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ON THE BICANONICAL MAP OF IRREGULAR VARIETIES

MIGUEL ANGEL BARJA, MARTÍ LAHOZ,
JUAN CARLOS NARANJO, AND GIUSEPPE PARESCHI

Abstract

From the point of view of uniform bounds for the birationality of pluricanonical maps, irregular varieties of general type and maximal Albanese dimension behave similarly to curves. In fact Chen-Hacon showed that, at least when their holomorphic Euler characteristic is positive, the tricanonical map of such varieties is always birational. In this paper we study the bicanonical map. We consider the natural subclass of varieties of maximal Albanese dimension formed by primitive varieties of Albanese general type. We prove that the only such varieties with non-birational bicanonical map are the natural higher-dimensional generalization to this context of curves of genus 2: varieties birationally equivalent to the theta-divisor of an indecomposable principally polarized abelian variety. The proof is based on the (generalized) Fourier-Mukai transform.

1. Introduction

Pluricanonical maps are an essential tool for understanding varieties of general type. In particular, given a variety of general type X, it is important to know, or at least bound, the minimal integer $m_0(X)$ such that the pluricanonical maps

$$\phi_m : X \dashrightarrow \mathbf{P}(H^0(X, \omega_X^m)^*)$$

are birational onto their image for each $m \geq m_0(X)$. In this paper we will deal with this sort of problem for complex *irregular* varieties (i.e. varieties such that $q(X) := h^1(\mathcal{O}_X) > 0$) of general type, mostly of *maximal Albanese dimension* (*m.A.d.* for short), i.e. such that their Albanese map $alb : X \to \operatorname{Alb} X$ is generically finite. In a sense these are the most basic irregular varieties since the Stein factorization of the Albanese map provides

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a canonical fibration onto a normal variety of the same irregularity, whose smooth models have m.A.d. From the point of view of the birationality of pluricanonical maps, Chen-Hacon [CH] showed that m.A.d. varieties of general type and $\chi(\omega_X) > 0$ behave like curves: their tricanonical map is always birational[1]. Such a result has been made more precise in [PP5], where it is proved that if X is m.A.d. and has positive generic vanishing index (see below), then the tricanonical map is an embedding outside the exceptional locus of the Albanese map. It is worth noting that ample line bundles on abelian varieties have the same behavior, since the third power of an ample line bundle is always very ample. In fact, the analogy has been explained in [PP3] and references therein, where it is showed that the two facts can be proved in the same way.

In this paper we address the next natural problem: *which are the m.A.d. varieties of general type whose bicanonical map is not birational?* To describe our view of the question, some preliminary remarks about irregular varieties are in order. According to the seminal work of Green-Lazarsfeld (see [GL1] and [GL2]) and Ein-Lazarsfeld [EL], key invariants of irregular varieties are their *cohomological support loci*

$$V^i(\omega_X) = \{\alpha \in \operatorname{Pic}^0 X \mid h^i(\omega_X \otimes \alpha) > 0\}.$$

These are intimately related to the geometry of X since, by the main theorem of [GL2], the positive dimensional components of $V^i(\omega_X)$ are translates of subtori arising, for $i > 0$, only in the presence of a morphism with connected fibres to a lower dimensional m.A.d. normal variety. It follows that, up to birational equivalence, a variety such that $\dim V^i(\omega_X) > 0$ for some $i > 0$ has a morphism onto a smooth lower-dimensional m.A.d. variety Y such that $\dim V^i(\omega_Y) = 0$ for all $i > 0$. This is the reason for the following.

Definition (Catanese [Ca, Def. 1.24])**.** A smooth projective irregular variety such that $\dim V^i(\omega_X) = 0$ for all $i > 0$ is called primitive.

Moreover, in a recent paper Pareschi and Popa introduced a single numerical invariant measuring the *codimension* of the various support loci, namely the *generic vanishing index*

$$gv(X) := \min_{i>0}\{\operatorname{codim}_{\operatorname{Pic}^0 X} V^i(\omega_X) - i\}.$$

In this terminology Green-Lazarsfeld's Generic Vanishing Theorem of [GL1] can be rephrased as follows: *if X has m.A.d., then $gv(\omega_X) \geq 0$.* The main feature of the invariant $gv(X)$ is that it governs the local sheaf-theoretic

[1] m.A.d. varieties of general type with $\chi(\omega_X) = 0$ were discovered by Ein-Lazarsfeld in [EL] and do not exist in dimension ≤ 2. However one knows, again by [CH], that their 6-canonical map is birational.

properties of the Fourier-Mukai transform of the structure sheaf (see [PP6], Def. 3.1 and Cor. 3.2). A variety verifying the extremal case $gv(X) = 0$ has quite special properties. From the Fourier-Mukai point of view, $gv(X) = 0$ means that the transform of the structure sheaf of X has torsion (see Theorem 2.6 below). From the geometric side, it means that the image of X via the Albanese map is fibered by subtori of the Albanese variety (see [EL], Proof of Theorem 3). For these reasons, m.A.d. varieties with $gv(\omega_X) = 0$ will be referred to as *special*.

Consequently, we are lead to divide m.A.d. varieties into four disjoint subclasses: the *non-special* varieties are further distinguished into *primitive* and *non-primitive* ones and similarly for the *special* varieties. An immediate computation shows that for *primitive* varieties, non-special (resp. special) means simply $\dim X < q(X)$ (resp. $\dim X = q(X)$, i.e. the Albanese map is surjective). We recall that in the literature there is a specific definition also for m.A.d. varieties with $\dim X < q(X)$: they are called *of Albanese general type* (Catanese [Ca, Def. 1.7]). Therefore *primitive non-special* is equivalent to *primitive of Albanese general type*.

Let us go back to the problem of describing m.A.d. varieties of general type such that their bicanonical map is not birational. In dimension 1 these are the curves of genus 2. The problem of classifying surfaces of general type whose bicanonical map is not birational has attracted considerable interest, starting from Du Val, later on Bombieri and more recently, Catanese, Ciliberto, Francia, Mendes Lopes, Pardini, Xiao Gang and others. We refer to the surveys [Ci] and (more recently) [BCP] for an account on this work. Although the general classification is still not fully achieved, things are much better behaved for surfaces of maximal Albanese dimension. In fact, from Theorems 8, 9 and 10 of [BCP], one extracts the following result due to the combined efforts of Catanese, Ciliberto and Mendes Lopes (see [CCM] and [CM]): minimal m.A.d. surfaces of general type whose bicanonical map is not birational are either fibered by curves of genus 2 (this is usually referred to as *the standard case*) or

(a) the symmetric product of a curve of genus 3, or

(b) the double cover of a principally polarized abelian surface branched along a divisor $D \in |2\Theta|$.

Symmetric products of curves of genus 3 are the minimal surfaces which are birational to the theta-divisors of indecomposable principally polarized abelian threefolds. On the other hand, curves of genus 2 are the theta-divisors of indecomposable principally polarized abelian surfaces. Moreover, bielliptic curves of genus 2 are the 1-dimensional version of case (b). The interesting conclusion is that both cases (a) and (b), as well as the standard case, can be seen as natural 2-dimensional generalizations of curves of genus 2.

The higher dimensional analogue of the *standard* case for surfaces is a variety fibred by primitive varieties with non-birational bicanonical map. Moreover, examples (a) and (b) also have a natural generalization to arbitrary dimension. Indeed, the bicanonical map of a variety X birational to either

(a′) a theta divisor of an indecomposible p.p.a.v., or

(b′) a double cover of a p.p.a.v. (A, Θ) branched along a smooth divisor of $|2\Theta|$

has degree 2. In the case (a′) it factors (up to birational equivalence) through the Kummer map and in the case (b′) it factors through the lifting to X of the involution -1_A. An immediate computation shows that both examples are *primitive* with Euler characteristic 1, the minimal value[2]. The case (a′) is *non-special* while the case (b′) is *special*.

In this paper we will focus on non-special primitive varieties. The somewhat surprising result is that, in arbitrary dimension, case (a′) is the only one, and so the picture is the same as in dimensions one and two.

Theorem A. *Let X be a smooth complex projective variety of general type. Assume, moreover, that X is primitive and* $\dim X < q(X)$*. The following are equivalent:*

(a) *The bicanonical map of X is non-birational.*

(b) *X is birationally equivalent to a theta-divisor of an indecomposable p.p.a.v.*

As a consequence of the above quoted main theorem of [GL2], it follows that:

Corollary B. *Let X be a smooth complex variety of general type with* $\dim X < q(X)$*. If the bicanonical map of X is non-birational, then either X has a morphism onto a lower dimensional irregular normal variety or X is birational to the theta-divisor of an indecomposable p.p.a.v..*

Theorem A leaves open the classification of m.A.d. varieties of general type with non-birational bicanonical map of the other three types. We conjecture the following:

- if *primitive and special* such varieties should have $\chi(\omega_X) = 1$ (but we don't have a clear idea about the possibility of other examples besides (b) above, and what they should be);

[2]m.A.d. varieties of general type satisfy the sharp lower bound $\chi(\omega_X) \geq 0$ [EL], but if X is, in addition, *primitive*, then $\chi(\omega_X) \geq 1$ (see Proposition 4.10 below for a more general statement).

- the *non-primitive* ones should all belong to the *standard case* (see the above discussion). Among these, the *non-special* ones should be birational to the product of a non-special m.A.d. variety and a theta-divisor.

The method of proof of Theorem A is different from the one of [CCM] and [CM], even if restricted to the 2-dimensional case. The basic framework is the (generalized) Fourier-Mukai transform and its relation with generic vanishing (the necessary background material is reviewed in §2). In particular, the equivalence

$$gv(\mathcal{F}) \geq 1 \quad \Leftrightarrow \quad \widehat{\mathbf{R}\Delta\mathcal{F}} \text{ torsion-free}$$

(see [PP5] and [PP6]; see also Theorem 2.6 below) is repeatedly used.

In §3 we prove a slight improvement of Hacon-Pardini's cohomological characterization of theta-divisors [HP, Thm. 2]. The result is Proposition 3.1 below, stating that: *primitive non-special varieties with* $\chi(\omega_X) = 1$ *are birational to theta-divisors*[3]. Interestingly, a slightly weaker version of the above result holds in any characteristic (we refer to Corollary 3.2 for the precise statement).

In §4, via the notion of *continuous global generation* (see [PP1] and [PP3] and references therein), we prove a birationality criterion asserting, roughly speaking, that *the non-birationality of the bicanonical map implies that, for general* $\alpha \in Pic^0 X$, *the linear series* $|\omega_X \otimes \alpha|$ *has a base divisor.* This works under hypotheses more general than those of Theorem A (we refer to Theorem 4.13 and Corollary 4.14 for the precise statements). In particular, it works for all non-special varieties and also for the primitive special ones. We have chosen (at the expense of considerable extra work) to prove the result in such generality. Moreover, it is worth to note that the above mentioned analogy with ample line bundles on abelian varieties persists. In fact our birationality criterion, and its proof, are similar to Ohbuchi's theorem, asserting that twice an ample line bundle L on an abelian variety is very ample unless L has a base divisor, as proved in [PP2].

The above birationality criterion is used in §5 where, by means of a geometric analysis of the paracanonical system, combined with the Fourier-Mukai transform, we show that a variety X satisfying (a) of Theorem A has $\chi(\omega_X) = 1$. At this point Theorem A is proved by applying the above characterization of theta-divisors of Hacon-Pardini's type.

Finally, in the Appendix we provide the proof of an useful technical fact about Fourier-Mukai transform that we couldn't find in the literature. It is our hope that some of the steps of the argument (namely, the birationality

[3]This result was proved independently (with a different proof) by Lazarsfeld-Popa [LP], building on the ideas of [HP].

criterion of §4, the decomposition of the Picard torus and the explicit description of the Poincaré line bundle of §5, the result of the Appendix) will be of independent interest.

Unless otherwise stated, throughout the paper the word *variety* will mean *smooth projective complex variety.* However, the proof of Theorem A is completely algebraic, and works over algebraically closed fields of any characteristic, assuming that X is a smooth projective variety of maximal Albanese dimension, primitive and such that $\dim X < \dim \mathrm{Alb}\, X$.

Given a line bundle α on a variety X, by abuse of language we will also denote α as the point of the Picard variety of X parametrizing α.

2. (Generalized) Fourier-Mukai transform

Here we review from [Mu2], [PP4], [PP5], and [PP6] the material about (generalized) Fourier-Mukai transforms and generic vanishing that will be needed in the sequel.

2.1. (Generalized) Fourier-Mukai transforms and generic vanishing.

Terminology/Notation 2.1 (Fourier-Mukai and generic vanishing)**.** Let X be a variety of dimension d, equipped with a morphism to an n-dimensional abelian variety

$$a : X \to A.$$

Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. We will denote

$$P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^*(\mathcal{P}).$$

When $a = alb$, the Albanese map of X, then the map alb^* identifies $\mathrm{Pic}^0(\mathrm{Alb} X)$ to $\mathrm{Pic}^0 X$ and the line bundle P_{alb} is identified to the Poincaré line bundle of X. We will denote $P = P_{alb}$. Letting p and q be the two projections of $X \times \mathrm{Pic}^0 A$, we consider the left-exact functor

$$\Phi_{P_a}(\mathcal{F}) = q_*(p^*(\mathcal{F}) \otimes P_a)$$

and its derived functor

$$\mathbf{R}\Phi_{P_a} : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A).$$

Sometimes we will have to consider the analogous derived functor $\mathbf{R}\Phi_{P_a^\vee} : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A)$ as well. Since $\mathcal{P}^{-1} \cong (1_A \times (-1)_{\mathrm{Pic}^0 A})^*\mathcal{P}$, we have that

$$\mathbf{R}\Phi_{P_a^\vee} = (-1_{\mathrm{Pic}^0 A})^* \mathbf{R}\Phi_{P_a}.$$

Finally, given a coherent sheaf $\mathcal{F}$ on X, its *i-th cohomological support locus with respect to a* is

$$V^i_a(\mathcal{F}) = \{\alpha \in \mathrm{Pic}^0 A \mid h^i(\mathcal{F} \otimes a^*\alpha) > 0\}.$$

Again, when a is the Albanese map of X, we will omit the subscript, simply writing $V^i(\mathcal{F})$. A natural measure of the size of the full package of the $V^i_a(\mathcal{F})$'s is provided by the *generic vanishing index* of $\mathcal{F}$ (with respect to a) (see [PP6, Def. 3.1])

$$gv_a(\mathcal{F}) := \min_{i>0}\{\mathrm{codim}_{\mathrm{Pic}^0 A} V^i_a(\mathcal{F}) - i\}.$$

A first basic result relates the generic vanishing index to the fact that the F-M transform of its Grothendieck dual is a sheaf (in cohomological degree d). In what follows, we will adopt the following notation for the dualizing functor

$$\mathbf{R}\Delta\mathcal{F} = \mathbf{R}\,\mathcal{H}om(\mathcal{F}, \omega_X).$$

Theorem 2.2 ([PP4, Thm. A] and [PP6, Thm. 2.2])**.** *The following are equivalent:*

(a) $gv_a(\mathcal{F}) \geq 0$.

(b) $R^i\Phi_{P_a}(\mathbf{R}\Delta\mathcal{F}) = 0$ *for all* $i \neq d$.

Terminology/Notation 2.3 (GV-sheaves)**.** If $gv_a(\mathcal{F}) \geq 0$, the sheaf $\mathcal{F}$ is said to be a *GV-sheaf (generic vanishing sheaf).* If this is the case, Theorem 2.2 tells us that the full transform $\mathbf{R}\Phi_{P_a}(\mathbf{R}\Delta\mathcal{F})$ is a sheaf concentrated in degree d:

$$\mathbf{R}\Phi_{P_a}(\mathbf{R}\Delta\mathcal{F}) = R^d\Phi_{P_a}(\mathbf{R}\Delta\mathcal{F})[-d].$$

Then one usually denotes

$$R^d\Phi_{P_a}(\mathbf{R}\Delta\mathcal{F}) = \widehat{\mathbf{R}\Delta\mathcal{F}}.$$

Note that, by (a) of Theorem 2.2, $H^i(\mathcal{F} \otimes a^*\alpha) = 0$ for all $i > 0$ and general $\alpha \in \mathrm{Pic}^0 A$. Therefore, by deformation-invariance of χ, the generic value of $H^0(\mathcal{F} \otimes a^*\alpha)$ equals $\chi(\mathcal{F})$. Since, by base-change, the fiber of $\widehat{\mathbf{R}\Delta\mathcal{F}}$ at a general point $\alpha \in \mathrm{Pic}^0 A$ is isomorphic to $H^d(\mathbf{R}\Delta\mathcal{F} \otimes a^*\alpha) \cong H^0(\mathcal{F} \otimes a^*\alpha^{-1})^*$, the (generic) rank of $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is

$$\mathrm{rk}\,(\widehat{\mathbf{R}\Delta\mathcal{F}}) = \chi(\mathcal{F}). \tag{1}$$

Via base-change Theorem 2.2 yields the following.

Corollary 2.4 ([Ha, Thm. 1.2] and [PP4, Prop. 3.13])**.** *If* $gv_a(\mathcal{F}) \geq 0$, *then*

$$V^d_a(\mathcal{F}) \subseteq \cdots \subseteq V^1_a(\mathcal{F}) \subseteq V^0_a(\mathcal{F}).$$

From Grothendieck duality and Theorem 2.2 it follows that:

Corollary 2.5 ([PP4, Rem. 3.12] and [PP6, Proof of Cor. 3.2])**.** *If* $gv_a(\mathcal{F}) \geq 0$*, then*

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathrm{Pic}^0 A}}(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 A}) \cong R^i\Phi_{P_a^\vee}(\mathcal{F}) \cong (-1_{\mathrm{Pic}^0 A})^* R^i\Phi_{P_a}(\mathcal{F}).$$

In [PP6] Pareschi-Popa established a dictionary between the value of $gv_a(\mathcal{F})$ and the local properties of the transform $\widehat{\mathbf{R}\Delta\mathcal{F}}$. Its first instance is the following theorem.

Theorem 2.6 ([PP6, Cor. 3.2])**.** *Assume that* $\mathcal{F}$ *is GV (with respect to* a*). Then the following are equivalent:*

(a) $gv_a(\mathcal{F}) \geq 1$.

(b) $\widehat{\mathbf{R}\Delta\mathcal{F}}$ *is a torsion-free sheaf.*

2.2. Mukai's equivalence of derived categories of abelian varieties. Assume that X coincides with the abelian variety A (and the map a is the identity). In this special case, according to Notation 2.1, $\mathcal{P}$ denotes the Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. Then Mukai's theorem asserts that $\mathbf{R}\Phi_{\mathcal{P}}$ is an equivalence of categories. More precisely, denoting $\mathbf{R}\Psi_{\mathcal{P}} : \mathbf{D}(\mathrm{Pic}^0 A) \to \mathbf{D}(A)$ and $n = \dim A$:

Theorem 2.7 ([Mu2, Thm. 2.2])**.**

$$\mathbf{R}\Psi_{\mathcal{P}} \circ \mathbf{R}\Phi_{\mathcal{P}} = (-1_A)^*[n], \qquad \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}\Psi_{\mathcal{P}} = (-1_{\mathrm{Pic}^0 A})^*[n].$$

Given a morphism $a : X \to A$, the functors $\mathbf{R}\Phi_{P_a}$ and $\mathbf{R}\Phi_{\mathcal{P}}$ (see Notation 2.1) are related by the following formula.

Proposition 2.8.

$$\mathbf{R}\Phi_{P_a} \cong \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}a_*.$$

Proof.

$$\begin{aligned}
\mathbf{R}\Phi_{P_a}(\,.\,) &= \mathbf{R}q_*(p_X^*(\,.\,) \otimes (a \times \mathrm{id})^*\mathcal{P}) \\
&\overset{L+PF}{\cong} \mathbf{R}q_*(\mathbf{R}(a \times \mathrm{id})_*(p_X^*(\,.\,)) \otimes \mathcal{P}) \\
&\overset{BC}{\cong} \mathbf{R}q_*(p_A^*(\mathbf{R}a_*(\,.\,)) \otimes \mathcal{P}) = \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}a_*(\,.\,)
\end{aligned}$$

where: $L =$ Leray, $PF =$ projection formula and $BC =$ Base Change. □

In the above terminology, the *Generic Vanishing Theorem* of Green-Lazarsfeld (see [GL1]; see also [EL, Rem. 1.6]) asserts that $gv_a(\omega_X) \geq 0$ if a is generically finite. The following proposition shows that the converse is also true.

Proposition 2.9. *Assume that* $gv_a(\omega_X) \geq 0$*. Then* a *is generically finite.*

More generally, one can prove in the same way the converse of the full Generic Vanishing theorem of [GL1], namely that if $gv(\omega_X) \geq -k$, then

$\dim a(X) \geq d - k$. Such statement was proved independently by Lazarsfeld-Popa [LP, Prop. 1.5].

Proof. Let $e = d - \dim a(X)$ and $n = \dim A$. By Kollár's splitting theorem [Ko1, Thm. 3.1], we have the following isomorphism in $\mathbf{D}(A)$,

$$\mathbf{R}a_*\omega_X \cong \sum_{i=0}^{e} R^i a_*\omega_X[-i].$$

Observe that Kollár's theorem is stated for surjective morphisms, but the surjectivity assumption can be dropped by composing with $\mathbf{R}i_*$, where i is the embedding of $a(X)$ into A. As in *loc. cit.*, Thm. 3.8, by Grothendieck duality it follows that

$$\mathbf{R}a_*\mathcal{O}_X \cong \sum_{i=0}^{e} \mathbf{R}\Delta_A(R^i a_*\omega_X)[n-d+i]$$

where $\mathbf{R}\Delta_A$ is the dualizing functor on A as in the notation before Theorem 2.2. By Proposition 2.8,

$$\mathbf{R}\Phi_{P_a}\mathcal{O}_X \cong \sum_{i=0}^{e} \mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\Delta_A(R^i a_*\omega_X))[n-d+i]. \tag{2}$$

Since $R^e a_*\omega_X$ is certainly non-zero, $\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\Delta_A(R^e a_*\omega_X))$ is non-zero (e.g. by Theorem 2.7). On the other hand, by Grothendieck-Serre duality, the hypercohomology group $H^j(A, \mathbf{R}\Delta_A(R^e a_*\omega_X \otimes \alpha)) = 0$ for all $j > n$ and all $\alpha \in \mathrm{Pic}^0(A)$. It follows from base change that $R^j\Phi_{\mathcal{P}}(\mathbf{R}\Delta_A(R^e a_*\omega_X))$ vanishes for $j > n$, so it must be non-zero for some $j \leq n$, call it k. By (2),

$$R^j\Phi_{P_a}\mathcal{O}_X \cong \bigoplus_{i=0}^{e} R^{n-d+i+j}\Phi_{\mathcal{P}}(\mathbf{R}\Delta_A(R^i a_*\omega_X)),$$

so $R^k\Phi_{\mathcal{P}}(\mathbf{R}\Delta_A(R^e a_*\omega_X)) \hookrightarrow R^{d-e-(n-k)}\Phi_{P_a}\mathcal{O}_X$, where $d-e-(n-k) \leq d-e$. Hence, $R^j\Phi_{P_a}\mathcal{O}_X$ is non-zero for some $j \leq d-e$. Therefore, if $gv_a(\omega_X) \geq 0$, Theorem 2.2 (applied to $\mathcal{F} = \omega_X$) yields to $e = 0$ (and $k = n$). □

Remark 2.10. Note that, with the exception of Proposition 2.9 (which uses Kollár's theorems on higher direct images of dualizing sheaves), all the results of the present section are algebraic, and work over algebraically closed fields of any characteristic.

3. On a cohomological characterization of theta-divisors

As a simple but instructive application of the results reviewed in the previous section, we prove the following slight improvement of Hacon-Pardini's cohomological characterization of theta-divisors [HP, Prop. 4.2]. Proposition 3.1 below is proved independently, with a different proof, also in [LP, Prop. 3.13]. An algebraic version, valid in any characteristic, is provided by Corollary 3.2 below.

Proposition 3.1. *Let X be a d-dimensional smooth projective variety (or a compact Kähler manifold) such that:* (a) $\dim V^i(\omega_X) = 0$ *for all* $i > 0$*;* (b) $d = \dim X < q(X)$ (i.e., in the terminology of the introduction, X is primitive and non-special)*, and* (c) $\chi(\omega_X) = 1$*. Then* $\operatorname{Alb} X$ *is a principally polarized abelian variety and the Albanese map* $alb : X \to \operatorname{Alb} X$ *maps X birationally onto a theta-divisor.*

Proof. We denote $q = q(X)$. By hypotheses (a) and (b) $gv(\omega_X) \geq 1$. Therefore, by Theorem 2.6, $\widehat{\mathcal{O}_X}$ is torsion-free. Since, by (1), $\operatorname{rk} \widehat{\mathcal{O}_X} = \chi(\omega_X) \overset{(c)}{=} 1$, we get that $\widehat{\mathcal{O}_X}$ is an ideal sheaf twisted by a line bundle on $\operatorname{Pic}^0 X$:

$$\widehat{\mathcal{O}_X} = \mathcal{I}_Z \otimes L.$$

By base change, the support of Z is contained in the union of the $V^i(\omega_X)$ for $i > 0$ which are assumed, by (a), to be finite sets. Therefore, $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathcal{E}xt^{i+1}(\mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X}) = 0$ for $i+1 \neq q$. On the other hand, by Proposition 6.1 of the Appendix, and Corollary 2.5 it follows that

$$\mathcal{E}xt^d(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong (-1_{\operatorname{Pic}^0 X})^* R^d \Phi_P(\omega_X) \cong \mathbb{C}(\hat{0}). \tag{3}$$

This implies:

(a) $d = q - 1$ (in fact this comes directly from the main inequality in [PP6]).

(b) $\mathcal{O}_Z = \mathbb{C}(\hat{0})$. (Indeed $\mathcal{E}xt^i(\mathbb{C}(\hat{0}), \mathcal{O}_{\operatorname{Pic}^0 X})$ is zero for $i < q(X)$ and equal to $\mathbb{C}(\hat{0})$ for $i = q(X)$. Since $\mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_{\operatorname{Pic}^0 X})$ is an involution, (b) follows from (3)). In conclusion

$$\widehat{\mathcal{O}_X} = \mathcal{I}_{\hat{0}} \otimes L,$$

where L is a line bundle on $\operatorname{Pic}^0 X$ and $\mathcal{I}_{\hat{0}}$ is the ideal sheaf of the (reduced) point $\hat{0}$. By Proposition 2.8,

$$\mathbf{R}\Phi_P(\mathcal{O}_X) = \mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\, alb_* \mathcal{O}_X) = \mathcal{I}_{\hat{0}} \otimes L[-q+1].$$

Therefore, by Mukai's Inversion Theorem 2.7,

$$\mathbf{R}\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = (-1)^*_{\operatorname{Pic}^0 X} \mathbf{R}\, alb_* \mathcal{O}_X[-1]. \tag{4}$$

In particular,

$$R^0\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0 \qquad \text{and} \qquad R^1\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) \cong alb_* \mathcal{O}_X. \tag{5}$$

Applying $\psi_{\mathcal{P}}$ to the standard exact sequence

$$0 \to \mathcal{I}_{\hat{0}} \otimes L \to L \to \mathcal{O}_{\hat{0}} \otimes L \to 0, \tag{6}$$

and using (5) we get,

$$0 \to R^0\Psi_{\mathcal{P}}(L) \to \mathcal{O}_{\mathrm{Alb}\,X} \to alb_*\mathcal{O}_X \tag{7}$$

whence $R^0\Psi_{\mathcal{P}}(L)$ is supported everywhere (since $alb_*\mathcal{O}_X$ is supported on a divisor). It is well known that this implies that L is *ample*. Therefore, $R^i\Psi_{\mathcal{P}}(L) = 0$ for $i > 0$. Therefore, by sequence (6), $R^i\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0$ for $i > 1$. By (4) and (5), this implies that $R^i alb_*(\mathcal{O}_X) = 0$ for $i > 0$. Furthermore, (7) implies easily that $h^0(L) = 1$, i.e. L is a *principal* polarization. Therefore, via the identification $\mathrm{Alb}(X) \cong \mathrm{Pic}^0 X$ provided by L, we have $R^0\Psi_{\mathcal{P}}(L) \cong L^{-1}$ (see [Mu2, Prop. 3.11(1)]). Since the arrow on the right in (7) is onto, it follows that $alb_*\mathcal{O}_X = \mathcal{O}_D$, where D is a divisor in $|L|$. Since we already know that alb is generically finite (Proposition 2.9), this implies that alb is a birational morphism onto D. □

Note that the proof is entirely algebraic, except for the use of Proposition 2.9 (see Remark 2.10). Therefore, the following statement holds.

Corollary 3.2. *Let X be a smooth projective variety (over any algebraically closed field) such that:* (a) $\dim V^i(\omega_X) = 0$ *for all* $i > 0$*;* (b) $d = \dim X < \dim \mathrm{Alb}\, X$ *and the Albanese map of X is generically finite, and* (c) $\chi(\omega_X) = 1$*. Then* $\mathrm{Alb}\, X$ *is a principally polarized abelian variety and the Albanese map* $alb : X \to \mathrm{Alb}\, X$ *maps X birationally onto a theta-divisor.*

4. Birationality criterion

First we review the necessary background about the notion of *continuous global generation*. Then we apply this machinery to the canonical bundle of a variety, under hypotheses relevant to the main theorem of the present paper. Finally, by applying the same machinery to the ideal sheaf of a point twisted by the canonical line bundle, we prove the birationality criterion we are aiming for, i.e. Theorem 4.13 and its Corollary 4.14.

4.1. Continuous global generation. The notion of *continuous global generation* was introduced by Pareschi-Popa in [PP1] and applied to various geometrical problems in [PP1], [PP2], and [PP5]; see also the survey [PP3].

Let $\mathcal{F}$ be a coherent sheaf on X, and let T be a subset of $\mathrm{Pic}^0 A$. Then we have the *continuous evaluation map associated to the pair* $(\mathcal{F}, T)$:

$$ev_T = ev_{T,\mathcal{F}} : \bigoplus_{\alpha \in T} H^0(X, \mathcal{F} \otimes a^*\alpha^{-1}) \otimes a^*\alpha \to \mathcal{F}.$$

Definition 4.1. Let p be a point of X. The sheaf $\mathcal{F}$ is said to be *continuously globally generated at p (CGG at p for short), with respect to the morphism a*, if the map ev_U is surjective *at p*, for all non-empty Zariski open subsets $U \subseteq \mathrm{Pic}^0 A$. When possible, we will omit the reference to the morphism a, and say simply *CGG* at p.

Remark 4.2. If L is a line bundle, then L is not *CGG* at p if and only if there is a non-empty Zariski open set $V \subseteq \mathrm{Pic}^0 A$ such that p is a base point of $L \otimes a^*\alpha^{-1}$ for all $\alpha \in V$. This is equivalent to the fact that p is a base point of $L \otimes a^*\alpha^{-1}$ for all $\alpha \in \mathrm{Pic}^0 A$ such that $h^0(L \otimes a^*\alpha^{-1})$ is minimal.

We will also need the following weaker version. This is in fact a variant of the *weak continuous global generation* of [PP2]. However, for technical reasons we prefer to give the following definition, which is natural in view of Proposition 4.4(b) and Theorem 4.5(b) below.

Definition 4.3. Let p be a point of X and let $\mathcal{F}$ be a GV-sheaf. Let $\widehat{\mathbf{R}\Delta\mathcal{F}}$ be the transform of the dual of $\mathcal{F}$ and let $\tau = \tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$ be its torsion sheaf. Then $\mathcal{F}$ is said to be *essentially continuously globally generated at p (ECGG at p for short), with respect to the morphism a*, if the map ev_T is surjective *at* p, for all subsets of the form $T = U \cup S$, where U is a non-empty Zariski open subset of $\mathrm{Pic}^0 A$ and S is the underlying subset of $\mathrm{supp}\,\tau$. As above, when possible, we will omit the reference to the morphism a, and say simply *ECGG* at p.

Obviously *CGG* at p implies *ECGG* at p. Moreover, we will say simply that a sheaf is *CGG* (resp. *ECGG*), when it is *CGG* (resp. *ECGG*) for all p.

The structure sheaf of an abelian variety X is an easy example of a line bundle which is not *CGG* at any point, since for any open subset U of $\mathrm{Pic}^0 X$ not containing the identity point $\hat{0}$, the map $ev_{U,\mathcal{O}_X}$ is zero, but it is *ECGG*. In fact, it is well known that in this case $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$ (see [Mu2]). Hence, the underlying subset of the support of $\tau(\widehat{\mathcal{O}_X}) = \mathbb{C}(\hat{0})$ is the identity point $\{\hat{0}\}$, and all evaluation maps $ev_{U \cup \{\hat{0}\},\mathcal{O}_X}$ are trivially surjective at all points. A generalization of this example is provided by Corollary 4.11(b) below.

A useful relation between continuous global generation and the usual *global generation* is provided by the following, where, given a line bundle M, $\mathrm{Bs}(M)$ will denote its base locus.

Proposition 4.4 ([PP2, Prop. 2.4])**.** *Let $\mathcal{F}$ and L be respectively a coherent sheaf and a line bundle on X, and let p be a point of X.*

(a) *If both $\mathcal{F}$ and L are continuously globally generated at p, then $\mathcal{F} \otimes L \otimes a^*\beta$ is globally generated at p for any $\beta \in \mathrm{Pic}^0 A$.*

(b) *Assume that L is CGG at p and that $\mathcal{F}$ is ECGG at p. Let $\tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$ be the torsion sheaf of $\widehat{\mathbf{R}\Delta\mathcal{F}}$, and assume that the underlying set S of the support of $\tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$ is finite. For any $\beta \in \mathrm{Pic}^0 A$, if $p \notin \bigcup_{\alpha \in S+\beta} \mathrm{Bs}(L \otimes a^*\alpha)$, then $\mathcal{F} \otimes L \otimes a^*\beta$ is globally generated at p.*

Proof. (a) By Remark 4.2 there is an open subset V_p of $\mathrm{Pic}^0 A$ such that p is not a base point of $L \otimes a^*\alpha$ for all $\alpha \in V_p$, i.e. the evaluation map at

p: $H^0(L\otimes a^*\alpha)\to(L\otimes a^*\alpha)_p$ is surjective for all $\alpha\in V_p$. Since $\mathcal{F}$ is CGG at p, it follows that the map

$$ev_{V_p}:\bigoplus_{\alpha\in V_p}H^0(\mathcal{F}\otimes a^*\beta\otimes a^*\alpha^{-1})\otimes H^0(L\otimes a^*\alpha)\to(\mathcal{F}\otimes L\otimes a^*\beta)_p$$

is surjective. This proves the assertion, since the above map factors through $H^0(\mathcal{F}\otimes L\otimes a^*\beta)$.

(b) The proof is the same with the difference that now if we use the continuous evaluation map ev_{T_p} with $T_p=V_p\cup(S+\beta)$. If $p\notin\bigcup_{\alpha\in S+\beta}\mathrm{Bs}(L\otimes a^*\alpha)$, then the evaluation map $H^0(L\otimes a^*\alpha)\to(L\otimes a^*\alpha)_p$ is surjective also for all $\alpha\in S+\beta$. □

As for the usual global generation, in many applications it is useful to have a criterion ensuring that, if the higher cohomology of a given sheaf $\mathcal{F}$ satisfies certain vanishing conditions, then $\mathcal{F}$ is CGG or $ECGG$. The following criterion, which applies to *sheaves on abelian varieties*, is due to Pareschi-Popa. In the reference a sheaf $\mathcal{F}$ on an abelian variety A such that $gv(\mathcal{F})\geq 1$ is called *M-regular*. The first part of the following theorem is [PP1, Prop. 2.13] or [PP5, Cor. 5.3]. The proof of the second part essentially follows the proof of [PP2, Thm. 4.1].

Theorem 4.5. *Let $\mathcal{F}$ be a sheaf on an abelian variety A.*

(a) *If $gv(\mathcal{F})\geq 1$, then $\mathcal{F}$ is CGG.*

(b) *If $\mathcal{F}$ is a GV-sheaf and* $\operatorname{supp}\tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$ *is a reduced scheme*[4]*, then $\mathcal{F}$ is $ECGG$.*

The main point is the following.

Lemma 4.6. *Let $\mathcal{F}$ be a GV-sheaf on an abelian variety A. Let $\widehat{\mathbf{R}\Delta\mathcal{F}}$ be the transform of the dual of $\mathcal{F}$. Let L be an ample line bundle on A. Then, for all sufficiently high $n\in\mathbb{N}$, and for any subset $T\subseteq\mathrm{Pic}^0A$, the Fourier-Mukai transform $\Phi_{\mathcal{P}}$ induces a canonical isomorphism*

$$H^0(A,\operatorname{coker} ev_{T,\mathcal{F}}\otimes L^n)\cong(\ker\psi_{T,\mathcal{F}})^*,$$

where ψ is the natural evaluation map,

$$\psi_{T,\mathcal{F}}:\mathrm{Hom}(\widehat{L^n},\widehat{\mathbf{R}\Delta\mathcal{F}})\to\prod_{\alpha\in T}\mathcal{H}om(\widehat{L^n},\widehat{\mathbf{R}\Delta\mathcal{F}})\otimes\mathbb{C}(\alpha).\tag{8}$$

Proof. Let $T\subseteq\mathrm{Pic}^0A$ be any subset. The map $H^0(ev_T\otimes L^n)$ is the "continuous multiplication map of global sections":

$$m^T_{\mathcal{F},L^n}:\bigoplus_{\alpha\in T}H^0(\mathcal{F}\otimes a^*\alpha^{-1})\otimes H^0(L^n\otimes a^*\alpha)\to H^0(\mathcal{F}\otimes L^n).$$

[4]We consider the annihilator support, instead of the Fitting support [E].

A standard argument with Serre vanishing shows that, if n is big enough,

$$H^0(\mathrm{coker}(ev_{T,\mathcal{F}}) \otimes L^n) \cong \mathrm{coker}(m^T_{\mathcal{F},L^n}).$$

By Serre duality, the dual of $m^T_{\mathcal{F},L^n}$ is

$$\mathrm{Ext}^q(L^n, \mathbf{R}\Delta\mathcal{F}) \to \prod_{\alpha\in T} \mathrm{Hom}_{\mathbb{C}}\big(H^0(L^n \otimes a^*\alpha), H^q((\mathbf{R}\Delta\mathcal{F}) \otimes a^*\alpha)\big). \tag{9}$$

Let us interpret such map via the Fourier-Mukai transform. Concerning the source, Mukai's Theorem 2.7 provides the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(A)}(L^n, \mathbf{R}\Delta\mathcal{F}[q]) &\cong \mathrm{Hom}_{\mathbf{D}(\mathrm{Pic}^0 A)}(\mathbf{R}\Phi_{\mathcal{P}}(L^n), \mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\Delta\mathcal{F})[q]) \\ &\cong \mathrm{Hom}_{\mathbf{D}(\mathrm{Pic}^0 A)}(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}})\end{aligned}$$

(here $q = \dim A$) i.e., since $\widehat{L^n}$ is locally free,

$$\mathrm{Ext}^q_A(L^n, \mathbf{R}\Delta\mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Pic}^0 A}(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}}). \tag{10}$$

Note that, besides $\widehat{L^n}$, also $\widehat{\mathbf{R}\Delta\mathcal{F}}$ has the base change property, since $H^{d+1}(\mathcal{F} \otimes a^*\alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0 A$; see [M, Cor. 3, p. 53]. This means that, in the target of the map (9), we have that $H^0(L^n \otimes a^*\alpha)$ (respectively $H^q((\mathbf{R}\Delta\mathcal{F}) \otimes a^*\alpha))$) are isomorphic to the fiber at the point α of $\widehat{L^n}$ (resp. of $\widehat{\mathbf{R}\Delta\mathcal{F}}$). Hence, also the sheaf $\mathcal{H}om(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}})$ has the base change property and the Fourier-Mukai isomorphism (10) identifies the map (9) to the evaluation map of the sheaf $\mathcal{H}om(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}})$ at points in T:

$$\mathrm{Hom}(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \to \prod_{\alpha\in T} \mathcal{H}om(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes \mathbb{C}(\alpha).$$

□

Now we are ready to prove the theorem.

Proof of Theorem 4.5. (a) By Theorem 2.2 we can assume that $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is torsion free. Then the evaluation map $\psi_{U,\mathcal{F}}$ is injective for *all* open subsets $U \subseteq \mathrm{Pic}^0 A$, so $H^0(\mathrm{coker}\, ev_{U,\mathcal{F}} \otimes L^n) = 0$ for $n \gg 0$. From Serre's theorem it follows that $\mathrm{coker}\, ev_{U,\mathcal{F}} = 0$.

(b) By Nakayama's Lemma, given a non-zero global section $s \in \mathrm{Hom}(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}})$, we have that $s(\alpha) \in \mathcal{H}om(\widehat{L^n}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes \mathbb{C}(\alpha)$ vanishes for all α in a dense subset $T \subset \mathrm{Pic}^0 A$, only if T does not meet a component of the support of the torsion part $\tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$ or this support is non-reduced. The second possibility is excluded by hypothesis. And the first possibility is excluded if we consider subsets of the form $T = U \cup S$, where S is the underlying set of $\mathrm{supp}\, \tau(\widehat{\mathbf{R}\Delta\mathcal{F}})$. Hence, the map $\psi_{T,\mathcal{F}}$ is injective. Therefore $H^0(\mathrm{coker}\, ev_{T,\mathcal{F}} \otimes L^n) = 0$ and, by Serre's theorem, $\mathrm{coker}\, ev_{T,\mathcal{F}} = 0$. □

4.2. The canonical bundle. Throughout the rest of the present section we will work under the following:

Hypotheses 4.7. *Let X be a variety of dimension d, equipped with a morphism to an abelian variety $a : X \to A$ and,*

(a) $\operatorname{codim} V^i_a(\omega_X) \geq i+1$ *for all i such that* $0 < i < d$ (note that this hypothesis is weaker than $gv_a(\omega_X) \geq 1$)*;*

(b) *the map $a^* \operatorname{Pic}^0 A \to \operatorname{Pic}^0 X$ is an embedding.*

Observe that (a), with $i = d-1$, implies that $\dim X \leq \dim A$.

Hypothesis (b) yields that $V^d_a(\omega_X) = \{\hat{0}\}$, where $\hat{0}$ denotes the identity point of $\operatorname{Pic}^0 A$. Combining (a) and (b) one gets that $gv_a(\omega_X) \geq 0$. Therefore, by Theorem 2.2, the transform of $\mathbf{R}\Delta\omega_X = \mathcal{O}_X$ is the sheaf $\widehat{\mathcal{O}_X}$ concentrated in degree d. Moreover, the morphism a is generically finite by Proposition 2.9. Assuming (b), we remark that hypothesis (a) is equivalent to $gv_a(\omega_X) \geq 1$ unless $\dim X = \dim A$, i.e. the morphism a is surjective. Therefore, by Theorem 2.6, the sheaf $\widehat{\mathcal{O}_X}$ has torsion if and only if $\dim X = \dim A$.

Proposition 4.8. *Under Hypotheses* 4.7*, assume moreover that* $\dim X = \dim A$*, i.e. that the morphism a is surjective. Then the torsion $\tau(\widehat{\mathcal{O}_X})$ is isomorphic to the one-dimensional skyscraper sheaf* $\mathbb{C}(\hat{0})$*.*

Proof. The argument is similar to the proof of Proposition 2.8 of [PP5]. Since $\operatorname{Pic}^0 A$ is smooth, the functor $R\mathcal{H}om(\,\cdot\,, \mathcal{O}_{\operatorname{Pic}^0 A})$ is an involution on $\mathbf{D}(\operatorname{Pic}^0 A)$. Thus there is a spectral sequence

$$E_2^{i,-j} := \mathcal{E}xt^i\Big(\mathcal{E}xt^j(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 A}), \mathcal{O}_{\operatorname{Pic}^0 A}\Big) \Rightarrow H^{i-j} = \mathcal{H}^{i-j}\widehat{\mathcal{O}_X} = \begin{cases} \widehat{\mathcal{O}_X}, & \text{if } i=j, \\ 0, & \text{otherwise.} \end{cases}$$

By duality (Corollary 2.5),

$$\dim \operatorname{supp}(\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 A})) = \dim \operatorname{supp} R^i\Phi_{P_a}(\omega_X)$$

and, by base-change, $\operatorname{supp} R^i\Phi_{P_a}(\omega_X) \subseteq V^i_a(\omega_X)$. Thus Hypothesis 4.7(a) yields that for all i such that $0<i<d$ we have $\operatorname{codim}\ \operatorname{supp}\big(\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 A})\big) > i$. Therefore $\mathcal{E}xt^i\Big(\mathcal{E}xt^j(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 A}), \mathcal{O}_{\operatorname{Pic}^0 A}\Big) = 0$ for all (i,j) such that $i-j \leq 0$, except for $(i,j) = (0,0)$ and $(i,j) = (d,d)$. Hence, the only non-zero $E^{i,-i}_\infty$ terms are $E^{d,-d}_\infty$ and $E^{0,0}_\infty$, and we have the exact sequence

$$0 \to E^{d,-d}_\infty \to H^0 = \widehat{\mathcal{O}_X} \to E^{0,0}_\infty \to 0. \tag{11}$$

The differentials coming into $E^{0,0}_p$ are always zero, so we get

$$E^{0,0}_\infty \subseteq E^{0,0}_2 = \widehat{\mathcal{O}_X}^{**}$$

and (11) is identified to the canonical exact sequence

$$0 \to \tau(\widehat{\mathcal{O}_X}) \to \widehat{\mathcal{O}_X} \to \widehat{\mathcal{O}_X}^{**}.$$

Hence the torsion $\tau(\widehat{\mathcal{O}_X})$ is canonically isomorphic to $E_\infty^{d,-d}$. By Proposition 6.1 of the Appendix, $R^d\Phi_{P_a}(\omega_X) \cong \mathbb{C}(\hat{0})$. Hence $E_2^{d,-d} = \mathcal{E}xt^d(R^d\Phi_{P_a}(\omega_X), \mathcal{O}_{\mathrm{Pic}^0 A}) \cong \mathcal{E}xt^d(\mathbb{C}(\hat{0}), \mathcal{O}_{\mathrm{Pic}^0 A}) = \mathbb{C}(\hat{0})$. Since the differentials going out of $E_p^{d,-d}$ are always zero, we get a surjection

$$E_2^{d,-d} \cong \mathbb{C}(\hat{0}) \to E_\infty^{d,-d} \cong \tau(\widehat{\mathcal{O}_X}) \to 0.$$

As we already know that $\tau(\widehat{\mathcal{O}_X}) \neq 0$, it follows that $\tau(\widehat{\mathcal{O}_X}) \cong \mathbb{C}(\hat{0})$. □

Remark 4.9. It is easily seen that the injection $\mathbb{C}(\hat{0}) \hookrightarrow \widehat{a_*\mathcal{O}_X}$ is $R^q\Phi_{\mathcal{P}}$ of the natural injection $\mathcal{O}_A \to a_*\mathcal{O}_X$.

As a first consequence we get the following proposition.

Proposition 4.10. *The following are equivalent:*

(a) *Hypotheses* 4.7 *holds, and X is not of general type.*

(b) *Hypotheses* 4.7 *holds, and $\chi(\omega_X) = 0$.*

(c) *The morphism $a : X \to A$ is birational.*

Proof. $(a) \Rightarrow (b)$: The morphism a is generically finite (Proposition 2.9). In this case it is well known that $\chi(\omega_X) \geq 1$ implies that X is of general type (by [CH], one knows even that the tricanonical map of X is birational).

$(b) \Rightarrow (c)$: Since $\chi(\omega_X) = 0$, then by (1), $\widehat{\mathcal{O}_X}$ is a torsion sheaf. By Theorem 2.6, it follows that $gv(\omega_X) = 0$. We already know that this, together with Hypotheses 4.7, is equivalent to the fact that the morphism a is surjective. Therefore, by Proposition 4.8, $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$. By Proposition 2.8,

$$\mathbb{C}(\hat{0}) = \widehat{\mathcal{O}_X} = \widehat{\mathbf{R}a_*\mathcal{O}_X}$$

where the hat on the left is the transform $\mathbf{R}\Phi_{P_a}$ (from X to $\mathrm{Pic}^0 A$) and the hat on the right is the transform $\mathbf{R}\Phi_{\mathcal{P}}$ (from A to $\mathrm{Pic}^0 A$). By Mukai's Theorem 2.7,

$$\mathcal{O}_A = \mathbf{R}\Psi_{\mathcal{P}}(\mathbb{C}(\hat{0})) = \mathbf{R}\Psi_{\mathcal{P}}(\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}a_*\mathcal{O}_X)) \overset{2.7}{=} (-1_A)^*\mathbf{R}a_*\mathcal{O}_X$$

whence $\mathbf{R}a_*\mathcal{O}_X = \mathcal{O}_A$. In particular, a has degree 1. □

Another useful consequence of the previous analysis is the following. We recall that in the first case ($gv(\omega_X) \geq 1$) everything is essentially already known [PP5, Prop. 5.5].

Corollary 4.11. *Assume Hypotheses* 4.7 *and that X is of general type.*

(a) *If $\dim X < \dim A$, then $a_*\omega_X$ is CGG.*

(b) *If $\dim X = \dim A$, i.e. a is surjective, then $a_*\omega_X$ is ECGG, with $\{\hat{0}\}$ as the underlying subset of* $\operatorname{supp}\tau(\mathbf{R}\Delta(a_*\omega_X))$.

Proof. By Grauert-Riemenschneider vanishing (and projection formula), $R^i a_*(\omega_X \otimes a^*\alpha) = 0$ for $i > 0$. Therefore the Leray spectral sequence degenerates giving

$$V^i_a(\omega_X) = V^i(a_*\omega_X). \tag{12}$$

(a) If $\dim X < \dim A$, then $gv(a_*\omega_X) \geq 1$. Therefore it is CGG by Theorem 4.5(a). As remarked before, this part of the result is well known [PP5, Prop. 5.5].

(b) If $\dim X = \dim A$, then $\tau(\widehat{\mathcal{O}_X}) \cong \mathbb{C}(\hat{0})$ (Proposition 4.8). Hence, by Theorem 4.5(b), $a_*\omega_X$ is $ECGG$. □

4.3. Birationality criterion. We introduce a final piece of notation.

Terminology/Notation 4.12. (a) Given a line bundle L, we denote $\mathrm{Bs}(L)$ its base locus.

(b) We denote U_0 the complement in $\mathrm{Pic}^0 A$ of the closed subset $V^1_a(\omega_X)$. Assume $gv_a(\omega_X) \geq 0$; then, by Corollary 2.4, $V^1_a(\omega_X) \supseteq \cdots \supseteq V^d_a(\omega_X)$, it follows that, for all $\alpha \in U_0$, $h^0(\omega_X \otimes a^*\alpha)$ takes the minimal value, i.e. $\chi(\omega_X)$.

(c) Given a point $p \in X$, we denote $\mathcal{B}_a(p)$ the subset (closed in U_0)

$$\mathcal{B}_a(p) = \{\alpha \in U_0 \mid p \in \mathrm{Bs}(\omega_X \otimes a^*\alpha)\}.$$

(d) We will say that a line bundle "is birational" to mean that the associated rational map to projective space is birational.

The statements we are aiming at are as follows.

Theorem 4.13. *Let X be a variety of general type satisfying Hypothesis 4.7. If, for general p in X, $\mathrm{codim}_{\mathrm{Pic}^0 A}\,\mathcal{B}_a(p) \geq 2$, then $\omega_X^2 \otimes a^*\alpha$ is birational for all $\alpha \in \mathrm{Pic}^0 A$. In particular, ω_X^2 is birational.*

Corollary 4.14. *Let X be a variety of general type satisfying Hypothesis 4.7, and assume that there exists an $\alpha \in \mathrm{Pic}^0 A$ such that $\omega_X^2 \otimes a^*\alpha$ is not birational. Then, for all $\beta \in U_0$, $\mathrm{Bs}(\omega_X \otimes a^*\beta)$ has codimension one. Moreover, X is covered by the divisorial components of $\mathrm{Bs}(\omega_X \otimes a^*\beta)$, for β varying in U_0.*

Note that it makes sense to speak of the base locus of $\omega_X \otimes a^*\alpha$ since, by Proposition 4.10, the hypotheses of both the theorem and the corollary imply that $\chi(\omega_X) > 0$, whence $h^0(\omega_X \otimes a^*\alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0 A$.

Proof of Corollary 4.14. Let $\mathcal{B}_a$ the closed subvariety of $X \times U_0$ defined as

$$\mathcal{B}_a = \{(p, \alpha) \in X \times U_0 \mid \ p \text{ is a base point of } \omega_X \otimes a^*\alpha\}.$$

If, for some $\alpha \in \mathrm{Pic}^0 X$, $\omega_X^2 \otimes a^*\alpha$ is not birational, then by Theorem 4.13, for general $p \in X$ we have that $\mathrm{codim}\,\mathcal{B}_a(p) = 1$. Since $\mathcal{B}_a(p)$ is the fiber of the projection $p : \mathcal{B}_a \to X$ it follows that $\mathrm{codim}_{X \times U_0}\,\mathcal{B}_a = 1$. This implies that the fibers of the other projection $q : \mathcal{B}_a \to \mathrm{Pic}^0 A$ have codimension 1 in X. □

Proof of Theorem 4.13. We first recall some basic facts about the Fourier-Mukai transform of the sheaves $\mathcal{I}_p \otimes \omega_X$. In the first place, if p does not belong to $exc(a)$, then

$$R^i a_*(\mathcal{I}_p \otimes \omega_X \otimes a^*\alpha) = 0 \qquad \text{for } i > 0. \tag{13}$$

This follows immediately from the exact sequence

$$0 \to \mathcal{I}_p \otimes \omega_X \to \omega_X \to \mathcal{O}_p \otimes \omega_X \to 0 \tag{14}$$

and the Grauert-Riemenschneider vanishing theorem. Hence, as for the canonical bundle (see the proof of Proposition 4.8), the Leray spectral sequence yields that

$$V^i_a(\mathcal{I}_p \otimes \omega_X) = V^i(a_*(\mathcal{I}_p \otimes \omega_X)). \tag{15}$$

By sequence (14), tensored by $a^*\alpha$, it follows that

$$V^i_a(\mathcal{I}_p \otimes \omega_X) = V^i_a(\omega_X) \qquad \text{for all } i \geq 2. \tag{16}$$

Concerning the case $i = 1$, we have the surjection

$$H^1(\mathcal{I}_p \otimes \omega_X \otimes a^*\alpha) \twoheadrightarrow H^1(\omega_X \otimes a^*\alpha),$$

which is an isomorphism if and only if p is not a base point of $\omega_X \otimes \alpha$. In other words,

$$V^1_a(\mathcal{I}_p \otimes \omega_X) = \mathcal{B}_a(p) \cup V^1_a(\omega_X).$$

Therefore the hypothesis about $\mathcal{B}_a(p)$ ensures that

$$\operatorname{codim} V^1_a(\mathcal{I}_p \otimes \omega_X) \geq 2. \tag{17}$$

Now we distinguish two cases:

(a) $\dim X < \dim A$. In this case, by Hypothesis 4.7, together with (15), (16) and (17), $gv(a_*(\mathcal{I}_p \otimes \omega_X)) \geq 1$. Hence, by Theorem 4.5(a), $a_*(\mathcal{I}_p \otimes \omega_X)$ is CGG. Therefore $\mathcal{I}_p \otimes \omega_X$ itself is *CGG* outside $exc(a)$ (with respect to a). Since the same is true for ω_X (Corollary 4.11(a)), it follows from Proposition 4.4(a) that, for p outside $exc(a)$ and for all $\alpha \in \operatorname{Pic}^0 A$, $\mathcal{I}_p \otimes \omega_X^2 \otimes a^*\alpha$ is globally generated outside $exc(a)$. This means that the projective map associated to $\omega_X^2 \otimes a^*\alpha$ is birational (in fact an isomorphism on the complement of $exc(a)$).

(b) $\dim X = \dim A$, i.e. the map a is surjective. Again by Hypotheses 4.7, (16) and (17), the sheaf $\mathcal{I}_p \otimes \omega_X$ satisfies,

$$\operatorname{codim} V^i(\mathcal{I}_p \otimes \omega_X) \geq i+1 \quad \text{for all } i \text{ such that } 0 < i < d$$

while

$$V^d(\mathcal{I}_p \otimes \omega_X) = \{\hat{0}\} \quad \text{and} \quad R^d\Phi_{P_a}(\mathcal{I}_p \otimes \omega_X) = R^d\Phi_{P_a}(\omega_X) = \mathbb{C}(\hat{0}).$$

Therefore the exact same arguments of Proposition 4.8 apply, proving that the torsion of $\mathbf{R}\,\mathcal{H}om(\widehat{a_*(\mathcal{I}_p \otimes \omega_X)}, \mathcal{O}_A)$ is $\mathbb{C}(\hat{0})$. Hence, by Theorem

4.5(b), $a_*(\mathcal{I}_p\otimes\omega_X)$ is ECGG, with $\{\hat{0}\}$ as an underlying subset of $\operatorname{supp}\tau(\mathbf{R}\Delta(\widehat{\mathcal{I}_p\otimes\omega_X}))$. It follows that, for p not belonging to $exc(a)$, the sheaf $\mathcal{I}_p\otimes\omega_X$ is *ECGG* away from $exc(a)$. Let W be the non-empty open set of points $p\in X$ such that ω_X is *CGG* at p. In view of Remark 4.2, W is the complement of the intersection of all base loci $\mathrm{Bs}(\omega_X\otimes a^*\alpha^{-1})$, for $\alpha\in\mathrm{Pic}^0 A$ such that $h^0(\omega_X\otimes a^*\alpha^{-1})$ is minimal, i.e. equal to $\chi(\omega_X)$. Now let $\alpha\in\mathrm{Pic}^0 A$. It follows from Proposition 4.4(b) that, if q is not a base point of $\omega_X\otimes a^*\alpha$ (and does not lie in $exc(a)$), then $\mathcal{I}_p\otimes\omega_X^2\otimes a^*\alpha$ is globally generated at q. Denoting U_α as the complement of $exc(a)\cup\mathrm{Bs}(\omega_X\otimes a^*\alpha)$, we conclude that for all $p\in U_\alpha\cap W$ the sheaf $\mathcal{I}_p\otimes\omega_X^2\otimes a^*\alpha$ is globally generated at all points of $U_\alpha\cap W$. As above, this means that the projective map associated to $\omega_X^2\otimes a^*\alpha$ is an isomorphism on $U_\alpha\cap W$. □

5. Proof of the theorem

Here we prove Theorem A of the Introduction.

(b) ⇒ (a) Let (A,Θ) be an indecomposable p.p.a.v., and let $X\to\Theta$ be a desingularization. Without loss of generality, we can assume that Θ is *symmetric*, i.e. $\Theta=(-1)^*\Theta$. The restriction map $H^0(A,\mathcal{O}_A(2\Theta))\to H^0(\Theta,\mathcal{O}_\Theta(2\Theta))=H^0(\Theta,\omega_\Theta^2)$ is surjective. Hence, the projective map associated to ω_Θ^2 has degree two, since x and $-x$ have the same image. By a result of Ein-Lazarsfeld, Θ is normal and has rational singularities [EL, Thm. 1]. Hence Θ has canonical singularities and therefore $H^0(X,\omega_X^2)\cong H^0(\Theta,\omega_\Theta^2)$. It follows that the bicanonical map of X has degree 2.

The rest of the section is devoted to the proof of the implication $(a)\Rightarrow(b)$. The argument is composed of a few steps.

5.1. Decomposition. The hypotheses of Theorem A are more than enough to imply that $alb: X\to\mathrm{Alb}\,X$, the Albanese map of X, satisfies Hypotheses 4.7. Keeping the Terminology/Notation 4.12, it follows that, by Corollary 4.14, $|\omega_X\otimes\alpha|$ has a base divisor for all $\alpha\in U_0$. More precisely, as in the Proof of Corollary 4.14, let us consider the relative base locus

$$\mathcal{B}=\{(p,\alpha)\in X\times U_0\mid\ p\text{ is a base point of }\omega_X\otimes\alpha\ \}$$

equipped with the projections on the two factors, p and q. $\mathcal{B}$ has a natural subscheme structure given by the image of the relative evaluation map $p^*(p_*\mathcal{L})\otimes\mathcal{L}^{-1}\to\mathcal{O}_{X\times U_0}$, where $\mathcal{L}=p^*\omega_X\otimes P_{|X\times U_0}$. We know from

Theorem 4.13 that $\mathcal{B}$ has codimension 1 and that its divisorial part dominates on X and surjects on U_0 via p and q. Let

$$\mathcal{Y} \subseteq \mathcal{B}$$

be the union of the divisorial components of $\mathcal{B}$.

For $\alpha \in U_0$, let F_α be the (scheme-theoretic) fiber of $q : \mathcal{Y} \to U_0$. Observe that $\mathcal{Y} \longrightarrow U_0$ is a flat family of algebraically equivalent divisors. At a *general* point $\alpha \in U_0$, F_α is the fixed divisor of $\omega_X \otimes \alpha$:

$$|\omega_X \otimes \alpha| = |M_\alpha| + F_\alpha$$

where $|M_\alpha|$ is the (possibly empty) mobile part.

We consider the Abel-Jacobi map

$$f_{\alpha_0} : U_0 \to \operatorname{Pic}^0 X \qquad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0})$$

where α_0 is fixed in U_0. Since f_{α_0} is a rational map to an abelian variety, it extends in a unique way to a morphism $\operatorname{Pic}^0 X \to \operatorname{Pic}^0 X$, which will be denoted f_{α_0} as well. By rigidity

$$f := f_{\alpha_0} - f_{\alpha_0}(\hat{0}) : \operatorname{Pic}^0 X \to \operatorname{Pic}^0 X$$

is a homomorphism. Note that f does not depend on α_0 since, given another suitably general $\alpha_1 \in \operatorname{Pic}^0 A$, $f_{\alpha_1} - f_{\alpha_0}$ is a translation. We have the following strong constraint on the Albanese and Picard variety of X:

Lemma 5.1. (a) *$f^2 = f$ and $\operatorname{Pic}^0 X$ decomposes as* $\operatorname{Pic}^0 X \cong \ker f \times \ker(\mathrm{id} - f)$. *Moreover,* $\dim \ker(\mathrm{id} - f) > 0$.

(b) *For $\gamma \in U_0 \cap \ker(\mathrm{id} - f)$ the line bundle $\mathcal{O}_X(F_\gamma) \otimes \gamma^{-1}$ does not depend on γ. Since it is effective by semicontinuity, we call it $\mathcal{O}_X(F)$. Then we have:*

- (i) *for all $(\beta, \gamma) \in \ker f \times \ker(\mathrm{id} - f) \cong \operatorname{Pic}^0 X$ such that $\beta \otimes \gamma \in U_0$, $|\mathcal{O}_X(F) \otimes \gamma|$ is contained in the fixed divisor of $\omega_X \otimes \beta \otimes \gamma$;*
- (ii) *for all (β, γ) such that $\beta \otimes \gamma$ is sufficiently general in U_0, $|\mathcal{O}_X(F) \otimes \gamma|$ is the fixed divisor of $\omega_X \otimes \beta \otimes \gamma$.*

(c) *Let $\mathcal{O}_X(M) := \omega_X(-F)$. Then $h^0(\mathcal{O}_X(M) \otimes \beta) = \chi(\omega_X)$ for all $\beta \in \ker f$; in particular, M is effective.*

Proof. (a) Let $\alpha, \beta \in \operatorname{Pic}^0 X$. We have that

$$\mathcal{O}_X(M_\alpha) \otimes \mathcal{O}_X(F_\beta) \cong \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1}). \tag{18}$$

This follows by definition of f since the left hand side is isomorphic to

$$\mathcal{O}_X(M_\alpha + F_\alpha) \otimes \mathcal{O}_X(F_\beta - F_\alpha) \cong \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1}).$$

Assuming that α and β are *general*, so that the fixed divisor of $\omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1})$ is $F_{\alpha \otimes f(\beta \otimes \alpha^{-1})}$ (i.e. $|M_{\alpha \otimes f(\beta \otimes \alpha^{-1})}|$ has no base divisors), it follows from (18) that

$$F_\beta = F_{\alpha\otimes f(\beta\otimes\alpha^{-1})},$$

i.e. $f(\beta) = f(\alpha \otimes f(\beta \otimes \alpha^{-1}))$. This means that $f(f(\beta\otimes\alpha^{-1})) = f(\beta\otimes\alpha^{-1})$ for general α and β, hence $f^2 = f$. This gives the splitting of the exact sequence $\hat{0} \to \ker f \to \mathrm{Pic}^0 X \to \mathrm{Im} f \to \hat{0}$ and the identification $\mathrm{Im} f \cong \ker(\mathrm{id} - f)$. The abelian subvariety $\ker(\mathrm{if} - f)$ is positive-dimensional since otherwise the fixed divisor of $\omega_X \otimes \alpha$ would be constant for general $\alpha \in U_0$, contradicting Corollary 4.14.

(b) We have that $U_0 \cap \ker(\mathrm{id} - f)$ is non-empty, since the complement of U_0 is a finite set. By definition of f, for $\gamma \in U_0 \cap \ker(\mathrm{id} - f)$ the line bundle $\mathcal{O}_X(F) = \mathcal{O}_X(F_\gamma)\otimes\gamma^{-1}$ does not depend on γ. For $(\beta,\gamma) \in \ker f \times \ker(\mathrm{id}-f)$ such that $\beta\otimes\gamma \in U_0$, we have that $\mathcal{O}_X(F_{\beta\otimes\gamma} - F_\gamma) = f(\beta\otimes\gamma)\otimes f(\gamma)^{-1} = \mathcal{O}_X$. Therefore $F_\gamma = F_{\beta\otimes\gamma}$, is, by definition, a fixed divisor of $\omega_X \otimes \beta \otimes \gamma$, and $\mathcal{O}_X(F_\gamma) = \mathcal{O}_X(F) \otimes \gamma$. If $\beta \otimes \gamma$ is sufficiently general in U_0, we know that there are no other base divisors, i.e. $F_\gamma = F_{\beta\otimes\gamma}$ is the fixed divisor.

(c) Let us choose $\bar{\gamma} \in \ker(\mathrm{id} - f)$ such that $\omega_X \otimes \beta \otimes \bar{\gamma} \in U_0$ for all $\beta \in \ker f$ (this is possible because $\ker(\mathrm{id} - f)$ is positive dimensional and the complementary set of U_0 in $\mathrm{Pic}^0 X$ is $V^1(\omega_X)$, which is assumed to be a finite set). Therefore $h^0(\omega_X \otimes \beta \otimes \bar{\gamma}) = \chi(\omega_X)$ for all $\beta \in \ker f$. But, by (b)(i), $h^0(\omega_X \otimes \beta \otimes \bar{\gamma}) = h^0(\mathcal{O}_X(M) \otimes \beta)$. ☐

5.2. The Poincaré line bundle on $X \times \mathrm{Pic}^0 X$. It follows from Lemma 5.1 that

$$\mathrm{Alb}\, X \cong B \times C, \tag{19}$$

where B and C are the dual abelian varieties of $\ker f$ and $\ker(\mathrm{id} - f)$. Consequently, concerning the Poincaré line bundle $\mathcal{P}$ on $\mathrm{Alb}\, X \times \mathrm{Pic}^0 X$, we have that

$$\mathcal{P} \cong \mathcal{P}_B \boxtimes \mathcal{P}_C.$$

Keeping in mind that P, the Poincaré line bundle on $X \times \mathrm{Pic}^0 X$ is $(alb, \mathrm{id}_{\mathrm{Pic}^0 X})^*\mathcal{P}$, a description of $(alb, \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times\mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C)$ is provided in the next lemma. In fact, let $\overline{\mathcal{Y}}$ be the closure of $\mathcal{Y}$ in $X \times \mathrm{Pic}^0 X$ (see the notation introduced in the previous subsection). Moreover, for $p \in X$ let $\mathcal{D}_p$ be the fiber of the projection $\overline{\mathcal{Y}} \to X$. For general $p \in X$, we have that $\mathcal{D}_p$ is the closure of the union of the divisorial components of the locus of $\alpha \in U_0$ such that $p \in \mathrm{Bs}(\omega_X \otimes \alpha)$. Since the Albanese map is defined up to a translation in $\mathrm{Alb}\, X$, we can assume that there is one such point, say $\bar{p}$, such that $alb(\bar{p}) = 0$ in $\mathrm{Alb}\, X$.

Lemma 5.2. $(alb, \mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times\mathrm{Pic}^0 B} \boxtimes \mathcal{P}_C) \cong \mathcal{O}_{X\times\mathrm{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^*(\omega_X^{-1} \otimes M) \otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar{p}})$.

Proof. By the definition of $\overline{\mathcal{Y}}$ and Lemma 5.1(b) we have that the line bundle

$$\mathcal{O}_{X\times \mathrm{Pic}^0 X}(\overline{\mathcal{Y}})\otimes p^*(\omega_X^{-1}(M))\otimes q^*\mathcal{O}_{\mathrm{Pic}^0 X}(-\mathcal{D}_{\bar p}),$$

- restricted to $X\times\{\beta\otimes\gamma\}$, is isomorphic to

$$\begin{aligned}\mathcal{O}_X(F_{\beta\otimes\gamma}-F) &= \mathcal{O}_X(F)\otimes\gamma\otimes\mathcal{O}_X(-F)=\gamma\\ &= (alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times\mathrm{Pic}^0 B}\boxtimes P_C)_{|X\times\{\beta\otimes\gamma\}},\end{aligned}$$

for all $\beta\otimes\gamma=\alpha$ sufficiently general in U_0;

- restricted to $\{\bar p\}\times\mathrm{Pic}^0 X$ is isomorphic to $\mathcal{O}_X(\mathcal{D}_{\bar p})\otimes\mathcal{O}_X(-\mathcal{D}_{\bar p})$, i.e. trivial.

The lemma follows from the see-saw principle. □

5.3. Conclusion of the proof of Theorem A.

Lemma 5.3. $\chi(\omega_X)=1$.

Having proved this, we can use the characterization of theta divisors given in Proposition 3.1 to prove the implication (a) ⇒ (b) of Theorem A. On a theta divisor, observe that $f=\mathrm{Id}$.

Proof of Lemma 5.3. Lemma 5.2 yields the standard short exact sequence on $X\times\mathrm{Pic}^0 X$,

$$\begin{aligned}0&\to (alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{O}_{B\times\mathrm{Pic}^0 B}\boxtimes\mathcal{P}_C^{-1})\xrightarrow{\cdot\overline{\mathcal{Y}}} p^*(\omega_X(-M))\otimes q^*(\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar p}))\\ &\to p^*(\omega_X(-M))\otimes q^*(\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar p}))_{|\overline{\mathcal{Y}}}\to 0.\end{aligned}$$

Applying the functor $R^dq_*(\,\cdot\,\otimes(alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes\mathcal{O}_{C\times\mathrm{Pic}^0 C}))$, recalling (19) and that $\mathbf{R}\Phi_{P^{-1}}\cong(-1_{\mathrm{Pic}^0 X})^*\mathbf{R}\Phi_P$ (see 2.1) we get

$$\begin{aligned}\cdots&\to R^d\Phi_{P^{-1}}(\mathcal{O}_X)\\ &\to R^dq_*(p^*(\omega_X(-M))\otimes(alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes\mathcal{O}_{C\times\mathrm{Pic}^0 C}))(\mathcal{D}_{\bar p})\\ &\to R^dq_*((p^*(\omega_X(-M))\otimes(alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes\mathcal{O}_{C\times\mathrm{Pic}^0 C})_{|\overline{\mathcal{Y}}})(\mathcal{D}_{\bar p})\to 0\end{aligned}$$

i.e.

$$0\to(-1)^*_{\mathrm{Pic}^0 X}\widehat{\mathcal{O}_X}\xrightarrow{\mu}\mathcal{E}(\mathcal{D}_{\bar p})\to\tau\to 0 \tag{20}$$

where:

(a) $\mathcal{E}=R^dq_*(p^*(\omega_X(-M))\otimes(alb,\mathrm{id}_{\mathrm{Pic}^0 X})^*(\mathcal{P}_B^{-1}\boxtimes\mathcal{O}_{C\times\mathrm{Pic}^0 C}))$ is a locally free sheaf of rank $\chi(\omega_X)$ on $\mathrm{Pic}^0 X$ because of Lemma 5.1(c);

(b) $\tau = R^dq_*(p^*(\omega_X\otimes M^{-1})\otimes q^*(\mathcal{O}_{\mathrm{Pic}^0 X}(\mathcal{D}_{\bar p})))_{|\overline{\mathcal{Y}}}$ is supported at the locus of $\alpha\in\mathrm{Pic}^0 X$ such that the fiber of the projection $q:\overline{\mathcal{Y}}\to\mathrm{Pic}^0 X$ has dimension d (i.e. the fibre of this projection coincides with X). This locus is contained in the union of the $V^i(\omega_X)$ for $i>0$, hence it is a finite set;

(c) the map μ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\mathrm{rk}\,\widehat{\mathcal{O}_X}=\chi(\omega_X)$), and, as $gv(\omega_X)\geq 1$, the source $\widehat{\mathcal{O}_X}$ is torsion free by Theorem 2.6).

(d) Since μ is $R^d q_*(m_s)$, where m_s is the multiplication for the section defining $\overline{\mathcal{Y}}$, μ is zero on the locus where the fiber of q has dimension d. Therefore, τ is a sheaf supported at a finite set of points contained in the finite set $\bigcup_{i>0} V^i(\omega_X) = V^1(\omega_X)$.

From (20), the fact that $\mathcal{E}$ is locally free, and the fact that $\operatorname{supp}\tau$ is a finite scheme, it follows that $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) \cong \mathcal{E}xt^{i+1}(\tau, \mathcal{O}_{\mathrm{Pic}^0 X}) = 0$ if $i \neq q(X) - 1$. On the other hand, by Corollary 2.5 and Proposition 6.1 of the Appendix, $\mathcal{E}xt^d(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) \cong \mathbb{C}(\hat{0})$. It follows that $d = q(X) - 1$ and that $\tau = \mathbb{C}(\hat{0})$. Lemma 5.3 follows since the length of τ is equal to $\chi(\omega_X)$. □

6. Appendix: A useful lemma on the generalized Fourier-Mukai transform of the canonical sheaf

As usual, let $a : X \to A$ be a morphism from a d-dimensional smooth projective variety to an abelian variety (over any algebraically closed field k). Let $P_a = (a \times \mathrm{id})^*\mathcal{P}$, where $\mathcal{P}$ is the Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. Assume furthermore that the induced morphism $a^* : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an *embedding*. Then the top cohomological support locus is $V^d_a(\omega_X) = \{\hat{0}\}$. By base change [M, Cor. 3, p. 53], it follows that, for $\alpha \in \mathrm{Pic}^0 A$,

$$R^d\Phi_{P_a}(\omega_X) \otimes k(\hat{\alpha}) \cong \begin{cases} k(\hat{0}), & \text{if } \alpha = \hat{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

It follows that $R^d\Phi_{P_a}(\omega_X)$ is a sheaf supported at $\hat{0}$. The result we are aiming at is as follows.

Proposition 6.1. $R^d\Phi_{P_a}(\omega_X) \cong k(\hat{0})$.[5]

Proof. Let $B = \mathcal{O}_{\mathrm{Pic}^0 A, \hat{0}}$ and $\mathfrak{m}$ its maximal ideal. By Nakayama's lemma, (21) implies that $R^d\Phi_{P_a}(\omega_X)$ is supported only at $\hat{0}$ and that $R^d\Phi_{P_a}(\omega_X) \cong B/J$, where J is an $\mathfrak{m}$-primary ideal.

Claim. $P_{a|X \times Spec\, B/J}$ is trivial.

Proof of the Claim. Let $\mathcal{E}$ be a sheaf on $\mathrm{Pic}^0 A$. By Grothendieck duality

$$\mathbf{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathbf{R}\Phi_{P_a}(\omega_X), \mathcal{E}) \cong \mathbf{R}q_*(\mathcal{H}om_{X \times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E}))[d]. \tag{22}$$

[5]When X itself is an abelian variety (or a complex torus) this statement is well known (see [BL, Cor. 14.1.6] and [Ke] for an elementary proof in the complex case and [Hu, p. 202] and [M, p. 128] for arbitrary characteristic). This fact is crucial in the proof of Mukai Inversion Theorem 2.7.

Indeed,

$$
\begin{aligned}
&\mathbf{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathbf{R}\Phi_{P_a}(\omega_X), \mathcal{E}) \\
&\qquad = \mathbf{R}\mathcal{H}om_{\mathrm{Pic}^0 A}(\mathbf{R}q_*(p^*\omega_X \otimes P_a), \mathcal{E}) \\
&\qquad \overset{GD}{\cong} \mathbf{R}q_*(\mathbf{R}\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(p^*\omega_X \otimes P_a, p^*\omega_X \otimes q^*\mathcal{E}[d])) \\
&\qquad \cong \mathbf{R}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E}))[d].
\end{aligned}
$$

Therefore we have a fourth quadrant spectral sequence

$$E_2^{i,-j} = \mathcal{E}xt^i_{\mathrm{Pic}^0 A}(R^j\Phi_{P_a}(\omega_X), \mathcal{E}) \Rightarrow R^{i-j+d}q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Clearly the term $E_2^{i,-j}$ is non-zero only if $i \geq 0$. Assuming $i \geq 0$, in the case $i-j+d = 0$, i.e. $j = i+d$ we have that $R^j\Phi_{P_a}(\omega_X)$ is non-zero if and only if $j = d$, i.e. $i = 0$. In conclusion, for $i - j + d = 0$ the only non-zero E_2-term is $E_2^{0,-d} = \mathcal{H}om_{\mathrm{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E})$. Since the differentials from and to $E_2^{0,-d}$ are zero, we get that

$$\mathcal{H}om_{\mathrm{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) = E_2^{0,-d} = E_\infty^{0,-d} \cong q_*(\mathcal{H}om_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Taking global sections, we get the isomorphism (functorial in $\mathcal{E}$),

$$\mathrm{Hom}_{\mathrm{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) \cong \mathrm{Hom}_{X\times \mathrm{Pic}^0 A}(P_a, q^*\mathcal{E}). \tag{23}$$

Using the previous isomorphism twice, once for $\mathcal{E} = B/J$ and the other for $\mathcal{E} = k(\hat{0})$, by functoriality we get the commutative diagram:

$$
\begin{array}{ccccc}
B/J & = & \mathrm{Hom}(B/J, B/J) & = & \mathrm{Hom}(P_{a|X\times Spec\, B/J}, \mathcal{O}_{X\times Spec\, B/J}) \\
\downarrow & & \downarrow & & \downarrow \\
k(\hat{0}) & = & \mathrm{Hom}(B/J, k(\hat{0})) & = & \mathrm{Hom}(P_{a|X\times\{\hat{0}\}}, \mathcal{O}_{X\times\{\hat{0}\}})
\end{array} \tag{24}
$$

We can take an isomorphism $h \in \mathrm{Hom}(P_{a|X\times\{\hat{0}\}}, \mathcal{O}_{X\times\{\hat{0}\}})$ since $P_{a|X\times\{\hat{0}\}}$ is trivial. By the diagram above, h lifts to a morphism $\bar{h} : P_{a|X\times Spec\, B/J} \to \mathcal{O}_{X\times Spec\, B/J}$. Since $\bar{h}$ is a map between invertible sheaves on $X \times Spec\, B/J$ which is an isomorphism when restricted to $X \times \{\hat{0}\}$, $\bar{h}$ is an isomorphism. Therefore $P_{a|X\times Spec\, B/J}$ is trivial. The claim is proved. □

At this point the proposition follows since *the smooth point* $\hat{0}$ *of* $\mathrm{Pic}^0 A$ *is the maximal subscheme* Z *of* $\mathrm{Pic}^0 A$ *such that* $P_{a|X\times Z}$ *is trivial* (see [M], §10). This in turn follows from the well known fact (see [M], §13) that the smooth point $\hat{0}$ of $\mathrm{Pic}^0 A$ is the maximal subscheme Z of $\mathrm{Pic}^0 A$ such that $\mathcal{P}_{|A\times Z}$ is trivial, combined with the fact that $a^*\,\mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an embedding. However, we provide an equivalent but self-contained argument. Let us consider, in analogy to Subsection 2.3, the functor $\mathbf{R}\Psi_a : \mathbf{D}(\mathrm{Pic}^0 A) \to \mathbf{D}(X)$, defined by $\mathbf{R}\Psi_a(\cdot) = \mathbf{R}p_*(P_a \otimes q^*(\cdot))$. Since $P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^*\mathcal{P}$, it follows that

$$\mathbf{R}\Psi_a \cong \mathbf{L}a^* \circ \mathbf{R}\Psi. \tag{25}$$

The claim implies, by the Künneth formula, that

$$\mathbf{R}\Psi_{P_a}(B/J) = R^0\Psi_{P_a}(B/J) = \mathcal{O}_X^{\bigoplus r} \tag{26}$$

where $r = \mathit{length}\, B/J$. On the other hand, by [Mu1, Lemma 4.8], $\mathbf{R}\Psi_{\mathcal{P}}(B/J) = R^0\Psi_{\mathcal{P}}(B/J) := U$, where U is a *unipotent* vector bundle on A of rank r, i.e. a vector bundle having a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_{r-1} \subset U_r = U$, such that $U_i/U_{i-1} \cong \mathcal{O}_A$. By (25) and (26) it follows that a^*U is trivial. The filtration of U, pulled back via a, induces the filtration of the trivial bundle:

$$0 \subset a^*U_1 \subset \cdots \subset a^*U_{r-1} \subset a^*U_r = \mathcal{O}_X^{\bigoplus r},$$

where $a^*U_i/a^*U_{i-1} \cong \mathcal{O}_X$. Since $h^0(X, a^*U_i) \leq i$ for all i, the fact that a^*U_r is trivial implies easily, by descending induction on i, that

$$h^0(X, a^*U_i) = i \quad \text{for all } i. \tag{27}$$

This in turn implies that the sequence

$$0 \to a^*U_{i-1} \to a^*U_i \to \mathcal{O}_X \to 0$$

splits for all i (the coboundary map $H^0(\mathcal{O}_A) \to H^1(U_{i-1})$ is zero). In particular, the extension

$$0 \to \mathcal{O}_X \to a^*U_2 \to \mathcal{O}_X \to 0$$

is split. But the natural pullback map

$$H^1(\mathcal{O}_A) \cong \mathrm{Ext}^1(\mathcal{O}_A, \mathcal{O}_A) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \cong H^1(\mathcal{O}_X) \tag{28}$$

is identified with the differential at $\hat{0}$ of the map $a^* : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$. Since a^* is assumed to be an embedding, (28) is injective. Hence, also the extension

$$0 \to \mathcal{O}_A \to U_2 \to \mathcal{O}_A \to 0$$

is split. This yields that $h^n(U) \geq 2$. But this is impossible since, by Mukai's inversion, $\mathbf{R}\Phi_{\mathcal{P}}(U) = \hat{U}[-n] \cong (-1_{\mathrm{Pic}^0 A})^* B/J[-n]$, and therefore, by base change, $H^n(U) \cong (B/J) \otimes k(\hat{0}) \cong k(\hat{0})$. In conclusion, $r = \mathit{length}(B/J) = 1$, i.e. $B/J \cong k(\hat{0})$. □

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Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, ETSEIB Avda. Diagonal 647, 08028 Barcelona, Spain
E-mail address: miguel.angel.barja@upc.edu

Departament d'Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain
E-mail address: marti.lahoz@ub.edu

Departament d'Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain
E-mail address: jcnaranjo@ub.edu

Dipartimento di Matematica, Università di Roma, Tor Vergata, V. le della Ricerca Scientifica, I-00133 Roma, Italy
E-mail address: pareschi@mat.uniroma2.it