Unit I: The $\mathbb{R}^n$ Vector Space
(Extra Material)

Exercise 1. Given the following vectors of $\mathbb{R}^3$

$$\vec{u} = (1,1,3), \quad \vec{v} = (2,-5,1), \quad \vec{w} = (0,2,-3)$$

compute

a) $(\vec{u}+\vec{v})+\vec{w}$

b) $2\vec{u}-3\vec{v}$

c) $\vec{u}-\vec{v}+2\vec{w}$

d) $\vec{v}+\vec{w}-\vec{u}$

Exercise 2. Check if the vector $\vec{u} = (2,-1,4)$ is a linear combination of $\{\vec{u}_1 = (2,2,1), \vec{u}_2 = (5,3,2)\}$.

Exercise 3. Check if the vector $\vec{u} = (4,2,3)$ is a linear combination of $\{\vec{u}_1 = (1,1,1), \vec{u}_2 = (3,2,-1), \vec{u}_3 = (4,3,0), \vec{u}_4 = (7,5,-1)\}$.

Exercise 4. For what values of the parameter k is the vector $\vec{u} = (4,2,-5)$ a linear combination of $\{\vec{u}_1 = (1,0,1), \vec{u}_2 = (0,1,k), \vec{u}_3 = (k,2,k)\}$?

Exercise 5. Study the linear dependence or independence of the following vectors of $\mathbb{R}^3$

$$\vec{u}_1 = (5,6,2), \quad \vec{u}_2 = (2,3,5), \quad \vec{u}_3 = (3,2,-1)$$

Exercise 6. For what values of the parameter k are the vectors

$$\vec{u}_1 = (2,0,k), \quad \vec{u}_2 = (k,-3,-1), \quad \vec{u}_3 = (k,1,k)$$

linearly independent?

Exercise 7. For what values of the parameter k are the vectors

$$\vec{u}_1 = (3,5,1), \quad \vec{u}_2 = (k,4,7), \quad \vec{u}_3 = (2,-k,0), \quad \vec{u}_4 = (k,k,3)$$

linearly independent?

Exercise 8. Reason out if the following statements are true or false

a) It is not possible to find more that three linearly independent vectors in the $\mathbb{R}^3$ vector space.

b) Every set of four vectors of the $\mathbb{R}^4$ vector space forms a basis of $\mathbb{R}^4$.

c) Every set of less than four vectors of the $\mathbb{R}^4$ vector space is linearly independent.

d) Every set of more than four vectors of the $\mathbb{R}^4$ vector space is a spanning set of $\mathbb{R}^4$.

Exercise 9. Check if the set $\{\vec{u}_1 = (2,3,1), \vec{u}_2 = (5,4,2), \vec{u}_3 = (5,0,1)\}$ is a spanning set of $\mathbb{R}^3$. That is, check if $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}) = \mathbb{R}^3$.

Exercise 10. Given the following vectors of $\mathbb{R}^3$

$$\vec{u}_1 = (-2,3,2), \quad \vec{u}_2 = (a+2,0,3), \quad \vec{u}_3 = (5,3,a)$$

a) For what values of the parameter a is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a spanning set of $\mathbb{R}^3$?

b) For what values of the parameter a are the three vectors linearly independent?

Exercise 11. Given the following vectors of $\mathbb{R}^3$

$$\vec{u}_1 = (0,1,4), \quad \vec{u}_2 = (2,1,0), \quad \vec{u}_3 = (7,-1,2)$$

a) Check that these three vectors form a basis of $\mathbb{R}^3$.

b) Compute the coordinates of the vector $\vec{u} = (-5,2,-2)$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Exercise 12. Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be a basis of $\mathbb{R}^3$.

a) Check that $\{\vec{w}_1 = \vec{u}_1, \vec{w}_2 = 2\vec{u}_2, \vec{w}_3 = 3\vec{u}_3\}$ is also a basis of $\mathbb{R}^3$.

b) Suppose that the coordinates of a vector $\vec{u}$ with respect to the basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are $(5,8,27)$. Which are the coordinates of $\vec{u}$ with respect to the basis $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$?

Exercise 13. Find a basis and the dimension of the vector subspace of $\mathbb{R}^2$ defined by

$$S = \{(x,y) \in \mathbb{R}^2 : x = 7y\}$$

Exercise 14. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0, x-2y = 0\}$$

Exercise 15. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : x+y = 0, x+z = 0\}$$

Exercise 16. Find a basis and the dimension of the vector subspace of $\mathbb{R}^3$ defined by

$$V = \{(x,y,z) \in \mathbb{R}^3 : 2x+y-z = 0, x+y+z = 0\}$$

Exercise 17. Given the following vectors of $\mathbb{R}^3$

$$\vec{u}_1 = (3, 2, 2), \quad \vec{u}_2 = (-4, 3, 1), \quad \vec{u}_3 = (-1, 5, 3)$$

Find a basis and the dimension of the vector subspace $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\})$.

Exercise 18. Given the following vectors of $\mathbb{R}^2$

$$\vec{u}_1 = (2, -3), \quad \vec{u}_2 = (1, 4), \quad \vec{u}_3 = (4, 5), \quad \vec{u}_4 = (7, 12)$$

Find a basis and the dimension of the vector subspace $Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\})$.

Exercise 19. Find the analytical expression of the vector subspace spanned by the vector

$$\vec{u} = (1, 2)$$

Exercise 20. Find the analytical expression of the vector subspace spanned by the vectors

$$\vec{u}_1 = (1, 0, 2), \quad \vec{u}_2 = (2, 2, -3)$$

Exercise 21. Reason out if the following statements are true or false

a) Every basis of an n-dimensional vector space is formed by n vectors.

b) In an n-dimensional vector space, every set of linearly independent vectors has exactly n vectors.

c) In an n-dimensional vector space, every set of linearly dependent vectors has at least n vectors.

Exercise 22. John has computed the basis of a vector subspace of $\mathbb{R}^3$ and has obtained the following set of vectors

$$\{(3, 2, 3), (1, -2, 1)\}$$

Mary has computed the basis of the same vector subspace and has obtained the following set of vectors

$$\{(7, 2, 7), (2, 4, 2)\}$$

Peter has also studied the same vector subspace and has obtained the following basis

$$\{(7, 2, 7)\}$$

We also know that the dimension of the aforementioned vector subspace is 2. Reason out if the following statements are true or false

a) The basis obtained by Peter is wrong.

b) The bases obtained by John and Mary correspond to the same vector subspace. However, we cannot say if they are correct because we do not know the vector subspace precisely.

c) The bases obtained by John and Mary correspond to different vector subspaces. Thus, at least one of them is wrong.

d) All the three basis are wrong.