Homology Stability for Spaces of Surfaces

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Introduction

The main subject of this thesis is the homology of the space of embedded compact, connected, oriented surfaces in a manifold M. We first show that, under some conditions, the homology groups of this space become stable with respect to the surface genus. Then we show that a natural map from the space of embedded surfaces to a space of sections over M is a homology equivalence in the stable range. The methods used follow recent advances on cobordism theory and the homology of diffeomorphism groups of surfaces, which we review in the following sections.

Background

Cobordism and mapping spaces

It was Pontryagin who first realized a crucial relationship between spaces of embedded submanifolds and spaces of maps into spheres. In 1938, he found that the set of normally framed closed submanifolds of dimension k of a given manifold M of dimension n up to normally framed cobordism is in bijection with the set of homotopy classes of compactly supported maps from M to the sphere S^{n-k} of dimension $n-k$:

$$\{\text{normally framed closed } k\text{-submanifolds of } M\}/\text{cobordism} \longleftrightarrow \pi_0 \mathrm{map}_c(M, S^{n-k}).$$

A *normally framed submanifold* of M of dimension k is a pair (W, ℓ), where W is a submanifold of M whose normal bundle is trivial and ℓ is a collection of $n-k$ linearly independent vector fields on the normal bundle of W. A *normally framed cobordism* between two such framed closed submanifolds $(W, \ell), (W', \ell')$ is a normally framed submanifold with boundary $(B, \mathfrak{l})$ in $M \times I$ such that $\partial B = B \cap (M \times \partial I)$ and such that $\partial B = W \times \{0\} \cup W' \times \{1\}$ and $\mathfrak{l}$ restricts to ℓ and ℓ' on its boundary.

If $M = \mathbb{R}^n$, then the resulting set of homotopy classes is the homotopy group $\pi_n(S^{n-k})$, which is one of the main objects of research in algebraic topology. According to [Pon50], the motivation of Pontryagin was the possibility to use the above equivalence to compute homotopy groups of spheres, in the belief that the space of normally framed cobordisms was easier to deal with.

Shortly after, Thom [Tho54] found that the set of cobordism classes of closed normally oriented k-submanifolds of a manifold M of dimension n is in bijection with the set of homotopy classes of compactly supported maps from M to the Thom space $\mathrm{Th}(\gamma_{n-k})$. Here γ_{n-k} is the tautological $(n-k)$-plane bundle over the Grassmannian $\mathrm{Gr}^+_{n-k}(\mathbb{R}^\infty)$ of oriented $(n-k)$-planes in the infinite union $\mathbb{R}^\infty = \bigcup_{n\geq 0}\mathbb{R}^n$, and $\mathrm{Th}(\gamma_{n-k})$ is the result of collapsing all vectors of length ≥ 1 in γ_{n-k} to a point:

$$\{\text{closed normally oriented } k\text{-submanifolds of } M\}/\text{cobordism} \\ \longleftrightarrow [M, \mathrm{Th}(\gamma_{n-k})]_c.$$

The identification is given by the Pontryagin–Thom map, described as follows. If W is a submanifold, we first choose a tubular neighbourhood U of W, that is, an embedding e of the normal bundle $\pi\colon NW \to W$ into M that restricts to the identity on W. Then W is sent to a map f that is constant with value the point at infinity outside U, and is defined as $f(p) = (N_{\pi(v)}W, v)$ if $e(v) = p$.

If $M = \mathbb{R}^n$, then $[M, \mathrm{Th}(\gamma_{n-k})]_c$ is precisely $\pi_n(\mathrm{Th}(\gamma_{n-k}))$. In this case M itself is canonically oriented, so a normal orientation of a submanifold is equivalent to a tangential orientation. The canonical inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ define compatible inclusions

$$\begin{tikzcd}
\{\text{closed oriented } k\text{-submanifolds of } \mathbb{R}^n\}/\text{cob.} \arrow[r, leftrightarrow] \arrow[d] & \pi_n(\mathrm{Th}(\gamma_{n-k})) \arrow[d] \\
\left\{\text{closed oriented } k\text{-submanifolds of } \mathbb{R}^{n+1}\right\}/\text{cob.} \arrow[r, leftrightarrow] & \pi_{n+1}(\mathrm{Th}(\gamma_{n+1-k})).
\end{tikzcd}$$

Letting n go to infinity, the space of embeddings of a manifold into $\mathbb{R}^n$ becomes more and more connected, so the space of embeddings into $\mathbb{R}^\infty$ is weakly contractible. On the other hand, if we denote by ϵ the trivial line bundle over $\mathrm{Gr}_{n-k}(\mathbb{R}^\infty)$, then the pullback of γ_{n-k+1} along the inclusion $\mathrm{Gr}_{n-k}(\mathbb{R}^\infty) \subset \mathrm{Gr}_{n-k+1}(\mathbb{R}^\infty)$ is isomorphic to $\gamma_{n-k} \oplus \epsilon$. Therefore, there are maps

$$\Sigma\mathrm{Th}(\gamma_{n-k}) \cong \mathrm{Th}(\gamma_{n-k} \oplus \epsilon) \longrightarrow \mathrm{Th}(\gamma_{n-k+1}),$$

which define a spectrum **MSO** whose $(n-k)$th space is $\mathrm{Th}(\gamma_{n-k})$. By construction, the adjoint of this map induces the vertical map on the right. Therefore, in the colimit one obtains

$$\{\text{closed oriented } k\text{-manifolds}\}/\text{cobordism} \longleftrightarrow \pi_k(\mathbf{MSO}). \tag{0.0.1}$$

This theory led to an enormous amount of results in algebraic topology that we skip to go directly to the application that concerns this thesis. The following is the main theorem of [GMTW09]. Its discovery was a consequence of several advances on the homology of mapping class groups that we will review in the next section.

Let us consider the category $\mathcal{C}_k$ whose objects are pairs (W, a) consisting of a closed oriented $(k-1)$-submanifold W of $\mathbb{R}^\infty$ and a number $a \in \mathbb{R}$, and whose morphisms between (W, a) and (W', a') are oriented cobordisms in $\mathbb{R}^\infty \times [a, a']$ between them. The set of objects and morphisms is topologised suitably. The connected components of the classifying space of this category are in bijection with the set of closed oriented $(k-1)$-manifolds up to cobordism. The main theorem of [GMTW09] establishes a homotopy equivalence

$$B\mathcal{C}_k \xrightarrow{\simeq} \Omega^{-1}\Omega^\infty \mathbf{MTSO}(k) \tag{0.0.2}$$

with the following spectrum $\mathbf{MTSO}(k)$. Let $\mathrm{Gr}_k^+(\mathbb{R}^n)$ be the Grassmannian of oriented k-planes in $\mathbb{R}^n$ and let $\gamma_{k,n}^\perp$ denote the complement of the tautological bundle $\gamma_{k,n}$ over $\mathrm{Gr}_k^+(\mathbb{R}^n)$, seen as a subbundle of the trivial vector bundle $\mathrm{Gr}_k^+(\mathbb{R}^n) \times \mathbb{R}^n$. If ϵ denotes the trivial line bundle over $\mathrm{Gr}_k^+(\mathbb{R}^n)$, then the pullback of $\gamma_{k,n+1}^\perp$ along the inclusion $\mathrm{Gr}_k^+(\mathbb{R}^n) \subset \mathrm{Gr}_k^+(\mathbb{R}^{n+1})$ is isomorphic to $\gamma_{k,n}^\perp \oplus \epsilon$. Hence, as before, we consider the spectrum $\mathbf{MTSO}(k)$ whose nth space is $\mathrm{Th}(\gamma_{k,n}^\perp)$, and whose structural maps are the composites $\Sigma(\mathrm{Th}(\gamma_{k,n}^\perp)) \cong \mathrm{Th}(\gamma_{k,n}^\perp \oplus \epsilon) \to \mathrm{Th}(\gamma_{k,n+1}^\perp)$. In the following, we recall how the equivalence (0.0.2) relates to Thom's bijection (0.0.1).

There is an inclusion $\mathrm{Gr}_k^+(\mathbb{R}^n) \subset \mathrm{Gr}_{k+1}^+(\mathbb{R}^{n+1})$ that sends an oriented plane P to the sum $P \oplus \mathbb{R}$ with the last factor in $\mathbb{R}^{n+1}$. This yields a map of spectra $\mathbf{MTSO}(k) \to \Sigma\mathbf{MTSO}(k+1)$ (here $(\Sigma\mathbf{X})_n := \mathbf{X}_{n+1}$). On the other hand, notice that $\mathrm{Gr}_k^+(\mathbb{R}^n)$ is canonically homeomorphic to $\mathrm{Gr}_{n-k}^+(\mathbb{R}^n)$ by sending a plane to its orthogonal complement. Composing this homeomorphism with the inclusion into the infinite Grassmannian gives a map

$$\mathrm{Gr}_k^+(\mathbb{R}^n) \xrightarrow{\cong} \mathrm{Gr}_{n-k}^+(\mathbb{R}^n) \longrightarrow \mathrm{Gr}_{n-k}^+(\mathbb{R}^\infty),$$

where the second map is k-connected. Moreover, the pullback of the tautological bundle $\gamma_{n-k,n}$ is $\gamma_{k,n}^\perp$. We have therefore induced maps from the nth space of $\mathbf{MTSO}(k)$ to the $(n-k)$th space of $\mathbf{MSO}$:

$$\mathrm{Th}(\gamma_{k,n}^\perp) \xrightarrow{\cong} \mathrm{Th}(\gamma_{n-k,n}) \longrightarrow \mathrm{Th}(\gamma_{n-k}),$$

where the second map is n-connected. These maps commute with the structural maps of $\Sigma^k\mathbf{MTSO}(k)$ and $\mathbf{MSO}(k)$, yielding a k-connected map of spectra $\Sigma^k\mathbf{MTSO}(k) \to \mathbf{MSO}$. By construction, the latter map factors as

$$\Sigma^k\mathbf{MTSO}(k) \longrightarrow \Sigma^{k+1}\mathbf{MTSO}(k+1) \longrightarrow \cdots \longrightarrow \mathbf{MSO},$$

hence the spectra $\Sigma^k\mathbf{MTSO}(k)$ give a filtration of the spectrum $\mathbf{MSO}$. As a consequence, $\pi_i(\mathbf{MTSO}(k)) \cong \pi_{i+k}(\mathbf{MSO})$ in degrees $i+k<k$, that is, $i<0$. For $i=-1$, we recover Thom's bijection (0.0.1) from the equivalence (0.0.2):

$$\pi_0 B\mathcal{C}_k \longrightarrow \pi_0(\Sigma\mathbf{MTSO}(k)) \cong \pi_{-1}(\mathbf{MTSO}(k)) \cong \pi_{k-1}(\mathbf{MSO}).$$

Stable homology of diffeomorphism groups

The moduli space $\mathcal{M}_g$ of connected, proper, smooth algebraic curves of genus g over $\mathbb{C}$ is rationally homotopy equivalent to the classifying space $B\mathrm{Diff}^+(\Sigma_g)$ of the orientation-preserving diffeomorphism group of a surface of genus g for $g \geq 2$. In turn, the topological group $\mathrm{Diff}^+(\Sigma_g)$ has contractible components [EE69] when $g \geq 2$, hence it is homotopy equivalent to its group of components $\Gamma(\Sigma_g)$, called the *mapping class group* of Σ_g:

$$\mathcal{M}_g \simeq_{\mathbb{Q}} B\mathrm{Diff}^+(\Sigma_g) \simeq B\Gamma(\Sigma_g), \quad g \geq 2.$$

Harer [Har85] found that the homology groups of $B\Gamma(\Sigma_g)$ were independent of g in the range $\leq \frac{1}{3}(g-1)$ (the *stable range*). In order to state his result more precisely, let us write $\Sigma_{g,b}$ for a generic compact connected oriented surface of genus g with b boundary components. The mapping class group of a surface with boundary is the group of connected components of the space of orientation-preserving diffeomorphisms of the surface that restrict to the identity on the boundary. If S and S' are oriented surfaces with boundary, and S is the union of $\Sigma_{g,b}$ and S', then the inclusion $\Sigma_{g,b} \subset S$ induces a map from the mapping class group of $\Sigma_{g,b}$ to the mapping class group of S by extending diffeomorphisms with the identity. Harer's theorem states that these maps induce isomorphisms in homology groups in degrees $\leq \frac{1}{3}(g-1)$. His estimation of the stable range was improved by Ivanov [Iva93] and later by Boldsen [Bol09] and Randal-Williams [RW10]. The improved range is known to be essentially optimal. If S' is a torus with two boundary components, then it induces a map $\Sigma_{g,1} \to \Sigma_{g+1,1}$, which we call *stabilisation map*. We write $B\Gamma_\infty$ for the colimit of $B\Gamma(\Sigma_{g,1})$ with respect to the maps induced by the stabilisation maps. It is the recipient of the stable homology of the mapping class groups.

At the same time of Harer's work, Mumford [Mum83] conjectured a complete description of the stable rational cohomology of $\mathcal{M}_g$. He defined classes

$\kappa_i \in H^{2i}(\mathcal{M}_g)$ and conjectured that the stable rational cohomology ring would be a polynomial algebra on these classes. The classes κ_i can be defined in the rational cohomology of $\mathrm{BDiff}^+(\Sigma_g)$ as follows: Let $\pi\colon E \to \mathrm{BDiff}^+(\Sigma_g)$ be the universal oriented surface bundle, and let $T^{\mathrm{vert}}E$ be the vertical tangent bundle of E (that is, the bundle of pairs (p, T_pW), where $p \in E$ and $W = \pi^{-1}(\pi(p))$). Let $e \in H^2(E)$ be the Euler class of $T^{\mathrm{vert}}E$. Then the class κ_i is defined as the pushforward $\pi_!(e^{i+1}) \in H^{2i}(\mathrm{BDiff}^+(\Sigma_g))$.

Using Harer's stability theorem, Miller [Mil86] and Morita [Mor87] proved independently that the map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \longrightarrow H^*(\mathrm{BDiff}^+(\Sigma_g); \mathbb{Q})$$

was injective in the stable range. As part of his proof, Miller constructed a product in the disjoint union $\coprod_{g\geq 0} \mathrm{BDiff}(\Sigma_{g,1})$, and the group completion with respect to this product has a double loop space structure. This group completion is homotopy equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$, where $^+$ denotes the plus construction. Inspired by this later result, Tillmann [Til97] proved that the space $\mathbb{Z} \times B\Gamma_\infty^+$ has an infinite loop space structure. In [MT01], Madsen and Tillmann conjectured that certain map

$$\alpha_\infty\colon \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbf{MTSO}(2)$$

was a homotopy equivalence. The rational cohomology of $\Omega^\infty\mathbf{MTSO}(2)$ is a polynomial algebra with one generator in each even degree; therefore this new conjecture implied Mumford's conjecture. In addition, it gave a description of the integral stable homology. They also showed that α_∞ was a map of infinite loop spaces. Finally, in [MW07], Madsen and Weiss proved that α_∞ was a homotopy equivalence, which in particular solved Mumford's conjecture.

The map α_∞ is defined as follows. The space of embedded oriented surfaces diffeomorphic to Σ_g in $\mathbb{R}^\infty$ is a model for $\mathrm{BDiff}^+(\Sigma_g)$, as it is the quotient of the space of embeddings of Σ_g into $\mathbb{R}^\infty$, which is contractible, by the free action of $\mathrm{Diff}^+(\Sigma_g)$. More generally, the space of cobordisms diffeomorphic to $\Sigma_{g,b}$ in $\mathbb{R}^\infty \times [0,1]$ that restrict to a fixed closed submanifold in the boundary of $\mathbb{R}^\infty \times [0,1]$ is a model for $\mathrm{BDiff}^+(\Sigma_{g,b})$. Recall that if W is a compact oriented surface in $\mathbb{R}^n$, then the Pontryagin–Thom collapse map is a compactly supported map

$$\mathbb{R}^n \longrightarrow \mathrm{Th}(\gamma_{n-2,n})$$

that restricts on W to the classifying map of the normal bundle of W in $\mathbb{R}^n$. Instead we construct a Pontryagin–Thom collapse map

$$\mathbb{R}^n \longrightarrow \mathrm{Th}(\gamma_{2,n}^\perp)$$

that restricts on W to the classifying map of the tangent bundle of W in $\mathbb{R}^n$. Alternatively, it is the composite of the map $\mathbb{R}^n \longrightarrow \mathrm{Th}(\gamma_{n-2,n})$ and the diffeomorphism $\mathrm{Th}(\gamma_{n-2,n}) \cong \mathrm{Th}(\gamma_{2,n}^{\perp})$. This construction assigns to each surface W in $\mathbb{R}^n$ a point in $\Omega^n\mathrm{Th}(\gamma_{2,n}^{\perp})$. Letting n go to infinity we obtain a map

$$\mathrm{BDiff}^+(\Sigma_g) \longrightarrow \operatorname*{colim}_n \Omega^n\mathrm{Th}(\gamma_{2,n}^{\perp}) = \Omega^\infty\mathbf{MTSO}(2).$$

Similarly, there is a map $\mathrm{BDiff}^+(\Sigma_{g,1}) \to \Omega^\infty\mathbf{MTSO}(2)$ that is compatible with the stabilisation maps, which defines a map

$$\mathrm{B}\Gamma_\infty \longrightarrow \operatorname*{colim}_n \Omega^n\mathrm{Th}(\gamma_{2,n}^{\perp}) = \Omega^\infty\mathbf{MTSO}(2)$$

that factors through the plus construction of $\mathrm{B}\Gamma_\infty$.

At this point we may give a hint on the construction of the map (0.0.2). The space of morphisms between two objects A and A' of the cobordism category $\mathcal{C}_k$ is the space of embedded surfaces with prescribed boundary. As already observed, this space of embedded surfaces is a model for the disjoint union $\coprod_g \mathrm{BDiff}^+(\Sigma_{g,b})$, where b is the number of components of the union of the submanifolds A and A'. Therefore we obtain a map

$$\mathcal{C}_k(A, A') \longrightarrow \Omega^\infty\mathbf{MTSO}(2)$$

that induces the map (0.0.2).

Stable homology of configuration spaces of points

The Harer stability theorem applies to diffeomorphism groups of surfaces; more specifically, of compact connected oriented smooth manifolds of dimension 2. An analogous result for diffeomorphism groups of compact manifolds of dimension 0, that is, symmetric groups, is due to Nakaoka [Nak60], who proved that the homology of the symmetric group on r letters is independent of r in degrees below $r/2$. A precedent of the Madsen–Weiss theorem in this setting was discovered earlier by Barratt and Priddy [BP72]. If we write $\mathrm{B}\Sigma_\infty$ for the colimit of $\mathrm{B}\Sigma_r$ and $\mathbf{S}^\infty$ for the sphere spectrum, then the Barratt–Priddy theorem states that there is a homology equivalence

$$\mathrm{B}\Sigma_\infty \longrightarrow \Omega^\infty_0(\mathbf{S}^\infty),$$

where the subscript 0 stands for the component of the constant loop. The sphere spectrum is precisely $\mathbf{MTO}(0)$ (the unoriented version of $\mathbf{MTSO}(0)$). Their proof is different from the Madsen–Weiss proof of Mumford's conjecture, because whereas Madsen and Weiss rely on Harer's stability theorem, Barratt and Priddy make no use of Nakaoka's theorem. We remark that a proof of the Madsen–Weiss theorem independent of Harer's theorem was found in [GRW10].

The symmetric group acts on the space of embeddings $\mathrm{Emb}([r], M)$ of a discrete space with r points into a manifold M. The quotient is the configuration space $C_r(M)$. It was proved by McDuff [McD75] that, if the dimension of M is at least 2 and M has boundary, then the homology groups of $C_r(M)$ are independent of r in low degrees with respect to r. Later, Segal [Seg79] showed that they are independent of r in degrees below $r/2$ (the *stable range*). In addition, McDuff constructed a map

$$\mathcal{S}_0 \colon C_r(M) \longrightarrow \Gamma_c(\mathrm{Th}^{\mathrm{fib}}(TM) \to M)_r$$

to the space of degree r compactly supported sections of the fibrewise one-point compactification of the tangent bundle of M. Such a section is said to be *compactly supported* if it sends each point p in the complement of some compact subset to the point at infinity on the fibre of p. McDuff's main theorem states that the map $\mathcal{S}_0$ is a homology equivalence in the stable range when M has dimension ≥ 2 (but may have empty boundary).

The main result of this thesis is joint work with Oscar Randal-Williams [CRW13] that generalizes the theorem of McDuff to configurations of connected surfaces in a manifold, stabilising with respect to the genus of the surface (instead of the number of connected components), as in Harer's theorem:

$$\begin{tikzcd} \begin{array}{c}\text{Nakaoka}\\ \text{Barratt–Priddy}\end{array} \arrow[r] \arrow[d] & \text{McDuff} \arrow[d, dashed] \\ \begin{array}{c}\text{Harer}\\ \text{Madsen–Weiss}\end{array} \arrow[r, dashed] & \text{this thesis.} \end{tikzcd}$$

One may also generalize McDuff's theorem by stabilising with respect to the number of components of the surfaces (instead of their genus). This has been done by Palmer in his main theorem of Chapter 5 in [Pal12]. A particular case of it states the following: Let M be a manifold of dimension n and let W and P be (possibly non-connected) k-manifolds. Let $\mathcal{E}_r^P(W, M)$ be the space of k-submanifolds diffeomorphic to the disjoint union of W and $\coprod_{i=1}^r P$ such that each factor P is isotopic to some fixed embedding $P \subset \partial M$. Then the homology of $\mathcal{E}_r^P(W, M)$ is independent of r in degrees $< r/2$ provided that M is path connected and $n \geq 2k+3$. In the particular case of surfaces one may take $n \geq 5$, as we do in this thesis.

Homology stability for spaces of embedded surfaces

Let M be a smooth manifold, not necessarily compact and possibly with boundary. Our object of study will be certain spaces of oriented surfaces in M, which we define as follows. Let Σ_g denote a connected closed oriented smooth surface of genus g, and let $\mathrm{Emb}(\Sigma_g, M)$ denote the space of all smooth embeddings of this surface into the interior of M, equipped with the C^∞ topology. The topological group $\mathrm{Diff}^+(\Sigma_g)$ acts continuously on $\mathrm{Emb}(\Sigma_g, M)$, and we define

$$\mathcal{E}(\Sigma_g, M) = \mathrm{Emb}(\Sigma_g, M)/\mathrm{Diff}^+(\Sigma_g)$$

to be the quotient space. As a set, $\mathcal{E}(\Sigma_g, M)$ is in bijection with the set of all subsets of M which are smooth manifolds diffeomorphic to Σ_g, which is why we refer to $\mathcal{E}(\Sigma_g, M)$ as the *moduli space of genus g surfaces in M*.

We will study the space $\mathcal{E}(\Sigma_g, M)$ using a technique called *scanning*, which compares this space of surfaces in M with a certain space of "formal surfaces in M".

For an inner product space $(V, \langle -,- \rangle)$, define

$$\mathcal{S}(V) = \mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(V)).$$

That is, we take the Grassmannian $\mathrm{Gr}_2^+(V)$ of oriented 2-planes in V, consider the tautological 2-plane bundle $\gamma_2 \subset V \times \mathrm{Gr}_2^+(V)$, and take its orthogonal complement using the inner product on V. Then we take the Thom space of this vector bundle. We will denote the point at infinity by $\infty \in \mathcal{S}(V)$.

If $V \to B$ is a vector bundle with metric, we let $\mathcal{S}(V) \to B$ be the fibre bundle obtained by performing this construction fibrewise to V. The constant section with value ∞ in each fibre gives a canonical section of this bundle.

We fix a Riemannian metric $\mathfrak{g}$ on M. The *space of formal surfaces* in M is defined to be

$$\Gamma_c(\mathcal{S}(TM) \to M; \infty),$$

the space of sections of $\mathcal{S}(TM) \to M$ which are compactly supported, i.e., agree with the canonical section ∞ outside of a compact set and on ∂M. Every such section chooses for each point $x \in M$ either an oriented affine 2-dimensional subset of T_xM or the empty subset. The scanning construction associates to each oriented surface $\Sigma \subset M$ such a section by —loosely speaking— assigning to each $x \in M$ the best approximation to Σ by an affine subset of T_xM.

To make this precise, we let $\mathcal{E}^\nu(\Sigma_g, M) \subset (0,\infty) \times \mathcal{E}(\Sigma_g, M)$ be the set of pairs $(\epsilon, \Sigma \subset M)$ such that the exponential map $\exp\colon \nu(\Sigma) \to M$ restricts to an embedding on each fibre of the subspace $\nu_\epsilon(\Sigma) \subset \nu(\Sigma)$ consisting of the vectors of length $< \epsilon$. We then define a map $M \times \mathcal{E}^\nu(\Sigma_g, M) \to \mathcal{S}(TM)$ by

$$(p, \epsilon, \Sigma) \longmapsto \begin{cases} \emptyset \in \mathcal{S}(T_pM) & \text{if } p \notin \exp(\nu_\epsilon(\Sigma)), \\ (D(\exp|_{T_wM}))(T_w\Sigma, \nu) \subset T_pM & \text{if } p = \exp(\nu) \text{ for } \nu \in \nu_\epsilon(\Sigma)_w, \end{cases}$$

where we consider the oriented 2-plane $T_w\Sigma$ and the vector ν as lying inside $T_\nu(T_wM)$ using the canonical isomorphism $T_\nu(T_wM) \cong T_wM$, and then apply the linear isomorphism $D(\exp|_{T_wM})\colon T_\nu(T_wM) \to T_pM$. The adjoint to this map,

$$\mathcal{S}_g\colon \mathcal{E}^\nu(\Sigma_g, M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M; \infty),$$

is the *scanning map*. As the forgetful map $\mathcal{E}^\nu(\Sigma_g, M) \to \mathcal{E}(\Sigma_g, M)$ is a weak homotopy equivalence, we often consider $\mathcal{S}_g$ as a map from $\mathcal{E}(\Sigma_g, M)$.

In Section 3.3 we construct a function $\chi\colon \Gamma_c(\mathcal{S}(TM) \to M; \infty) \to \mathbb{Z}$ such that $\chi \circ \mathcal{S}_g$ takes constant value $2 - 2g$; we think of χ as sending a formal surface to its Euler characteristic, and write $\Gamma_c(\mathcal{S}(TM) \to M; \infty)_g = \chi^{-1}(2 - 2g)$. The simplest form of our theorem is then as follows.

Theorem A. *If M is simply connected and of dimension at least 5, then the scanning map $\mathcal{S}_g\colon \mathcal{E}(\Sigma_g, M) \to \Gamma_c(\mathcal{S}(TM) \to M; \infty)_g$ induces an isomorphism in integral homology in degrees smaller than or equal to $\frac{1}{3}(2g - 2)$.*

Theorem A follows from two rather more technical results. First, a homology stability theorem analogous to Harer stability [Har85]. To make sense of such a result it is essential to discuss surfaces with boundary, and we will shortly define spaces of surfaces with boundary inside a manifold M with non-empty boundary. A large part of our work is devoted to proving this homology stability theorem, which requires new techniques. The second technical result is analogous to the Madsen–Weiss theorem [MW07], and identifies the stable homology of these spaces of surfaces. To describe these results we must introduce more terminology.

Surfaces with boundary

Now we suppose that M has non-empty boundary ∂M and that we are given a collar $C\colon \partial M \times [0, 1) \hookrightarrow M$. For $0 < \epsilon < 1$, we write C_ϵ for the restriction of C to $\partial M \times [0, \epsilon)$. Let $\Sigma_{g,b}$ be a fixed smooth oriented surface of genus g with b boundary components, and let $c\colon \partial\Sigma_{g,b} \times [0, 1) \hookrightarrow \Sigma_{g,b}$ be a collar. We write c_ϵ for the restricted collar. We also fix an embedding $\delta\colon \partial\Sigma_{g,b} \hookrightarrow \partial M$, which we call a *boundary condition*.

Let $\mathrm{Emb}_\epsilon(\Sigma_{g,b}, M; \delta)$ be the set of ϵ-collared embeddings, that is, embeddings e such that $e \circ c_\epsilon = C_\epsilon \circ (\mathrm{Id}_{[0,\epsilon)} \times \delta)$. We topologise $\mathrm{Emb}^\epsilon(\Sigma_{g,b}, M)$ with the C^∞ topology as before. Let $\mathrm{Diff}^+_\epsilon(\Sigma_{g,b})$ be the topological group of diffeomorphisms φ of $\Sigma_{g,b}$ such that $\varphi \circ c_\epsilon = c_\epsilon$. This group acts on $\mathrm{Emb}_\epsilon(\Sigma_{g,b}, M)$, and we define

$$\mathcal{E}_\epsilon(\Sigma_{g,b}, M; \delta) = \mathrm{Emb}_\epsilon(\Sigma_{g,b}, M; \delta)/\mathrm{Diff}^+_\epsilon(\Sigma_{g,b}).$$

If $\epsilon < \epsilon'$, then there is a continuous map $\mathcal{E}_{\epsilon'}(\Sigma_{g,b}, M; \delta) \hookrightarrow \mathcal{E}_\epsilon(\Sigma_{g,b}, M; \delta)$, and we write $\mathcal{E}(\Sigma_{g,b}, M; \delta)$ for the colimit taken over all ϵ.

Let $Q\colon \partial M \rightsquigarrow N$ be a cobordism which is collared at both boundaries. Then we can glue M and Q along ∂M to obtain a new manifold $M \circ Q$ (using the collars to obtain a smooth structure). Similarly, if $e\colon \Sigma_{b+b'} \hookrightarrow Q$ is an embedding of a surface with b boundary components in ∂M and b' in N (which we call the *incoming* and *outgoing* boundaries ∂_{in}, ∂_{out} respectively) such that every component of the surface intersects the incoming boundary, and if $e(\partial_{\mathrm{in}}\Sigma_{b+b'}) = \mathrm{Im}(\delta)$, we obtain a *gluing map*

$$\begin{aligned} \mathcal{E}(\Sigma_{g,b}, M; \delta) &\longrightarrow \mathcal{E}(\Sigma_{h,b'}, M \circ Q; e|_{\partial_{\mathrm{out}}\Sigma_{b+b'}}) \\ (W \subset M) &\longmapsto (W \cup e(\Sigma_{b+b'}) \subset M \circ Q) \end{aligned}$$

where the value of h depends on the combinatorics of the topology of $\Sigma_{b+b'}$ (note that, as $\Sigma_{g,b}$ is connected and every component of $\Sigma_{b+b'}$ intersects the incoming boundary, $\Sigma_{g,b} \cup \Sigma_{b+b'}$ is connected).

In particular, if we let $Q = \partial M \times [0,1]$ and choose a diffeomorphism $M \circ Q \cong M$ (for example by reparametrising the collar in M), we obtain gluing maps $\mathcal{E}(\Sigma_{g,b}, M; \delta) \to \mathcal{E}(\Sigma_{h,b'}, M; \delta')$.

Homological stability

There are three basic types of gluing maps which generate all general gluing maps under composition. These occur when $\Sigma_{b+b'}$ is

(i) the disjoint union of a pair of pants with the legs as incoming boundary and a collection of cylinders;

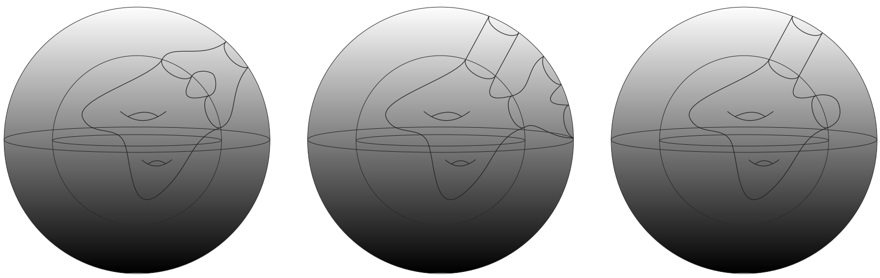
(ii) the disjoint union of a pair of pants with the waist as incoming boundary and a collection of cylinders;

(iii) the disjoint union of a disc with its boundary incoming and a collection of cylinders.

When these surfaces are embedded in $\partial M \times [0,1]$, we call them *stabilisation maps*, and we denote them by

$$\begin{aligned}
\alpha_{g,b} = \alpha_{g,b}(M;\delta,\delta')\colon \mathcal{E}(\Sigma_{g,b},M;\delta) &\longrightarrow \mathcal{E}(\Sigma_{g+1,b-1},M;\delta') \\
\beta_{g,b} = \beta_{g,b}(M;\delta,\delta')\colon \mathcal{E}(\Sigma_{g,b},M;\delta) &\longrightarrow \mathcal{E}(\Sigma_{g,b+1},M;\delta') \\
\gamma_{g,b} = \gamma_{g,b}(M;\delta,\delta')\colon \mathcal{E}(\Sigma_{g,b},M;\delta) &\longrightarrow \mathcal{E}(\Sigma_{g,b-1},M;\delta').
\end{aligned}$$

As a warning to the reader, we remark that *the notation does not determine the map*: we will often write, for example, $\beta_{g,b}$ to denote *any* gluing map of this type. There are many because there can be many non-isotopic embeddings of $\Sigma_{b+b'}$ into $\partial M \times [0,1]$.



(a) $\alpha_{2,2}(D^3)$ (b) $\beta_{2,2}(D^3)$ (c) $\gamma_{2,2}(D^3)$

Figure 1: The three basic types of stabilisation maps acting on a surface in the space $\mathcal{E}(\Sigma_{2,2},D^3;\delta)$ of surfaces in the disc of dimension 3.

The following is our main result concerning homological stability.

Theorem B. *Let M be a simply connected manifold of dimension at least 5. If the dimension of M is 5, we assume that all the pairs of pants defining stabilisation maps are contractible in $\partial M \times [0,1]$.*

(i) *Every map $\alpha_{g,b}$ induces an isomorphism in homology in degrees less than or equal to $\frac{1}{3}(2g-2)$, and an epimorphism in the next degree.*

(ii) *Every map $\beta_{g,b}$ induces an isomorphism in homology in degrees less than or equal to $\frac{1}{3}(2g-3)$, and an epimorphism in the next degree. If one of the newly created boundaries of the pair of pants is contractible in ∂M, then the map $\beta_{g,b}$ is also a monomorphism in all degrees.*

(iii) *Every map $\gamma_{g,b}(M;\delta,\delta')$ induces an isomorphism in homology in degrees less than or equal to $\frac{2}{3}g$, and an epimorphism in the next degree. If $b \geq 2$, then it is always an epimorphism.*

Note that we do *not* require that ∂M be simply connected, but only that M be. Thus there can be many non-isotopic boundary conditions.

The proof of this theorem follows the proof of the main result of [RW10], where it is proven that the moduli space of surfaces with tangential structures satisfies homological stability. It is important to observe that embeddings are not determined by a local condition, hence they do not arise as a tangential structure. The detailed survey [Wah12] has been of great aid too.

The space of embeddings of a compact surface into $\mathbb{R}^\infty = \bigcup_{n\geq 0}\mathbb{R}^n$ is weakly contractible by the Whitney embedding theorem. On the other hand, the action of $\mathrm{Diff}^+(\Sigma_{g,b})$ on this space of embeddings is free, hence the space $E(\Sigma_{g,b}, \mathbb{R}^\infty\times[0,1];\delta)$ is a model for the classifying space $B\mathrm{Diff}^+(\Sigma_{g,b})$ (note that the space of boundary conditions is contractible, so the embedding $\delta\colon \partial\mathcal{S}_{g,b}\to \partial(\mathbb{R}^\infty\times[0,1])$ is irrelevant). The statement of Theorem B in this setting is precisely the content of the Harer stability theorem.

These stability ranges are known to be essentially optimal for Harer's theorem (cf. [Bol09]), hence they are also optimal in this general setting.

Stable homology

To identify the stable homology groups resulting from Theorem B, we require a version of the scanning map for surfaces with boundary. We will not describe it in full detail here, but just say that it is a map

$$\mathcal{S}_{g,b}\colon \mathcal{E}(\Sigma_{g,b}, M;\delta)\longrightarrow \Gamma_c(\mathcal{S}(TM)\to M;\bar\delta)_g$$

to the space of sections of the bundle $\mathcal{S}(TM)\to M$ which are compactly supported and which in addition are equal to a fixed section $\bar\delta\colon \partial M\to \mathcal{S}(TM)|_{\partial M}$ on the boundary.

Theorem C. *If M is simply connected and of dimension at least 5, then the map*

$$\mathcal{S}_{g,b}\colon \mathcal{E}(\Sigma_{g,b}, M;\delta)\longrightarrow \Gamma_c(\mathcal{S}(TM)\to M;\bar\delta)_g$$

induces an isomorphism in integral homology in degrees $\leq \frac{1}{3}(2g-2)$.

This theorem will be proven during the first two sections of Chapter 3 for manifolds M with non-empty boundary. An additional argument in the third section establishes this theorem in full generality. Theorem A follows as the particular case when $b=0$.

The proof of this theorem relies on [GMTW09] and also on [RW11], where the homotopy type of the cobordism category of submanifolds in a background manifold was found, using an h-principle argument. This homotopy type in the case $M=\mathbb{R}^n$ was previously found by Galatius at the end of [Gal11].

Applications and related results

During this section we assume that the surfaces Σ have non-empty boundary for simplicity. If X is a paracompact space, then the set of homotopy classes of maps $[X, \mathcal{E}(\Sigma, M)]$ are in one-to-one correspondence with the set of subsurface bundles of $X \times M$ up to concordance, that is, surface bundles that are fibrewise included in the trivial bundle $X \times M$ up to concordances that are included in the trivial bundle $(X \times I) \times M$:

$$\{\text{subsurface bundles of } X \times M\}/\text{concordance} \longleftrightarrow [X, \mathcal{E}(\Sigma, M)].$$

As a consequence, the cohomology ring of $\mathcal{E}(\Sigma, M)$ is the ring of *characteristic classes of subsurface bundles of* $X \times M$.

Characteristic classes of fibre bundle pairs and diffeomorphism groups of manifolds

The topological group $\mathrm{Diff}_\partial(M)$ acts on the space $\mathcal{E}(\Sigma, M; \delta)$, and if $O_2(M)$ denotes the set of orbits of this action, then the orbit map

$$\coprod_{[W] \in O_2(M)} \mathrm{Diff}_\partial(M) \times \{W\} \longrightarrow \mathcal{E}(\Sigma, M; \delta)$$

is a fibration whose fibre over a surface W is the group $\mathrm{Diff}_\partial(M; W)$ of those diffeomorphisms of M that are the identity on the boundary and restrict to an orientation-preserving diffeomorphism of W. Taking classifying spaces we obtain

$$\coprod_{O_2(M)} \mathcal{E}(\Sigma, M; \delta) \longrightarrow \coprod_{O_2(M)} \mathrm{BDiff}_\partial(M; W) \longrightarrow \mathrm{BDiff}_\partial(M).$$

The stabilisation maps $- \cup P$ of our Theorem B give stabilisation maps

$$\begin{tikzcd}
\coprod_{O_2(M)} \mathcal{E}(\Sigma, M; \delta) \arrow[r] \arrow[d] & \mathrm{BDiff}_\partial(M; W) \arrow[d] \arrow[dr] & \\
\coprod_{O_2(M_1)} \mathcal{E}(\Sigma \cup P, M_1; \delta) \arrow[r] & \mathrm{BDiff}_\partial(M_1; W \cup P) \arrow[r] & \mathrm{BDiff}_\partial(M_1)
\end{tikzcd}$$

where $M_1 := M \cup (\partial M \times I)$. Hence it follows that the relative homology groups of the map in the fibre are zero in the range $* \leq \frac{2}{3}(g-1)$. An argument with the relative Serre spectral sequence shows that the homology groups of the map between total spaces are also zero in those degrees. There are maps of fibrations

$$\begin{tikzcd}
\coprod_{O_2(M)} \mathcal{E}(\Sigma, M; \delta) \arrow[r] \arrow[dd] & \mathrm{BDiff}_\partial(M; W) \arrow[dr] \arrow[dd] & \\
& & \mathrm{BDiff}_\partial(M_1; W \cup P), \\
\Gamma_c(\mathcal{S}(TM) \to M; \bar{\delta})_g \arrow[r] & \Gamma_c(\mathcal{S}(TM) \to M; \bar{\delta})_g /\!/ \mathrm{Diff}_\partial(M) \arrow[ur] &
\end{tikzcd} \tag{0.0.3}$$

where // denotes the homotopy quotient. A direct consequence of the main theorem is

Corollary D. *If* M *is simply connected and of dimension at least* 5*, then the map between total spaces in* (0.0.3)

$$\mathrm{BDiff}_\partial(M; W) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M; \bar{\delta})_g /\!/ \mathrm{Diff}_\partial(M)$$

is a homology equivalence in degrees $\leq \frac{2}{3}(g-1)$*. If* M *has non-empty boundary, then the space on the right is independent of the genus of the surface* W*.*

If Σ is an oriented surface, let $\Phi(M, \Sigma) := \pi_0 \mathcal{E}(\Sigma, M)/\mathrm{Diff}_\partial(M)$. The set of homotopy classes of maps from a paracompact space X to

$$\coprod_{[W] \in \Phi(M, \Sigma)} \mathrm{BDiff}_\partial(M; W)$$

is in bijection with the set of pairs of fibre bundles with fibres M and Σ over X up to concordance. Each such pair consists of a triple $(E \subset E', p)$, where E is a subspace of the space E' and $p \colon E' \to X$ is a smooth fibre bundle with fibre M such that the restriction $p_{|E}$ is an oriented surface bundle with fibre Σ. Two such pairs are concordant if there is a pair over $X \times I$ that restricts to the given pairs over $X \times \{0, 1\}$. Therefore, the cohomology ring of $\mathrm{BDiff}_\partial(M; W)$ is the ring of characteristic classes of pairs of fibre bundles as above.

Moduli spaces of surfaces in $\mathbb{R}^n$ and rational homotopy theory

In Section 4.1 we compute the rational stable cohomology of the space of subsurfaces in $\mathbb{R}^n$. If we denote by $\gamma_{2,n}^{\perp}$ the complement of the tautological bundle over $\mathrm{Gr}_2^+(\mathbb{R}^n)$, then the main theorem in this case gives a rational homology isomorphism in the stable range

$$\mathcal{E}(\Sigma_g, \mathbb{R}^n) \longrightarrow \Omega_0^n \mathrm{Th}(\gamma_{2,n}^{\perp}).$$

We compute the rational cohomology of the space on the right in Theorem 4.1.5. We also describe the homomorphism induced in cohomology by the classifying map $\mathcal{E}(\Sigma_g, M) \longrightarrow \mathrm{BDiff}^+(\Sigma_g)$ in the stable range (Corollary 4.1.8).

As part of these computations, we prove that the spaces $\mathrm{Th}(\gamma_{2,n}^{\perp})$ are intrinsically formal and give explicit Sullivan and Lie minimal models, which is a first step in the calculation of the stable rational homology of other spaces of surfaces with different background manifolds [Hae82, BS97, BFM09].

In addition, in Section 4.2 we apply these calculations in the spirit of [BM12] to deduce that

Theorem E. *If M is a closed parallelizable manifold, then the rational homology of $\mathcal{E}(\Sigma_g, M)$ is independent of g in degrees $\leq \frac{2}{3}(g-1)$.*

A metric for the space of submanifolds of Galatius and Randal–Williams

One of the tools of the present work is the space $\Psi_k(\mathbb{R}^n)$ of submanifolds of dimension k of $\mathbb{R}^n$ that are closed as subsets (but may be noncompact). This space was introduced by Galatius and Randal-Williams in [GRW10] and its main use is homotopical, through the following theorem:

Proposition. *The space $\Psi_k(\mathbb{R}^n)$ is homotopy equivalent to the Thom space $\mathrm{Th}(\gamma_{k,n}^{\perp})$.*

In [BM11] Bökstedt and Madsen proved that a C^1 version of $\Psi_k(\mathbb{R}^n)$ is metrizable as a step in their proof of the main theorem of that article. Their proof is not constructive, but shows that the space is regular and second countable. In Section 4.3 we give an actual metric for this C^1 version, which, loosely speaking, is given by the Hausdorff distance in the one-point compactification of the Grassmannian of k-planes of the tangent bundle of $\mathbb{R}^n$, plus a correction term that measures the volume of the submanifolds in compact subsets. We also give an interpretation of the topology given by the plain Hausdorff distance.

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I dedicate this thesis to my parents, whose perseverance has always been an inspiration to me, and to my wife and children, who gave me every day the joy and happiness not so often provided by research, generously letting me return to mathematics the day after.

Chapter 1

Preliminaries

This chapter is a recollection of terminology and results that are used in the proofs of the main theorems. We do not claim any originality in the presentation, except for several lemmas about local retractibility in Section 1.5.

1.1 Manifolds and submanifolds

An m-*dimensional topological manifold* is a Hausdorff second countable topological space locally modeled on $\mathbb{R}^m$ and its homeomorphisms. A *topological embedding* $f\colon M \to N$ is a continuous map that induces a homeomorphism between M and its image.

An m-*dimensional smooth manifold* M is an m-dimensional topological manifold locally modeled on $\mathbb{R}^m$ and its diffeomorphisms. Smooth maps between smooth manifolds are required to preserve the local structure; in particular they induce maps between tangent bundles. A smooth map $f\colon M \to N$ is a *smooth embedding* if it is a topological embedding and the fibrewise linear maps induced between tangent bundles are injective.

A subset $W \subset M$ is a *smooth submanifold* if there is a smooth manifold Σ and an embedding $f\colon \Sigma \to M$ whose image is W. The homeomorphism of Σ onto its image endows W with a smooth structure, which does not depend on the embedding [Hir94].

An *orientation* on a manifold M is a compatible choice of orientations in the tangent spaces of M. A manifold is *orientable* if it admits an orientation, in which case there are exactly two possible orientations for each connected component of M. An orientable submanifold of M is not naturally oriented, even if M itself is oriented.

If Σ is an oriented manifold and M is a manifold, then there is a map

$$p\colon \mathrm{Emb}(\Sigma, M) \longrightarrow \mathcal{E}(\Sigma, M)$$

from the space of embeddings of Σ into M (with the Whitney C^∞ topology [Hir94]) to the set of oriented submanifolds of M that are diffeomorphic to Σ through an orientation-preserving diffeomorphism, given by sending each embedding to its image, with the orientation induced by the embedding. The inverse image $p^{-1}(W)$ of a submanifold W has a free action of the group $\mathrm{Diff}^+(\Sigma)$ of orientation-preserving diffeomorphisms of Σ, given by precomposition. As the smooth structure on a submanifold W does not depend on the embedding, if $f, g\colon \Sigma \to M$ are two embeddings whose image is W, the composite fg^{-1} is a diffeomorphism of Σ. Therefore this action is also transitive.

Hence, there is a bijection between the quotient of the space of embeddings of Σ by the action of $\mathrm{Diff}^+(\Sigma)$ and the set of oriented submanifolds of M that are diffeomorphic to Σ. This bijection endows $\mathcal{E}(\Sigma, M)$ with a topology.

During the proof of Theorem B we will need to consider embedded surfaces with boundary. We will also need to consider spaces of embedded surfaces in the complement of some submanifold of M. Such a complement in general will not be a manifold, but will be a *manifold with corners*. We start by recalling their definition in this section, and in Section 1.2 we will define spaces of embedded surfaces in a manifold with corners.

Manifolds with corners

Although the main theorems of this thesis can be formulated in the realm of smooth manifolds as defined above, at certain places of the proof it is unavoidable the use of manifolds with boundary, and, what is more, manifolds with corners, which we next define.

Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, and let $f : A \to B$ be a continuous map. We say that f is *smooth* if it extends to a smooth map from an open neighbourhood of A to $\mathbb{R}^m$. If f is a homeomorphism, we say that it is a *diffeomorphism* if it is smooth and has a smooth inverse. In particular, this defines the notion of diffeomorphism between subsets of $\mathbb{R}^m_+ := [0, \infty)^m$.

A *manifold with corners* M of dimension m is a Hausdorff, second countable topological space locally modeled on the space $\mathbb{R}^m_+$ and its diffeomorphisms [Cer61, Lau00].

In detail, a *smooth atlas of* M is a family of topological embeddings (called *charts*) $\{\varphi_i\colon U_i \to [0, \infty)^m\}_{i\in I}$, where each U_i is an open subset of M and if $U_i \cap U_j \neq \emptyset$ then the composite $\varphi_j \varphi_i^{-1}$ is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to

$\varphi_j(U_i \cap U_j)$. A *smooth structure* on a Hausdorff, second countable topological space is a smooth atlas which is maximal with respect to inclusion.

A k-*submanifold* of a manifold with corners is a subset $W \subset M$ such that for each point $p \in W$ there is a chart (U, φ) of M with $p \in U$ such that $W \cap U = \varphi^{-1}(v + \mathbb{R}^k_+)$, where $v \in \mathbb{R}^m_+$ (see [Cer61, Definition 1.3.1]). An *embedding* is a smooth map whose image is a submanifold and that induces a diffeomorphism onto its image.

The *tangent space* $T_p\mathbb{R}^m_+$ at a point $p \in \mathbb{R}^m_+$ is defined to be $T_p\mathbb{R}^m$. Since diffeomorphisms of $\mathbb{R}^m_+$ induce isomorphisms between tangent spaces of $\mathbb{R}^m_+$, we can define the tangent bundle of a manifold with corners following the usual procedure. A smooth map $f\colon N \to M$ induces a fibrewise linear map $Df\colon TN \to TM$ between tangent bundles.

If (U, φ) is a chart and $p \in U$, then the number $c(p)$ of zero coordinates in $\varphi(p)$ is independent of the chart. The boundary of M is the subspace

$$\partial M = \{p \in M \mid c(p) > 0\}.$$

A *connected* k-*face* of M is the closure of a component of the subspace

$$\{p \in M \mid c(p) = k\}.$$

A manifold with corners M is a *manifold with faces* if each $p \in M$ belongs to $c(p)$ connected 1-faces. A *face* is a (possibly empty) union of pairwise disjoint connected k-faces, for some k, and is itself a manifold with faces. Observe that the boundary of a k-face is a union of $(k-1)$-faces.

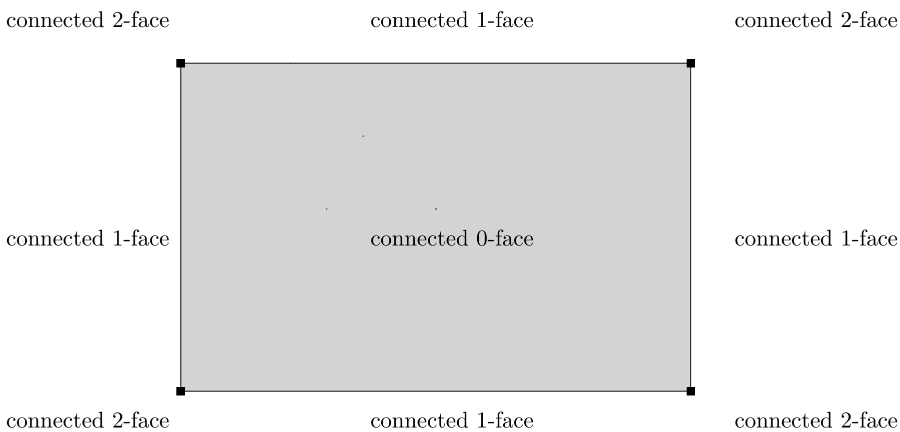


Figure 1.1: A rectangle is a manifold with faces.

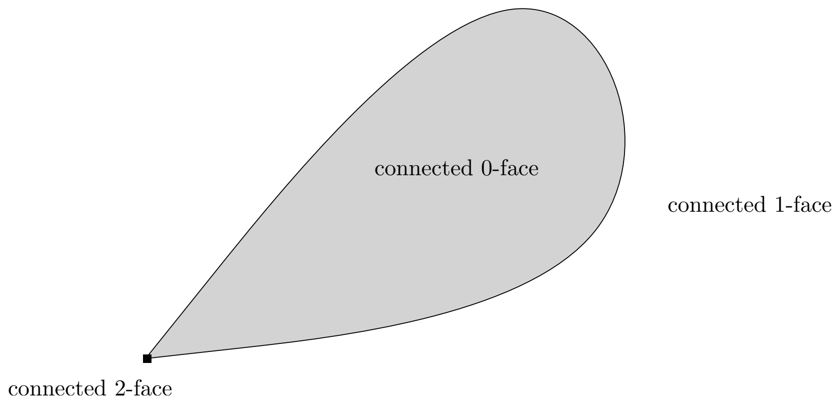


Figure 1.2: A drop is not a manifold with faces.

Notation 1.1.1. During Chapters 1 and 2, the word manifold will be used synonymously with manifold with faces. In Chapter 3 we will turn back to the usual definition of a manifold.

Definition 1.1.2. If M is a manifold and $\partial^0 M$ is a 1-face in M, a *collar* of $\partial^0 M$ is an embedding c of the manifold $\partial^0 M \times [0,1)$ into M such that $c(x,0) = x$ and such that $c_{|(F \cap \partial^0 M) \times [0,1)}$ is a collar of $\partial^0 M \cap F$ in F for any other 1-face F. A manifold is *collared* if a 1-face $\partial^0 M$ and a collar of $\partial^0 M$ are given.

If B and M are manifolds, and c and C are collars of $\partial^0 B$ and $\partial^0 M$, then an embedding $e\colon B \to M$ is said to be ϵ-*collared* if $ec(x,t) = C(e(x),t)$ for all $t < \epsilon$. In particular, $e(\partial^0 B) \subset \partial^0 M$. An embedding e is said to be *collared* if it is ϵ-collared for some $\epsilon > 0$.

A smooth map $f\colon B \to M$ between manifolds is *transverse* to a submanifold $W \subset M$ if for each $p \in B$ such that $f(p) \in W$ we have $Df(T_p B) + T_{f(p)} W = T_{f(p)} M$. A smooth map $f\colon B \to M$ to a manifold M is *transverse to* ∂M if it is transverse to any connected face. A *neat* embedding f is an embedding that is transverse to the boundary and that is collared if B is collared. A neat submanifold is the image of a neat embedding.

If A and W are submanifolds of manifolds B and M, a *neat embedding of the pair* (B, A) *into the pair* (W, M) is an embedding $e\colon B \to M$ such that $e(B) \cap W = e(A)$ and such that $\dim(e(T_p B) + T_{e(p)} W) = \min\{\dim B + \dim W, \dim M\}$.

1.2 Spaces of embeddings and spaces of submanifolds

In this section we define carefully the spaces of embeddings of manifolds with corners and the spaces of embedded surfaces with corners.

Boundary conditions for spaces of embeddings

We say that a pair of embeddings $f, g \colon B \to M$ have the same *incidence relation* if $f(x)$ and $g(x)$ belong to the interior of the same connected face of M for every $x \in B$. To each embedding $f \colon B \to M$ we can associate a function $[f]$ from B to the set of faces of M that sends a point x to the minimal face to which $f(x)$ belongs. Then f and g have the same incidence relation if and only if $[f] = [g]$. We denote by

$$\mathrm{Emb}(B, M; [f])$$

the space of neat embeddings g of B in M such that $[g] = [f]$, endowed with the Whitney C^∞ topology, as in [Cer61]. If $f \colon (B, A) \to (M, W)$ is an embedding of a pair, we denote by

$$\mathrm{Emb}((B, A), (M, W); [f]) \subset \mathrm{Emb}(B, M; [f])$$

the subspace of neat embeddings of the pair (B, A) into the pair (M, W) that have the same incidence relation as f.

A *boundary condition* for $\mathrm{Emb}(B, M; [f])$ is a function q that assigns to each point of B a neat submanifold of $[f](x)$ and is constant on each connected face. We denote by

$$\mathrm{Emb}(B, M; q) \subset \mathrm{Emb}(B, M; [f])$$

the subspace of those embeddings g such that $g(x) \in q(x)$. If $A \subset B$ and $W \subset M$, then we denote by

$$\mathrm{Emb}((B, A), (M, W); q) \subset \mathrm{Emb}(B, M; q)$$

the subspace of embeddings of the pair (B, A) into the pair (M, W). We say that a boundary condition is *constant* if $q(x)$ is diffeomorphic to the connected face that contains x.

We denote by $\mathrm{Diff}(M) \subset \mathrm{Emb}(M, M; [\mathrm{Id}])$ the space of diffeomorphisms of M that restrict to diffeomorphisms of each connected face. If W is a submanifold, we denote by

$$\mathrm{Diff}(M; W) \subset \mathrm{Diff}(M)$$

the subspace of those diffeomorphisms of M that restrict to a diffeomorphism of W, which is orientation-preserving if W is oriented. We denote by $\mathrm{Diff}_\partial(M)$ the space of diffeomorphisms of M that restrict to the identity on the boundary.

If $f, g\colon B \to M$ are neat embeddings, we say that f and g *have the same jet along* ∂M if $f^{-1}(\partial M) = g^{-1}(\partial M)$ and all the partial derivatives of f and g at all points in $f^{-1}(\partial M)$ agree. This defines an equivalence relation, and we define the set of jets $J_\partial(B, M)$ to be the quotient of the set of all neat embeddings of B into M by the relation of having the same jet. If $d \in J_\partial(B, M)$, we denote by

$$\mathrm{Emb}(B, M; d)$$

the space of all neat embeddings of B into M whose jet is d endowed with the Whitney C^∞ topology.

Boundary conditions for spaces of submanifolds

As sketched in the Introduction, we will join the surfaces in M with cobordisms defined in a collar of M. This operation is not defined in the space of all embedded surfaces, so we need to define the space of all surfaces W that glue well with a given cobordism P. Since we are in the realm of smooth manifolds, it is not enough to take all submanifolds that intersect the boundary of M in the same submanifold as the cobordism does: We need to impose that the partial derivatives of W and P at their common boundaries agree.

Let us denote by T^kM the k-fold tangent space of M, which is recursively defined as $T^kM := T\,T^{k-1}M$. Let $T^\infty(M)$ be the inverse limit of the projections

$$\ldots \longrightarrow T^kM \longrightarrow T^{k-1}M \longrightarrow \ldots \longrightarrow TM \longrightarrow M,$$

and write $T^\infty_\partial M := T^\infty M_{|\partial M}$. If $e \in \mathrm{Emb}(B, M; d)$ is an embedding with $e(\partial B) \subset \partial M$, then it induces a map $T^\infty e\colon T^\infty_\partial B \to T^\infty_\partial M$. By construction, the image of this map only depends on d, and this defines a map

$$j_B\colon J_\partial(B, M) \longrightarrow \mathcal{P}(T^\infty M)$$

to the power set of $T^\infty M$. We define the space of boundary conditions $\Delta(B, M)$ as the image of j_B. Let $\Delta_n(M)$ be the union of the images of j_B, where B runs along diffeomorphism classes of n-dimensional manifolds. If $\delta \in \Delta(B, M)$, we denote by $\mathcal{E}(B, M; \delta)$ the set of submanifolds W in M diffeomorphic to B such that $T^\infty_\partial W = \delta$. If $j_B(d) = \delta$, we endow $\mathcal{E}(B, M; \delta)$ with a topology via the bijection

$$\mathrm{Emb}(B, M; d)/\mathrm{Diff}_\partial(B) \longrightarrow \mathcal{E}(B, M; \delta)$$

that sends a class of an embedding e to its image. If W is orientable, we write $\Delta^+(B, M)$ for the space of pairs (δ, θ) where $\delta \in \Delta(B, M)$ and θ is an orientation of δ. If $\delta \in \Delta^+(B, M)$, we write $\mathcal{E}^+(B, M; \delta)$ for the set of oriented submanifolds W of M diffeomorphic to B such that $T_\partial^\infty W = \delta$, and we topologize it using the bijection

$$\mathrm{Emb}(B, M; d)/\mathrm{Diff}_\partial^+(B) \longrightarrow \mathcal{E}^+(B, M; \delta).$$

If $B = \Sigma$ is a collared compact connected oriented surface, this topology does not depend on the chosen d, because if $j(d) = j(d')$ then there exists a diffeomorphism h (that does not need to fix the boundary) of Σ such that the map $\mathrm{Emb}(\Sigma, M; d) \to \mathrm{Emb}(\Sigma, M; d')$ given by composing with h is a homeomorphism. Similarly, a diffeomorphism $h\colon \Sigma \to \Sigma'$ induces a homeomorphism $\mathrm{Emb}(\Sigma, M; d) \to \mathrm{Emb}(\Sigma', M; d \circ h^{-1})$, which is equivariant with respect to the induced homeomorphism $\mathrm{Diff}^+(\Sigma) \to \mathrm{Diff}^+(\Sigma')$. Therefore, this space is determined by the genus of Σ and the boundary condition δ (which describes the boundary of Σ).

Notation 1.2.1. If Σ has genus g and b boundary components, we use the shorter notation $\mathcal{E}^+_{g,b}(M; \delta) := \mathcal{E}^+(\Sigma, M; \delta)$, although specifying b is redundant, as it is already given by δ. We write $\delta^0 := \delta \cap \partial^0 M$, and observe that δ^0 is determined by its underlying submanifold (hence, if M is a manifold with boundary, i.e., $\partial M = \partial^0 M$, then $\Delta_2(M)$ is the set of compact submanifolds of dimension 1 in ∂M). We denote by $\mathrm{Diff}(M; \delta)$ the space of those diffeomorphisms that fix δ.

1.3 Haefliger's approximation for embeddings

In the proof of Theorem B we will need to compute $\pi_0\mathrm{Emb}(B, M; q)$ several times. Our approach to this problem has been through the following theorem of Haefliger, which we state in its relative version. Thus, homotopies and isotopies are relative to the boundary of B.

Theorem 1.3.1 ([Hae61]). *Let B be a compact connected manifold of dimension n and let M be a manifold of dimension m. Let $g\colon \partial B \to M$ be a smooth embedding and let $f\colon B \to M$ be a $(k+1)$-connected map with $f_{|\partial B} = g$. Then*

(i) *f is homotopic to an embedding if and only if $m \geq 2n - k$ and $n > 2k + 2$ (alternatively, $m \geq 2n - k$ and $2m \geq 3(n+1)$);*

(ii) *two embeddings of B into M that are homotopic to f are isotopic if $m > 2n - k$ and $n \geq 2k + 2$ (alternatively, $m > 2n - k$ and $2m > 3(n+1)$).*

Corollary 1.3.2. *If* B *is a connected surface,* q *is a constant boundary condition for embeddings of* B *into* M *and either* $m > 5$ *or* $m = 5$ *and* $\pi_1(M)$ *is trivial, then the inclusion* $\mathrm{Emb}(B, M; q) \to \mathrm{map}(B, M; q)$ *induces a bijection on connected components. If* $\pi_1(M)$ *is not trivial, then it induces a surjection on connected components.*

An alternative approach to this problem is through the Goodwillie–Weiss calculus of embeddings [GKW01]. Roughly speaking, their theory constructs a sequence of spaces $T_k\mathrm{Emb}(B, M)$ and maps $\mathrm{Emb}(B, M) \to T_k\mathrm{Emb}(B, M)$ that are $(3-\dim(M)+k(\dim(M)-\dim(B)-2))$-connected. One might ask whether this approach gives a better approximation of $\pi_0\mathrm{Emb}(B, M)$. A first remark in this direction is that this approximation only works if $\dim(M) - \dim(B) \geq 3$, so the only improvement we could expect would be the removal of the assumption that M is simply connected. By the above theorem, as the dimension of M grows, the existence of homology stability for embedded surfaces is equivalent to the existence of homology stability with respect to genus for surfaces with maps to a background space, that is, the homotopy quotient $\mathrm{map}(\Sigma, M)/\!/\,\mathrm{Diff}^+(\Sigma)$ (i.e., the Borel construction [Hus94]). When M is simply connected, homology stability for this quotient was established by Cohen and Madsen [CM09, CM10] and later improved by Randal-Williams [RW10]. On the other hand, if M is not simply connected, then the problem of homology stability for the quotient is still open.

If we let B be a surface with non-empty boundary and M be non-simply connected, then the cardinality of $\pi_0\mathrm{map}(B, M; q)$ will in general grow to infinity with the genus of B, and so will the cardinality of $\pi_0\mathrm{Emb}(B, M; q)$. We point out that a higher version of the results of this thesis, that is, a study of homology stability properties of the space of embedded high-dimensional manifolds would find better approximations via the Goodwillie–Weiss theory.

1.4 Tubular neighbourhoods

Let $V \subset M$ be a neat submanifold and denote by $N_M V = TM_{|V}/TV$ the normal bundle of V in M. A *tubular neighbourhood* of V in M is a neat embedding

$$f\colon N_M V \longrightarrow M$$

such that the restriction $f_{|V}$ is the inclusion $V \subset M$ and the composition

$$TV \oplus N_M V \longrightarrow T(N_M V)_{|V} \xrightarrow{Df} TM_{|V} \xrightarrow{\text{proj.}} N_M V$$

agrees with the projection onto the second factor. If $W \subset M$ is a submanifold and the inclusion $V \subset M$ is an embedding of pairs of $(V, V\cap W)$ into $(M, M\cap W)$, we define a *tubular neighbourhood of* V *in the pair* (M, W) as a tubular neighbourhood f of V in M such that $f \cap W$ is a tubular neighbourhood of $V \cap W$.

We may compactify $N_M V$ fibrewise by adding a sphere at infinity to each fibre, obtaining the *closed normal bundle* $\overline{N}_M V$ of V in M. We denote by

$$S(\overline{N}_M V) \subset \overline{N}_M V$$

the subbundle of spheres at infinity. We define a *closed tubular neighbourhood* of a collared submanifold V to be an embedding of $\overline{N}_M V$ into M whose restriction to $N_M V$ is a tubular neighbourhood.

Note that V determines an incidence relation f for $\overline{N}_M V$ in M, by assigning to each vector (x, v) the minimal face to which x belongs. We denote by

$$\overline{\mathrm{Tub}}(V, M) \subset \mathrm{Emb}((\overline{N}_M V, V), (M, V); f)$$

the subspace of tubular neighbourhoods. A boundary condition q_N for a tubular neighbourhood of V is a boundary condition for V in M, and we denote by

$$\overline{\mathrm{Tub}}(V, M; q_N) \subset \overline{\mathrm{Tub}}(V, M)$$

the subspace of those tubular neighbourhoods t such that $t(x, v) \subset q_N(x)$. We denote by

$$\overline{\mathrm{Tub}}(V, (M, W); q_N) \subset \overline{\mathrm{Tub}}(V, M; q_N)$$

the subspace of tubular neighbourhoods of V in the pair (M, W). Finally, if $q \subset q_N$ is a pair of boundary conditions, we denote by

$$\overline{\mathrm{Tub}}(V, M; (q_N, q))$$

the space of tubular neighbourhoods of V in M such that the restriction to each face F is a tubular neighbourhood in the pair $(q_N(x), q(x))$, where x is any point in F.

The following lemma follows from the proof of [God07, Proposition 31], where it is stated for the space of all tubular neighbourhoods of compact submanifolds.

Lemma 1.4.1. *If* V *and* W *are compact submanifolds of* M *and* q_N *is a boundary condition for* V *in* M *such that* $q_N(x)$ *is a neighbourhood of* x *in the face* $[V \subset M](x)$, *then the spaces* $\overline{\mathrm{Tub}}(V, M; q_N)$ *and* $\overline{\mathrm{Tub}}(V, (M, W); q_N)$ *are contractible.*

Proof. Let us denote by $\mathrm{Tub}(V, M; q_N)$ the space of all non-closed, collared tubular neighbourhoods of V in M. The proof in [God07] has two steps. In the first, a tubular neighbourhood f is fixed and a weak deformation retraction

H of $\mathrm{Tub}(V, M; q_N)$ is constructed into the subspace T_f of all tubular neighbourhoods whose image is contained in $\mathrm{Im}\, f$. The second step provides a contraction of T_f to the point $\{f\}$. It is easy to see that if f is a closed, collared tubular neighbourhood of a pair, then both homotopies define homotopies for $\overline{\mathrm{Tub}}(V, (M, W); q_N)$, and the argument there applies verbatim. □

1.5 Local retractibility

During the proof of Theorem B, we will need to prove that certain restriction maps onto spaces of embeddings or spaces of embedded surfaces are Serre fibrations. We approach this problem in this section, following Palais and Cerf, through Lemma 1.5.3, wich says that if E and X are spaces with an action of a group G, then an equivariant map $f\colon E \to X$ between them is a locally trivial fibration (in particular a Serre fibration) provided that X is "G-locally retractile". We point out that the role of the group G here is merely auxiliar to prove that a map is a fibration. The rest of the section is devoted to verify that the spaces of embeddings and embedded surfaces that will concern us are G-locally retractile when G is the diffeomorphism group of M.

Definition 1.5.1. ([Cer61, Pal60]) Let G be a topological group. A G-space X is G-*locally retractile* if for any $x \in X$ there is a neighbourhood U of x and a continuous map $\xi\colon U \to G$ (called the G-*local retraction around* x) such that $y = \xi(y) \cdot x$ for all $y \in U$.

Lemma 1.5.2. *If X is a G-locally retractile locally connected space and $G_0 \subset G$ denotes the connected component of the identity, then X is also G_0-locally retractile.*

Proof. If $\xi\colon U \to G$ is a local retraction around $x \in X$, and $x \in U_0 \subset U$ is a connected neighbourhood of x, then, since $\xi(x) = \mathrm{Id}$, we deduce that $\xi_{|U}$ factors through G_0 and defines a G_0-local retraction around x. □

Lemma 1.5.3 ([Cer61])**.** *A G-equivariant map $f\colon E \to X$ onto a G-locally retractile space is a locally trivial fibration.*

Proof. For each $x \in X$ there is a neighbourhood U and a G-local retraction ξ that gives a homeomorphism

$$\begin{aligned} f^{-1}(x) \times U &\longrightarrow f^{-1}(U) \\ (z, y) &\longmapsto \xi(y) \cdot z. \end{aligned}$$

□

Lemma 1.5.4. *If $f\colon X \to Y$ is a G-equivariant map that has local sections and X is G-locally retractile, then Y is also G-locally retractile. In particular, f is a locally trivial fibration.*

Proof. The composite of a local section for f and a G-local retraction for X gives a G-local retraction for Y. $\square$

Let $P \to B$ be a principal G-bundle and let F be a left G-space. The space of compactly supported sections $\Gamma_c(P \to B)$ does not act on the space of compactly supported sections of the associated bundle $\Gamma_c(P \times_G F \to B)$. On the other hand, the space of compactly supported sections of the adjoint bundle $P \times_{\mathrm{adj} G} G \to B$ —classified by the composition $B \to G \to \mathrm{Aut}(G)$ of the classifying map of the principal bundle and the adjoint representation— does act on $\Gamma_c(P \times_G F \to B)$. Recall that a space F is locally-equiconnected if the inclusion of the diagonal into $F \times F$ is a cofibration. This condition is satisfied by CW-complexes.

Lemma 1.5.5. *If B is compact and locally compact, and F is a locally equiconnected G-space that is also G-locally retractile, then $\mathrm{map}(B, F)$ is $\mathrm{map}(B, G)$-locally retractile. For a fibre bundle $P \times_G F \to B$, the space of sections $\Gamma(E \to B)$ is $\Gamma(P \times_{\mathrm{adj} G} G \to B)$-locally retractile.*

Proof. Here is a proof of the first part. A proof of the second part is similar to this one, working with spaces over B.

Let $\Phi\colon G \times F \to F \times F$ be defined as $\Phi(x, g) = (x, g \cdot x)$. If we prove that there is a neighbourhood V of the diagonal $D = f(B) \times f(B) \subset F \times F$ and a global section φ of the restriction $\Phi_{|V}\colon \Phi^{-1}(V) \to V$, then we obtain a $\mathrm{map}(B, G)$-local retraction ψ around any point $f_0 \in \mathrm{map}(B, F)$ as the composition

$$A = \{f \mid (f_0 \times f)(B) \subset V\} \xrightarrow{f \mapsto f_0 \times f} \mathrm{map}(B, V) \xrightarrow{\varphi \circ -} \mathrm{map}(B, G \times F) \xrightarrow{\pi} \mathrm{map}(B, G),$$

where the last map is induced by the projection onto the second factor. To see that this is a local retraction, we first notice that A is an open neighbourhood of f_0 because B is compact and locally compact. Second, write $\Psi\colon \mathrm{map}(B, G) \times \{f_0\} \to \mathrm{map}(B, F)$, and

$$\begin{aligned}(\Psi\psi(f))(x) &= (\Psi\pi\varphi(f_0 \times f))(x) = \Psi\pi\varphi(f_0(x), f(x)) \\ &= (\pi\varphi(f_0(x), f(x))) \cdot f_0(x) = f(x).\end{aligned}$$

So we only need to find V and φ. Let us denote $G_{x,y} = \{g \in G \mid gx = y\}$. Let $x_0 \in F$ and let $\xi\colon U_{x_0} \to G \times \{x_0\}$ be a local retraction around x_0. Then there is a homeomorphism

$$U_{x_0} \times U_{x_0} \times \Phi^{-1}(x_0, x_0) \cong U_{x_0} \times U_{x_0} \times G_{x_0, x_0}$$

that sends a triple (x, y, g) to the triple $(x, y, \xi(y)^{-1} g \xi(x))$. Define V' to be the open set $\bigcup_{x \in F} U_x \times U_x$. The restriction $\Phi_{|V'}\colon \Phi^{-1}(V') \to V'$ is locally trivial, hence a fibre bundle.

As we have assumed that F is locally equiconnected, we may take a neighbourhood deformation retract V of $D \subset F \times F$, and, since $f(B)$ is compact, we may also impose that $V \subset V'$. Then in the following pullback square of fibre bundles

$$\begin{tikzcd} \Phi^{-1}(D) \arrow[r] \arrow[d] & \Phi^{-1}(V) \arrow[d] \\ D \arrow[r] & V \end{tikzcd}$$

the bottom map is a homotopy equivalence and the left vertical map has a global section that sends a pair (x, x) to the pair (x, e), where $e \in G$ is the identity element. Hence the right vertical map has a global section ϕ too. $\square$

If $p \colon E \to B$ is a rank n vector bundle over a manifold B, we denote by

$$\mathrm{Vect}_k(E) := \Gamma(\mathrm{Gr}_k(E) \to B)$$

the space of rank k vector subbundles of E. If $P \to B$ is the associated principal bundle, let

$$\mathrm{GL}(E) := \Gamma(P \times_{\mathrm{adj}(\mathrm{GL}_n(\mathbb{R}))} \mathrm{GL}_n(\mathbb{R}) \to B).$$

be the space of sections of its adjoint bundle. If $Z \subset \partial B$ is a submanifold and $L_\partial \in \mathrm{Vect}_k(E_{|Z})$ is a vector subbundle, we denote by $\mathrm{Vect}_k(E; L_\partial) \subset \mathrm{Vect}_k(E)$ the subspace of those vector bundles whose restriction to Z is L_∂. Similarly, we denote by $\mathrm{GL}_Z(E)$ the group of bundle automorphisms of E that restrict to the identity over Z. The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is compact, locally equiconnected and $\mathrm{GL}(n)$-locally retractile. As a consequence,

Corollary 1.5.6. *If B is compact and E is a vector bundle over B, then the space $\mathrm{Vect}_k(E)$ is $\mathrm{GL}(E)$-locally retractile and the space $\mathrm{Vect}_k(E; L_\partial)$ is $\mathrm{GL}_Z(E)$-locally retractile.*

The following lemma and its corollary are a consequence of a more general theorem proved by Cerf [Cer61, 2.2.1 Théorème 5, 2.4.1 Théorème 5′] in full generality for manifolds with faces. Palais [Pal60] proved it for manifolds without corners and Lima [Lim63] gave later a shorter proof.

Proposition 1.5.7 ([Cer61]). *If $f \colon B \to M$ is an embedding of a compact manifold B into a manifold M, then $\mathrm{Emb}(B, M; [f])$ is $\mathrm{Diff}(M)$-locally retractile. If d is a jet of an embedding of B into M, then $\mathrm{Emb}(B, M; d)$ is $\mathrm{Diff}_\partial(M)$-locally retractile.*

Applying Lemma 1.5.3 to the restriction map between spaces of embeddings, we obtain

Corollary 1.5.8 ([Cer61]). *If* $A \subset B$ *is a compact submanifold and* $f\colon B \to M$ *is an embedding, then the restriction map*

$$\mathrm{Emb}(B, M; [f]) \longrightarrow \mathrm{Emb}(A, M; [f_{|A}])$$

is a locally trivial fibration.

We will need local retractibility for the space of surfaces in a manifold:

Proposition 1.5.9 ([BF81, Mic80]). *If* B *and* M *are manifolds,* d *is a jet of* B *in* M *and* $\delta = j(d)$*, then the quotient map*

$$\mathrm{Emb}(B, M; d) \longrightarrow \mathrm{Emb}(B, M; d)/\mathrm{Diff}_\partial(B) := \mathcal{E}(B, M; \delta)$$

is a fibre bundle. $\mathrm{Diff}_\partial(B)$ *may be replaced by* $\mathrm{Diff}^+_\partial(B)$ *if* B *is oriented.*

From Lemmas 1.5.4 and 1.5.9 we deduce that

Corollary 1.5.10. *The space* $\mathcal{E}^+_{g,b}(M; \delta)$ *is* $\mathrm{Diff}_\partial(M)$*-locally retractile.*

We will also need local retractibility for two more spaces.

Proposition 1.5.11. *Let* $W \subset M$ *be a submanifold, let* $A \subset B$ *be manifolds, and let* $f\colon (B, A) \to (M, W)$ *be an embedding. Then the space of embeddings of pairs* $\mathrm{Emb}((B, A), (M, W); [f])$ *is* $\mathrm{Diff}(M; W)$*-locally retractile.*

Proof. Let e_0 be one such embedding, and consider the diagram

$$\begin{tikzcd}
& \mathrm{Diff}(M; W) \times \{e_0\} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Diff}(M) \times \{e_0\} \arrow[dl] \arrow[dd] \\
\mathrm{Diff}(W) \times \{e_{0|A}\} \arrow[rr] \arrow[dd] & & \mathrm{Emb}(W, M; [W \subset M]) \arrow[dd, "\circ e_{0|A}"'] & \\
& X \arrow[rr] \arrow[dl] & & \mathrm{Emb}(B, M; [f]) \arrow[dl] \\
\mathrm{Emb}(A, W; [f_{|A}]) \arrow[rr] & & \mathrm{Emb}(A, M; [f_{|A}]) &
\end{tikzcd}$$

where the space $X \subset \mathrm{Emb}(B, M; [f])$ is the subspace of those embeddings e such that $e(A) \subset W$. Hence the bottom square is a pullback square and has a natural action of $\mathrm{Diff}(M; W, [f])$. The space $\mathrm{Emb}((B, A), (M, W); [f])$ is a subspace of X and is invariant under the action of $\mathrm{Diff}(M; W)$. All the vertical maps except $\mathrm{Emb}(W, M; [W \subset M]) \to \mathrm{Emb}(A, M; [f_{|A}])$ are orbit maps. All the vertical maps except possibly $h\colon \mathrm{Diff}(M; W) \times \{e_0\} \to X$ are locally trivial fibrations by Lemmas 1.5.3 and 1.5.7. Moreover, h is the pullback of the other three vertical maps, hence is also a locally trivial fibration, so it has local sections. As the subspace $\mathrm{Emb}((B,A),(M,W);[f]) \subset X$ is $\mathrm{Diff}(M;W)$-invariant, any $\mathrm{Diff}(M;W)$-local retraction around e_0 in X gives, by restriction, a local retraction around e_0 in $\mathrm{Emb}((B,A),(M,W);[f])$. ☐

Corollary 1.5.12. *The space* $\mathrm{Emb}(B,M;q)$ *is* $\mathrm{Diff}(M;q)$*-locally retractile and the space* $\mathrm{Emb}((B,A),(M,W);q)$ *is* $\mathrm{Diff}(M;W,q)$*-locally retractile.*

Proof. Both results are obtained by applying Lemma 1.5.11 recursively to the submanifolds in the image of q. ☐

Let B and M be manifolds and let $C \subset B$ be a submanifold. Let q and q_N be boundary conditions for B in M. The set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B,M;q,q_N)$ is defined as the set of triples (e,t,L), where

(i) $e \in \mathrm{Emb}(B,M;q)$ is an embedding;

(ii) $t \in \overline{\mathrm{Tub}}(e(B),M;(q_N,q))$;

(iii) $L \in \Gamma(\mathrm{Gr}_k(N_M e(C)) \to e(C)); N_{q(\partial^0 C)} e(\partial^0 C))$.

If $k=0$, the subset C is irrelevant and we will write $\overline{\mathrm{T}}\mathrm{Emb}(B,W;q,q_N)$ for $\overline{\mathrm{T}}\mathrm{Emb}_{0,C}(B,M;q,q_N)$. Note that, if $\partial^0 C \neq \emptyset$, then the last condition forces $k = \dim q(\partial^0 C) - \dim \partial^0 C$.

The set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B,M;q,q_N)$ has a natural action of the discretization of $\mathrm{Diff}(M;q,q_N)$ given as follows: If g is a diffeomorphism of M and t is a tubular neighbourhood as above, then g induces isomorphisms $TM_{|e(B)} \to TM_{|ge(B)}$ and $Te(B) \to Tge(B)$, hence an isomorphism $g_*\colon N_M e(B) \to N_M ge(B)$. We define $g(t)$ as the composite

$$N_M ge(B) \xrightarrow{g_*^{-1}} N_M e(B) \xrightarrow{t} M \xrightarrow{g} M.$$

We define $g(L)$ as $g_*(L)$.

Endow the set $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B,M;q,q_N)$ with the following topology: For a point (e_0,t_0,L_0), pick a $\mathrm{Diff}(M;q,q_N)$-local retraction $\xi\colon U_0 \to \mathrm{Diff}(M;q,q_N)$ around e_0 using Lemma 1.5.7 and write

$$U_{e_0} = \{(e,t,L) \in \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B,M;q,q_N) \mid e \in U_0\}.$$

There is an injective map

$$j\colon U_{e_0} \longrightarrow \mathrm{Emb}(\overline{N}_M e_0, M; q_N)$$

given by $j(e,t,L) = (\xi(e), t \circ \xi(\overline{N}_M e(B)), \xi(\overline{N}_M e(C))(L))$ and we give U_{e_0} the subspace topology. The U_e's cover $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$, so they define a topology.

This discussion applies to the subspace $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}((B,A),(M,W); q, q_N)$ of tubular neighbourhoods of embeddings of a pair (B,A) in a pair (M,W), and this latter space has a natural action of $\mathrm{Diff}(M; W, q, q_N)$.

Proposition 1.5.13. *The space* $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ *is* $\mathrm{Diff}(M; q, q_N)$*-locally retractile.*

Proof. Let $(e_0, t_0, L_0) \in \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$. Let

$$\xi_{e_0} : U_{e_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

be a local retraction around e_0 in $\mathrm{Emb}(B, M; q)$. Let

$$\xi_{t_0} : U_{t_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

be a local retraction around t_0 in $\overline{\mathrm{Tub}}(e_0(B), M; (q_N, q))$. Let

$$\xi'_{L_0} : U_{L_0} \longrightarrow \mathrm{GL}_{\partial^0 C}(N_M e(C))$$

be a local retraction around L_0 in $\mathrm{Vect}_k(N_M e_0(C); N_{q(\partial^0 C)} e_0(\partial^0 C))$. There is a canonical embedding $\mathrm{GL}_{\partial^0 C}(N_M e_0(C)) \to \mathrm{Diff}(t_0(\overline{N}_M e(C)); q, q_N)$ induced by the diffeomorphism t_0, and using a bump function we may construct a non-canonical embedding $\mathrm{Diff}(t_0(\overline{N}_M e_0(C)); q, q_N) \to \mathrm{Diff}(M; q, q_N)$. We denote by

$$\xi_{L_0} : U_{L_0} \longrightarrow \mathrm{Diff}(M; q, q_N)$$

the composite of ξ'_{L_0} and these embeddings. Note that $\xi_{t_0}(t)(e_0) = e_0$, that $\xi_{L_0}(L)(e_0) = e_0$, and that $\xi_{L_0}(L)(t_0) = t_0$, by the definition of tubular neighbourhood and the definition of the action of $\mathrm{Diff}(M; q, q_N)$ on the space of tubular neighbourhoods. Let $U \subset \overline{\mathrm{T}}\mathrm{Emb}_{k,C}(B, M; q, q_N)$ be the intersection of the inverse images of U_{e_0}, U_{t_0} and U_{L_0} under the projection maps. For a point (e, t, L) in U, let us denote $t_1 = \xi_{e_0}(e)^{-1}(t)$ and $L_1 = \xi_{t_0}(t_1)^{-1} \circ \xi_{e_0}(e)^{-1}(L)$, and define a map

$$\xi \colon U \longrightarrow \mathrm{Diff}(M; q, q_N)$$

given by $\xi(e,t,L) = \xi_{e_0}(e) \circ \xi_{t_0}(t_1) \circ \xi_{L_0}(L_1)$. □

Using the proof of Lemma 1.5.11, and the previous lemma we obtain

Corollary 1.5.14. *The spaces* $\overline{\mathrm{T}}\mathrm{Emb}_{k,C}((B,A),(M,W); q, q_N)$ *are* $\mathrm{Diff}(M; q, q_N)$*-locally retractile.*

1.6 Homotopical resolutions

A *semi-simplicial space*, also called Δ-*space*, is a contravariant functor

$$X_\bullet \colon \Delta_{\mathrm{inj}}^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

from the category Δ_{inj} whose objects are non-empty finite ordinals and whose morphisms are injective order-preserving inclusions to the category **Top** of topological spaces. The image of the ordinal n is written X_n and we denote by $\partial_j \colon X_{n+1} \to X_n$ the image of the inclusion $n = \{0, 1, \dots, n-1\} \hookrightarrow \{0, 1, \dots, n\} = n+1$ that misses the element $j \in \{0, 1, \dots, n\}$. These are called *face maps* and the whole structure of $X_\bullet$ is determined by specifying the spaces X_n for each n together with the face maps in each level.

A *semi-simplicial space augmented over a topological space* X is a semi-simplicial space $X_\bullet$ together with a continuous map $\epsilon \colon X_0 \to X$ (the *augmentation* or 0-*augmentation*) such that $\epsilon\partial_0 = \epsilon\partial_1 \colon X_1 \to X$. Alternatively, an *augmented semi-simplicial space* is a contravariant functor

$$X_\bullet \colon \Delta_{\mathrm{inj},0}^{\mathrm{op}} \longrightarrow \mathbf{Top}$$

from the category $\Delta_{\mathrm{inj},0}$ whose objects are (possibly empty) ordinals and whose morphisms are injective order-preserving inclusions to the category **Top**. As above, we denote by $\partial_j \colon X_n \to X_{n+1}$ the image of the inclusion that misses j and we denote by ϵ the image of the unique inclusion $\emptyset \to 0$. We denote by $\epsilon_i \colon X_i \to X$ the unique composition of face maps and the augmentation map.

A *semi-simplicial map* between (augmented) semi-simplicial spaces is a natural transformation of functors. If $\epsilon_\bullet \colon X_\bullet \to X$ and $\epsilon'_\bullet \colon Y_\bullet \to Y$ are augmented semi-simplicial spaces, a semi-simplicial map $f_\bullet$ is equivalent to a sequence of maps $f_n \colon X_n \to Y_n$ such that $d_i \circ f_n = f_{n-1} \circ d_i$ for all i and all n, together with a map $f \colon X \to Y$ such that $\epsilon' \circ f_0 = f \circ \epsilon$.

There is a realization functor [Seg74]

$$|\cdot| \colon \text{Semi-simplicial spaces} \longrightarrow \mathbf{Top}$$

and we say that an augmented semi-simplicial space $X_\bullet$ over a space X is a *resolution* of X if the map induced by the augmentation

$$|\epsilon_\bullet| \colon |X_\bullet| \longrightarrow X$$

is a weak homotopy equivalence. It is an n-*resolution* if the map induced by the augmentation is n-connected (i.e., the relative homotopy groups $\pi_i(X, |X_\bullet|)$ vanish when $i \leq n$).

We will use the spectral sequences given by the skeletal filtration associated with augmented semi-simplicial spaces as they appear in [RW10]. For each augmented semi-simplicial space $\epsilon_\bullet\colon X_\bullet \to X$ there is a spectral sequence defined for $t \geq 0$ and $s \geq -1$:

$$E^1_{s,t} = H_t(X_s) \Longrightarrow H_{s+t+1}(X, |X_\bullet|),$$

and for each map between augmented semi-simplicial spaces $f_\bullet\colon X_\bullet \to Y_\bullet$ there is a spectral sequence defined for $t \geq 0$ and $s \geq -1$:

$$E^1_{s,t} = H_t(Y_s, X_s) \Longrightarrow H_{s+t+1}((|\epsilon^Y_\bullet|), (|\epsilon^X_\bullet|)),$$

where, for a continuous map $f\colon A \to B$, we denote by M_f the mapping cylinder of f and by (f) the pair (M_f, A).

The following criteria will be widely used throughout the paper.

Criterion 1.6.1 ([RW10, Lemma 2.1]). *Let $\epsilon_\bullet\colon X_\bullet \to X$ be an augmented semi-simplicial space. If each ϵ_i is a fibration and $\mathrm{Fib}_x(\epsilon_i)$ denotes its fibre at x, then the realization of the semi-simplicial space $\mathrm{Fib}_x(\epsilon_\bullet)$ is weakly homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$ at x.*

An *augmented topological flag complex* is an augmented semi-simplicial space $\epsilon_\bullet\colon X_\bullet \to X$ such that

(i) the product map $X_i \to X_0 \times_X \cdots \times_X X_0$ is an open embedding;

(ii) a tuple $(x_0, \ldots, x_i)$ is in X_i if and only if for each $0 \leq j < k \leq i$ the pair $(x_j, x_k) \in X_0 \times_X X_0$ is in X_1.

Criterion 1.6.2 ([GRW12, Theorem 6.2]). *Let $\epsilon_\bullet\colon X_\bullet \to X$ be an augmented topological flag complex. Suppose that*

(i) *$X_0 \to X$ has local sections, that is, ϵ is surjective and for each $x_0 \in X_0$ such that $\epsilon(x_0) = x$ there is a neighbourhood U of x and a map $s\colon U \to X_0$ such that $\epsilon(s(y)) = y$ and $s(x) = x_0$;*

(ii) *given any finite collection $\{x_1, \ldots, x_n\} \subset X_0$ in a single fibre of ϵ over some $x \in X$, there is an x_∞ in that fibre such that each (x_i, x_∞) is a 1-simplex.*

Then $|\epsilon_\bullet|\colon |X_\bullet| \to X$ is a weak homotopy equivalence.

Remark 1.6.3. For the second condition, we could also ask that there is an x_0 such that each (x_0, x_i) is a 1-simplex, and the conclusion still holds.

Resolutions of a single surface

In the proof of [RW10, Proposition 4.1] the following semi-simplicial space was introduced: If W is a compact connected oriented surface with non-empty boundary, and k_0, k_1 are embedded intervals in ∂W, the semi-simplicial space $O(W; k_0, k_1)_\bullet$ is defined as follows: An i-simplex is a tuple $(a_0, \dots, a_i)$ of pairwise disjoint embeddings of the interval $[0, 1]$ in W such that

(i) $a_j(0) \in k_0$ and $a_j(1) \in k_1$;

(ii) the complement of $a_0 \cup \ldots \cup a_i$ in W is connected;

(iii) the ordering at the endpoints of the arcs is $(a_0(0), \dots, a_i(0))$ in k_0 and $(a_i(1), \dots, a_0(1))$ in k_1, where k_0 and k_1 are ordered according to the orientation of δ.

The jth face map forgets the jth arc. In order to simplify the notation we will write $[i]$ for $\{0, \dots, i\}$. The set of i-simplices is topologized as a union of components of the space $\mathrm{Emb}(I \times [i], W; q)$ with $q(x) = k_j$ if $x \in \{j\} \times [i]$ and $q(x) = M$ otherwise.

Proposition 1.6.4 ([RW10, Proposition 4.1])**.** *The realization* $|O(W; k_0, k_1)_\bullet|$ *is* $(g-2)$*-connected, where* g *is the genus of* W*.*

This proposition is the hardest step in the proof of homological stability for diffeomorphism groups of surfaces, and the change in the definition of this complex of curves yielded various improvements [Iva93, Bol09, RW10] of the original complex of Harer [Har85]. A detailed proof of the above proposition may be found in [Wah12].

1.7 A criterion for homological stability

Criterion 1.7.1. *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a map of augmented semi-simplicial spaces such that* $|\epsilon_\bullet^X| : |X_\bullet| \to X$ *is* $(l-1)$*-connected and* $|\epsilon_\bullet^Y| : |Y_\bullet| \to Y$ *is* l*-connected. Suppose there is a sequence of path connected based spaces* (B_i, b_i) *and maps* $p_i \colon Y_i \to B_i$*, and form the map*

$$g_i \colon \mathrm{hofib}_{b_i}(p_i \circ f_i) \longrightarrow \mathrm{hofib}_{b_i}(p_i)$$

induced by composition with f_i*. Suppose that there is a* $k \leq l+1$ *such that*

$$H_q(g_i) = 0 \text{ when } q + i \leq k\text{, except if } (q, i) = (k, 0).$$

Then the map induced in homology by the composition of the inclusion of the fibre and the augmentation map

$$H_q(g_0) \longrightarrow H_q(f_0) \xrightarrow{\epsilon} H_q(f)$$

is an epimorphism in degrees $q \leq k$.

If in addition $H_k(g_0) = 0$*, then* $H_q(f) = 0$ *in degrees* $q \leq k$.

Given the data in this criterion, we will call the map of pairs $(g_i) \to (f_i)$ the *approximation over* b_i, and we will call the composition $(g_i) \to (f_i) \to (f)$ the *approximate augmentation over* b_i.

Proof. We have a homotopy fibre sequence of pairs $(g_i) \to (f_i) \to B_i$, and so a relative Serre spectral sequence

$$\widetilde{E}^2_{p,q} = H_p(B_i; \mathcal{H}_q(g_i)) \Longrightarrow H_{p+q}(f_i),$$

where $\mathcal{H}_q(g_i)$ denotes homology with twisted coefficients. Since $H_q(g_i) = 0$ for all $q \leq k - i$ except $(q, i) = (k, 0)$, we have that $H_q(f_i) = 0$ for all $q + i \leq k$ except $(q, i) = (k, 0)$. Moreover, if $i = 0$, all differentials with target or source $H_0(B; \mathcal{H}_q(g_0))$ for $q \leq k$ are trivial, and these are the only possibly non-trivial groups with total degree $p + q \leq k$. Hence

$$H_q(g_0) \longrightarrow H_0(B; \mathcal{H}_q(g_0)) \longrightarrow H_q(f_0)$$

is the composition of two epimorphisms if $q \leq k$.

The first page of the spectral sequence for the resolution $(f_\bullet) \to (f)$ is

$$E^1_{p,q} = H_q(f_p), \quad p \geq -1,$$

and it converges to zero in total degrees $p + q \leq l$. Since $H_q(f_p) = 0$ for all $p + q \leq k$ except $(p, q) = (0, k)$, any differential with target $E^r_{-1,q}$ for $q \leq k$ and $r \geq 2$,

$$d_r : E^r_{r-1,q-r+1} \longrightarrow E^r_{-1,q},$$

has source a quotient of $H_{q-r+1}(f_{r-1})$, which is trivial. As $k - 1 \leq l$, and the spectral sequence converges to zero in total degrees $p + q \leq l$, we have that for each $q \leq k$ there is an $r \geq 1$ such that $E^r_{-1,q} = 0$, hence the homomorphisms induced by the augmentation map $d_1 \colon H_q(f_0) \to H_q(f)$ are epimorphisms in degrees $q \leq k$.

For the second part, note that in that case all epimorphisms $H_q(g_0) \to H_q(f_0) \to H_q(f)$ have trivial source when $q \leq k$, hence the target is also trivial is those degrees. ☐

There is one final concept that we will use rather often to describe the homotopy type of the fibre of a fibration. Recall that each map $f\colon B \to C$ admits a factorization $B \to B' \to C$, where the first map is a homotopy equivalence and the second map is a fibration. Here B' is the space of pairs (b, γ), where $b \in B$ and γ is a path in C with $\gamma(0) = f(b)$. The map $B' \to C$ sends a pair (b, γ) to $\gamma(1)$. The fibre of this map over $c \in C$ is called the *homotopy fibre* of f and is denoted by $\mathrm{hofib}_c(f)$.

We say that a pair of maps

$$A \xrightarrow{g} B \xrightarrow{f} C$$

is a *homotopy fibre sequence* if $f \circ g$ is homotopic to the constant map to a point $c \in C$, and the induced map $A \to \mathrm{hofib}_c(f)$ is a weak homotopy equivalence. For our purposes, such data can be treated as if f were a fibration and f were the inclusion of the fibre over c.

Chapter 2

Homology Stability

This chapter contains the proof of Theorem B. In the first section we define two kinds of gluing maps between spaces of surfaces. In the second section we construct resolutions of the stabilisation maps. The third section contains the proof of the first two assertions of the theorem, except for a claim that is proved in Sections 2.4 and 2.5. The proof is by induction and uses the resolutions constructed in Section 2.2 via Criterion 1.7.1. The last section proves the last assertion of the theorem.

2.1 Stabilisation maps

For a collared manifold M, we will use two kinds of maps between spaces of surfaces of the form $\mathcal{E}^+_{g,b}(M;\delta)$. The first map glues a collar $\partial^0 M \times I$ to M and a surface $P \subset \partial^0 M \times I$ to the surfaces in $\mathcal{E}^+_{g,b}(M;\delta)$. For the second map, we remove a submanifold $u' \subset M$ from M. If a surface $u'' \subset u'$ is given, we may glue u'' to the surfaces in $M \setminus u'$, obtaining a map from $\mathcal{E}^+_{h,c}(M \setminus u';\delta')$ to $\mathcal{E}_{g,b}(M;\delta)$, where h, c, δ' depend on the surfaces u''. In the following paragraphs these constructions are explained in detail.

The manifold M_1 is defined as the union of the manifold $\partial^0 M \times [0,1]$ and the manifold M along $\partial^0 M \times \{0\}$ using the collar of M. The collar of M gives a canonical collar both to $\partial^0 M_1 := \partial^0 M \times \{1\}$ and to $\partial^0(\partial^0 M \times I) := \partial^0 M \times \{0,1\}$. The boundary condition δ also gives boundary conditions

$$\tilde{\delta} = T^\infty_\partial(\delta^0 \times I) \in \Delta_2(\partial^0 M \times I)$$
$$\overline{\delta} = T^\infty_\partial(W \cup (\delta^0 \times I)) \in \Delta_2(M_1)$$

where W is any surface in $\mathcal{E}^+_{g,b}(M;\delta)$. Let Σ' be another collared surface with

$\partial\Sigma' = \partial(\partial^0\Sigma \times I)$. For each $P \in \mathcal{E}(\Sigma', \partial^0 M \times [0,1]; \tilde{\delta})$, there is a continuous map

$$- \cup P\colon \mathcal{E}(\Sigma, M; \delta) \longrightarrow \mathcal{E}(\Sigma \cup \Sigma', M_1; \bar{\delta})$$

that sends a submanifold W to the union $W \cup P$. These are *maps of type* I.

If Σ is a compact connected oriented surface of genus g and b boundary components, then, in some cases, we will denote the map $- \cup P$ by

$$\begin{aligned}
\alpha_{g,b}(M;\delta,\bar{\delta})\colon \mathcal{E}_{g,b}(M;\delta) &\longrightarrow \mathcal{E}_{g+1,b-1}(M;\bar{\delta})\\
\beta_{g,b}(M;\delta,\bar{\delta})\colon \mathcal{E}_{g,b}(M;\delta) &\longrightarrow \mathcal{E}_{g,b+1}(M;\bar{\delta})\\
\gamma_{g,b}(M;\delta,\bar{\delta})\colon \mathcal{E}_{g,b}(M;\delta) &\longrightarrow \mathcal{E}_{g,b-1}(M;\bar{\delta})
\end{aligned}$$

depending on the genus and the number of boundary components of the surfaces in the target. Note that, if Σ has no corners, then P will be a disjoint union of connected surfaces, one of them a pair of pants or a disc, and the rest diffeomorphic to cylinders. If Σ has corners, then P will be a disjoint union of strips, one of them meeting δ in two intervals, and the rest meeting δ in a single interval. Often we will write $\alpha_{g,b}(M)$, $\beta_{g,b}(M)$ and $\gamma_{g,b}(M)$ when the boundary condition is clear from the context or when we are talking about arbitrary boundary conditions.

Now we define the second type of gluing map. Let $s \subset \Sigma'$ be either empty or a closed tubular neighbourhood of an arc or a point in Σ. The complement $\Sigma \setminus s$ is again a collared manifold, but if s is an arc and M is a manifold with boundary, its complement is no longer a manifold with boundary. This justifies working with manifolds with corners.

Consider a tuple $u = (u', u'', u''')$ consisting on a neat embedding $u''' \in \mathrm{Emb}(B, M; q)$, a closed tubular neighbourhood u' of u''' in M, and a (possibly empty) surface $u'' \in \mathcal{E}(s, u'; \delta[u])$ such that $\delta[u] \cap \delta = \delta[u]^0$. Then $\mathrm{cl}\,(M \setminus u')$ is a collared manifold with $\partial^0\,\mathrm{cl}\,(M \setminus u') = \mathrm{cl}\,(\partial^0 M \setminus \partial^0 u')$. The boundary conditions δ and $\delta[u]$ give rise to a boundary condition

$$\delta(u) = T^\infty_\partial(\mathrm{cl}\,(W \setminus u'')) \in \Delta_2(\mathrm{cl}\, M \setminus u')$$

where $W \in \mathcal{E}_{g,b}(M;\delta)$ is any surface that contains u''.

The triple $u = (u', u'', u''')$ defines a map

$$\mathcal{E}(\mathrm{cl}\,\Sigma \setminus s, \mathrm{cl}\, M \setminus u'; \delta(u)) \longrightarrow \mathcal{E}(\Sigma, M; \delta)$$

that sends a submanifold W to the union $W \cup u''$. These are *maps of type II*.

Notation 2.1.1. First, since the map defined above is completely determined by the tuple u, we will use the notation $M(u)$ for $\mathrm{cl}\, M \setminus u'$. Second, for maps of type I, we denote with a tilde ˜ the objects that we glue to the space of surfaces and with a dash ¯ the objects obtained by removing or gluing surfaces to $\mathcal{E}_{g,b}(M;\delta)$. For maps of type II, we denote with brackets [ ] the objects that we remove from the space of surfaces and with parentheses ( ) the result of removing those objects. In addition, we denote with ‴ the submanifold, with ′ the tubular neighbourhood and with ″ the surface in the tubular neighbourhood. We will be consistent with these notations. Third, for maps of type I, the triple u defines triples $\tilde{u}$ and $\overline{u}$ in the manifolds $\partial^0 M\times I$ and M_1 given by $\tilde{u}=\partial^0 u\times I$ and $\overline{u}=u\cup\tilde{u}$. If we assume in addition that $P\cap\tilde{u}'=\tilde{u}''$ and that $(\partial^0 s)\times I\subset\Sigma'$, then in the diagram

$$\begin{tikzcd}
\mathcal{E}(\Sigma(s),M(u);\delta(u)) \arrow[r, dashed] \arrow[d] & \mathcal{E}((\Sigma\cup\Sigma')(\overline{s}),M_1(\overline{u});\overline{\delta}(\overline{u})) \arrow[d] \\
\mathcal{E}(\Sigma,M;\delta) \arrow[r] & \mathcal{E}(\Sigma\cup\Sigma',M_1;\overline{\delta})
\end{tikzcd}$$

we may construct the upper horizontal arrow as $-\cup P\setminus\tilde{u}''$. As before, we will use the notation $P(\tilde{u})=P\setminus\tilde{u}''$. We will apply these constructions three times:

In Section 2.2, u''' will be an embedding u''' of an interval with $\partial^0 I=\{0,1\}$, u'' will be a strip, $\delta^0[u]$ a pair of intervals and $\tilde{u}''\subset P$ a pair of strips.

In Section 2.4, we will use the letter ν instead of u, and ν''' will be an embedding of a half disc $D_+=D^2\cap\mathbb{R}\times\mathbb{R}_+$ with $\partial^0 D_+=D^2\cap\mathbb{R}\times\{0\}$. The surface ν'' will be empty, therefore $\delta^0[u]$ and $\tilde{\nu}''$ will also be empty.

In Section 2.6, we will use the letter p instead of u, and p''' will be an embedding of an interval with $\partial^0 I=\{0\}$. The surface p'' will be a disc with $\delta^0[u]$ empty, hence $\tilde{p}''$ will be empty too.

2.2 Resolutions of pairs of pants and their fibrations

In this section we construct two $(g-1)$-resolutions of the space $\mathcal{E}_{g,b}(M;\delta)$, where M is a collared manifold with non-empty boundary, and $\delta\in\Delta_2(M)$ is a non-empty boundary condition (in particular $b\geq 1$). We will also characterize the space of i-simplices of each resolution as the total space of a certain homotopy fibration. Afterwards we will explain how these $(g-1)$-resolutions give rise to a $(g-1)$-resolution or a g-resolution of the stabilisation maps (the connectivity of each resolution depends on the stabilisation map), and how to characterize their spaces of i-simplices.

Resolutions of spaces of surfaces

Let $\ell_0, \ell_1 \subset \partial^0 M$ be a pair of disjoint open balls that intersect δ^0 in two intervals and write $\ell = (\ell_0, \ell_1)$. There is a semi-simplicial space $\mathcal{O}_{g,b}(M;\delta,\ell)_\bullet$ (for which we write $\mathcal{O}_{g,b}(M;\delta)_\bullet$ for brevity) whose i-simplices are tuples $(W, u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$, where

(i) $W \in \mathcal{E}_{g,b}(M;\delta)$;

(ii) $(u_0''', \dots, u_i''') \in O(W, \ell_0 \cap \delta, \ell_1 \cap \delta)_i$;

(iii) (u_j', u_j'') is a closed tubular neighbourhood of u_j''' in the pair (M, W), such that $u_j'(0,-) \in \ell_0$ and $u_j'(1,-) \in \ell_1$;

(iv) $u_0', \dots, u_i'$ are pairwise disjoint.

The jth face map forgets u_j, that is, it sends an i-simplex $(W, u_0, \dots, u_i)$ to the $(i-1)$-simplex $(W, u_0, \dots, \hat{u}_j, \dots, u_i)$. There is an augmentation map $\epsilon_\bullet$ to the space $\mathcal{E}_{g,b}(M;\delta)$ that forgets everything but W. This defines a semi-simplicial set, and we topologise the set of i-simplices as a subspace of

$$\mathcal{E}_{g,b}(M;\delta) \times \overline{\mathrm{T}}\mathrm{Emb}(I \times [i], M; q, q_N),$$

where q and q_N stand for the boundary conditions

$$\begin{aligned} &q(\{0\} \times [i]) = \ell_0 \cap \delta, \quad &&q(\{1\} \times [i]) = \ell_1 \cap \delta, \quad &&q(x) = M \text{ otherwise},\\ &q_N(\{0\} \times [i]) = \ell_0, &&q_N(\{1\} \times [i]) = \ell_1, &&q(x) = M \text{ otherwise}. \end{aligned}$$

If we want to stress that ℓ_0 and ℓ_1 intersect the same component of δ, we denote the semi-simplicial space by $\mathcal{O}^1_{g,b}(M;\delta,\ell)_\bullet$. If we want to stress that ℓ_0 and ℓ_1 intersect the different components of δ, we denote the semi-simplicial space by $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$.

Proposition 2.2.1. *$\mathcal{O}_{g,b}(M;\delta,\ell)_\bullet$ is a $(g-1)$-resolution of $\mathcal{E}_{g,b}(M;\delta)$.*

Proof. In order to find the connectivity of the homotopy fibre of $\epsilon_\bullet$, we use Criterion 1.6.1 to assure that the semi-simplicial fibre $\mathrm{Fib}_W(\epsilon_\bullet)$ of $\epsilon_\bullet$ over a surface W is homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$: the space $\mathcal{E}^+_{g,b}(M;\delta)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Corollary 1.5.10, and, as the group $\mathrm{Diff}(M;\delta,\ell)$ also acts on this space, any local retraction for $\mathrm{Diff}_\partial(M)$ gives also a local retraction for $\mathrm{Diff}(M;\delta,\ell)$. In addition, the augmentation maps ϵ_i are $\mathrm{Diff}(M;\delta,\ell)$-equivariant for all i. Therefore, by Lemma 1.5.3, they are locally trivial fibrations.

The i-simplices of $\mathrm{Fib}_W(\epsilon_\bullet)$ are tuples $(u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$ where u_j''' are embeddings of an interval in W and (u_j', u_j'') are pairwise disjoint closed tubular neighbourhoods of u_j''' in the pair (W, M). Forgetting the closed tubular neighbourhoods gives a levelwise $\mathrm{Diff}(M; W, \delta, \ell)$-equivariant semi-simplicial map

$$r_\bullet \colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow O(W; \ell_0 \cap \delta, \ell_1 \cap \delta)_\bullet,$$

and the space of i-simplices of $O(W; \ell_0 \cap \delta, \ell_1 \cap \delta)_\bullet$ is $\mathrm{Diff}(W)$-locally retractile, and also $\mathrm{Diff}(M; W, \delta, \ell)$-locally retractile by Corollary 1.5.12. Therefore, by Lemma 1.5.3, $r_\bullet$ is a levelwise locally trivial fibration. The fibre of $r_\bullet$ over an i-simplex is a space of closed tubular neighbourhoods of arcs in the pair (M, W), which is contractible by Lemma 1.4.1, so $r_\bullet$ is a homotopy equivalence. As the space on the right is $(g-2)$-connected by Proposition 1.6.4, the result follows. □

Define $A_i(M; \delta, \ell)$ to be the set of tuples $(u_0, \dots, u_i)$ with $u_j = (u_j', u_j'', u_j''')$ and

(i) the $u_j''' \colon I \to M$ are pairwise disjoint embeddings with $u_j'''(0) \in \ell_0 \cap \delta$ and $u_j'''(1) \in \ell_1 \cap \delta$, and $u_j'''(0) > u_k'''(0)$, $u_j'''(1) < u_k'''(1)$ if $j > k$;

(ii) u_j' is a closed tubular neighbourhood of u_j''' disjoint from u_k' if $j \neq k$ whose restriction to $u_j'''(\{0,1\})$ is a closed tubular neighbourhood in the pair $(\partial^0 M, \delta \cap \ell)$;

(iii) u_j'' is the restriction of u_j' to some subbundle $L_j \subset N_M u_j'''$ such that $L_{j|\partial u_j'''} = N_{\delta\cap\ell}(\partial u_j''')$, i.e., $L_j \in \mathrm{Gr}_1^+(N_M u_j'''; N_{\delta\cap\ell}(\partial u_j'''))$.

This space is in canonical bijection with a union of components of the space $\overline{\mathrm{T}}\mathrm{Emb}_{1, I\times[i]}(I \times [i], M; q, q_N)$. The bijection sends a triple (u', u'', u''') to the triple (u''', u', L_j), and we use it to give a topology to $A_i(M; \delta, \ell)$.

There are restriction maps

$$\mathcal{O}_{g,b}(M; \delta, \ell)_i \longrightarrow A_i(M; \delta, \ell)$$

that send $(W, u_0, \dots, u_i)$ to $(u_0, \dots, u_i)$.

Proposition 2.2.2. *The restriction maps*

$$\mathcal{O}_{g,b}(M; \delta, \ell)_i \longrightarrow A_i(M; \delta, \ell)$$

are fibrations, and, using the notation of Section 2.1*, their fibres over a point* $\mathbf{u} = (u_0, \dots, u_i)$ *in* $A_i(M; \delta, \ell)$ *are given by*

$$\begin{aligned} \mathcal{E}_{g-i-1, b+i+1}(M(\mathbf{u}); \delta(\mathbf{u})) &\longrightarrow \mathcal{O}^1_{g,b}(M; \delta, \ell)_i \longrightarrow A_i(M; \delta, \ell) \\ \mathcal{E}_{g-i, b+i-1}(M(\mathbf{u}); \delta(\mathbf{u})) &\longrightarrow \mathcal{O}^2_{g,b}(M; \delta, \ell)_i \longrightarrow A_i(M; \delta, \ell), \end{aligned}$$

depending on how many components of δ *intersect* ℓ_0 *and* ℓ_1*.*

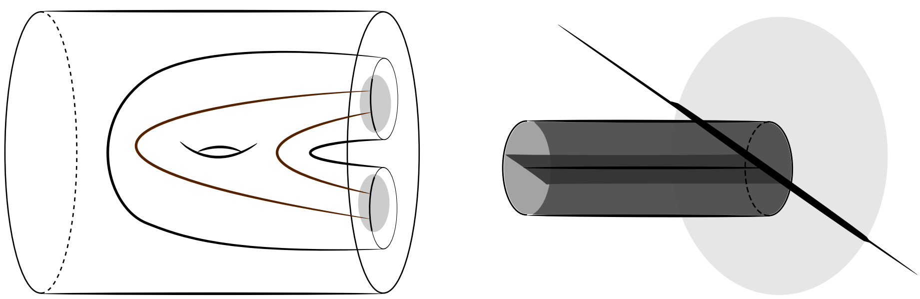
Proof. The restriction maps are $\mathrm{Diff}(M;\delta,\ell)$-equivariant and, by Lemma 1.5.13, the space $A_i(M;\delta,\ell)$ is $\mathrm{Diff}(M;\delta,\ell)$-locally retractile, hence the restriction maps are locally trivial fibrations by Lemma 1.5.3.

The fibre over a point $\mathbf{u}$ is the space of surfaces W in M that contain the strips $(u_0'',\dots,u_i'')$ and such that $W\backslash(u_0'',\dots,u_i'')$ lies outside $u_0'\cup\dots\cup u_i'$. If we take a parametrization $f\colon \Sigma\to W$ of any surface and write $\mathbf{a}=(a_0,\dots,a_i)=(f^{-1}\circ u_0'',\dots,f^{-1}\circ u_i'')$, then this space is canonically homeomorphic to the space $\mathcal{E}(\Sigma(\mathbf{a}),M(\mathbf{u});\delta(\mathbf{u}))$, so we just need to classify $\Sigma(\mathbf{a})$.

Removing a strip from Σ is the same as removing a 1-cell, up to homotopy equivalence, hence $\chi(\Sigma(\mathbf{a}))=\chi(\Sigma)+i+1$. Now, let us say that a strip a_j'' in Σ is of type I if both components of $\partial a_j''$ are contained in a single component of $\partial\Sigma((a_0'',\dots,a_{j-1}''))$, and that it is of type II otherwise.

- If a_j'' is of type I, then $\partial\Sigma(a_0'',\dots,a_j'')$ has one more boundary component than δ and, as a consequence of the last condition of the definition of $O(\Sigma)_\bullet$, the strip a_{j+1} is again of type I.
- If a_j'' is of type II, then $\partial\Sigma(a_0'',\dots,a_j'')$ has one less boundary component than δ and, as a consequence of the last condition of the definition of $O(\Sigma)_\bullet$, the strip a_{j+1}'' is of type I.

Hence, the only strip of type II that may occur in the construction of $\partial\Sigma(\mathbf{a})$ is the one given by a_0 in $O^2(\Sigma)_\bullet$. Hence $\partial\Sigma(\mathbf{a})$ has $b+i+1$ components if $\mathbf{a}\in\overline{O}^1(\Sigma)_\bullet$ and $b+i-1$ components if $\mathbf{a}\in\overline{O}^2(\Sigma)_\bullet$. Finally, we obtain the genus of $\Sigma(\mathbf{a})$ from the formula $g=\frac{1}{2}(2-\chi-b)$. $\square$



(a) (b)

Figure 2.1: (a) A 2-simplex in the boundary resolution. The grey shadows are the balls ℓ_0 and ℓ_1. (b) Detail of one of the closed tubular neighbourhoods u_j' with its strip u_j'' near the boundary.

Stabilisation maps between resolutions

In this subsection we show how to extend the stabilisation maps defined in Subsection 2.1 to maps between the resolutions we have constructed.

$$\begin{tikzcd} \mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d, "\epsilon_i"] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d, "\epsilon_i"] & \mathcal{O}^1_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d, "\epsilon_i"] & \mathcal{O}^2_{g,b+1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d, "\epsilon_i"] \\ \mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})"] & \mathcal{E}^+_{g+1,b-1}(M_1;\overline{\delta}) & \mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\beta_{g,b}(M;\delta,\overline{\delta})"] & \mathcal{E}^+_{g,b+1}(M_1;\overline{\delta}). \end{tikzcd} \tag{2.2.1}$$

In Section 2.1, we defined the map $\alpha_{g,b}(M;\delta,\overline{\delta})$ by gluing a cobordism $P \subset \partial^0 M \times I$ to each surface in $\mathcal{E}_{g,b}(M;\delta)$. As we did there, in the following constructions we will assume, without loss of generality, that

(i) $\overline{\ell}_0 = \ell_0 \times \{1\}$ and $\overline{\ell}_1 = \ell_1 \times \{1\}$;

(ii) $P \cap (\ell \times I) = (\ell \cap \delta) \times I$, in particular $\overline{\ell} \cap \overline{\delta} = (\ell \cap \delta) \times \{1\}$.

These assumptions make the extension of the stabilisation map canonical: Let us define $\tilde{u}_j = \partial u_j \times I$. Then, joining the closed tubular neighbourhoods, strips and arcs in $(u_0, \dots, u_i)$ that live in $A_i(M;\delta,\ell)$ to the products $(\tilde{u}_0, \dots, \tilde{u}_i)$ that are subsets of $\partial^0 M \times I$, we obtain new triples $(\overline{u}_0, \dots, \overline{u}_i)$ that live in $A_i(M_1;\overline{\delta},\overline{\ell})$, defined as $\overline{u}_j = u \cup \tilde{u}_j$. This yields the dashed maps $\alpha_{g,b}(M;\delta,\overline{\delta})_i$ in the first diagram. These maps commute with the face maps and with the augmentation maps, so they define a map of semi-simplicial spaces

$$\alpha_{g,b}(M;\delta,\overline{\delta})_\bullet \colon \mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet \longrightarrow \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_\bullet$$

which is augmented over $(\alpha_{g,b}(M;\delta,\overline{\delta}))$. Analogously, we can define a map

$$\beta_{g,b}(M;\delta,\overline{\delta})_\bullet \colon \mathcal{O}^1_{g,b}(M;\delta,\ell)_\bullet \longrightarrow \mathcal{O}^2_{g,b+1}(M_1;\overline{\delta},\overline{\ell})_\bullet$$

which is augmented over $(\beta_{g,b}(M;\delta,\overline{\delta}))$.

Corollary 2.2.3 (To Proposition 2.2.1). *The semi-simplicial pair $(\alpha_{g,b}(M;\delta,\overline{\delta})_\bullet)$ together with the natural augmentation map to $(\alpha_{g,b}(M;\delta,\overline{\delta}))$ is a g-resolution. The semi-simplicial pair $(\beta_{g,b}(M;\delta,\overline{\delta})_\bullet)$ together with the natural augmentation map to $(\beta_{g,b}(M;\delta,\overline{\delta}))$ is a $(g-1)$-resolution.*

There is a commutative square

$$\begin{tikzcd} \mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[d] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta,},\overline{\ell})_i \arrow[d] \\ A_i(M;\delta,\ell) \arrow[r, "u \mapsto \overline{u}"] & A_i(M_1;\overline{\delta},\overline{\ell}) \end{tikzcd} \tag{2.2.2}$$

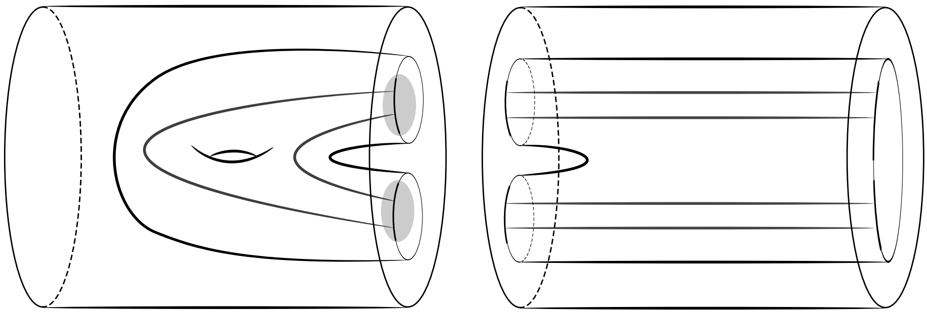


Figure 2.2: The map $\alpha_{1,2}(M)$ acting on a 2-simplex in the boundary resolution.

where the lower map is a homotopy equivalence. Hence we obtain a map between the fibres over the points $\mathbf{u}$ and $\overline{\mathbf{u}}$ of the fibrations of Proposition 2.2.2,

$$\mathcal{E}^{+}_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u})) \longrightarrow \mathcal{E}^{+}_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})). \tag{2.2.3}$$

More concretely, this is a map of type I given by the cobordism $P \setminus \tilde{\mathbf{u}}'' \subset \partial^0 M(\mathbf{u}') \times I$, which is denoted $\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$, as can be seen from the difference between the genus of the surfaces in the source and target spaces.

As the map $A_i(M;\delta,\ell) \to A_i(M_1;\overline{\delta},\overline{\ell})$ is a homotopy equivalence, the space $\mathcal{E}^{+}_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))$ is homotopy equivalent to the homotopy fibre of the composition of the augmentation map of $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ with this map. Moreover, we have shown that the map between the fibres of the locally trivial fibrations of diagram (2.2.2) is a stabilisation map $\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$.

As a consequence, we have a diagram

$$\begin{tikzcd}
\mathcal{E}^{+}_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u})) \arrow[rr, "\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))"] \arrow[d, "\simeq"] & & \mathcal{E}^{+}_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})) \arrow[d, "\simeq"] \\
\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho) \arrow[rr] \arrow[d] & & \mathrm{hofib}_{\overline{\mathbf{u}}}(\rho') \arrow[d] \\
\mathcal{O}^2_{g,b}(M;\delta,\ell)_i \arrow[rr, "\alpha_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[dr, "\rho"] & & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[dl, "\rho'"'] \\
& A_i(M_1;\overline{\delta},\overline{\ell}). &
\end{tikzcd}$$

This gives that the pair $(\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho'),\mathrm{hofib}_{\overline{\mathbf{u}}}(\rho))$ is homotopy equivalent to the pair $(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}})))$.

Following the same procedure with the map $\beta_{g,b}(M;\delta,\overline{\delta})$, we obtain that the pair given by the map from the homotopy fibre of

$$\mathcal{O}^1_{g,b}(M;\delta,\ell)_i \longrightarrow A_i(M_1;\overline{\delta},\overline{\ell})$$

to the fibre of the composition

$$\mathcal{O}^1_{g,b}(M;\delta)_i \longrightarrow A_i(M_1;\overline{\delta},\overline{\ell}) \longrightarrow A_i(M_1;\overline{\delta},\overline{\ell})$$

is homotopy equivalent to the pair given by

$$\mathcal{E}^+_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u})) \longrightarrow \mathcal{E}^+_{g-i,b+i}(M_1(\overline{\mathbf{u}});\overline{\delta}(\overline{\mathbf{u}})),$$

which is a map of type $\alpha_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u}),\overline{\delta}(\overline{\mathbf{u}}))$.

Corollary 2.2.4 (To Proposition 2.2.2)**.** *There are homotopy fibre sequences*

$$\begin{array}{ccccc}(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) & \longrightarrow & (\alpha_{g,b}(M;\delta)_i) & \longrightarrow & A_i(M_1;\overline{\delta},\overline{\ell}),\\ (\alpha_{g-i-1,b+i+1}(M(\mathbf{u});\delta(\mathbf{u}))) & \longrightarrow & (\beta_{g,b}(M;\delta)_i) & \longrightarrow & A_i(M_1;\overline{\delta},\overline{\ell}),\end{array}$$

that is, the homotopy fibre over $\overline{\mathbf{u}}$ is homotopy equivalent to the pair shown.

2.3 Proof of homology stability for α and β maps

In this section we prove the first two assertions of Theorem B, leaving some details until Sections 2.4 and 2.5. The proof of the last assertion will be deferred to Section 2.6.

Proposition 2.3.1. *Let M be a simply connected manifold of dimension at least 5. If the dimension of M is 5, we assume in addition that the pairs of pants defining the stabilisation maps are contractible in $\partial^0 M \times [0,1]$. Then*

(i) $H_k(\alpha_{g,b}(M)) = 0$ *for* $k \leq \frac{1}{3}(2g+1)$*;*

(ii) $H_k(\beta_{g,b}(M)) = 0$ *for* $k \leq \frac{2}{3}g$.

This proposition will be proven by induction: Lemma 2.3.5 gives the starting step and Lemma 2.3.3 gives the inductive step.

Remark 2.3.2. The proof of Proposition 2.3.1 follows the proof by induction of Theorem 7.1 in [RW10]. Following the language in that paper, stabilisation on π_0 is covered by Lemma 2.3.5 and 1-triviality will be the subject of Sections 2.4 and 2.5.

One important difference with [RW10] is that the fibrations in Proposition 2.2.2 have as fiber a space of surfaces in $M(\mathbf{u})$ —the complement of i arcs in

M—, instead of M. In [RW10, Propositions 4.2 and 4.4] the fiber of the corresponding fibrations are moduli spaces of surfaces with the *same* tangential structure as the moduli space for which the $(g-1)$-resolution was constructed. This problem is solved in Section 2.4, where an additional resolution is introduced. This is the main technical departure from [RW10].

The inductive step

We define the following statements, which we will prove by simultaneous induction: Firstly, for the stabilisation maps,

F_g : $H_k(\alpha_{h,b}(M)) = 0$ for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{1}{3}(2h+1)$;

G_g : $H_k(\beta_{h,b}(M)) = 0$ for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{2}{3}h$.

Second, for the approximated augmentations for the g-resolution $\alpha_{g,b}(M)_\bullet$ and the $(g-1)$-resolution $\beta_{g,b}(M)_\bullet$,

X_g : $H_k(\beta_{h,b-1}(M(u))) \to H_k(\alpha_{h,b}(M))$ is an epimorphism for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{1}{3}(2h+1)$;

A_g : $H_k(\beta_{h,b-1}(M(u))) \to H_k(\alpha_{h,b}(M))$ is zero for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{1}{3}(2h+2)$;

Y_g : $H_k(\alpha_{h-1,b+1}(M(u))) \to H_k(\beta_{h,b}(M))$ is an epimorphism for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{2}{3}h$;

B_g : $H_k(\alpha_{h-1,b+1}(M(u))) \to H_k(\beta_{h,b}(M))$ is zero for all simply connected manifolds M of dimension at least 5 with non-empty boundary, for $h \leq g$ and $k \leq \frac{1}{3}(2h+1)$.

Lemma 2.3.3. *If* M *satisfies the hypotheses of Proposition* 2.3.1, *then*

$$\text{(i) } X_g; A_g \Rightarrow F_g; \quad \text{(iii) } G_g \Rightarrow X_g; \quad \text{(v) } G_g; X_{g-1} \Rightarrow A_g;$$
$$\text{(ii) } Y_g; B_g \Rightarrow G_g; \quad \text{(iv) } F_{g-1} \Rightarrow Y_g; \quad \text{(vi) } F_{g-1}; Y_{g-1} \Rightarrow B_g.$$

Proof. (i) The morphism induced in homology by the approximate augmentation

$$H_k(\beta_{g,b-1}(M(u))) \longrightarrow H_k(\alpha_{g,b}(M))$$

is both zero and an epimorphism in all degrees $k \leq \frac{1}{3}(2g+1)$ (since X_g and A_g hold), so $H_k(\alpha_{g,b}(M)) = 0$ in these degrees. Similarly for (ii).

(iii) Consider the g-resolution $\alpha_{g,b}(M;\delta)_\bullet$ of $\alpha_{g,b}(M;\delta)$ given by Corollary 2.2.3, together with the homotopy fibre sequences

$$(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) \longrightarrow (\alpha_{g,b}(M;\delta)_i) \longrightarrow A_i(M_1;\overline{\delta})$$

of Corollary 2.2.4. For all $i \geq 1$ we have the inequality $\frac{1}{3}(2g+1)-i \leq \frac{2}{3}(g-i)$, and so, as $M(\mathbf{u})$ is simply connected, by inductive assumption

$$H_q(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$$

for $q \leq \frac{1}{3}(2g+1)-i$. When $i=0$, we have the inequality $\frac{1}{3}(2g+1)-1 \leq \frac{2}{3}g$ so $H_q(\beta_{g,b-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$ for $q \leq \frac{1}{3}(2g+1)-1$. In total we deduce that we have $H_q(\beta_{g-i,b+i-1}(M(\mathbf{u});\delta(\mathbf{u}))) = 0$ for $q \leq \frac{1}{3}(2g+1)-i$ except $(q,i) = (\frac{1}{3}(2g+1),0)$. As $\lfloor \frac{1}{3}(2g+1) \rfloor \leq g+1$, and $A_i(M_1,\overline{\delta})$ is path connected, Criterion 1.7.1 shows that the approximate augmentations are epimorphisms for $k \leq \frac{1}{3}(2g+1)$. Similarly for (iv).

The implications (v) and (vi) will be proven in Sections 2.4 and 2.5. $\square$

Stability on connected components

In this section we prove that the assertions X_g, Y_g, A_g and B_g hold for $g=0$. This, together with Lemma 2.3.3, will finish the proof of Proposition 2.3.1. For that, we first construct a (non-canonical) bijection between $\pi_0(\mathcal{E}^+_{g,b}(M;\delta))$ and the second homology group of the manifold M.

Suppose that M is a simply connected manifold of dimension $d \geq 5$, and let us describe an action of the abelian group $H_2(M;\mathbb{Z})$ on the set $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$ of isotopy classes of surfaces of genus g in M with boundary condition δ. Let $\hat{e} : \Sigma_{g,b} \hookrightarrow M$ be an embedding with boundary condition δ, representing an element $e \in \pi_0\mathcal{E}^+_{g,b}(M;\delta)$. Let $x \in \pi_2(M) \cong H_2(M;\mathbb{Z})$ be a homotopy class of maps from S^2 to M.

As M has dimension at least 5, x may be represented by an embedding $\hat{x} : S^2 \hookrightarrow M$ disjoint from the image of $\hat{e}$, and we can then choose an embedded path from the image of $\hat{e}$ to the image of $\hat{x}$. Forming the ambient connected sum along this path we obtain a new embedding $\hat{x}\cdot\hat{e} : \Sigma_{g,b} \hookrightarrow M$.

Lemma 2.3.4. *The map*

$$\begin{aligned} H_2(M;\mathbb{Z}) \times \pi_0\mathcal{E}^+_{g,b}(M;\delta) &\longrightarrow \pi_0\mathcal{E}^+_{g,b}(M;\delta) \\ (x,e) &\longmapsto [\hat{x}\cdot\hat{e}] \end{aligned}$$

is well defined and gives a free and transitive action of $H_2(M;\mathbb{Z})$ *on* $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$.

If $\partial\colon H_2(M,\delta;\mathbb{Z}) \to H_1(\delta;\mathbb{Z})$ *denotes the boundary homomorphism and* $[\delta] \in H_1(\delta;\mathbb{Z})$ *denotes the fundamental class, then the map*

$$\begin{aligned} \pi_0\mathcal{E}^+_{g,b}(M;\delta) &\longrightarrow \partial^{-1}([\delta]) \\ [\hat{e}] &\longmapsto \hat{e}_*([\Sigma_{g,b}, \partial\Sigma_{g,b}]) \end{aligned}$$

is an isomorphism of $H_2(M;\mathbb{Z})$*-sets.*

Proof. Consider the natural $\mathrm{Diff}(\Sigma_{g,b})$-equivariant inclusion

$$\varphi\colon \mathrm{Emb}(\Sigma_{g,b}, M;\delta) \longrightarrow \mathrm{map}(\Sigma_{g,b}, M;\delta).$$

As the dimension of M is at least 5 and it is simply connected, the main result of [Hae61] says that φ induces a bijection

$$\pi_0\mathrm{Emb}(\Sigma_{g,b}, M;\delta) \cong \pi_0\mathrm{map}(\Sigma_{g,b}, M;\delta).$$

Consider the cofiber sequence $S^1 \xrightarrow{i} \Sigma^1_{g,b} \to \Sigma_{g,b}$, where $\Sigma^1_{g,b}$ denotes a 1-skeleton of $\Sigma_{g,b}$ to which just a single 2-cell needs to be attached to obtain $\Sigma_{g,b}$. The second inclusion gives the locally trivial fibration

$$\mathrm{map}(\Sigma_{g,b}, M;\delta) \longrightarrow \mathrm{map}(\Sigma^1_{g,b}, M;\delta),$$

whose base space is connected because M is simply connected. The fiber over a point $\phi \in \mathrm{map}(\Sigma^1_{g,b}, M;\delta)$ is the space $\mathrm{map}(D^2, M;\phi)$ of maps from the 2-disc D^2 to M that restrict to $\phi \circ i$ on the boundary.

By considering the long exact sequence on homotopy groups for this fibration, in low degrees we find that $\pi_1(\mathrm{map}(\Sigma^1_{g,b}, M;\delta), \phi)$ acts on the set $\pi_0(\mathrm{map}(D^2, M;\phi))$ with quotient $\pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta))$.

We have the composition

$$\pi_0(\mathrm{map}(D^2, M;\phi)) \twoheadrightarrow \pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta)) \twoheadrightarrow \partial^{-1}([\delta])$$

where the source has a free transitive $\pi_2(M)$-action and the target has a free transitive $H_2(M;\mathbb{Z})$-action, and the map is equivariant with respect to the Hurewicz homomorphism. This shows that both maps are in fact bijections, and that the induced $\pi_2(M)$-action on the set $\pi_0(\mathrm{map}(\Sigma_{g,b}, M;\delta))$ is free and transitive.

Given this calculation, it is clear that the group $\mathrm{Diff}^+(\Sigma_{g,b})$ acts trivially on the set $\pi_0\mathrm{Emb}(\Sigma_{g,b}, M;\delta))$, so there is an induced bijection

$$\pi_0\mathcal{E}^+_{g,b}(M;\delta) \longrightarrow \partial^{-1}([\delta]).$$

It is then easy to see that the $H_2(M;\mathbb{Z})$-action on $\partial^{-1}([\delta])$ corresponds to the one that we constructed on $\pi_0\mathcal{E}^+_{g,b}(M;\delta)$. □

This lemma allows us to begin the inductive proof of Proposition 2.3.1, as it tells us what the zeroth homology of $\mathcal{E}^+_{g,b}(M;\delta)$ is.

Lemma 2.3.5. *If* M *is simply connected of dimension at least* 5*, then the statements* F_0 *and* G_0 *hold. As a consequence, the statements* X_0*,* Y_0*,* A_0 *and* B_0 *hold too.*

Proof. Each stabilisation map glues on a cobordism P with incoming boundary δ and outgoing boundary $\overline{\delta}$. With the notation of Lemma 2.3.4, adding on the 2-chain representing the relative fundamental class of P defines an isomorphism of $H_2(M;\mathbb{Z})$-sets $\partial^{-1}([\delta]) \to \partial^{-1}([\overline{\delta}])$ between the inverse images of the fundamental classes $[\delta]$ and $[\overline{\delta}]$, and hence F_0 and G_0 hold. □

2.4 Resolutions of complements of arcs and their fibrations

In the following two sections we prove statements (v) and (vi) of Lemma 2.3.3. These say that the approximate augmentations over a 0-simplex u for the resolutions $(\alpha_{g,b}(M)_\bullet)$ and $(\beta_{g,b}(M)_\bullet)$, that is, the maps

$$(\beta_{g,b-1}(M(u))) \longrightarrow (\alpha_{g,b}(M))$$
$$(\alpha_{g-1,b+1}(M(u))) \longrightarrow (\beta_{g,b}(M)),$$

induce the zero homomorphism in homology in degrees $\leq \frac{1}{3}(2g+2)$ and $\leq \frac{1}{3}(2g+1)$ respectively.

Let us explain the strategy of our proof of these statements, where as an example we consider the approximate augmentation for $\alpha_{g,b}(M)_\bullet$. In this section (in particular in Corollary 2.4.5) we construct a resolution of pairs, called the *relative disc resolution*,

$$(\mathcal{D}\beta_{g,b}(M(u))_\bullet) \longrightarrow (\beta_{g,b}(M(u))),$$

and in Corollary 2.4.6 we show that its space of 0-simplices fits into a homotopy fibre sequence

$$(\beta_{g,b}(M(v))) \longrightarrow (\mathcal{D}\beta_{g,b}(M(u))_0) \longrightarrow D_i(M(u)),$$

where $M(v)$ denotes a manifold obtained from $M(u)$ by cutting out a 2-disc v spanning the arc u. Assuming that G_g holds, in Lemma 2.4.7 we show that the *approximate augmentation for* $(\mathcal{D}\beta_{g,b}(M(u))_\bullet)$*,*

$$(\beta_{g,b}(M(v))) \longrightarrow (\mathcal{D}\beta_{g,b}(M(u))_0) \longrightarrow (\beta_{g,b}(M(u))),$$

induces an epimorphism in homology in degrees $\leq \frac{1}{3}(2g+3)$.

In Section 2.5 we apply the relative disc resolution as follows. Assuming that X_{g-1} holds, we show that the composite of the approximate augmentation for the resolution $(\mathcal{D}\beta_{g-1,b+1}(M(u))_\bullet)$ and the approximate augmentation for the resolution $(\alpha_{g,b}(M)_\bullet)$,

$$(\beta_{g,b-1}(M(v))) \longrightarrow (\beta_{g,b-1}(M(u))) \longrightarrow (\alpha_{g,b}(M)),$$

induce the zero homomorphism in homology in degrees $\leq \frac{1}{3}(2g+2)$. Since we have shown that the first map induces an epimorphism in homology in at least these degrees, we deduce that the approximate augmentations for the resolution $(\alpha_{g,b}(M)_\bullet)$ must induce the zero homomorphism in homology in those degrees, as required.

Thick disc resolution

Suppose that we have a resolution of a stabilisation map $\alpha_{g,b}(M;\delta,\overline{\delta})$ and a 0-simplex u as in Section 2.2 (the stabilisation map $\beta_{g,b}(M;\delta,\overline{\delta})$ can be treated identically). This means that we are given a pair of open balls $\ell = (\ell_0,\ell_1)$ in $\partial^0 M$ and a triple $u = (u', u'', u''') \in A_0(M;\delta,\ell)$ where u' is a tubular neighbourhood of an arc u''' in M and u'' is a strip in u' that contains u'''. In addition, the stabilisation map is a map of type I that glues a cobordism $P \subset \partial^0 M \times I$ to the surfaces in $\mathcal{E}^+_{g,b}(M;\delta)$, and P contains the strips $\tilde{u}'' = \partial u'' \times I$.

Recall that the space $\mathcal{E}^+_{g,b}(M(u);\delta(u))$, consisting of surfaces in $M(u')$ with boundary condition given by δ and the boundary of u'', was defined in Section 2.1 and it appeared in Proposition 2.2.2 as the fibre of the map

$$\mathcal{O}_{g,b}(M;\delta,\ell)_0 \longrightarrow A_0(M;\delta,\ell) \tag{2.4.1}$$

that forgets the surfaces. In Section 2.2, we showed how to extend the stabilisation map to a semi-simplicial map between resolutions, which in particular gives a map on 0-simplices

$$\mathcal{O}^2_{g,b}(M;\delta,\ell)_0 \xrightarrow{\alpha_{g,b}(M;\delta,\overline{\delta})_0} \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_0.$$

This map also fits in the commutative square

$$\begin{tikzcd}
\mathcal{O}^2_{g,b}(M;\delta,\ell)_0 \arrow[r, "\alpha_{g,b}(M;\delta,\overline{\delta})_0"] \arrow[d] & \mathcal{O}^1_{g+1,b-1}(M_1;\overline{\delta},\overline{\ell})_0 \arrow[d] \\
A_0(M;\delta,\ell) \arrow[r, "u\mapsto\overline{u}"] & A_0(M_1;\overline{\delta},\overline{\ell}),
\end{tikzcd}$$

and the maps between the fibres are

$$\beta_{g,b-1}(M;\delta,\overline{\delta})\colon \mathcal{E}^+_{g,b-1}(M(u);\delta(u)) \longrightarrow \mathcal{E}^+_{g,b}(M_1(\overline{u});\overline{\delta}(\overline{u})), \tag{2.4.2}$$

where $\overline{u} = u \cup \tilde{u}$ and $M_1 = M \cup \partial^0 M \times I$. This is a map of type I obtained by gluing the cobordism $P(\tilde{u}) := P \setminus \tilde{u}''$ to each surface. We will now construct a resolution of the individual spaces in (2.4.2), and later show how to extend the map of (2.4.2) to a map of resolutions.

Construction 2.4.1. *Let φ be a path in $P(\tilde{u})$ from $\ell_0 \cap \delta(u)$ to $\ell_1 \cap \delta(u)$ that cannot be contracted to the boundary of P. Then $\overline{\delta}$ is isotopic in $\partial^0 M$ to the oriented ambient connected sum $\#_\varphi \delta$ of δ along φ (when φ is homotoped in $\partial^0 M \times [0,1]$ to lie in $\partial^0 M \times \{0\}$).*

Let η be a path in $\partial^0 M$ from $u'''(0)$ to $u'''(1)$ that cancels the surgery done by φ, that is, so that $\#_\eta \#_\varphi \delta$ is ambient isotopic to δ.

Let $D_+ = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 \leq 1, b \geq 0\}$ be the half disc with some collar of $\partial^0 D_+ = D_+ \cap \mathbb{R} \times \{0\}$, and write $\partial^1 D_+$ for the other 1-face of D_+. Let us fix a neat embedding (using that M is assumed to be simply connected and of dimension at least 5 [Hae61])

$$y\colon D_+ \longrightarrow M(u)$$

disjoint from δ, such that $y(\partial^1 D_+)$ is contained in $u'(S(N_M u'''))\subset \partial M(u)$, and $y_{|\partial^0 D_+} \simeq \eta$ rel $(\ell_0 \cup \ell_1)$.

We remind the reader that, by the definition of neat embedding, we have $y(\partial^0 D_+) \subset \partial^0 M(u)$. Choose, once for all, an (open) tubular neighbourhood Y of y disjoint from δ and write $Y^0 = Y_{|y(\partial^0 D_+)}$ and $Y^1 = Y_{|y(\partial^1 D_+)}$.

Definition 2.4.2. Let $\mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(W, v_0, \dots, v_i)$ where $v_j = (v'_j, v'''_j)$ and

(i) $W \in \mathcal{E}^+_{g,b}(M(u);\delta(u))$;

(ii) $v'''_j\colon D_+ \to M$ is an embedding isotopic to y with $v'''_j(\partial^0 D_+) \subset Y^0$, and $v'''_j(\partial D_+ \setminus \partial^0 D_+) \subset Y^1$;

(iii) v'_j is a closed tubular neighbourhood of v'''_j disjoint from W, whose restriction to $v'''_j(\partial D_+)$ is contained in $Y^0 \cup Y^1$.

The jth face map forgets v_j, and there is an augmentation map $\epsilon_\bullet$ to the space $\mathcal{E}^+_{g,b}(M(u);\delta(u))$ given by forgetting all the v_j. We topologise this set as a subspace of

$$\mathcal{E}^+_{g,b}(M(u);\delta(u)) \times \overline{\mathrm{T}}\mathrm{Emb}(D_+ \times [i], M; q, q_N),$$

where

$$q(\partial^0 D_+) = Y^0, \quad q(\partial^1 D_+) = Y^1, \quad q(x) = M \text{ otherwise}, \quad q_N = q.$$

Proposition 2.4.3. *If* M *is simply connected and of dimension at least* 5*, then the augmentation*

$$\mathcal{D}_{g,b}(M(u); \delta(u), Y)_\bullet \longrightarrow \mathcal{E}^+_{g,b}(M(u); \delta(u))$$

is a resolution of $\mathcal{E}^+_{g,b}(M(u); \delta(u))$.

Proof. In order to find the connectivity of the homotopy fibre of $\epsilon_\bullet$, we use Criterion 1.6.1 to assure that the semi-simplicial fibre $\mathrm{Fib}_W(\epsilon_\bullet)$ of the augmentation map $\epsilon_\bullet$ over a surface W is homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$. The space $\mathcal{E}^+_{g,b}(M(u); \delta(u))$ is $\mathrm{Diff}_\partial(M(u))$-locally retractile by Corollary 1.5.10, hence also $\mathrm{Diff}(M(u); \delta(u), \partial Y)$-locally retractile. In addition, the augmentation maps ϵ_i are $\mathrm{Diff}(M; \delta(u), \partial Y)$-equivariant for all i, therefore they are also locally trivial fibrations by Lemma 1.5.3.

The space of i-simplices of the semi-simplicial fibre is the connected component of $\overline{\mathrm{T}}\mathrm{Emb}((D_+ \times [i], \emptyset), (M, W); q, q_N)$ to which (y, Y) belongs. Let us define the semi-simplicial space $D(W, M)_\bullet$ whose space of i-simplices is the space of embeddings $\mathrm{Emb}((D_+ \times [i], \emptyset), (M, W); q)$, and the face maps are given by forgetting half-discs. Forgetting the tubular neighbourhoods gives a map

$$r_\bullet \colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow D(W, M)_\bullet.$$

The space on the right is *not* levelwise $\mathrm{Diff}(M; \partial Y)$-locally retractile, because an arbitrary diffeomorphism may change the isotopy class of each embedding, but is $\mathrm{Diff}_0(M; Y)$-locally retractile by Lemma 1.5.2 and Corollary 1.5.12. In addition, $r_\bullet$ is levelwise $\mathrm{Diff}(M; \partial Y)$, hence it is a levelwise fibration by Lemma 1.5.3. The fibre over an i-simplex $\mathbf{v} = (v_0''', \dots, v_i''')$ is the space

$$\overline{\mathrm{Tub}}((\mathbf{v}'''(I \times [i]), \emptyset), (M, W); q_N),$$

which is contractible by Lemma 1.4.1.

The semi-simplicial space $D(W, M)_\bullet$ is a topological flag complex, and we will apply Criterion 1.6.2 to prove that it is contractible. Suppose we are given a (possibly empty) collection $v_0''', \dots, v_i'''$ of 0-simplices in $D(W, M)_\bullet$. As the dimension of M is at least 5, we may perturb the embedding y and obtain a map that is transverse to $v_0''', \dots, v_i'''$ and to W (hence disjoint). By [Hae61] and the assumption on the dimension of M, this map is homotopic to an embedding, which is in turn isotopic to y, giving a 0-simplex orthogonal to $v_0''', \dots, v_i'''$. $\square$

Recall that $\overline{\mathrm{T}}\mathrm{Emb}(D_+ \times [i], M; q, q_N)$ is the space of tuples $(\nu_0, \dots, \nu_i)$, with $\nu_j = (\nu_j', \nu_j''')$, where ν_j''' is an embedding of D_+ into M and ν_j' is a tubular neighbourhood of ν_j'' as above, and such that the tubular neighbourhoods are pairwise disjoint. We denote by $D_i(M(u); Y) \subset \overline{\mathrm{T}}\mathrm{Emb}(D_+ \times [i], M; q, q_N)$ the connected component of any tubular neighbourhood of the embedding y.

There is a restriction map

$$\mathcal{D}_{g,b}(M(u); \delta(u), Y)_i \longrightarrow D_i(M(u); Y) \tag{2.4.3}$$

that sends a tuple $(W, \nu_0, \dots, \nu_i)$ to the tuple $(\nu_0, \dots, \nu_i)$.

Proposition 2.4.4. *The map* (2.4.3) *is a locally trivial fibration, and its fibre over an i-simplex $\mathbf{v} = (\nu_0, \dots, \nu_i)$ is the space $\mathcal{E}^+_{g,b}(M(\mathbf{v}); \delta(u))$ where $M(\mathbf{v}) := \mathrm{cl}\,(M(u) \setminus \mathbf{v})$.*

Proof. Since the restriction map is $\mathrm{Diff}(M(u); \delta, Y)$-equivariant, and the space $D_i(M(u); Y)$ is $\mathrm{Diff}(M(u); \delta, Y)$-locally retractile by Lemma 1.5.7, we deduce from Lemma 1.5.3 that the map (2.4.3) is a fibration.

The fibre is the space of oriented surfaces of genus g with boundary condition $\delta(u)$ in $M(u)$ that do not intersect the closed tubular neighbourhoods $\nu_0', \dots, \nu_i'$, which is the space $\mathcal{E}^+_{g,b}(M(\mathbf{v}); \delta(u))$. □

Stabilisation maps between resolutions

In this section we show how to extend the stabilisation maps of Section 2.2 to maps between the resolutions that we have constructed. We will follow closely the methods of that section. As recalled in Section 2.4, to define the bottom map in the diagram

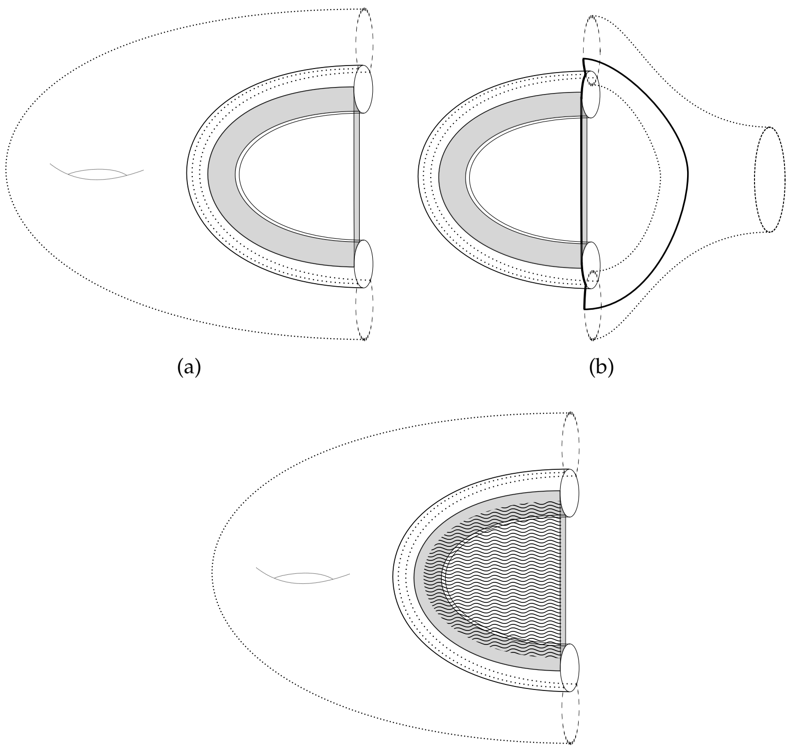
$$\begin{tikzcd}
\mathcal{D}_{g,b}(M(u); \delta(u), Y)_\bullet \arrow[r, dashed] \arrow[d] & \mathcal{D}_{g+1,b-1}(M_1(\bar{u}); \overline{\delta}(\overline{u}), \overline{Y})_\bullet \arrow[d] \\
\mathcal{E}^+_{g,b}(M(u); \delta(u)) \arrow[r, "\alpha_{g,b}(M(u);\delta(u),\overline{\delta}(\overline{u}))"] & \mathcal{E}^+_{g+1,b-1}(M_1(\overline{u}); \overline{\delta}(\overline{u}))
\end{tikzcd}$$

we joined each surface with a cobordism $P(\tilde{u})$ in $\partial^0 M(u) \times I$. We now impose, without loss of generality, the following conditions:

(i) $P(\tilde{u}) \cap (Y^0 \times I) = \emptyset$, and

(ii) $\overline{Y}^0 = Y^0 \times \{1\}$, $\overline{Y}^1 = Y^1 \cup (\partial Y^1 \times I)$.

These assumptions make the extension of the stabilisation map canonical: Let us define $\tilde{\nu}_j = \partial^0 \nu_j \times I$. Then, joining the discs in $(\nu_0, \dots, \nu_i)$ that live in



(a) (b) (c)

Figure 2.3: Figure 2.3a shows the tubular neighbourhood u' and, in light grey, the boundary condition for the discs, that is, Y^0 and Y^1. Figure 2.3b corresponds to the condition that y^0 has to be homotopic to η, which in the picture can be seen as the requirement that the dark circle has to be contractible in $\partial^0 M(u)$. Figure 2.3c shows a typical 0-simplex in $\mathcal{D}_{1,1}(M(u); \delta(u))_\bullet$.

$D_i(M(u);Y)$ to the products $(\tilde{v}_0,\dots,\tilde{v}_i)$ that are subsets of $\partial^0 M \times I$, we obtain new triples $(\bar{v}_0,\dots,\bar{v}_i)$ that live in $D_i(M_1;\bar{\delta})$, where $\bar{v}_j = v \cup \tilde{v}_j$. This rule defines the dashed maps $\mathcal{D}\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_i$ in the first diagram.

These maps commute with the face maps and with the augmentation maps, so they define a map (the *resolution of* $\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))$) of semi-simplicial spaces

$$\mathcal{D}\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_\bullet\colon \mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet \longrightarrow \mathcal{D}_{g+1,b-1}(M(\bar{u});\bar{\delta}(\bar{u}),Y)_\bullet$$

that extends $\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))$. Analogously, we may also define the *resolution of* $\beta_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))$,

$$\mathcal{D}\beta_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_\bullet\colon \mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet \longrightarrow \mathcal{D}_{g,b+1}(M(\bar{u});\bar{\delta}(\bar{u}),Y)_\bullet.$$

Corollary 2.4.5 (To Proposition 2.4.3)**.** *The pair $(\mathcal{D}\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_\bullet)$ together with the natural augmentation map to the pair $(\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u})))$ is an ∞-resolution. The pair $(\mathcal{D}\beta_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_\bullet)$ together with the natural augmentation map to the pair $(\beta_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u})))$ is an ∞-resolution.*

There is a commutative square

$$\begin{tikzcd}
\mathcal{D}_{g,b}(M(u);\delta(u),Y)_i \arrow[r, "\mathcal{D}\alpha_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_i"] \arrow[d] & \mathcal{D}_{g+1,b-1}(M_1(u);\bar{\delta}(\bar{u}),Y)_i \arrow[d] \\
D_i(M(u);Y) \arrow[r, "\mathbf{v}\mapsto\bar{\mathbf{v}}"] & D_i(M_1(\bar{u});Y),
\end{tikzcd} \tag{2.4.4}$$

where the vertical maps are the fibrations of Proposition 2.4.4 and the lower map is a homotopy equivalence. Hence we obtain a map between the fibres over the points $\mathbf{v}$ and $\bar{\mathbf{v}}$ of the fibrations of (2.4.4),

$$\mathcal{E}^+_{g,b}(M(\mathbf{v});\delta(u)) \longrightarrow \mathcal{E}^+_{g-1,b+1}(M_1(\bar{\mathbf{v}});\bar{\delta}(\bar{u})),$$

which is obtained by gluing the cobordism $P(\tilde{u}) \subset \partial^0 M(\mathbf{v}) \times I$ to each surface. This is a map of type $\alpha_{g,b}(M(\mathbf{v});\delta(u),\bar{\delta}(\bar{u}))$.

Following the same procedure with the map $\mathcal{D}\beta_{g,b}(M(u);\delta(u),\bar{\delta}(\bar{u}))_\bullet$, we obtain a map on the fibres over the points $\mathbf{v}$ and $\bar{\mathbf{v}}$ of the analogous diagram (2.4.4)

$$\mathcal{E}^+_{g,b}(M(\mathbf{v});\delta(u)) \longrightarrow \mathcal{E}^+_{g,b+1}(M_1(\bar{\mathbf{v}});\bar{\delta}(\bar{u})),$$

which is of type $\beta_{g,b}(M(\mathbf{v});\delta(u),\bar{\delta}(\bar{u}))$. Since the inclusion

$$D_i(M(u),Y) \longrightarrow D_i(M_1(\bar{u}),Y)$$

is a homotopy equivalence, we may compose the right-hand side fibration in diagram (2.4.4) with the bottom map, obtaining a map of homotopy fibrations over the same base space.

Corollary 2.4.6 (To Proposition 2.4.4). *There are homotopy fibre sequences of pairs*

$$\begin{array}{ccccc} \alpha_{g,b}(M(\mathbf{v});\delta(u)) & \longrightarrow & \mathcal{D}\alpha_{g,b}(M(u);\delta(u))_i & \longrightarrow & D(M_1(\overline{u});\overline{\delta}(\overline{u}))_i \\ \beta_{g,b}(M(\mathbf{v});\delta(u)) & \longrightarrow & \mathcal{D}\beta_{g,b}(M(u);\delta(u))_i & \longrightarrow & D(M_1(\overline{u});\overline{\delta}(\overline{u}))_i. \end{array}$$

Proof. As in Corollary 2.2.3, the map between the base spaces is a weak equivalence, hence we obtain a fibrewise map of fibrations. □

Homology of approximate augmentations of the thick disc resolution

Lemma 2.4.7. *If F_g holds, then the approximate augmentations for $\mathcal{D}\alpha_{g,b}(M(u))_\bullet$ induce epimorphisms in homology up to degree $\frac{1}{3}(2g+4)$.*

If G_g holds, then the approximate augmentations for $\mathcal{D}\beta_{g,b}(M(u))_\bullet$ induce epimorphisms in homology up to degree $\frac{1}{3}(2g+3)$.

Proof. We apply Criterion 1.7.1. Consider the map of semi-simplicial spaces

$$\mathcal{D}\alpha_{g,b}(M(u);\delta(u))_\bullet \longrightarrow \alpha_{g,b}(M(u);\delta(u)$$

given in Corollary 2.4.5, which is a homotopy equivalence (so we take $l = \infty$ in the criterion). Consider also the sequence of path connected based spaces $D(M_1(\overline{u});\overline{\delta}(\overline{u}))_i$, for which we choose an arbitrary basepoint $\mathbf{v} = (v_0, \dots, v_i)$. By Corollary 2.4.6, the map called g_i in the criterion is of type $\alpha_{g,b}(M(\mathbf{v});\delta(u))$.

Now, if we let $k = \lfloor \frac{1}{3}(2g+1) \rfloor + 1$ in the criterion and

$$H_q(\alpha_{g,b}(M(\mathbf{v});\delta(u))) = 0 \tag{2.4.5}$$

when

$$q + i \leq k \quad \text{(except for}(q,i) = (k,0)), \tag{2.4.6}$$

then the approximate augmentation is an epimorphism in degrees $q \leq k$. Because F_g holds, we have that the equality (2.4.5) holds for all $q \leq \frac{1}{3}(2g+1)$ and all i, in particular for the (q,i) in (2.4.6). The proof for $\mathcal{D}\beta_{g,b}(M(u))_\bullet$ is analogous. □

2.5 Triviality

This section is divided in two parts. In the first part we prove Lemma 2.5.1, which is geometric in nature (in contrast to the previous sections). In the language of [RW10], it says that the space of embedded subsurfaces is 1-trivial. In the second part, we apply this lemma to prove assertions (v) and (vi) in Lemma 2.3.3, hence finishing the proof of Proposition 2.3.1. The second part follows [RW10] too, and we slightly improve the argument in that article avoiding the use of homomorphisms that only exist at the level of homology.

Composition of approximate augmentations

Suppose we have a resolution of a stabilisation map $\alpha_{g,b}(M;\delta,\bar{\delta})$, a 0-simplex u in $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ and a 0-simplex v in $\mathcal{D}_{g,b}(M(u);\delta(u),Y)_\bullet$ as in Section 2.4.

Let us denote by $\mathfrak{b}_{g,b-1}(v)$ the composition of the approximate augmentation for the resolution $\mathcal{O}^2_{g,b}(M;\delta,\ell)_\bullet$ over u and the approximate augmentation for the resolution $\mathcal{D}_{g,b-1}(M(u);\delta(u),Y)_\bullet$ over v, and by $\mathfrak{a}_{g,b}(v)$ the analogue composite when we start with $\mathcal{O}^1_{g,b}(M;\delta)_\bullet$. With the notation of Sections 2.2 and 2.4, we have the following commutative square:

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v);\delta(u)) \arrow[r, "\beta_{g,b-1}(M(v);\delta(u))"] \arrow[d, "\mathfrak{b}_{g,b-1}(v)"'] & \mathcal{E}^+_{g,b}(M_1(\bar{v});\bar{\delta}(\bar{u})) \arrow[d, "\mathfrak{a}_{g,b}(v)"] \arrow[dl, dashed] \\
\mathcal{E}^+_{g,b}(M;\delta) \arrow[r, "\alpha_{g,b}(M;\delta)"] & \mathcal{E}^+_{g+1,b-1}(M_1;\bar{\delta}).
\end{tikzcd} \tag{2.5.1}$$

The first result of this section will be the construction of a dotted map with certain properties making both triangles commute up to homotopy. We will construct it in Lemma 2.5.1. All maps in the diagram are maps between spaces of surfaces of the two kinds of maps constructed in Section 2.1:

$$\begin{aligned}
\alpha_{g,b}(M;\delta) &= - \cup P && \text{with} \quad P \subset \partial^0 M \times I && \text{(type I)},\\
\beta_{g,b-1}(M(v);\delta(u)) &= - \cup P(\tilde{u}'') && \text{with} \quad P(\tilde{u}) \subset \partial^0 M(v) \times I && \text{(type I)},\\
\mathfrak{b}_{g,b-1}(v) &= - \cup u'' && \text{with} \quad u'' \subset u' \cup v' && \text{(type II)},\\
\mathfrak{a}_{g,b}(v) &= - \cup \bar{u}'' && \text{with} \quad \bar{u}'' \subset \bar{v}' \cup \bar{u}' && \text{(type II)}.
\end{aligned}$$

Let us denote by $M_i = M \cup \partial^0 M \times [0,i]$. We first prolong the vertical maps in the diagram to obtain

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v);\delta(u)) \arrow[r, "\beta_{g,b-1}(M(v);\delta(u))", "-\cup P(\tilde{u}'')"'] \arrow[d, "b_{g,b-1}(v)"', "-\cup u''"] & \mathcal{E}^+_{g,b}(M_1(\overline{v});\overline{\delta}(\overline{u})) \arrow[d, "a_{g,b}(v)"', "-\cup \overline{u}''"] \\
\mathcal{E}^+_{g,b}(M;\delta) \arrow[r, "\alpha_{g,b}(M;\delta)", "-\cup P"'] \arrow[d, "i_0(\delta)"', "-\cup \delta^0\times[0,2]"] & \mathcal{E}^+_{g+1,b-1}(M_1;\overline{\delta}) \arrow[d, "i_1(\overline{\delta})"', "-\cup \overline{\delta}^0\times[1,3]"] \\
\mathcal{E}^+_{g,b}(M_2;\delta+2) \arrow[r, "\alpha_{g,b}(M_2)", "-\cup(P+2)"'] & \mathcal{E}^+_{g+1,b-1}(M_3;\overline{\delta}+2),
\end{tikzcd} \tag{2.5.2}$$

using the maps $i_j(\delta) = -\cup\delta^0\times[j,j+2]$, with $\delta^0\times[j,j+2]\subset\partial^0M\times[j,j+2]$ and the map $\alpha_{g,b}(M_2) = -\cup(P+2)$, where $P+2\subset\partial^0M\times[2,3]$ is the cobordism P translated 2 units. The boundary condition $\delta+2$ is $(\delta\setminus\delta^0)\cup(\partial\delta^0\times[0,2])\cup(\delta^0\times\{2\})$. In the diagram we have specified both the name of the map and the cobordism that defines the map (see Figures 2.4 and 2.5).

The maps $i_0(\delta)$ and $i_1(\overline{\delta})$ are homotopy equivalences and the bottom square commutes up to homotopy. The purpose of enlarging the diagram is to make room to define a map $-\cup Q$ from the upper right corner to the lower left corner.

In (2.5.2), we have that

$$M_1(\overline{v})\cup\big(\overline{u}\cup\overline{v}\cup\partial^0M\times[1,2]\big) = M_2,$$

and we want to find a cobordism $Q\subset N := \overline{u}\cup\overline{v}\cup\partial^0M\times[1,2]$ defining a map

$$-\cup Q\colon \mathcal{E}^+_{g,b}(M_1(\overline{v});\overline{\delta}(\overline{u})) \longrightarrow \mathcal{E}^+_{g,b}(M_2;\delta+2). \tag{2.5.3}$$

The boundary of N is divided in three parts:

$$\partial N = \ \partial M_1(\overline{v})\ \cup\ \partial\partial^0M_1\times[1,2]\ \cup\ \partial^0M\times\{2\}$$

and the boundary condition which Q must satisfy is

$$\xi = \quad \overline{\delta}(\overline{u})\ \cup\ \partial\delta^0\times[1,2]\ \cup\ \delta^0\times\{2\}.$$

Lemma 2.5.1. *If the dimension of* M *is at least* 6*, then there is a disjoint union of strips and cylinders* Σ'*, a cobordism* $Q\in\mathcal{E}(\Sigma',N;\xi)$*, and isotopies*

$$\begin{array}{rcccl} P(\tilde{u})\cup Q & \overset{G}{\simeq} & u''\cup(\delta^0\times[0,2]) & \subset & u\cup v\cup\partial^0M\times[0,2] \\ Q\cup(P+2) & \overset{H}{\simeq} & \overline{u}''\cup(\overline{\delta}^0\times[1,3]) & \subset & \overline{u}\cup\overline{v}\cup\partial^0M\times[1,3] \end{array}$$

relative to the boundaries. If there is a simply connected open subset Y *of* ∂^0M *with* $P\subset Y\times[0,1]$*, then the result also holds for manifolds of dimension* 5*.*

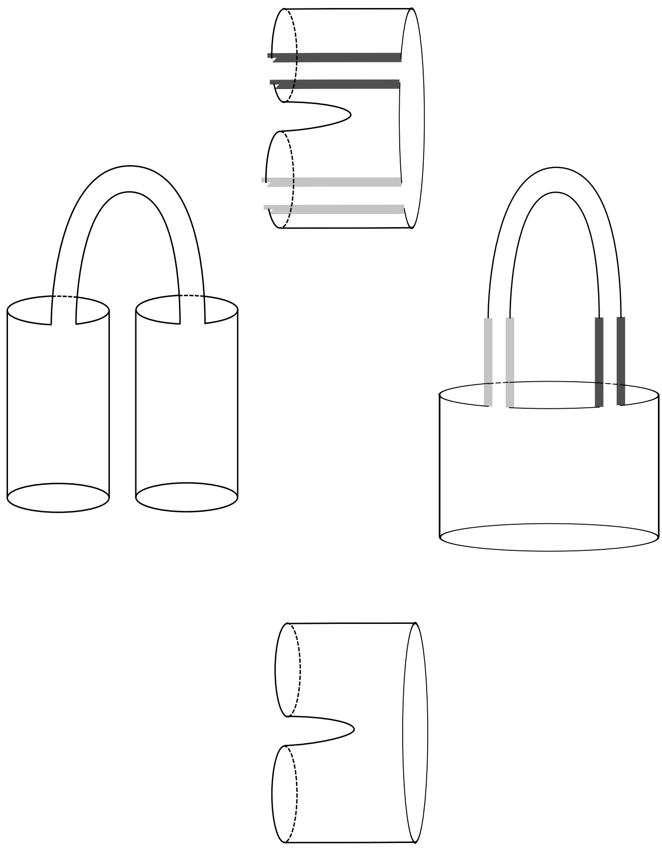


Figure 2.4: The cobordisms in diagram 2.5.2 when $\delta = \delta^0$. The grey colourings in the boundary show how these pieces of the boundary glue together. The cylinders attached to the other boundary components have not been drawn.

Proof. First we show that the space $\mathcal{E}(\Sigma', N; \xi)$ is non-empty. Recall that, by the assumption made in Section 2.2 (see also Section 2.1), $\partial\overline{\delta}(\overline{u})^0 = \partial\delta^0(u)^0$. From Construction 2.4.1, we know that $\overline{\delta}(\overline{u}) \simeq \#_\varphi \delta(u)$. Moreover, $\delta(u) = \#_{u'''}\delta$, and u''' is isotopic in N to η relative to their boundaries, as both are contained in the interior of the thick disc with corners $u \cup v$, hence $\delta(u) = \#_\eta \delta$. In addition, all these isotopies are constant on $\partial\overline{\delta}(\overline{u})^0$. Therefore we conclude that

$$\overline{\delta}(\overline{u}) \simeq \#_\varphi \delta(u) \simeq \#_\varphi \#_\eta \delta \simeq \delta \text{ rel } \partial\overline{\delta}(\overline{u})^0,$$

where we have used the natural identifications

$$\partial M(u) \cong \partial M_1(\overline{u}) \quad \text{and} \quad \partial M \cong \partial M_1.$$

As the dimension of M is at least 5, this isotopy can be realized as a union of embedded strips $Q \subset N$ whose boundary is ξ and a union of cylinders in the

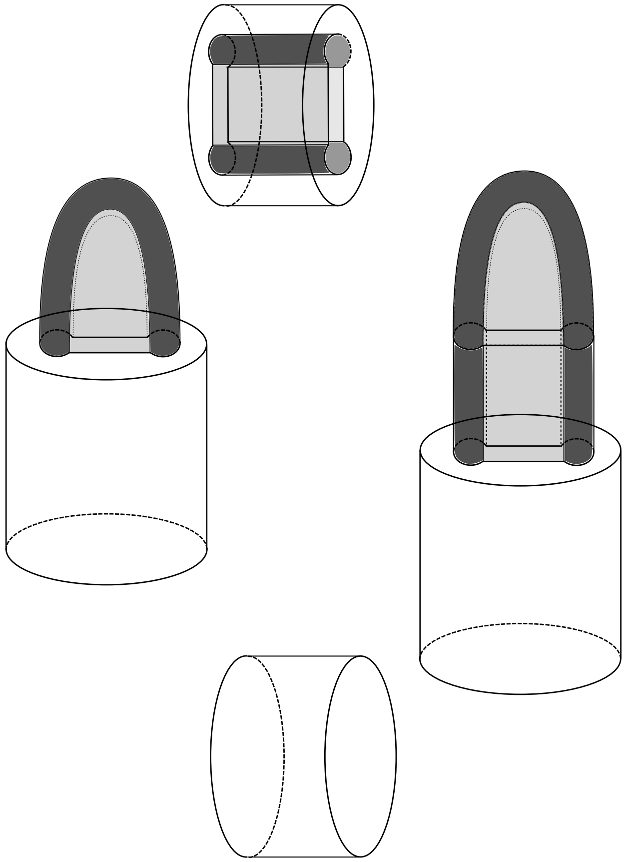


Figure 2.5: Background manifolds in diagram 2.5.2, in the case when $\partial^0 M$ is a disc. The upper picture should be interpreted as the complement of the coloured figure in the cylinder.

components of δ that are disjoint from u'''. This shows that there exists a $Q \subset N$ giving a map (2.5.3).

Now, for each triangle of (2.5.1), we show how to choose a Q so that the triangle commutes up to homotopy (and these homotopies satisfy certain extra properties). Note that all the cobordisms in diagram (2.5.2) contain some part of the product $(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,3]$:

$$(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,1] \subset P(\tilde{u}), \quad (\delta^0 \cap (\ell_0 \cup \ell_1)) \times [0,2] \subset u'' \cup \delta^0 \times [0,2],$$

$$(\delta^0 \cap (\ell_0 \cup \ell_1)) \times [2,3] \subset P+2, \quad (\delta^0 \cap (\ell_0 \cup \ell_1)) \times [1,3] \subset \overline{u}'' \cup \overline{\delta}^0 \times [1,3].$$

Fix an interval l_0 in $\delta^0(u) \cap \ell_0$ and an interval l_1 in $\delta^0(u) \cap \ell_1$, and let $L_0 = l_0 \times [0,3]$ and $L_1 = l_1 \times [0,3]$, and write L for their union. The extra property

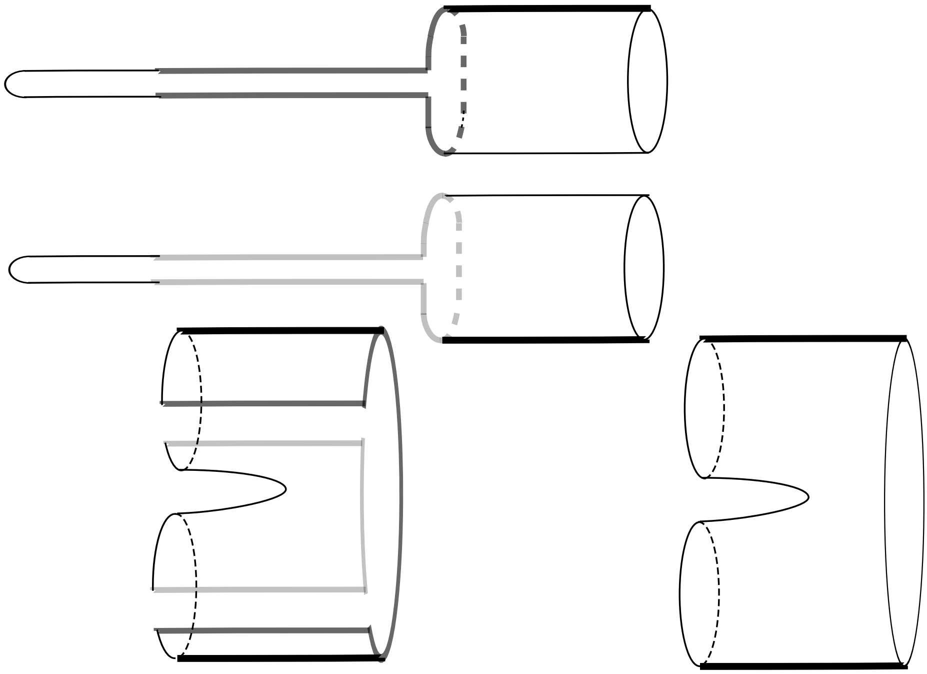


Figure 2.6: From left to right, a picture of the cobordisms $P(\tilde{u})$, Q and $(P+2)$ when P is a pair of pants (this covers the case $\partial M = \partial^0 M$). The thick lines are the strips L, and the grey lines indicate how $P(\tilde{u})$ and Q glue together.

satisfied by the homotopies that we construct is that they are constant on L.

The four cobordisms in diagram (2.5.2) are discs with corners or pairs of pants. Hence the complement of L in each of these cobordisms is a union of discs with corners, that we denote with a superscript L.

Observe first that the inclusion $N \subset u' \cup v' \cup \partial^0 M \times [0,2]$ is an isotopy equivalence, and second that the inclusion $Q \subset P(\tilde{u})^L \cup Q$ is surjective on components (see Figure 2.7). As each component is a disc (with corners), we deduce that the isotopy type of the discs $P(\tilde{u})^L \cup Q$ is fully determined by the isotopy type of the discs Q. Let us choose a $Q_1 \in \mathcal{E}(\Sigma', N; \xi)$ for which $P(\tilde{u})^L \cup Q_1$ is isotopic to $u'' \cup \delta^0 \times [0,2]$. Similarly, the inclusion $N \subset \overline{u}' \cup \overline{v}' \cup \partial^0 M \times [1,3]$ is an isotopy equivalence and the inclusion $Q \subset Q \cup (P+2)^L$ is surjective on components, hence we may choose a $Q_2 \in \mathcal{E}(\Sigma', N; \xi)$ for which $Q_2 \cup (P+2)^L$ is isotopic to $\overline{u}'' \cup \overline{\delta}^0 \times [1,3]$.

Finally, we prove that Q_1 is in the same isotopy class as Q_2, hence we may take $Q = Q_1$, finishing the proof of the lemma. For this, observe first that the

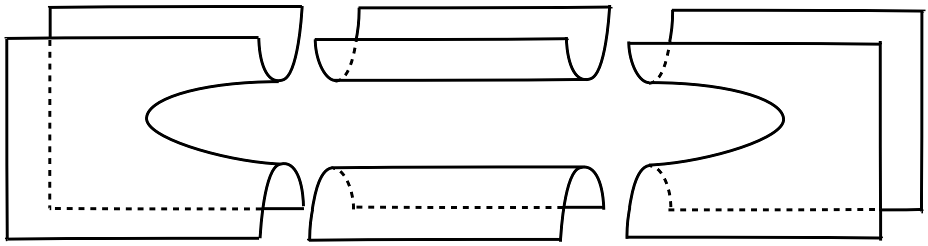


Figure 2.7: An abstract picture (that rounds some corners, for simplicity) of the cobordisms $P(\tilde{u})^L$, Q and $(P+2)^L$. The inclusion $Q \subset P(\tilde{u})^L \cup Q$ is a surjection on π_0, and when restricted to each component of Q, it is an isotopy equivalence onto the component it hits. The same holds for the inclusion $Q \subset Q \cup (P+2)^L$.

composite cobordisms

$$(u'' \cup \delta^0 \times [0,2])^L \cup (P+2)^L, \qquad P(\tilde{u})^L \cup (\overline{u}'' \cup \overline{\delta}^0 \times [1,3])^L,$$

which are the complements of L of the two ways in diagram 2.5.2, are isotopy equivalent. Therefore, by the choices made,

$$R_1 := P(\tilde{u})^L \cup Q_1 \cup (P+2)^L, \qquad R_2 := P(\tilde{u})^L \cup Q_2 \cup (P+2)^L$$

are isotopy equivalent and so are the complements $R_1 \setminus \mathring{Q}_1$ and $R_2 \setminus \mathring{Q}_2$. Take now isotopy equivalent parametrizations $f_1, f_2 \colon \Sigma_{1,2} \to \partial^0 M \times [0,3]$ of R_1 and R_2 with the same jet d near their boundaries and such that $f_1^{-1}(Q_1) = f_2^{-1}(Q_2) =: S$. In addition, fix an isotopy between them. The restriction of these parametrizations to S defines a pair of points in $\pi_0(\mathrm{Fib}_{f_{1|\Sigma_{1,2}\setminus S}}(p))$ of the restriction map

$$p \colon \pi_0\mathrm{Emb}(\Sigma_{1,2}, \partial^0 M \times [0,3]; d) \longrightarrow \pi_0\mathrm{Emb}(\Sigma_{1,2} \setminus S, \partial^0 M \times [0,3]; d_{|\Sigma_{1,2}\setminus S}).$$

These points are mapped to the same point $[f_1] = [f_2]$, and Lemma 2.5.2 implies that they are in the same isotopy class, hence $[f_{1|S}] = [f_{2|S}]$, so $[Q_1] = [Q_2]$, as we wanted.

If there is a simply connected subset $Y \subset \partial^0 M \times [0,1]$ with $P \subset Y$, we may perform the same proof replacing $\partial^0 M$ by Y. $\square$

Lemma 2.5.2. *Let X be a manifold of dimension at least 6 or a simply connected manifold of dimension 5. Let Σ be an oriented surface, let $S \subset \Sigma$ be a disc and write $\Sigma' := \mathrm{cl}(\Sigma \setminus S)$. Let $\overline{f} \colon \Sigma \to X$ be an embedding with jet d. Then, the inclusion of the fiber $\mathrm{Fib}_f(p)$ of the locally trivial fibration $p \colon \mathrm{Emb}(\Sigma, X; d) \to \mathrm{Emb}(\Sigma', X; d_{|\Sigma'})$ over $f_{|\Sigma'}$ is injective on π_0.*

Proof. Let τ be a tubular neighbourhood of $f(\Sigma')$, and let d' be the jet of the restriction $f_{|S}\colon S \to X \setminus \tau$ along the boundary ∂S. Let

$$q\colon \mathrm{map}(\Sigma, X; d) \longrightarrow \mathrm{map}(\Sigma', X; d_{|\Sigma'})$$

be the restriction map, which is also a locally trivial fibration. There is a commutative square

$$\begin{tikzcd}
 & \pi_0\mathrm{Fib}_f(p) \arrow[r, "i"] & \pi_0\mathrm{Emb}(\Sigma, X; d) \arrow[ddd, "k"] \\
 & \pi_0\mathrm{Emb}(S, X\setminus\tau; d') \arrow[u, "a"] \arrow[d, "b"] & \\
 & \pi_0\mathrm{map}(S, X\setminus\tau; d') \arrow[d, "c"] & \\
 \pi_0\mathrm{Fib}_f(q) = & \pi_0\mathrm{map}(S, X; d') \arrow[r, "j"] & \pi_0\mathrm{map}(\Sigma, X; d).
\end{tikzcd} \tag{2.5.4}$$

The map a is induced by a homotopy equivalence, and c is a bijection due to the high dimension of X and the fact that $X \setminus \tau$ is homotopy equivalent to the complement in X of a union of submanifolds of dimension 1 (namely, the image of a 1-skeleton of Σ'). In addition, $X \setminus \tau$ remains simply connected if X is simply connected and of dimension 5, hence the map b is injective (and surjective) by Haefliger's theorem [Hae61].

Moreover, the lower map fits in the exact sequence of pointed sets

$$\pi_1\mathrm{map}(\Sigma', X)_f \xrightarrow{y} \pi_0\mathrm{Fib}_f(q) \xrightarrow{j} \pi_0\mathrm{map}(\Sigma, X; d), \tag{2.5.5}$$

and we claim that the action of $\pi_1\mathrm{map}(\Sigma', X)_f$ on $\pi_0\mathrm{Fib}_f(q)$ is trivial: Given $A \in \pi_0(\mathrm{Fib}_{f_{|\Sigma'}}(p))$ and $B \in \pi_1(map(\Sigma', X), f|'_{\Sigma})$, that is, a map $A\colon D^2 \to X$ from a disc (possibly with corners) D^2 satisfying a boundary condition and a map $B\colon [0,1] \times \Sigma' \to X$ whose restriction to $\{0,1\} \times \Sigma'$ is $\{0,1\} \times f_{|\Sigma'}$, the element $y(B, A)$ is obtained by gluing the map $B_{|[0,1]\times\partial\Sigma'}$ to A to get a new map from the disc. As $B_{|[0,1]\times\partial\Sigma'}$ extends to B (by construction), both A and $y(B, A)$ are cobordant, so the two (relative) cycles A and $y(B, A)$ are homologous. But as X is simply connected, the (relative) Hurewicz map is a bijection, so A and $y(B, A)$ are homotopic.

As a consequence, the map j is injective, hence jcb is injective, and so is kia, and since a is a bijection, we deduce that i is injective. □

Composing the map $- \cup Q$ with an inverse of the homotopy equivalence

$i_0(\delta)$ we obtain a diagonal map for the square (2.5.1), and the isotopies found in Lemma 2.5.1 yield the following:

Corollary 2.5.3 (To Lemma 2.5.1)**.** *In the square* (2.5.1) *the dashed diagonal map exists and both triangles commute up to homotopy.*

Zero in homology

During this section, if $A \to X$ is a map, we will denote by (X, A) its mapping cone. We will use the letter Σ for unreduced suspension, and write $CX = [0,1] \times X/\{1\} \times X$.

Lemma 2.5.4. *If*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ A' \arrow[r, "j"] & X' \end{tikzcd}$$

is a map of pairs and there is a map $t\colon X \to A'$ *making the bottom triangle commute up to a homotopy* $H\colon f \simeq jt$*, then the induced map between mapping cones* $(f,g)\colon (X,A) \to (X',A')$ *factors as* $(X,A) \xrightarrow{p} CA \cup_i CX \xrightarrow{h} (X',A')$*, where* p *comes from the Puppe sequence. In addition, if there is also a homotopy* $G\colon g \simeq ti$*, then the composite* $CA \cup_i CX \xrightarrow{h} (X',A') \xrightarrow{p'} CA' \cup_j CX'$ *is nullhomotopic.*

Proof. The map $h\colon CA \cup_i CX \to CA' \cup_j X'$ is given by

$$\begin{aligned} h(a,s) &= (g(a), s) \in CA' && \text{if } (a,s) \in CA \\ h(b,s) &= H(b, 2s) \in X' && \text{if } (b,s) \in CX \text{ and } 0 \le s \le 1/2 \\ h(b,s) &= (t(b), 2s-1) \in CA' && \text{if } (b,s) \in CX \text{ and } 1/2 \le s \le 1, \end{aligned}$$

and it restricts to (f,g) in the mapping cone $CA \cup_i X$, hence $hp = (f,g)$.

For the second part, let $C_{\frac{1}{2}}Y = \{(y,s) \in CY \mid 0 \le s \le 1/2\}$, notice that

$$(CA \cup_i CX)/C_{\frac{1}{2}}X \cong \Sigma A \vee \Sigma X,$$

and consider the diagram

$$\begin{tikzcd} CA \cup CA = \Sigma A \arrow[d, "\simeq"', "\mathrm{Id} \cup Ci"] & & & \\ CA \cup_i CX \arrow[r, "h"] \arrow[d, "\text{collapse } C_{\frac{1}{2}}X"] & CA' \cup_j X' \arrow[r, "p'"] & CA' \cup_j CX' \arrow[r, "\text{collapse } CX'", "\simeq"'] & \Sigma A' \\ (CA \cup_i CX)/C_{\frac{1}{2}}X \cong \Sigma A \vee \Sigma X \arrow[rrr, "\Sigma g \vee \Sigma t"] & & & \Sigma A' \vee \Sigma A' \arrow[u, "\nabla"'] \end{tikzcd}$$

which is easily checked to commute. As ti is homotopic to g, the lower composition is homotopic to $\nabla \circ (\Sigma g \vee \Sigma g) \circ \vee$, i.e. $\Sigma g - \Sigma g$, so it is nullhomotopic, as required. □

We now return to (2.5.1), where we had chosen an arc $u_0 = u \in \mathcal{O}^2_{g,b}(M;\delta)_0$ and a disc $v_0 = v \in \mathcal{D}_{g,b}(M(u_0);\delta(u_0))_0$. Suppose that we have another arc $u_1 \in \mathcal{O}^1_{g,b-1}(M(v_0);\delta(u_0))_0$ and another disc $v_1 \in \mathcal{D}_{g-1,b}(M(v_0)(u_1),\delta(u_0))_0$ (notice that we can also consider u_1 as a point in $\mathcal{O}^2_{g,b}(M;\delta)_0$ and u_0 as a point in $\mathcal{O}^1_{g,b-1}(M(v_1);\delta(u_1))_0$). The diagram below shows the various maps which can be constructed from these data using Lemma 2.5.4:

$$\begin{tikzcd}
\mathcal{E}^+_{g,b-1}(M(v_0)) \arrow[r] \arrow[d, "\mathfrak{b}_{g,b-1}(v_0)"'] \arrow[dr, phantom, "(1)"] & \mathcal{E}^+_{g,b}(M(v_0)) \arrow[r] \arrow[d, "\mathfrak{a}_{g,b}(v_0)"] & \beta_{g,b-1}(M(v_0)) \arrow[r, "p"] \arrow[d] & \Sigma\mathcal{E}^+_{g,b-1}(M(v_0)) \arrow[dl, "h"] \\
\mathcal{E}^+_{g,b}(M) \arrow[r] & \mathcal{E}^+_{g+1,b-1}(M) \arrow[r] & \alpha_{g,b}(M) & \\
\mathcal{E}^+_{g,b-1}(M(v_1)) \arrow[r] \arrow[u, "\mathfrak{b}_{g,b-1}(v_1)"] \arrow[ur, phantom, "(2)"] & \mathcal{E}^+_{g,b}(M(v_1)) \arrow[r] \arrow[u, "\mathfrak{a}_{g,b}(v_1)"'] & \beta_{g,b-1}(M(v_1)) \arrow[r, "p'"] \arrow[u] & \Sigma\mathcal{E}^+_{g,b-1}(M(v_1)) \arrow[ul, "h'"'] \\
\mathcal{E}^+_{g-1,b}(M(v_0,v_1)) \arrow[r] \arrow[u, "\mathfrak{a}_{g-1,b}(v_0)"] \arrow[ur, phantom, "(3)"] & \mathcal{E}^+_{g,b-1}(M(v_0,v_1)) \arrow[r] \arrow[u, "\mathfrak{b}_{g,b-1}(v_0)"'] & \alpha_{g-1,b}(M(v_0,v_1)) \arrow[r] \arrow[u] & \Sigma\mathcal{E}^+_{g-1,b}(M(v_0,v_1)). \arrow[ul, "h''"'] \arrow[uuu, bend right, "\Sigma\mathfrak{a}_{g-1,b}(v_0,v_1)"']
\end{tikzcd}$$

The second line of the diagram is the Puppe sequence for the stabilisation map $\alpha_{g,b}(M;\delta,\bar{\delta})$, and the first and third lines are the Puppe sequences for the approximate augmentations corresponding to the data (u_0,v_0) and (u_1,v_1) respectively. The fourth line is the Puppe sequence for the approximate augmentation obtained by using the data (u_0,v_0) on the map $\beta_{g,b-1}(M(v_1))$. Importantly, it may also be considered to be the Puppe sequence for the approximate augmentation obtained by using the data (u_1,v_1) on the map $\beta_{g,b-1}(M(v_0))$.

We use Corollary 2.5.3 in order to provide a diagonal map and isotopies in the square (3), which gives the map h'' such that $p' \circ h''$ is nullhomotopic. We use the same diagonal map and isotopies in the square (1) to obtain the map h, because in square (1) the problem of finding such data is the same as in square (3), but without the requirement that the isotopies have to fix the cylinders $\tilde{u}'_1$. We choose any diagonal map and isotopies for the square (2) to obtain the map h'. After doing this, it follows from the definition of the map h that

Lemma 2.5.5. *The maps*

$$(\mathfrak{a}_{g,b}(v_1), \mathfrak{b}_{g,b-1}(v_1)) \circ h'' \quad \textit{and} \quad h \circ \Sigma\mathfrak{a}_{g-1,b}(v_0,v_1)$$

from $\Sigma\mathcal{E}^+_{g-1,b}(M(v_0,v_1))$ to $\alpha_{g,b}(M)$ are homotopic.

Proof. The map $\Sigma\mathfrak{a}_{g-1,b}(\nu_0,\nu_1)$ is the suspension of the map that glues back the strip u_1'' corresponding to ν_1. The map $(\mathfrak{a}_{g,b}(\nu_1), \mathfrak{b}_{g,b-1}(\nu_1))$ glues back the strips u_1'' and $\bar{u}_1''$. The homotopies in square (3) fix these strips by their very definition, and we have chosen the homotopies in (1) to be the ones in square (3), so they fix the strips too. Therefore the manifolds in which the cobordisms that define h and the cobordisms that define $\Sigma\mathfrak{a}_{g-1,b}(\nu_0,\nu_1)$ live are disjoint. The maps h and h'' are obtained by gluing to the spaces of manifolds the same cobordisms and performing the same isotopies on them. ☐

Proposition 2.5.6. *Let M be simply connected and of dimension at least 5. If the dimension is 5, then we assume that all the stabilisation maps in what follows are induced by pairs of pants that are contractible in $\partial^0 M \times I$. If X_{g-1} holds, then the map $(\beta_{g,b-1}(M(\nu_0))) \to (\alpha_{g,b}(M))$ induces the zero homomorphism in homology degrees $\leq \frac{1}{3}(2g+2)$. If Y_{g-1} holds, then the map $(\alpha_{g,b-1}(M(u,\nu))) \to (\beta_{g,b}(M))$ induces the zero homomorphism in homology degrees $\leq \frac{1}{3}(2g+1)$.*

Proof. We find homotopies

$$h \circ \Sigma\mathfrak{a}_{g-1,b}(\nu_0,\nu_1) \simeq (\mathfrak{a}_{g,b}(\nu_1), \mathfrak{b}_{g,b-1}(\nu_1)) \circ h'' \simeq h' \circ p' \circ h'' \simeq *$$

by applying Lemmas 2.5.5 and 2.5.4. Since X_{g-1} holds, the map $\Sigma\mathfrak{a}_{g-1,b}(\nu_0,\nu_1)$ induces an epimorphism in homology degrees $\leq \frac{1}{3}(2(g-1)+1)+1$ (although the map $\mathfrak{a}_{g-1,b}(\nu_0,\nu_1)$ is not a map of type $\alpha_{g,b}(M(\nu_0,\nu_1)$, it is isotopic to such a map after rounding the corners of M), hence h must induce the zero homomorphism in those degrees, and so must hp. The second part is proven similarly, by rewriting all of this section in the analogous way. ☐

The following finishes the proof of parts (ν) and (νi) of Proposition 2.3.3.

Corollary 2.5.7. *Let M be a simply connected manifold of dimension at least 5. If the dimension of M is 5, we assume in addition that the pairs of pants defining the stabilisation maps are contractible in $\partial^0 M \times [0,1]$. If X_{g-1} and G_g hold, then the map*

$$(\beta_{g,b-1}(M(u_0))) \longrightarrow (\alpha_{g,b}(M))$$

induces the zero homomorphism in homology degrees $\leq \frac{1}{3}(2g+2)$. If Y_{g-1} and F_{g-1} hold, then the map

$$(\alpha_{g-1,b+1}(M(u_0))) \longrightarrow (\beta_{g,b}(M))$$

induces the zero homomorphism in homology degrees $\leq \frac{1}{3}(2g+1)$.

Proof. In the first case, by the previous proposition we have seen that if X_{g-1} holds, then the composition

$$(\beta_{g,b-1}(M(\nu))) \longrightarrow (\beta_{g,b-1}(M(u_0))) \longrightarrow (\alpha_{g,b}(M))$$

induces the zero homomorphism in degrees $\leq \frac{1}{3}(2g+2)$, while in Proposition 2.4.7 we have proven that if G_g holds, then the left arrow is an epimorphism in degrees $\leq \frac{1}{3}(2g+3)$. Thus the composition is zero in the range of degrees claimed. The second case is completely analogous. □

2.6 Closing the last boundary

We have to prove the last assertion of Theorem B, as well as the injectivity in homology of the maps of type β for which one of the newly created boundaries is contractible in ∂M. That $\beta_{g,b}(M;\delta)$ induces a monomorphism in homology in this case and that $\gamma_{g,b}(M;\delta)$ induces an epimorphism in homology if $b \geq 2$ is a consequence of the fact (cf. Remark 2.6.5) that for each such $\beta_{g,b}(M;\delta)$ (resp. each $\gamma_{g,b}(M;\delta)$), there is a $\gamma_{g,b+1}(M;\delta')$ (resp. $\beta_{g,b-1}(M;\delta')$) and homotopy retractions

$$\gamma_{g,b+1}(M;\delta')\beta_{g,b}(M;\delta) \simeq \mathrm{Id}, \quad \beta_{g,b-1}(M;\delta')\gamma_{g,b}(M;\delta) \simeq \mathrm{Id}.$$

Moreover, by Proposition 2.3.1 the β-maps induce isomorphisms in homology in degrees $\leq \frac{2}{3}g - 1$ and an epimorphism in the next degree, but since the β maps are monomorphisms, it follows that they also induce isomorphisms in homology up to degree $\frac{2}{3}g$. Finally, this implies that $\gamma_{g,b}(M;\delta)$ is an isomorphism in those degrees too.

Therefore, it only remains to prove the third assertion of Theorem B when $b = 1$. This is the purpose of this section.

Resolutions and fibrations

Consider the space $\mathcal{E}^+_{g,b}(M;\delta)$, and let $\ell \subset \partial^0 M$ be a subset diffeomorphic to a ball disjoint from δ. There is a semi-simplicial space $\mathcal{P}_{g,b}(M;\delta,\ell)_\bullet$ whose i-simplices are tuples $(W, p_0, \ldots, p_i)$, where $p_j = (p'_j, p''_j, p'''_j)$ and

(i) $W \in \mathcal{E}^+_{g,b}(M;\delta)$;

(ii) $p'''_j\colon ([0,1],\{1/2\}) \to (M,W)$ is an embedding of pairs with $p'''_j(0) \in \ell$ and $p'''_j(1) \in \mathring{M}$;

(iii) (p'_j, p''_j) is a closed tubular neighbourhood of $p'''_j([0,1])$ in the pair (M,W);

(iv) the neighbourhoods $p'_0, \ldots, p'_i$ are disjoint.

The jth face map forgets p_j and there is an augmentation map to $\mathcal{E}^+_{g,b}(M;\delta)$ that forgets all the p_j. We topologise the space of i-simplices as a subset of

$\mathcal{E}^+_{g,b}(M;\delta) \times \overline{\mathrm{T}}\mathrm{Emb}(I \times [i], M; q, q_N)$, where

$$q(x) = q_N(x) = \ell \text{ if } x = 0, \qquad q(x) = q_N(x) = M \text{ if } x \neq 0.$$

Proposition 2.6.1. *If M is connected and of dimension at least 3, then the semi-simplicial space $\mathcal{P}_{g,b}(M;\delta)_\bullet$ is a resolution of $\mathcal{E}^+_{g,b}(M;\delta)$.*

Proof. The space $\mathcal{E}^+_{g,b}(M;\delta)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Lemma 1.5.7, and therefore it is also $\mathrm{Diff}(M;\delta,\ell)$-locally retractile. For each i, the augmentation map

$$\epsilon_i \colon \mathcal{P}_{g,b}(M;\delta)_i \longrightarrow \mathcal{E}^+_{g,b}(M;\delta)$$

is $\mathrm{Diff}(M;\delta,\ell)$-equivariant for all i, therefore it is also a locally trivial fibration by Lemma 1.5.3. As a consequence, the semi-simplicial fibre $\mathrm{Fib}_W(\epsilon_\bullet)$ is homotopy equivalent to the homotopy fibre of $|\epsilon_\bullet|$ by Criterion 1.6.1. The space of i-simplices of the semi-simplicial fibre is $\overline{\mathrm{T}}\mathrm{Emb}((I\times[i],\{1/2\}\times[i]),(M,W);q,q_N)$. Let us define the semi-simplicial space $P(W,M)_\bullet$ whose space of i-simplices is

$$\mathrm{Emb}((I \times [i], \{1/2\} \times [i]), (M, W); q),$$

and the face maps are given by forgetting embeddings. Forgetting the tubular neighbourhoods yields a map

$$r_\bullet \colon \mathrm{Fib}_W(\epsilon_\bullet) \longrightarrow P(W,M)_\bullet$$

that is levelwise $\mathrm{Diff}(M;W,\ell)$-equivariant onto the space $P(W,M)_\bullet$, which is levelwise $\mathrm{Diff}(M;W,\ell)$-locally retractile by Corollary 1.5.12, hence this map is a levelwise fibration by Lemma 1.5.3. The fibre of the map r_i over an i-simplex $\mathbf{p} = (p_0''', \dots, p_i''')$ is the space $\overline{\mathrm{Tub}}(\mathbf{p}'''(I),(M,W);q_N)$, which is contractible by Lemma 1.4.1.

The semi-simplicial space $P(W,M)_\bullet$ is a topological flag complex, and we will apply Criterion 1.6.2 to prove that it is contractible. As M is connected, for each tuple $(W,p_0'''),\dots,(W,p_{i-1}''')$ of 0-simplices over a surface W there is another 0-simplex (W,p_i''') over W orthogonal to them all, by general position. Hence it is contractible by Criterion 1.6.2. □

Let $B_i(M;\ell)$ be the set of tuples $(p_0,\dots,p_i)$ with $p_j = (p_j', p_j'', p_j''')$, where p_j''' is an embedding of an interval in M, p_j' is a tubular neighbourhood of p_j'' in M and p_j'' is the restriction of p_j' to some vector subspace $L_j \subset \overline{N}_M p_j'''(1/2)$ of dimension 2. Moreover, we require that p_j' be disjoint from p_k'. This space is in canonical bijection with $\overline{\mathrm{T}}\mathrm{Emb}_{2,\{1/2\}\times[i]}(I \times [i], M; q, q_N)$, and we use this bijection to topologize it.

There is a map

$$\mathcal{P}_{g,b}(M;\delta,\ell)_i \longrightarrow B_i(M;\ell)$$

that sends a tuple $(W, p_0, \ldots, p_i)$ to the tuple $(p_0, \ldots, p_i)$.

Proposition 2.6.2. *For a point* $\mathbf{p} \in B_i(M;\ell)$*, with the notation of Section* 2.1*, there is a homotopy fibre sequence*

$$\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p})) \longrightarrow \mathcal{P}_{g,b}(M;\delta)_i \longrightarrow B_i(M;\ell).$$

Proof. The map is $\mathrm{Diff}(M;\delta,\ell)$-equivariant and $B_i(M;\ell)$ is $\mathrm{Diff}(M;\delta,\ell)$-locally retractile by Lemma 1.5.13, therefore this map is a locally trivial fibration by Lemma 1.5.3. The fibre over a point $\mathbf{p}$ is the space of surfaces W in M that meet the tubular neighbourhoods p'_j in the image of p''_j. This space is canonically homeomorphic to the space $\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}))$. ☐

Stabilisation maps between resolutions

In this section we will show how to extend the stabilisation map $\gamma_{g,b}(M;\delta)$ to a map between resolutions

$$\begin{tikzcd}
\mathcal{P}_{g,b}(M;\delta,\ell)_i \arrow[r, dashed] \arrow[d] & \mathcal{P}_{g,b-1}(M_1;\overline{\delta},\overline{\ell})_i \arrow[d] \\
\mathcal{E}^+_{g,b}(M,\delta) \arrow[r, "\gamma_{g,b}(M_1;\delta,\overline{\delta})"] & \mathcal{E}^+_{g,b-1}(M_1,\overline{\delta}).
\end{tikzcd}$$

To define the maps $\gamma_{g,b}(M;\delta,\overline{\delta})\colon \mathcal{E}^+_{g,b}(M;\delta) \to \mathcal{E}^+_{g,b-1}(M_1,\overline{\delta})$, we joined each surface with a cobordism P in $\partial^0 M \times I$. We will assume, without loss of generality, that

(i) $\overline{\ell} = \ell \times \{1\}$;

(ii) $(\ell \times I) \cap P = \emptyset$.

As in previous constructions, we define $\tilde{p}'_j = \partial^0 p'_j \times I$ and $\overline{p}'_j = p'_j \cup \tilde{p}'_j$ and similarly $\tilde{p}'_j$ and $\overline{p}'_j$. There is a map $\gamma_{g,b}(M;\delta,\overline{\delta})_i$ making the diagram commute, that sends a tuple $(W,\mathbf{p})$ to the tuple $(W \cup P, \overline{\mathbf{p}})$. These maps commute with the face maps and with the augmentation maps, so they define a map of semi-simplicial spaces (the *resolution of* $\gamma_{g,b}(M;\delta,\overline{\delta})$),

$$\gamma_{g,b}(M;\delta,\overline{\delta})_\bullet\colon \mathcal{P}_{g,b}(M;\delta)_\bullet \longrightarrow \mathcal{P}_{g,b-1}(M_1;\overline{\delta})_\bullet,$$

extending $\gamma_{g,b}(M;\delta,\overline{\delta})$.

Corollary 2.6.3 (To Proposition 2.6.1). *The pair* $(\gamma_{g,b}(M)_\bullet, \delta, \overline{\delta})$ *together with the natural augmentation to the pair* $\gamma_{g,b}(M; \delta, \overline{\delta})$ *is a resolution.*

The diagram

$$\begin{tikzcd}
\mathcal{P}_{g,b}(M;\delta)_i \arrow[r, "\gamma_{g,b}(M;\delta,\overline{\delta})_i"] \arrow[d] & \mathcal{P}_{g,b-1}(M;\overline{\delta})_i \arrow[d] \\
B_i(M;\ell) \arrow[r, "\mathbf{p}\mapsto\overline{\mathbf{p}}"] & B_i(M_1,\ell)
\end{tikzcd}$$

is an extension of the homotopy equivalence $B_i(M;\ell) \to B_i(M_1;\overline{\ell})$. Hence we obtain a well-defined map on the homotopy fibres over the points $\mathbf{p}$ and $\overline{\mathbf{p}}$ of the fibrations of Proposition 2.6.2,

$$\mathcal{E}^+_{g,b+i+1}(M(\mathbf{p}); \delta(\mathbf{p})) \longrightarrow \mathcal{E}^+_{g,b+i}(M(\overline{\mathbf{p}}); \overline{\delta}(\overline{\mathbf{p}})),$$

obtained by gluing the cobordism P to each surface. This is a map of type $\gamma_{g,b+i+1}(M(\mathbf{p}); \delta(\mathbf{p}), \overline{\delta}(\overline{\mathbf{p}}))$.

Corollary 2.6.4 (To Proposition 2.6.2). *There is a relative homotopy fibre sequence*

$$(\gamma_{g,b+i+1}(M(\mathbf{p}); \delta(\mathbf{p}), \overline{\delta}(\overline{\mathbf{p}}))) \longrightarrow (\gamma_{g,b}(M; \delta, \overline{\delta})_i) \longrightarrow B_i(M;\ell).$$

Homological stability

Remark 2.6.5. Consider a stabilisation map $\gamma_{g,b}(M;\delta,\overline{\delta})$, which is given by closing off one of the boundaries b of δ (which must necessarily be nullhomotopic in $\partial^0 M$). If δ has another boundary component b' in the same component of $\partial^0 M$ as b, then there exists a stabilisation map $\beta_{g,b-1}(M;\delta_0,\delta)$ creating the boundaries b and b'. In this case we may enlarge collars, and we have the composition

$$\mathcal{E}^+_{g,b-1}(M;\delta_0) \xrightarrow{\beta_{g,b-1}(M;\delta_0,\delta)} \mathcal{E}^+_{g,b}(M_1;\delta) \xrightarrow{\gamma_{g,b}(M_1;\delta,\overline{\delta})} \mathcal{E}^+_{g,b-1}(M_2;\overline{\delta})$$

which is homotopic to a stabilisation map which takes the union with a cylinder inside $\partial^0 M \times [0,2]$. This map may not be homotopic to the identity —the cylinder may be embedded in a non-trivial way— but it is a homotopy equivalence (as we may find an inverse cylinder), so the map $\gamma_{g,b}(M_1;\delta,\overline{\delta})$ is split surjective in homology. By the same argument, any map $\beta_{g,b}(M_1;\delta,\overline{\delta})$ which creates a boundary which is nullhomotopic in $\partial^0 M$ is split injective in homology.

The following proposition finishes the proof of Theorem B.

Proposition 2.6.6. *Let* M *be a simply connected manifold of dimension at least* 5 *with non-empty boundary, and* δ *be a boundary condition. Then, for any boundary condition* $\overline{\delta}$*,*

(i) *for every map* $\gamma_{g,b}(M;\delta,\overline{\delta})$ *we have* $H_k(\gamma_{g,b}(M;\delta,\overline{\delta})) = 0$ *for* $k \leq \frac{2}{3}g + 1$*;*

(ii) *every map* $\beta_{g,b}(M;\delta,\overline{\delta})$ *for which one of the newly created components of* $\overline{\delta}$ *is contractible in* $\partial^0 M$ *induces a monomorphism in all homology degrees;*

(iii) *every map* $\gamma_{g,b}(M;\delta,\overline{\delta})$ *for which there is another component of* δ *in the same component of* $\partial^0 M$ *as the one one which is closed induces an epimorphism in all homology degrees.*

Proof. We have already shown the last two statements above. Regarding the first statement, suppose first that there is another component of δ in the same component of $\partial^0 M$ as the one which is closed by $\gamma_{g,b}(M;\delta,\overline{\delta})$, and choose a $\beta_{g,b-1}(M;\delta_0,\delta)$ as in Remark 2.6.5. By Proposition 2.3.1, we know that the map $\beta_{g,b-1}(M;\delta_0,\delta)$ induces an epimorphism in homology degrees $\leq \frac{2}{3}g$, and it also induces a monomorphism in all degrees: thus it induces an isomorphism in degrees $\leq \frac{2}{3}g$. As $\gamma_{g,b}(M;\delta,\overline{\delta})$ is a left inverse to it, this also induces an isomorphism in these degrees. Hence $H_k(\gamma_{g,b}(M;\delta,\overline{\delta})) = 0$ for $k \leq \frac{1}{3}(2g+3)$, as $\gamma_{g,b}(M;\delta,\overline{\delta})$ induces an epimorphism in all degrees.

Now suppose that there is no other component of δ in the same component of $\partial^0 M$ as the one which is closed by $\gamma_{g,b}(M;\delta,\overline{\delta})$. We choose a ball $\ell \subset \partial^0 M$ and form the resolution of $\gamma_{g,b}(M;\delta,\overline{\delta})$ given by Corollary 2.6.3. Using Corollary 2.6.4 to identify the space of i-simplices in this resolution, the pair $(\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}))$ is a map of type γ for surfaces with (after rounding the corners of M) at least $i+1$ extra boundary components of $\delta(\mathbf{p})$ in the component of ∂M containing the boundary which is closed off, so the discussion above applies and shows that $H_k(\gamma_{g,b+i+1}(M(\mathbf{p});\delta(\mathbf{p}))) = 0$ for $k \leq \frac{1}{3}(2g+3)$. Applying the second result of Criterion 1.7.1 to this resolution gives the result. □

Chapter 3

Stable Homology

In this chapter we prove Theorem C. The first section proves the theorem in the case where M has non-empty boundary, except for the proof of a proposition (Proposition 3.1.9), which we defer to the second section. We show how to deduce Theorem A for manifolds without boundary in the third section. In this chapter we only work with manifolds and manifolds with boundary, but not manifolds with corners, as we did in the previous chapter.

3.1 Group completion

This section gravitates around a group completion argument that takes place in Proposition 3.1.10. Roughly speaking, the space of all compact connected oriented surfaces $\coprod_{g,\delta} \mathcal{E}^+_{g,b}(M;\delta)$ in M is a module over the cobordism category of cobordisms in $\partial M \times I$, and the group completion will invert the operation "gluing a torus in $\partial M \times I$". We will compare the homotopy type of these modules with the homotopy type of certain spaces of sections in order to deduce Theorem C for background manifolds with non-empty boundary.

Spaces of manifolds and scanning maps

Galatius and Randal-Williams introduced a topology on the set $\Psi_2(\mathbb{R}^n)$ of all smooth oriented 2-dimensional submanifolds of $\mathbb{R}^n$ that are closed as subsets of $\mathbb{R}^n$. This topology is discussed in detail in Section 4.3. More generally, for any real vector space V we can consider the space $\Psi_2(V)$ of smooth oriented 2-dimensional manifolds in V. If $\langle -, - \rangle$ is an inner product on V, we can define the Thom space of the orthogonal complement of the tautological bundle over the oriented Grassmannian $\mathrm{Gr}_2^+(V)$,

$$\mathcal{S}(V) := \mathrm{Th}(\gamma_m^\perp(V)),$$

as in page 8 in the Introduction. There is an inclusion

$$i\colon \mathcal{S}(V) \longrightarrow \Psi_m(V) \tag{3.1.1}$$

given by sending a pair $(L \in \mathrm{Gr}_2^+(V), v \in L^\perp)$ to the oriented surface $v + L \subset V$, and sending the point at infinity to the empty surface.

Proposition 3.1.1 ([GRW10]). *The inclusion* i *is a weak homotopy equivalence.*

Let M be a smooth manifold of dimension d, possibly with boundary. Let $\Psi_2(M)$ be the set of all smooth oriented 2-dimensional submanifolds of $\mathring{M}$ which are closed as subsets of M. Due to the fact that $\Psi_2(\mathbb{R}^n)$ is a sheaf of topological spaces, there is a unique way of promoting the sheaf of sets $\Psi_2(M)$ to a sheaf of topological spaces [RW11, Section 3].

Let $\mathfrak{g}$ be a Riemannian metric on M, not necessarily complete. There is an associated partially defined exponential map $\exp\colon TM \dashrightarrow M$. The *injectivity radius* of $\mathfrak{g}$ at $p \in M$ is the supremum of the real numbers $r \in (0, \infty)$ such that exp is defined on T_pM on vectors of length $< r$, and exp is injective when restricted to the open ball of radius r in T_pM.

Let $a\colon M \to (0, \infty)$ be a smooth map whose value at each point is strictly less than the injectivity radius of the metric $\mathfrak{g}$ at that point —such functions exist by a partition of unity argument. If V is an inner product space, define an endomorphism h of V by $v \mapsto \left(\frac{1}{\pi}\arctan\|v\|\right)v$. Let $\exp_a\colon TM \to M$ be the composition of the endomorphism of TM given by $v \mapsto a(p)h(v)$ if $v \in T_pM$ and the exponential map.

Let $\Psi(TM)$ denote the space of pairs (p, W) with $p \in M$ and $W \in \Psi(T_pM)$, i.e., the space obtained by performing the construction $\Psi(-)$ fibrewise to TM. There is a map

$$\Psi(M) \times M \longrightarrow \Psi(TM),$$

given by $(W, p) \mapsto ((\exp_a|_{T_pM})^{-1}(W) \subset T_pM)$, whose adjoint

$$s_a\colon \Psi(M) \longrightarrow \Gamma(\Psi(TM) \to M),$$

a map to the space of sections of the bundle $\Psi(TM) \to M$, is called *non-affine scanning map*.

Proposition 3.1.2 ([RW11]). *If* M *has no compact components, then the non-affine scanning map* s_a *is a weak homotopy equivalence.*

On the other hand, let $\mathcal{S}(TM)$ denote the result of performing the construction $\mathcal{S}(-)$ fibrewise to the tangent bundle of M. Let $\Psi^{\nu}(M)$ denote the space of pairs (W, t) consisting of a submanifold $W \in \Psi(M)$ and a map $t\colon W \to (0, \infty)$ such that the exponential map restricted to the subspace

$$\nu_t(W) := \{(w, v) \in NW \mid \|v\| < t(w)\} \subset NW$$

is an embedding. There is a map

$$\Psi(M)^{\nu} \times M \longrightarrow \mathcal{S}(TM)$$

given by

$$(W, t, p) \longmapsto \begin{cases} \infty & \text{if } p \notin \exp(\nu_t(W)), \\ (D(\exp|_{T_w M}))(T_w W, v) \subset T_p M & \text{if } p = \exp(w, v) \\ & \text{for } (w, v) \in \nu_t(W), \end{cases}$$

where we consider the oriented 2-plane $T_w W$ and the vector v as lying inside $T_v(T_w M)$ using the canonical isomorphism $T_v(T_w M) \cong T_w M$, and then apply the linear isomorphism $D(\exp|_{T_w M})\colon T_v(T_w M) \longrightarrow T_p M$. The adjoint to this map,

$$s\colon \Psi^{\nu}(M) \longrightarrow \Gamma(\mathcal{S}(TM) \to M),$$

is called the *scanning map*.

Because the inclusion (3.1.1) is $O(n)$-equivariant, it follows that the inclusion

$$i\colon \Gamma(\mathcal{S}(TM) \to M) \longrightarrow \Gamma(\Psi(TM) \to M)$$

is a weak homotopy equivalence. The projection $\pi\colon \Psi^{\nu}(M) \to \Psi(M)$ is also a weak homotopy equivalence, by Lemma 1.4.1. The following proposition shows that the scanning map is also a weak homotopy equivalence if M has no compact components.

Proposition 3.1.3. *The square*

$$\begin{tikzcd} \Gamma(\mathcal{S}(TM) \to M) \arrow[r] & \Gamma(\Psi(TM) \to M) \\ \Psi^{\nu}(M) \arrow[u, "s"] \arrow[r, "\pi"] & \Psi(M) \arrow[u, "s_a"'] \end{tikzcd}$$

commutes up to homotopy, so if M has no compact components then

$$s\colon \Psi^{\nu}(M) \longrightarrow \Gamma(\mathcal{S}(TM) \to M)$$

is a weak homotopy equivalence.

Proof. For $(V, \langle -, - \rangle)$ an inner product space, let us define an auxiliary subspace

$$\widetilde{\Psi}(V) = \{W \in \Psi(V) \mid W \text{ is empty or has a unique closest point to the origin}\}$$

of $\Psi(V)$, which is again natural in $(V, \langle -, - \rangle)$. This subspace contains $\mathcal{S}(V)$, and we wish to show that the inclusion $\mathcal{S}(V) \hookrightarrow \widetilde{\Psi}(V)$ is a homotopy equivalence. Let $\widetilde{\Psi}(V)_0$ be the subspace consisting of the non-empty manifolds. There is a continuous function

$$c\colon \widetilde{\Psi}(V)_0 \longrightarrow V$$

which picks out the unique closest point to the origin. This function extends to a continuous function $c^+\colon \widetilde{\Psi}(V) \to V^+$ by sending the empty manifold to the point at infinity.

We define a homotopy $H_V\colon [1, \infty] \times \widetilde{\Psi}(V) \to \widetilde{\Psi}(V)$ by the rule

$$(t, W) \longmapsto \begin{cases} c(W) + t \cdot (W - c(W)) & \text{if } t < \infty \text{ and } W \text{ is non-empty,} \\ c(W) + T_{c(W)}W & \text{if } t = \infty \text{ and } W \text{ is non-empty,} \\ \emptyset & \text{if } W \text{ is empty,} \end{cases}$$

which is easily checked to be continuous.

The above discussion is completely natural in the inner product vector space $(V, \langle -, - \rangle)$, so we may apply it fibrewise to any vector bundle with metric. In particular, the homotopies H_{T_pM} fit together to give a deformation retraction H of $\widetilde{\Psi}(TM)$ onto $\mathcal{S}(M)$. The map $s_a \circ \pi$ lands in $\Gamma(\widetilde{\Psi}(TM) \to M)$, and applying the deformation retraction H gives a homotopic map $H(\infty, -) \circ s_a \circ \pi$, which by inspection is equal to the map s. $\square$

Scanning maps with boundary conditions

We will also often need scanning maps when M has a boundary, and surfaces are required to satisfy a boundary condition, as in Section 1.2. We formalise this as follows.

Let M be a manifold with boundary, $c\colon (-1, 0] \times \partial M \to M$ be a collar, and $\xi \subset \partial M$ be a compact oriented 1-manifold. Write $M(\infty) = M \cup_{\partial M} ([0, \infty) \times \partial M)$ for the manifold obtained by attaching an infinite collar to M. Let also

$$\Psi(M; \xi) := \{W \in \Psi(M(\infty)) \mid W \cap ((-1, \infty) \times \partial M) = (-1, \infty) \times \xi\}.$$

By choosing a Riemannian metric $\mathfrak{g}$ on $M(\infty)$ (which is a product on $(-1, \infty) \times \partial M$) and a function a as above, we obtain a scanning map s_a for $M(\infty)$. If the function a is chosen so that

$$\exp_a(TM(\infty)|_{[0,\infty) \times \partial M}) \subset (-1, \infty) \times \partial M,$$

then for $W \in \Psi(M;\xi)$ the section $s_a(W)$ is a product when restricted to $[0,\infty) \times \partial M$, and is independent of W. By a slight abuse of notation, we call this product section $s_a([0,\infty)\times\xi)$, and write $\Gamma(\Psi(TM) \to M; s_a(\xi))$ for the space of sections of $\Psi(TM) \to M$ which agree with $s_a([0,\infty)\times\xi)|_{\partial M}$ over ∂M. In this case there is a scanning map

$$s_a \colon \Psi(M;\xi) \longrightarrow \Gamma(\Psi(TM) \to M; s_a(\xi))$$

by construction. As in Proposition 3.1.2, if M has no compact components then this scanning map is a weak homotopy equivalence.

Adding tails to M

Suppose that M is a *compact* manifold with collared boundary, and let $N, L \subset \partial M$ be open codimension 0 submanifolds, with L diffeomorphic to a ball.

Definition 3.1.4. We define the following subspaces of $\partial M \times [0,\infty)$:

$$N_{[a,b]} := N \times [a,b], \qquad L_{[a,b]} := L \times [a,b],$$

and we also write $N_{[0,\infty)} = N \times [0,\infty)$ and $N_a = N_{[a,a]}$, and similarly for L.

We then write

$$M_{a,b} := M \cup N_{[0,a]} \cup L_{[0,b]},$$

and let $M_{a,\infty}$, $M_{\infty,b}$ or $M_{\infty,\infty}$ have their obvious meaning.

Note that the boundary component $N_a \subset M_{a,b}$ has a canonical collar inside $((-1,0] \times \partial M) \cup_{\partial M} (N \times [0,\infty))$; similarly for the boundary component $L_b \subset M_{a,b}$.

For $\delta \subset N$ and $\xi \subset L$ compact oriented 1-manifolds, we define

$$\Psi(M_{a,b};\delta,\xi) := \Psi(M_{a,b};\delta\cup\xi) \subset \Psi(M_{\infty,\infty}),$$

as in Section 3.1. A careful examination of the topology of $\Psi(M_{\infty,\infty})$ shows that $\Psi(M_{a,b};\delta,\xi)$ is homeomorphic to the disjoint union $\coprod_{[\Sigma]} \mathcal{E}(\Sigma, M_{a,b};\delta\cup\xi)$ where $[\Sigma]$ runs along a set of compact oriented surfaces with boundary diffeomorphic to $\delta\cup\xi$, one in each diffeomorphism class.

Restricting the scanning map

$$s_a \colon \Psi(M_{a,b};\delta,\xi) \longrightarrow \Gamma_c(\mathcal{S}(TM_{a,b}) \to M_{a,b}; s_a(\delta), s_a(\xi))$$

to the subspace of connected genus g surfaces gives a map

$$\mathcal{S}_{g,c}(\delta,\xi) \colon \mathcal{E}_{g,c}(M_{a,b};\delta,\xi) \longrightarrow \Gamma_c(\mathcal{S}(TM_{a,b}) \to M_{a,b}; s_a(\delta), s_a(\xi)). \qquad (3.1.2)$$

Semi-simplicial models

In order to show that the map (3.1.2) induces an isomorphism in homology in a range of degrees, we will pass through certain auxiliary semi-simplicial spaces.

Definition 3.1.5. For $\xi \subset L_b$ a boundary condition, let $D(M_{\infty,b};\xi)_p$ be the set of tuples $(a_0, a_1, \dots, a_p, W)$ where

(i) $0 < a_0 < a_1 < \cdots < a_p$ are real numbers;

(ii) $W \in \Psi(M_{\infty,b};\xi)$ is a surface satisfying the boundary condition ξ, and the a_i are regular values for the projection $p_W \colon W \cap N_{[0,\infty)} \to [0,\infty)$.

We give it the subspace topology from $(\mathbb{R}^\delta)^{p+1} \times \Psi(M_{\infty,b};\xi)$. The collection of all the spaces $D(M_{\infty,b};\xi)_p$ for $p \geq 0$ forms a semi-simplicial space, where the jth face map is given by forgetting a_j, and it is augmented over $\Psi(M_{\infty,b};\xi)$.

Definition 3.1.6. Let $D(N_{(0,\infty)})_p$ be the set of tuples $(a_0, a_1, \dots, a_p, W)$ where

(i) $0 < a_0 < a_1 < \cdots < a_p$ are real numbers;

(ii) $W \in \Psi(N_{(0,\infty)})$ and the a_i are regular values for the projection $p_W \colon W \to [0,\infty)$.

We topologise this as a subspace of $(\mathbb{R}^\delta)^{p+1} \times \Psi(N_{(0,\infty)})$. The collection of all these spaces forms a semi-simplicial space where the jth face map forgets a_j. It is *not* augmented.

Let $\widehat{D}(N_{(0,\infty)})_p$ be the quotient space of $D(N_{(0,\infty)})_p$ by the relation

$$(a_0, a_1, \dots, a_p, W) \sim (a_0', a_1', \dots, a_p', W')$$

if $a_j = a_j'$ for all j and $p_W^{-1}([a_0,\infty)) = p_{W'}^{-1}([a_0,\infty))$. These form again a semi-simplicial space by forgetting the a_j.

There is a semi-simplicial map

$$\pi \colon D(M_{\infty,b};\xi)_\bullet \longrightarrow \widehat{D}(N_{(0,\infty)})_\bullet$$

given by sending a tuple $(a_0, a_1, \dots, a_p, W)$ to $[a_0, a_1, \dots, a_p, W \cap N_{(0,\infty)}]$, which factors through the quotient map $r \colon D(N_{(0,\infty)})_\bullet \to \widehat{D}(N_{(0,\infty)})_\bullet$. In addition to these semi-simplicial spaces, we require another pair with stricter requirements.

Definition 3.1.7. Let $D_\partial(M_{\infty,b};\xi)_\bullet \subset D(M_{\infty,b};\xi)_\bullet$ be the sub-semi-simplicial space where, in addition,

(i) $W \cap M_{a_0,b}$ is connected, and

(ii) each pair $(p_W^{-1}[a_i, a_{i+1}], p_W^{-1}(a_i))$ is connected.

Similarly, let $\widehat{D}_\partial(N_{(0,\infty)})_\bullet \subset \widehat{D}(N_{(0,\infty)})_\bullet$ be the sub-semi-simplicial space where, in addition, each pair $(p_W^{-1}[a_i, a_{i+1}], p_W^{-1}(a_i))$ is connected. As before, there is a semi-simplicial map $\pi_\partial\colon D_\partial(M_{\infty,b};\xi)_\bullet \to \widehat{D}_\partial(N_{(0,\infty)})_\bullet$ given by restriction.

If $\Sigma \subset L_{[b,c]}$ is a surface satisfying the boundary condition $\xi \subset L_b$ and the boundary condition $\xi' \subset L_c$, we obtain a semi-simplicial map

$$- \cup \Sigma\colon D(M_{\infty,b};\xi)_\bullet \longrightarrow D(M_{\infty,c};\xi')_\bullet$$

over π, and if $(\Sigma, \Sigma \cap L_b)$ is connected then we also obtain a semi-simplicial map

$$- \cup \Sigma\colon D_\partial(M_{\infty,b};\xi)_\bullet \longrightarrow D_\partial(M_{\infty,c};\xi')_\bullet$$

over π_∂.

Proof of Theorem C when $\partial M \neq \emptyset$

Let us choose once and for all a surface $\Sigma \subset L \times [0,3]$ which satisfies the boundary condition $\xi \subset L$ at both ends (with respect to the obvious collars), is connected, and has positive genus. We define

$$D(M_{\infty,\infty};\xi)_\bullet := \operatorname*{colim}_{b\to\infty} D(M_{\infty,b};\xi)_\bullet,$$

where the colimit is formed using the maps

$$- \cup \Sigma\colon D(M_{\infty,b};\xi)_\bullet \longrightarrow D(M_{\infty,b+3};\xi)_\bullet.$$

We define $D_\partial(M_{\infty,\infty};\xi)_\bullet$ in the same way. Similarly, we define

$$\Psi(M_{\infty,\infty};\xi) := \operatorname*{colim}_{b\to\infty} \Psi(M_{\infty,b};\xi),$$

where the maps in the colimit are again given by union with Σ.

There is a commutative diagram

$$\begin{tikzcd}
D_\partial(M_{\infty,\infty};\xi)_\bullet \arrow[r] \arrow[d, "\pi_\partial"] & D(M_{\infty,\infty};\xi)_\bullet \arrow[r, equal] \arrow[d, "\pi"] & D(M_{\infty,\infty};\xi)_\bullet \arrow[r, "\epsilon_\bullet"] \arrow[d] & \Psi(M_{\infty,\infty};\xi) \arrow[d] \\
\widehat{D}_\partial(N_{(0,\infty)})_\bullet \arrow[r] & \widehat{D}(N_{(0,\infty)})_\bullet & D(N_{(0,\infty)})_\bullet \arrow[l, "r_\bullet"'] \arrow[r, "\epsilon_\bullet"] & \Psi(N_{(0,\infty)})
\end{tikzcd} \tag{3.1.3}$$

which we will use to compare the leftmost and rightmost vertical maps after geometric realisation. The first step in doing so is the following.

Lemma 3.1.8. *The map $r_\bullet$ is a levelwise weak homotopy equivalence, and the two augmentation maps labelled $\epsilon_\bullet$ are weak homotopy equivalences after geometric realisation.*

Proof. The map $r_\bullet$ can be treated with the techniques of [GRW10, Theorem 3.9], and the two augmentation maps can be treated with the techniques of [GRW10, Theorem 3.10]. □

The second step in comparing the leftmost and rightmost vertical maps of (3.1.3) is to show that the unlabelled horizontal maps are weak homotopy equivalences after geometric realisation. This is much more complicated and it is deferred to Section 3.2, although we state the result here.

Proposition 3.1.9. *The maps*

$$|D_\partial(M_{\infty,\infty};\xi)_\bullet| \longrightarrow |D(M_{\infty,\infty};\xi)_\bullet| \quad \textit{and} \quad |\widehat{D}_\partial(N_{(0,\infty)})_\bullet| \longrightarrow |\widehat{D}(N_{(0,\infty)})_\bullet|$$

are weak homotopy equivalences.

Before moving on to the proof of this proposition, let us show how we will apply it. We choose a Riemannian metric $\mathfrak{g}$ on $M_{\infty,\infty}$, an $a_0 \in (0,\infty)$, and a function $a\colon M_{\infty,\infty} \to (0,\infty)$ bounded above by the injectivity radius, and so that $\exp_a(TM_{\infty,\infty}|_{N_{[a_0,\infty)}}) \subset N_{(0,\infty)}$. The non-affine scanning map gives the following commutative diagram:

$$\begin{array}{ccc} \Psi(M_{\infty,b};\xi) & \xrightarrow{s_a} & \Gamma(\Psi(TM_{\infty,b}) \to M_{\infty,b}; s_a(\xi)) \\ \big\downarrow & & \big\downarrow \pi_{a_0,b} \\ \Psi(N_{(0,\infty)}) & \xrightarrow{s_a} & \Gamma(\Psi(TN_{[a_0,\infty)}) \to N_{[a_0,\infty)}), \end{array} \tag{3.1.4}$$

where both vertical maps are given by restriction. By Proposition 3.1.2, the two non-affine scanning maps are weak homotopy equivalences. (For the lower one, we must use that the restriction map

$$\rho\colon \Gamma(\Psi(TN_{(0,\infty)}) \to N_{(0,\infty)}) \longrightarrow \Gamma(\Psi(TN_{[a_0,\infty)}) \to N_{[a_0,\infty)})$$

is an equivalence, and that if we choose a different function a' bounded above by the injectivity radius of $\mathfrak{g}|_{N_{(0,\infty)}}$, then the functions s_a and $\rho \circ s_{a'}$ are homotopic.)

Finally, as $N_{[a_0,\infty)} \hookrightarrow M_{\infty,b}$ is a cofibration, the rightmost vertical map is a fibration, so its homotopy fibre over a section f is equivalent to

$$\Gamma(\Psi(TM_{a_0,b}) \to M_{0,b}; f|_{N_{a_0}}, s_a(\xi)).$$

The following group completion argument lets us understand the homotopy fibre of $|\pi_\partial|$. Recall that a map $f\colon X \to Y$ is a *homology fibration* if for each point $y \in Y$, the natural map $\mathrm{fib}(y) \to \mathrm{hofib}(y)$ to the homotopy fibre (cf. page 36) is a homology equivalence, that is, induces isomorphisms in homology groups.

Proposition 3.1.10. *If $x = (a_0, W) \in \widehat{D}_\partial(N_{(0,\infty)})_0$ then the fibre of $|\pi_\partial|$ over x is*

$$F(a_0, W) := \operatorname*{colim}_{b\to\infty}\left(\coprod_{g\geq 0}\mathcal{E}_{g,c}(M_{a_0,b}; p_W^{-1}(a_0), \xi)\right),$$

where the colimit is formed by $-\cup\Sigma$, and c denotes the number of components of $p_W^{-1}(a_0)\cup\xi$. Furthermore, the map $|\pi_\partial|$ is a homology fibration.

Proof. Identifying the fibre is elementary. To show that $|\pi_\partial|$ is a homology fibration we wish to apply [MS76, Proposition 4]. To do this, we observe that $(\pi_\partial)_p$ is a fibration, and that its fibre over $[a_0, a_1, \dots, a_p, W]$ is $F(a_0, W)$. Thus face maps d_i for $i > 0$ induce homeomorphisms on fibres, but the face map d_0 induces a map

$$\operatorname*{colim}_{b\to\infty}\left(\coprod_{g\geq 0}\mathcal{E}_{g,c}(M_{a_0,b}; p_W^{-1}(a_0), \xi)\right) \longrightarrow \operatorname*{colim}_{b\to\infty}\left(\coprod_{g\geq 0}\mathcal{E}_{g,c'}(M_{a_1,b}; p_W^{-1}(a_1), \xi)\right)$$

on fibres, given by union with the cobordism $p_W^{-1}([a_0, a_1])$. As this cobordism is connected relative to $p_W^{-1}(a_0)$, union with it may be expressed as a composition of maps of type α, β and γ, so by Theorem B the induced map on homology is an isomorphism. □

In all, taking geometric realisation and the colimit of diagrams (3.1.3) and (3.1.4) over stabilisation of the top row by $-\cup\Sigma$, we obtain a diagram where all horizontal maps are homotopy equivalences. A choice of point $(a_0, W) \in D(N_{(0,\infty)})_0$ such that $p_W^{-1}(a_0) = \emptyset$ gives a compatible collection of basepoints in all the spaces on the bottom row, and we obtain a zig-zag of weak homotopy equivalences between the homotopy fibres of all the vertical maps, taken at this compatible collection of basepoints. In particular, we obtain a zig-zag of homology equivalences between the actual fibres of of $|\pi_\partial|$ and Π_∞,

$$\operatorname*{colim}_{b\to\infty}\left(\coprod_{g\geq 0}\mathcal{E}_{g,c}(M_{a_0,b}; \emptyset, \xi)\right) \tag{3.1.5}$$

and

$$\operatorname*{colim}_{b\to\infty}\left(\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))\right). \tag{3.1.6}$$

Lemma 3.1.11. *The stabilisation maps between the spaces of sections*

$$\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))$$

are homotopy equivalences.

Proof. The stabilisation map is given by union with the section

$$s_a(\Sigma) \in \Gamma_c(\Psi(T(L \times [0,1])) \to L \times [0,1]; s_a(\xi), s_a(\xi)) =: X$$

obtained by scanning the surface Σ. The space X is a homotopy associative H-space, by concatenating intervals and reparametrising. As L was chosen to be diffeomorphic to $\mathbb{R}^{d-1}$, we may choose such a diffeomorphism; this identifies X with

$$\mathrm{map}_c(\mathbb{R}^{d-1} \times [0,1], \Psi(\mathbb{R}^d); s_a(\xi), s_a(\xi)) \simeq \Omega_{s_a(\xi)}(\Omega^{d-1}\Psi(\mathbb{R}^d))$$

as an H-group. In particular, $\pi_0(X)$ is a group. Thus there is a section f such that $s_a(\Sigma) \cdot f$ is homotopic to the constant section $s_a(\xi) \times [0,1]$, but then union with the section f gives a homotopy inverse to the stabilisation map. □

Corollary 3.1.12. *There is a bijection*

$$\pi_0(\Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))) \cong \mathbb{Z} \times H_2(M; \mathbb{Z}).$$

Proof. The set of path components of (3.1.5) is isomorphic to $\mathbb{Z} \times H_2(M;\mathbb{Z})$, by Lemma 2.3.4. □

In Section 3.3 we give a concrete description of this bijection. Combining the homology equivalence between (3.1.5) and (3.1.6), Lemma 3.1.11, and Theorem B, we see that the scanning map

$$\mathcal{E}_{g,c}(M_{a_0,b}; \emptyset, \xi) \longrightarrow \Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))$$

is a homology isomorphism in degrees $\leq \frac{2}{3}(g-1)$. (We have used the fact that L is contractible, so when we write $\Sigma \subset L \times [0,3]$ as the composition of α and β maps, the β maps are always gluing on a pair of pants with nullhomotopic outgoing boundary, so Theorem B (ii) gives a stability range $\leq \frac{2}{3}g$ for gluing β maps.)

Extending surfaces and sections cylindrically from M to $M_{a_0,b}$ gives a commutative square

$$\begin{tikzcd}
\mathcal{E}_{g,c}(M;\xi) \arrow[r] \arrow[d] & \Gamma(\Psi(TM) \to M; s_a(\xi)) \arrow[d] \\
\mathcal{E}_{g,c}(M_{a_0,b}; \emptyset, \xi) \arrow[r] & \Gamma(\Psi(TM_{a_0,b}) \to M_{a_0,b}; s_a(\emptyset), s_a(\xi))
\end{tikzcd}$$

where the vertical maps are clearly homotopy equivalences; this proves the first part of Theorem C. The second part of Theorem C, *in the case where the manifold* M *has non-empty boundary*, follows from the commutative square

$$\begin{tikzcd}
\mathcal{E}_{g,1}(M;\xi) \arrow[r] \arrow[d, "\gamma_{g,1}"] & \Gamma(\Psi(TM) \to M; s_a(\xi)) \arrow[d] \\
\mathcal{E}_{g}(M_1) \arrow[r] & \Gamma(\Psi(TM_1) \to M_1; s_a(\emptyset))
\end{tikzcd}$$

where $\xi \subset \partial M$ is a single nullhomotopic circle, $\gamma_{g,1}$ is the map that glues on a collar $[0,1] \times \partial M$ containing a disc, and the right-hand map is given by union with the section obtained by scanning the disc. The right-hand map is an equivalence by an argument analogous to that of Lemma 3.1.11, and the left-hand map is an isomorphism in homology in degrees $\leq \frac{2}{3}g$ by Theorem B. This finishes the proof of Theorem C in the case where the manifold M has non-empty boundary. In Section 3.3 we show how to deduce Theorem C in the case where M has empty boundary.

3.2 Surgery on semi-simplicial spaces

In this section we prove Proposition 3.1.9, following the methods of [GMTW09] and [GRW12]. There is an improvement in the way we deal with the complex of surgery data in the sense that the maps used are always simplicial (cf. [GRW12, Section 6]). We will prove in detail that the map

$$|D_\partial(M_{\infty,\infty};\xi)_\bullet| \longrightarrow |D(M_{\infty,\infty};\xi)_\bullet| \tag{3.2.1}$$

is a weak homotopy equivalence, and then briefly explain the changes in the argument to show that

$$|\widehat{D}_\partial(N_{(0,\infty)})_\bullet| \longrightarrow |\widehat{D}(N_{(0,\infty)})_\bullet| \tag{3.2.2}$$

is a weak homotopy equivalence. We first introduce two more auxiliary semi-simplicial spaces.

Definition 3.2.1. Define a semi-simplicial space $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ whose space of i-simplices is the space of tuples $(W, a_0, \ldots, a_i)$ such that

(i) $0 < a_0 < a_1 < \cdots < a_i \in \mathbb{R}$;

(ii) $W \in \Psi(M_{\infty,b};\xi)$;

(iii) each a_j is either a regular value of $p_W \colon W \cap N_{[0,\infty)} \to [0,\infty)$, or $p_W^{-1}(a_j)$ contains only Morse critical points of index at least 1. We denote $\delta_j = p_W^{-1}(a_j)$;

(iv) for each j, the map $\pi_0(\delta_j) \to \pi_0(W \cap N_{[a_j, a_{j+1})})$ induced by the inclusion is a surjection;

(v) $W \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]})$ is path connected.

Similarly, we let $D^\natural(M_{\infty,b};\xi)_\bullet$ have as i-simplices those tuples $(W, a_0, \dots, a_i)$ which satisfy just the first three conditions above. In both cases, the simplices are topologised as a subspace of $(\mathbb{R}^\delta)^{i+1} \times \Psi(M_{\infty,b};\xi)$ and the face maps are given by forgetting the a_j.

Lemma 3.2.2. *The inclusions*

$$|D_\partial(M_{\infty,b};\xi)_\bullet| \longrightarrow |D^\natural_\partial(M_{\infty,b};\xi)_\bullet| \quad \text{and} \quad |D(M_{\infty,b};\xi)_\bullet| \longrightarrow |D^\natural(M_{\infty,b};\xi)_\bullet|$$

are weak homotopy equivalances.

Proof. The argument is the same in both cases; to be specific, we treat the first case. Let $\mathcal{J}_{\bullet,\bullet}$ be the bi-semi-simplicial space whose (i,j)-simplices consist of the tuples $(W, a_0, \dots, a_i, b_0, \dots, b_j)$ such that $(W, a_0, \dots, a_i)$ is an i-simplex in $D_\partial(M_{\infty,b};\xi)_\bullet$ and $(W, a_0, \dots, a_i, b_0, \dots, b_j)$ is an $(i+j+1)$-simplex in $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$.

The $(p,\bullet)$-face map forgets the value a_p and the $(\bullet,q)$-face map forgets the value b_q. It has an augmentation $\epsilon_{-,\bullet}$ to $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ given by forgetting all the values $a_0, \dots, a_i$ and an augmentation $\epsilon_{\bullet,-}$ to $D_\partial(M_{\infty,b};\xi)_\bullet$ given by forgetting all the values $b_0, \dots, b_i$. The triangle

$$\begin{tikzcd}
 & {|\mathcal{J}_{\bullet,\bullet}|} \arrow[dl, "|\epsilon_{\bullet,-}|"'] \arrow[dr, "|\epsilon_{-,\bullet}|"] & \\
{|D_\partial(M_{\infty,b};\xi)_\bullet|} \arrow[rr] & & {|D^\natural_\partial(M_{\infty,b};\xi)_\bullet|}
\end{tikzcd}$$

commutes up to homotopy, by construction.

The augmentation maps have local sections. We try to define a section of $\epsilon_{i,-} \colon \mathcal{J}_{i,0} \to D_\partial(M_{\infty,b};\xi)_i$ through the point $(W, a_0, \dots, a_i, b_0)$ on the open neighbourhood U of $(W, a_0, \dots, a_i)$ consisting of those W' such that $a_0, \dots, a_i$ are still regular values and b_0 contains only Morse critical points of index at least 1, by the formula $(W', a_0, \dots, a_i) \mapsto (W', a_0, \dots, a_i, b_0)$. To see that this defines a section, we must check that $p_{W'}^{-1}([a_j, a_{j+1}))$ and $p_{W'}^{-1}([a_i, b_0))$ all satisfy the connectivity requirement (iv). The first case is immediate: as the a_j

remain regular values, $p_{W'}^{-1}([a_j, a_{j+1})) \cong p_W^{-1}([a_j, a_{j+1}))$ and

$$W \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]}) \cong W' \cap (M \cup N_{[0,a_0)} \cup L_{[0,b]}).$$

In the second case, $p_{W'}^{-1}([a_i, b_0))$ differs from $p_W^{-1}([a_i, b_0))$ by adding 1- or 2-handles, but this does not change the connectivity property with respect to the lower boundary. We show that the augmentation map $\epsilon_{-,j}$ has local sections in a similar (but easier) way.

The fibre $F_\bullet$ of $\epsilon_{\bullet,-}$ over $(W, a_0, \ldots, a_i)$ has p-simplices those tuples of real numbers $(b_0, \ldots, b_p)$ such that $(W, a_0, \ldots, a_i, b_0, \ldots, b_p)$ is a simplex of $D_\partial^\natural(M_{\infty,b}; \xi)_\bullet$, i.e., $p_W^{-1}(b_j)$ contains only Morse critical points of index at least 1, and $p_W^{-1}([a_i, b_0))$ and each $p_W^{-1}([b_j, b_{j+1}))$ are connected relative to its lower boundary. These conditions only involve pairs of b_j's, so this is a topological flag complex (whose topology is discrete). Given a finite collection $b_1, \ldots, b_n$ of elements of F_0, we may choose $a_i < c < \min(b_j)$ such that $[a_i, c]$ consists of regular values of p_W. Then c is also in F_0, and $(c, b_j) \in F_1$ for each b_j. It follows from Criterion 1.6.2 (and Remark 1.6.3) that $|\epsilon_{\bullet,-}|$ is a weak homotopy equivalence.

The fibre $F'_\bullet$ of $\epsilon_{-,\bullet}$ over $(W, b_0, \ldots, b_j)$ has p-simplices those tuples of real numbers $(a_0, \ldots, a_p)$ which are regular values of p_W, such that

$$(W, a_0, \ldots, a_p, b_0, \ldots, b_j)$$

is a simplex of $D_\partial^\natural(M_{\infty,b}; \xi)_\bullet$, which is again seen to be a topological flag complex (whose topology is discrete). For a finite collection $a_1, \ldots, a_n$ of elements of F'_0, choose $\max(a_j) < c < b_0$ such that $[c, b_0)$ consists of regular values of p_W. Then c is also in F'_0, and we claim that each (a_j, c) is a 1-simplex of $F'_\bullet$, i.e., that $p_W^{-1}([a_j, c))$ is path connected relative to $p_W^{-1}(a_j)$. To see this, first note that $p_W^{-1}([a_j, b_0))$ is path connected relative to $p_W^{-1}(a_j)$ by assumption, so there is a path from any point of $p_W^{-1}([a_j, c))$ to $p_W^{-1}(a_j)$ inside of $p_W^{-1}([a_j, b_0))$, but as $[c, b_0)$ consists of regular values $p_W^{-1}([c, b_0))$ is a cylinder, so this path may be homotoped into $p_W^{-1}([a_j, c))$ relative to its ends. It follows from Criterion 1.6.2 that $|\epsilon_{-,\bullet}|$ is a weak homotopy equivalence. □

Local surgery move

Let $w = (W, a_0, \ldots, a_i)$ be a simplex in $D^\natural(M_{\infty,b}; \xi)_\bullet$. We first construct a path from this i-simplex to a i-simplex $w' = (W', a_0, \ldots, a_i)$ in $D_\partial^\natural(M_{\infty,b}; \xi)_\bullet$. In particular, this will prove that the inclusion $D_\partial^\natural(M_{\infty,b}; \xi)_\bullet \to D^\natural(M_{\infty,b}; \xi)_\bullet$ is levelwise 0-connected. In the last section we use this path to show that it is in fact a homotopy equivalence after geometric realisation.

Let $R(w) = \{W_1, \dots, W_k\}$ be the set of connected components of $W \cap M_{a_0,b}$, and let W_0 be the connected component that contains ξ. Define

$$P_{a,b}(W) = \{\omega \in \pi_0(p_W^{-1}[a,b)) \mid a \notin p_W(\omega)\}, \qquad P(w) = \bigcup_{k=0}^{i-1} P_{a_k,a_{k+1}}(W).$$

Observe that w *is in* $D_\partial^\natural(M;\xi)_\bullet$ *if and only if* $R(w) \cup P(w) = \emptyset$. We define the following subsets of $\mathbb{R}^3$:

$$\begin{aligned} T' &= (\{0\} \times [-3,3]) \cup ((0,5] \times \{0\}) \subset \mathbb{R}^2 \subset \mathbb{R}^3 \\ T &= \{(x,y,z) \in \mathbb{R}^3 \mid d(T',(x,y,z)) < 1,\ |x| \leq 3,\ y < 5\} \end{aligned}$$

and let $x_1, x_2 \colon T \to \mathbb{R}$ be the first and second coordinate functions.

Definition 3.2.3. Let $w = (W, a_0, \dots, a_i)$ be an i-simplex in $D^\natural(M;\xi)_\bullet$. A *local surgery datum for* w is a pair $Q = (\Lambda, e)$ where Λ is a set and $e \colon \Lambda \times T \to M_{\infty,b}$ is a closed embedding, whose restriction $e_{|\{\lambda\}\times T}$ we denote by e_λ, such that:

(i) $e^{-1}(W \cap M_{a_i,b}) = \Lambda \times (T \cap x_2^{-1}(\{-3,3\}))$;

(ii) $(\mathrm{Id}_\Lambda \times x_1)(e^{-1}(W \cap N_{[a_i,\infty)})) \subset \Lambda \times (4,5)$;

(iii) for each $\lambda \in \Lambda$, $e_\lambda(x_2^{-1}(-3)) \subset W_0 \cap M_{a_0,b}$;

(iv) for each $\omega \in P(w) \cup R(w)$, there is a $\lambda \in \Lambda$ such that $e_\lambda(x_2^{-1}(3)) \subset \omega$;

(v) $\lim_{x\to 5} e_\lambda(x,y,z) = \infty$ for all $\lambda \in \Lambda$ and all (y,z) such that $\sqrt{y^2+z^2} < 1$;

(vi) for each $\lambda \in \Lambda$ and for each $j = 0, \dots, i$ there is an $\epsilon > 0$ such that for all $a \in (a_j - \epsilon, a_j + \epsilon)$, either $x_2 e_\lambda^{-1}(N_{a_j}) \in (-2,-1)$ or $x_1 e_\lambda^{-1}(N_{a_j}) \in (2,3)$.

Proposition 3.2.4. *A local surgery datum* Q *for an* i*-simplex* w *of* $D^\natural(M;\xi)_\bullet$ *determines a path* $\Phi_Q(t)$ *that starts at* w *and ends in an* i*-simplex of* $D_\partial^\natural(M_{\infty,b};\xi)_\bullet$.

Proof. Consider the 1-parameter family of diffeomorphisms of

$$Y = T \cup \{(x,y,z) \in \mathbb{R}^3 \mid x \leq 5, \|(y,z)\| < 1\}$$

given by

$$h_t(x,y,z) = \begin{cases} (y, x + (x-3)te^{2-\frac{1}{1-\|(y,z)\|^2}}, z) & \text{if } \|(y,z)\| < 1, x \leq 3, \\ (x,y,z) & \text{otherwise.} \end{cases}$$

The properties of this family which we will use are the following:

(i) h_0 is the identity;

(ii) if $x \in (4,5)$ and $\|(y,z)\| < 1/\sqrt{2}$, then $x_1 h_1^{-1}(x,y,z) \in (3,4)$;

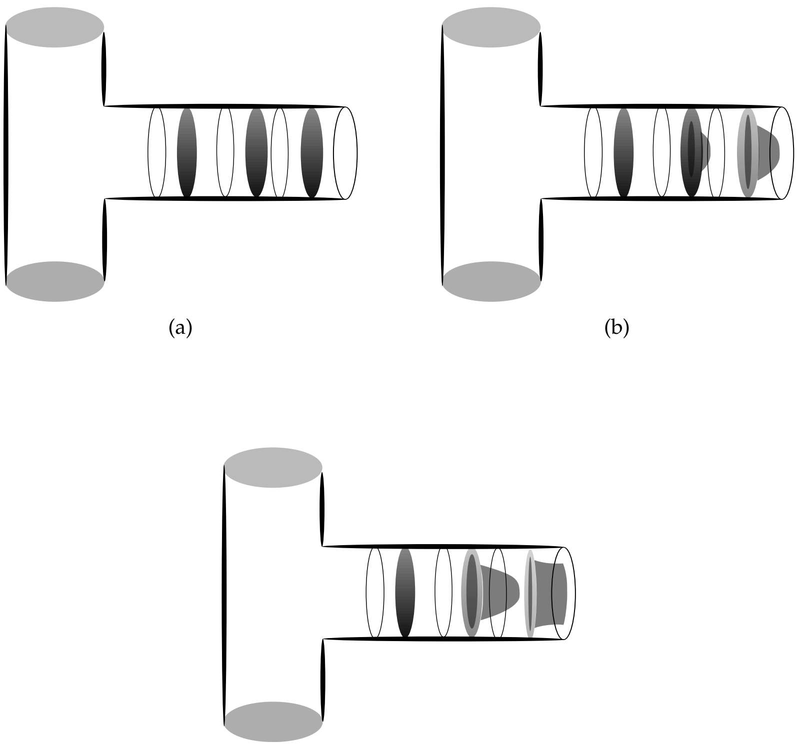
(iii) h_t is the identity on $T \cap x_1^{-1}([-\infty, 3))$;

(iv) h_t extends to $\mathbb{R}^3$ with the identity outside T.

The family h_t induces a 1-parameter family of maps

$$H_t \colon \Psi(T) \longrightarrow \Psi(T)$$

given by sending a submanifold W to $h_t(W) \cap T$. From the first property of h_t it follows that H_0 is the identity. In Figures 3.1b and 3.1c we give a picture of the action of H_t on the dark disc at the bottom of Figure 3.1a.



(a) (b) (c)

Figure 3.1: The effect of the family H_t in the surgery movement on discs in $x_1^{-1}((2,3))$, $x_1^{-1}((3,4))$ and $x_1^{-1}((4,5))$.

Consider now the path η in $\Psi(T)$ given in Figure 4.1 that starts with the surface $x_2^{-1}(\{-3,3\})$, which is the disjoint union of two open balls in T. It pushes both balls to infinity, joins the balls there and then pulls them backwards. In

Figure 3.2a, a picture at time 0 is given. The three vertical circles represent the balls $x_1^{-1}(2)$, $x_1^{-1}(3)$ and $x_1^{-1}(4)$ and the horizontal circle represents the ball $x_2^{-1}(-1)$. The planes in the figure will be given an interpretation later. In Figures 3.2b, 3.2c and 3.2d, the ball is pushed to infinity, and in Figures 3.2e and 3.2f the surface returns in the shape of a (non-compact) pair of pants. The main properties of this movement are the following:

(i) $\eta(0) = x_2^{-1}(\{-3,3\})$;

(ii) all the values in $(1,2)$ are regular values or Morse critical values of index 2 for the restriction of x_2 to $\eta(t)$;

(iii) all the values in $(2,3)$ are regular values for the restriction of x_1 to $\eta(t)$ or Morse critical values of Morse index 1 or 2 (the former possibility happens only in the step from 3.2e to 3.2f);

(iv) $\eta(t) \cap x_1^{-1}((4,5)) \subset \{(x,y,z) \in T \mid \|(y,z)\| < 1/\sqrt{2},\ y \in (4,5)\}$;

(v) in the surface $\eta(1)$, the circles $x_2^{-1}(\{-3,3\}) \cap \eta(1)$ are in the same connected component.

Now, let $V \in \Psi(T)$ be the union of the balls $V_0 = x_2^{-1}(\{-3,3\})$ and some surface $V_1 \subset \{(x,y,z) \in T \mid x \in (4,5)\}$. We define a path $\phi_V \colon I \to \Psi(T)$ as

$$\phi_V(t) = \begin{cases} H_{2t}(V) & \text{if } t \in [0,1/2], \\ H_1(V_1) \cup \eta(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

Property (iii) of h_t assures that both paths glue well: $H_1(V) = H_1(V_1) \cup V_0 = H_1(V_1) \cup \eta(0)$. Property (ii) for h_t and property (iv) for η assure that the union $H_1(V_1) \cup \eta(2t-1)$ is a union of disjoint surfaces, hence a surface. Hence the path is well-defined. We will use the following properties of this path:

(i) $\phi_V(0) = V$ by Property (i) of h_t;

(ii) $\phi_V(1) \cap x_2^{-1}(\{-3,3\})$ is connected, by property (v) of η.

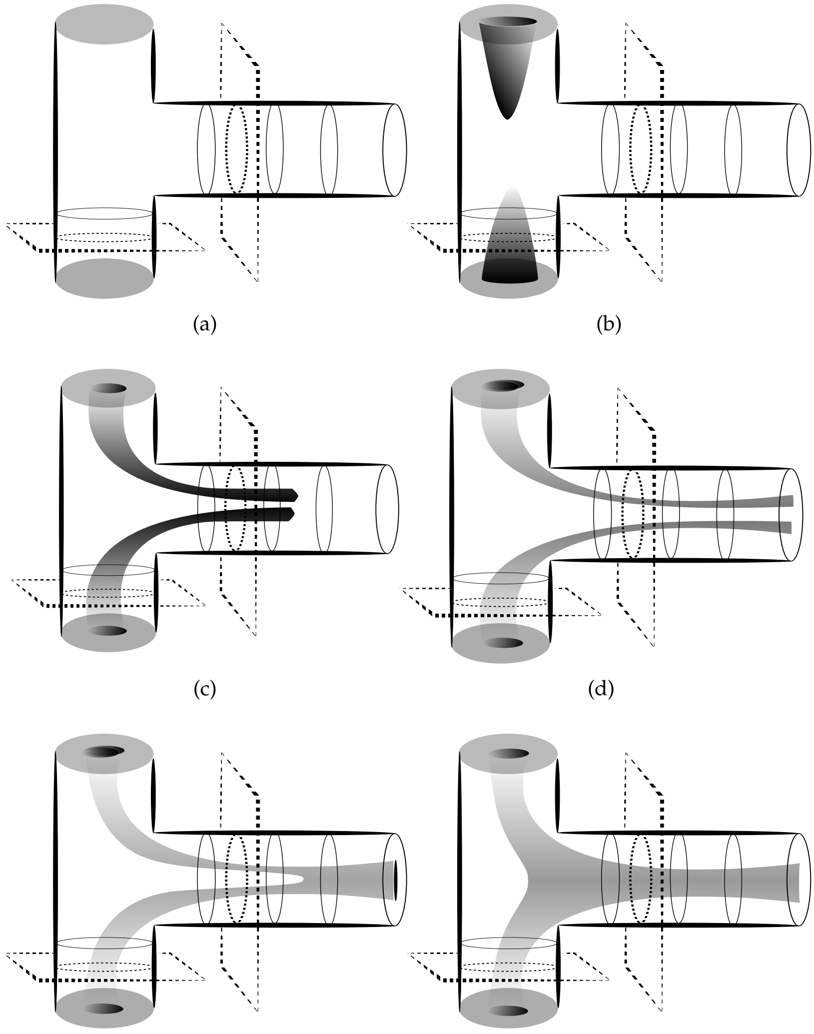
If we are given a set V_Λ of surfaces $V_\lambda \subset \Psi(\{\lambda\} \times T)$ indexed by λ, we denote by $\phi_{V_\Lambda}(t)$ the result of performing $\phi_{V_\lambda}(t)$ in each $\lambda \times T$.

Now suppose we are given a surgery datum Q for w, and let us define a path Φ_Q in $D^\natural(M_{\infty,b};\xi)_\bullet$ starting at $w = (W, a_0, \ldots, a_i)$ as

$$\Phi_Q(t) = (W_Q(t), a_0, \ldots, a_i),$$

where

$$W_Q(t) \cap e = e\phi_{e^{-1}(W)}(t), \qquad W_Q(t) \setminus e = W \setminus e.$$



(a) (b)

(c) (d)

(e) (f)

Figure 3.2: The path η in the surgery movement. The shadowed surface at the top of (3.2a) is $\gamma(t)$, starting with $\gamma(0) = x_2^{-1}(\{-3, 3\})$. The dotted planes are $e^{-1}(N_{a_j})$, and are still planes because of condition (vi) of the local surgery data.

There are five things to check for each $\lambda \in \Lambda$ in order to verify that this path is well-defined. First, that $e_\lambda^{-1}(W)$ is the union of V_0 and some surface V_1 as above is granted by conditions (i) and (ii) of the surgery data, hence $\phi_{e_\lambda^{-1}(W)}$ is well-defined. Second, that $\Phi_Q(0) = w$ follows from property (i) of ϕ_V. Third, that the union of the two pieces of $W_Q(t)$ is indeed a surface is guaranteed by Property (iv) of h_t. Fourth: as described, the embedding e_λ does not induce a map $\{\lambda\} \times \Psi(T) \to \Psi(M_{\infty,b})$. Condition (v) of the surgery data and properties (iii) and (iv) of h_t grant that the precomposition $I \to \Psi(T) \to \Psi(M_\infty, b)$ with $\phi_{e_\lambda^{-1}(W)}$ is continuous. In other words, they grant that the surface $W_Q(t) \subset M_{\infty,b}$ is closed in $M(\infty)$ and that $W \cap M_{a,b}$ is compact. Fifth, that $(a_0, \dots, a_i)$ are regular values or Morse critical points of index 1 or 2 is a consequence of properties (ii) and (iii) of the path η, together with the following consequences of conditions (v) and (vi) of the surgery data:

(i) If $x_2 e_\lambda^{-1}(a_j) \in (-2,-1)$, then $p^{-1}_{W_Q(t)}(a_j) = x_2^{-1}(b_j)$ for some $b_j \in (1,2)$.

(ii) If $x_1 e_\lambda^{-1}(a_j) \in (2,3)$, then $p^{-1}_{W_Q(t)}(a_j) = x_1^{-1}(b_j)$ for some $b_j \in (2,3)$.

(iii) $\frac{\partial}{\partial x_1} p_W e_\lambda(x,y,z) > 0$ if $y \in (-2,-1)$.

(iv) $\frac{\partial}{\partial x_2} p_W e_\lambda(x,y,z) < 0$ if $x \in (2,3)$.

Finally, from conditions (ii) and (iii) in the definition of surgery datum and property (ii) of ϕ_V, it follows that $P(\Phi_Q(1)) \cup R(\Phi_Q(1))$ is the empty set, hence $\Phi_Q(1) \in D^\natural(M_{\infty,b};\xi)_\bullet$. $\square$

Remark 3.2.5. This move is a simplified version of the one used in [GMTW09]. The one used there is more powerful and extends to surfaces with any tangential structure. Sadly, that move needs to push parts of the surface to both $+\infty$ and $-\infty$, while here we are only allowed to push things to $+\infty$.

Global surgery move

We will now construct a bi-semi-simplicial space $\mathcal{H}_{\bullet,\bullet}$ with an augmentation to $D^\natural(M_{\infty,b};\xi)_\bullet$ which, over each simplex of $D^\natural(M_{\infty,b};\xi)_\bullet$, consists of certain tuples of local surgery data. This will allow us to compare it to $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ by "doing surgery" in an appropriate way.

Definition 3.2.6. Let $\mathcal{H}_{\bullet,\bullet}$ be the bi-semi-simplicial space whose space of (i,j)-simplices is the space of tuples $(w, Q_0, \dots, Q_j, s_0, \dots, s_j)$ where

(i) w is an i-simplex in $D^\natural(M_{\infty,b};\xi)_\bullet$;

(ii) each Q_q is a local surgery datum for w;

(iii) the embeddings in $Q_0, \dots, Q_j$ are pairwise disjoint;

(iv) $(s_0, \dots, s_j) \in [0,1]^{j+1}$.

The $(p, \bullet)$th face map forgets the regular value $a_p \in w$ and the $(\bullet, q)$th face map is

$$\partial_{\bullet,q}(w, Q_0, \dots, Q_i, s_0, \dots, s_i) = \\ (\Phi_{Q_q}(s_q), Q_0, \dots, \hat{Q}_q, \dots, Q_i, s_0, \dots, \hat{s}_q, \dots, s_i).$$

There is an augmentation map $\epsilon_{\bullet,\bullet}$ to $D^\natural(M_{\infty,b};\xi)_\bullet$ given by performing the surgery Q_q on w up to time s_q for all q and forgetting all the surgery data. Let $\mathcal{H}^1_{\bullet,\bullet}$ be the bi-semi-simplicial subspace of those simplices such that $s_0 = \dots = s_j = 1$. Note that by Proposition 3.2.4 the restriction $\epsilon^1_{\bullet,\bullet}$ of $\epsilon_{\bullet,\bullet}$ to this subspace gives an augmentation onto $D^\natural_\partial(M_{\infty,b};\xi)_\bullet$ and the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}^1_{\bullet,\bullet} \arrow[r] \arrow[d, "\epsilon^1_{\bullet,\bullet}"] & \mathcal{H}_{\bullet,\bullet} \arrow[d, "\epsilon_{\bullet,\bullet}"] \\ D^\natural_\partial(M_{\infty,b};\xi)_\bullet \arrow[r] & D^\natural(M_{\infty,b};\xi)_\bullet. \end{tikzcd} \tag{3.2.3}$$

Proposition 3.2.7. *If M has dimension at least 4, the inclusion of $\mathcal{H}^1_{\bullet,\bullet}$ into $\mathcal{H}_{\bullet,\bullet}$ and the augmentation maps are weak homotopy equivalences after geometric realisation.*

The first part of Proposition 3.1.9 now follows from the commutative diagram

$$\begin{tikzcd} |D_\partial(M_{\infty,b};\xi)_\bullet| \arrow[r] \arrow[d] & |D(M_{\infty,b};\xi)_\bullet| \arrow[d] \\ |D^\natural_\partial(M_{\infty,b};\xi)_\bullet| \arrow[r] & |D^\natural(M_{\infty,b};\xi)_\bullet|, \end{tikzcd}$$

after taking the limit when $b \to \infty$, since the vertical maps are equivalences by Lemma 3.2.2 and the lower map is an equivalence by (3.2.3) and Proposition 3.2.7. As we remarked earlier, the second part of Proposition 3.1.9 is proved similarly.

Proof of Proposition 3.2.7. It is clear that the inclusion $\mathcal{H}^1_{\bullet,\bullet} \to \mathcal{H}_{\bullet,\bullet}$ is a levelwise equivalence. To see that the augmentation map $\epsilon^1_{\bullet,\bullet}$ is a homotopy equivalence after geometric realisation, we notice that the augmented semi-simplicial space $\epsilon^1_{i,\bullet} : \mathcal{H}^1_{i,\bullet} \to D^\natural_\partial(M_{\infty,b};\xi)_i$ has a simplicial contraction, by adding the empty surgery data.

For the map $\epsilon_{\bullet,\bullet}$, let $\mathcal{H}^0_{\bullet,\bullet}$ be the semi-simplicial subspace of $\mathcal{H}_{\bullet,\bullet}$ where the simplices are required to have all s_i equal to 0, and let $\mathcal{H}'_{\bullet,\bullet}$ be the semi-simplicial space defined as $\mathcal{H}^0_{\bullet,\bullet}$, but replacing Condition (iii) in the definition of $\mathcal{H}_{\bullet,\bullet}$ by

(iii′) The restrictions of the embeddings e_q in each $Q_q = (\Lambda_q, e_q)$ to the subspace $\Lambda \times T' \subset \Lambda \times T$ are pairwise disjoint.

Notice that $\mathcal{H}^0_{\bullet,\bullet} \subset \mathcal{H}'_{\bullet,\bullet}$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{H}'_{\bullet,\bullet} \arrow[dr, "\epsilon'_{\bullet,\bullet}"'] & \mathcal{H}^0_{\bullet,\bullet} \arrow[l] \arrow[r] \arrow[d, "\epsilon^0_{\bullet,\bullet}"] & \mathcal{H}_{\bullet,\bullet} \arrow[dl, "\epsilon_{\bullet,\bullet}"] \\
& D^\natural(M_{\infty,b};\xi)_\bullet. &
\end{tikzcd}$$

We next prove that the following statements are true, concluding that the augmentation map $\epsilon_{\bullet,\bullet}$ for $\mathcal{H}_{\bullet,\bullet}$ is a homotopy equivalence after geometric realisation, hence finishing the proof of this proposition.

(i) The inclusion of $\mathcal{H}^0_{\bullet,\bullet}$ into $\mathcal{H}_{\bullet,\bullet}$ is a levelwise homotopy equivalence.

(ii) The inclusion of $\mathcal{H}^0_{\bullet,\bullet}$ into $\mathcal{H}'_{\bullet,\bullet}$ is a levelwise homotopy equivalence.

(iii) The augmentation map $\epsilon'_{\bullet,\bullet}$ is a homotopy equivalence.

Statement (i) is clear. For statement (ii), we will prove that the inclusion $\mathcal{H}^0_{\bullet,\bullet} \to \mathcal{H}'_{\bullet,\bullet}$ is a levelwise weak homotopy equivalence. Consider the deformation $h\colon \mathcal{H}'_{i,j} \times (0,1] \to \mathcal{H}'_{i,j}$ that sends a tuple $(w, Q_0, \dots, Q_i)$ to the tuple $(w, h_t(Q_0), \dots, h_t(Q_i))$, where $h_t(Q_q) = (\Lambda_q, h_t(e_q))$ and $h_t(e_q)(x,y,z) = h_t(tx, y, tz)$. Under this deformation, any point eventually ends up, and stays, in the subspace $\mathcal{H}^0_{i,j}$. If $f\colon (D^n, S^{n-1}) \to (\mathcal{H}'_{i,j}, \mathcal{H}^0_{i,j})$ represents a relative homotopy class, then because D^n is compact the map $h(-,t)\circ f$ has image in $\mathcal{H}^0_{i,j}$ for some t, so the homotopy class of f is trivial.

For statement (iii), we notice that $\mathcal{H}'_{i,\bullet} \to D^\natural(M_{\infty,b};\xi)_i$ is an augmented topological flag complex, so we may apply Criterion 1.6.2 to show that it is a weak homotopy equivalence. Then $\epsilon'_{\bullet,\bullet}$ will be a levelwise equivalence in the i-direction, hence a weak homotopy equivalence after realization.

We will prove in Lemma 3.2.8 that the augmentation map is surjective and has local sections. Moreover, given $w \in D^\natural_\partial(M_{\infty,b};\xi)_i$ and a non-empty finite collection $(w, Q_0), \dots, (w, Q_j)$ of $(i,0)$-simplices over w, as the dimension of M is > 2, we can perturb the restriction $e_{0|\Lambda\times T'}$ of $e_0 \in Q_0$ to be disjoint from $Q_0, \dots, Q_j$, and any extension e_{j+1} to $\Lambda \times T$ of this perturbation will define a 0-simplex orthogonal to the given ones. □

Lemma 3.2.8. *The augmentation map* $\epsilon'_{i,0}\colon \mathcal{H}'_{i,0} \to D^\natural(M_{\infty,b};\xi)_i$ *is surjective and has local sections.*

Proof. First we show that $\epsilon'_{i,0}$ is surjective: if $w \in D^\natural_\partial(M_{\infty,b};\xi)_i$, let $\Lambda = P(w) \cup R(w)$. As M is connected, it is clear that we may take a smooth map $e'\colon \Lambda \times T' \to M_{\infty,b}$ satisfying the restriction of conditions (i), (ii), (iii), (iv) and (v) of the local surgery data to T', except that of being an embedding and that of being disjoint from W outside $e(x_1^{-1}((4,5)))$, $e(0,-3,0)$ and $e(0,3,0)$. As the dimension of M is at least 4, a small perturbation makes it satisfy the latter properties. Again, as the dimension is greater than 3, we may thicken the embedding e' to an embedding $e\colon \Lambda \times T \to M_{\infty,b}$ that satisfies all conditions except (vi), and we may deform e to satisfy this last condition.

Next, we show that $\epsilon'_{i,0}$ has local sections. Let $(w,(\Lambda,e)) \in \mathcal{H}'_{i,0}$. We need to find a neighbourhood U of w in $D^\natural(M_{\infty,b};\xi)_i$ and a section $s\colon U \to \mathcal{H}'_{i,0}$ so that $s(w) = (w,(\Lambda,e))$. Write $w = (W, b_0, \dots, b_i)$ and choose a regular value $a > b_i$ of p_W. Let U be an open neighbourhood of w in $D^\natural(M_{\infty,b};\xi)_i$ for which a remains regular. The space

$$E := \{((W', b_0, \dots, b_i), x \in W' \cap M_{a,b}) \in U \times M_{a,b}\}$$

over U is a fibre bundle, and so it is locally trivial. Choosing a trivialisation on a smaller neighbourhood U' of w, we obtain a map

$$\psi\colon U' \longrightarrow \mathrm{Emb}(W \cap M_{a,b}, M_{\infty,b}),$$

and using the $\mathrm{Diff}_c(M_{\infty,b})$-locally retractile property of $\mathrm{Emb}(W \cap M_{a,b}, M_{\infty,b})$ we obtain an even smaller neighbourhood U'' and a map

$$\phi\colon U'' \longrightarrow \mathrm{Diff}_c(M_{\infty,b})$$

such that $\phi(W', b_0, \dots, b_i)(W \cap M_{a,b}) = W' \cap M_{a,b}$ for $(W', b_0, \dots, b_i) \in U''$.

We now attempt to define a section $s\colon U'' \to \mathcal{H}'_{i,0}$ by

$$s(W', b_0, \dots, b_i) = ((W', b_0, \dots, b_i), (\Lambda, \phi(W', b_0, \dots, b_i) \circ e)).$$

To check that this is indeed a section, we must verify the six properties of Definition 3.2.3 for these data. Properties (i) and (iii) are immediate from the fact that inside $M_{a,b}$ the data $(W', (\Lambda, \phi(W', b_0, \dots, b_i) \circ e))$ agree with the data (W, Q) modified by a diffeomorphism of $M_{\infty,b}$. Property (v) is automatic, and property (ii) holds at the point w and is an open condition, so it also holds on some neighbourhood $w \in U''' \subset U''$. Property (vi) holds after perhaps shrinking U'', as then the diffeomorphisms $\phi(U'')$ may be assumed to be supported away from $e \cap p^{-1}(\{b_0, \dots, b_i\})$.

This leaves Property (iv), which follows from the important observation that if w' is sufficiently close to w, then $P(w')$ and $R(w')$ can only be *smaller* than $P(w)$ and $R(w)$; i.e., the amount of surgery we must do to obtain suitably connected surfaces is upper semi-continuous. More precisely, if w' is sufficiently close to w then $W' \cap (M \cup N_{[0,b_0)} \cup L_{[0,b]})$ is obtained from $W \cap (M \cup N_{[0,b_0)} \cup L_{[0,b]})$ by attaching 1- and 2-handles at b_0, and $p_{W'}^{-1}([b_i, b_{i+1}))$ is obtained from $p_{W'}^{-1}([b_i, b_{i+1}))$ by attaching 1- and 2-handles at b_{i+1}, or subtracting 1- and 2-handles at b_i, neither of which change the required connectivity properties. $\square$

3.3 Manifolds without boundary

In this section we prove Theorem C for manifolds M with empty boundary. Before doing so, we briefly study the set of path components of the space of sections $\Gamma_c(\mathcal{S}(TM) \to M)$ for such manifolds. We fix a complete Riemannian metric $\mathfrak{g}$ on M.

Path components of $\Gamma_c(\mathcal{S}(TM) \to M)$

The space $\mathcal{S}(TM)$ is a bundle of Thom spaces over M, with fibre over $p \in M$ given by $\mathrm{Th}(\gamma_2^{\perp} \to \mathrm{Gr}_2^+(T_pM))$, the Thom space of the orthogonal complement to the tautological bundle over the Grassmannian of oriented 2-planes in T_pM. Similarly, we can form the bundle of Grassmannians $q\colon \mathrm{Gr}_2^+(TM) \to M$, which comes equipped with a bundle injection $\gamma_2 \hookrightarrow q^*TM$ from the tautological bundle to the pullback of the tangent bundle of M. We let $\gamma_2^{\perp} \to \mathrm{Gr}_2^+(TM)$ denote the orthogonal complement to γ_2 in q^*TM.

There is a map

$$c\colon \mathcal{S}(TM) \longrightarrow \mathrm{Th}(\gamma_2^{\perp} \to \mathrm{Gr}_2^+(TM))$$

given by identifying all the points at infinity. If we choose an orientation of TM there is an induced orientation of $\gamma_2^{\perp}$, hence a Thom class

$$u \in H^{d-2}(\mathrm{Th}(\gamma_2^{\perp} \to \mathrm{Gr}_2^+(TM)); \mathbb{Z}).$$

There is also an Euler class $e = e(\gamma_2) \in H^2(\mathrm{Gr}_2^+(TM); \mathbb{Z})$, and so a class

$$u \cdot e \in H^{d}(\mathrm{Th}(\gamma_2^{\perp} \to \mathrm{Gr}_2^+(TM)); \mathbb{Z}).$$

By abuse of notation, we use the names u and $u \cdot e$ for the cohomology classes on $\mathcal{S}(TM)$ given by $c^*(u)$ and $c^*(u \cdot e)$ respectively.

There are maps

$$\begin{array}{rcccl} \pi\colon \Gamma_c(\mathcal{S}(TM) \to M) & \longrightarrow & H^{d-2}_c(M;\mathbb{Z}) & \longrightarrow & H_2(M;\mathbb{Z}) \\ s & \longmapsto & s^*(u) & \longmapsto & \pi(s) \\ \chi\colon \Gamma_c(\mathcal{S}(TM) \to M) & \longrightarrow & H^{d}_c(M;\mathbb{Z}) & \longrightarrow & H_0(M;\mathbb{Z}) \\ s & \longmapsto & s^*(u \cdot e) & \longmapsto & \chi(s) \end{array}$$

obtained by pulling back the classes e or $u \cdot e$ along a section, and then applying Poincaré duality.

Lemma 3.3.1. *If M is connected then under the scanning map*

$$\mathcal{S}\colon \mathcal{E}_g(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)$$

we have

$$\pi(\mathcal{S}([f\colon \Sigma_g \hookrightarrow M])) = f_*([\Sigma_g]) \in H_2(M;\mathbb{Z})$$
$$\chi(\mathcal{S}([f\colon \Sigma_g \hookrightarrow M])) = 2-2g \in \mathbb{Z} = H_0(M;\mathbb{Z}).$$

Proof. The cohomology class $u \in H^{d-2}(\mathcal{S}(TM);\mathbb{Z})$ is Poincaré dual to the class of the submanifold $Gr_2^+(TM) \subset \mathcal{S}(TM)$, so if s is a (suitably transverse) section then $s^*(u)$ is Poincaré dual to the submanifold $s^{-1}(Gr_2^+(TM))$.

The cohomology class $u \cdot e \in H^d(\mathcal{S}(TM);\mathbb{Z})$ is Poincaré dual to the class of the submanifold $Z \subset Gr_2^+(TM) \subset \mathcal{S}(TM)$, which is the zero set of a transverse section of $\gamma_2 \to Gr_2^+(TM)$. Thus if s is a (suitably transverse) section, then the class $s^*(u \cdot e)$ is Poincaré dual to the set of zeroes of a section of $Ts^{-1}(Gr_2^+(TM))$ which is transverse to the zero section. The latter is $\chi(s^{-1}(Gr_2^+(TM)))$ by the Poincaré–Hopf theorem.

The map obtained by scanning an embedded submanifold $f(\Sigma_g)$ is suitably transverse, and $s^{-1}(Gr_2^+(TM)) = f(\Sigma_g)$, so the claimed identities hold. □

Proposition 3.3.2. *If M is connected, so $H_0(M;\mathbb{Z}) = \mathbb{Z}$, then the map χ takes values in $2\mathbb{Z}$. If M is simply connected and of dimension $d \geq 5$, then the map*

$$\chi \times \pi\colon \pi_0(\Gamma_c(\mathcal{S}(TM) \to M)) \longrightarrow 2\mathbb{Z} \times H_2(M;\mathbb{Z})$$

is a bijection.

Proof. The space of compactly supported sections is the space of compactly supported lifts along $p\colon \mathcal{S}(TM) \to M$ of the identity map of M. We will use the notation $F = \mathcal{S}(\mathbb{R}^d) = Th(\gamma_2^\perp \to Gr_2^+(\mathbb{R}^d))$ for the fibre of the map p, and suppose for simplicity that M is compact.

The map $\mathrm{Gr}_2^+(\mathbb{R}^d) \to \mathrm{Gr}_2^+(\mathbb{R}^\infty)$ induces an isomorphism in cohomology in degrees $\leq d-1$, so

$$\mathbb{Z}[e(\gamma_2)] \longrightarrow H^*(\mathrm{Gr}_2^+(\mathbb{R}^d);\mathbb{Z})$$

is an isomorphism in degrees $\leq d-1$, and hence

$$u \cdot \mathbb{Z}[e(\gamma_2)] \longrightarrow \widetilde{H}^*(\mathcal{S}(\mathbb{R}^d);\mathbb{Z})$$

is an isomorphism in degrees $\leq 2d-3$. As there are cohomology classes $u \cdot e^i \in H^*(\mathcal{S}(TM), M;\mathbb{Z})$ restricting to $u \cdot e(\gamma_2)^i$ on the fibre, the bundle p satisfies the conditions of the (relative) Leray–Hirsch theorem in degrees $\leq 2d-3$, so

$$H^*(M;\mathbb{Z}) \otimes (u \cdot \mathbb{Z}[e(\gamma_2)]) \longrightarrow H^*(\mathcal{S}(TM), M;\mathbb{Z})$$

is an isomorphism in this range of degrees.

Let us show that χ takes even values. As $q^*TM = \gamma_2 \oplus \gamma_2^\perp$, we calculate in the $\mathbb{F}_2$-cohomology of $\mathrm{Th}(\gamma_2^\perp \to \mathrm{Gr}_2^+(TM))$

$$Sq^2(u) = u \cdot w_2(\gamma_2^\perp) = u \cdot (w_2(\gamma_2) + q^*w_2(M)) = u \cdot e + u \cdot q^*w_2(M)$$

and so pulling back via c we have

$$Sq^2(u) = u \cdot e + u \cdot p^*w_2(M) \in H^d(\mathcal{S}(TM);\mathbb{F}_2).$$

Thus for any section s we have $Sq^2(s^*u) = s^*(u \cdot e) + s^*u \cdot w_2(M)$ in the $\mathbb{F}_2$-cohomology of M. However $Sq^2(s^*u) = v_2(M) \cdot s^*u = w_2(M) \cdot s^*u$ as M is simply connected, and so $s^*(u\cdot e) = 0 \in H^d(M;\mathbb{F}_2)$. Thus $s^*(u\cdot e) \in H^d(M;\mathbb{Z}) = \mathbb{Z}$ is even, as claimed.

We will be required to know $\pi_k(\mathcal{S}(\mathbb{R}^d))$ for $k \leq d$. By considering the cohomology calculation above in degrees $\leq 2d-3$, we see that as long as $d \geq 4$ then $\mathcal{S}(\mathbb{R}^d)$ has a cell structure whose $(d+1)$-skeleton X consists of a $(d-2)$-cell and a d-cell. Because

$$Sq^2(u) = u \cdot w_2(\gamma_2^\perp) = u \cdot w_2(\gamma_2) \neq 0,$$

we see that the d-cell is attached along a non-trivial map $S^{d-1} \to S^{d-2}$, which must be the Hopf map as long as $d \geq 5$. Thus $X \simeq \Sigma^{d-4}\mathbb{CP}^2$, and it remains to calculate the homotopy groups of this space in degrees $\leq d$. By the Blakers–Massey theorem, the map of pairs $\pi_k(S^{d-2}, S^{d-1}) \to \pi_k(X,*)$ is an isomorphism for $k \leq 2d-5$, so for $k \leq d$ as we have assumed that $d \geq 5$. Calculating by means of the known stable homotopy groups of spheres in this range shows that

$$\pi_{d-2}(\mathcal{S}(\mathbb{R}^d)) \cong \mathbb{Z}, \qquad \pi_{d-1}(\mathcal{S}(\mathbb{R}^d)) = 0, \qquad \pi_d(\mathcal{S}(\mathbb{R}^d)) \cong \mathbb{Z},$$

and also that the Hurewicz map is injective in these degrees. (When $d = 5$ we must use that $\pi_5(S^3) = \mathbb{Z}/2\langle\eta^2\rangle$, even though it is not in the stable range; this may be found in Toda's book [Tod62].)

Let s_0 and s_1 be two sections of p which have the same value of the invariants π and χ, and let us show that they are fibrewise homotopic. We obtain a diagram

$$\begin{tikzcd}
\{0,1\} \times M \arrow[r, "s_0 \cup s_1"] \arrow[d] & \mathcal{S}(TM) \arrow[r, "p \times u"] \arrow[d, "p"] & M \times K(\mathbb{Z}, d-2) \arrow[d, "\mathrm{proj}"] \\
{[0,1] \times M} \arrow[r, "\mathrm{proj}"] \arrow[ur, dashed] & M \arrow[r, equal] & M
\end{tikzcd} \tag{3.3.1}$$

and we must supply the dashed arrow. By obstruction theory, the first possible obstruction lies in

$$H^{d-1}([0,1] \times M, \{0,1\} \times M; \pi_{d-2}(\mathcal{S}(\mathbb{R}^d))) \cong H^{d-2}(M; \mathbb{Z})$$

and it must be $\pi(s_0) - \pi(s_1)$, as it agrees with the first possible obstruction for the (trivial) right-hand fibration in (3.3.1). But we have assumed that $\pi(s_0) - \pi(s_1)$ is zero, so there is no obstruction at this stage. The next possible obstruction lies in

$$H^{d+1}([0,1] \times M, \{0,1\} \times M; \pi_d(\mathcal{S}(\mathbb{R}^d))) \cong H^d(M; \mathbb{Z}),$$

and by comparing it with the trivial bundle $M \times K(\mathbb{Z}, d) \to M$ via $p \times (u \cdot e)$, as above, and using the injectivity of the Hurewicz map, we see that this obstruction vanishes if and only if $\chi(s_0) - \chi(s_1)$ does; we have assumed this. As M has dimension d, there are no higher obstructions to constructing the dotted map, which gives a fibrewise homotopy between the two sections. $\square$

Proof of Theorem C when $\partial M = \emptyset$

Recall that we have fixed a complete Riemannian metric $\mathfrak{g}$ on M. Let $\mathcal{E}_g^{\nu}(M) \subset \mathcal{E}_g^{+}(M) \times (0, \infty)$ be the space of pairs (W, ϵ), where $W \in \mathcal{E}_g^{+}(M)$ and ϵ is smaller than the injectivity radius of the exponential map $\exp\colon \nu(W) \to M$; we denote by W^{ϵ} the image of this embedding. The forgetful map $\mathcal{E}_g^{\nu}(M) \to \mathcal{E}_g^{+}(M)$ is a weak homotopy equivalence. We denote by ∞_p the point at infinity of $\mathcal{S}(T_pM)$ and write $\infty = \bigcup_p \infty_p$, and we define the support of $f \in \Gamma_c(\mathcal{S}(TM) \to M)$, denoted $\mathrm{supp}\, f$, to be the closure of $M \setminus f^{-1}(\infty)$.

Definition 3.3.3. Let $\mathcal{G}_g(M)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(W, \epsilon; d_0, \ldots, d_i)$, where

(i) $(W, \epsilon) \in \mathcal{E}_g^{\nu}(M)$;

(ii) $d_0, \dots, d_i$ are disjoint embeddings of the closed unit disc into M;

(iii) the geodesic distance from $d_j(0)$ to W is at least ϵ, for all j.

The semi-simplicial structure is as usual given by forgetting data, which gives a semi-simplicial space augmented over $\mathcal{E}_g^{\nu}(M)$.

Proposition 3.3.4. *If the dimension of* M *is at least* 3*, then* $\mathcal{G}_g(M)_\bullet$ *is a resolution of* $\mathcal{E}_g^{\nu}(M)$.

Proof. Let $G_\bullet$ be the semi-simplicial space constructed similarly to the above, with i-simplices consisting of those tuples $(W, \epsilon; d_0, \dots, d_i)$ such that condition (i) above holds, as well as

(ii′) $d_0, \dots, d_i \colon D^d \hookrightarrow M$ are embeddings of the closed unit disc into M such that the $d_j(0)$ are distinct;

(iii′) $d_j(0) \cap W = \emptyset$, for all j.

There is an inclusion $\mathcal{G}_g(M)_\bullet \hookrightarrow G_\bullet$, which is a levelwise weak homotopy equivalence, by shrinking the discs and ϵ. Now $G_\bullet$ is an augmented topological flag complex over $\mathcal{E}_g^{+}(M)$, so we apply Criterion 1.6.2. The augmentation map is a fibration by Corollary 1.5.8, hence has local sections, and given any finite (possibly empty) collection $(W, \epsilon, d_0), \dots, (W, \epsilon, d_i)$ of 0-simplices over (W, ϵ), the complement $M \setminus (W \cup \bigcup d_j(0))$ is a non-empty manifold of dimension at least 3, so there is an embedding d of a closed d-ball into it. Then (W, d) is orthogonal to all the former 0-simplices. □

Proposition 3.3.5. *There are fibrations*

$$\mathcal{E}_g^{\nu}(M \setminus \cup d_j(0)) \longrightarrow \mathcal{G}_g(M)_i \longrightarrow C_i(M) =: \mathrm{Emb}(\{0, 1, \dots, i\} \times D^d, M)$$

where the fibre is taken over the point $(d_0, \dots, d_i)$.

Proof. This is a consequence of Corollary 1.5.8. □

In the notation of the last section, we let $\Gamma_c(\mathcal{S}(TM) \to M)_g$ denote the collection of path components $\chi^{-1}(2-2g)$. Thus it consists of those sections which have "formal genus g".

Definition 3.3.6. Let $\mathcal{F}_g(M)_\bullet$ be the semi-simplicial space whose i-simplices are tuples $(f, (d_0, h_0), \dots, (d_i, h_i))$, where

(i) $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$;

(ii) $d_0, \dots, d_i \colon D^d \hookrightarrow M$ are disjoint embeddings of the closed unit disc of dimension d into M;

(iii) $h_0, \dots, h_i \colon [0,1] \times M \to \mathcal{S}(TM)$ are homotopies of sections such that

$$h_j(0,-) = f(-), \qquad d_j(0) \notin \operatorname{supp} h_j(1,-),$$

and the homotopy h_j is constant outside of the set $d_j(D^d)$.

The jth face map forgets (d_j, h_j), and forgetting everything but f gives an augmentation to the space $\Gamma_c(\mathcal{S}(TM) \to M)_g$.

Proposition 3.3.7. *If M has dimension at least 3, then $\mathcal{F}_g(M)_\bullet$ is a resolution of $\Gamma_c(\mathcal{S}(TM) \to M)_g$.*

Proof. Let us define $F_g(M)_\bullet$ as the semi-simplicial space whose i-simplices are tuples $(f, (d_0, h_0), \dots, (d_i, h_i))$ such that conditions (i) and (iii) above hold and condition (ii) is replaced by

(ii′) $d_0, \dots, d_i \colon D^d \hookrightarrow M$ are embeddings such that the $d_j(0)$ are distinct,

and whose face maps are given by forgetting data, and it has an augmentation to $\Gamma_c(\mathcal{S}(TM) \to M)_g$ that forgets everything but f. There is an obvious semi-simplicial inclusion $\mathcal{F}_g(M)_\bullet \hookrightarrow F_g(M)_\bullet$ over $\Gamma_c(\mathcal{S}(TM) \to M)_g$, and the lemma will follow from the following statements:

(1) the semi-simplicial inclusion is a levelwise weak homotopy equivalence and

(2) the augmentation of $F_g(M)_\bullet$ is a weak homotopy equivalence.

For the first statement, take a smooth function $\lambda \colon [0,\infty) \to [0,1]$ with

$$\lambda([1,\infty)) = 0, \ \lambda([0,1/2]) = 1.$$

Consider the following deformation $H_s \colon F_g(M)_i \times (0,1] \to F_g(M)_i$ (which restricts to a deformation of $\mathcal{F}_g(M)_i$):

$$H_s(f) = f, \ H_s(d_j)(y) = d_j(sy), \ H_s(h_j)(t,x) = \begin{cases} h_j(\lambda(\|sy\|)t, sy) & \text{if } x = d_j(sy) \\ x & \text{otherwise.} \end{cases}$$

Under this deformation, every i-simplex eventually ends up, and stays, in the subspace $\mathcal{F}_g(M)_i$. If $f \colon (D^n, S^{n-1}) \to (F_g(M)_i, \mathcal{F}_g(M)_i)$ represents a relative homotopy class, then because D^n is compact the map $h(-,t) \circ f$ has image in $\mathcal{F}_g(M)_i$ for some t, so the homotopy class of f is trivial.

For the second statement, note that $F_g(M)_\bullet$ is a topological flag complex augmented over $\mathcal{E}_g^+(M)$ whose augmentation is a fibration by Lemmas 1.5.10 and 1.5.3. Given a possibly empty finite collection of 0-simplices

$$(f, d_0, h_0), \dots, (f, d_i, h_i)$$

over f, we may find an embedding of a disc d_{i+1} such that $d_{i+1}(0)$ is different from the points $d_0(0), \dots, d_i(0)$. We may also find a homotopy h_{i+1} satisfying condition (iii) for the embedding d_{i+1} and the section f, because the space $\mathcal{S}(T_{d_{i+1}(0)}M)$ is path connected. Hence the conditions of Criterion 1.6.2 hold, so the augmentation for $F_g(M)_\bullet$ is a weak homotopy equivalence. $\square$

Proposition 3.3.8. *There are homotopy fibrations*

$$\Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g \longrightarrow \mathcal{F}_g(M)_i \longrightarrow C_i(M),$$

where the fibre is taken over the point $(d_0, \dots, d_i)$.

Proof. The space $C_i(M)$ is $\mathrm{Diff}_\partial(M)$-locally retractile by Lemma 1.5.7, and the map is equivariant for the action of $\mathrm{Diff}_\partial(M)$; hence, by Lemma 1.5.3, this is a locally trivial fibration. The fibre is the space Fib_i of tuples $(f, h_0, \dots, h_i)$ where $f \in \Gamma_c(\mathcal{S}(TM) \to M)_g$ and h_j is a homotopy of f supported in d_j such that $h_j(1, -)$ is a section supported away $d_j(0)$. Since the homotopies h_j have disjoint support, we may compose them. There is a homotopy

$$\begin{aligned} H\colon I \times \mathrm{Fib}_i &\longrightarrow \mathrm{Fib}_i \\ (t, (f, (h_0, \dots, h_i))) &\longmapsto (H_t(f), H_t(h_0), \dots, H_t(h_i)) \end{aligned}$$

where

$$\begin{aligned} H_t(f)(-) &= h_0(t, -) \circ \dots \circ h_i(t, -) \\ H_t(h_j)(s, -) &= h_j(t + s(1-t), -). \end{aligned}$$

This homotopy deformation retracts Fib_i into the subspace Y of those tuples $(f, h_0, \dots, h_i)$ such that $d_j(0) \notin \operatorname{supp} h_j$ and h_j is the constant homotopy. Finally, there is a map

$$Y \longrightarrow \Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g$$

given by sending $(f, h_0, \dots, h_i)$ (recall that these homotopies are all constant) to $f_{|M \setminus \cup d_j(0)}$, and this map is a homeomorphism. $\square$

By condition (iii) of Definition 3.3.3, the scanning map

$$\mathcal{S}\colon \mathcal{E}_g^\nu(M) \longrightarrow \Gamma_c(\mathcal{S}(TM) \to M)_g$$

constructed using the previously chosen Riemannian metric $\mathfrak{g}$ extends to a semi-simplicial map $\mathcal{S}_\bullet \colon \mathcal{G}_g(M)_\bullet \to \mathcal{F}_g(M)_\bullet$ given on i-simplices by sending each tuple $(W, \epsilon, d_0, \dots, d_i)$ to the tuple $(\mathcal{S}(W, \epsilon), (d_0, \mathrm{Id}), \dots, (d_i, \mathrm{Id}))$, where Id denotes the constant homotopy.

Proposition 3.3.9. *The resolution $\mathcal{S}_\bullet$ of the scanning map is a levelwise homology equivalence in degrees $\leq \frac{1}{3}(2g-2)$. Hence the scanning map is also a homology equivalence in those degrees.*

Proof. The induced map on the space of i-simplices is a map of fibrations over $C_i(M)$, and the induced map on fibres is

$$\mathcal{S}_i \colon \mathcal{E}_g^\nu(M \setminus \cup d_j(0)) \longrightarrow \Gamma_c(\mathcal{S}(TM \setminus \cup d_j(0)) \to M \setminus \cup d_j(0))_g.$$

As $\mathcal{S}_i$ is a scanning map, Theorem C for surfaces in a manifold with boundary (which was proven in the preceding two sections) asserts that $\mathcal{S}_i$ is a homology equivalence in degrees $\leq \frac{1}{3}(2g-2)$. Note that although $M \setminus \cup d_j(0)$ does not have boundary, it does admit a boundary. □

Chapter 4

Applications

4.1 The case $M = \mathbb{R}^n$

Recall from the Introduction (page 12), that the colimit of the spaces of compact connected oriented embedded surfaces of genus g in $\mathbb{R}^n$ is a model for the homotopy type of $\mathrm{BDiff}^+(\Sigma_g)$:

$$\operatorname*{colim}_n \mathcal{E}(\Sigma_g, \mathbb{R}^n) \simeq \mathrm{BDiff}^+(\Sigma_g),$$

and Theorem A is in this case the Madsen–Weiss theorem, which establishes a homology equivalence

$$\mathrm{BDiff}^+(\Sigma_g) \longrightarrow \Omega_0^\infty \mathbf{MTSO}(2)$$

in degrees $\leq \frac{1}{3}(2g-2)$. Here $\mathbf{MTSO}(2)$ is a spectrum obtained as follows: If $\gamma_{2,n}^\perp$ is the orthogonal complement of the tautological bundle on the Grassmannian of oriented 2-planes in $\mathbb{R}^n$, then there is a pullback square

$$\begin{tikzcd}
\gamma_{2,n}^\perp \oplus \mathbb{R} \arrow[r] \arrow[d] & \gamma_{2,n+1}^\perp \arrow[d] \\
\mathrm{Gr}_2^+(\mathbb{R}^n) \arrow[r] & \mathrm{Gr}_2^+(\mathbb{R}^{n+1})
\end{tikzcd}$$

which induces a map on Thom spaces $\Sigma\mathrm{Th}(\gamma_{2,n}^\perp) \to \mathrm{Th}(\gamma_{2,n+1}^\perp)$. The spaces $\mathrm{Th}(\gamma_{2,n}^\perp)$ together with these maps define the spectrum $\mathbf{MTSO}(2)$.

The rational cohomology of the component $\Omega_0^\infty \mathbf{MTSO}(2)$ of the constant loop of $\Omega^\infty \mathbf{MTSO}(2)$ is a polynomial algebra with a single generator in each even dimension. One may take as generators the *Miller–Morita–Mumford classes* $\kappa_i \in H^{2i}(\Omega_0^\infty \mathbf{MTSO}(2); \mathbb{Q})$, also called κ-*classes* [Mum83, Mil86, Mor86]:

$$H^*(\Omega_0^\infty \mathbf{MTSO}(2); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots].$$

The κ-classes may be constructed as follows. The integral cohomology of the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^n)$ in degrees $< n-2$ is a truncated polynomial algebra generated by the Euler class e_2 of the tautological bundle. In addition, the inclusion $\mathrm{Gr}_2^+(\mathbb{R}^n) \to \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$ preserves the Euler class by naturality and is therefore $(n-2)$-connected. The Thom isomorphism

$$\phi\colon H^{2i}(\mathrm{Gr}_2^+(\mathbb{R}^n);\mathbb{Q}) \xrightarrow{\cong} H^{n-2+2i}(\mathrm{Th}(\gamma_{2,n}^\perp);\mathbb{Q})$$

sends the power e_2^i to a class u_{n-2+2i}, and it follows that the induced map between Thom spaces $\mathrm{Th}(\gamma_{2,n}^\perp) \to \mathrm{Th}(\gamma_{2,n+1}^\perp)$ is $(2n-4)$-connected. The class u_{n-2+2i} transgresses in turn to a class k_{2i-2} in the cohomology of $\Omega^n\mathrm{Th}(\gamma_{2,n}^\perp)$. The map of loop spaces

$$\Omega^n\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \Omega^{n+1}\mathrm{Th}(\gamma_{2,n+1}^\perp) \tag{4.1.1}$$

is $(n-4)$-connected, and, if $2i-2 < n-4$, then the homomorphism induced in cohomology sends the class k_{2i-2} in the right-hand side to the class k_{2i-2} in the left-hand side, hence the classes k_{2i-2} stabilise to a class in the cohomology of $\Omega_0^\infty\mathbf{MTSO}(2)$. This is the class κ_{i-1}.

The cohomology of $\Omega_0^\infty\mathbf{MTSO}(2)$ with coefficients in a finite field was computed in [Gal04].

In this section we apply our main theorem to compute the rational stable cohomology of each of the spaces $\mathcal{E}(\Sigma_g,\mathbb{R}^n)$. Recall that our main theorem (Theorem A on page 9) applied to the background manifold $M = \mathbb{R}^n$ establishes that the scanning map

$$\mathcal{E}(\Sigma_g,\mathbb{R}^n) \longrightarrow \Gamma_c(\mathcal{S}(T\mathbb{R}^n) \to \mathbb{R}^n)_g \tag{4.1.2}$$

is a homology equivalence in degrees $\leq \frac{2}{3}(g-1)$. By Proposition 3.3.2, the connected components of the space of sections $\Gamma_c(\mathcal{S}(T\mathbb{R}^n) \to \mathbb{R}^n)$ are indexed by the even numbers, and the subindex g in (4.1.2) indicates that we take the component labeled by $2-2g$. Since $\mathbb{R}^n$ admits a boundary, all connected components of this space of sections have the same homotopy type by Lemma 3.1.11, so it will be enough to compute the rational homotopy type of any component, say $\Gamma_c(\mathcal{S}(T\mathbb{R}^n) \to \mathbb{R}^n)_0$.

As the tangent bundle of $\mathbb{R}^n$ is trivial, the space of compactly supported sections $\Gamma_c(\mathcal{S}(T\mathbb{R}^n) \to \mathbb{R}^n)$ is homotopy equivalent to the space of based maps from the one-point compactification of $\mathbb{R}^n$ to the space $\mathcal{S}(\mathbb{R}^n) := \mathrm{Th}(\gamma_{2,n}^\perp)$, which is the n-fold loop space $\Omega^n\mathrm{Th}(\gamma_{2,n}^\perp)$. The component $\Omega_0^n\mathrm{Th}(\gamma_{2,n}^\perp)$ of the constant loop corresponds to the component $\Gamma_c(\mathcal{S}(T\mathbb{R}^n) \to \mathbb{R}^n)_0$.

The computation of $H^*(\Omega^n\mathrm{Th}(\gamma_{2,n}^\perp);\mathbb{Q})$ will be done in the following steps. First, we recall the multiplicative structure of the cohomology of the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^n)$. We give details in the case when n is odd, as we have not found any reference. Then, we deduce the cohomology ring structure of $\mathrm{Th}(\gamma_{2,n}^{\perp})$. Afterwards, we prove that the spaces $\mathrm{Th}(\gamma_{2,n}^{\perp})$ are intrinsically formal. This is immediate in the odd-dimensional case, but requires an argument when n is even.

We recall that a cochain complex is *formal* if it is quasi-isomorphic to its cohomology with zero differential. A graded ring R is *intrinsically formal* if any two simply connected spaces with rational cohomology rings isomorphic to R are rationally homotopy equivalent. We say that a simply connected topological space is *formal* if its rational cochain complex is formal. This implies, for simply connected spaces, that the rational homotopy type of the space can be deduced from the rational cohomology ring. We say that a simply connected space is *intrinsically formal* if its rational cohomology ring is intrinsically formal.

Then, using formality, we give an explicit minimal Lie model for $\mathrm{Th}(\gamma_{2,n}^{\perp})$, which is used to find the rational homotopy type of $\Omega\mathrm{Th}(\gamma_{2,n}^{\perp})$; cf. [Pet67, Bak12]. A loop space of a simply connected space is rationally homotopy equivalent to a product of rational Eilenberg–Mac Lane spaces, hence from the knowledge of the rational homotopy type of $\Omega\mathrm{Th}(\gamma_{2,n}^{\perp})$, the rational homotopy type of $\Omega^n\mathrm{Th}(\gamma_{2,n}^{\perp})$ follows, and so its cohomology ring.

We will also compute the homomorphism induced in rational cohomology by the maps in (4.1.1) and by the map

$$\Omega_0^n\mathrm{Th}(\gamma_{2,n}^{\perp}) \longrightarrow \Omega_0^{\infty}\mathbf{MTSO}(2)$$

into the colimit. From these computations we deduce also a cohomological description in the stable range of the maps

$$\mathcal{E}(\Sigma_g, \mathbb{R}^n) \longrightarrow \mathcal{E}(\Sigma_g, \mathbb{R}^{n+1}),$$
$$\mathcal{E}(\Sigma_g, \mathbb{R}^n) \longrightarrow \mathrm{BDiff}^+(\Sigma_g).$$

In this section, all cohomology rings are meant with $\mathbb{Q}$ coefficients if unspecified. As we have already done, we will use subindices to indicate the degree of cohomology classes.

Case n odd

If n is odd, the rational cohomology ring of $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is easy to compute by means of the Serre spectral sequence associated with the fibrations

$$\begin{tikzcd}
S^{n-2} \arrow[d] & & \\
V_2(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Gr}_2^+(\mathbb{R}^n) \arrow[r] & BSO(2), \\
S^{n-1} & &
\end{tikzcd}$$

where $V_2(\mathbb{R}^n)$ is the Stiefel manifold of 2-frames in $\mathbb{R}^n$. The vertical fibration is the unit sphere bundle of the tangent bundle of S^{n-1}. The second page of the associated Serre spectral sequence has only four non-trivial groups, and the transgression

$$H^0(S^{n-1}; H^{n-2}(S^{n-2})) \longrightarrow H^{n-1}(S^{n-1}; H^0(S^{n-2})) \tag{4.1.3}$$

is multiplication by the Euler characteristic of S^{n-1}, which is 2 since n is odd, hence (4.1.3) is an isomorphism. As a consequence, the rational cohomology of $V_2(\mathbb{R}^n)$ is isomorphic to the rational cohomology of S^{2n-3}. After writing the Serre spectral sequence for the horizontal fibration, one realizes that, since $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is compact, it has finite-dimensional cohomology, so the transgression

$$H^0(BSO(2); H^{2n-3}(V_2(\mathbb{R}^n))) \longrightarrow H^{2n-2}(BSO(2); H^0(V_2(\mathbb{R}^n)))$$

has to be an isomorphism. Hence, if we denote by e_2 the Euler class of the tautological bundle over $\mathrm{Gr}_2^+(\mathbb{R}^n)$ (which is the pullback of the generator of the cohomology ring of $BSO(2)$), then

$$H^*(\mathrm{Gr}_2^+(\mathbb{R}^n)) \cong \mathbb{Q}[e_2]/e_2^{n-1}.$$

Now we use the Thom isomorphism to compute $H^*(\mathrm{Th}(\gamma_{2,n}^{\perp}))$. Recall that if $E \to B$ is a rank n vector bundle and E_0 is the complement of the zero section, then there is a relative cup product

$$H^*(E) \otimes H^*(E, E_0) \longrightarrow H^*(E, E_0)$$

which we denote with a dot. By excision, $H^*(E, E_0)$ is the cohomology of the Thom space of E. The Thom isomorphism theorem establishes that if E is oriented, then there is a class $u \in H^n(E, E_0)$ such that the homomorphism

$$H^*(E) \xrightarrow{\cong} H^{*+n}(E, E_0)$$

that sends a class a to the product $a \cdot u_{n-2}$ is an isomorphism. Turning back to our situation, if u_{n-2} is the Thom class of $\gamma_{2,n}^{\perp}$ and $u_{n+2k} = e_2^{k+1} \cdot u_{n-2}$, then the Thom isomorphism theorem gives an isomorphism

$$H^*(\mathrm{Th}(\gamma_{2,n}^{\perp})) \cong \mathbb{Q}\langle 1, u_{n-2}, u_n, \dots, u_{3n-6}\rangle$$

of graded vector spaces. Moreover, $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is an intrinsically formal space by the following theorem, from which we deduce also the ring structure on its cohomology.

Theorem 4.1.1 ([HS79, Theorem 1.5], [Bau77]). *If the rational cohomology groups of a simply connected space* X *vanish in even degrees, then* X *is formal, therefore rationally homotopy equivalent to a wedge of spheres.*

Hence, $\mathrm{Th}(\gamma_{2,n}^{\perp}) \simeq_{\mathbb{Q}} \bigvee_{k=-1}^{n-3} S^{n+2k}$. If we write $\tilde{u}_{n+2k-1} = s^{-1}u_{n+2k}^*$, then a minimal Lie model of $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is the free Lie algebra on the vector space $\bigoplus_{k=-1}^{n-3} \mathbb{Q}\langle \tilde{u}_{n+2k-1}\rangle$, with zero differential.

Case n even

The integral cohomology ring of $\mathrm{Gr}_2^+(\mathbb{R}^n)$ when $n = 2r$ was computed by Lai in [Lai74], using that $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is canonically diffeomorphic to the even-dimensional complex quadric. The standard isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ defines an embedding of $\mathbb{C}P^{r-1}$ into $\mathrm{Gr}_2^+(\mathbb{R}^n)$ that sends a complex line in $\mathbb{C}^r$ to the underlying oriented 2-plane in $\mathbb{R}^n$. Write $c_{n-2} \in H^{n-2}(\mathrm{Gr}_2^+(\mathbb{R}^n); \mathbb{Z})$ for the Poincaré dual of the fundamental class of the image of this embedding, e_2 for the Euler class of the tautological bundle $\gamma_{2,n}$ and f_{n-2} for the Euler class of the orthogonal complement $\gamma_{2,n}^{\perp}$ of the tautological bundle.

Proposition 4.1.2 ([Lai74]). *The ring* $H^*(\mathrm{Gr}_2^+(\mathbb{R}^n); \mathbb{Z})$ *for* $n = 2r$ *is isomorphic to*

$$\mathbb{Z}[e_2, c_{n-2}]/\ (e_2^r = 2e_2c_{n-2},\ e_2^{n-1} = 0,\ c_{n-2}^3 = 0).$$

In addition $f_{n-2} = 2c_{n-2} - e_2^{r-1}$ *and* $c_{n-2}f_{n-2}$ *is the dual of the fundamental class of* $\mathrm{Gr}_2^+(\mathbb{R}^n)$*. As a consequence, the following relations also hold:*

$$e_2^{n-2} = (-1)^{r-1} f_{n-2}^2, \quad e_2 f_{n-2} = 0.$$

Let us describe now the cohomology ring of the Thom space $\mathrm{Th}(\gamma_{2,n}^{\perp})$ with rational coefficients. If u_{n-2} denotes the Thom class of $\gamma_{2,n}^{\perp}$, then $H^*(\mathrm{Th}(\gamma_{2,n}^{\perp}))$ is generated as a vector space by the classes

$$u_{n+2k} := e_2^{k+1} \cdot u_{n-2}, \quad v_{2n-4} := f_{n-2} \cdot u_{n-2},$$

where k ranges from -1 to $n-3$. In order to find the ring structure, we use that $(a \cdot u_{n-2})(b \cdot u_{n-2}) = (abf_{n-2}) \cdot u_{n-2}$ for all a and b. From the relations given in Proposition 4.1.2, the only non-trivial products are

$$u_{n-2}^2 = f_{n-2} \cdot u_{n-2} = v_{2n-4},$$
$$u_{n-2}^3 = f_{n-2}^2 \cdot u_{n-2} = (-1)^{r-1} u_{3n-6}.$$

Hence, the ring structure on the rational cohomology of $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is given by

$$H^*(\mathrm{Th}(\gamma_{2,n}^{\perp})) \cong \mathbb{Q}\langle 1, u_{n-2}, u_{n-2}^2, u_{n-2}^3, u_n, u_{n+2} \ldots, u_{3n-8} \rangle,$$

and the class u_{3n-6} is non-zero and equal to $(-1)^{r-1} u_{n-2}^3$.

Theorem 4.1.3 ([HS79, Corollary 5.16]). *If X is a simply connected rational space such that $H^p(X) = 0$ for $1 \leq p \leq l$ and for $p > 3l + 1$ for some l, then X is formal.*

Note that it actually follows from this theorem that such an X is intrinsically formal.

Proposition 4.1.4. *The Thom space $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is intrinsically formal.*

Proof. The space $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is $(n-3)$-connected and has nontrivial rational cohomology up to degree $3n-6$, which is bigger than $3(n-3)+1 = 3n-8$, hence Theorem 4.1.3 does not apply. Nevertheless, it applies to its $(3n-8)$-skeleton Sk_{3n-8} (which is therefore intrinsically formal). The space $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is the result of adjoining a single cell of dimension $3n-6$ to Sk_{3n-8}.

The attaching map is an element of $\pi_{3n-7}(\mathrm{Sk}_{3n-8}) \otimes \mathbb{Q}$, and this vector space is isomorphic to the degree $3n-7$ part of the Sulivan minimal model $(\Lambda V, d)$ of Sk_{3n-8}. It is immediate to see that

- a basis for the degree $3n-6$ part is given by the cube u_{n-2}^3 and the double products $u_k u_l$ such that $k + l = 3n-6$.

The first generator different from u_k is some a in degree $2n-3$, with $d(a) = u_{n-2}u_n$. Therefore, since each u_k has even degree, the first decomposable with odd degree is $u_{n-2}a$, with degree $2n-5$. In particular, as n is even,

- all elements in degree $3n-7$ are indecomposable, therefore

- all elements in degree $3n-8$ have zero differential.

Since the homology of Sk_{3n-8} in degree $3n-6$ vanishes, a basis for the degree $3n-7$ part of ΛV is given by an element b and elements $a(k,l)$ for each product $u_k u_l$ with $d(b) = u_{n-2}^3$ and $d(a(k,l)) = u_k u_l$. Following [FHT01, Proposition 13.12], a commutative model for the result of attaching a $(3n-6)$-cell to Sk_{3n-8} with a map $f\colon S^{3n-7} \to \mathrm{Sk}_{3n-8}$ that is in the class of a linear combination

$$\beta b + \sum_{k,l} \alpha_{k,l} a(k,l) \in \pi_{3n-7}(\mathrm{Sk}_{3n-8}) \otimes \mathbb{Q}$$

is given by $(\Lambda V \oplus \mathbb{Q}z, \bar{d})$, where z has degree $3n-6$, all products with z are trivial, and there are numbers $\beta, \alpha_{k,l} \in \mathbb{Q}$ such that

$$\begin{aligned}\bar{d}(a(k,l)) &= d(a(k,l)) + \alpha_{k,l} z = u_k u_l + \alpha_{k,l} z,\\ \bar{d}(b) &= d(b) + \beta z = u_{n-2}^3 + \beta z,\\ \bar{d}(z) &= 0.\end{aligned}$$

In our case $u_{2n-4}^3 = z$, hence $\beta = -1$, and $u_k u_l = 0$ if $k + l = 3n-6$, therefore $\alpha_{k,l} = 0$ for all k, l. As a consequence, the homotopy class of the attaching map f is determined by the rational cohomology ring of $\mathrm{Th}(\gamma_{2,n}^{\perp})$. Finally, if X is any other simply connected space with the same rational cohomology ring as $\mathrm{Th}(\gamma_{2,n}^{\perp})$, then the same argument shows that $X \simeq_{\mathbb{Q}} \mathrm{Th}(\gamma_{2,n}^{\perp})$. Hence $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is intrinsically formal. □

The argument used in the previous proof shows that there is only one simply connected rational homotopy type, which we denote by C_3^{n-2}, such that

$$H^*(C_3^{n-2}) \cong \mathbb{Q}[u_{n-2}]/u_{n-2}^4.$$

(Alternatively, one may use that this cohomology ring is *hyperformal*, hence C_3^{n-2} is intrinsically formal by [FH82].) Since spheres are formal and a wedge of formal spaces is formal [HS79], we deduce that

$$\mathrm{Th}(\gamma_{2,n}^{\perp}) \simeq_{\mathbb{Q}} C_3^{n-2} \vee \bigvee_{k=0}^{n-4} S^{n+2k}.$$

Now we compute the minimal Lie model of $\mathrm{Th}(\gamma_{2,n}^{\perp})$ to obtain the rational homotopy groups of its loop space. Recall that the minimal Lie model of the sphere S^k is the free Lie algebra on one generator of dimension $k-1$. In order to find a minimal model of C_3^{n-2} we notice that its Sullivan minimal model has three non-trivial cohomology groups in dimensions $n-2, 2n-4$ and $3n-6$, hence its minimal Lie model has three generators in dimensions $n-3, 2n-5$ and $3n-7$. On the other hand, its Sullivan minimal model has two generators in dimensions $n-2$ and $4n-9$, therefore the homology of the minimal Lie model has only two non-trivial groups in dimensions $n-3$ and $4n-10$. This completely determines the differentials of the three generators in its minimal Lie model, namely, if $\tilde{u}_{n-3}$ denotes the desuspension of u_{n-2},

$$(\mathbb{L}(\tilde{u}_{n-3}, b_{2n-5}, c_{3n-7}),\ d(\tilde{u}) = 0,\ d(b) = [\tilde{u}, \tilde{u}],\ d(c) = [\tilde{u}, b]) .$$

Therefore, denoting by $\tilde{u}_{n+2k-1}$ the desuspension of u_{n+2k}, a minimal Lie model $L(\mathrm{Th}(\gamma_{2,n}^{\perp}))$ of $\mathrm{Th}(\gamma_{2,n}^{\perp})$ is

$$\begin{aligned}(\mathbb{L}(&\tilde{u}_{n-3}, b_{2n-5}, c_{3n-7}, \tilde{u}_{n-1}, \tilde{u}_{n+1}, \dots, \tilde{u}_{3n-9}),\\ &d(\tilde{u}_k) = 0,\ d(b_{2n-5}) = [\tilde{u}_{n-3}, \tilde{u}_{n-3}],\ d(c_{3n-7}) = [\tilde{u}_{n-3}, b_{2n-5}]).\end{aligned}$$

Conclusion

Let V_n be the homology of the minimal Lie model of $\mathrm{Th}(\gamma_{2,n}^{\perp})$ computed in the previous sections, and let us denote by $\Lambda(s^{1-n}V_n^*)_{>0}$ the free commutative algebra (where, as usual, we mean *graded*-commutative) on the dual of the graded vector space $(s^{1-n}V_n)_{>0}$ of vectors with positive degree.

Theorem 4.1.5. *There is an isomorphism of graded algebras*

$$H^*(\Omega_0^n \mathrm{Th}(\gamma_{2,n}^{\perp})) \cong \Lambda(s^{1-n}V_n^*)_{>0}.$$

Proof. If (L, ∂) is a minimal Lie model of a simply connected space X, then the desuspension of the homology of (L, ∂) with zero differential and trivial products is a model for ΩX, which is a product of Eilenberg–Mac Lane spaces indexed by a basis of $H(L, \partial)$. The cohomology of ΩX is generated by the dual of $H(L, \partial)$, and the cohomology of $\Omega^n X$ is generated by the positive degree part of the dual of the $(1-n)$th desuspension of $H(L, \partial)$. □

We now summarize the results so far attained, both in the odd and the even-dimensional cases. From the construction given at the beginning of this section, the class κ_i is the dual of the desuspension $s^{1-n}\tilde{u}_{n+2i-1}$, which can be also denoted by $s^{-n}u_{n+2i}$. Observe that when n is even, if $u_{n+2k} \neq 0$, then $s^{-1}u_{n+2k}^* = \tilde{u}_{n+2k-1} \neq 0$ in V_n except for the top class u_{3n-6}. In the minimal Lie model $s^{-1}u_{3n-6}^* = c_{3n-7}$, which is zero after taking homology.

Corollary 4.1.6. *In the cohomology ring of $\Omega_0^n \mathrm{Th}(\gamma_{2,n}^{\perp})$, the class κ_i is non-trivial if and only if $i \leq n-3$ when n is odd and $i \leq n-4$ when n is even. There are many additional classes that are not κ-classes, and the first of them is $s^{1-n}[\tilde{u}_{n-3}, \tilde{u}_{n-1}]^*$, in dimension $n-3$.*

Below is a table with the dimensions of the graded vector space $s^{1-n}V_n^*$ in low degrees, for $n = 5, 6, 7$. A bold number indicates the presence of a κ-class.

n	1	2	3	4	5	6	7	8	9
5	–	**2**	–	**3**	–	5	–	6	–
6	–	**1**	1	**1**	2	–	2	2	3
7	–	**1**	–	**2**	–	**2**	–	**4**	–

The map $\Omega^n\mathrm{Th}(\gamma_{2,n}^\perp) \to \Omega^{n+1}\mathrm{Th}(\gamma_{2,n+1}^\perp)$ in cohomology

The inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ induces an inclusion $\mathrm{Gr}_2^+(\mathbb{R}^n) \subset \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$ that is covered by a bundle map $\gamma_{2,n}^\perp \oplus \mathbb{R} \to \gamma_{2,n+1}^\perp$. This induces a map between Thom spaces:

$$\Sigma\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \mathrm{Th}(\gamma_{2,n+1}^\perp) \tag{4.1.4}$$

and also an adjoint map $\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \Omega\mathrm{Th}(\gamma_{2,n+1}^\perp)$. Now, recall that the cohomology of $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is generated by the Euler class e_2 of the tautological bundle and the Euler class f_{n-2} of the complement of the tautological bundle. The induced homomorphism in cohomology

$$H^*(\mathrm{Gr}_2^+(\mathbb{R}^{n+1})) \longrightarrow H^*(\mathrm{Gr}_2^+(\mathbb{R}^n))$$

sends e_2 to itself and f_{n-2} to zero. Using the naturality of the Thom isomorphism, this implies that the homomorphism induced in cohomology by

$$\varphi\colon \Sigma\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \mathrm{Th}(\gamma_{2,n+1}^\perp)$$

sends the classes u_k to the suspensions su_{k-1}. From the previous computations, we deduce that, if $u_k \neq 0$, then

$$\varphi^*(u_k) = \begin{cases} 0 & \text{if } n+1 \text{ is odd and } k = 3(n+1)-6, \\ (-1)^r su_{n-2}^3 & \text{if } n+1 \text{ is odd and } k = 3(n+1)-8, \\ \text{is indecomposable} & \text{otherwise.} \end{cases}$$

Its adjoint $\phi\colon \mathrm{Th}(\gamma_{2,n}^\perp) \to \Omega\mathrm{Th}(\gamma_{2,n+1}^\perp)$ factors as

$$\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \Omega\Sigma\mathrm{Th}(\gamma_{2,n}^\perp) \longrightarrow \Omega\mathrm{Th}(\gamma_{2,n+1}^\perp).$$

A minimal Lie model $L(\mathrm{Th}(\gamma_{2,n}^\perp))$ of the leftmost space has been computed in the previous section and is the free Lie algebra on the cohomology of $\mathrm{Th}(\gamma_{2,n}^\perp)$ seen as a graded vector space, with a possibly non-trivial differential. A minimal Lie model $L(\Omega\Sigma\mathrm{Th}(\gamma_{2,n}^\perp))$ of the middle space is the same free Lie algebra with trivial differentials. A minimal Lie model $L(\mathrm{Th}(\gamma_{2,n+1}^{\perp}))$ of the rightmost space is the Lie algebra with trivial differential and trivial product on the desuspension of the homology of a minimal model $L(\mathrm{Th}(\gamma_{2,n+1}^{\perp}))$. The latter has been computed also in the previous section.

The first map sends the generators of $L(\mathrm{Th}(\gamma_{2,n-1}^{\perp}))$ to the generators of $L(\Omega\Sigma\mathrm{Th}(\gamma_{2,n-1}^{\perp}))$, and all decomposables to zero. The second map is the loop of the map φ, and is determined by it on generators. Therefore, if we denote by ϕ_* the homomorphism induced in Lie models, then

$$\begin{aligned}\phi_*(\tilde{u}_{n+2k-1}) &= s^{-1}\tilde{u}_{n+2k},\\ \phi_*(b_{2n-5}) &= 0,\\ \phi_*(c_{3n-7}) &= s^{-1}\tilde{u}_{3n-6},\\ \phi_*([x,y]) &= 0,\end{aligned}$$

where b_{2n-5} and c_{3n-7} only exist when n is even. The desuspension of the homology of $L(\mathrm{Th}(\gamma_{2,n}^{\perp}))$ with trivial differential and trivial product is a Lie model of $\Omega(\mathrm{Th}(\gamma_{2,n}^{\perp}))$, and the homomorphism

$$\Omega\phi_*\colon L(\Omega(\mathrm{Th}(\gamma_{2,n}^{\perp}))) \longrightarrow L(\Omega^2(\mathrm{Th}(\gamma_{2,n+1}^{\perp})))$$

is the desuspension of the homomorphism $H(\phi_*)$ induced in homology by ϕ_*. The classes b_{2n-5} and c_{3n-7} are not cycles in the homology of $L(\mathrm{Th}(\gamma_{2,n}^{\perp}))$, which is therefore generated by the classes $s^{-1}\tilde{u}_{n+2k-1}$ and desuspensions of indecomposables. Hence, $\Omega\phi_*$ is given by $\Omega\phi_*(s^{-1}\tilde{u}_{n+2k-1}) = s^{-2}\tilde{u}_{n+2k}$ and is trivial on desuspensions of decomposables. Analogously, the homomorphism

$$\Omega^n\phi_*\colon L(\Omega^n(\mathrm{Th}(\gamma_{2,n}^{\perp}))) \longrightarrow L(\Omega^{n+1}(\mathrm{Th}(\gamma_{2,n+1}^{\perp})))$$

is given in generators by $\Omega^n\phi_*(s^{-n}\tilde{u}_{n+2k-1}) = s^{-n-1}\tilde{u}_{n+2k}$ and is trivial on decomposables too.

From the last corollary of the previous section we deduce that

Corollary 4.1.7. *The homomorphisms induced in rational cohomology by the maps*

$$\begin{aligned}\Omega^n\phi_{n+1}\colon \Omega^n\mathrm{Th}(\gamma_{2,n}^{\perp}) &\longrightarrow \Omega^{n+1}\mathrm{Th}(\gamma_{2,n+1}^{\perp})\\ \Omega^n\phi_\infty\colon \mathrm{Th}(\gamma_{2,n}^{\perp}) &\longrightarrow \Omega_0^\infty\mathbf{MTSO}(2)\end{aligned}$$

send the classes κ_i to themselves and any other class to zero. Therefore:

(i) *The kernel of $\Omega^n\phi_{n+1}^*$ contains κ_{n-2} and κ_{n-3} if n is even. If n is odd, there is no κ-class in the kernel. The first class in this kernel is $s^{-n}[\tilde{u}_{n-2},\tilde{u}_n]^*$, in dimension $n-2$.*

(ii) *The kernel of* $\Omega^n \phi^*_\infty$ *consists of all* κ_i *with* $i > n-3$ *if* n *is odd and* $i > n-4$ *if* n *is even.*

(iii) *The cokernels of both homomorphisms contain no* κ*-classes. The first class in the cokernels is* $s^{1-n}[\tilde{u}_{n-3}, \tilde{u}_{n-1}]^*$*, in dimension* $n-3$*.*

In particular, both homomorphisms are $(n-3)$*-connected.*

Geometric interpretation

The cohomology ring of $\mathcal{E}(\Sigma_g, \mathbb{R}^n)$ is the ring of characteristic classes of Σ_g-subsurface bundles of the trivial $\mathbb{R}^n$-bundle. In detail, if X is a topological space and Σ is a manifold, then a Σ-manifold bundle is a fibre bundle with fibre Σ. A Σ-submanifold bundle of the trivial $\mathbb{R}^n$-bundle is a subset $E \subset X \times \mathbb{R}^n$ such that the restriction to E of the projection $\pi \colon X \times \mathbb{R}^n \to X$ is a Σ-manifold bundle. Two such k-submanifold bundles E, E' are *concordant* if there is a Σ-submanifold bundle Q of the trivial $\mathbb{R}^n$-bundle over $X \times [0,1]$ such that $Q \cap X \times \{0\} \times \mathbb{R}^n = E \times \{0\}$ and $Q \cap X \times \{1\} \times \mathbb{R}^n = E \times \{1\}$. In order to short the notation, a Σ-submanifold bundle of the trivial $\mathbb{R}^n$-bundle over X will be called (Σ, n)-submanifold bundle over X.

A (Σ, n)-submanifold bundle E over X has a classifying map

$$\Phi_E \colon X \longrightarrow \mathcal{E}(\Sigma, \mathbb{R}^n)$$

defined by sending a point $x \in X$ to the submanifold $\pi^{-1}_{|E}(x) \subset \mathbb{R}^n$. Two (Σ, n)-submanifold bundles are concordant if and only if their classifying maps are homotopic. Consider now the inclusion $\mathcal{E}(\Sigma, \mathbb{R}^n) \subset \mathcal{E}(\Sigma, \mathbb{R}^{n+1})$ induced by the inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Then a $(\Sigma, n+1)$-submanifold bundle E over X can be compressed to a (Σ, n)-submanifold bundle over X if and only if the classifying map of E lifts along the inclusion. In particular, the kernel of the homomorphism

$$H^*(\mathcal{E}(\Sigma, \mathbb{R}^{n+1})) \longrightarrow H^*(\mathcal{E}(\Sigma, \mathbb{R}^n)),$$

gives obstructions to compress a $(\Sigma, n+1)$-submanifold bundle into $X \times \mathbb{R}^n$. On the other hand, the cokernel gives obstructions to compress a concordance of $(\Sigma, n+1)$-submanifold bundles to a concordance of (Σ, n)-submanifold bundles.

In the stable range the relative cohomology of $\mathcal{E}(\Sigma, \mathbb{R}^n) \to \mathcal{E}(\Sigma, \mathbb{R}^{n+1})$ is isomorphic to the relative cohomology of $\Omega^n \mathrm{Th}(\gamma^\perp_{2,n})$, therefore, in that range the classes in the kernel and the cokernel of the second map of Corollary 4.1.7 give obstructions to compress $(\Sigma, n+1)$-manifold bundles and their concordances. We state it as a corollary:

Corollary 4.1.8. *The rational relative cohomology of the map*

$$\mathcal{E}(\Sigma_g, \mathbb{R}^n) \longrightarrow \mathcal{E}(\Sigma_g, \mathbb{R}^{n+1}) \tag{4.1.5}$$

vanishes up to degree $\min\{n-3, \frac{2}{3}(g-1)\}$*. If* $n-3 \leq \frac{2}{3}(g-1)$*, then the first generator in the cokernel of the homomorphism induced in rational cohomology is the class* $s^{1-n}[\tilde{u}_{n-3}, \tilde{u}_{n-1}]$ *in degree* $n-3$*. If* $n-2 \leq \frac{2}{3}(g-1)$*, then the first generator in the kernel is* $s^{-n}[\tilde{u}_{n-2}, \tilde{u}_n]$ *in degree* $n-1$

From the Madsen–Weiss theorem, the previous corollary and Theorem A, we deduce that

Corollary 4.1.9. *The rational relative cohomology of the map*

$$\mathcal{E}(\Sigma_g, \mathbb{R}^n) \longrightarrow \mathrm{BDiff}^+(\Sigma_g) \tag{4.1.6}$$

vanishes up to degree $\min\{n-3, \frac{2}{3}(g-1)\}$*. If* $n-3 \leq \frac{2}{3}(g-1)$*, then the first generator in the cokernel of the homomorphism induced in rational cohomology is the class* $s^{1-n}[\tilde{u}_{n-3}, \tilde{u}_{n-1}]$ *in degree* $n-3$*. If* n *is even (odd) and* $2(n-3) \leq \frac{2}{3}(g-1)$ *(resp.* $2(n-2)$*), then the first class in the kernel is* κ_{n-3} *(resp.* κ_{n-2}*).*

The connectivity of the map (4.1.6) can also be studied by means of the connectivity of its fibre, which is the space of embeddings of Σ_g in $\mathbb{R}^n$. From Corollary 1.4.7 in [GKW01], it follows that $\mathrm{Emb}(\Sigma_g, \mathbb{R}^n)$ is $(\min\{n-4, 2n-10\})$-connected, hence (4.1.6) is $(\min\{n-3, 2n-9\})$-connected.

The connectivity of the map (4.1.5) can also be studied by means of the following map of fibrations:

$$\begin{tikzcd}
\mathrm{Emb}(\Sigma_g, \mathbb{R}^n) \arrow[r] \arrow[d, "h"] & \mathcal{E}(\Sigma_g, \mathbb{R}^n) \arrow[r] \arrow[d, "f"] & \mathrm{BDiff}^+(\Sigma_g) \arrow[d, "="] \\
\mathrm{Emb}(\Sigma_g, \mathbb{R}^{n+1}) \arrow[r] & \mathcal{E}(\Sigma_g, \mathbb{R}^{n+1}) \arrow[r] & \mathrm{BDiff}^+(\Sigma_g).
\end{tikzcd}$$

Since the map between the base spaces is the identity, the maps g and h have the same connectivity. If $n \geq 6$, then h goes from an $(n-4)$-connected space to an $(n-3)$-connected space. Therefore h is at least $(n-3)$-connected, hence f is at least $(n-3)$-connected. If $n=5$, then $2n-10 = 0 < 1 = n-4$, so we know that $\mathrm{Emb}(\Sigma_g, \mathbb{R}^5)$ is 0-connected and that $\mathrm{Emb}(\Sigma_g, \mathbb{R}^6)$ is 2-connected. Therefore h is at least 1-connected, and so is f.

Note that if $n-3 \leq \frac{2}{3}(g-1)$, then Corollary 4.1.5 asserts that f is not $(n-2)$-connected, hence g is not $(n-2)$-connected.

Corollary 4.1.10. *The maps in Corollaries* 4.1.8 *and* 4.1.9 *are* $(n-3)$*-connected.*

Example 4.1.11. Let us discuss this problem in low dimensions. Let $X \to S^1$ be a line subbundle of the product bundle $S^1 \times \mathbb{R}^n$. Such a line bundle is classified by a map to $\mathbb{R}P^{n-1}$. Its unit sphere bundle is a 2-sheeted covering space of S^1, fibrewise contained in the product $S^1 \times \mathbb{R}^n$, hence classified by a map g to $C_2(\mathbb{R}^n)$. Moreover, the map

$$\mathbb{R}P^{n-1} \longrightarrow C_2(\mathbb{R}^n)$$

that sends each line to its intersection with S^{n-1} is a homotopy equivalence. In particular the characteristic classes of line bundles and 2-sheeted covering spaces in $\mathbb{R}^n$ coincide. The only non-trivial characteristic classes of a covering space over S^1 have degree 1, and we have:

$$H^1(C_2(\mathbb{R}^1);\mathbb{Q}) = 0, \quad H^1(C_2(\mathbb{R}^2);\mathbb{Q}) \cong \mathbb{Q}, \quad H^1(C_2(\mathbb{R}^3);\mathbb{Q}) = 0. \tag{4.1.7}$$

Therefore, if h is the universal characteristic class in $H^1(C_2(\mathbb{R}^2);\mathbb{Q})$, and the classifying map $g\colon S^1 \to C_2(\mathbb{R}^2) \cong S^1$ is non-trivial, then $g^*(h) \neq 0$ gives an obstruction to compress X to a subcovering space of $S^1 \times \mathbb{R}$. If $Y \to S^1$ is another subcovering space of $S^1 \times \mathbb{R}^2$ classified by a map $f\colon S^1 \to C_2(\mathbb{R}^2)$, then the difference $g^*(h) - f^*(h)$ gives an obstruction to compress an isotopy equivalence between X and Y viewed as subcovering spaces of $S^1 \times \mathbb{R}^3$ to an isotopy equivalence between X and Y.

Alternatively, the isomorphisms in (4.1.7) for $\mathbb{R}^2$ and $\mathbb{R}^3$ can be obtained by applying McDuff's theorem [McD75] (see also page 7), which establishes a homology equivalence

$$C_k(\mathbb{R}^n) \longrightarrow \Omega^n_k S^n$$

in degrees $\leq \frac{1}{2}k$. The rational cohomology of the loop space on the right can be computed as in the previous section: A minimal Lie model for S^n is the free Lie algebra on one generator $\tilde{u}$ in dimension n with trivial differential. Therefore

$$\pi_k(\Omega S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}\langle \tilde{u}_{n-1} \rangle & \text{if } k = n-1, \\ \mathbb{Q}\langle [\tilde{u}_{n-1}, \tilde{u}_{n-1}] \rangle & \text{if } k = 2n-2 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise}, \end{cases}$$

and the non-trivial class in (4.1.7) is $s^{-1}[\tilde{u}_1, \tilde{u}_1]^*$.

4.2 Rational stability for spaces of surfaces in closed manifolds

Recall that Theorem B states that the homology groups of $\mathcal{E}(\Sigma_g, M)$ are independent of g in the stable range $* \leq \frac{2}{3}(g-1)$ when M is simply connected, of dimension ≥ 5 and *with non-empty boundary*. In this section we prove that the same result is true for parallelizable manifolds with empty boundary if we take rational coefficients.

It was first discovered by Church [Chu12] (see also [RW13]) that configuration spaces of points in a closed manifold satisfy rational homological stability (this fact for manifolds with boundary follows from McDuff's theorem (see page 7)). Following the results in [MR85] on the rational homotopy type of spaces of maps into spheres, Miller and Bendersky [BM12] gave a new proof of Church's result as follows: McDuff's theorem states that there is a rational homology equivalence

$$C_r(M) \longrightarrow \Gamma(\mathrm{Th}^{\mathrm{fib}}(TM) \to M)_r.$$

Miller and Bendersky proved that the path components of the right-hand side space are rationally homotopy equivalent in most cases, which implies that the rational homology of $C_r(M)$ is independent of r in the stability range $\frac{1}{2}r$. We follow their argument in what follows.

If M is parallelizable, then the space $\Gamma_c(\mathcal{S}(TM) \to M)$ (see page 9) is homotopy equivalent to the mapping space $\mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))$. As proven in Proposition 3.3.2, there is a bijection

$$\pi_0 \mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp})) \longleftrightarrow 2\mathbb{Z} \times H_2(M;\mathbb{Z}) \tag{4.2.1}$$

defined as follows: We let e_2 be the Euler class of the tautological bundle $\gamma_{2,n}$ over $\mathrm{Gr}_2^+(\mathbb{R}^n)$. We denote by $u_k^{\mathbb{Z}}$ (resp. u_k) the image of e^{k-1} under the Thom isomorphism with integral (resp. rational) coefficients. After choosing an orientation on M, the correspondence (4.2.1) assigns to each map f the pair given by the Poincaré dual of $f^*(u_n^{\mathbb{Z}}) \in H_c^n(M;\mathbb{Z})$ and the Poincaré dual of $f^*(u_{n-2}^{\mathbb{Z}}) \in H_c^{n-2}(M;\mathbb{Z})$. The mapping subspace $\mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g$ consists of those maps f such that the Poincaré dual of $f^*(u_n^{\mathbb{Z}})$ is $2-2g$. We will write $\mathrm{map}_{c,f}(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g$ for the component that contains f. Our main theorem in this case establishes a homology equivalence

$$\mathcal{E}(\Sigma_g, M) \longrightarrow \mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g.$$

Lemma 4.2.1. *For a parallelizable manifold M the rational cohomology of the space $\mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g$ is independent of g, as long as $g \neq 1$.*

Proof. Recall that, by the previous section,

$$\mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}} \simeq \bigvee_{k=-1}^{n-3} S_{\mathbb{Q}}^{n+2k}, \qquad \mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}} \simeq C_3^{n-2} \vee \bigvee_{k=0}^{n-4} S_{\mathbb{Q}}^{n+2k},$$

depending on whether n is odd or even. In addition, composing with the rationalization map yields a rational homotopy equivalence

$$L_{\mathbb{Q},f}\colon \mathrm{map}_{c,f}(M, \mathrm{Th}(\gamma_{2,n}^{\perp})) \longrightarrow \mathrm{map}_{c,f_{\mathbb{Q}}}(M, \mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}})$$

where we denot by $f_{\mathbb{Q}}$ the image of a map f. We remark that if $H_2(M;\mathbb{Z})$ has torsion, then there are maps f and f' in different connected components but whose rationalizations $f_{\mathbb{Q}}$ and $f'_{\mathbb{Q}}$ are in the same component.

For a non-zero rational number q and a natural number n, there are rational homotopy equivalences $\alpha_{q,n}\colon S^n_{\mathbb{Q}} \to S^n_{\mathbb{Q}}$ such that $\alpha^*_{q,n}(u_n) = qu_n$. The maps $\alpha_{q,n}$ induce homotopy equivalences

$$\beta_{q,n}\colon \bigvee_{k=-1}^{n-3} S^{n+2k}_{\mathbb{Q}} \longrightarrow \bigvee_{k=-1}^{n-3} S^{n+2k}_{\mathbb{Q}} \qquad \text{if } n \text{ is odd,}$$

$$\beta_{q,n}\colon C_3^{n-2} \vee \bigvee_{k=0}^{n-4} S^{n+2k}_{\mathbb{Q}} \longrightarrow C_3^{n-2} \vee \bigvee_{k=0}^{n-4} S^{n+2k}_{\mathbb{Q}} \qquad \text{if } n \text{ is even,}$$

by extending $\alpha_{q,n}$ with the identity on the other factors. By construction, if γ is a compactly supported map from M to $\mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}}$, then $(\beta_{q,n}\gamma)^*(u_n) = q\gamma^*(u_n)$ and $(\beta_{q,n}\gamma)^*(u_{n-2}) = \gamma^*(u_{n-2})$. Hence, if g and h are numbers different from 1 so that $q = \frac{2-2h}{2-2g}$ is well-defined and different from 0, and $f \in \mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g$, then composing with $\beta_{q,n}$ defines a homotopy equivalence between connected components

$$B_{q,n}\colon \mathrm{map}_{c,f_{\mathbb{Q}}}(M, \mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}})_g \longrightarrow \mathrm{map}_{c,\beta_{q,n}f_{\mathbb{Q}}}(M, \mathrm{Th}(\gamma_{2,n}^{\perp})_{\mathbb{Q}})_h.$$

Let $f'\colon M \to \mathrm{Th}(\gamma_{2,n}^{\perp})$ be a map such that the Poincaré dual of $f'^*(u_n^{\mathbb{Z}})$ is $2-2h$ and such that $f'^*(u_{n-2}^{\mathbb{Z}}) = f^*(u_{n-2}^{\mathbb{Z}})$. Then $\beta_{q,n}f_{\mathbb{Q}}$ is in the same component as $f'_{\mathbb{Q}}$, therefore the zig-zag

$$\mathrm{map}_{c,f}(M, \mathrm{Th}(\gamma_{2,n}^{\perp})) \xrightarrow{L_{\mathbb{Q},f}} \bullet \xrightarrow{B_{q,n}} \bullet \xleftarrow{L_{\mathbb{Q},f'}} \mathrm{map}_{c,f'}(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))$$

gives a rational homotopy equivalence. Considering all components at the time, one obtains an isomorphism in rational cohomology

$$H^*(\mathrm{map}_c(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_h) \longrightarrow H^*(\mathrm{map}_{c,}(M, \mathrm{Th}(\gamma_{2,n}^{\perp}))_g). \qquad \square$$

From this lemma and Theorem A (and also Lemma 2.3.4 for the case $g = 1$) we deduce Theorem E, which reads as follows:

Corollary 4.2.2. *If M is a closed parallelizable manifold, then the rational homology of $\mathcal{E}(\Sigma_g, M)$ is independent of g in degrees $\leq \frac{2}{3}(g-1)$.*

4.3 Distances on spaces of submanifolds

Galatius and Randal-Williams defined a topology in the set of closed submanifolds of $\mathbb{R}^n$ in [GRW10]. Bökstedt and Madsen in [BM11] proved that a C^1 version of this topology is metrizable by showing that it is regular and second countable. In this section we give an explicit metric to the space considered by Bökstedt and Madsen.

The space $\Psi_m(\mathbb{R}^n)$ and its topology

Let $\Psi_m(\mathbb{R}^n)$ be the set of all submanifolds of $\mathbb{R}^n$ of dimension m that are closed as subsets of $\mathbb{R}^n$ (including the empty set). We will denote it by $\Psi(\mathbb{R}^n)$ if m is clear from the context. If $A \subset \mathbb{R}^n$ is an open subset, then $\Psi_m(A)$ denotes the set of all closed submanifolds of A of dimension m.

The GRW-topology

The topology of Galatius and Randal-Williams [GRW10] is defined in three steps. First, if $W \in \Psi_m(A)$ is a submanifold, then the space of compactly supported sections of its normal bundle $\Gamma_c(NW \to W)$ is a subspace of the space of sections of the restriction of $T\mathbb{R}^n$ to W, which is in turn canonically identified with the product $\prod_{i=1}^n C_c^\infty(W)$ of the space $C_c^\infty(W)$ of compactly supported smooth functions on W. This latter space is endowed with the Whitney C^∞ topology. Recall that a subbasic neighbourhood in the Whitney C^∞ topology of a function $f\colon W \to \mathbb{R}$ is given by a pair $((X_1, \dots, X_r), \epsilon)$, where $\epsilon\colon W \to (0, \infty)$ is a smooth function and $X_1, \dots, X_r$ is a finite set of smooth vector fields on NW. A smooth function g is in the neighbourhood determined by such data if

$$|X_1 \cdots X_r(f - g)(x)| < \epsilon(x) \quad \text{for all } x \in W.$$

We endow $\Gamma_c(NW \to W)$ with the subspace topology. Finally, let us denote by $\exp_W\colon NW \to A$ the exponential map, which is partially defined on NW. A subbasic neighbourhood of W is the image under $\exp_W$ of a subbasic neighbourhood of the zero section. This defines the *compactly supported topology*, and we denote by $\Psi_m(A)^{cs}$ the set $\Psi_m(A)$ equipped with this topology.

Second, if K is a compact subset of A, we define $\Psi_m(K \subset A)^{cs}$ as the quotient space that results from identifying two submanifolds $W, W' \in \Psi_m(A)^{cs}$ if there is an open neighbourhood U of K such that $U \cap W = U \cap W'$. The K-*topology* on $\Psi_m(A)$ is the coarsest topology that makes the quotient map $\Psi_m(A) \to \Psi_m(K \subset A)^{cs}$ continuous. We denote by $\Psi_m(A)^{cs,K}$ the set $\Psi_m(A)$ endowed with the K-topology.

Finally, the *GRW-topology* on $\Psi_m(A)$ is the initial topology with respect to the identity maps $\Psi_m(A) \to \Psi_m(A)^{cs,K}$, for all compact subsets K of A.

The C^1 GRW-topology

Bökstedt and Madsen considered instead the topology on $\Psi_m(A)$ obtained by the same procedure, but endowing the set of C^∞-sections $\Gamma_c(NW \to W)$ with the C^1 topology, instead of the C^∞ topology. For that, we view again $\Gamma_c(NW \to W)$ as a subset of $\mathrm{map}_c(W, \mathbb{R}^n) \cong \prod_{i=1}^n C_c^\infty(W)$, and a subbasic neighbourhood of $C_c^\infty(W)$ with the C^1 topology is given by a pair (X, ϵ), where $\epsilon\colon W \to [0,\infty)$ is a smooth function and X is a smooth vector field on NW. A submanifold W' belongs to the neighbourhood given by (X, ϵ) if $W = \exp \circ f$ for some section f of the normal bundle of W and $|X(f)(x)| < \epsilon(x)$ for all $x \in W$. We denote by $\Psi_m^1(A)^{cs}$ the set $\Psi_m(A)$ endowed with this topology.

We then follow the steps above to define the topologies $\Psi_m^1(K \subset A)^{cs}$ and $\Psi_m^1(A)^{cs,K}$. Finally, we let $\Psi_m^{gs}(A)$ be the set $\Psi_m(A)$ endowed with the initial topology with respect to the maps $\Psi_m(A) \to \Psi_m^1(A)^{cs,K}$ for all compact subsets K of A. Theorem 2.1 in [BM11] proves that this space is metrizable. In what follows we construct a metric for $\Psi_m^{gs}(A)$.

The metric

We start by introducing some terminology and conventions on metric spaces. Recall that a *pseudo-metric* on a set X is a symmetric function $d\colon X \times X \to [0,\infty)$ satisfying the triangle inequality and such that $d(x,x) = 0$ for all $x \in X$. The balls of a pseudo-metric on X define a basis of a topology, and that topology is Hausdorff if and only if d is a metric (that is, $d(x,y) \neq 0$ whenever $x \neq y$). If (X, d) is a metric space and $f\colon Y \to X$ is a function from a set Y, then $d \circ (f \times f)$ is a pseudo-metric on Y, which is a metric if f is injective. It is called the *restriction (pseudo-)metric of* d.

For a metric space (X, d), we will denote by $\mathbf{B}_\epsilon^d(x)$ the ball of radius ϵ centered at x with respect to the distance d. If $K \subset X$ is a subset, we denote

$$d(x, K) = \inf_{y \in K} d(x, y).$$

We also recall that the set of compact subsets $\mathcal{P}_c(X)$ of a metric space X is naturally endowed with the Hausdorff distance, which is defined as

$$d_H(K, K') = \max\left\{\max_{x \in K}\{d(x, K')\},\ \max_{x \in K'}\{d(x, K)\}\right\}.$$

The distance $\bar{d}_H$

Let $A \subset \mathbb{R}^n$ be an open subset, and let d_0 be the Euclidean distance on $\mathbb{R}^n$. Let us endow the sphere with the distance given by the angle between points. Let the Grassmannian $\mathrm{Gr}_{m,n}$ of m-planes in $\mathbb{R}^n$ be endowed with the distance

$$d_1(L, L') = (d_0)_H(L \cap S^{n-1}, L' \cap S^{n-1}).$$

Then d_0 and d_1 define pseudo-metrics on $\mathrm{Gr}_m(TA) := A \times \mathrm{Gr}_{m,n}$, and we obtain a new metric on $\mathrm{Gr}_m(TA)$ as

$$d((a, L), (a', L')) = d_0(a, a') + d_1(L, L').$$

We endow its one-point compactification $\overline{\mathrm{Gr}_m(TA)}$ with the following metric (cf. [Pet, Man89]): Here we let 0 denote the origin in $\mathbb{R}^n$ and

$$\begin{aligned}\bar{d}(x, \infty) &= \min\left\{\frac{1}{1 + d_0(x, 0)},\ d(x, \mathrm{Gr}_m(T\mathbb{R}^n) \setminus \mathrm{Gr}_m(TA))\right\},\\ \bar{d}(x, y) &= \min\left\{d(x, y),\ \bar{d}(x, \infty) + \bar{d}(y, \infty)\right\}.\end{aligned}$$

The notation $d_0(x, 0)$ stands for the distance d_0 between the first coordinate of x and the origin. The reader is invited to examine the metric on the one-point compactification of A given in page 121, which is simpler to grasp than this one. An advantage of these metrics on one-point compactifications is that the inclusion of $\mathrm{Gr}_m(TA)$ into $\overline{\mathrm{Gr}_m(TA)}$ is a local isometry.

Given a submanifold $W \in \Psi(A)$, the Gauss map

$$\phi\colon W \longrightarrow \mathrm{Gr}_m(TA)$$

sends a point $p \in W$ to the pair (p, T_pW). We write $\bar{\phi}$ for the composite of the Gauss map and the inclusion into $\overline{\mathrm{Gr}_m(TA)}$. There is an inclusion

$$\Psi_m(A) \longrightarrow \mathcal{P}_c(\overline{\mathrm{Gr}_m(TA)})$$

that sends a (possibly empty) submanifold W to the subset $\bar{\phi}(W) \cup \{\infty\}$, which endows $\Psi_m(A)$ with the restriction of the Hausdorff metric $\bar{d}_H$.

The pseudo-distance $\bar{v}_H$

Let us endow the set $F((0, 1], \mathbb{R}^n)$ of non-increasing real-valued functions on $(0, 1]$ with a topology. For that, consider the composite

$$F((0, 1], \mathbb{R}) \xrightarrow{\text{graph}} \mathcal{P}((0, 1] \times \mathbb{R}) \overset{\tau}{\dashrightarrow} \mathcal{P}_c(([0, 1] \times \mathbb{R})/(\{0\} \times \mathbb{R})). \tag{4.3.1}$$

The first function sends each function to its graph and fills in the discontinuities:

$$\mathrm{graph}(f) = \{(x_1, x_2) \in (0,1] \times \mathbb{R} \mid f(x_1^-) \le x_2 \le f(x_1^+)\}.$$

The space $([0,1] \times \mathbb{R})/(\{0\} \times \mathbb{R})$ is a partial compactification of $(0,1] \times \mathbb{R}$. We consider on $(0,1] \times \mathbb{R}$ the restriction of the distance on $\mathbb{R}^2$ given by the norm $\|-\|_\infty$, and we endow the quotient $([0,1] \times \mathbb{R})/(\{0\} \times \mathbb{R})$ with a distance $\bar{v}$ as we did above with the one-point compactification: If $\bar{0}$ denotes the added point, then we define

$$\bar{v}((x_1, x_2), \bar{0}) = x_1, \qquad \bar{v}(x, y) = \min\{\|x - y\|_\infty,\ x_1 + y_1\}.$$

Again, its set of compact subsets carries the Hausdorff distance. The function τ sends a subset S to the union $S \cup \{\infty\}$, which in general is not a compact subset, but it is so if S is the graph of a function. Since the composite (4.3.1) is injective, it endows $F((0,1], \mathbb{R})$ with the restriction metric, which we denote by $\bar{v}_H$. It may be observed that this distance defines the compact-open topology on its subspace of continuous functions.

Let $(\bar{A}, \bar{d}_0)$ be the one-point compactification of (A, d_0) as above:

$$\begin{aligned} \bar{d}_0(x, \infty) &= \min\left\{\frac{1}{1 + d_0(x, 0)},\ d_0(x, \mathbb{R}^n \setminus A)\right\}; \\ \bar{d}_0(x, y) &= \min\left\{d_0(x, y),\ \bar{d}_0(x, \infty) + \bar{d}_0(y, \infty)\right\}. \end{aligned}$$

For each $W \in \Psi_m(A)$, let us denote $W_r = \left\{x \in W \mid \bar{d}_0(x, \infty) \ge r\right\}$ and let $\mathrm{vol}(W_r)$ be the volume of W_r. The function $r \mapsto \mathrm{vol}(W_r)$, which is defined in the half-open interval $(0,1]$, is clearly decreasing, but it is not continuous as the following example shows: If $W = S^m \subset \mathbb{R}^m \subset \mathbb{R}^n$, then, as the points of the sphere are at distance 1 from the origin,

$$\mathrm{vol}(S^m_r) = \begin{cases} 0 & \text{if } r > \bar{d}_0(\infty, S^m) = \frac{1}{2}, \\ \mathrm{vol}(S^m) & \text{if } r \le \bar{d}_0(\infty, S^m) = \frac{1}{2}. \end{cases}$$

The function

$$\mathrm{vol}(-)\colon \Psi_m(\mathbb{R}^n) \longrightarrow F((0,1], \mathbb{R})$$

that assigns to each submanifold W the function $r \mapsto \mathrm{vol}(W_r)$ is not injective, hence $\Psi_m(\mathbb{R}^n)$ does not inherit a metric from it, but it does inherit a pseudometric that we denote also by $\bar{v}_H$. The sum

$$\bar{d}_\psi(W, W') = \bar{d}_H(W, W') + \bar{v}_H(W, W')$$

is a metric on $\Psi_m(A)$.

Theorem 4.3.1. *The distance $\bar{d}_\psi$ is a metric for the space $\Psi_m^{gs}(A)$.*

One might expect the distance $\bar{d}_H$ to be a metric for $\Psi_m^{gs}(A)$, but it is not. To show why, we take a compact submanifold W. A submanifold W' is close to W in $\Psi^{gs}(A)$ if there is a *global* section of the normal bundle of W that is close to the zero section and $W' = f(W)$. On the other hand, W' is close W in $(\Psi(A), \bar{d}_H)$ if each point of W is close to W' and each point of W' is close to W. Therefore, if W'' and W' are disjoint submanifolds that are both at distance δ from W, then the union of W' and W'' is a submanifold that is at distance δ from W but is not the image of a global section. Nevertheless, we will prove that it is *locally* the image of local sections.

In order to make this explicit, we introduce in the following section a new topology on $\Psi_m(A)$ for which $\bar{d}_H$ is a metric. In this topology, a submanifold W' is close to a submanifold W if, when restricted to a compact subset, W' is *locally* the image of a section of the normal bundle of NW that is close to the zero section. We will denote by $\Psi_m^{ls}(A)$ the set $\Psi_m(A)$ endowed with this topology.

The next section is devoted to prove that $\bar{d}_H$ is a metric for $\Psi_m^{ls}(A)$. Relying on this result, in the subsequent section we prove that $\bar{d}_\psi$ is a metric for $\Psi_m^{gs}(A)$.

The space $\Psi_m^{ls}(A)$ and the $\bar{d}_H$-topology

We start by giving an equivalent definition of the space $\Psi^{gs}(A)$ and introducing a new space $\Psi^{ls}(A)$. Consider the normal bundle NW as a subspace of $T\mathbb{R}^n_{|W} = W \times \mathbb{R}^n$. For each neighbourhood U of W, there is an inclusion

$$\Gamma(NW_{|U} \to U) \longrightarrow \mathrm{map}(U, \mathbb{R}^n),$$

and we consider the C^1 operator norm of a section f at a point $x \in U$, namely

$$\|f\|_x = \|f(x)\| + \|Df(x)\|.$$

Recall that the operator norm of a linear map A defined on an Euclidean space is defined as $\|A\| = \max_{\|v\|=1} A(v)$.

The topology on $\Psi^{gs}(A)$ is given as follows: If W is a non-empty submanifold, then each compact subset $K \subset A$ and each $\epsilon > 0$ define a neighbourhood $(K, \epsilon)^{gs}$ of W; a submanifold W' belongs to it if there is a section f of the normal bundle $NW \to W$ such that $\exp_W(f(W)) \cap K = W' \cap K$ and $\|f - \mathrm{Id}\|_x < \epsilon$ for all $x \in W$ such that $\exp_W \circ f(x) \in W' \cap K$. If W is the empty submanifold, then each compact subset K defines a neighbourhood K^{gs} of $\emptyset$; a submanifold W' belongs to it if $W' \cap K = \emptyset$.

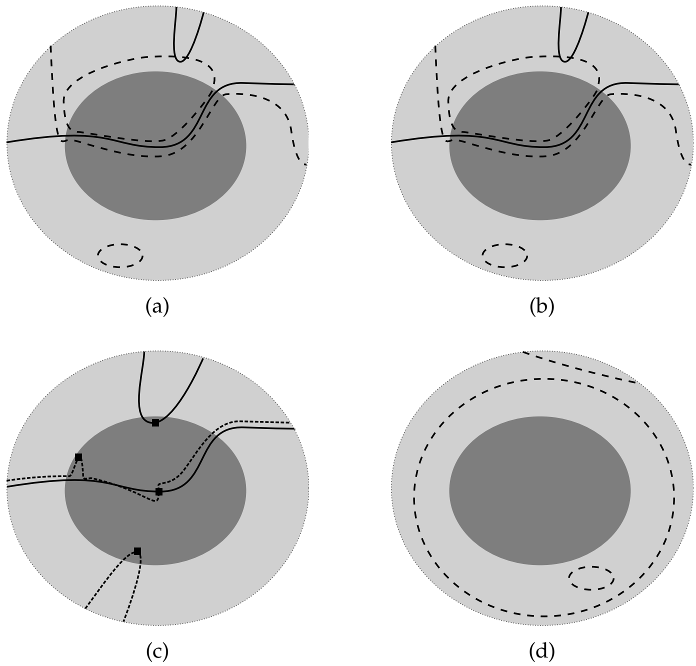


Figure 4.1: The light grey area is an open subset A while the dark area is a compact subset $K \subset A$. We denote the dashed submanifold by W' and the non-dashed submanifold by W. In Figure 4.1a, W' is close to W both in $\Psi^{gs}(A)$ and $\Psi^{ls}(A)$. In Figure 4.1b, W' is close to W in Ψ^{ls} but not in $\Psi^{gs}(A)$. In Figure 4.1c, W' is far from W in $\Psi^{ls}(A)$, hence in $\Psi^{gs}(A)$ too. The small squares indicate points that are far from the other submanifold. In Figure 4.1d, W' is close to $\emptyset$.

Let $\Psi^{ls}(A)$ be the set $\Psi(A)$ endowed with the following topology. If W is a non-empty submanifold, then each compact subset $K \subset A$ and each $\epsilon > 0$ define a neighbourhood $(K, \epsilon)^{ls}$ of W: A submanifold W' belongs to it if

(i) there is a finite-sheeted covering $\pi_{|X} \colon X \subset NW \to Y$ onto a neighbourhood Y of $W \cap K$ such that $\exp_W(X) \cap K = W' \cap K$;

(ii) for each local section f of $\pi_{|X}$, it holds that $\|f - \mathrm{Id}\|_x < \epsilon$ for all x in the domain of f.

If W is the empty submanifold, then each compact subset K defines a neighbourhood K^{ls} of W; a submanifold W' belongs to it if $W' \cap K = \emptyset$.

Remark 4.3.2. Note that $W' \in (K, \epsilon)^{gs}$ if and only if $W' \in (K, \epsilon)^{ls}$ and $\pi_{|W \cap K}$ has a single sheet.

In this section we prove the following:

Theorem 4.3.3. *The distance $\bar{d}_H$ is a metric for the topological space $\Psi^{ls}(A)$.*

We will use the following notation and facts:

(i) If $x \in W'$ and $y \in W$, we will write $d(x, y)$ for $d(\phi(x, W'), \phi(y, W))$.

(ii) If W is a submanifold of A, we let $\pi\colon A \dashrightarrow W$ be the partially defined map that sends a point to its d_0-closest point in W. A point $x \in A$ belongs to the domain of π if there is a *unique* point $y^* \in W$ that minimizes the distance $d_0(x, W)$.

(iii) An injectivity radius of a compact submanifold $V \subset A$ (in our case V will be of the form W_r) is a number $i > 0$ such that the exponential map $\exp_V$ is well defined when restricted to the vector subspace $N_i W \subset NW$ of vectors of norm $< i$. The image of $N_i W$ under the exponential map lays in the domain of π.

(iv) For all x and y we have

$$\begin{aligned} d_0(x, y) \leq d(x, y), \quad \bar{d}_0(x, y) \leq \bar{d}(x, y), \quad \bar{d}_0(x, y) \leq d_0(x, y), \\ \bar{d}(x, y) \leq d(x, y), \quad \bar{d}_0(x, \infty) = \bar{d}(x, \infty). \end{aligned}$$

(v) If $x \in W'_r$ and $\bar{d}(x, W) < \delta < r$, then $\bar{d}(x, W) = d(x, W)$. If y^* minimizes the distance, then $y^* \in W'_{r-\delta}$.

(vi) The distances d and d_0 restricted to the compact subset W_r define the same topology, hence $d(-,-)\colon W_r \times W_r \to (0, \infty)$ is uniformly continuous with respect to the metric $d_0 \times d_0$. Therefore for each $\delta > 0$ there is a $\lambda(\delta, r) > 0$ such that if $x, y \in W_r$ and $d_0(x, y) < \lambda(\delta, r)$, then $d(x, y) < \delta$.

We now give two lemmas that are needed to show that the $\bar{d}_H$-topology is finer than the topology in $\Psi^{ls}(A)$. Let us suppose that $x \in W'$ and x belongs to the domain of π. In principle, the knowledge of $\bar{d}_H(W, W')$ does not imply knowledge about $\bar{d}(x, \pi(x))$: It may very well happen that the distance $\bar{d}(x, W')$ is realized in a point different from $\pi(x)$. The first lemma shows that $\bar{d}(x, \pi(x))$ can be taken as small as desired if we assume that $\bar{d}_H(W, W')$ is small enough. The second lemma will prove that if $K \subset A$ is a compact subset and $\bar{d}_H(W, W')$ is small enough, then Condition (i) in the definition of $\Psi^{ls}(A)$) holds.

Lemma 4.3.4. *Let $W, W' \in \Psi(A)$, let $r > 0$, let $\delta < r$, and let $x \in W'_r$ be in the domain of π. Let $\sigma\colon (0, 1]^2 \to \mathbb{R}$ be a function such that*

$$\sigma(r, \delta) < \min\left\{\delta, \frac{\lambda(\delta, r-\delta)}{2}\right\}.$$

If $\bar{d}(x, W) < \sigma(r, \delta)$*, then* $d(x, \pi(x)) < 2\delta$*.*

Proof. As $x \in W_r$ and $\delta < r$, by fact (v) above, $\bar{d}(x, W) = d(x, W)$. Let $y^* \in W_{r-\delta}$ be a point minimizing $d(x, W) < \delta$. By its very definition, $\pi(x)$ minimizes $d_0(x, W)$, which is smaller than $d(x, W) < \delta$, hence $\pi(x) \in W_{r-\delta}$ too. From the inequalities

$$d_0(x, \pi(x)) \leq d_0(x, y^*) < \frac{\lambda(\delta, r-\delta)}{2}$$

we get

$$d_0(\pi(x), y^*) \leq d_0(x, \pi(x)) + d_0(x, y^*) < 2\frac{\lambda(\delta, r-\delta)}{2},$$

therefore $d(\pi(x), y^*) < \delta$, so

$$d(x, \pi(x)) \leq d(x, y^*) + d(\pi(x), y^*) < \delta + \delta = 2\delta. \qquad \square$$

Lemma 4.3.5. *Let* $W, W' \in \Psi(A)$ *and* $r > 0$*. Let* $\delta < \{\frac{1}{7}r, \frac{1}{4}\pi\}$ *and let* $i_{r-6\delta}$ *be an injectivity radius for* $W_{r-6\delta}$*. Let* $\rho\colon (0,1] \to \mathbb{R}$ *be a function such that*

$$\rho(r) < \min\{\sigma(\delta, r-5\delta), i_{r-6\delta}, \delta\}.$$

If $\bar{d}(W, W') < \rho(r)$*, then* π *is defined in the whole* $W'_{r-5\delta}$ *and* $\pi_{|W'_{r-5\delta}}$ *is a proper submersion and every point in* $W_{r-\delta}$ *belongs to its image. So there exist neighbourhoods* $W'_r \subset X \subset W'$ *and* $W_r \subset Y \subset W$ *such that* $\pi_{|X}\colon X \to Y$ *is a finite-sheeted covering map.*

Proof. For a point $x \in W_{r-5\delta}$, we have that $\bar{d}_0(x, W) < \bar{d}(x, W) < \delta$, so if y^* minimizes $\bar{d}_0(x, W)$, then $y^* \in W_{r-6\delta}$ (hence $y^* \neq \infty$ so $d_0(x, y^*) = \bar{d}_0(x, y^*)$). We also have that $d_0(x, y^*) = \bar{d}_0(x, y^*) \leq \bar{d}_0(x, W) \leq \bar{d}(x, W) < i_{r-6\delta}$, hence y^* is unique, so π is defined on x.

Let us assume that there is a singular point x in $W'_{r-5\delta}$ for $\pi_{W'_{r-5\delta}}$. Then the planes $T_x W'$ and $T_{\pi(x)} W$ are perpendicular, hence $d_1(x, \pi(x)) = \frac{1}{2}\pi$, so $d(x, \pi(x)) \geq \frac{1}{2}\pi$, but this is impossible, because by Lemma 4.3.4 the hypothesis $\bar{d}(x, W) < \sigma(\delta, r-5\delta)$ implies that $d(x, \pi(x)) < 2\delta < \frac{1}{2}\pi$. Hence the map $\pi_{|W'_{r-5\delta}}$ is a submersion, which is proper because $W_{r-5\delta}$ is compact.

Finally, let us prove that any $y \in W_{r-\delta}$ has a preimage. Again by fact (v), we have that $d(y, W') = \bar{d}(y, W')$, therefore $d_0(y, W') < \{\delta, i_{r-6\delta}\}$. Let $x^* \in W'_{r-2\delta}$ be its d_0-closest point, and recall that we have already proven that $\pi(x^*) \in W_{r-3\delta}$ exists. Let $\eta := d_0(y, x^*) \geq d_0(x^*, \pi(x^*))$, and observe that both $\mathbf{B}^{d_0}_\eta(y) \cap W$ and $\mathbf{B}^{d_0}_\eta(\pi(x)) \cap W$ are connected because $\eta < i_{r-6\delta}$, and also that both are contained in $W_{r-4\delta}$ because $\eta < \delta$. Let γ be a path from $\pi(x^*)$ to y contained in the union of the balls. Then there is a lift of $\gamma(0) = \pi(x^*)$ along $\pi_{|W_{r-5\delta}}$, and, because $\pi_{|W_{r-5\delta}}$ is a submersion from the compact manifold with boundary $W_{r-5\delta}$, we will be able to lift the path globally provided that the local lifts stay far from $\partial W'_{r-5\delta} = \{x \in W' \mid \bar{d}(x, \infty) = r - 5\delta\}$. This is clear, because if $\pi(x) = \gamma(t) \subset W_{r-4\delta}$ then x belongs to the interior of $W_{r-5\delta}$. $\square$

Now that we know that, for a given compact subset $K \subset A$, all submanifolds that are $\bar{d}_H$-close enough to W satisfy Condition (i) in the definition of $\Psi^{ls}(A)$, we move on to the second condition. If $x \in W'$ is in the image of a local section of the normal bundle of W, then the following lemma relates $d(x, \pi(x))$ and $\|f_x\|_{\pi(x)}$, where f is a local section of $\pi_{|W'}$ whose domain contains $\pi(x)$.

Lemma 4.3.6. *Let $x \in W$ and let f be a local section of NW. Then $\|(f - \mathrm{Id})(x)\| = d_0(x, f(x))$ and $\|D(f - \mathrm{Id})(x)\| = \tan(d_1(x, f(x)))$, where $f(x)$ is viewed as a point in $f(W)$.*

Proof. The first equality is evident. For the second, notice that $(D(f - \mathrm{Id})(x))^{-1}$ is the projection of $T_{f(x)} f(W) \subset \mathbb{R}^n$ onto $T_x W$. Thus, the vector v in $T_x W \cap S^{n-1}$ that realizes $\|D(f - \mathrm{Id})(x)\|$ also realizes $d_1(x, f(x))$. By definition, $d_1(x, f(x))$ is the angle θ between v and $\frac{Df(x)(v)}{\|Df(x)(v)\|}$ while $\|D(f - \mathrm{Id})(x)\|$ is the distance between v and $Df(x)(v)$, which is $\tan(\theta)$. $\square$

It is clear that the empty submanifold has the same neighbourhoods in $(\Psi(A), \bar{d}_H)$ and in $\Psi^{ls}(A)$. The following lemma shows that the same holds with any submanifold $W \in \Psi(A)$, thus finishing the proof of Theorem 4.3.3.

Proposition 4.3.7. *Let W be a non-empty submanifold in $\Psi(A)$.*

(i) *For any pair K, ϵ there is a δ such that $\mathbf{B}_\delta^{\bar{d}_H}(W) \subset (K, \epsilon)^{ls}$.*

(ii) *For any δ, there is a pair K, ϵ such that $(K, \epsilon)^{ls} \subset \mathbf{B}_\delta^{\bar{d}_H}(W)$.*

Proof. Let $r = \bar{d}_0(\infty, K)$, let $\delta = \min\{\rho(r), \sigma(r, \frac{1}{2}\arctan(\frac{1}{2}\epsilon))\}$ and suppose that $W' \in \mathbf{B}_\delta^{\bar{d}_H}(W)$. Since $\delta < \rho(r)$, by Lemma 4.3.5 there exist neighbourhoods $W'_r \subset X \subset W'$ and $W_r \subset Y \subset W$ such that $\pi_{|X}$ is a finite-sheeted covering map onto Y, that is, Condition (i) holds. In addition, if f is a local section of $\pi_{|W' \cap K}$, then, by Lemmas 4.3.6 and 4.3.4,

$$\|(f - \mathrm{Id})(y)\| = d_0(y, f(y)) \leq d(y, f(y)) < \arctan\left(\frac{\epsilon}{2}\right) \leq \frac{\epsilon}{2};$$
$$\|D(f - \mathrm{Id})(y)\| = \tan(d_1(y, f(y))) \leq \tan(d(y, f(y))) < \frac{\epsilon}{2},$$

for all y in the domain of f, hence $\|f - \mathrm{Id}\|_y < \epsilon$ for all local sections of $\pi_{|W' \cap K}$ and all y in their domains, so Condition (ii) holds. Therefore $W' \in (K, \epsilon)^{ls}$.

Conversely, let $K = A \setminus \mathbf{B}_\delta^{\bar{d}_0}(\infty)$, let $\epsilon = \frac{1}{2}\delta$ and suppose that $W' \in (K, \epsilon)^{ls}$. Then,

$$\bar{d}(x, W') \le \bar{d}(x, \infty) < \delta, \text{ for all } x \in W \setminus K;$$
$$\bar{d}(x, W) \le \bar{d}(x, \infty) < \delta, \text{ for all } x \in W' \setminus K.$$

On the other hand, if $f(y) \in W' \cap K$ is in the image of a local section f of $\pi_{|W'}$,

$$d_0(y, f(y)) = \|(f - \mathrm{Id})(y)\| < \frac{\delta}{2},$$
$$d_1(y, f(y)) = \arctan(\|D(f - \mathrm{Id})(y)\|) < \|D(f - \mathrm{Id})(y)\| < \frac{\delta}{2},$$

hence $d(x, f(x)) < \delta$. And the same shows that if $x \in W \cap K$, then $d(x, f(x)) < \delta$. As a consequence $\bar{d}_H(W, W') < \delta$. □

The space $\Psi^{gs}(A)$ and the $\bar{d}_\psi$-topology

Let us write $W'(r) := \pi^{-1}_{|W'}(W_r)$ and $\pi_r \colon W'(r) \to W_r$ for the restriction of π.

In this section we prove that the distance $\bar{d}_\psi$ is a metric for $\Psi^{gs}(A)$. Observe that we already know by Theorem 4.3.3 that, given a compact subset $K \subset A$ and a number ϵ, if $\bar{d}_H(W, W')$ is small enough, then $W' \in (K, \epsilon)^{ls}$. Recall in addition that a submanifold W' belongs to $(K, \epsilon)^{gs}$ if and only if it belongs to $(K, \epsilon)^{ls}$ and the projection $\pi_{|W' \cap K}$ has a single sheet.

Therefore we face the problem of counting the number of sheets of the projection $\pi_{|W' \cap K}$. We will solve it by adding the pseudo-metric $\bar{v}_H$ to the metric $\bar{d}_H$. Roughly speaking, the pseudo-metric $\bar{v}_H$ measures the difference between the volumes of W_r and W_s for r close to s. Lemma 4.3.8 shows that if W is close to W' in $(\Psi(A), \bar{d}_H)$ and we know the difference of the volumes of $W'(r)$ and W_r, then we can count the number of sheets of π_r.

In practice, the pseudo-metric $\bar{v}_H$ compares only the volume of W'_s with the volume of W_r, for some number s that is close to r. In Lemma 4.3.10 we show that if $\bar{v}_H(W, W')$ is small enough, then we can approximate as much as we want the difference of the volumes of $W'(r)$ and W_r. Therefore we may count the number of sheets of π_r, and the proof that the $\bar{d}_\psi$-topology is finer than the topology in $\Psi^{gs}(A)$ follows immediately as the first part of Proposition 4.3.13. The second part of the proposition, i.e., that the topology in $\Psi^{gs}(A)$ is finer than the $\bar{d}_\psi$-topology, is independent of the above discussion and only requires a sufficient understanding of the pseudo-metric $\bar{v}_H$.

Lemma 4.3.8. *Suppose $W, W' \in \Psi(A)$, $r > 0$ and that $\pi_r \colon W'(r) \to W_r$ is a finite-sheeted covering map. Let c be the cardinality of the fibres of π_r. If $\mu > 0$, then there* *is a* $\eta(\mu, r)$ *and a* $\zeta(\mu, r)$ *such that if* $\|f - \mathrm{Id}\|_y < \zeta(\mu, r)$ *for any local section* f *of* $\pi_{|W'(s)}$ *and any* y *in the domain of* f*, or* $\bar{d}_H(W, W') < \eta(\mu, r)$*, then*

$$|\mathrm{vol}(W'(s)) - c \cdot \mathrm{vol}(W_s)| < \mu$$

for all $s \leq r$*.*

Proof. Since π_s is a local diffeomorphism, we may choose, for each $x \in W'(s)$, a local section f_x of π_s such that x belongs to its image. Writing J_y for the Jacobian at point y,

$$\int_{x \in W'(s)} 1 dx = \int_{y \in W_r} \sum_{\pi(x)=y} f_x^*(dy) = \int_{y \in W_r} \sum_{\pi(x)=y} \det(J_y(f_x)) dy$$

and the difference $|\mathrm{vol}(W'(s)) - c \cdot \mathrm{vol}(W_s)|$ is

$$\left| \int_{y \in W_s} 1 dx - \int_{y \in W_s} \sum_{\pi(x)=y} \det(J_y(f_x)) dx \right|,$$

which is bounded by $\max_{x \in W_s} |\sum_{\pi(x)=y} \det(J_y(f_x)) - 1| \cdot \mathrm{vol}(W_s)$, that can be taken to be as small as desired if $\|f - \mathrm{Id}\|_y$ is small enough for any section f of π_s and any point in its domain. Also, by Lemmas 4.3.4 and 4.3.6, $\|f - \mathrm{Id}\|_y$ can be bounded as desired by taking $\bar{d}(W, W')$ small enough, and so can the difference $|\sum_{\pi(x)=y} \det(J_y(f_x)) - 1|$. □

Remark 4.3.9. If $f \in F((0, 1], \mathbb{R})$, $r \in \mathbb{R}$ and $\xi > 0$, then for each neighbourhood O of r there exists an open subset $O' \subset O$ with $\mathrm{diam}\, f(O') < \xi$.

Lemma 4.3.10. *If* $W, W' \in \Psi(A)$*,* $r > 0$ *and* $\bar{d}_H(W, W') < \rho(r)$*, so that* π_r *is a finite-sheeted covering map, then there exists a* $\delta > 0$ *such that if* $\bar{d}_\psi(W, W') < \delta$*, then* π_r *has a single sheet.*

Proof. Apply Remark 4.3.9 to the function $s \mapsto \mathrm{vol}(W_s)$, the number r and $\xi = \frac{1}{4}\mathrm{vol}(W_r)$ to produce a new $s > r$, and a neighbourhood $(s - 3\delta, s + 3\delta)$ such that $|\mathrm{vol}(W_s) - \mathrm{vol}(W_{s \pm 3\delta})| < \frac{1}{4}\mathrm{vol}(W_s)$. Assume in addition that $\delta < \frac{1}{4}\mathrm{vol}(W_s)$ and $2\delta < s$. Then, as $\bar{v}_H(W, W') \leq \bar{d}_\psi(W, W') < \delta$, it follows that

$$\mathrm{vol}(W'(s)) \leq \mathrm{vol}(W'_{s-\delta}) \leq \mathrm{vol}(W_{s-2\delta}) + \delta \leq \mathrm{vol}(W_s) + \frac{1}{4}\mathrm{vol}(W_s) + \delta$$

$$\mathrm{vol}(W'(s)) \geq \mathrm{vol}(W'_{s+\delta}) \geq \mathrm{vol}(W_{s+2\delta}) - \delta \geq \mathrm{vol}(W_s) - \frac{1}{4}\mathrm{vol}(W_s) - \delta,$$

so the difference $|\mathrm{vol}(W'(s)) - \mathrm{vol}(W_s)|$ is bounded by $\frac{1}{2}\mathrm{vol}(W_s)$.

Take $\mu = \frac{1}{2}\mathrm{vol}(W_s)$ and let $\eta(s, \mu)$ be given by Lemma 4.3.8. Assume also that $\delta < \eta(s, \mu)$, so $\bar{d}_H(W, W') < \eta(s, \mu)$. Then, if the cardinality of the fibres of π_r is $c > 1$, then the fibres of π_s have also cardinality c, and we obtain the following contradiction:

$$\begin{aligned}\frac{1}{2}\mathrm{vol}(W_s) = \mu > |\mathrm{vol}(W'(s)) - c \cdot \mathrm{vol}(W_s)| \\ &= |\mathrm{vol}(W'(s)) - \mathrm{vol}(W_s) - (c-1)\cdot \mathrm{vol}(W_s)| \\ &\geq (c-1)\cdot \mathrm{vol}(W_s) - |\mathrm{vol}(W'(s)) - \mathrm{vol}(W_s)| \\ &> (c-1)\cdot \mathrm{vol}(W_s) - \frac{1}{2}\mathrm{vol}(W_s) \\ &\geq \frac{1}{2}\mathrm{vol}(W_s). \qquad \square\end{aligned}$$

There is an inclusion $W'_{r+\delta} \subset W'(r)$, hence redefining $r := r+\delta$ and keeping $\delta := \delta$ in the previous lemma we get the following result

Corollary 4.3.11. *For each $r > 0$, there is a $\theta(r) > 0$ such that if $\delta < \theta(r)$ and $W' \in \mathbf{B}_\delta^{\bar{d}_\psi}(W)$, then π_r has a single sheet. In particular, W'_r is the intersection of $A \setminus \mathbf{B}_r^{\bar{d}_0}(\infty)$ with the image of a section of the normal bundle of W.*

Lemma 4.3.12. *If $f \in F((0,1],\mathbb{R})$, then* $\mathrm{graph}(f)$ *is an embedded half-open interval.*

Proof. Let $D \subset \mathbb{R}^2$ be the line $x + y = 0$. Then the projection of $\mathrm{graph}(f)$ into D is injective, because if two points in the graph are projected to the same point in D, then f is not non-increasing. The graph is connected, hence the image is connected and its image is a half-open interval. Moreover, the projection is continuous and open, hence homeomorphic to its image. $\square$

It is clear that the empty submanifold has the same neighbourhoods in $(\Psi(A), \bar{d}_\psi)$ and in $\Psi^{gs}(\mathbb{R}^n)$. The following proposition shows that the same holds for non-empty submanifolds, finishing the proof of the main result of this section (Theorem 4.3.1).

Proposition 4.3.13. *Let W be a non-empty submanifold in $\Psi(A)$.*

(i) *For each pair K, ϵ, there is a δ such that $\mathbf{B}_\delta^{\bar{d}_\psi}(W) \subset (K,\epsilon)^{gs}$.*

(ii) *For each δ, there is a pair K, ϵ such that $(K,\epsilon)^{gs} \subset \mathbf{B}_\delta^{\bar{d}_\psi}(W)$.*

Proof. Let us construct a ball $\mathbf{B}_{\delta'}^{\bar{d}_H}(W) \subset (K,\epsilon)^{ls}$ using Proposition 4.3.7, and let us assume that $\delta < \delta'$. It follows from Corollary 4.3.11 that if in addition $\delta < \theta(\bar{d}(\infty, K))$, then $\mathbf{B}_\delta^{\bar{d}_\psi}(W) \subset (K,\epsilon)^{gs}$.

Conversely, let $\mu = s = \frac{1}{4}\delta$ and let $K = A \setminus \mathbf{B}_s^{\bar{d}_0}(\infty)$. Let $\epsilon < \zeta(\mu, s)$ be so small that $(K,\epsilon)^{ls} \subset \mathbf{B}_s^{\bar{d}_H}(W)$. If $W' \in (K,\epsilon)^{gs}$, then $\bar{d}_H(W,W') < s \leq \frac{1}{2}\delta$. In the rest of the proof we show that $\bar{v}_H(W,W') < \frac{1}{2}\delta$ as well, and therefore

$$\bar{d}_\psi(W,W') = \bar{d}_H(W,W') + \bar{v}_H(W,W') < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

If we write $\nu(W_r) := \{(r,t) \mid \mathrm{vol}(W_{r^-}) \le t \le \mathrm{vol}(W_{r^+})\}$, then $\bar{\nu}_H(W, W')$ was defined as the maximum of

$$\max_r \bar{\nu}(\nu(W'_r), \nu(W)), \qquad \max_r \bar{\nu}(\nu(W_r), \nu(W')), \tag{4.3.2}$$

whose first term is the maximum among

$$\max_{r \ge \frac{1}{2}\delta} \bar{\nu}(\nu(W'_r), \nu(W)), \qquad \sup_{r < \frac{1}{2}\delta} \bar{\nu}(\nu(W'_r), \nu(W)).$$

The second term of this formula is bounded by $\frac{1}{2}\delta$. For the first term, consider the following diagram:

$$W_{r+s} \quad \sim \quad W'(r+s) \quad \subset \quad W'_r \quad \subset \quad W'(r-s) \quad \sim \quad W_{r-s},$$

where the inclusions hold because $W' \in \mathbf{B}_s^{\bar{d}_H}(W)$, therefore $\bar{d}_H(W, W') < s$. The $\sim$ signs between two compact submanifolds indicate that their volumes are very close, meaning precisely that

$$|\mathrm{vol}(W(r)) - \mathrm{vol}(W_r)| < \mu$$

for all $r \le s$, by Lemma 4.3.8 above. As a consequence,

$$\mathrm{vol}(W_{r+s}) - \mu \le \mathrm{vol}(W'(r+s)) \le \mathrm{vol}(W'_r) \le \mathrm{vol}(W'(r-s)) \le \mathrm{vol}(W_{r-s}) + \mu,$$

therefore $\nu(W'_r)$ is contained in the segment

$$\{r\} \times [\mathrm{vol}(W_{r+s}) - \mu, \mathrm{vol}(W_{r-s}) + \mu].$$

On the other hand, by Lemma 4.3.12, the graph $\nu(W')$ contains a continuous path between $\nu(W'_{r-s})$ and $\nu(W'_{r+s})$ inside the square

$$[r-s, r+s] \times [\nu(W_{r+s}), \nu(W_{r-s})].$$

A simple inspection shows that the distance between the path and the segment is at most $s = \mu$, so the second term in equation (4.3.2) is bounded by $\frac{1}{4}\delta$.

For the first term in equation (4.3.2), we follow the same steps, but we approximate the value of $\mathrm{vol}(W_r)$ following the diagram

$$\begin{array}{ccccc} W'_{r+s} & \subset & W'(r) & \subset & W'_{r-s} \\ & & \wr & & \\ & & W_r, & & \end{array}$$

that is,

$$\mathrm{vol}(W'_{r+s}) - \mu \le \mathrm{vol}(W'(r)) - \mu \le \mathrm{vol}(W_r) \le \mathrm{vol}(W'(r)) + \mu \le \mathrm{vol}(W'_{r-s}) + \mu,$$

so $\nu(W'_r)$ is contained in the segment

$$\{r\} \times [\mathrm{vol}(W'_{r+s}) - \mu, \mathrm{vol}(W'_{r-s}) + \mu].$$

On the other hand, by Lemma 4.3.12, the graph $\nu(W)$ contains a continuous path between $\nu(W'_{r-s})$ and $\nu(W'_{r+s})$ inside the square

$$[r-s, r+s] \times [\nu(W'_{r+s}), \nu(W'_{r-s})].$$

Again a simple inspection shows that the distance between the path and the segment is at most $s = \mu$, so the first term in (4.3.2) is bounded by $\frac{1}{4}\delta$. □

The sheaf and the metric

After introducing a topology in $\Psi(A)$, in [GRW10] it is proven that Ψ is a sheaf on $\mathbb{R}^n$. We review these results from the perspective of the distance $\bar{d}_\psi$.

It is immediate to verify that if $A \subset B$ are open subsets of $\mathbb{R}^n$, then the restriction map $\Psi(B) \to \Psi(A)$ sends a ball of radius δ into a ball of radius δ, so it is continuous, and also 1-Lipschitz continuous (hence uniformly continuous).

It is clear that, if $A = A_0 \cup \ldots \cup A_q$, then the map to the equalizer

$$\varphi\colon \Psi(A) \longrightarrow \mathrm{eq}\left(\prod_{j=0}^{q} \Psi(A_j) \to \prod_{i,j=0}^{q} \Psi(A_i \cap A_j)\right)$$

given by sending a submanifold W to $(W \cap A_0, \ldots, W \cap A_q)$ is a bijection. If δ is so small that for all $x \in W_\delta$ at least one of $\bar{d}(x, \infty_{A_j})$ is greater than δ, then the balls of radius δ and centre W in $\Psi(A_0 \cup \ldots \cup A_q)$ are balls of radius δ in $\Psi(A_0) \times \ldots \times \Psi(A_q)$, with the distance of the maximum. As a consequence, φ is a bijection and a local isometry, hence a homeomorphism.

Also, the map induced by the inclusion $\Psi(\mathbb{R}^m) \to \Psi(\mathbb{R}^n)$ is an isometry.

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