Unit I: The $\mathbb{R}^n$ Vector Space

Exercise 1. Given the vectors $\vec{u} = (1, 0, 3)$, $\vec{v} = (-2, 1, 2)$, and $\vec{w} = (1, -2, -4)$ compute the following linear combinations

a) $\vec{u} + (\vec{v} + \vec{w})$

b) $2\vec{u} - 3\vec{v}$

c) $2\vec{v} - \vec{u} + \vec{w}$

Exercise 2. Draw and describe geometrically (line, plane...) all linear combinations of

a) $\{(1, 3)\}$

b) $\{(1, -1), (-3, 3)\}$

c) $\{(0, 1), (1, 0)\}$

Exercise 3. Draw and describe geometrically (line, plane...) all linear combinations of

a) $\{(-1, -1, -1), (-4, -4, -4)\}$

b) $\{(2, 0, 0), (1, 2, 2)\}$

c) $\{(2, 2, 2), (1, 0, 2), (3, 2, 3)\}$

d) $\{(2, -1, 3), (1, 4, 1), (5, 2, 7)\}$

Exercise 4. Is the vector $\vec{u} = (2, 1, 4)$ a linear combination of $\{\vec{u}_1 = (2, 2, 1), \vec{u}_2 = (5, 3, 2)\}$?

Exercise 5. Is the vector $\vec{u} = (1, -1, 3)$ a linear combination of $\{\vec{u}_1 = (2, -1, 0), \vec{u}_2 = (3, 1, 1), \vec{u}_3 = (0, -1, 1)\}$?

Exercise 6. For what values of the parameter k is the vector $\vec{u} = (k, 2, 1)$ a linear combination of $\{\vec{v} = (5, 2, 0), \vec{w} = (3, 0, 1)\}$?

Exercise 7. Let $\vec{u}, \vec{v}$, and $\vec{w}$ be three vectors of $\mathbb{R}^3$. Suppose that the determinant of the matrix build from the three vectors is different from 0. Is $\vec{w}$ a linear combination of $\{\vec{u}, \vec{v}\}$?

Exercise 8. Find the equation of the plane that contains $(2, -1, 3), (1, 4, 1)$, and passes through the origin. Check if the point $(5, 2, 7)$ is contained in the plane. Is the vector $(5, 2, 7)$ a linear combination of $\{(2, -1, 3), (1, 4, 1)\}$?

Exercise 9. Let $\vec{u}_1 = (1, 0, 0)$, $\vec{u}_2 = (0, 1, 0)$, $\vec{u}_3 = (0, 0, 1)$, and $\vec{u}_4 = (-2, 3, 1)$. Show that the four vectors are linearly dependent

a) By writing one of the vectors as a linear combination of the rest.

b) Using the rank of the matrix built from the vectors.

Exercise 10. Explain why any set containing the $\vec{0}$ vector is linearly dependent.

Exercise 11. Study the linear dependence/independence of the following sets:

a) $\{(1,-2,3),(3,1,2),(2,-3,1)\}$

b) $\{(-2,-3,3),(3,4,1),(1,2,-7)\}$

c) $\{(5,6,2),(2,3,5),(3,2,-1)\}$

When the set is linearly dependent, write one of the vectors as a linear combination of the rest.

Exercise 12. Let $\vec{u}=(0,2,3)$ and $\vec{v}=(-1,-4,1)$. Find a vector $\vec{w}$ such that $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent.

Exercise 13. Find the largest number of linearly independent column vectors of:

$$A=\begin{pmatrix}-2&1&2&3\\0&0&6&-1\\0&5&0&0\\0&0&0&0\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}-3&0&-6&3\\2&0&4&2\\1&0&2&1\\1&0&2&7\end{pmatrix}$$

Exercise 14. Show that if $a=0$, $d=0$, or $f=0$, the column vectors of the matrix

$$A=\begin{pmatrix}a&b&c\\0&d&e\\0&0&f\end{pmatrix}$$

are linearly dependent.

Exercise 15. Let $\vec{u}=(2,2,1)$, $\vec{v}=(k,1,2)$, and $\vec{w}=(-3,k,1)$. For what value of k is $\{\vec{u},\vec{v},\vec{w}\}$ linearly independent?

Exercise 16. Let $\vec{u}_1$ and $\vec{u}_2$ be two linearly independent vectors. Reason out if the following statements are true or false.

a) The vectors $\vec{w}_1$ and $\vec{w}_2$ given by $\vec{w}_1=\vec{u}_1$ and $\vec{w}_2=\vec{u}_1+\vec{u}_2$ are always linearly independent.

b) The vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ given by $\vec{w}_1=\vec{u}_1$, $\vec{w}_2=\vec{u}_1+\vec{u}_2$, and $\vec{w}_3=\vec{u}_1-\vec{u}_2$ are always linearly independent.

Exercise 17. Let $\vec{u}_1=\left(3,\frac{1}{2}\right)$, $\vec{u}_2=\left(\frac{9}{2},\frac{3}{4}\right)$, and $\vec{u}_3=\left(-\frac{3}{2},2\right)$.

a) Is the linear span of $\{\vec{u}_1\}$ a line in $\mathbb{R}^2$?

b) Is $\{\vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

c) Is $\{\vec{u}_1,\vec{u}_2\}$ a spanning set of $\mathbb{R}^2$?

d) Is $\{\vec{u}_1,\vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

e) Is $\{\vec{u}_2,\vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

f) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^2$?

g) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

Exercise 18. Let $\vec{u}_1 = (3, 2, -1)$, $\vec{u}_2 = (0, 2, 2)$, $\vec{u}_3 = (3, 0, -3)$, and $\vec{u}_4 = (0, 3, 3)$.

a) Is the linear span of $\{\vec{u}_1, \vec{u}_2\}$ a plane in $\mathbb{R}^3$?

b) Is the linear span of $\{\vec{u}_2, \vec{u}_4\}$ a plane in $\mathbb{R}^3$?

c) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

d) Is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ a spanning set of $\mathbb{R}^3$? If not, what is the linear span of the set of vectors?

Exercise 19. Is $\{(2, 1, 3), (4, 3, 1), (0, 2, 2)\}$ a spanning set of $\mathbb{R}^3$?

Exercise 20. Reason out if the following statement is true or false:
"A set of vectors of $\mathbb{R}^n$ is a basis of the $\mathbb{R}^n$ vector space if every vector of $\mathbb{R}^n$ is a linear combination of the vectors of the set and if no vector of the set is a linear combination of the rest".

Exercise 21. Let $\vec{u}_1 = (a, 0, a)$, $\vec{u}_2 = (1, 1, 1)$, and $\vec{u}_3 = (2, 5, a)$. For what value of a is the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a basis of $\mathbb{R}^3$?

Exercise 22. Let $\vec{u}_1 = (-1, 2, 3)$, $\vec{u}_2 = (0, 2, 2)$, and $\vec{u}_3 = (5, -2, -3)$.

a) Check that the three vectors above form a basis of $\mathbb{R}^3$.

b) Find the coordinates of $\vec{u} = (4, 2, 2)$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Exercise 23. Let $\vec{u}_1 = (0, 1, 4)$, $\vec{u}_2 = (2, 1, 0)$, and $\vec{u}_3 = (7, -1, 2)$.

a) Check that the three vectors above form a basis of $\mathbb{R}^3$.

b) Find the coordinates of $\vec{u} = (-9, 2, 6)$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Exercise 24. Is it possible to have a basis of $\mathbb{R}^3$ composed of four vectors? Is it possible to have a spanning set of $\mathbb{R}^3$ composed of four vectors?

Exercise 25. Explain why the columns of every n-by-n invertible (non-singular) matrix form a basis of $\mathbb{R}^n$.

Exercise 26. Reason out if the following statements are true or false.

a) In the vector space $\mathbb{R}^3$, every set of more than three vectors is linearly dependent.

b) In the vector space $\mathbb{R}^4$, every set of four vectors is a basis of $\mathbb{R}^4$.

c) In the vector space $\mathbb{R}^4$, every set composed of less than four vectors is linearly independent.

d) In the vector space $\mathbb{R}^4$, every set of more than four vectors is a spanning set of $\mathbb{R}^4$.

Exercise 27. Show that if $\{\vec{u_1}, \vec{u}_2, \ldots, \vec{u}_k\}$ is a spanning set of $\mathbb{R}^n$, then $\{\vec{u_1}, \vec{u}_2, \ldots, \vec{u}_k, \vec{u}_{k+1}\}$ is also a spanning set of $\mathbb{R}^n$.

Exercise 28. Check that the linear span of

$$\{\vec{u}_1 = (2,2,1), \vec{u}_2 = (1,0,-2)\}$$

is a plane in $\mathbb{R}^3$ and find its analytical expression.

Exercise 29. Find the analytical expression of the following vector subspace of $\mathbb{R}^2$:

$$Span\left(\left\{\left(\frac{1}{2}, -1\right)\right\}\right).$$

Exercise 30. Describe the subspace of $\mathbb{R}^3$ spanned by $\vec{u} = (-1,1,2)$ and find its analytical expression.

Exercise 31. Given the following vectors of $\mathbb{R}^2$

$$\vec{u}_1 = (-1,-1), \quad \vec{u}_2 = (0,5), \quad \vec{u}_3 = (2,3), \quad \vec{u}_4 = (4,-6)$$

Find a basis and the dimension of $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u_4}\})$.

Exercise 32. Find the analytical expression of the following vector subspace of $\mathbb{R}^3$:

$$Span\left(\{(-1,0,3),(2,2,-1)\}\right).$$

Exercise 33. Find a basis of the vector subspace of $\mathbb{R}^3$ that consists of all vectors of $\mathbb{R}^3$ whose coordinates coincide.

Exercise 34. Find a basis of the vector subspace of $\mathbb{R}^3$ that consists of all vectors of $\mathbb{R}^3$ whose coordinates add up to 0.

Exercise 35. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : x - y = 0, z = 0\}.$$

Exercise 36. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : x - y = 0, x - 2z = 0, y - 2z = 0\}.$$

Exercise 37. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : 4x - y = 0, x + y - 2z = 0\}.$$

Exercise 38. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : 3x + 2y - z = 0\}.$$

Exercise 39. Given the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = 0\}$$

a) Show that S is a vector subspace of $\mathbb{R}^3$.

b) Find a basis of S and compute $dim(S)$.

Exercise 40. Given the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x - z = 0, x + 3y = 0\}$$

a) Show that S is a vector subspace of $\mathbb{R}^3$.

b) Find a basis of S and compute $dim(S)$.

Exercise 41. Check if the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$$

is a vector subspace of $\mathbb{R}^3$.

Exercise 42. Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : \frac{x}{y} = 0\}$$

is not a vector subspace of $\mathbb{R}^2$.

Exercise 43. Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$$

is a vector subspace of $\mathbb{R}^2$.