Unit I: The $\mathbb{R}^n$ Vector Space. Solutions

Exercise 1.

a) $(0, -1, 1)$

b) $(8, -3, 0)$

c) $(-4, 0, -3)$

Exercise 2.

a) A line passing through the points $(1, 3)$ and $(0, 0)$

b) A line passing through the points $(1, -1)$ and $(0, 0)$ (also through $(-3, 3)$.

c) The whole 2-dimensional space, hence it is a plane.

Exercise 3.

a) A line passing through the points $(-1, -1, -1)$ and $(0, 0, 0)$ (also through $(-4, -4, -4)$).

b) A plane passing through the points $(2, 0, 0)$,$(1, 2, 2)$, and $(0, 0, 0)$.

c) A plane passing through the points $(2, 2, 2)$, $(1, 0, 2)$, and $(3, 2, 3)$ (also through $(0, 0, 0)$). It is difficult to realize that the three points are in a plane that passes through the origin graphically. However, we can conclude this taking into account that the rank of the matrix built from the three vectors is 2.

d) A plane passing through the points $(2, -1, 3)$, $(1, 4, 1)$, and $(5, 2, 7)$ (also through $(0, 0, 0)$). This is checked in Exercise 10.

Exercise 4. No. The rank of the matrix built from $\{\vec{u}_1, \vec{u}_2\}$ is different from the rank of the matrix built from $\{\vec{u}_1, \vec{u}_2, \vec{u}\}$. This also means that the three vectors do not lie in the same plane.

Exercise 5. Yes.

Exercise 6. For $k = 8$.

Exercise 7. No. We will check that the equation $\lambda_1\vec{u} + \lambda_2\vec{v} = \vec{w}$ has no solution. The equation above gives rise to the following system of equations:

$$\lambda_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \lambda_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

From the statement of the problem we know that the rank of the augmented matrix is 3. Since the rank of the coefficient matrix is at most two, by the Rouché-Frobenius theorem we know that the system above has no solution. Hence, $\vec{w}$ is not a linear combination of $\{\vec{u}, \vec{v}\}$.

Exercise 8. The plane is defined by the equation $-13x+y+9z=0$ (you may want to look on the internet how to find the equation of a plane passing through 3 points of $\mathbb{R}^3$). Substituting $(5,2,7)$ in the equation we get $13\cdot 5+2+9\cdot 7=0$. Since the equation is satisfied we conclude that $(5,2,7)$ is a point of the plane.

Exercise 9.

a) The vector $\vec{u}_4$ is a linear combination of the rest of vectors. To show this claim we can solve the following system of equations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix},$$

which trivially has the solution $(\lambda_1,\lambda_2,\lambda_3)=(-2,3,1)$. We can also check that the solution is correct by computing the linear combination $-2\vec{u}_1+3\vec{u}_2+\vec{u}_3$ and seeing that it equals $\vec{u}_4$.

b) It is easy to see that

$$rk \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 3,$$

computing any of the 4 minors of the matrix.

Exercise 10. The zero vector, $\vec{0}$, is a linear combination of any set of vectors. For any $\{\vec{u}_1,\dots,\vec{u}_k\}$,

$$\vec{0}=\lambda_1\vec{u}_1+\cdots+\lambda_k\vec{u}_k,$$

taking the coefficients $\lambda_1=\cdots=\lambda_k=0$.

Exercise 11. In order to decide if a set of vectors is linearly dependent/independent we only need to compute the rank of the matrix whose columns are the considered vectors. Recall that the rank of a matrix indicates the number of linearly independent column vectors. Thus, if the rank of the matrix equals the number of vectors we can conclude that the set of vectors is linearly independent, otherwise the set would be linearly dependent.

a) The set is linearly independent.

b) The set is linearly dependent. Solving the system of equations $\lambda_1(-2,-3,3)+\lambda_2(3,4,1)=(1,2,-7)$ we obtain $\lambda_1=-2$ and $\lambda_2=-1$. Hence we can write the third vector as a linear combination of the first two as follows:

$$(1,2,-7)=-2\cdot(-2,-3,3)-1\cdot(3,4,1).$$

c) The set is linearly independent.

Exercise 12. We need to find a vector $\vec{w} = (a, b, c)$ such that the rank of matrix A is 3 where

$$A = \begin{pmatrix} 0 & -1 & a \\ 2 & -4 & b \\ 3 & 1 & c \end{pmatrix}.$$

In other words, we need to find the values of a, b, and c that make the determinant of A not zero. Computing the determinant of A, we obtain

$$det(A) = 14a - 3b + 2c.$$

Hence, the vector $(0, 0, 1)$ is a possible answer.

Exercise 13. The rank of a matrix determines the maximum number of linearly independent column vectors contained on it.

$$rk(A) = 3 \quad \text{and} \quad rk(B) = 2.$$

Exercise 14. It is easy to see that if $a = 0$, $d = 0$ or $f = 0$ the rank of the matrix is less than 3 and hence, the set of column vectors is linearly dependent.

Exercise 15. For any value of k different from 7 and -1.

Exercise 16.

a) We will proof by contradiction that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent. Suppose that $\vec{w}_1$ and $\vec{w}_2$ are NOT linearly independent. Then, one of the vectors is a linear combination of the other. That is,

$$\vec{w}_1 = \lambda \vec{w}_2, \tag{1}$$

for some $\lambda \in \mathbb{R}$. Next, taking into account that $\vec{w}_1 = \vec{u}_1$ and $\vec{w}_2 = \vec{u}_1 + \vec{u}_2$, we rewrite equation (1) as

$$\vec{u}_1 = \lambda(\vec{u}_1 + \vec{u}_2), \tag{2}$$

if $\lambda \neq 0$ from equation (2) we obtain

$$\vec{u}_2 = \frac{1-\lambda}{\lambda}\vec{u}_1,$$

which contradicts the fact that $\vec{u}_1$ and $\vec{u}_1$ are linearly independent vectors. In case $\lambda = 0$, equation (2) implies that $\vec{u}_1 = \vec{0}$ which again contradicts the fact that $\vec{u}_1$ and $\vec{u}_1$ are linearly independent vectors. Since the original two vectors ($\vec{u}_1$ and $\vec{u}_2$) are linearly independent, the supposition that $\vec{w}_1$ and $\vec{w}_2$ were linearly dependent must be false and then $\vec{w}_1$ and $\vec{w}_2$ are linearly independent.

b) The vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are NEVER linearly independent. It can be easily seen that $\vec{w}_3$ is a linear combination of $\vec{w}_1$ and $\vec{w}_2$. Indeed,

$$\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2.$$

Exercise 17.

a) Yes. The line passing through $(0,0)$ and $(3,\frac{1}{2})$.

b) No. A single vector is not enough to span a 2-dimensional space. We need at least two vectors.

c) No. The two vectors are linearly dependent. Thus, their linear span is a line.

d) Yes. The vectors are independent. Then, they span a 2-dimensional space.

e) Yes. The same reason as above.

f) Yes. Because among the three there are two linearly independent vectors.

g) No. The vectors are in $\mathbb{R}^2$ and then, they cannot span $\mathbb{R}^3$.

Exercise 18.

a) Yes. The two vectors are linearly independent. Hence, they span a 2-dimensional space, that is, a plane.

b) No. The two vectors are linearly dependent, which means that they span a line.

c) No. The matrix built from the vectors has rank 2. This means that they span a plane.

d) No. The rank of the matrix built from the vectors has rank 2, which means that their linear span is a plane in $\mathbb{R}^3$.

Exercise 19. Yes.

Exercise 20. True. In order a set of vectors to be a basis of $\mathbb{R}^n$ we need it to be a spanning set of $\mathbb{R}^n$ and to be a linearly independent set. Since any vector of $\mathbb{R}^n$ is a linear combination of the vectors in the set it is a spanning set of $\mathbb{R}^n$. Since no vector of the set is a linear combination of the rest the set is linearly independent.

Exercise 21. For any value of a different from 0 and 2.

Exercise 22.

a) $rk(A) = 3$.

b) The coordinates of $\vec{u}$ with respect to the basis are $(1,1,1)$.

Exercise 23.

a) $rk(A) = 3$.

b) The coordinates of $\vec{u}$ with respect to the basis are $(2,-1,-1)$.

Exercise 24. No. 4 vectors cannot be linearly independent in $\mathbb{R}^3$.
Yes. It only needs to contain 3 linearly independent vectors.

Exercise 25. First, since the matrix is n-by-n the dimension of the space equals the number of vectors. Second, since the matrix is invertible its determinant is non zero, that is, the rank of the matrix is n.

Exercise 26.

a) Yes. The maximum number of linearly independent vectors in $\mathbb{R}^3$ is 3.

b) No. Only in case the vectors are linearly independent.

c) No. Take for instance $\{(1,-1,1,-1),(-2,2,-2,2)\}$.

d) No. Only if it contains 4 linearly independent vectors.

Exercise 27. Suppose that $Span(\{\vec{u}_1,\ldots,\vec{u}_k\}) = \mathbb{R}^n$, then every vector in $\mathbb{R}^n$ is a linear combination of $\{\vec{u}_1,\ldots,\vec{u}_k\}$. That is, for every $\vec{u} \in \mathbb{R}^n$ there are $\lambda_1,\ldots,\lambda_k \in \mathbb{R}$ such that

$$\vec{u} = \lambda_1\vec{u}_1 + \cdots + \lambda_k\vec{u}_k.$$

It is easy to check that every vector in $\mathbb{R}^n$ can be written as a linear combination of $\{\vec{u}_1,\ldots,\vec{u}_k,\vec{u}_{k+1}\}$ because $\vec{u} = \lambda_1\vec{u}_1 + \cdots + \lambda_k\vec{u}_k = \lambda_1\vec{u}_1 + \cdots + \lambda_k\vec{u}_k + 0\vec{u}_{k+1}$.

Exercise 28. The two vectors are linearly independent, so they span a 2-dimensional subspace of $\mathbb{R}^3$, that is, a plane. The analytical expression of the subspace is

$$Span(\{\vec{u}_1,\vec{u}_2\}) = \{(x,y,z) \in \mathbb{R}^3 : 4x - 5y + 2z = 0\}.$$

Exercise 29. One vector (different from the zero vector) spans a 1-dimensional vector subspace, that is a line. The analytical expression of the subspace is

$$Span(\{\vec{u}\}) = \{(x,y) \in \mathbb{R}^2 : 2x + y = 0\}.$$

Exercise 30. One vector (different from the zero vector) spans a 1-dimensional vector subspace, that is a line.

Let $(x,y,z) \in \mathbb{R}^3$ be an arbitrary vector of $\mathbb{R}^3$. We need to find the conditions that the coordinates x, y, and z have to satisfy in order the vector (x,y,z) to be in $Span(\{(-1,1,2)\})$. In other words, we need to find the conditions that guarantee the existence of a $\lambda \in \mathbb{R}$ such that

$$(x,y,z) = \lambda(-1,1,2).$$

The equation above has a solution whenever $rk(A) = rk(A;b)$ where

$$A = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad (A;b) = \begin{pmatrix} -1 & x \\ 1 & y \\ 2 & z \end{pmatrix}.$$

Since $rk(A) = 1$, we need that the three minors of $(A; b)$ have a null determinant. Hence, we need that

$$\begin{cases} -y - x = 0 \\ -z - 2x = 0 \\ z - 2y = 0 \end{cases}$$

There are several ways to describe the conditions above. For instance, if we demand $x+y = 0$ and $2x + z = 0$ the desired conditions are guaranteed. Then, an analytical expression of the vector subspace of $\mathbb{R}^3$ spanned by $(-1, 1, 2)$ is

$$Span(\{(-1, 1, 2)\}) = \{(x, y, z) \in \mathbb{R}^3 : x + y = 0 \text{ and } 2x + z = 0\}.$$

Exercise 31. Among the four vectors it is easy to find two linearly independent vectors. Take for instance, $\vec{u}_1$ and $\vec{u}_2$. This means that the vector subspace spanned by $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is 2-dimensional. Then, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a spanning set of $\mathbb{R}^2$. That is $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}) = \mathbb{R}^2$. A basis consists of any pair of independent vectors. For example $\{\vec{u}_1, \vec{u}_2\}$ is a basis and the dimension is 2.

Exercise 32.

$$Span(\{\vec{u}_1, \vec{u}_2\}) = \{(x, y, z) \in \mathbb{R}^3 : 6x - 5y + 2z = 0\}.$$

Exercise 33. A basis of the vector subspace

$$\{(x, y, z) \in \mathbb{R}^3 : x = y = z\} = \{(x, y, z) \in \mathbb{R}^3 : x = y \text{ and } x = z\}$$

consists of any vector whose coordinates are equal (except for the zero vector). For example $\{(1, 1, 1)\}$ is a basis.

Exercise 34. A possible basis of the vector subspace

$$\{(x, y, z) \in \mathbb{R}^3 : x = y = z\} = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

is $\{(1, 0, -1), (0, 1, -1)\}$.

Exercise 35. $\{(1, -1, 0)\}$ is a basis of V and hence, $dim(V) = 1$.

Exercise 36. Note that the analytical expression of V has a redundant equation and then,

$$V = \{(x, y, z) \in \mathbb{R}^3 : x-y = 0, x-2z = 0, y-2z = 0\} = \{(x, y, z) \in \mathbb{R}^3 : x-y = 0, x-2z = 0\}.$$

In any case, $\{(2, 2, 1)\}$ is a basis of V and $dim(V) = 1$.

Exercise 37. From the analytical expression that defines the vector subspace we can obtain that (note that this is not the only possibility)

$$(x, y, z) \in S \Leftrightarrow \begin{cases} y = 4x \\ z = \frac{5}{2}x \end{cases}$$

Then, every vector of the vector subspace can be written as follows:

$$(x, y, z) = \left(x, 4x, \frac{5}{2}x\right) = x\left(1, 4, \frac{5}{2}\right).$$

Hence, every vector of the vector subspace is a linear combination of $\left(1, 4, \frac{5}{2}\right)$, which means that S is the linear span of $\left\{\left(1, 4, \frac{5}{2}\right)\right\}$. Finally, $\left\{\left(1, 4, \frac{5}{2}\right)\right\}$ is a basis of S because it is linearly independent (it is different from $\vec{0}$). Therefore, $dim(S) = 1$.

Exercise 38. $\{(1, 0, 3), (0, 1, 2)\}$ is a basis of V and hence, $dim(V) = 2$.

Exercise 39.

a) It is trivial to check that $S \subseteq \mathbb{R}^3$ and $S \neq \emptyset$ (because $\vec{0} \in S$). Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be two elements of S. Then, we know that $2u_1 + 3u_2 = 0$ and $2v_1 + 2v_2 = 0$. We have to check that $\vec{u} + \vec{v} \in S$. So, we need to check that 2 times the first component of $\vec{u} + \vec{v}$ plus 3 times the second component of $\vec{u} + \vec{v}$ equals 0, that is, we need to check that $2(u_1 + v_1) + 3(u_2 + v_2) = 0$. Which follows directly if we rearrange the equation above and use what we know:

$$2(u_1 + v_1) + 3(u_2 + v_2) = (2u_1 + 3u_2) + (2v_1 + 3v_2) = 0 + 0 = 0.$$

Finally, we have to check that for every $\lambda \in \mathbb{R}$, $\lambda\vec{u} = (\lambda u_1, \lambda u_2, \lambda u_3) \in S$. That is, if $2(\lambda u_1) + 3(\lambda u_2) = 0$. Which follows directly if we rearrange the equation above and use what we know:

$$2(\lambda u_1) + 3(\lambda u_2) = \lambda(2u_1) + \lambda(3u_2) = \lambda 0 + \lambda 0 = 0.$$

b) $\{(0, 0, 1), (-3, 2, 0)\}$ is a basis of S and hence, $dim(S) = 2$.

Exercise 40.

a) It is trivial to check that $S \subseteq \mathbb{R}^3$ and $S \neq \emptyset$ (because $\vec{0} \in S$). Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be two elements of S. Then, we know that $u_1 - u_3 = v_1 - v_3 = 0$ and $u_1 + 3u_3 = v_1 + 3v_3 = 0$. We check that $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ satisfies the equations that determine the subspace. That is,

$$(u_1 + v_1) - (u_3 + v_3) = (u_1 - u_3) + (v_1 - v_3) = 0 + 0 = 0$$
$$(u_1 + v_1) + 3(u_2 + v_2) = (u_1 + 3u_2) + (v_1 + 3v_2) = 0 + 0 = 0$$

Next, for $\lambda \in \mathbb{R}$ we check that $\lambda\vec{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$ satisfies the conditions to be in the subspace. That is,

$$(\lambda u_1 - \lambda u_3) = \lambda(u_1 - u_3) = 0$$
$$(\lambda u_1 + 3(\lambda u_2) = \lambda(u_1 + 3u_2) = 0$$

b) $\{(3,-1,3)\}$ is a basis of S and hence, $dim(S) = 1$.

Exercise 41. S is not a vector subspace because $\vec{0} \notin S$.

Exercise 42. S is not a vector subspace because $\vec{0} \notin S$.

Exercise 43. At a first glance S does not look linear at all. However, it is not difficult to realize that $S = \{(0,0)\}$. And it is trivial to check that the set that consists of only the zero vector is a vector subspace. In general (for any n), the dimension of the vector subspace of $\mathbb{R}^n$ spanned by $\vec{0}$ is 0.