Unit IV: Optimization
(Extra Material)

Exercise 1. Given the real function of two variables $f(x,y) = \dfrac{e^{x+y}}{x^2-4}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1$ and $-3 \leq y \leq 3\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : -3 \leq x \leq 3$ and $-1 \leq y \leq 1\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : y \geq x^2 - 6x + 18\}$

Exercise 2. Given the real function of two variables $f(x,y) = x^2 + y^2$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1$ and $-1 \leq y \leq 1\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$

Exercise 3. Given the real function of two variables $f(x,y) = e^{\frac{x^2+y^2}{x^2-9}}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2$ and $-4 \leq y \leq 4\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : y - x^2 + 4x - 5 \geq 0\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : y - x^2 + 4x - 5 \geq 0$ and $y \leq 6\}$

Exercise 4. Find the stationary points of the real function $f(x,y,z) = 2x^2 + y^2 - z^2 + x + 5$

Exercise 5. Suppose that studying the local extreme points of a real function f, there is a point p at which the gradient is not canceled but the Hessian matrix is positive definite. What can we say about it? Is it a local extreme point of the function?

Exercise 6. Given the real function $f(x,y,z) = -4x^2 - y^2 + 6x + 3y + z$. Study the possible local extreme point of it.

Exercise 7. A company produces two commodities and sells them at prices $p_1 = 51$€ and $p_2 = 42$€. Compute the production level that maximizes the benefits of the company if the cost of producing q_1 units of the first commodity and q_2 units of the second is described by,

$$C(q_1, q_2) = 2q_1q_2 + \frac{3}{2}q_1^2 + q_2^2$$

Exercise 8. For each of the real functions below, find the stationary points and classify[1] them whenever possible.

a) $f(x, y) = x^2 + y^2 - 6xy - 39x + 18y + 20$
b) $f(x, y) = 2x + 3y - x^2 - 2y^2 + xy$
c) $f(x, y) = x^2 + y^2 - 2\ln(x) - 18\ln(y)$
d) $f(x, y) = (1 + e^y)\cos(x) - ye^y$
e) $f(x, y, z) = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 12yz$

Exercise 9. Let $f(x, y) = 2xy+x^2+xy^2+y^3$. Decide whether the following statements are true or false:

a) $(0, 0)$ satisfies the necessary condition to be a local extreme point of f.
b) $(1, 1)$ is a local minimum of f because $Hf(1, 1)$ is positive semidefinite.
c) $(-1 - \sqrt{2}, \sqrt{2})$ is a stationary point of f.
d) $(0, 0)$ is a saddle point of f.

Exercise 10. Is $(0, 0)$ a local extreme point of the real function $z = 12y^2 - 8x^2$?

Exercise 11. Let $f(x, y) = x + y$, then decide whether the following statements are true or false:

a) f attains both a global minimum and maximum in its domain.
b) $(0, 0)$ is a global minimum of f.
c) f does not attain a maximum nor a minimum in its domain.
d) $(0, 0)$ is a stationary point.

Exercise 12. Let f be a real function of two variables such that

$$\frac{\partial f}{\partial x} = 2x + y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x + 2y + 3,$$

What can we say about the stationary point $(1, -2)$?

Exercise 13. Given the real function $f(x, y) = x^2 + y^2 + axy$ where $a \in \mathbb{R}$, study for what values of the parameter a is f convex/concave and for what values it is strictly convex/concave.

Exercise 14. Find the global extreme points of the real function $f(x, y) = x^2+y^2+xy$.

Exercise 15. Find the global extreme points of the real function $f(x, y) = x^4 - 8x^3 + 24x^2 - 32x + y^2 + 4y + 36$.

[1] Local maximum, local minimum, or saddle point

Exercise 16. A company produces two types of paper handkerchiefs (small and big) in lots of 1 000 units. The selling price of a lot of small handkerchiefs is 1 000 pesetas and the selling price of a lot of big handkerchiefs is 2 000 pesetas. The cost of producing x lots of small handkerchiefs per hour is $5x^2$ pesetas and the cost of producing y lots of big handkerchiefs per hour is $5y^2$ pesetas. Finally, the fixed administrative costs of the company amounts to 10 000 pesetas per hour. If the company wants to maximize the benefits

a) find the quantities of each type of handkerchiefs that should be sold per hour.

b) and the production of big handkerchiefs has to be three times the production of small handkerchiefs, find the quantities of each type of handkerchiefs that should be sold per hour.

Exercise 17. A company produces two goods A and B. Last week 400 units of good A were sold at a price of 720 pesetas each. A market research has concluded that for each decrease of 10 pesetas in the price of good A 2 more units would be sold and that the price of good B should be kept at 1 480 pesetas. The overall cost function of the company is given by $C(q_1, q_2) = 5q_1^2 + 10q_2^2 + 50000$ where q_1 and q_2 stand for the amounts of goods A and B produced, respectively. Taking into account that the company wants to maximize the benefits

a) find the quantities of each type of product that should be sold weekly.

b) and that the joint weekly production of both goods cannot exceed 100 units, find the quantities of each good that should be sold weekly.