Herd behavior and social contagion

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We consider the voter model dynamics in random networks with mean-field approach. We apply a Poisson process in the election of the neighbor whose state will be copied by an active node, which is also chosen according to the same process at each time step. For simplification, we consider only two possible Poisson rates distributed in two groups, a fast minority and a slow majority. We find that, for a critical set of parameters, the system exhibit characteristic patterns, with abrupt alternation between two consensus states in the fast group. After the analysis of Langevin equation, an effective potential for the fast group is found that models the transition between the state of two alternate consensus and another state where the fast minority oscillates around majority opinion trend.

I. INTRODUCTION

Statistical physics have proven useful in the modeling of so called complex systems. In this context, it may be interesting to detect behavioral patterns in systems where it is necessary to handle high levels of disorder. In other words, it may be interesting to model social systems [1]. Some ingredients are necessary for this purpose, the first of them being heterogeneity. Indeed, interactions between humans usually lack symmetry, thus giving rise to emerging patterns, society itself being its main example. But it is also true that sometimes people leave its idiosyncratic believes, adopting the so called herding behavior. This can be seen in many context, including business, stock market, and general social networks. In this work we study a well known case of majority rule, the *voter model*. First appeared in 1973 for modeling competition between species, now it is perhaps on of the simplest and more paradigmatic model of opinion dynamics, and it has produced many interesting applications. For example, in [2] is used to reproduce empirical statistical data from US presidential elections, a system with binary opinion.

In this work we analyze the effects of introducing heterogeneity in both the activation and neighbor selection rules. In this case active agents revise their opinion with higher frequencies, but at the same time their state is copied with higher probability by neighbors. We consider a set of agents divided in two homogeneous groups with different activation rates, a fast paced minority and a slow majority. Their opinions are competing in a random mean-field network, and our aim is to study the emergence of patterns in both groups, exploring in which conditions the influence of an active minority produces macroscopic effects in global dynamics.

II. THE VOTER MODEL

The general idea for the simple voter model is the following: we have a set of agents with a binary state of opinion (sell or buy, right-handed or left-handed,up or down, etc) controlled by an activation rule which make them interact purely by imitation. In the current work, this rule is determined by a Poisson process with an individual rate λ_i. The whole system activation process will be that of a Poisson with a total rate of $\lambda_T = \sum_{i=1}^{N} \lambda_i$. This implies that the inter-activation time of the process will depend inversely on the size of the system, $\Delta t \propto \frac{1}{N}$.

In our simulations, we have followed the standard update rules known as Random Asynchronous Update (RAU) . That is, at each simulation step (activation) one agent will update its opinion (the probability of being chosen is directly proportional to the Poisson rate of each agent) and will choose randomly one of his neighbors and copy its state. For considerations about the effects of non-Poissonian update rules, please see [3].

We start by defining a network of N nodes, where connections between agents are ruled by the matrix $P\{j|i\}$, which determines the conditional probability for agent j to be copied by the previously chosen agent i. Is easy to see that for a generic network (including lattices), $P\{j|i\} = \frac{a_{ij}}{k_i}$, where a_{ij} is the adjacency matrix and k_i the node's degree. For a mean-field random network this is $P\{j|i\} = \frac{1}{N-1}$. In these cases we are assuming that there is no priority when choosing a neighbor.

The current binary opinion of each agent is coded by $n_i(t)$. After an infinitesimal time step dt, the opinion of each agent will be determined either by its previous opinion or by herd behavior. The probability of herding depends on the dichotomic variable $\xi_i(dt)$:

$$\xi_i(dt) = \begin{cases} 1 & \text{with probability } \lambda_i dt \\ 0 & \text{with probability } 1 - \lambda_i dt \end{cases} \tag{1}$$

In case of herding, the opinion of a neighbor will be chosen according to $P\{i|j\}$:

$$\eta_i(t) = \begin{cases} 1 & \text{with probability } \sum_{j\neq i} P\{j|i\} n_j(t) \\ 0 & \text{with probability } 1 - \sum_{j\neq i} P\{j|i\} n_j(t) \end{cases} \tag{2}$$

The opinion of agent i in the instant $t + dt$ will be [4]:

$$n_i(t + dt) = n_i(t)(1 - \xi_i) + \eta_i(t)\xi_i. \tag{3}$$

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After taking the expected value conditioned by the state of the system $\vec{n}(t)$ and averaged over the ensemble, we can define $\rho_i(t) = \langle n_i(t) \rangle_{\vec{n}}$. Time evolution of the average opinion can be expressed, from equation (3), as:

$$\frac{d\rho_i}{dt} = \lambda_i \left(\sum_{j=1}^{N} P\{j|i\}\rho_j - \rho_i \right) \tag{4}$$

A. Global magnitude conservation

Here we shall prove that finding the conservation law in the voter model is equivalent to find the eigenvector $\phi(j)$ of eigenvalue 1 of $P\{j|i\}$, which satisfies:

$$\phi(j) = \sum_{i=1}^{N} \phi(i)P\{j|i\}. \tag{5}$$

Indeed, it is clear that by multiplying (4) by $\phi(i)$ and summing over every agent in the system, the right side of the equation vanishes. Therefore, we find the conservation of the following magnitude:

$$\sum_{i=1}^{N} \frac{\phi(i)}{\lambda_i}\rho_i(t). \tag{6}$$

For a network with no priority in the choice of neighbors, $\phi(i) \propto k_i$, where k_i is the degree of agent i. This will be a constant for a mean-field random network (or a lattice).

B. Probability of absorption

In a finite size system, a frozen consensus is always the final fate of the dynamics. Interestingly, the conserved quantity derived in the previous subsection allows us to evaluate the probability of the system being trapped in one of the two possible absorbing states. Then $\rho_i(t = \infty) = P_1$, where P_1 is the probability of ending in the "up" state. Hence, it follows from (6) that

$$P_1 = \frac{\sum_{i=1}^{N} \frac{\phi(i)}{\lambda_i}\rho_i(t=0)}{\sum_{i=1}^{N} \frac{\phi(i)}{\lambda_i}}. \tag{7}$$

When initial probability of being "up" is uniform for every node, $P_1 = \rho_i(t = 0)$, regardless of topology. Figure 1 exemplifies this conservation in some systems.

All this results have already been derived with other methods,as can be seen for example in [1], pag.9, where one-body correlation function is used.

III. HETEROGENEITY IN ACTIVATION RATES

As explained before, the activation rate of each agent depends on its λ_i. In the homogeneous case, all agents will update its state with the same pace. But it can be imagined that in real life social systems, decisions are taken by agents who typically behave with different temporal frequencies. It could also be asserted that, in general, high-frequency agents are outnumbered by slow ones, since their positions are often more radical or risky. This differences can be seen in stock market between noise-traders and funds, or in more everyday situations, like a debate with very active speakers and more reflexive ones. In this last scenario, a minority of active speakers will contrast their opinion between them in a shorter time scale than those reflexive orators. The question is, how will these differences affect global opinion dynamics?

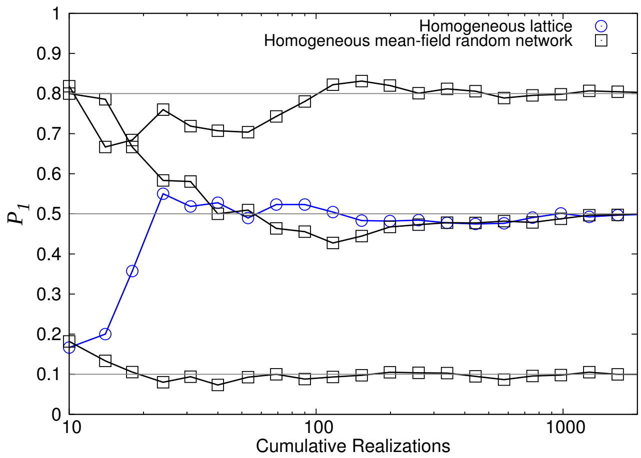

FIG. 1: Convergence of $P1$ for different situations: after 2000 realizations, probability converges to the initial condition.

A. Two-compounded heterogeneous networks

We can easily model this effect by assuming a discrete distribution of λ_i's with only two possible values, λ_f (fast) and λ_s (slow), with $\lambda_f \gg \lambda_s$. It is clear that λ_f will imply a shorter inter-activation time of fast agents and, when $\lambda_f \gg \lambda_s$, we can consider that time scales are completely separated for each kind of agent.

Besides, we consider nodes placed in a mean-field random network with homogeneity in degree distribution (for a deep analysis of the effect of heterogeneous degree distribution, please see [5]).

In this case, we set new rules for the choice of the copied neighbor. Instead of taking one of them randomly, we assume that neighbors will also be chosen with a probability proportional to their rates. Therefore:

$$P\{j|i\} = \frac{\lambda_j}{\sum_{i=1}^{N} \lambda_i}, \tag{8}$$

which implies that $\phi(i) \propto \lambda_i$ and, thus, that $\sum_i \rho_i(t)$ is the conserved quantity.

Figure 2 shows particular realizations of the process for different values of λ_s/λ_f. As it can be seen, periods of

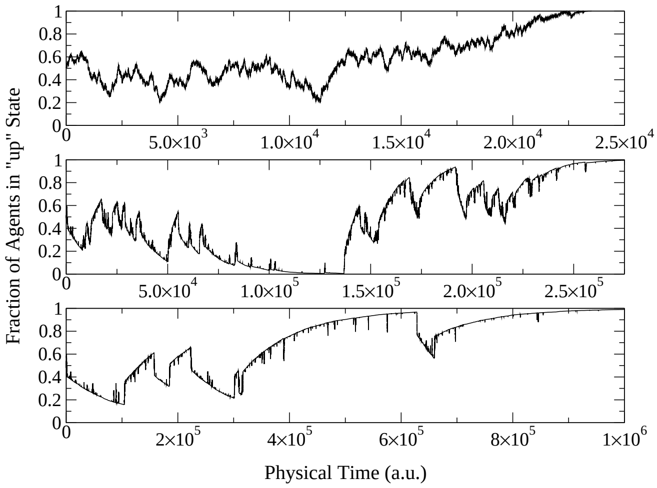

FIG. 2: Evolution of the fraction of agents in "up" state of a two-compounded heterogeneous system, in a mean-field random network of N=5000 agents, where 20% of them are fast.In all cases $\lambda_f = 1$.**Top:** $\lambda_s = 10^{-2}$. **Center:** $\lambda_s = 10^{-4}$. **Bottom:** $\lambda_s = 10^{-5}$

regular growing appears as we increase the separation of time scales. This continuous evolution is suddenly broken by sharp peaks. Although the system ends up absorbed in one of the two absorbing states, the peculiar pathway to reach consensus cannot be observed in the standard voter model (for example Figure 1 in [3]).

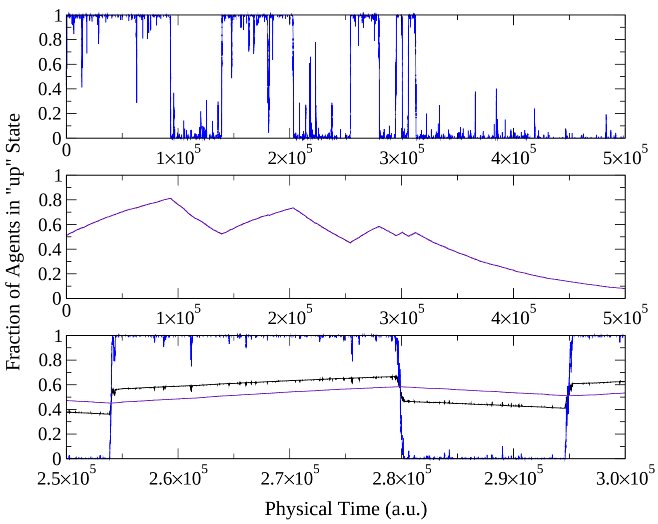

FIG. 3: Evolution of the fraction of agents in "up" state of a two-compounded heterogeneous system, in a mean-field random network of N=5000 agents, where 20% of them are fast. In this case $\lambda_f = 1$ and $\lambda_s = 10^{-5}$.**Top:** Evolution of the fast group. **Center:** Evolution of the slow group. **Bottom:** Expanded time period showing fast (blue), slow (indigo) and global (black) dynamics.

For a better understanding of this phenomenon, we show in Figure 3 the separated temporal evolution of both groups. It appears then clear that this sharp peaks and this exponential decays are not casual but are the result of very differentiated dynamics between fast and slow agents. As mentioned before the global dynamics is, in general, governed by exponential increasing or decreasing decays, which are closely related to the slow group evolution. On the other hand, transitions between two consensus states in the fast group produces sharp peaks in both global and slow temporal series. Due to the huge differences between time scales, from the fast group perspective slow agents will seem as if being frozen in their state, while from the slow group perspective fast agents will be in the absorbing state and thus they will also seem at rest. We shall show that this differences are responsible of the emergence of the observed behavior.

Exponential decay of system's opinion is consistent with the interaction of the slow group with the fast one. In fact, in the slow time scale the subgroup of fast agents will be a coherent nucleus in consensus, which will almost always determine the copied opinion for any slow agent. In the case of this nucleus being condensed at "up" state, it can be asserted that the total of agents in this state (N_1) will follow $\frac{dN_1}{dt} \propto N_0 \propto (N - N_1)$. This implies an increasing exponential decay saturating in N. In the opposite case, is easy to see that $\frac{dN_1}{dt} \propto -N_1$, then we will have a decreasing exponential decay collapsing at zero.

On the other hand, transitions in the small group can be explained by the opposite interaction. Despite the tinny probability of a fast agent to copy a slow one, its time scale is small enough to visualize this interactions many times during the process. When they occur, fast agents (which are in a consensus context) may copy an opposite opinion from a slow outsider, thus introducing some noise in the small subsystem preventing it to be trapped in the consensus states. When the fast nucleus is moved out from consensus, it can evolve to the opposite state, in a similar way as an homogeneous voter model system could do. This change of consensus must be abrupt as long as the probability of transition diminish with the time necessary to carry it on. As we have seen, this change of nucleus consensus will change the sign of the exponential decay of global dynamics.

B. Characteristic consensus time

In this subsection, we proceed with an analysis of fast agents transitions in the context of Langevin equation. As will be seen, competition between diffusion and drift plays a main role in the presence of ordered behavior.

As seen in [4], the Langevin equation for a general stochastic process $X(t)$ can be written as:

$$dX(t) = -V'(x)dt + \sqrt{D(x)}dW, \tag{9}$$

where dW is the differential Wiener process, and the drift and the diffusion term are respectively defined as:

$$V'(x) = -\frac{\langle X(t+dt)|X(t)\rangle - X(t)}{dt}, \quad (10)$$
$$D(x) = \frac{\langle X^2(t+dt)|X(t)\rangle - \langle X(t+dt)|X(t)\rangle^2}{dt}.$$

In general, voter models are pure diffusive processes. But in this case, $P\{j|i\}$ introduces an interaction that generates a drift in both slow and fast dynamics.

Here we redefine $\rho \equiv \frac{1}{N}\sum_{i\in N} n_i(t)$, so it can be considered a stochastic variable following (9). Working with the expressions of (10), and using (3),(4) and with the prescription of (8), we can find the terms of our Langevin equation. We only show results for the fast group, since we want to study its transitions, although the results for the slow group are equivalent:

$$V'_f(\rho_f) = -\frac{d\rho_f}{dt} = \frac{\lambda_f}{1+k_{fs}}(\rho_f - \rho_s) \quad (11)$$

$$D^f(\rho_f) = \frac{\lambda_f}{N_f}\left(\rho_f + \frac{(1-2\rho_f)(k_{fs}\rho_f + \rho_s)}{1+k_{fs}}\right) \quad (12)$$

where we define $k_{fs} \equiv \frac{\lambda_f}{\lambda s}\frac{N_f}{N_s}$. We will assume that from one group the density of "up" states of the other is a constant of time, a quasistatic approximation that makes sense for very differentiated timescales. Dependence with fast ratio can be omitted by considering time units consistent with $\lambda_f = 1$.

A first approach to the calculation of mean transition time comes when considering the fast nucleus as an independent and homogeneous voter system moving between two barriers, on of them absorbing and the other reflecting. The reflecting barrier models the interaction between fast and slow paced agents, since this interaction can move fast agents out of consensus. In this case, the diffusion term is $D^f(\rho_f) = \frac{2\lambda_f}{N_f}\rho_f(1-\rho_f)$.

Now, we are going to study the transition time for the case of fast agents beginning near the zero state consensus, at $\rho_f \sim 1/N_f$ (the other case is symmetric). First Passage Time $T(a,b,\rho)$ is defined [6] as the mean time necessary for a system to escape from a space enclosed between two barriers a and b ($a < b$) beginning from ρ. We will consider an absorbing barrier $b = 1$ in the "up" consensus and a reflecting one $a = \rho = \frac{1}{N_f}$. Therefore:

$$T(a,b,\rho) \propto \frac{N_f}{\lambda_f}\ln N_f \quad (13)$$

We have taken some measurements of transition times in two-compounded heterogeneous systems for different fast subsystems sizes. To asses the goodness of our assumptions, we have performed this operation with a system with no further restrictions and also with a system with an artificial reflecting barrier.

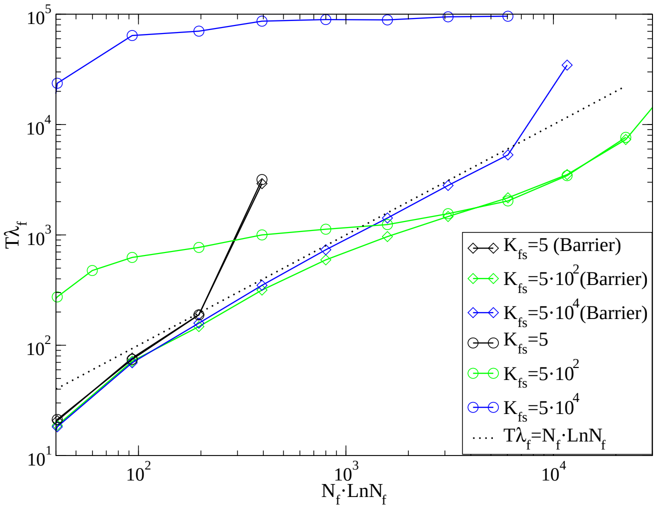

FIG. 4: Experimental mean First Passage Time (T) for fast nucleus starting at $\rho_f = 1/N_f$ and slow agents beginning at $\rho_s = 1/2$. Diamond points correspond to measurements in simulations with an imposed barrier in $1/N_f$. Circle points simulations don't have explicit barrier. High values of k_{fs} correspond to low coupling situations. The dashed line correspond to predictions of equation (13).

It is clear from Figure 4 that the assumption of the fast nucleus as an homogeneous independent system with barriers does not fit with the real model. Even in the case of imposing a reflecting barrier, the expected lineal tendency of time respect $N_f LnN_f$ is only observed in a certain range of sizes.

An important effect we are not considering here is the feedback mechanism between the fast and slow subgroups. In fact, when the fast nucleus is in consensus, it attracts slow dynamics to its position in exponential decay. During this process, slow agents also influence the nucleus with their mean opinion. In case that slow agents are far from fast ones, the small system will suffer a greater noise that could eventually produce a transition to approach both systems. On the contrary, if fast and slow agents are near the same state, the transition will become improbable. Thus, when drift is more relevant than diffusion transition times have long tails which are difficult to observe. If we don't consider them, we obtain smaller mean values than it could be expected.

As fast group diffusion diminish with its size (12), this could explain why in some systems (for example $k_{fs} = 5\cdot 10^2$ with and without barrier) transition time growth rate decays,just before rising again. This increase can be seen as a phase transitions for N_f sufficiently high. For example, in the system with barrier with $k_{fs} = 5\cdot 10^4$, passage time suffers a sudden increase around $N_f \sim 200$. This suggests that competition between diffusion and drift becomes critical for some value of system size and coupling. This can be easily modeled by finding the effective potential of fast agent dynamics, where we can naturally study this phase transition.

C. Effective potential

We are now interested in finding the effective potential for our model. Using the corresponding drift and diffusion coefficients for fast agents in an heterogeneous two-compounded system, we can solve the Fokker-Planck equation for this model in the stationary condition, thus obtaining the effective potential V_{eff}. We will consider the particular case where slow agents are frozen in $\rho_s = 1/2$, and we will select the constant so that V_{eff} is zero when $\rho_f = 1/2$. It is easy to see then that:

$$V_{eff} = \left(1 - \frac{N_f}{2k_{fs}}\right) \ln \left\{ \frac{(k_{fs}+1) - k_{fs}(1-2\rho_f)^2}{k_{fs}+1} \right\}. \quad (14)$$

Consequently, V_{eff} will always have an extremum at $\rho_f = 1/2$, which will be a maximum (minimum) if $\left(1 - \frac{N_f}{2k_{fs}}\right)$ is positive (negative). When there is a maximum, $\rho_s = \{0, 1\}$ are minima of energy system, so we will find the observed transition regime (the potential barrier between both states will depend on k_{fs}). Otherwise, minority opinion will oscillate around majority's with an amplitude also depending of the coupling. Therefore, we can estimate a critical relation, $\frac{N^c_{fs}}{2k^c_{fs}} \sim 1$.

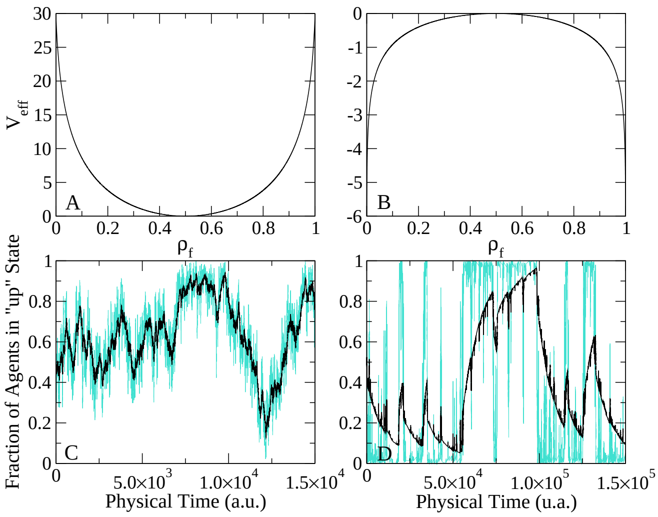

FIG. 5: **A:** Effective Potential for $N_f = 500$ and $k_{fs} = 25$. **B:** Effective Potential for $N_f = 500$ and $k_{fs} = 2500$. **C:** Simulation of process with conditions in A. **D:** Simulation of process with conditions in B. For this parameters, $k^c_{fs} = 250$.

IV. CONCLUSIONS

We have shown here how heterogeneity plays a special role in random networks when applied to the process of neighbor selection. Using a discrete distribution of Poisson rates we have found the presence of a phase transition between two clearly distinct regimes. If k_{fs} (which is inversely proportional to the coupling strength) is great enough compared to the size of the fast minority group, this core alternates between two low energy states of consensus. As a result, global opinion exhibits sharp discontinuities followed by changes in growing tendency that can be very relevant in a macroscopic scale. By contrast, when we have a sufficiently large coupling strength the consensus solution disappears from fast autonomous dynamics. Instead they are engaged to the opinion of majority, oscillating around an equilibrium state centered in majority state.

In further considerations it could be important to consider how the effective potential changes when slow dynamics are not centered at 50%. Furthermore, it would be interesting to find a way to relax the quasi-static hypothesis and consider the feedback mechanism between fast and slow agents in macroscopic timescales.

Next obvious generalization would come by considering more complex distributions in the Poisson rate. If this is done, for example, with a very skew Weibull distribution of rates, it is easy to see that we will not be able to consider two differentiated groups. This can be seen as a way of introducing heterogeneity in the fast and slow group, thus establishing a hierarchy in the process of activation and in the process of being chosen. It can be tested with simulations that in this case the global behavior will have much resemblances to our discrete case, with more sharpened and less frequent discontinuities in global opinion dynamics.

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