Unit III: Real Functions of Several Variables. Solutions

Exercise 1. $Dom(f) = \{(x, y) \in \mathbb{R}^2 : xy(x^2 + y^2 - 4) \geq 0$ and $x^2 + y^2 \neq 4\}$. The domain is an open set. Check the graph of the domain in Exercise 1. ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus.

Exercise 2. Check the solutions in Exercise 2. ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus. Note that *b)* and *c)* are switched.

a) Not open, not closed, not bounded, not compact, and not convex set.

b) Not open, closed, not bounded, not compact, and not convex set.

c) Open, not closed, not bounded, not compact, and convex set.

d) Not open, closed, not bounded, not compact, and convex set.

Exercise 3. It is an open, not closed, not bounded, not compact, and convex set.

Exercise 4. It is a not open, not closed, bounded, not compact, and convex set.

Exercise 5. *c)* is true, the rest are false.

Exercise 6. *d)* is false, the rest are true.

Exercise 7. Check the solutions in Exercise 7. ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus.

Exercise 8. $\|\nabla f(0, \pi, \pi/2)\| = \sqrt{64\pi^2 + 2}$

Exercise 9. $\nabla f(2, 1) = (4, 1)$

Exercise 10.

a) Check the solutions in Exercise 10. (a) ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus.

b) $\nabla f(0, 0) = (0, 1/2)$

c) $\dfrac{\partial f}{\partial(0, -1)}(0, 0) = \dfrac{-1}{2}$

Exercise 11. The function satisfies the equation for any value of a.

Exercise 12.

a) Check the solutions in Exercise 12. (a) ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus.

b) $\nabla f(1/2, 1) = (2, -1)$

c) $\dfrac{\partial f}{\partial(1,1)}\left(\dfrac{1}{2}, 1\right) = 1$

d) $\dfrac{\partial f}{\partial \frac{(4,3)}{\|(4,3)\|}}\left(\dfrac{1}{2}, 1\right) = \dfrac{\partial f}{\partial\left(\frac{4}{5}, \frac{3}{5}\right)}\left(\dfrac{1}{2}, 1\right) = 1$

Exercise 13.

a) Check the solutions in Exercise 13. (a) ("Material de treball autònom $\rightarrow$ Funcions de n-variables $\rightarrow$ Solucions") of the Meatacampus.

b) $\nabla f(1, 1) = (1/2, -1/2)$

c) $\dfrac{\partial f}{\partial(-3,4)}(1,1) = \dfrac{-7}{2}$

d) $\dfrac{\partial f}{\partial \frac{(4,3)}{\|(4,3)\|}}(1,1) = \dfrac{\partial f}{\partial\left(\frac{4}{5}, \frac{3}{5}\right)}(1,1) = \dfrac{1}{10}$

Exercise 14. $\dfrac{\partial f}{\partial(4,5)}(0,3) = 15$

Exercise 15. $\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\right)(x,y,z) = 0.$

Exercise 16. $\nabla f(x,y) = (e^x \cos(y), -e^x \sin(y))$. $\nabla f(0,\pi) = (-1, 0)$.

Exercise 17. $Hf(x,y) = \begin{pmatrix} 2y & 2x-5 \\ 2x-5 & 0 \end{pmatrix}, \quad Hf(1,1) = \begin{pmatrix} 2 & -3 \\ -3 & 0 \end{pmatrix}$

Exercise 18. $Hf(x,y) = \begin{pmatrix} 3xye^{x^3y}(2+3x^3y) & 3x^2e^{x^3y}(1+x^3y) \\ 3x^2e^{x^3y}(1+x^3y) & x^6e^{x^3y} \end{pmatrix}, \quad Hf(1,3) = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$

Exercise 19. We compute the second-order partial derivatives corresponding to the upper triangle of the Hessian matrix:

$$\frac{\partial f}{\partial x}(x,y) = 2xy + yz \quad \longrightarrow \quad \begin{cases} \frac{\partial^2 f}{\partial x^2}(x,y) = 2y \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = 2x + z \\ \frac{\partial^2 f}{\partial x \partial z}(x,y) = y \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 - 2yz^2 + xz \quad \longrightarrow \quad \begin{cases} \frac{\partial^2 f}{\partial y^2}(x,y) = -2z^2 \\ \frac{\partial^2 f}{\partial y \partial z}(x,y) = -4yz + x \end{cases}$$

$$\frac{\partial f}{\partial z}(x,y) = -2y^2 z + xy \quad \longrightarrow \quad \frac{\partial^2 f}{\partial z^2}(x,y) = -2y^2$$

Now, we can check that the conditions to apply the Schwartz Theorem are satisfied because all the second-order partial derivatives computed above are continuous in their whole domain ($\mathbb{R}^2$). Then, using the Schwartz Theorem we have that

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2x+z \qquad \frac{\partial^2 f}{\partial z \partial x}(x,y) = y \qquad \frac{\partial^2 f}{\partial z \partial y}(x,y) = -4yz+x$$

Finally,

$$Hf(x,y,z) = \begin{pmatrix} 2y & 2x+z & y \\ 2x+z & -2z^2 & -4yz+x \\ y & -4yz+x & -2y^2 \end{pmatrix} \qquad Hf(-2,1,0) = \begin{pmatrix} 2 & -4 & 1 \\ -4 & 0 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

Exercise 20. $z = bx + ay - ab$.

Exercise 21. $z = 3x + 5y - 6$. $f(1, 1'05) \simeq 2'25$

Exercise 22. $z = 2x + y - 6$. $f(1'1, 2'01) \simeq -1'79$

Exercise 23.

a) $E_x f(5) = 6$

b) $E_x f(0) = 0$

c) $E_x f(x_0) = -1$ if $x_0 \in \{1, 2\}$, $E_x f(0)$ does not exist

d) $E_x f(3) = 27/7$

Exercise 24.

a) $E_x f(x) = 1$

b) $E_x f(x) = 2$

c) $E_x f(x) = n + x$

d) $E_x f(x) = \dfrac{4x^4}{x^4+3}$

Exercise 25.

a) $f(10, 5) = 750$

b) $z = 175x + 100y - 1500$. $f(10'1, 5) \simeq 767'5$

c) $E_x f(10, 5) = 7/3$. If the value of the second variable is fixed ($y = 5$), a change of $1\,\%$ in the variable x causes a change of $2'3333\,\%$ in the output of f.

d) $f(10'1, 5) = 767'6$. Comparing this value with the result *b)* above we see that the error of the approximation is just $0'1$ units. Finally, note that moving from point $(10, 5)$ to point $(10'1, 5)$ we only change the value of x in $1\,\%$. From *c)* above we know that the output of the function changes about $2'3333\,\%$. Since the $2'3333\,\%$ of 750 is $17'5$, the output of the function at point $(10'1, 5)$ is about $17'5 + 750 = 767'5$.

Exercise 26.

a) $E_x f(1,1,1) = 12/7$ *b*) $E_y f(1,1,1) = 1$ *c*) $E_z f(1,1,1) = -1/7$

Exercise 27. The daily production will be incremented in 10 units approximately. The exact increment is $9'997$ units.

Exercise 28. The monthly benefit of the company will increase $3\,360$€ approximately.

Exercise 29. $\dfrac{\partial z}{\partial u} = \left(\dfrac{2}{u+v} + v\right) z$ and $\dfrac{\partial z}{\partial v} = \dfrac{2+u^2+uv}{u+v} \cdot z.$

Exercise 30. $\dfrac{\partial z}{\partial u}(1,1,-1) + \dfrac{\partial z}{\partial v}(1,1,-1) + \dfrac{\partial z}{\partial w}(1,1,-1) = 0$

Exercise 31. $0.$

Exercise 32. $-e^2.$

Exercise 33. $-1.$

Exercise 34. $c = 4.$ $\dfrac{\partial z}{\partial x}(0,e) = \dfrac{2}{e^2-4}$ and $\dfrac{\partial z}{\partial y}(0,e) = \dfrac{2e}{e^2-4}$

Exercise 35.

a) Homogeneous of degree 2.

b) Homogeneous of degree $-1/2$.

Exercise 36.

a) f is homogeneous of degree $m_1 + m_2 + \cdots + m_n$. Euler's Theorem says that

$$x_1 \frac{\partial f}{\partial x_1} + \ldots x_n \frac{\partial f}{\partial x_n} = (m_1 + \cdots + m_n) f,$$

which can be easily checked. Note that for every $i \in \{1, \ldots, n\}$, $\dfrac{\partial f}{\partial x_i} = \dfrac{m_i}{x_i} f(x_1, \ldots, x_n)$

b) f is homogeneous of degree 1. It is easy to check that $K\dfrac{\partial f}{\partial K} + L\dfrac{\partial f}{\partial L} = f.$

c) f is homogeneous of degree 1. Computing the partial derivatives, $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = f.$

d) f is homogeneous of degree 0. We can check that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = 0.$