+.$$

If $\Re\zeta \leq 0$, then $\operatorname{dist}(\zeta, B_n) \geq \xi_n - r_n \geq g(\xi_n) - r_n$; if $\Im\zeta \leq 0$, then $\operatorname{dist}(\zeta, B_n) \geq g(\xi_n) - r_n$, and if $\zeta \in \Pi \setminus A_k^+$, $\Im\zeta > 0$, then

$$\operatorname{dist}(\zeta, B_n) \geq |\zeta - \zeta_n| - r_n \geq |\zeta_n|\sin\varphi - r_n \geq g(\xi_n)\sin\varphi - r_n,$$

where φ is the sector in Π with sides $\eta = 2\xi$, $\eta = k\xi$. Thus, for a $c_k > 0$,

$$(41) \qquad\qquad\qquad \operatorname{dist}(\zeta, B_n) \geq c_k g(\xi_n)$$

for all $\zeta \in \mathbb{C}^+ \setminus A_k^+$ and $n = 1, 2, \ldots$. Combining (40), (41) and (37) we see that f_+ is analytic and bounded in $\mathbb{C} \setminus A_k^+$, $k > 2$. The same proof applies to f_-, so that f_- is bounded in $\mathbb{C} \setminus \overline{A_k^+}$, $k > 2$. Putting $A_k = A_k^+ \cup \overline{A_k^+} \cup (0, +\infty)$ we conclude that f_2 is bounded in $\mathbb{C} \setminus A_k$, and $f_1 = f - f_2$ *is bounded in* $\Pi \setminus A_k$. But $f_1(\zeta) = O(1)$ ($|\zeta| \to \infty, \zeta \in \Pi$), and we may fix a large $R > 0$ making f_1 bounded in $\Pi \setminus s$ where $s = A_k \cap \{|\zeta| < R\}$ (we fix a $k > 2$, say $k = 3$). It remains to estimate f_1 in s. Being analytic in s, f_1 is bounded on its sides and the arc $A_k \cap \{|\zeta| = R\}$ whereas

$$(42) \qquad\qquad |f_1(\zeta)| = O(\log|\zeta|), \ |\zeta| \to 0, \zeta \in s$$

and (a very weak form of) Phragmen-Lindelöf applies whence f_1 *is bounded in* s, and (39) holds. (To get (42) write (using $|\eta| < k\xi$)

$$|f_1(\zeta)| \lesssim \int_{-\infty}^{+\infty} \frac{dy}{(1+|y|)(\xi + |\eta - y|)} \lesssim \int_{|y|<2k\xi} + \int_{|y|>2k\xi} =$$

$$= O(1) + O\left(\int_{2k\xi}^{+\infty} \frac{dy}{(1+y)y}\right) = O(|\log\xi|) = O(|\log\zeta|),$$

since $|\zeta| \simeq \xi$ for $\zeta \in s$).

(b) From (a) we see that $f_2 = f_+ + f_-$ is bounded in $\Pi \setminus S$ (and, in fact, in $\mathbb{C} \setminus (S \cup \{0\})$). We already know f_+ is bounded in $\mathbb{C} \setminus (A_k^+ \cup \{0\})$. We only have to show f_+ is bounded in $A_k^+ \setminus S_+$. But in $A_k^+ \setminus S_+$

$$f_+ = f - f_1 - f_-;$$

f is bounded in $\Pi \setminus S$, f_1 is bounded in Π (see (39)) and we have proved f_- is bounded in $\mathbb{C}^+$. Thus $f_+ \in H^\infty(A_k^+ \setminus S_+)$, $f_+ \in H^\infty(\mathbb{C} \setminus (S^+ \cup \{0\}))$. The boundedness of f_- in $\Pi \setminus S_-$ follows now from $f_- = f_2 - f_+$. $\qquad\square$

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