BANDLIMITED LIPSCHITZ FUNCTIONS

YURII LYUBARSKII AND JOAQUIM ORTEGA-CERDÀ

Abstract. We study the space of bandlimited Lipschitz functions in one variable. In particular we provide a geometrical description of interpolating and sampling sequences for this space. We also give a description of the trace of such functions to sequences of critical density in terms of a cancellation condition.

1. Introduction

A standard model for one-dimensional bandlimited signals is the space of functions (or distributions) that have Fourier transform $\hat{f}$ supported on a finite interval. According to the Paley-Wiener-Schwartz theorem the functions with compact frequency support can be extended from the real line into the whole complex plane $\mathbb{C}$ as entire functions of exponential type.

The size of a signal is usually measured either in terms of its energy, i.e. the $L^2(\mathbb{R})$ norm, or in the supremum $L^\infty(\mathbb{R})$ norm. In the first case we encounter the familiar Paley-Wiener space of entire functions and in the second the Bernstein space (its definition is reminded on the next page).

One objection to the use of the Bernstein space as a model for bandlimited signals is that a very common operation in signal processing, the filtering, does not preserve the space. By filtering we mean the operation that to f corresponds a function $T(f)$ with Fourier transform $\hat{f}\chi_{w<0}$. Here $\chi_{w<0}$ denotes the characteristic function of the negative axis. The content of the signal in all frequencies bigger than 0 has been filtered out. The fact that the Bernstein space is not preserved by filtering is due to the unboundedness of the Hilbert transform in $L^\infty(\mathbb{R})$:

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for $f \in L^\infty(\mathbb{R})$ its Hilbert transform belongs to the space of functions of bounded mean oscillation $BMO(\mathbb{R})$. We refer the reader to [G07, Ch. 6] for the definition and basic properties of functions in this space. In view of this, a natural substitute for the Bernstein space has been proposed in [T11]. It consists of entire functions of exponential type that, when restricted to $\mathbb{R}$, belong to BMO.

It was observed, see [T11, Thm 7], that the bandlimited functions in BMO enjoy a much better regularity than expected, they have bounded derivative on $\mathbb{R}$, i.e., they are Lipschitz. We provide a short argument showing this: from the Fourier transform side taking derivative and applying the Hilbert transform corresponds to multiplying the function by $-i\omega$ and by $\operatorname{sgn}(\omega)$ respectively. If the function f has Fourier transform supported on $[-\pi,\pi]$, say, then $f' = f \star \phi$, where $\hat\phi$ is any compactly supported smooth function which coincides with $i\omega$ on $[-\pi,\pi]$. If f is a BMO function then $f = Hg + h$, where $g, h \in L^\infty$ and H is the Hilbert transform. If, in addition, $\operatorname{supp} \hat f \subset [-\pi,\pi]$, then $f' = \psi \star g + \phi \star h$, where ϕ is as above and $\hat\psi(\omega) = \operatorname{sgn}(\omega)\hat\phi(w)$. Both ϕ and ψ belong to $L^1(\mathbb{R})$ and f' is therefore bounded.

Thus it seems natural to consider the following spaces: The Bernstein space of entire functions:

$$B_\pi = \{F \in L^\infty(\mathbb{R}), \operatorname{supp} \hat F \subset [-\pi,\pi]\}$$

endowed with its natural norm $\|F\|_{B_\pi} := \sup_{\mathbb{R}} |F(x)|$ and the space

$$B_\pi^1 = \{F,\ F' \in B_\pi\},$$

endowed with its natural seminorm: $\|F\|_{B_\pi^1} := \sup_{\mathbb{R}} |F'(x)|$. Clearly by the Bernstein inequality $B_\pi \subset B_\pi^1$ but the converse is not true, the function $f(x) = x$ belongs to $B_\pi^1 \setminus B_\pi$.

A fundamental problem in the study of bandlimited functions is the process of discretization of signals. This problem can be decoupled in two:

- The problem of stable reconstruction of a signal from the set of its samples at a given sequence of points $\Lambda \subset \mathbb{R}$. If this is possible we say that Λ is a sampling sequence.
- Its companion problem of prescribing an arbitrary set of values on a sequence $\Lambda \subset \mathbb{R}$. If this is possible we say that Λ is an interpolating sequence.

Beurling in [B89, Chapt. IV, V] considered both problems in the Bernstein space and provided a complete geometric description of sampling and interpolating sequences. We aim to do such description for the space B_π^1. The major difference between the two settings is related to the fact that the set of traces of functions in B_π^1 is now defined by the divided differences of the values rather than the values themselves, so the (now) classical machinery from [B89] cannot be applied directly. We need to combine this machinery with additional tools in order to deal with spaces defined through the derivatives. In particular we use ideas from [BN04], this article deals with the Bloch space and also [LM05], this article studies functions whose derivatives are in the Paley-Wiener space.

We now introduce the corresponding spaces of sequences.

Given $\Lambda = \{\lambda_k\}_k \subset \mathbb{R}$, $\lambda_k < \lambda_{k+1}$, and a function $a : \Lambda \to \mathbb{C}$ we denote the divided differences:

$$\Delta_a(\lambda_k) = \frac{a(\lambda_{k+1}) - a(\lambda_k)}{\lambda_{k+1} - \lambda_k},$$

and consider the space

$$\ell_1^\infty(\Lambda) = \{a : \Lambda \to \mathbb{C}; \|a\|_{\ell_1^\infty(\Lambda)} := \sup_k |\Delta_a(\lambda_k)| < \infty\},$$

Equivalently it can be defined as

$$\ell_1^\infty(\Lambda) = \{a : \Lambda \to \mathbb{C}; \|a\|_{\ell_1^\infty(\Lambda)} := \sup_{\lambda,\lambda'\in\Lambda,\ \lambda\neq\lambda'} \left\{ \left| \frac{a(\lambda) - a(\lambda')}{\lambda - \lambda'} \right| \right\} < \infty\}.$$

It is clear that, given $F \in B_\pi^1$ we have that $a \in \ell_1^\infty$ if $a(\lambda_k) = F(\lambda_k)$.

Definition 1. We say that a sequence Λ is *sampling* for B_π^1 if

$$\|F\|_{B_\pi^1} \leq C\|\{F(\lambda_k)\}\|_{\ell_1^\infty},\ F \in B_\pi^1$$

with some $C < \infty$ independent of the choice of F.

By $K = K(\Lambda)$ we denote the *sampling constant*, i.e. the smallest possible value of C in the above inequality.

Definition 2. We say that a sequence Λ is *interpolating* for B_π^1 if for each $a \in \ell_1^\infty(\Lambda)$ there is an $F \in B_\pi^1$ such that

$$F(\lambda_k) = a(\lambda_k),\ k \in \mathbb{Z}. \tag{1}$$

If a sequence Λ is interpolating, then by the closed graph theorem it is possible to interpolate with a size control. That is, there is a constant C such that, for each $a \in \ell_1^\infty(\Lambda)$, one can choose F interpolating a on Λ and in addition $\|F\|_{B_\pi^1} \leq C\|a\|_{\ell_1^\infty(\Lambda)}$. The smallest constant possible in this inequality is called the *interpolation constant* and it will be denoted by $K_0(\Lambda)$.

The problem of describing of sampling and interpolating sequences sequences in B_π^1 presents an interesting challenge because the routine interpolation tools such as Lagrange interpolation series cannot be applied directly to interpolation by divided differences. We develop technique of interpolation related to $\bar\partial$ problem and combine them with the classical techniques in [B89].

Our first aim is to provide a complete geometric characterization of interpolating and sampling sequences in B_π^1. We introduce now the geometric concepts that are used in the description.

We say that a sequence $\Lambda \subset \mathbb{R}$ is *separated* whenever

$$\alpha := \inf\{|\lambda - \lambda'|,\ \lambda, \lambda' \in \Lambda,\ \lambda \neq \lambda'\} > 0,$$

we say that α is the *separation constant* for Λ.

We will also use the classical notions of upper and lower Beurling densities:

$$D^+(\Lambda) = \lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, x+R))}{R},$$
$$D^-(\Lambda) = \lim_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, x+R))}{R}.$$

Theorem 1. *A sequence $\Lambda \subset \mathbb{R}$ is interpolating for B^1_π if and only if $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1, Λ_2 are two separated sequences, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $D^+(\Lambda) < 1$.*

Theorem 2. *A sequence $\Lambda \subset \mathbb{R}$ is sampling for B^1_π if and only if there are separated sequences $\Lambda_1, \Lambda_2 \subset \Lambda$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $D^-(\Lambda_1 \cup \Lambda_2) > 1$.*

Remark. The fact that that $\Lambda_1 \cap \Lambda_2 = \emptyset$ is not relevant. This is due to the fact that for the sake of simplicity in the exposition we are not considering points with multiplicity. If we did, we would have to introduce derivatives at the multiple points replacing divided differences.

We will also study two further problems in these spaces. As one can see from Theorem 1, the trace of a function $F \in B^1_\pi$ on a sequence Λ of density one is not arbitrary sequence with bounded divided differences. Apart from this necessary condition it must also satisfy a certain cancellation property.

This was studied by Levin in [L56, Appendix VI], see also [L96], in the case of the Bernstein space and Λ being the set of integers. We carry out the analogous result in the context of bandlimited Lipschitz functions. The characterization of the traces on a sequence which is a zero set of a sine-type function (the integers is the main example) is achieved through the use of the discrete and regularized Hilbert transform, in a similar spirit as for the Bernstein space and integer nodes, yet an additional regularization is needed. One consequence of our result is that it is possible to reconstruct the function from its values in the zeros of a sine-type function plus the value in any other given point. Of course the reconstruction is not stable in view of Theorem 2. This is the case also in the Bernstein space but it is curious that this is possible in the strictly bigger space B^1_π. This fact has already been observed in [T11].

Remark It is interesting to know how wide a space X of functions of exponential type π can be such that the zero set of a sine-type function plus one point are sets of uniqueness for X. We do not know the general answer to this question. Yet we observe that this property is more related to the regularity of the Fourier transform (in the distributional sense) of functions in X near the endpoints of the spectra, rather than their growth properties. In particular the spaces considered in [LM05] possess this property yet contain functions of polynomial growth.

The structure of the paper is as follows. In Section 2 we prove Theorem 2 providing a description of sampling sequences in B^1_π. In Section 3 we prove the necessity part of the interpolating Theorem 1 and in Section 4 we prove the sufficiency of the geometric description. In Section 5 we provide a description of the traces of functions in B^1_π on the zero sets of sine-type functions.

We will use the following notation: given two positive quantities a and b we write $a \lesssim b$ or $b \gtrsim a$ if there is a constant $C > 0$ such that $a \le Cb$ for all possible values of parameters. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

2. Sampling sequences

The strategy for the sampling part is to reduce the problem to the analogous problem in B_π.

Definition 3. A sequence $\Lambda \subset \mathbb{R}$ is called *relatively dense* if there is $R > 0$ such that $\Lambda \cap (\xi - R, \xi + R) \neq \emptyset$ for each $\xi \in \mathbb{R}$.

Claim 1. *Let $\Lambda \subset \mathbb{R}$ be a sampling sequence for B^1_π. Then Λ is relatively dense.*

Proof. Let the opposite be true: for any $R > 0$ there is a $\xi \in \mathbb{R}$ such that $\Lambda \cap (\xi - R, \xi + R) = \emptyset$. Consider the function

$$F_\xi(z) := \int_\xi^z \frac{\sin(\zeta - \xi)}{\zeta - \xi} d\zeta \ \in B_\pi.$$

We clearly have $\|F_\xi\|_{B^1_\pi} = 1$. On the other hand $\|F|_\Lambda\|_{l^\infty_1} \lesssim R^{-1}$ since $|F'_\xi(x)| \lesssim R^{-1}$ as $x \notin (\xi - R, \xi + R)$ and also $\|F_\xi\|_{B_\pi} \lesssim 1$. This contradicts the sampling property of Λ. □

Claim 2. *If a separated sequence Λ is sampling for B^1_π then $D^-(\Lambda) > 1$.*

Proof. We will prove that in this case Λ is sampling for B_π and we may use then the results by Beurling, [B89].

It suffices to prove that, for any function $F \in B_\pi$ the inequality $\|F|_\Lambda\|_{l^\infty} < 1$ yields $\|F\|_{B_\pi} < C$ with some constant C independent of the choice of F. Indeed since Λ is separated then $\|F|_\Lambda\|_{l^\infty} < 1$ yields $\|F|_\Lambda\|_{\ell^\infty_1} \lesssim 1$ and $F' \in B^1_\pi$, $\|F'\|_{B_\pi} \lesssim 1$ because Λ is sampling for B^1_π. The desired inequality follows now from the fact that, being a sampling sequence for B^1_π, the sequence Λ is relatively dense. □

Given two sequences $\Lambda, \Gamma \subset \mathbb{R}$ we say that $\mathrm{dist}_H(\Lambda, \Gamma) < \epsilon$ if $\#\{[\mu - \epsilon, \mu + \epsilon] \cap \Lambda\} \ge 1$ for each $\mu \in \Gamma$ and $\#\{[\lambda - \epsilon, \lambda + \epsilon] \cap \Gamma\} \ge 1$ for each $\lambda \in \Lambda$. Here dist_H stands for the Hausdorff distance.

We say that the sequences Λ_k *converge weakly to* Λ if, for each $N > 0$

$$\mathrm{dist}_H((\Lambda \cap [-N, N]) \cup \{-N, N\}, (\Lambda_k \cap [-N, N]) \cup \{-N, N\}) \to 0, \text{ as } k \to \infty.$$

In this case we write $\Lambda_k \rightharpoonup \Lambda$.

Given a sequence Λ we denote by $W(\Lambda)$ the set of all its weak limits of translates, i.e. sequences M such that

$$\Lambda + x_k \rightharpoonup \mathrm{M}, \text{ for some } \{x_k\} \subset \mathbb{R}.$$

We will use the two following stability results. Their proof mimics the one of [B89, Chapter iV, Thm. 2] with the natural modifications: the divided differences should be approximated by the first derivatives and then Bernstein's theorem for the *second* derivative should be applied.

Claim 3 (First stability result)**.** *Let $\epsilon > 0$ and let the sequences Λ and M be such that each $\mu \in \mathrm{M}$ has at least two ϵ-neighbors in Λ, i.e. $\#\big\{(\Lambda \setminus \mathrm{M}) \cap [\mu - \varepsilon, \mu + \varepsilon]\big\} \geq 2$ for all $\mu \in \mathrm{M}$. Then*

$$|K(\Lambda)^{-1} - K(\Lambda \cup \mathrm{M})^{-1}| < 10\epsilon$$

Corollary 1. *If Λ is sampling then there is an $\epsilon > 0$ and a subsequence $\Sigma \subset \Lambda$ which is also sampling and such that $\sharp\Sigma \cap (x, x+\varepsilon) \leq 2$ for each $x \in \mathbb{R}$. In other words each sampling sequence Λ contains a sampling sequence Σ which is a union of two separated sequences.*

Claim 4 (Second stability result)**.** *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}$ be two separated sequences $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Sigma = \Gamma_1 \cup \Gamma_2$. Let also Γ_2' be a separated sequence $\Gamma_1 \cap \Gamma_2' = \emptyset$, $\Sigma' = \Gamma_1 \cup \Gamma_2'$, and $\operatorname{dist}_H(\Gamma_2, \Gamma_2') < \epsilon$. Then*

$$\left| \frac{1}{K(\Sigma)} - \frac{1}{K(\Sigma')} \right| \leq 10\varepsilon.$$

Corollary 2. *If Λ is a sampling sequence which is a union of two separated sequences, then, for some $\epsilon > 0$ there is a separated sampling sequence Λ' such that $\operatorname{dist}_H(\Lambda, \Lambda') < \epsilon$.*

Theorem 3. *Λ is a sampling sequence for B^1_π if and only if it contains a subsequence Σ which is the union of two separated sequences with $D^-(\Sigma) > 1$.*

Proof. The *necessity* part is just a compilation of the previous claims.

Now let $D^-(\Sigma) > 1$ and Σ be a union of two separated sequences. We follow the arguments in [B89, Chapter IV, theorem 3]: it suffices to prove that each $\Sigma' \in W(\Sigma)$ is a uniqueness set for B^1_π.

Take any $\Sigma' \in W(\Sigma)$. We still have $D^-(\Sigma) > 1$, here one has to count points according to their multiplicities, some points in a weak limit may have multiplicity two. The corresponding divided difference should be replaced by the derivative then. It suffices to prove uniqueness for functions $F \in B^1_\pi$ such that $F(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Each such function has zero increments on Σ' then its derivative is vanishing at some intermediate points, the set of intermediate points has density bigger than one so $F' = 0$. □

Corollary 3. *If Λ is sampling for B^1_π then there is an $\varepsilon > 0$ such that $(1+\varepsilon)\Lambda$ is still sampling for B^1_π.*

3. Interpolation theorem. Necessity.

Let Λ be an interpolating sequence for B^1_π. First we prove that $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1, Λ_2 are separated sequences, $\Lambda_1 \cap \Lambda_2 = \emptyset$. This follows from the simple statement below.

Claim 5. *There is $\epsilon > 0$ such that, for each $\lambda \in \Lambda$*

$$\#(\Lambda \cap \{\zeta; |\zeta - \lambda| < \epsilon|\}) \leq 2.$$

Proof. Indeed, if this were not the case then for each $\epsilon > 0$ there would exist points $\lambda_{k-1}, \lambda_k, \lambda_{k+1} \in \Lambda$, $\lambda_{k+1} - \lambda_{k-1} < \epsilon$. One can construct $a \in \ell^\infty_1$, $\|a\|_{\ell^\infty_1} = 1$ so that $a(\lambda_k) = 0$, $a(\lambda_{k-1}) = \lambda_k - \lambda_{k-1}$, $a(\lambda_{k+1}) = \lambda_{k+1} - \lambda_k$. Let $F \in B^1_\pi$ be a solution of (1) satisfying $\|F\|_{B^1_\pi} \leq K_0(\Lambda)$ and, hence, $\|F''\|_{B_\pi} \leq \pi K_0(\Lambda)$. On the other hand the choice of interpolation data yields $\|F''\|_{B_\pi} \geq \epsilon^{-1}$, so ϵ cannot be chosen arbitrary small. □

We can now split the sequence Λ into blocks $\Lambda = \cup_j \Lambda^{(j)}$ containing at most two points each: either $\Lambda^{(j)} = \{\lambda_{k_j}\}$ or $\Lambda^{(j)} = \{\lambda_{k_j}, \lambda_{k_j+1}\}$, in addition $k_j < k_{j+1}$ and $\operatorname{dist}(\Lambda^{(j)}, \Lambda^{(j+1)}) > a > 0$ for all $j \in \mathbb{Z}$.

Denote $\Gamma^{(j)} = \Lambda^{(j)}$ if $\#\Lambda^{(j)} = 1$ and $\Gamma^{(j)} = \{\lambda_{k_j}, \lambda_{k_j+1} + \epsilon\}$ otherwise, the number ϵ will be chosen later. Let $\Gamma = \cup_j \Gamma^{(j)} = \{\mu_k\}$ with enumeration corresponding to those of Λ.

Claim 6. *If ϵ is sufficiently small then Γ is an interpolating sequence for B^1_π.*

Proof. Let for definiteness $\lambda_0 \in \Lambda \cap \Gamma$. Given any $a \in \ell^\infty_1(\Gamma)$ we may assume $a(\lambda_0) = 0$. We use induction to construct the sequences $a^{(p)} \in \ell^\infty_1(\Gamma)$, $c^{(p)} \in \ell^\infty_1(\Lambda)$ $p = 0, 1, \ldots$.

Set $a^{(0)} = a$. Given $a^{(p)}$ construct $c^{(p)} \in \ell^\infty_1(\Lambda)$ such that $c^{(p)}(\lambda_0) = 0$,

$$\frac{a^{(p)}(\mu_{k+1}) - a^{(p)}(\mu_k)}{\mu_{k+1} - \mu_k} = \frac{c^{(p)}(\lambda_{k+1}) - c^{(p)}(\lambda_k)}{\lambda_{k+1} - \lambda_k}$$

and let $F_p \in B^1_\pi$ solve the interpolation problem $F_p(\lambda_k) = c^{(p)}(\lambda_k)$ and $\|F_p\|_{B^1_\pi} \leq K_0(\Lambda)\|a^{(p)}\|_{\ell^\infty_1(\Gamma)}$. Further let

$$a^{(p+1)}(\mu) = a^{(p)}(\mu) - F_p(\mu), \ \mu \in \Gamma.$$

We claim that for sufficiently small ϵ there is $q \in (0, 1)$, such that $qK_0(\Lambda) < 1$ and

$$\|a^{(p+1)}\|_{\ell^\infty_1(\Gamma)} \leq q\|a^{(p)}\|_{\ell^\infty_1(\Gamma)}, \ \|F_{p+1}\|_{B^1_\pi} < qK_0(\Lambda)\|F_p\|_{B^1_\pi}. \tag{2}$$

If this is proved we will use that $F_p(\lambda_0) = 0$ for all p's, hence the series $F = \sum F_p$ converges on each compact set in $\mathbb{C}$ and delivers a solution to the interpolation problem

$$F(\mu) = a(\mu), \ \mu \in \Gamma, \ F \in B^1_\pi.$$

It remains to prove (2). Without loss of generality we may assume that all data $a^{(p)}$ are real and also the functions F_p are real on $\mathbb{R}$. Let $\mu_k = \lambda_k$, $\mu_{k+1} = \lambda_{k+1} + \epsilon$ We have

$$\begin{aligned}
&\frac{a^{(p+1)}(\mu_{k+1}) - a^{(p+1)}(\mu_k)}{\mu_{k+1} - \mu_k} = \\
&= \frac{a^{(p)}(\mu_{k+1}) - a^{(p)}(\mu_k)}{\mu_{k+1} - \mu_k} - \frac{F_p(\mu_{k+1}) - F_p(\mu_k)}{\mu_{k+1} - \mu_k} = \\
&= \frac{c^{(p)}(\lambda_{k+1}) - c^{(p)}(\lambda_k)}{\lambda_{k+1} - \lambda_k} - \frac{F_p(\mu_{k+1}) - F_p(\mu_k)}{\mu_{k+1} - \mu_k} = \\
&= \frac{F_p(\lambda_{k+1}) - F_p(\lambda_k)}{\lambda_{k+1} - \lambda_k} - \frac{F_p(\mu_{k+1}) - F_p(\mu_k)}{\mu_{k+1} - \mu_k}.
\end{aligned} \tag{3}$$

Let $\lambda_{k+1} - \lambda_k < 1/10\pi$. Then

$$\frac{a^{(p+1)}(\mu_{k+1}) - a^{(p+1)}(\mu_k)}{\mu_{k+1} - \mu_k} = F_p'(\tilde{\lambda}_k) - F_p'(\tilde{\mu}_k),$$

for some $\tilde{\lambda}_k \in (\lambda_k, \lambda_{k+1})$, $\tilde{\mu}_k \in (\lambda_k, \lambda_{k+1} + \epsilon)$. For $\epsilon < 1/10\pi$ we obtain

$$|\tilde{\mu}_k - \tilde{\lambda}_k| \le \frac{1}{5\pi} \text{ and } \left| \frac{a^{(p+1)}(\mu_{k+1}) - a^{(p+1)}(\mu_k)}{\mu_{k+1} - \mu_k} \right| < \frac{1}{5} \|F_p\|_{B^1_\pi}.$$

In the case $\lambda_{k+1} - \lambda_k \ge 1/10\pi$ we have

$$\begin{aligned}
&\frac{a^{(p+1)}(\mu_{k+1}) - a^{(p+1)}(\mu_k)}{\mu_{k+1} - \mu_k} = \\
&\qquad \frac{F_p(\lambda_{k+1}) - F_p(\lambda_k)}{\mu_{k+1} - \mu_k} \frac{\mu_{k+1} - \lambda_{k+1}}{\lambda_{k+1} - \lambda_k} - \frac{F_p(\lambda_{k+1}) - F_p(\mu_{k+1})}{\mu_k - \lambda_k}
\end{aligned}$$

and an explicit estimate shows

$$\left| \frac{a^{(p+1)}(\mu_{k+1}) - a^{(p+1)}(\mu_k)}{\mu_{k+1} - \mu_k} \right| \lesssim \epsilon \|F\|_{B^1_\pi}.$$

Relation (2) now follows. □

Corollary 4. *Without loss of generality we can assume that the sequence Λ is separated.*

We will prove that if Λ is an interpolating set for B^1_π, then one can refer to the classical Beurling result in order to get $D^+(\Lambda) < 1$.

It suffices to construct a sequence of functions $\{f_\lambda\}_{\lambda\in\Lambda} \subset B_\pi$ such that

$$f_\lambda(\mu) = \delta_{\lambda,\mu},\ \ \mu \in \Lambda, \tag{4}$$

and

$$|f_\lambda(x)| \lesssim \frac{1}{|x-\lambda|^2 + 1},\ \ x \in \mathbb{R}. \tag{5}$$

Then Λ will be an interpolating sequence for B_π since the solution to the interpolation problem

$$F(\lambda) = b(\lambda);\ b \in \ell^\infty(\Lambda),\ F \in B_\pi,$$

can be achieved by the function

$$F_b = \sum b(\lambda) f_\lambda,$$

and according to [B89, Chaprer V, Theorem 1] $D^+(\Lambda) < 1$.

The construction of the functions $\{f_\lambda\}_{\lambda\in\Lambda}$ relies on the following statement:

Claim 7. *Let Λ be a separated interpolating sequence for B^1_π. For each $\xi \in \mathbb{R} \setminus \Lambda$ the sequence $\Lambda \cup \{\xi\}$ is also interpolating for B^1_π. Moreover the constant $K_0(\Lambda \cup \{\xi\})$ depends only on* $\operatorname{dist}(\Lambda, \xi)$.

Let us take this claim for granted for the moment being. Let α be the separation constant for Λ. For each $\lambda \in \Lambda$, choose the points $\xi_\pm = \lambda \pm \alpha/8$, $\xi_1 = \lambda + \alpha/4$. The set $\Lambda_\lambda := \Lambda \cup \{\xi_+, \xi_-, \xi_1\}$ is B^1_π-interpolating and also $K_0(\Lambda_\lambda) < C$, C being independent of the choice of $\lambda \in \Lambda$.

Define the sequence $a_\lambda \in \ell^\infty_1(\Lambda_\lambda)$ as

$$a_\lambda(\mu) = \begin{cases} 1, & \mu = \lambda; \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and take $g_\lambda \in B^1_\pi$ such that $g_\lambda(\mu) = a(\mu)$, $\mu \in \Lambda_\lambda$, $\|g_\lambda\|_{B^1_\pi} \lesssim 1$. It is straightforward that one can find numbers c_λ such that $|c_\lambda| \sim 1$ and the functions

$$f_\lambda(z) = c_\lambda \frac{g_\lambda(z)}{(z-\xi_-)(z-\xi_+)(z-\xi_1)}$$

satisfy (4) and (5).

In order to verify Claim 7 it suffices to prove that, for each $\xi \in \mathbb{R}\setminus\Lambda$, there is a function $h_\xi \in B^1_\pi$ such that

$$h_\xi|_\Lambda = 0,\ h_\xi(\xi) = 1.$$

and $\|h_\xi\|_{B^1_\pi}$ can be estimated by a quantity which only depends on $\operatorname{dist}(\Lambda, \xi)$.

We mimic the proof of the corresponding fact in [B89], Chapter V.

Claim 8. *If Λ is a separated interpolating sequence for B^1_π then each $\Gamma \in W(\Lambda)$ is also an interpolating sequence for B^1_π, in addition $K_0(\Gamma) \le K_0(\Lambda)$.*

The proof follows that in [B89, Chapter V, Lemma 5.].

Given $\xi \in \mathbb{R} \setminus \Lambda$ denote

$$\rho_\Lambda(\xi) = \sup\{|F(\xi)|, F \in B^1_\pi,\ F|_\Lambda = 0,\ \|F\|_{B^1_\pi} \le 1\}.$$

Claim 9. *Let Λ be an α-separated interpolating sequence. Then for each $\alpha' \in (0, \alpha/2)$ there is $\kappa > 0$ such that*

$$\rho_\Lambda(\xi) > \kappa, \text{ if } \operatorname{dist}(\xi, \Lambda) > \alpha' \tag{7}$$

Proof. First we mention that $\rho_\Lambda(\xi) > 0$, $\xi \notin \Lambda$. Indeed otherwise $F \in B^1_\pi$, $F|_\Lambda = 0$ yields $F = 0$ in other words the mapping $T : F \mapsto F|_\Lambda$, $T : B^1_\pi \to \ell^\infty_1(\Lambda)$ has zero kernel. Since Λ is an interpolating sequence T acts onto and hence is invertible. This means that Λ is also a sampling set and, by Theorem 3, $D^-(\Lambda) > 1$. Take any three points $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ and denoting $\Lambda' = \Lambda \setminus \{\lambda_i\}_{i=1,2,3}$ we have $D^-(\Lambda') > 1$, hence Λ' is a sampling set as well, this contradicts the fact that Λ is an interpolating sequence.

It follows now that, if Λ is an interpolating sequence and $\xi \notin \Lambda$, then $\Lambda \cup \{\xi\}$ is also an interpolating sequence.

Assume that there is a sequence of points ξ_n, $\operatorname{dist}(\xi_n, \Lambda) > \alpha'$ and $\rho_\Lambda(\xi_n) \to 0$ as $n \to \infty$. Let $\Lambda_n = \Lambda - \xi_n$. Each Λ_n is an α-separated sequence and also $K_0(\Lambda_n) = K_0(\Lambda)$, $\rho_{\Lambda_n}(0) \to 0$. We may assume that $\Lambda_n \rightharpoonup \Gamma$, then $\operatorname{dist}(0, \Gamma) \geq \alpha'$. Fix two points $t_1, t_2 \notin \Gamma \cup \{0\}$. The set $\Gamma' = \Gamma \cup \{t_1, t_2\}$ is also an interpolating sequence, hence $\gamma := \rho_{\Gamma'}(0) > 0$.

Therefore there exists $F \in B^1_\pi$ such that $\|F\|_{B^1_\pi} = 1$, $F|_{\Gamma'} = 0$, and $F(0) = \gamma$. Then

$$G(z) := \frac{F(z)}{(z - t_1)(z - t_2)} \in B^1_\pi,\ G|_\Gamma = 0,\ G(0) = \gamma t_1^{-1} t_2^{-1},$$

in addition $G'(x) \to 0$, as $x \to \infty$.

We have now $\|G|_{\Lambda_n}\|_{l^\infty_1(\Lambda_n)} \to 0$ as $n \to \infty$: for large values of the argument this follows from the decay of G' (we remind that all Λ_n are α-separated) for limited values of the argument this follows from the fact that $G|_\Gamma = 0$ and $\Lambda_n \rightharpoonup \Gamma$. Since $K_0(\Lambda_n) = K_0(\Lambda)$ we can find a function $H_n \in B^1_\pi$ such that $H_n|_{\Lambda_n} = G|_{\Lambda_n}$ and also $\|H_n\|_{B^1_\pi} \to 0$ as $n \to \infty$. In addition $H_n(0) \to 0$, $n \to \infty$. Now the functions

$$\Phi_n(z) = G(z) - H_n(z)$$

satisfy $\Phi_n|_{\Lambda_n} = 0$, $\|\Phi_n\|_{B^1_\pi} < C$ and also $\Phi_n(0) \to \gamma t_1^{-1} t_2^{-1}$. The latter is incompatible with $\rho_{\Lambda_n}(0) \to 0$. □

4. Interpolation theorem. Sufficiency.

Let $\Lambda = \Lambda_1 \cup \Lambda_2 \subset \mathbb{R}$ satisfy the hypothesis of Theorem 1, i.e. the subsequences Λ_1, Λ_2 are separated, $\Lambda_1 \cap \Lambda_2 = \emptyset$, and $D^+(\Lambda) < 1$. We are going to prove that Λ is interpolating for B^1_π.

Claim 10. *Let a sequence $\Sigma \subset \mathbb{R}$, $\Sigma \cap \Lambda = \emptyset$ be such that $\Gamma = \Lambda \cup \Sigma$ is interpolating for B^1_π. Then Λ is also interpolating for B^1_π with the same constant of interpolation.*

Proof. Given any data $a \in \ell^\infty_1(\Lambda)$ we may extend it on $\Lambda \cup \Sigma$ with the same Lipschitz constant as

$$\widetilde{a}(x) = \inf_{\lambda\in\Lambda}(a(\lambda) + Lip(a)|x - \lambda|),\ \forall x \in \Lambda \cup \Sigma$$

where $Lip(a)$ is the Lipschitz constant of a.

We have then $a \in \ell^\infty_1(\Gamma)$ and $\|a\|_{\ell^\infty_1(\Gamma)} = \|a\|_{\ell^\infty_1(\Lambda)}$. Any solution of the corresponding problem on Γ gives now a solution on Λ. □

Corollary 5. *Without loss of generality we may assume $D^-(\Lambda) > 0$.*

Indeed, were this not the case one can add a relatively dense sequence of points Σ such that the union $\Gamma = \Lambda \cup \Sigma$ still has the property $D^-(\Gamma) < 1$ and we apply the previous Claim 10.

We start with proving the sufficiency assuming in addition that Λ is a separated sequence. The corresponding machinery is related to the notion of sine-type functions.

Definition 4. An entire function S is a sine-type function if it is of exponential type, its zeros are simple and separated, and there is a constant C such that

$$|S(z)| \simeq e^{\pi|\operatorname{Im} z|}, \qquad \forall z,\ |\operatorname{Im} z| > C.$$

This definition was introduced by Levin, see e.g. [L96] who proved that the zero set of a sine-type function, which lies in a strip around the real axis, is both an interpolating and a sampling sequence for the Paley-Wiener space.

Lemma 1. *Let $\Lambda = \{\lambda_k\} \subset \mathbb{R}$ be separated, $D^+(\Lambda) < 1$ and $D^-(\Lambda) > 0$. Then Λ is interpolating for B^1_π.*

Proof. Since the upper density of the sequence Λ is strictly smaller than one, it is possible to find a sequence $\Sigma \subset \mathbb{R}$ and a sine-type function S with zero set $\Lambda \cup \Sigma$. This is done in [OCS98, Lemma 3].

Moreover for for each $\lambda \in \mathbb{R}$ one can consider the sequences $\Lambda_\lambda = \Lambda \cup \{\lambda + i\}$ and, with the same proof as in [OCS98], one can find sequences $\Sigma(\lambda)$ and functions S_λ with zero sets $\Lambda \cup \{\lambda + i\} \cup \Sigma(\lambda)$ and such that

$$|S_\lambda(z)| \simeq e^{\pi|\operatorname{Im} z|}, \qquad \forall z,\ |\operatorname{Im} z| > C, \tag{8}$$

with the constants implicit in (8) being uniform for all $\lambda \in \Lambda$. We split the remaining construction into several steps

Elementary solutions: The hypothesis imply that there is an $\alpha > 0$ such that $\lambda_{k+1} - \lambda_k > \alpha > 0$.

Denote $\gamma_k = \lambda_k + \alpha/10 + i\mathbb{R}$ and let

$$\chi_k(z) = \begin{cases} 0, & \text{if } \operatorname{Re} z < \lambda_k + \alpha/10; \\ 1, & \text{if } \operatorname{Re} z > \lambda_k + \alpha/10. \end{cases}$$

Consider the functions

$$\Phi_k(z) = \frac{1}{2i\pi}\ \frac{S_{\lambda_k}(z)}{z-(\lambda_k+i)} \int_{\gamma_k} \frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)} \frac{d\zeta}{\zeta - z} + \chi_k(z) + d_k, \tag{9}$$

where the constants d_k are chosen so that $\Phi_k(0) = 0$. Convergence of the integral in the right-hand side follows from the estimate (8), Φ_k are well-defined and analytic functions outside γ_k. Furthermore, Φ_k can be extended as an entire function of exponential type π: it follows from the Sokhotskii-Plemelj formula that Φ_k are continuous on γ_k, so the singularities along γ_k can be removed. The growth estimates are straightforward. We also have

$$\Phi(\lambda_j) = 0,\ j \le k, \quad \Phi(\lambda_j) = 1,\ j > k, \tag{10}$$

and respectively

$$\Phi_k(\lambda_{j+1}) - \Phi_k(\lambda_j) = \delta_{k,j}. \tag{11}$$

Formal solution to the interpolation problem:

Given a sequence $a = \{a(\lambda_k)\} \in \ell^\infty_1(\Lambda)$ denote

$$\Delta_a(\lambda_k) = \frac{a(\lambda_{k+1}) - a(\lambda_k)}{\lambda_{k+1} - \lambda_k},$$

then the function

$$F(z) = \sum_k (\lambda_{k+1} - \lambda_k)\Delta_a(\lambda_k)\Phi_k(z). \tag{12}$$

yields a solution to the interpolation problem (1) provided that the series in the right-hand side is convergent to a function in B^1_π.

Solution to the interpolation problem. Convergence:

Claim 11. *The series* (12) *converges uniformly on compact sets in* $\mathbb{C}$, *to an entire function* $F \in B^1_\pi$. *This function provides a solution to the interpolation problem* (1).

Proof. It suffices to prove the convergence of the sum $\sum \Phi_k'(z)$ on each compact set in $\mathbb{C}$, to a function in B_π. The convergence of (12) will then follow due to the normalization $\Phi_k(0) = 0$.

We remind that a set $E \subset \mathbb{R}$ is called *relatively dense* if, for some $L > 0$,

$$\inf_{x\in\mathbb{R}} \operatorname{mes}(E \cup (x, x+L))$$

For example the set

$$E = \Big\{x \in \mathbb{R}; |x - (\lambda_k + \frac{\alpha}{10})| < \frac{\alpha}{20}\Big\} \tag{13}$$

is relatively dense, since $D^-(\Lambda) > 0$. Observe also that $\operatorname{dist}(E, \cup\gamma_k) > 0$, here dist stands for the usual Euclidean distance.

We will use the following fact (see e.g. [Ka73, LS74])

Given a relatively dense set $E \subset \mathbb{R}$ there exist a constant C such that

$$\sup_{\mathbb{R}} |F(x)| \le C \sup_{E} |F(x)|$$

for each entire function F of exponential type π.

That is why it suffices to prove the uniform convergence of the series $\sum \Phi_k'(z)$ only on the set E defined by (13).

Let $b(\lambda_k) = (\lambda_{k+1} - \lambda_k)\Delta_a(\lambda_k)$. Since $\{\Delta_a\} \in l^\infty(\Lambda)$ and also $D^-(\Lambda) > 0$ we have $b \in \ell^\infty(\Lambda)$ and

$$\sum_k \Phi_k'(x) = \sum_k \frac{b(\lambda_k)}{2i\pi} \frac{S'_{\lambda_k}(x)}{x-(\lambda_k+i)} \int_{\gamma_k} \frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)} \frac{d\zeta}{\zeta - x} -$$
$$\sum_k \frac{b(\lambda_k)}{2i\pi} \frac{S_{\lambda_k}(x)}{(x-(\lambda_k+i))^2} \int_{\gamma_k} \frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)} \frac{d\zeta}{\zeta - x} +$$
$$\sum_k \frac{b(\lambda_k)}{2i\pi} \frac{S_{\lambda_k}(x)}{x-(\lambda_k+i)} \int_{\gamma_k} \frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)} \frac{d\zeta}{(\zeta - x)^2} =$$
$$= \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x).$$

We consider just the first sum, the rest can be treated similarly. It follows from the construction of the sine-type functions S_{λ_k} that for some $C > 0$ and all k

$$|S_{\lambda_k}(x)| < C,\ x \in \mathbb{R}, \qquad \left|\frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)}\right| < Ce^{-\pi|\operatorname{Im}\zeta|/2},\ \zeta \in \gamma_k,$$

so the Cauchy inequality gives

$$\left|\frac{S'_{\lambda_k}(x)}{x-(\lambda_k+i)} \int_{\gamma_k} \frac{\zeta-(\lambda_k+i)}{S_{\lambda_k}(\zeta)} \frac{d\zeta}{\zeta - x}\right| \le$$
$$C_1|x-(\lambda_k+i)|^{-1}|x-(\lambda_k+\alpha/10)|^{-1/2}.$$

Now the proof of the claim is straightforward. □

This claim completes the proof of Lemma 1. □

Splitting of the sequence: In order to complete the sufficiency proof we split the sequence into two interlacing parts: $\Lambda = \Gamma_1 \cup \Gamma_2$ so that

$$\Gamma_i \text{ are separated, } D^-(\Gamma_i) > 0,\ D^+(\Gamma_i) < 1/2,\ \operatorname{dist}(\Gamma_1, \Gamma_2) > 0.$$

We remind that Λ already admits the representation $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_i are separated sequences, not necessarily satisfying the density restrictions. Our goal is to rearrange this splitting. We enumerate the sequence Λ in an increasing order, i.e. $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ with $\lambda_k < \lambda_{k+1}$. We define $\Gamma_1 = \{\lambda_{2k}\}_{k\in\mathbb{Z}}$ and $\Gamma_2 = \{\lambda_{2k+1}\}_{k\in\mathbb{Z}}$. This splitting satisfy the desired properties.

Now we complete the proof of the general case. We use a trick from [BN04]. Consider the splitting $\Lambda = \Gamma_1 \cup \Gamma_2$ as above. It follows from

Lemma 1 that each sequence Γ_i is interpolating in $B^1_{\pi/2}$. They are also interpolating for $B_{\pi/2}$ by Beurling theorem. Since $D^+(\Gamma_1) < 1/2$ we can construct a sine-type function S of type $\pi/2$ vanishing on Γ_1 and, perhaps, at some other points. Without loss of generality one may assume that the distance between Γ_2 and the zero set of S is positive, if need be on can move extra zeros form the real axis by increasing their imaginary parts by one, say. In particular we obtain

$$\inf_{\mu\in\Gamma_2} |S(\mu)| > 0.$$

Now given a sequence $a \in \ell^\infty_1$ we observe that the sequences $a|_{\Gamma_i}$ belong to the spaces $l^\infty_1(\Gamma_i)$, $i = 1, 2$. We look for the solution of the problem (1) in the form

$$F(z) = H_1(z) + S(z)H_2(z),\ H_1 \in B^1_{\pi/2},\ H_2 \in B_{\pi/2}. \tag{14}$$

Let $H_1 \in B^1_{\pi/2}$ solve the interpolation problem

$$H_1|_{\Gamma_1} = a|_{\Gamma_1},$$

then H_2 should satisfy

$$H_2(\mu) = \frac{a(\mu) - H_1(\mu)}{S(\mu)},\ \mu \in \Gamma_2. \tag{15}$$

We will prove that the right-hand side of (15) is bounded, then, since Γ_2 is separated the equation has a solution in $B_{\pi/2}$. The boundedness is straightforward: let μ' be the nearest point to μ in Γ_1. We then have

$$\frac{a(\mu) - H_1(\mu)}{S(\mu)} = \frac{a(\mu) - a(\mu')}{S(\mu)} - \frac{H_1(\mu) - H_1(\mu')}{S(\mu)}$$

It follows from the definition of the sine-type function that there are $\epsilon > 0$, and $c > 0$ such that $|S(x) - S(\mu')| \simeq |x - \mu'|$ if $\mu' \in \Gamma_1$, $|x - \mu'| < \epsilon$, and that$|S(x)| > c$ if $\mathrm{dist}(x, \Gamma_1) > \epsilon$. Since $|\mu - \mu'| < R$, $a \in \ell^\infty_1(\Lambda)$, and $H_1 \in B^1_{\pi/2}$ the boundedness of the right-hand side in (15) follows.

5. Traces

Let S be a sine-type function such that its zero set $\Lambda = \{\lambda_n\}^\infty_{-\infty} \subset \mathbb{R}$, for simplicity assume that $0 \notin \Lambda$. In this section we study the traces of functions in B^1_π on Λ. In the classical case of the space B_π and $\Lambda = \mathbb{Z}$ the traces can be described in terms of boundedness of the corresponding discrete Hilbert transform see e.g. [L56], Appendix VI, [L96], Lecture 21 when $S(z) = \sin \pi z$. We refer the reader to [E95, F98] for other spaces of entire functions of exponential type. In the case of the space B^1_π one needs in addition a regularization of the Hilbert transform.

We introduce some additional notation. Given $x \in \mathbb{R}$ we denote by $\rfloor\ x\ \lfloor = \max\{n; \lambda_n < x\}$ and, for a sequence $\mathbf{a} = \{a_n\}$ we define the usual and regularized Hilbert transforms (with respect to $S(\lambda)$) at any point $x \notin \Lambda$) as

$$(\mathcal{H}\mathbf{a})(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{a_n}{S'(\lambda_n)} \left(\frac{1}{x-\lambda_n} + \frac{1}{\lambda_n} \right), \tag{16}$$

and

$$(\tilde{\mathcal{H}}\mathbf{a})(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left[\frac{a_n}{S'(\lambda_n)} \left(\frac{1}{x-\lambda_n} + \frac{1}{\lambda_n} \right) - \frac{a_{\rfloor x \lfloor}}{S(x)} \right] \tag{17}$$

assuming that the limits exist. The additional term in the right-hand side of (17) regularizes the behavior at ∞: the sequence $\mathbf{a}$ needs not to be bounded, we will consider the cases when $(\tilde{\mathcal{H}}\mathbf{a})(x)$ is bounded for large values of x, in contrast to $(\mathcal{H}\mathbf{a})(x)$.

Theorem 4. *Let $B \subset \mathbb{R}$ be any separated sequence such that*

$$\operatorname{dist}(B, \Lambda) > 0, \quad D^-(B) > 1.$$

Given a sequence $a \in \ell_1^\infty(\Lambda)$ there exists a function $f \in B_\pi^1$ such that

$$a_k = f(\lambda_k) \tag{18}$$

if and only if

$$\{\tilde{\mathcal{H}}\mathbf{a}(\beta)\}_{\beta \in B} \in \ell^\infty(B). \tag{19}$$

Remark. We will see that this relation is independent of the choice of the sequence B.

Proof. Let a sequence $a \in \ell_1^\infty(\Lambda)$ satisfy (19). We replace the problem (18) by

$$f(\lambda_{k+1}) - f(\lambda_k) = a_{k+1} - a_k =: b_k, \ f \in B_\pi^1, \tag{20}$$

and look for the solution of this problem in the form

$$f(z) = \sum_n b_n \Psi_n(z), \tag{21}$$

where the functions $\Psi_n \in B_\pi^1$ satisfy the equations

$$\Psi_n(\lambda_{k+1}) - \Psi_n(\lambda_k) = \delta_{k,n} \tag{22}$$

and, in addition,

$$\Psi_n(0) = 0. \tag{23}$$

The idea of constructing such functions is the same as in the previous section, yet its realization is slightly different. Take $\alpha_n \in (\lambda_n, \lambda_{n+1})$ so that $3\kappa := \operatorname{dist}(\{\alpha_n\}, \Lambda \cup B) > 0$ and such that $(\lambda_n, \alpha_n) \cap B = \emptyset$. Denote $\gamma_n = \alpha_n + i\mathbb{R}$ and let

$$\chi_n(z) = \begin{cases} 1, & \operatorname{Re} z > \alpha_n; \\ 0, & \text{otherwise.} \end{cases}$$

The functions

$$\Psi_n(z) = \frac{S(z)}{2i\pi}\int_{\gamma_n}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right)\frac{d\zeta}{S(\zeta)} + \chi_n(z) + d_n, \tag{24}$$

belong to B^1_π and satisfy (22), (23) for an appropriate choice of d_n's.

Claim 12. *Given any sequence* $\mathbf{b} = \{b_n\} \in \ell^\infty$ *the series* (21) *converges uniformly on compact sets in* $\mathbb{C}$ *to an entire function* f *of exponential type* π.

Proof. We will prove the convergence of the series

$$\sum_n b_n \Psi'_n \tag{25}$$

and then use (23). We write

$$\Psi'_n(z) = \underbrace{\frac{S'(z)}{2i\pi}\int_{\gamma_n}\left(\frac{1}{z-\zeta} + \frac{1}{\zeta}\right)\frac{d\zeta}{S(\zeta)}}_{I_n(z)} + \underbrace{\frac{S(z)}{2i\pi}\int_{\gamma_n}\frac{1}{(z-\zeta)^2}\frac{d\zeta}{S(\zeta)}}_{J_n(z)} \tag{26}$$

and consider the series

$$\sum b_n I_n(x) \quad \text{and} \quad \sum b_n J_n(x). \tag{27}$$

separately. Given a compact set $K \subset \mathbb{C}$ we prove the convergence in the set $\{z \in K;\ \operatorname{dist}(K \cup (\cup\gamma_n)) > \kappa/2\}$, for the remaining piece of K we replace the lines γ_n in (24) by $\gamma'_n = \gamma_n + \kappa$. This does not change Ψ_n, and we repeat the same reasonings.

Now the uniform convergence on K follows from the estimates

$$|z - \zeta| \gtrsim \operatorname{dist}(\gamma_n, K),\ z \in K, \zeta \in \gamma_n, \tag{28}$$

and

$$|S(\zeta)| \gtrsim e^{\pi|\operatorname{Im}\zeta|},\ \zeta \in \cup_n\gamma_n. \tag{29}$$

□

The function f defined by (21) may not always belong to B^1_π.

Claim 13. *Let* $\mathbf{a} \in \ell^\infty_1$ *satisfy* (19), $\mathbf{b} = \{b_n\}$, *where* $b_n = a_{n+1} - a_n$ *and* f *is defined by* (21). *Then* $f \in B^1_\pi$.

Proof. It is straightforward that, for any $\mathbf{b} \in \ell^\infty$, the sum $\sum b_n J_n(x)$ in (27) is uniformly bounded on B. So it suffices to prove that the first sum $\sum b_n I_n(x)$ is also bounded on B. Then the function f' itself will be also bounded on B, so one can once again refer to [B89] to get boundedness everywhere.

In order to estimate the first sum in (27) we observe that one can apply the residue theorem in the halfplane $\operatorname{Re}\zeta > \alpha_n$ in order to express $I_n(x)$:

$$I_n(x) = S'(x) \sum_{j>n} \left(\frac{1}{x-\lambda_j} + \frac{1}{\lambda_j} \right) \frac{1}{S'(\lambda_j)} + \frac{\chi_n(x)}{S(x)}, \tag{30}$$

Respectively

$$A(x) := \sum_{-\infty}^{\infty} b_n I_n(x) = \tag{31}$$

$$\lim_{N\to\infty} \left(\underbrace{\sum_{-N}^{N} b_n \sum_{j>n} \left(\frac{1}{x-\lambda_j} + \frac{1}{\lambda_j} \right) \frac{1}{S'(\lambda_j)}}_{A_1(x)} + \underbrace{\frac{1}{S(x)} \sum_{-N}^{N} b_n \chi_n(x)}_{A_2(x)} \right)$$

Since $b_n = a_{n+1} - a_n$ we have

$$A_1(x) = \sum_{-N+1}^{N} a_n \left(\frac{1}{x-\lambda_n} + \frac{1}{\lambda_n} \right) \frac{1}{S'(\lambda_n)} - \tag{32}$$

$$a_{-N} \sum_{j>-N} \left(\frac{1}{x-\lambda_j} + \frac{1}{\lambda_j} \right) \frac{1}{S'(\lambda_j)} +$$

$$a_{N+1} \sum_{j>N} \left(\frac{1}{x-\lambda_j} + \frac{1}{\lambda_j} \right) \frac{1}{S'(\lambda_j)} = B_1(x) + B_2(x) + B_3(x).$$

For large N we have $\chi_N(x) = 0$ therefore, using (30) once again and then applying the dominated convergence theorem we obtain

$$B_3(x) = S'(x)^{-1} a_N I_N(x) \to 0 \quad \text{as} \quad N \to \infty$$

Besides, since

$$a_{-N} \left(\frac{1}{x-\lambda_{-N}} + \frac{1}{\lambda_{-N}} \right) = o(1), \text{ as } N \to \infty,$$

we may assume that the summation in $B_1(x)$ is taken from $-N$ to N.
Similarly we have

$$A_2(x) = \tag{33}$$

$$\left[\sum_{-N+1}^{N} a_n(\chi_{n-1}(x) - \chi_n(x)) - a_{-N}\chi_{-N}(x) + a_{N+1}\chi_{N+1}(x) \right] \frac{1}{S(x)}.$$

Recall that we denote $k = \lfloor x \rfloor \in \mathbb{Z}$ so that $\gamma_k < x < \gamma_{k+1}$. Then the only non-zero summand in the first term in (33) is $a_{\lfloor x \rfloor}$. Besides $\chi_{N+1}(x) = 0$ for sufficiently big N.

We substitute this together with (32) in (31) and observe (using (30) once again) that

$$B_2(x) + a_{-N}\chi_{-N}(x) = I_{-N}(x) \to 0 \text{ as } N \to \infty.$$

Finally we have

$$(34) \quad A(x) = \lim_{N\to\infty} (B_1(x) - \frac{a_{\lfloor x \rfloor}}{S(x)}) = \\ \lim_{N\to\infty} \left\{ \sum_{-N}^{N} a_n \left(\frac{1}{x-\lambda_n} + \frac{1}{\lambda_n} \right) \frac{1}{S'(\lambda_n)} - \frac{a_{\lfloor x \rfloor}}{S(x)} \right\}$$

Boundedness of this expression in $x \in B$ is equivalent to the boundedness of $(\tilde{\mathcal{H}}\mathbf{a})_{\lfloor x \rfloor}$. Thus, the function f defined by (21) belongs to B^1_π and solves the interpolation problem (1) modulo an additive constant. This proves the "if" part of Theorem 4.

In order to prove the "only if" part of Theorem 4 we need an auxiliary statement.

Lemma 2. *Given an $\epsilon > 0$, each function $f \in B^1_\pi$ admits the representation:*

$$(35) \qquad f = f_1 + f_2,$$

where $f_1 \in B^1_\pi$ is of exponential type at most ϵ and $f_2 \in B_\pi$.

We postpone the proof of this lemma until the end of the section.

Now, given $f \in B^1_\pi$ denote $\mathbf{a} = \{f(\lambda_n)\}$. In order to prove (19) we chose $\epsilon = \pi/2$ and we use the representation (35). Then

$$\mathbf{a} = \mathbf{a}^1 + \mathbf{a}^2, \quad \mathbf{a}^i = \{f_i(\lambda_n)\}.$$

That $\tilde{\mathcal{H}}\mathbf{a}^2 \in \ell^\infty(B)$ is straightforward since $\mathbf{a}^2$ is the trace on B of a function from B_π and we can use the known results from [L96].

In order to study the Hilbert transform of $\mathbf{a}^1$ we consider the integral

$$I_R(x) = \frac{1}{2i\pi} \int_{\Gamma_R} \frac{f_1(\zeta)}{S(\zeta)} \left(\frac{1}{x-\zeta} + \frac{1}{\zeta} \right) d\zeta,$$

where $\Gamma_R = \{\zeta; |\zeta| = R\}$. We consider only those values of R for which $\operatorname{dist}(\Gamma_R, \Lambda) > \alpha$ for some fixed $\alpha > 0$.

Let $\zeta = \xi + i\eta$. We have $|f_1(\zeta)| = O(|\zeta| e^{\pi|\eta|/2})$, $\zeta \in \mathbb{C}$, $|S(\zeta)| \gtrsim e^{\pi|\eta|}$, $\zeta \in \Gamma_R$ and, by the Jordan lemma,

$$I_R(x) \to 0, \text{ as } R \to \infty.$$

On the other hand the residue theorem gives

$$I_R(x) = \sum_{|\lambda_n|<R} \sum_{-N}^{N} f_1(\lambda_n) \left(\frac{1}{x-\lambda_n} + \frac{1}{\lambda_n} \right) \frac{1}{S'(\lambda_n)} - \frac{f(x)}{S(x)} + \frac{f_1(0)}{S(0)}.$$

We obtain

$$\tilde{\mathcal{H}}\mathbf{a}^1(x) = -\frac{f_1(0)}{S(0)}$$

for all $x \in \mathbb{R}$. This of course suffices to complete the proof of the theorem. □

We proceed now with the proof of Lemma 2. We say that a function f has a spectral gap at the origin if there is an $\epsilon > 0$ such that $\operatorname{supp} \hat{f} \cap (-\varepsilon, \varepsilon) = \emptyset$. We start with the following Lemma:

Lemma 3. *Let $f \in B^1_\pi$ have a spectral gap at the origin, then $f \in B_\pi$.*

Proof. We start by observing that if $f \in B^1_\pi$ then $g = f'$ is bounded on $\mathbb{R}$ as observed in the Introduction. Since $\hat{g}(\omega) = 2\pi i\omega \hat{f}(\omega)$, g has also a spectral gap. Let ϕ be a compactly supported smooth function such that $\phi(w) = 1/(2\pi i w)$ when $\varepsilon < |w| < \pi$. Thus $\hat{f}(\omega) = \phi(\omega)\hat{g}(w)$. Therefore $f = -\hat{\phi} \star g$. Since $-\hat{\phi}$ belongs to the Schwartz space and g is bounded then f is itself bounded. □

Proof of Lemma 2. Given $\varepsilon > 0$ it is possible to find, with a partition of unity, two smooth, compactly supported functions ϕ and ψ such that $\operatorname{supp}\phi \subset [-2\varepsilon, 2\varepsilon]$, $\phi(\omega) = 1$ if $|\omega| < \varepsilon$ and $\phi(\omega) + \psi(\omega) = 1$ for all $\omega \in [-\pi, \pi]$. Let Φ, Ψ be the two functions in the Schwartz space such that $\hat{\Phi} = \phi$ and $\hat{\Psi} = \psi$. Now given $f \in B^1_\pi$ we can decompose

$$f = \Phi \star f + \Psi \star f = f_1 + f_2.$$

The function $f_1' = \Phi \star f'$, thus it is bounded on the real line. Moreover its spectrum lies in $[-2\varepsilon, 2\varepsilon]$ because $\hat{f}_1 = \phi \hat{f}$. Finally $f_2' = \Psi \star f'$, and therefore it is bounded on the real line. The spectrum of f_2 is contained in the spectrum of f, thus $f_2 \in B^1_\pi$. Moreover $\hat{f}_2 = \psi \hat{f}$ and $\psi(\omega) = 0$ when $|\omega| < \varepsilon$, thus f_2 has a spectral gap at the origin. By Lemma 3, f_2 is bounded. □

We finish by observing some elementary remarks on the zero sets of functions in B^1_π. When the functions are in the Bernstein class, its zeros have been studied, [K11]. The novel case is when $f \in B^1_\pi \setminus B_\pi$. In this setting we can prove

Claim 14. *Given any $f \in B^1_\pi \setminus B_\pi$ and $A > 0$ let $Z_A(f)$ be the set of zeros of f located in the strip $\{z \in \mathbb{C}; |\operatorname{Im} z| < A\}$. Then $D^-(Z_A(f)) = 0$.*

Remark. Here we use a slightly modified notion of density which counts the number of points in a strip rather than on the real line.

Proof. Suppose that $|f(x)| = M$. Since f is Lipschitz there is an interval I centered at x and of size comparable to M such that f is zero free in I. Thus if $|f(x_n)| \to \infty$ there are arbitrary big gaps I_n without zeros. Therefore $D^-(Z_A(f)) = 0$. □

On the other hand given any $\varepsilon > 0$, since $\Lambda = (1-\varepsilon)\mathbb{Z}$ is an interpolating sequence it is possible to construct a function $f \in B^1_\pi \setminus B_\pi$ vanishing on most points of Λ and such that

$$\limsup_{R\to\infty} \frac{\#\{Z(f)\cap(-R,R)\}}{2R} > (1-\varepsilon).$$

This can be done prescribing the value 0 in very long intervals of Λ alternating with shorter intervals in Λ (but still of increasing lenth) where the values are bigger and bigger.

References

[B89] A. Beurling, *The collected works of Arne Beurling. Vol. 2*, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989.

[BN04] B. Bøe and A. Nicolau, *Interpolation by functions in the Bloch space*, J. Anal. Math. **94** (2004), 171–194.

[E95] C.Eoff, *The discrete nature of the Paley-Wiener spaces*, Proc. Amer. Math. Soc. 123 (1995), no. 2, 505–512.

[F98] K.Flornes, *Sampling and interpolation in the Paley-Wiener spaces* L^p_π, $0 < p \le 1$, Publ. Mat. 42 (1998), no. 1, 103–118.

[G07] J. Garnett, *Bounded analytic functions*, Springer, New York, 2007. xiv+459 pp.

[K11] B. Khabibullin *Distribution of zero subsequences for Bernstein space and criteria of completeness for exponential system on a segment*, arXiv:1104.2683

[Ka73] V.Kacnelson, *Equivalent norms in spaces of entire functions*, (Russian) Mat. Sb. (N.S.) 92(134) (1973), 34–54. (English translation: Math. USSR-Sb. 21 (1973), 33–55).

[L56] B. Levin, *Distribution of zeros of entire functions* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956, 632 pp. English translation: American Mathematical Society, Providence, R.I. 1964.

[L96] B. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996.

[LS74] V.Logvinenko, Ju. Sereda, *Equivalent norms in spaces of entire functions of exponential type*, (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp. 20 (1974), 102–111.

[LM05] Yu. Lyubarskii and W. Madych, *Interpolation of functions from generalized Paley-Wiener spaces*, J. Approx. Theory 133 (2005), no. 2, 251 - 268.

[OCS98] J. Ortega-Cerdà and K. Seip, *Beurling-type density theorems for weighted* L^p *spaces of entire functions*, J. Anal. Math. **75** (1998), 247–266.

[T11] G. Thakur, *Bounded mean oscillation and bandlimited interpolation in the presence of noise*, J. Funct. Anal. **260** (2011), no. 8, 2283–2299.

Dept. Mathematical Sciences, Norwegian University of Science and Technology,NO-7491 Trondheim, Norway
E-mail address: yura@math.ntnu.no

Dept. Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail address: jortega@ub.edu