Unit III: Real Functions of Several Variables

Exercise 1. Represent graphically the following real functions of one variable:

a) $f(x) = 3x - 2$

b) $f(x) = -x^2 + 6x - 7$

c) $f(x) = \dfrac{2}{x}$

d) $f(x) = \sqrt{x}$

Exercise 2. Compute the derivatives of the following real functions of one variable.

a) $f(x) = x^4 - 2x^3 + \dfrac{1}{2}x^2 - 4x + 5$

b) $f(x) = (3x - 4)^2$

c) $f(x) = e^x$

d) $f(x) = \ln(x^2 + 1)$

Exercise 3. Given $f(x) = \sqrt{x^2 + 1}$ and $g(x) = e^x$, compute:

a) $f \circ g$ and $(f \circ g)'$.

b) $g \circ f$ and $(g \circ f)'$.

Exercise 4. Compute the domain of f in the following cases:

a) $f(x) = \dfrac{2x - 1}{x^2 - 1}$

b) $f(x) = \sqrt{-x^2 - 3x + 4}$

c) $f(x) = \ln\big((x - 2)^2 - 4\big)$

d) $f(x) = \dfrac{x}{x^2 - 9}$

Exercise 5. For each of the domains computed in Exercise 4, check if it is a closed, open, bounded, compact, or convex set.

Exercise 6. Draw the set $A \subseteq \mathbb{R}^2$ and discuss its topological properties in the following cases:

a) $A = \{(x, y) \in \mathbb{R}^2 : y^2 - 2x + 1 < 0\}$

b) $A = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + (y + 1)^2 \leq 1\}$

c) $A = \big\{(x, y) \in \mathbb{R}^2 : xy > 1\big\}$

d) $A = \big\{(x, y) \in \mathbb{R}^2 : xy \geq 1 \text{ or } x^2 + y^2 \leq 1\big\}$

e) $A = \big\{(x, y) \in \mathbb{R}^2 : xy \geq 1 \text{ and } x^2 + y^2 \leq 1\big\}$

f) $A = \big\{(x, y) \in \mathbb{R}^2 : y \geq (x - 2)^2\big\}$

g) $A = \big\{(x, y) \in \mathbb{R}^2 : |y| \geq (x - 2)^2\big\}$

The first five Exercises are about real functions of one variable. This topic is a previous requirement that all students are supposed to know.

Exercise 7. Compute and describe graphically the domain of the following real functions of two variables. For each of the computed domains discuss the topological properties of the sets.

a) $f(x,y) = \dfrac{3x-1}{x^2-y-1}$
b) $f(x,y) = \sqrt{6x^2+3y-9}$
c) $f(x,y) = \sqrt{x^2+y^2-9}$
d) $f(x,y) = \ln(x+y)$
e) $f(x,y) = \dfrac{\sqrt{xy}}{x}$
f) $f(x,y) = \sqrt{1-|xy|}$

Exercise 8. Compute and describe graphically the level curves of the following real functions of two variables.

a) $f(x,y) = \dfrac{x}{y}$
b) $f(x,y) = 2x^2+4y^2$
c) $f(x,y) = 2x^2+2y^2$
d) $f(x,y) = 4xy$
e) $f(x,y) = x^2+y+1$
f) $f(x,y) = \dfrac{\sqrt{x^2-y}}{3x}$

Exercise 9. Compute all the partial derivatives of the following real functions.

a) $f(x,y) = 5x^3y^2-2x$
b) $f(x,y) = \cos\left(x^3+y^4\right)$
c) $f(x,y) = \dfrac{x^2y-xy^2}{x+y}$
d) $f(x,y) = xe^{x+y^2}$
e) $f(x,y,z) = \sqrt{xe^{x+y}+yz^3}$
f) $f(x,y,z) = \ln\left(\dfrac{xy-6}{x^2-z}\right)$

Exercise 10. Compute the gradient of the real function f and evaluate it at the given point in the following cases:

a) $f(x,y) = \sin(x-y)+\cos(x+y)$ at point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.
b) $f(x,y) = x^3e^{x-y}+xy^3$ at point $(2,2)$.
c) $f(x,y) = \ln\left(x^2+y^4\right)$ at point $(1,3)$.
d) $f(x,y,z) = \dfrac{x+y+z^3}{x^3+y^2+z}$ at point $(1,0,1)$
e) $f(x,y,z) = \sqrt{2yz-xy^2}$ at point $(1,1,1)$.
f) $f(x,y,z,t) = 4x^3+yz^2-6t^5$ at point $(2,3,1,4)$.

Exercise 11. Compute the directional derivative of the real function $f(x,y)=3x^2y-y$ along the vector $(1,0)$ at point $(2,3)$.

Exercise 12. For each of the real functions in Exercise 10, compute the Hessian matrix of f and evaluate it at the given point.

Exercise 13. Let $f(x,y,z) = 5x^4 + 3xy^3 - 6yz$ and $p = (0,1,2)$.

a) Find the direction of the greatest rate of increase of f at point p.

b) Compute the directional derivative of f along the direction of the vector obtained in a) at point p.

c) Evaluate the Hessian matrix of f at p.

Exercise 14. Compute the equation of the tangent plane of f at p in the following cases:

a) $f(x,y) = \dfrac{e^{x^2}}{x+y}$, $p = (4,-3)$

b) $f(x,y) = \dfrac{(y-x^2)(y-2x^2)}{xy}$, $p = (1,1)$

Exercise 15. Given a real function of two variables, $f(x,y)$, such that

$$f(100,100) = 5, \qquad \frac{\partial f}{\partial x}(100,100) = 2, \qquad \text{and} \qquad \frac{\partial f}{\partial y}(100,100) = 4,$$

compute the approximate value of $f(101,100)$ using the concept of marginality.

Exercise 16. A given company produces three different products, A, B, and C. The overall daily benefit of the company is described by the following function (in €)

$$B(x,y,z) = e^{\frac{x}{100} + \frac{y^2}{1000} + \frac{z^2}{10000}} - 1,$$

where x, y, and z are the quantities of products A, B, and C produced daily.

Nowadays, 100 items of each of the three products are produced daily. Study how would the benefit of the company vary if the production of good C is incremented in one unit.

Exercise 17. Compute the partial elasticity of $f(x,y) = Ax^5y^3$, where A is a fixed parameter, with respect to y.

Exercise 18. Suppose that the production cost (in €) of producing x units of good A and y units of good B is given by the following real function

$$C(x,y) = \ln(x+1) + \ln(y+1) + 2x + 3y.$$

Compute and interpret the elasticity of the production cost with respect to good A at the production level $(x,y) = (100,150)$.

Exercise 19. Given the Cobb-Douglas production function $Q(K,L) = K^{\frac{1}{2}}L^{\frac{1}{2}}$, where Q is the production, K is the capital, and L is the working time. If the capital and working time are functions of the time, $K = K(t) = 100e^{\frac{-t}{2}}$ and $L = L(t) = 50t^{\frac{1}{2}}$, compute dQ/dt and evaluate it at $t = 4$.

Exercise 20. Compute $\partial z/\partial t$ and $\partial z/\partial s$ where $z = \cos(x+y^2) - \sin(x^2 - y)$, $x = \ln(s-t)$, and $y = \ln(s+t)$.

Exercise 21. Let f be a real function of three variables $f(x, y, z)$, where $x = \ln(u + v)$, $y = u - v$, and $z = v - u$. Compute $\dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v}$.

Exercise 22. The equation $(x+y)^2 + \ln(2x - y - 2) = 0$ defines y as a function of x ($y = y(x)$) in a neighborhood of the point $(1, -1)$.

a) Find the value of dy/dx at this point.

b) Compute the equation of the tangent line of $y(x)$ at $x = 1$.

Exercise 23. The equation $x^3 + x^2y - 2y^2 - 10y = 0$ defines y as a function of x in a neighborhood of the point $(2, 1)$. Find the equation of the tangent line of the curve $y(x)$ at that point.

Exercise 24. The equation $xyz + x^2\ln(z) + y - 2 = 0$ defines z as a function of x and y in a neighborhood of the point $(1, 1, 1)$. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$ at this point.

Exercise 25. Decide whether the real function f is homogeneous or not and find its degree of homogeneity in case it is homogeneous in the following cases:

a) $f(x, y) = \sqrt{xy}$

b) $f(x, y) = \sqrt{x^2 + y^2}$

c) $f(x, y, z) = x^4 - xy^2 + z^2$

d) $f(x, y, z) = x^4 - xy^2z + y^3z$

e) $f(x, y, z) = xe^{\frac{x^2+y^2}{z^4}}$

f) $f(x, y, z) = \dfrac{x^3y + xy^2z}{x(y - z)}$

Exercise 26. Decide whether the real functions of Exercise 7 are homogeneous or not using Euler's Theorem.