Unit II: The Euclidean Space. Solutions

Exercise 1.

a) $\vec{u} \cdot \vec{v} = 17$

b) $\vec{u} \cdot \vec{v} = 32$

c) Impossible

d) $\vec{u} \cdot \vec{v} = 0$

Exercise 2.

a) $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = -20$

b) $(\vec{u} \cdot \vec{v}) \cdot \vec{v} = (-45, 0, -27)$

c) $(\lambda\vec{u}) \cdot (\mu\vec{v}) = -72$

d) $[\lambda(\vec{u} + \vec{v})] \cdot (\mu\vec{v}) = 200$

Exercise 3.

a) Not orthogonal

b) Not orthogonal

c) Orthogonal

d) Not orthogonal

e) Orthogonal

f) Not orthogonal

g) Not orthogonal

h) Not orthogonal

Exercise 4. $k = 1$ or $k = 2$.

Exercise 5. $k = 1$.

Exercise 6. $k = -1$ or $k = 5$.

Exercise 7. $k = 3$ or $k = -2$

Exercise 8. We show by contradiction that $\{(a, b), (c, d)\}$ is linearly independent. Suppose that $\{(a, b), (c, d)\}$ is linearly dependent. Then one of the vectors is a linear combination of the other, that is, there is a $\lambda \in \mathbb{R}$ such that $(c, d) = \lambda(a, b)$. Moreover, $\lambda \neq 0$ because $(c, d) \neq (0, 0)$. From the orthogonality of (a, b) and (c, d) we have that $(a, b) \cdot (c, d) = 0$. However,

$$(a, b) \cdot (c, d) = (a, b) \cdot \lambda(a, b) = \lambda\|(a, b)\|^2 \neq 0,$$

because $\lambda \neq 0$ and $(a, b) \neq (0, 0)$. So, we have obtained a contradiction which proves that the hypothesis in the beginning ($\{(a, b), (c, d)\}$ is linearly dependent) is false.

Exercise 9.

a) $\|\vec{u}\| = \sqrt{14}$ and $\|\vec{v}\| = \sqrt{17}$

b) $\|2\vec{u}\| = \sqrt{56} = 2\sqrt{14}$

c) $\|\vec{u} - \vec{v}\| = \sqrt{17}$

d) $\|2\vec{u} + \vec{v}\| = \sqrt{101}$

Exercise 10.

a) $\|\vec{u}\| = \sqrt{29}$ and $\|\vec{v}\| = \sqrt{22}$

b) $\|3\vec{u}\| = \sqrt{261} = 3\sqrt{29}$

c) $\|-3\vec{u}\| = 3\sqrt{29}$

d) $\|\vec{u}+\vec{v}\| = \sqrt{97}$

Exercise 11.

a) Schwartz inequality: $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|$
For the particular vectors given by the exercise we compute: $|\vec{u}\cdot\vec{v}| = 7$ and $\|\vec{u}\|\cdot\|\vec{v}\| = \sqrt{238}$. Then, since $7 < \sqrt{238}$ the Schwartz inequality is satisfied.

b) Triangle inequality: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
For the particular vectors given by the exercise we compute: $\|\vec{u}+\vec{v}\| = 3\sqrt{5}$ and $\|\vec{u}\|+\|\vec{v}\| = \sqrt{14}+\sqrt{17}$. Then, the Triangle inequality holds because $3\sqrt{5} \simeq 6,71 < 7{,}86 \simeq \sqrt{14}+\sqrt{17}$.

Exercise 12.

a) The Schwartz inequality holds since $24 < 5\sqrt{26}$.

b) The triangle inequality is a consequence of $\sqrt{3} < \sqrt{26}+5$.

Exercise 13.

$$\frac{1}{\|\vec{u}\|}\cdot\vec{u} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right) \quad \text{and} \quad \frac{1}{\|\vec{v}\|}\cdot\vec{v} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Exercise 14.

$$\frac{1}{\|\vec{u}\|}\cdot\vec{u} = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right) \quad \text{and} \quad \frac{1}{\|\vec{v}\|}\cdot\vec{v} = \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right).$$

Exercise 15. It is an orthogonal basis of $\mathbb{R}^3$.

Exercise 16. The vectors form an orthogonal basis but not an orthonormal basis because $\vec{u}_2$ and $\vec{u}_3$ are not unit vectors.

Exercise 17. It is an orthonormal basis of $\mathbb{R}^3$.

Exercise 18. $k = 3$ or $k = -3$.

Exercise 19. $55°33'$.

Exercise 20. $k = 3\sqrt{43}$ or $k = -3\sqrt{43}$.

Exercise 21. $\alpha \simeq 59°31'48''$.

Exercise 22. $\alpha = 60°$.

Exercise 23. $0°$.

Exercise 24. $d(\vec{u}, \vec{v}) = 5$.

Exercise 25. $k = 1$.

Exercise 26. $d(\vec{u}, \vec{v}) = 6$.

Exercise 27. $k = -3$ and $k = 5$.

Exercise 28. $k = \sqrt{\frac{7}{2}}$ and $k = -\sqrt{\frac{7}{2}}$.

Exercise 29. $k = 6$ or $k = -56/10$.