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MÀSTER DE MATEMÀTICA AVANÇADA

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On the local and global phase portrait of the 1-dimensional complex equation $\dot{z} = f(z)$ and perturbations

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Barcelona, 13 de setembre de 2014

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1 Introduction

This work consists of studying the complex first order differential equation

$$\dot{z} = \frac{dz}{dt} = f(z), \qquad z \in \mathbb{C}, t \in \mathbb{R}, \tag{1.1}$$

where f is an analytic function of $\mathbb{C}$ except, possibly, at isolated singularities. This is a rather general family of complex functions that includes polynomial, rational, holomorphic and entire functions, and functions with isolated essential singularities.

My previous knowledge was the qualitative theory of planar differential equations (see [1]), that is the theory of a simple differential systems in two real variables

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y), \end{cases}$$

where P and Q are C^r functions defined on an open subset U of $\mathbb{R}^2$, with r=1,2,...,∞.

It is clear that if we write $z = x + iy$ (1.1) can be written as a system of planar ($\mathbb{R}$) differential equations

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y), \end{cases}$$

where $x, y \in \mathbb{R}$, that is x={real part of z} and y ={image part of z}. We denote by $\Re(.) = Re(.)$ and $\Im(.) = Im(.)$. Then

$$\begin{cases} P(x, y) = \Re(f(z)) = \Re(f(x + iy)) \\ Q(x, y) = \Im(f(z)) = \Im(f(x + iy)). \end{cases}$$

Hilbert's 16th problem was posed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900, as part of his list of 23 problems in mathematics.
The original problem was posed as the Problem of the topology of algebraic curves and surfaces.
Actually the problem consists of two similar problems in different branches of mathematics:
* An investigation of the relative positions of the branches of real algebraic curves of degree n (and similarly for algebraic surfaces).
* The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n and their possible relative positions.

Associated with this equation we have a local flow defined near the origin and having the origin as a fixed point.

Our main result on local conformal conjugation is Theorem 3.5. In case of f being rational it is also considered the behavior of the vector field near infinity by means of the Poincaré Compactification. We also consider the stereographic projection of a sphere onto a plane in which case the infinity corresponds to the equator, and similarly we consider the Poincaré disk, if we project the Poincaré sphere to a disc, representing the infinity is its boundary.

As usual, after studying the local phase portrait near a regular or singular point, the next step is to determine the topology of the orbits in a neighborhood of a periodic orbit.

We get the non-existence of limit cycles and able to extend this preserving area phenomenon to a neighborhood of any monodromic graph. We are summarize these implications in Theorem 5.4.

We give a topological classification of the phase portrait of the family of rational functions of the form

$$\dot{z} = f(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials in the z variable such that deg$P$ $\leq$2 and deg$Q$ $\leq$2 without common factors.

Finally we study non holomorphic perturbations for the special case $f(z) = iz + z^2$ and we study the Poincaré map near the separatrix between the two centers of the non perturbed system. We use some results in [10] we give an integral expression whose zeros control the limit cycles that bifurcate form the periodic orbits of the period annulus of p. The result is applied to control the simultaneous bifurcation of limit cycles from the two period annuli of $\dot{z} = iz + z^2$, after a polynomial perturbation.

The paper is organized as follows. In Section 2 there are some preliminaries. In Section 3 and Section 4, we put and prove our main theorems, we also discus the phase portrait around a singular point and at infinity. In Section 5 we show the Hamiltonian structure of equation (1.1) and prove Theorem 5.4. In Section 6 we give a topological classification of the phase portraits of the family of rational functions of the form $\dot{z} = P(z)/Q(z)$ such that deg$P$ $\leq$2 and deg$Q$ $\leq$2 without common factors. Finally, in Section 7 we study non holomorphic perturbations for the special case $f(z) = iz + z^2$.

2 Preliminaries

2.1 Locally structure of singular points

We consider differential systems in two real variables

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y), \end{cases}$$

where P and Q are C^r functions defined on an open subset U of $\mathbb{R}^2$, with r=1,2,...,∞ or $r = \omega$. And we frequently associate with the equation the vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

on U∈ $\mathbb{R}^2$.

Let p be a singular point of a planar C^r vectors field X=(P,Q). In general the study of the local behavior of the flow near p requires the consideration of several cases. Already the linear systems show different classes, even for local topological equivalence. We say that

$$DX(p) = \begin{pmatrix} \frac{\partial P}{\partial x}(p) & \frac{\partial P}{\partial y}(p) \\ \frac{\partial Q}{\partial x}(p) & \frac{\partial Q}{\partial y}(p) \end{pmatrix},$$

is the linear part of vector field X at the singular point p.

The singular point p is called $non-degenerate$ if 0 is not an eigenvalue of $DX(p)$.

The singular point p is called *hyperbolic* if both eigenvalues of DX(p) have real part different from 0.

The singular point p is called $semi-hyperbolic$ if exactly one eigenvalues of DX(p) is equal to 0. Hyperbolic and semi-hyperbolic singularities are also called elementary singular points.

The singular point p is called *linearly* nilpotent if both eigenvalues of DX(p) are equal to 0 but DX(p) is not equal to 0.

The singular point p is called *linearly zero* if DX(p)≡0.

The singular point p is called a *center* if there is an open neighborhood of it consisting, besides the singularity, of periodic orbits. The singularity is said to be linearly a center if the eigenvalues of DX(p) are purely imaginary without being zero.

It is obvious that if $p = (x_0, y_0)$ is a singular point of the differential system

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y), \end{cases}$$

then the point (0,0) is a singular point of the system

$$\begin{cases} \dot{\bar{x}} = P(\bar{x}, \bar{y}) \\ \dot{\bar{y}} = Q(\bar{x}, \bar{y}), \end{cases}$$

where $x = \bar{x} + x_0$ and $y = \bar{y} + y_0$, and now the functions $P(\bar{x}, \bar{y})$ and $Q(\bar{x}, \bar{y})$ start with terms of order 1 in $\bar{x}$ and $\bar{y}$. In other words, we can always move a singular point to the origin of coordinates in which case system

$$\begin{cases} \dot{\bar{x}} = P(\bar{x}, \bar{y}) \\ \dot{\bar{y}} = Q(\bar{x}, \bar{y}), \end{cases}$$

becomes (dropping the bars over x and y)

$$\begin{cases} \dot{x} = ax + by + F(x, y) \\ \dot{y} = cx + dy + G(x, y), \end{cases}$$

where F and G together with their first partial derivatives at (0,0) vanish. We also write $X \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + H.O.T$ by a linear change of coordinates the linearization A=DX(0,0)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

can be transformed to real Jordan canonical form.

If the singularity is hyperbolic, the Jordan form is either

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

with $\lambda_1 \lambda_2 \neq 0, \alpha \neq 0$ and $\beta \geq 0$.

In the semi-hyperbolic case

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix},$$

with $\lambda \neq 0$ and $\beta \geq 0$, while we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

in the zero-nilpotent case or the linear zero case, respectively.

2.2 Infinite Singular Points

In order to study the behavior of the trajectories of a planar differential system near infinity it is possible to use a compactification of the plane. One of the possible constructions relies on the stereographic projection of the sphere onto the plane, in which case a single point at infinity is adjoined to the plane. A better approach for studying the behavior of trajectories near infinity is to use the so called Poincaré sphere.

It has the advantage that the singular points at infinity are spread out along the equator of the sphere and are therefore of a simpler nature than the singular points of the Bendixson sphere. However, some of the singular points at infinity on the Poincaré sphere may still be very complicated.

In order to draw the phase portrait of a vector field, we would have to work over the complete real plane $\mathbb{R}^2$, which is not practical. If the functions defining the vector field are polynomials, we can apply the Poincaré Compactification, which will help us to draw it in a finite region. Even more, it controls the orbits which tend or come from infinity.

Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be a polynomial vector field (the functions P and Q are polynomials of arbitrary degree in the variables x and y), or in other words :

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y), \end{cases}$$

We say that the degree of X is d if d is the maximum of the degrees of P and Q.

Poincaré Compactification works as follows.

First we consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We consider the sphere

$$\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\},$$

which we will call here Poincaré sphere. Note that it is tangent to $\mathbb{R}^2$ at the point (0,0,1). We may divide this sphere into

the northern hemisphere

$$H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\},$$

the southern hemisphere

$$H_- = \{y \in \mathbb{S}^2 : y_3 < 0\},$$

and the equator

$$\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}.$$

Now we consider the projection of the vector field X from $\mathbb{R}^2$ onto $\mathbb{S}^2$ given by the central projections $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$ and $f^- : \mathbb{R}^2 \to \mathbb{S}^2$

$$f^+(x) = (\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}),$$

$$f^-(x) = (\frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)}),$$

where

$$\Delta(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

We notice that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator $\mathbb{S}^1$. Let $\bar{X}$ be the transported vector field to $\mathbb{S}^2 \backslash \mathbb{S}^1$. Now we would like to extend the induced vector field $\bar{X}$ from $\mathbb{S}^2 \backslash \mathbb{S}^1$ to $\mathbb{S}^2$. The extended vector field on $\mathbb{S}^2$ is called the Poincaré Compactification of the vector field X on $\mathbb{R}^2$, and it is denoted by $p(X)$.

For $\mathbb{S}^2$ we use the six local charts given by

$$U_k = \{y \in \mathbb{R}^2 : y_k \geq 0\},$$

$$V_k = \{y \in \mathbb{R}^2 : y_k \leq 0\},$$

for $k = 1, 2, 3$. The corresponding local maps $\phi : U_k \to \mathbb{R}^2$ and $\psi : V_k \to \mathbb{R}^2$ are defined as $\phi_k(y) = -\psi(y) = (\frac{y_m}{y_k}, \frac{y_n}{y_k})$ for $m \leq n$ and $m, n \neq k$. We denote by $z = (u, v)$ the value of $\phi_k(y)$ or $\psi_k(y)$ for any k, so that (u, v) will play different roles depending on the local chart we are considering. Geometrically the coordinates (u, v) can be seen in Figure 1.1. The points of $\mathbb{S}^1$ in any chart correspond to $v = 0$. The singular points at $\mathbb{S}^1$ will be called infinite singular points.

In what follows we make a detailed calculation of the expression of p(X) only in the local chart U_1. We have $X = (P(x, y), Q(x, y))$. We obtain a vector field on $\mathbb{S}^2$ which is C^ω-equivalent to X on each of the hemispheres H_+, H_-

The expression for $p(X)$ in the local chart (U_1, ϕ_1) is given by

$$\dot{u} = v^d[-uP(\frac{1}{v}, \frac{u}{v}) + Q(\frac{1}{v}, \frac{u}{v})],$$
$$\dot{v} = -v^{d+1}Q(\frac{1}{v}, \frac{u}{v}).$$

The expression in (U_2, ϕ_2) is given by

$$\dot{u} = v^d[-uP(\frac{u}{v}, \frac{1}{v}) + Q(\frac{u}{v}, \frac{1}{v})],$$
$$\dot{v} = -v^{d+1}Q(\frac{u}{v}, \frac{1}{v}).$$

and the one in (U_3, ϕ_3) is

$$\dot{u} = P(u, v),$$
$$\dot{v} = Q(u, v).$$

The expression for $p(X)$ in the charts (V_k, ψ_k) is the same as for (U_k, ϕ_k) but multiplied by $(-1)^{d-1}$, for $k = 1, 2, 3$.

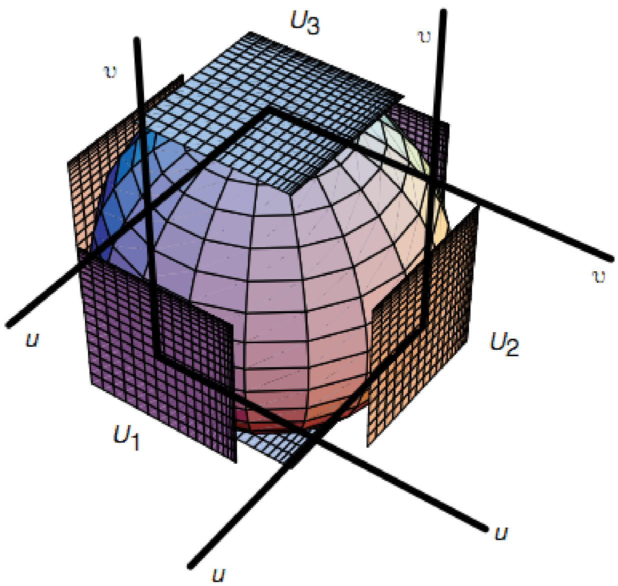

Figure 2.1: The local charts (U_k, ϕ_k) for k=1,2,3 of the Poincaré sphere.

We want to study the local phase portrait at infinite singular points. For this we choose an infinite singular point $(u, 0)$ and start by looking at the expression of the linear part of the vector field $p(X)$. We denote by P_i and Q_i the homogeneous polynomials of degree i for $i = 0, 1, ..., d$ such that $P = P_0 + P_1 + ... + P_d$ and $Q = Q_0 + Q_1 + ... + Q_d$. Then $(u, 0) \in \mathbb{S}^1 \cap (U_1 \cup V_1)$ is an infinite singular point of $p(X)$ if and only if

$$F(u) = Q_d(1, u) - uP_d(1, u) = 0.$$

Similarly $(u, 0) \in \mathbb{S}^1 \cap (U_2 \cup V_2)$ is an infinite singular point of $p(X)$ if and only if

$$G(u) = P_d(u, 1) - uQ_d(u, 1) = 0.$$

Also we have that the Jacobian of the vector field $p(X)$ at the point $(u, 0)$ after being reparmeterized is

$$\begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix}$$

or

$$\begin{pmatrix} G'(u) & P_{d-1}(u, 1) - uQ_{d-1}(u, 1) \\ 0 & -Q_d(u, 1) \end{pmatrix}.$$

If $(u, 0)$ belongs to $U_1 \cup V_1$ or $U_2 \cup V_2$, respectively.

2.3 Limit cycle

We consider the system of differential equations

$$\dot{x} = \frac{dx}{dt} = P(x, y), \qquad \dot{y} = \frac{dy}{dt} = Q(x, y), \tag{2.1}$$

where x, y, and t are real variables, and P and Q are C^1 functions of x and y. Then the existence and uniqueness of its maximal solutions is guaranteed.

If a solution $x = x(t)$, $y = y(t)$ of system (1.2) is a nonconstant periodic function of t, then

$$\gamma = \{(x, y) : x = x(t), y = y(t)\},$$

is called a periodic orbit of system (1.2)

If for some neighborhood of the periodic orbit γ there does not exist other periodic orbits, then γ is called a limit cycle.

We are going to assume that Δ is an open subset of $\mathbb{R}^2$ and X is a vector field on Δ of class C^r with $r \geq 1$. Also, in Δ, γ_p^+ denotes a positive semi-orbit passing through the point p.

Given a real dynamical system with flow φ, a point x and an orbit γ through x, we say that a point y is an $\omega - limit$ point of γ if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that

$$\lim_{n\to\infty} t_n = \infty,$$
$$\lim_{n\to\infty} \varphi(t_n, x) = y.$$

Theorem 2.1. *(Poincaré-Bendixson theorem)*
Let $\Phi(t) = \Phi(t, p)$ be an integral curve of X defined for all t≥0, such that γ_p^+ is contained in a compact set K⊂ Δ. Assume that the vector field X has at most a finite number of singularities in K. Then one of the following statements holds.
(i) If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbit.
(ii) If $\omega(p)$ contains both regular and singular points, then $w(p)$ is formed by a set of orbits, every one of which tends to one of the singular points in $w(p)$ as $t \to \infty$
(iii) If $\omega(p)$ does not contain regular points, then $\omega(p)$ is a unique singular point.

Analytic vector fields have a finite sectorial decomposition property at their singularities, and, under mild conditions, C^∞ vector fields also have it (see [1]). The Poincaré-Bendixson theorem also holds for vector fields on the two-dimensional sphere $\mathbb{S}^2$.

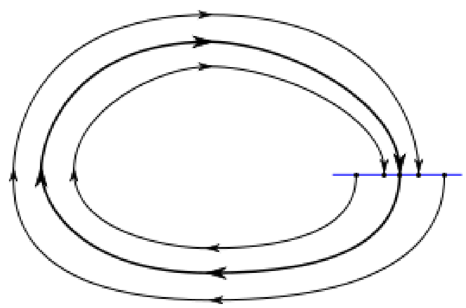

Figure 2.2: Limit cycle.

2.4 Hamiltonian systems

A Hamiltonian system is a dynamical system governed by Hamilton's equations. Informally, a Hamiltonian system is a mathematical formalism developed by Hamilton to describe the evolution equations of certain conservative physical systems. The advantage of this description is that it gives important insight into the dynamics, even if the initial value problem cannot be solved analytically.

Formally, a Hamiltonian system is a dynamical system completely determined by a scalar function $H(x, y)$ called Hamiltonian. The state of the system, r, is described by the generalized coordinates 'momentum' x and 'position' y, where both x and y are vectors of the same dimension N. So, the system is completely described by the 2N dimensional vector

$$r = (x, y),$$

and the evolution equation is given by the so-called Hamilton's equations:

$$\dot{x} = \frac{dx}{dt} = +\frac{\partial H}{\partial y}, \tag{2.2}$$

$$\dot{y} = \frac{dy}{dt} = -\frac{\partial H}{\partial x}. \tag{2.3}$$

A trajectory r(t) is a solution of the initial value problem defined by the Hamilton's equations and the initial condition $r(0) = r_0 \in \mathbb{R}^{2N}$.

The Hamiltonian function is a first integral of the system. Indeed,

$$\frac{d}{dt}H(x(t), y(t)) = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial y} \cdot \dot{y} = \frac{\partial H}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial H}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} - \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x} = 0.$$

3 Local normal forms

3.1 Local Theorem

We start by presenting and discussing our main results at the local level. Assume f is a holomorphic function in a punctured neighborhood of $z = 0$, i.e. $z = 0$ is either a regular point, a singular point, a pole, or an essential singularity.

We start with the notation of the conformal conjugacy (see [4]).

For a given z close to 0, we denote by $\varphi_f(t, z)$ the solution of equation (1.1) passing through z at $t = 0$.

Definition 3.1. If $f(0) = g(0) = 0$ we say that $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are **conformally conjugated** near the origin if there exists a conformal function $\Phi : U \to V$, where U and V are two open neighborhood of the origin, such that $\Phi(0) = 0$ and

$$\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z)), \qquad z \in U$$

and for all t for which the above expressions are well defined and the corresponding point are in U and V.

Remark 3.2. The notion of conformal conjugacy is a stronger notion than topological conjugacy. In particular, under conformal conjugacies the angles in the tangent space are preserved.

Now let us see some properties of conformal conjugacy :

First we give a useful characterization and establish two necessary conditions for equation $\dot{z} = f(z)$ and $\dot{z} = g(z)$ to be conformally conjugated near the origin.

Lemma 3.3. The equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated near the origin if and only if there exists a conformal function $\Phi : U \to V$, where U and V are two open neighborhoods of the origin, such that $\Phi(0) = 0$ and

$$\Phi'(z)f(z) = g(\Phi(z)), \qquad z \in U.$$

Proof. Suppose $\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z))$ holds. Then differentiating and using that φ_f and φ_g satisfy the corresponding differential equations we obtain

$$\Phi'(\varphi_f(t, z))f(\varphi_f(t, z)) = g(\varphi_g(t, \Phi(z))) = g(\Phi(\varphi_f(t, z))).$$

Taking $t = 0$ we obtain the conclusion.

Conversely, let us assume $\Phi'(z)f(z) = g(\Phi(z))$. Then

$$(\frac{d}{dt})(\Phi(\varphi_f(t, z))) = \Phi'(\varphi_f(t, z)))f(\varphi_f(t, z)) = g(\Phi(\varphi_f(t, z))).$$

Therefore $\Phi(\varphi_f(t, z))$ and $\varphi_g(t, \Phi(z))$ satisfy the same initial value problem, so that $\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z))$ follows from uniqueness.

□

Lemma 3.4. Let $f(z)$ and $g(z)$ be two analytic functions in some punctured neighborhood of the origin. If the corresponding equations $\dot{z} = f(z)$ and $\dot{z} = g(z)$ are conformally conjugated near the origin, then
(a) $\text{Res}(\frac{1}{f}, 0) = \text{Res}(\frac{1}{g}, 0)$,
(b) the order of vanishing of f and g at the origin coincide, if one of them is finite.

Proof. (a) There exists $\Phi : U \to V$ such that $\Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z))$ and by the lemma, $\Phi'(z)f(z) = g(\Phi(z))$ holds.

To prove (a), we choose $\epsilon > 0$ and define γ a small circle around the origin contained in U

$$\gamma(t) = \epsilon e^{it}, \qquad 0 \leq t \leq 2\pi.$$

Then, if ϵ is sufficiently small, we have

$$\text{Res}(\frac{1}{g}, 0) = \frac{1}{2\pi i} \int_{\Phi(\gamma)} \frac{dw}{g(w)} = \frac{1}{2\pi i} \int_{\gamma} \frac{\Phi'(z)dz}{g(\Phi(z))},$$

since $w = \Phi(z) \Rightarrow dw = \Phi'(z)dz$ and this is equal to

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{f(z)},$$

since f, g are conformally conjugated, and this is also equal to

$$\text{Res}(\frac{1}{f}, 0).$$

This proves (a).

(b) To see the equalality of the order of vanishing at the origin we just check the orders in the expression $\Phi'(z)f(z) = g(\Phi(z))$. If Φ exists and is conformal, we have $\Phi(z) = O(z)$ and $\Phi'(z) = O(1)$. Thus, the orders of vanishing of f and g, if next finite, must be equal. □

We will use next theorem to study a holomorphic function in a punctured neighborhood of $z = 0$:

Theorem 3.5. *Let $f(z)$ be an analytic function in a punctured neighborhood of $p\,(p = 0)$. By shrinking this neighborhood of, if necessary, the corresponding equation $z' = f(z)$ is conformally conjugated, near $z = 0$, to*
(a) $z' = 1$, if $f(0) \neq 0$.
(b) $z' = f'(0)z$, if $z = 0$ is a zero of f of order 1.
(c) $z' = z^n - cz^{2n-1}$, where $c = \text{Res}(\frac{1}{f}, 0)$, if $z = 0$ is a zero of f of order $n \geq 1$.
(d) $z' = \frac{1}{z^n}$, if $z = 0$ is a pole of order n.

Before proving this theorem we need another lemma.

Lemma 3.6. Let u and v be two analytic functions in a punctured neighborhood of the origin such that $v(z) = o(u(z))$. Assume that

$$h(z,s) := (u(z) + sv(z)) \int_0^z \frac{v(\xi)}{(u(\xi) + sv(\xi))^2} d\xi, \tag{3.1}$$

defines, for all s∈[0,1], a holomorphic function in a whole neighborhood of the origin (including $z = 0$) and that $h(0,s) = 0$. Then the equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$ are conformally conjugated near the origin.

We will first introduce some properties of the Lie brackets we will use to prove Lemma 2.6.

Let

$$X = \begin{cases} \dot{z} = X_1(z,s) \\ \dot{s} = X_2(z,s) \end{cases}$$

and

$$Y = \begin{cases} \dot{z} = Y_1(z,s) \\ \dot{s} = Y_2(z,s) \end{cases}$$

be two systems of complex differential equation in $\mathbb{C}^2$. Denote by $\varphi(t; z_0, s_0)$ and $\phi(\tau; z_0, s_0)$ the solutions of systems X and Y satisfing $\varphi(0; z_0, s_0) = (z_0, s_0)$ and $\phi(t; z_0, s_0) = (z_0, s_0)$, respectively.

Definition 3.7. We say that systems X and Y **commute** if

$$\boxed{\phi(\tau; \varphi(t; z_0, s_0)) = \varphi(t; \phi(\tau; z_0, s_0))}$$

To obtain a useful tool for deciding whether the systems commute, the Lie bracket [X,Y] is introduced as follows

$$\boxed{[X,Y] = (DX)Y - (DY)X},$$

where D denotes the differential matrix. Specifically, it can be shown that two systems commute if and only if their corresponding Lie bracket vanishes (see [6]).

Proof of Lemma 2.6. In the assumptions of the lemma we consider the equations $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$. We want to prove that $\dot{z} = u(z)$ and $\dot{z} = u(z) + v(z)$ are conformally conjugated near the origin. We also build the system X as

$$X = \begin{cases} \dot{z} = u(z) + v(z) \\ \dot{s} = 0 \end{cases},$$

where $s \in [0,1]$, and system Y as

$$Y = \begin{cases} \dot{z} = H(z,s) \\ \dot{s} = 1 \end{cases},$$

where H is to be determined later on. The idea of the proof is to use the following diagram

$$\begin{bmatrix} \text{If X and Y commute} & \Longleftrightarrow & \phi(\tau; \varphi(t; z_0, s_0)) = \varphi(t; \phi(\tau; z_0, s_0)) \\ \Updownarrow & & \Updownarrow \\ [X,Y] = 0 & & \dot{z} = u(z), \dot{z} = u(z) + v(z) \text{ are conf. conj.} \end{bmatrix}$$

The holomorphic map (Φ in the definition) that conjugates both equations is given by the holomorphic solutions of system Y at 'time' $\tau = s = 1$.

We now show that under that condition that (3.1) is holomorphic there exists a holomorphic function H defined in the whole neighborhood of the origin such that $[X, Y] = 0$. We have that

$$[X, Y] = \begin{pmatrix} u_z + sv_z & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} H \\ 1 \end{pmatrix} - \begin{pmatrix} H_z & H_s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u + sv \\ 0 \end{pmatrix}.$$

We see that $[X, Y] = 0$ equivalent to the first order linear equation $(u_z + sv_z)H + v - H_z(u + sv) = 0$. Clearly, one of its solutions is $H(z, s) = h(z, s)$, where h is the holomorphic function given in the statement of the lemma. The fact that $h(0, s) = 0$ implies that the solution $z = 0$ of system Y in a neighborhood of the initial condition $z = 0$ is well defined for $\tau = 1$. $\square$

Now we can prove Theorem 3.5

Proof of Theorem 3.5. We consider each case separately. Independent proofs of this result can be found in [4] and [5].

Proof of (b) If we write $u(z) = f'(0)z$ and $v(z) = f(z) - f'(0)z = O(z^2)$, statement (b) is equivalent to showing that equation $\dot{z} = u(z)$ is conformally conjugate to equation $\dot{z} = u(z) + v(z)$ (near z=0).

Applying Taylor Theorem, we can write $f(z)$ as

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + ...$$

In this case $z = 0$ is a zero of order 1, this means $f(0) = 0$, so we can write

$$f(z) = f'(0)z + O(z^2).$$

In this case we can use Lemma 3.6. We easily check that u and v are two analytic functions in a punctured neighborhood of the origin such that $v(z) = o(u(z))$ and

$$h(z, s) := (u(z) + sv(z)) \int_0^z \frac{v(\xi)}{(u(\xi) + sv(\xi))^2} d\xi = (f'(0)z + sO(z^2)) \int_0^z \frac{O(\xi^2)}{(f'(0)\xi + sO(\xi^2))^2} d\xi,$$

defines a holomorphic map around $z = 0$ for all $s \in [0, 1]$ and $h(0, s) = 0$.

Proof of (c) This case follows in three steps

First we notice that the equations

$$\dot{z} = f(z) = az^n + a_1 z^m + O(z^{m+1}), \quad a \neq 0 \tag{3.2}$$

$$\dot{w} = w^n + b_1 w^m + O(w^{m+1}) \tag{3.3}$$

are conformally conjugated where $b_1 = a_1 \sqrt[n-1]{\frac{1}{a^{m-1}}}$ via the linear change of variable $z = \sqrt[n-1]{\frac{1}{a}}w$. Indeed, by Lemma 3.3 we check that $\Phi(w) = \sqrt[n-1]{\frac{1}{a}}w$ satisfies Φ

$$\Phi'(w)g(z) = f(\Phi(w)). \tag{3.4}$$

We write $f(z) = az^n + a_1z^m + O(z^{m+1})$ and $g(w) = w^n + b_1w^m + O(w^{m+1})$

$$\begin{aligned} RHS \quad of \quad (3.4) = f(\Phi(w)) &= a(\sqrt[n-1]{\frac{1}{a}}w)^n + a_1(\sqrt[n-1]{\frac{1}{a}}w)^m + O(z^{m+1}) \\ &= a(\frac{1}{a})^{\frac{n}{n-1}}w^n + a_1(\frac{1}{a})^{\frac{m}{n-1}}w^m + O(z^{m+1}) \\ &= (\frac{1}{a})^{\frac{1}{n-1}}w^n + a_1(\frac{1}{a})^{\frac{m}{n-1}}w^m + O(z^{m+1}), \end{aligned}$$

and

$$LHS \quad of \quad (3.4) = \Phi'(w)g(z) = \sqrt[n-1]{\frac{1}{a}}(w^n + b_1w^m + O(w^{m+1})) + O(z^{m+1}).$$

Using that $b_1 = a_1 \sqrt[n-1]{\frac{1}{a^{m-1}}}$, then

$$= \sqrt[n-1]{\frac{1}{a}}(w^n + a_1 \sqrt[n-1]{\frac{1}{a^{m-1}}}w^m + O(w^{m+1})) = (\frac{1}{a})^{\frac{1}{n-1}}w^n + a_1(\frac{1}{a})^{\frac{m}{n-1}}w^m + O(z^{m+1}).$$

and we see that RHS of (3.4)=LHS of (3.4).

We rename the variable of the second equation as z. We claim that equations

$$\dot{z} = z^n + a_m z^m + O(z^{m+1}) \tag{3.5}$$

$$\dot{w} = w^n + cw^{2n-1} + O(w^{2n}) \tag{3.6}$$

$n \leq m \leq 2n-1$, are conformally conjugated where c is Res$(1/f, 0)$. To see the claim, we consecutively apply the change of variable $z = w(1 + \alpha w^{m-n})$, where $\alpha = \frac{-a_m}{2n-m-1}$ to erase the mth degree monomial up to $m = 2n-2$:

$$\begin{aligned} z = w(1 + \alpha w^{m-n}) \Rightarrow \dot{z} &= \dot{w}(1 + \alpha w^{m-n}) + w((m+n)\alpha w^{m-n-1}\dot{w}) \\ \dot{z} &= \dot{w}(1 + \alpha w^{m-n} + (m+n)\alpha w^{m-n}) \\ \dot{z} &= \dot{w}(1 + w^{m-n}\alpha(1+m+n)). \end{aligned} \tag{3.7}$$

We substitute this into equation (3.5) and we have

$$\dot{z} = w^n(1 + \alpha w^{m-n})^n + a_m(w(1 + \alpha w^{m-n}))^m + O(w(1 + \alpha w^{m-n})^{m+1}). \tag{3.8}$$

From (3.7) we get

$$\dot{w} = \frac{\dot{z}}{(1 + w^{m-n}\alpha(1+m+n))}.$$

And using equation (3.8), then

$$\dot{w} = \frac{w^n(1+\alpha w^{m-n})^n + a_m(w(1+\alpha w^{m-n}))^m}{(1+w^{m-n}\alpha(1+m+n))}$$

$$= \frac{1}{(1+w^{m-n}\alpha(1+m+n))}(w^n(1+\alpha w^{m-n})^n + a_m(w(1+\alpha w^{m-n}))^m).$$

We know that $\frac{1}{1+x} \approx 1 - x + x^2$ so we can write the above equation, using $x = w^{m-n}\alpha(1+m+n)$), in the form

$$= (1 - w^{m-n}\alpha(1+m+n))(w^n(1+\alpha w^{m-n})^n + a_m(w(1+\alpha w^{m-n}))^m).$$

We use the binomial Newton's formula $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + ...$

$$= (1-w^{m-n}\alpha(1+m+n))(w^n(1+n\alpha w^{m-n}+\beta\alpha^2 w^{2(m-n)}+...)+a_m w^m(1+m\alpha w^{m-n}+\delta\alpha^2 w^{2(m-n)}+...)).$$

As $2m - n \geq m+1$ and $3m - n \geq m+1$ we can cut this higher order part, so we have

$$= (1 - w^{m-n}\alpha(1+m+n))(w^n + n\alpha w^m + a_m w^m) = w^n + w^m(n\alpha + a_m + \alpha(1+m-n)) + O(w^{m+1}).$$

From the condition that the coefficient of w^m is zero, we can get the value of α

$$n\alpha + a_m + \alpha(1+m-n) = 0 \to \alpha = \frac{-a_m}{2n-m-1}.$$

The monomial w^{2n-1} cannot be erased since it is a resonant term. Moreover, using Lemma 3.4 it is easy to prove that the coefficient of this monomial coincides with c. Hence, again renaming as z the variable of the equation, we assume that the initial equation becomes $\dot{z} = z^n - cz^{2n-1} + O(z^{2n})$.

We have arrived to equation $\dot{z} = g(z)$ with $g(z) = z^n - a_{2n-1}z^{2n-1} + O(z^{2n})$ and we know that $\mathrm{Res}(\frac{1}{g}, 0) = -a_{2n-n}$ so $c = -a_{2n-n}$.

Finally, we show that equations

$$\dot{z} = z^n + cz^{2n-1} + O(z^{2n}) \tag{3.9}$$

$$\dot{z} = z^n + cz^{2n-1} \tag{3.10}$$

are conformally conjugated. To do so, we apply Lemma 3.6, with $u(z) = z^2 - cz^{2n-1}$ and $v(z) = O(z^{2n})$. We have

$$h(z,s) := z^2(1 - cz^{n-1} + sO(z^n)) \int_0^z \frac{\xi^{2n}(1+O(\xi^{n+1}))}{\xi^2(1-c\xi^{n-1}+sO(\xi^n))^2} d\xi$$

$$= z^2(1 - cz^{n-1} + sO(z^n)) \int_0^z \frac{\xi^{2n-2}(1+O(\xi^{n+1}))}{(1-c\xi^{n-1}+sO(\xi^n))^2} d\xi,$$

which defines a holomorphic function, for all $s \in [0,1]$, in a neighborhood of the origin. Again $h(0,s) = 0$ and (c) follows.

Proof of (a) and (d)
To unify the regular case and the case when f has a pole we consider $n \in \mathbb{N} \cup \{0\}$. As in the previous case, it is easy to see that equations

$$\dot{z} = f(z) = az^{-n} + O(z^{1-n}) \tag{3.11}$$
$$\dot{z} = z^{-n} + O(z^{1-n}) \tag{3.12}$$

are conformally conjugated via a linear change of variable.

We look for Φ such that $\Phi'(w)f(z) = g(\Phi(w))$ with $f(z) = az^{-n}$ and $g(z) = z^{-n}$. We have

$$\Phi'(z)az^{-n} = (\Phi(z))^{-n} \Rightarrow \Phi^n(z)\Phi'(z) = \frac{1}{a}z^n.$$

We integrate both sides

$$\int \Phi^n(z)\Phi'(z) = \int \frac{1}{a}z^n,$$

and we have the formula of the function $\Phi(z) = \sqrt[n+1]{\frac{1}{a}}z$

Now we claim that equations

$$\dot{z} = z^{-n} + O(z^{1-n}) \tag{3.13}$$
$$\dot{z} = z^{-n} \tag{3.14}$$

are conformally conjugated. To prove this claim, we use Lemma 3.6, with $u(z) = z^{-n}$ and $v(z) = O(z^{1-n})$. Precisely, equation (3.1) reads now

$$h(z,s) := (z^{-n} + sO(z^{1-n})) \int_0^z \frac{O(\xi^{1-n})}{(\xi^{-n} + sO(\xi^{1-n}))^2} d\xi = (z^{-n} + sO(z^{1-n})) \int_0^z O(\xi^{n+1}) d\xi,$$

which implies that $h(z,s) = O(z^2)$ is holomorphic in a sufficiently small neighborhood of the origin and $h(0,s) = 0$.

The particular case $n = 0$ proves (a). □

3.2 Local classification of singular points

In this subsection we use Theorem 3.5 to obtain the phase portraits of $\dot{z} = f(z)$ in a punctured neighborhood of a singular point ($f(0) = 0$) or a singularity given by a pole.

Corollary 3.8. Let $f(z)$ be an analytic function in a punctured neighborhood of p
(a) If $f'(p) \neq 0$, according to $\Re(f'(p)) < 0$, $\Re(f'(p)) > 0$ or $\Re(f'(p)) = 0$, the phase portrait of equation $\dot{z} = f(z)$ in a neighborhood of p is a stable focus, an unstable focus or an isochronous center, respectively. In all cases, the index of p is 1.
(b) If p is a zero of f of order $n > 1$, then the phase portrait of equation $\dot{z} = f(z)$ in a neighborhood of p is the union of $2(n-1)$ elliptic sectors, and so the index of p is n.
(c) If p is a pole of f of order $n \geq 1$, then the phase portrait of equation $\dot{z} = f(z)$ in a neighborhood of p is the union of $2(n+1)$ hyperbolic sectors, and so the index of p is $-n$.

Proof. Statement (a) follows from Theorem 3.5(b), since equation (1.1) near p is conformally conjugated to its linear part, $\dot{z} = f'(p)z$

$$\begin{cases} if \quad \Re(f'(p)) = 0, \quad \dot{z} = i\beta z, \beta \in \mathbb{R}, \\ if \quad \Re(f'(p)) < 0, \quad \dot{z} = (\alpha + i\beta)z, \alpha, \beta \in \mathbb{R}, \\ if \quad \Re(f'(p)) > 0, \quad \dot{z} = (\alpha + i\beta)z, \alpha, \beta \in \mathbb{R}. \end{cases}$$

Moreover, if $\Re(f'(p)) = 0$, $\dot{z} = i\beta z, \beta \in \mathbb{R}$ and $z = \gamma e^{i\theta}$

$$z = \gamma e^{i\theta} \Rightarrow \dot{z} = \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i\dot{\theta} \Rightarrow i\beta\gamma e^{i\theta} = \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i\dot{\theta} \Rightarrow \dot{\gamma} + \gamma i\dot{\theta} = i\beta\gamma.$$

Then the equation reads, in polar coordinates

$$\dot{\gamma} = 0, \quad \dot{\theta} = \beta.$$

Then the topological phase portrait in a neighborhood of the origin is given by Figure 3.1.

If $\Re(f'(p)) \neq 0$, $\dot{z} = (\alpha + i\beta)z, \alpha, \beta \in \mathbb{R}$.

$$z = \gamma e^{i\theta} \Rightarrow \dot{z} = \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i\dot{\theta} \Rightarrow \dot{\gamma} + \gamma i\dot{\theta} = (\alpha + i\beta)\gamma e^{i\theta},$$

then

$$\dot{\gamma} = \alpha\gamma, \quad \dot{\theta} = \beta.$$

Then the topological phase portraits in a neighborhood of the origin are given by Figure 3.1.

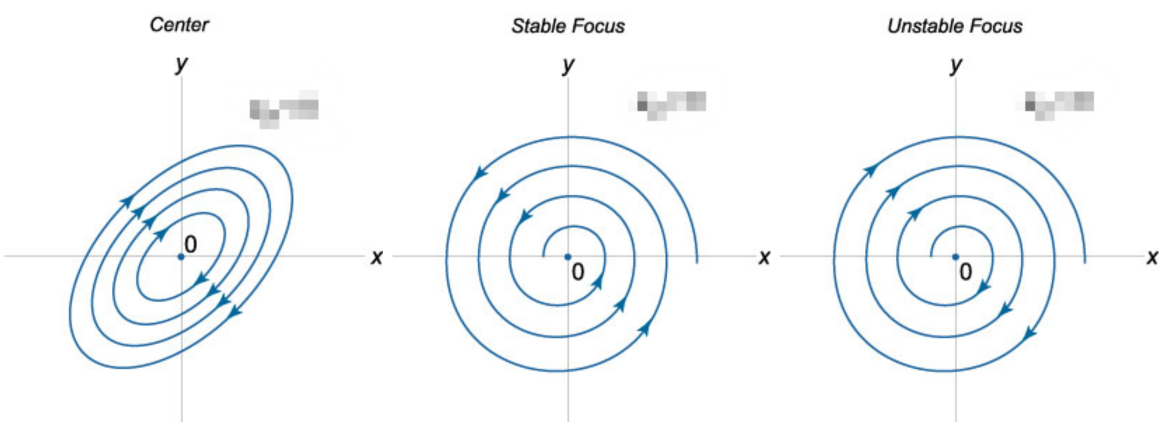

Figure 3.1: a center ($\alpha = 0, \beta < 0$), a stable focus ($\alpha < 0, \beta > 0$)and an unstable focus ($\alpha > 0, \beta < 0$).

Statements (b) and (c) follow by performing the polar $blow-up$ $z = \gamma e^{i\theta}$ and a recalling of time (see [1]).

We use a basic tool for studying not elementary singularities of a differential system in the plane. This tool is based on changes of variables called $blow-ups$. $Blow-ups$ are used to show that at isolated singularities an analytic system has a finite sectorial decomposition.

(b) We limit ourselves to the simple case $n = 2$, $\dot{z} = z^2$. The polar blow-up $z = \gamma e^{i\theta} \Rightarrow \dot{z} = \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i \dot{\theta}$ and

$$z^2 = \gamma^2 e^{2i\theta} \Rightarrow \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i \dot{\theta} = \gamma^2 e^{i\theta} (\cos(\theta) + i \sin(\theta)).$$

Comparing real and imaginary parts

$$\begin{cases} \dot{\gamma} = \gamma^2 \cos(\theta) \\ \dot{\theta} = \gamma \sin(\theta). \end{cases}$$

Then neighborhood of p is the union of $2(n-1) = 2(2-1) = 2$ elliptic sectors, and so the index of p is 2.

Then the topological phase portrait in a neighborhood of the origin is given by Figure 3.2.

(c) If p is a pole of f of order $n \geq 1 \Rightarrow \dot{z} = f(z)$ is conformally conjugated to $\dot{z} = \frac{1}{z^n}$.

We limit ourselves to the simple case $n = 2$, $\dot{z} = \frac{1}{z^2}$. The polar blow-up $z = \gamma e^{i\theta} \Rightarrow \dot{z} = \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i \dot{\theta}$ and

$$\frac{1}{z^2} = \frac{1}{\gamma^2 e^{2i\theta}} \Rightarrow \dot{\gamma} e^{i\theta} + \gamma e^{i\theta} i \dot{\theta} = \frac{1}{\gamma^2 e^{2i\theta}} \Rightarrow \begin{cases} \dot{\gamma} = \gamma^{-2} \cos(-3\theta) \\ \dot{\theta} = \gamma^{-3} \sin(-3\theta). \end{cases}$$

Then neighborhood of p is the union of $2(n+1) = 2(2+1) = 6$ hyperbolic sectors, and so the index of p is -2.

Then the topological phase portraits in a neighborhood of the origin are given by Figure 3.2.

□

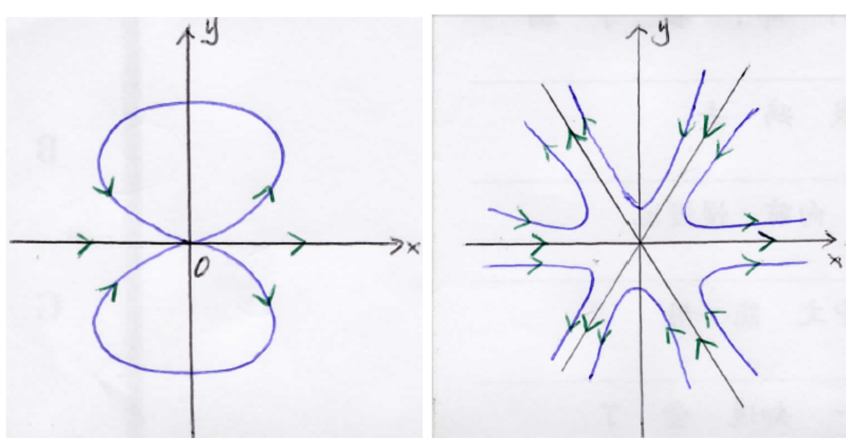

Figure 3.2: elliptic sectors and hyperbolic sectors

4 The dynamics at infinity

4.1 Global Theorem

Using mainly Theorem 3.5, we also prove the following result for the conformally conjugacy classes in $\mathbb{C}\backslash\overline{\mathbb{D}(0,R)}, R > 1$, i.e. in a sufficiently small neighborhood of infinity. We study the phase portrait at infinity at the topological level.

To better understand the global phase portrait of equation $\dot{z} = f(z)$ we need to study the behavior of the orbits in a neighborhood of infinity i.e. we need to know how the orbits escape to infinity, if indeed they do.

The main result is summarized in Theorem 4.1.

Theorem 4.1. *Let $f(z)$ be a rational function, i.e., $f(z) = \frac{P(z)}{Q(z)}$, where $P(z) = a_n z^n + ... + a_0$ and $Q(z) = b_m z^m + ... + b_0$ are polynomials in z of degree n and m, respectively. Let c be the residue of $g(z) = -\frac{Q(\frac{1}{z})}{z^2 P(\frac{1}{z})}$. Then, there exists $R > 0$ such that the corresponding equation $z' = f(z)$ is conformally conjugated, in $\mathbb{C}\backslash\overline{\mathbb{D}(0,R)}$, to*
(a) $\dot{z} = (\frac{1}{z})^{m-n} + c(\frac{1}{z})^{2(m-n)+1}$, if $n \leq m+1$,
(b) $\dot{z} = -(\frac{a_n}{b_m})z$, if $n = m+1$,
(c) $\dot{z} = z^2$, if $n = m+2$,
(d) $\dot{z} = z^{n-m}$, if $n \geq m+2$.

In the context of planar (real) vector fields, there are two natural compactification that allow us to extend the flow to infinity under certain hypotheses (see [7]).

1 Riemann Compactification: infinity is represented by the north pole of the Riemann sphere.

2 Poincaré Compactification: infinity is represented by the equator of the Poincaré sphere, which is invariant by the extension of the flow projected on the sphere.

In this second case, we have two copies of the flow, one on each hemisphere. By direct projection, the vector field defined in the north hemisphere of the Poincaré sphere can be drawn in the so-called Poincaré disc (as we explain in the introduction), where infinity now is given by the boundary $\mathbb{S}^1$ of the Poincaré disc, which corresponds to the blow-up of the north pole of Riemann sphere.

Here, we can consider a similar approach but taking into account the complex structure of equation $\dot{z} = f(z)$. It is easy to see that infinity can be formally interpreted by $z = \infty$ (the north pole of the Riemann sphere), so it can be directly studied by considering the change of variables $w = \frac{1}{z}$. Thus $z = \infty$ for equation $\dot{z} = f(z)$ becomes $w = 0$ in the new variable. To study the dynamics (and normal forms) around $w = 0$, we use Theorem 3.5. Finally, using the inverse change of variables $z = \frac{1}{w}$, we will find the normal form of the original equation $\dot{z} = f(z)$ in a neighborhood of infinity.

Proof. We consider equation $\dot{z} = f(z)$ and we assume that f is a rational function, i.e.

$$\dot{z} = \frac{P(z)}{Q(z)} = \frac{a_n z^n + \cdots + a_0}{b_m z^m + \cdots + b_0}, \qquad a_n b_m \neq 0. \tag{4.1}$$

After doing the change of variables $w = \frac{1}{z} \Longrightarrow \dot{z} = (\frac{1}{w})' = -\frac{1}{w^2}\dot{w}$, we obtain $\dot{w} = -w^2 \dot{z}$. Then, we have

$$\dot{w} = -w^2 \frac{P(\frac{1}{w})}{Q(\frac{1}{w})} = -w^2 \frac{a_n(\frac{1}{w})^n + \cdots + a_0}{b_m(\frac{1}{w})^m + \cdots + b_0} = -w^{2+m-n} \frac{a_n + \cdots + a_0 w^n}{b_m + \cdots + b_0 w^m} = -w^{2+m-n}(\frac{a_n}{b_m} + O(w)). \tag{4.2}$$

Clearly, the dynamics of equation (4.2) in a neighborhood of $w = 0$ represents the dynamics of equation (4.1) in the neighborhood of $z = \infty$. From Theorem 3.5 we know that, in some open neighborhood, say $D(0, \epsilon_0) = \{w \in \mathbb{C}, |w| < \epsilon_0\}$ of $w = 0$, the normal form for equation (4.2) is given, respectively, by

(a) $\dot{w} = w^{2+m-n} + cw^{2(2+m-n)}$, if $n \leq m+1$,
(b) $\dot{w} = (\frac{a_n}{b_m})w$, if $n = m+1$,
(c) $\dot{w} = 1$, if $n = m+2$,
(d) $\dot{w} = w^{-n+m+2}$, if $n \geq m+2$,

where c is the residue at the origin of the origin of function $g(w) = -\frac{Q(1/w)}{w^2 P(1/w)}$. We know that there is a conformal conjugacy Φ between (4.2) and its corresponding normal form.

We now go back to the origin of the z-variable by applying the inverse change of variable

$$\tilde{z} = \frac{1}{w} \Rightarrow w = \frac{1}{\tilde{z}} \Rightarrow \dot{w} = -\frac{1}{\tilde{z}^2}\dot{\tilde{z}} \Rightarrow \dot{\tilde{z}} = -\tilde{z}^2 \dot{w}$$

to the above normal forms. Consequently, in a suitable neighborhood of infinity (precisely, $\mathbb{C} \backslash \overline{D(0,R)}$, with $R = 1/\epsilon_0$), the normal form of equation (4.2) becomes

(a) $\dot{\tilde{z}} = -\tilde{z}^2 \dot{w} = -\tilde{z}^2(w^{2+m-n} + cw^{2(2+m-n)}) = -\tilde{z}^2({\frac{1}{\tilde{z}}}^{2+m-n} + c{\frac{1}{\tilde{z}}}^{2(2+m-n)}) = -\tilde{z}^{n-m} - c(\tilde{z})^{2(n-m)-1}$, if $n \leq m+1$,

(b) $\dot{\tilde{z}} = -\tilde{z}^2 \dot{w} = -\tilde{z}^2(\frac{a_n}{b_m})w = -\tilde{z}^2(\frac{a_n}{b_m})\frac{1}{\tilde{z}} = -(\frac{a_n}{b_m})\tilde{z}$, if $n = m+1$,

(c) $\dot{\tilde{z}} = -\tilde{z}^2 \dot{w} = -\tilde{z}^2 \cdot 1 = -\tilde{z}^2$, if $n = m+2$,

(d) $\dot{\tilde{z}} = -\tilde{z}^2 \dot{w} = -\tilde{z}^2(w^{-n+m+2}) = -\tilde{z}^2({\frac{1}{\tilde{z}}}^{-n+m+2}) = -\tilde{z}^{n-m}$, if $n \geq m+2$.

Furthermore, the linear change of variable $z = \lambda \tilde{z}$ (with $\lambda = (-1)^{1/(m-n+1)}$, $\lambda = 1$, $\lambda = -1$, and $\lambda = (-1)^{1/(1-m+n)}$ depending on the cases (a), (b), (c) and (d)) gives the normal forms stated in the theorem. Indeed

$$z = \lambda \tilde{z} \Rightarrow \dot{z} = \lambda \dot{\tilde{z}} \quad \text{and} \quad \tilde{z} = \frac{z}{\lambda}.$$

(a) If $\lambda = (-1)^{1/(m-n+1)} \Rightarrow \dot{z} = \lambda\dot{\tilde{z}} = (-1)^{1/(m-n+1)}(-\tilde{z}^{n-m} - c(\tilde{z})^{-2(1+m-n)}) = (-1)^{1/(m-n+1)}(-\frac{z}{\lambda}^{n-m} - c(\frac{z}{\lambda})^{2(n-m)}) = (\frac{1}{z})^{m-n} + c(\frac{1}{z})^{2(m-n)+1}$, if $n \leq m+1$,

(b) If $\lambda = 1 \Rightarrow \dot{z} = \lambda\dot{\tilde{z}} = -(\frac{a_n}{b_m})\tilde{z} = (-\frac{a_n}{b_m})z$, if $n = m+1$,

(c) If $\lambda = -1 \Rightarrow \dot{z} = \lambda\dot{\tilde{z}} = \tilde{z}^2 = z^2$, if $n = m+2$,

(d) If $\lambda = (-1)^{1/(1-m+n)} \Rightarrow \dot{z} = \lambda\dot{\tilde{z}} = (-1)^{1/(1-m+n)}\tilde{z}^{n-m} = z^{n-m}$, if $n \geq m+2$.

Finally, we observe that the above normal forms and the original equation (3.2) are conformally conjugated on $\mathbb{C}\backslash\overline{D(0,R)}$ by the conjugacy under the map $H(z) = \lambda/\Phi(1/z)$. □

To end this section we particularize the above result for a concrete family of rational systems for which we will study, their global phase portrait.

4.2 Classification of singular points at infinity

Corollary 4.2. Consider equation (4.1), where $P(z)$ and $Q(z)$ are polynomials of one complex variable with deg(P)≤ 2 and deg(Q)≤ 1. Its phase portrait in a neighborhood of infinity is conformally conjugated to the phase portrait in a neighborhood of infinity of one of the following normal forms
(a) $\dot{z} = 1$ if $\deg(P) = 0$ and $\deg(Q) = 0$. (see Figure 4.1)
(b) $\dot{z} = kz$ if $\deg(P) = 1$ and $\deg(Q) = 0$, or, $\deg(P) = 2$ and $\deg(Q) = 1$, depending on whether $k = \alpha$ (see Figure 4.2(b1)), $k = i\beta$ (see Figure 4.2 (b2)) or $k = \alpha + i\beta$ (see Figure 4.2 (b3))
(c) $\dot{z} = z^2$ if $\deg(P) = 2$ and $\deg(Q) = 0$.(see Figure 4.3)
(d) $\dot{z} = 1 + \frac{c}{z}$ if $\deg(P) = 1$ and $\deg(Q) = 1$.(see Figure 4.1)
(e) $\dot{z} = \frac{1}{z} + c\frac{1}{z}^3$ if $\deg(P) = 0$ and $\deg(Q) = 1$.(see Figure 4.4)

Proof. We use the Poincaré Compactification :

(a) $\dot{z} = 1 \Rightarrow \dot{x} + i\dot{y} = 1$ and we can write that

$$\begin{cases} \dot{x} = 1 = P(x,y) \\ \dot{y} = 0 = Q(x,y) \end{cases}.$$

We recall that the degree of X is d if d is the maximum of the degrees of P and Q, so we have that $d = 0$.

We compute the singular points in the chart U_1.

$F(u) = Q_d(1,u) - uP_d(1,u) = 0 - u \cdot 1 = -u \Rightarrow u = 0$. It means that there is a singular point at infinity on U_1.

Then we compute singular points in the chart U_2.

$G(u) = P_d(u,1) - uQ_d(u,1) = 1 - u \cdot 0 = 1 \neq 0$ its means that there are not singular points at infinity on U_2.

We study the character of $u = 0$ in U_1:

$$F'(u) = -1$$
$$Q_{d-1}(1,u) - uP_{d-1}(1,u) = 0$$
$$-P_d(1,u) = -1,$$

$$A(u) = \begin{pmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{pmatrix} \Rightarrow A(0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clear this is only one eigenvalue is -1 (double). Hence the point is stable node. Since $d = 0$ is even there is another point which is an unstable node.

Then the topological phase portrait in a neighborhood of infinity is given by Figure 4.1.

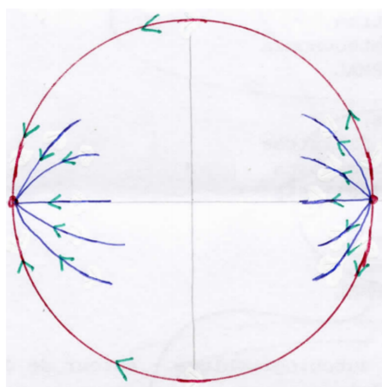

Figure 4.1: case (a) : $\dot{z} = 1$

(b) $\dot{z} = kz$ we have 3 types depending on the value of $k \in \mathbb{C}$.

(b1) if $\Im(k) = 0 \Rightarrow k = \alpha \in \mathbb{R}$ and then $\dot{z} = \dot{x} + i\dot{y} = \alpha(x + iy)$ and we can write that

$$\begin{cases} \dot{x} = \alpha \cdot x = P(x,y) \\ \dot{y} = \alpha \cdot y = Q(x,y) \end{cases}.$$

In this case we have d=1.

$$\begin{cases} F(u) = Q_d(1,u) - uP_d(1,u) = \alpha \cdot u - u \cdot \alpha = 0 \\ G(u) = P_d(u,1) - uQ_d(u,1) = \alpha \cdot u - u \cdot \alpha = 0 \end{cases}$$

Then all of points at infinity are singular points.

Then the topological phase portrait in a neighborhood of the infinity is given by Figure 4.2 when $\alpha < 0$.

When α positive we have the same phase portrait with the signs of the arrows changed.

(b2) if $\Re(k) = 0 \Rightarrow k = i\beta$ and $\beta \in \mathbb{R}$ and $\beta \neq 0$ then $\dot{z} = \dot{x} + i\dot{y} = i\beta(x+iy) = i\beta x - \beta y$ and we can write that

$$\begin{cases} \dot{x} = -\beta \cdot y = P(x,y) \\ \dot{y} = \beta \cdot x = Q(x,y) \end{cases}.$$

In this case we have d=1 and we compute:

$$\begin{cases} F(u) \equiv Q_d(1,u) - uP_d(1,u) = \beta + u \cdot \beta u = \beta(1+u^2) \neq 0 \\ G(u) \equiv P_d(u,1) - uQ_d(u,1) = -\beta - u \cdot \beta u = -\beta(1+u^2) \neq 0 \end{cases}$$

Then we do not have singular points in U_1 and U_2 at infinity. Then infinity is a cycle.

Then the topological phase portrait in a neighborhood of infinity is given by Figure 4.2 when $\beta > 0$.

(b3) if $\Re(k)\Im(k) \neq 0 \Rightarrow k = \alpha + i\beta$ and, $\alpha, \beta \in \mathbb{R}$ and, $\alpha, \beta \neq 0$ then $\dot{z} = \dot{x} + i\dot{y} = (\alpha + i\beta)(x+iy) = \alpha x + \alpha i y + i\beta x - \beta y = (\alpha x - \beta y) + i(\alpha y + \beta x)$ and we can write that

$$\begin{cases} \dot{x} = \alpha x - \beta y = P(x,y) \\ \dot{y} = \alpha y + \beta x = Q(x,y) \end{cases}.$$

Now d=1, and we can compute:

$$\begin{cases} F(u) \equiv Q_d(1,u) - uP_d(1,u) = \alpha u + \beta - u \cdot (\alpha - \beta u) = \alpha u + \beta - u\alpha + \beta u^2 = \beta(1+u^2) \neq 0 \\ G(u) \equiv P_d(u,1) - uQ_d(u,1) = \alpha u - \beta - u \cdot (\alpha + \beta u) = \alpha u - \beta - u\alpha - \beta u^2 = -\beta(1+u^2) \neq 0 \end{cases}$$

Then we do not have singular points in U_1 and U_2 at infinity. Then infinity is a cycle.

Then the topological phase portrait in a neighborhood of infinity is given by Figure 4.2 when $\beta > 0$ and $\alpha > 0$.

For both (b2) and (b3), infinity is a cycle. To distinguish (b2) and (b3) we make the change of variable $w = \frac{1}{z}$ in the original equation. The transformed equation is $\dot{w} = -kw$. From Corollary 3.8(a) we know that if $f'(p) \neq 0$, according to $\Re(f'(p)) < 0$, $\Re(f'(p)) > 0$ or $\Re(f'(p)) = 0$, then the phase portrait of equation $\dot{z} = f(z)$ in a neighborhood of p is a stable focus, an unstable focus or an isochronous center, respectively. In all cases, the index of p is 1.

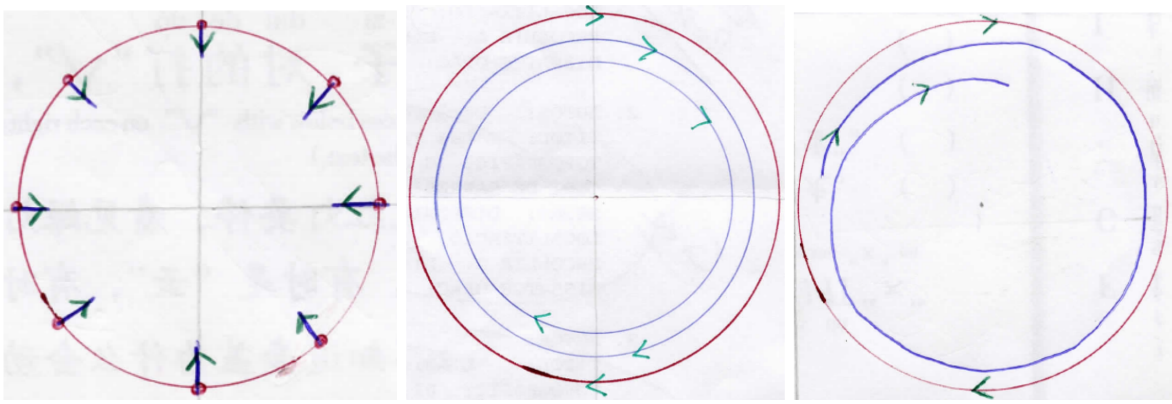

Figure 4.2: case (b1), (b2) and (b3)

Then we get (b1) is a center and (b3) is a focus.

(c) $\dot{z} = z^2$. In this case:

$\dot{z} = \dot{x} + i\dot{y} = (x+iy)^2 = x^2 - y^2 + 2ixy$ and we can write that

$$\begin{cases} \dot{x} = x^2 - y^2 = P(x,y) \\ \dot{y} = 2xy = Q(x,y) \end{cases}.$$

Now $d = 2$, and we compute:

$$\begin{cases} F(u) \equiv Q_d(1,u) - uP_d(1,u) = 2u - u\cdot(1-u^2) = 2u - u + u^3 = u + u^3 = u(u^2+1) \\ G(u) \equiv P_d(u,1) - uQ_d(u,1) = u^2 - 1 - u\cdot 2u = -1 - u^2 \neq 0 \end{cases}$$

$$\begin{cases} F(u) = 0 \Rightarrow u(u^2+1) = 0 \Rightarrow u = 0 \\ G(u) \neq 0 \end{cases}$$

We obtain the singular point $u = 0$ in U_1:

$$\begin{aligned} &F'(u) = 1 + 3u^2, \\ &Q_{d-1}(1,u) - uP_{d-1}(1,u) = 0, \\ &-P_d(1,u) = -1 + u^2. \end{aligned}$$

Moreover

$$A(u) = \begin{pmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{pmatrix} \Rightarrow A(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since matrix A(0) is diagonal then the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, hence the singular points at infinity is a saddle.

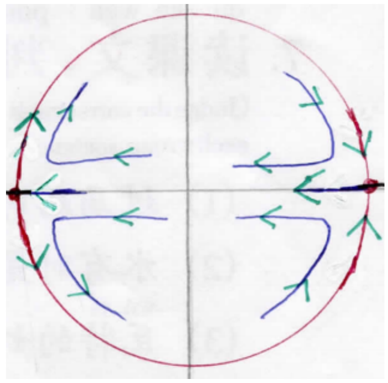

Figure 4.3: case (c):$\dot{z} = z^2$

Then the topological phase portrait in a neighborhood of the infinity is given by Figure 4.3.

(d)$\dot{z} = 1 + \frac{c}{z}$
$\dot{z} = \dot{x} + i\dot{y} = 1 + \frac{c}{x+iy} = 1 + \frac{c(x-iy)}{x^2+y^2}$ and we can write

$$\begin{cases} \dot{x} = 1 + \frac{cx}{x^2+y^2} \\ \dot{y} = -\frac{cy}{x^2+y^2} \end{cases}.$$

We multiply the vector field by $x^2 + y^2$ and we obtain a equivalent polynomial vector field,

$$\begin{cases} \dot{x} = x^2 + y^2 + cx = P(x, y) \\ \dot{y} = -cy = Q(x, y) \end{cases}.$$

In this case we have $d = 2$, we compute the singular points in the charts U_1 and U_2 :

$$\begin{cases} F(u) \equiv Q_d(1, u) - uP_d(1, u) = 0 - u \cdot (1^2 + u^2) = -u(1^2 + u^2) \\ G(u) \equiv P_d(u, 1) - uQ_d(u, 1) = 1^2 + u^2 - 0 \cdot u \end{cases} \begin{cases} if \quad F(u) = 0 \Rightarrow u = 0 \\ and \quad G(u) \neq 0 \end{cases}$$

We obtain the singular point $u = 0$ in U_1

$$F'(u) = -1,$$
$$Q_{d-1}(1, u) - uP_{d-1}(1, u) = -cu - u \cdot c,$$
$$-P_d(1, u) = -1.$$

And we compute:

$$A(u) = \begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix} = \begin{pmatrix} -1 & -2cu \\ 0 & -1 \end{pmatrix} \Rightarrow A(0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly there is only one eigenvalue which is -1 (double). Hence the point is a stable node. Since $d = 2$ is even another point is unstable node (as the same as (a)).

Then the topological phase portrait in a neighborhood of infinity is given by Figure 4.1.

(e)$\dot{z} = \frac{1}{z} + \frac{c}{z^3}$
$\dot{z} = \dot{x} + i\dot{y} = \frac{1}{x+iy} + \frac{c}{(x+iy)^3} = \frac{((x+iy)^2+c)(x-iy)^3}{(x^2+y^2)^3}$ and we can write

$$\begin{cases} \dot{x} = \frac{x^5-10x^3y^2-xy^4+cx^3-3cy^2}{(x^2+y^2)^3} \\ \dot{y} = \frac{-y^5+5x^4y+4x^2y^3+cy^3+3cx^2y}{(x^2+y^2)^3} \end{cases}$$

We multiply the vector field by $(x^2+y^2)^3$ and we get a equivalent polynomial vector field,

$$\begin{cases} \dot{x} = x^5 - 10x^3y^2 - xy^4 + cx^3 - 3cy^2 = P(x,y) \\ \dot{y} = -y^5 + 5x^4y + 4x^2y^3 + cy^3 + 3cx^2y = Q(x,y) \end{cases}$$

In this case we have $d = 5$. We compute the singular points in the charts U_1 and U_2 :

$$\begin{cases} F(u) \equiv Q_d(1,u) - uP_d(1,u) = -(1+u^2)^2 - u \cdot (1+u^2)2 = -2u(1+u^2)^2 \\ G(u) \equiv P_d(u,1) - uQ_d(u,1) = (u^2+1)^2 - u \cdot (u^2+1)^2 = 2u(1+u^2) \end{cases}$$

$$\begin{cases} if \quad F(u) = 0 \Rightarrow 4u(1+c) + 2u^3(7+2c) = 0 \Rightarrow u = 0 \\ and \quad G(u) = 0 \Rightarrow 4u(1+c) + 2u^3(7+2c) = 0 \Rightarrow u = 0 \end{cases}$$

We obtain the singular point $u = 0$ in U_1:

$$A(u) = \begin{pmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{pmatrix} \Rightarrow A(0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the singular point $u = 0$ in U_2:

$$B(u) = \begin{pmatrix} G'(u) & P_{d-1}(u,1) - uQ_{d-1}(u,1) \\ 0 & -Q_d(u,1) \end{pmatrix} \Rightarrow B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, we have 4 singulars points at infinity, two stable nodes two are unstable ones

Then the topological phase portrait in a neighborhood of infinity is given by Figure 4.4.

□

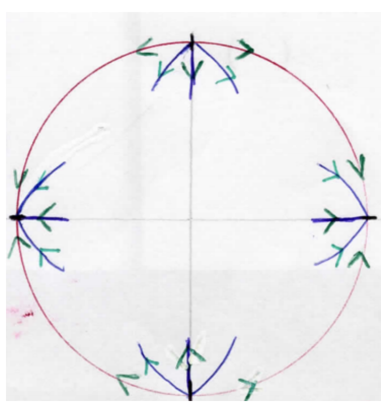

Figure 4.4: case (e) : $\dot{z} = \frac{1}{z} + \frac{c}{z^3}$

5 Limit cycles and polycycles

As usual, after studying the local phase portrait near a regular or singular point, the next step is to determine the topological behavior of the orbits in a neighborhood of a periodic orbit. The study of the phase portrait of equation (1.1) near a periodic orbit is a global, rather than a local, problem. A periodic orbit of equation (1.1) can be either in an annulus of periodic orbits or an isolated periodic orbit. In the later case we say that the periodic orbit is a limit cycle. A key discussion on planar (polynomial) differential equations is to know the existence (or not) and possible number of limit cycles.

The non-existence of limit cycles for equation (1.1) is studied by showing that such equation is a Hamiltonian-like differential equation. Precisely we will show that equation (1.1) has an integrating factor defined in a certain region of the plane which prevents the existence of isolated periodic orbits.

Even more, using this integrating factor we may also exclude the existence of graphs. We say that Γ is a graph of equation $\dot{z} = f(z)$ if it is a compact connected invariant set such that the α and ω-limit set of every point in the regular orbits of Γ is a singular point, a pole or an essential singularity of Γ. Note that the flow is not defined on the singularities associated with Γ. We say that Γ is a polycycle if it is a monodromic graph, i.e. a graph for which the Poincaré map is well defined on at least one of its half-neighborhoods. A **half-neighborhood** means a ring-shape open set with Γ or some subset of Γ as one of its boundaries. For instance, if Γ has a number eight shape, then we can construct 3 such half-neighborhoods.

In fact, it is known that for an analytic or a meromorphic function f, the equation $\dot{z} = f(z)$ has no limit cycles (see [8], [9]). An explanation of 'why' is the following. For analytic f, every periodic orbit of $\dot{z} = f(z)$ is contained in a continuous family of periodic orbits. Let C be a periodic orbit of period τ. We take an appropriate tubular neighborhood U of C in which the function f is holomorphic (there are no poles or essential singularities). The solutions of the differential equation as well as the Poincaré map P and the τ-time function T are therefore also analytic functions in U. Let $z_1 \in C$ and fix V such that $z_1 \in V \in U$. Since $T(z) = z$ for all points in $V \cap C$, then $T(z) = z$ in V. Consequently, $P(z) = z$. Therefore, there are no limit cycle and all periodic orbits near C have the same period.

We summarize this discussion in the following theorem.

Theorem 5.1. *Let equation $\dot{z} = f(z)$. The following statements hold.*

(a) There are no limit cycles. Moreover, in any neighborhood of a periodic orbit C in which the return map is defined, all orbits are periodic and have he same period as C. Furthermore, this period is

$$T = \int_C \frac{1}{f(z)} dz.$$

(b) If Γ is a monodromic graph, then in any ring-shape neighborhood of Γ in which the return map is defined, all orbits of the differential equation are periodic.

The proof of this Theorem 5.1 follows from some preliminary results and two propositions

Theorem 5.2 (Residues Theorem)**.** *Suppose U is a simply connected open subset of the complex plane, and $a_1, ..., a_n$ are finitely many points of U and f is a function which is defined and holomorphic on $U \setminus \{a_1, ..., a_n\}$. If γ is a rectifiable curve in U which does not meet any of the a_k, and whose start point equals its endpoint, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k) \operatorname{Res}(f, a_k).$$

If γ is a positively oriented simple closed curve, $I(\gamma, a_k) = 1$ if a_k is in the interior of γ, and 0 if not, so

$$\int_\gamma f(z)dz = 2\pi i \sum \operatorname{Res}(f, a_k)$$

with the sum over those k for which a_k is inside γ.

Proposition 5.3. Let f, g be vector fields on $U \in \mathbb{C}$ such that

$$\vartheta(x) := \det(f(x), g(x))$$

has no zeros on U. Then we have the following.

(a) $[g, f] = (\frac{L_g(\vartheta)}{\vartheta} - \operatorname{div}(g))f + (\frac{-L_f(\vartheta)}{\vartheta} - \operatorname{div}(f))g$.

(b) $[g, f] = \lambda f$ for some analytic λ if and only if ϑ^{-1} is an integrating factor for $\dot{x} = f(x)$.

Proof. Part (b) is an immediate consequence of part (a). For any $y \in U$, Cramer's rule show $[g, f](y) = \lambda f(y) + \mu g(y)$, with $\lambda = \frac{\det([g,f](y), g(y))}{\vartheta(y)}$, and a similar expression for μ.

(To a given differential equation $\dot{x} = f(x)$ on an open non-empty subset U of $\mathbb{C}^2$, one associates the derivation L_f, assigning to a function $\phi : U \to \mathbb{C}$ its Lie derivative $L_f(\phi)$, with $L_f(\phi)(x) := D\phi(x)f(x)$.)

In this case, we want to calculate $L_g(\vartheta)(x)$. We can write :

$$L_g(\vartheta)(x) = D\vartheta(x)g(x).$$

We know that $\vartheta(x) := \det(f(x), g(x))$ then:

$$L_g(\vartheta)(x) = D\vartheta(x)g(x) = D(\det(f(x), g(x)))g(x) = \det(Df(x)g(x), g(x)) + \det(f(x), Dg(x)g(x)) \quad (5.1)$$

while

$$\det([g, f](x), g(x)) = \det(Df(x)g(x), g(x)) - \det(Dg(x)f(x), g(x)) \quad (5.2)$$

Therefore from (6.1)-(5.2)

$$\begin{aligned} &L_g(\vartheta)(x) - \det([g, f](x), g(x)) \\ &= \det(Df(x)g(x), g(x)) + \det(f(x), Dg(x)g(x)) - \det(Df(x)g(x), g(x)) + \det(Dg(x)f(x), g(x)) \\ &= \det(f(x), Dg(x)g(x)) + \det(Dg(x)f(x), g(x)) = trDg(x)\vartheta(x) = \operatorname{div}(g(x))\vartheta(x). \end{aligned}$$

□

Proposition 5.4. Consider the differential equation $\dot{z} = f(z)$. The following holds:
(a) The above differential equation and $\dot{z} = if(z)$ commute, i.e. $[f, if] = 0$, whenever the function f is defined.
(b) Let C be a periodic orbit of period T. All orbits in a neighborhood of C are also periodic and have the same period T. Moreover

$$\int_C \frac{1}{f(z)} dz.$$

(c) The equation $\dot{z} = f(z)$ has $(f\bar{f})^{-1}$ as an integrating factor. That is, its phase portrait is topologically equivalent to the one of equation $\dot{z} = \frac{1}{f(z)}$ whenever the integrating factor is defined (outside zeroes and essential singularities of f).

Proof. (a) Let the differential equations $\dot{z} = f(z, \bar{z})$, $\dot{z} = g(z, \bar{z})$, where f and g are two analytic maps (not necessarily holomorphic). We recall the Lie bracket operator

$$[X, Y] = (DX)Y - (DY)X.$$

In $(z, \bar{z})$-coordinates we have

$$[f, g] = (Df)g - (Dg)f = f_z g + \bar{g} f_{\bar{z}} - \bar{f} g_{\bar{z}} - f g_z,$$

$$[f, g] = f g_z - f_z g + \bar{f} g_{\bar{z}} - \bar{g} f_{\bar{z}}, \tag{5.3}$$

where the subscripts denote the corresponding derivatives of f and g with respect to z and $\bar{z}$. Easily, if f and g are holomorphic, then

$$\bar{f} g_{\bar{z}} = \bar{g} f_{\bar{z}} = 0,$$

and (5.3) simplifies to

$$[f, g] = f g_z - f_z g.$$

Thus if $g = if$ it is straightforward that

$$[f, if] = f \cdot i f_z - f_z \cdot i f = 0,$$

whenever the function f is defined.

(b) Let γ be a periodic orbit and let A an annulus neighborhood of it (full of periodic orbits). The period function T defined in A gives the return time at each point in A. Using the analyticity of the map f in A it is clear that the period function is also an analytic map in A since T being is constant in γ implies T is constant in A. Let us compute this. Consider any periodic orbit $\gamma \in U$. Its period T_γ is given by

$$\int_\gamma \frac{1}{f(z)} dz = \int_0^{T_\gamma} \frac{z'(t)}{f(z(t))} dt = \int_0^{T_\gamma} dt = T_\gamma.$$

The left hand side of this equality is determined by the sum of the residues of $\frac{1}{f(z)}$ over of the zeros of f surrounded by γ, which of course is independent of γ. Hence, the above equality also gives that $T_\gamma = T$, *i.e.* another proof of the isochronism of all the periodic orbits in U.

(c) It is a straightforward computation that

$$\dot{z} = \frac{f(z)}{f(z)\bar{f(z)}} = \frac{1}{\bar{f(z)}},$$

is Hamiltonian wherever it is defined. We check that equation $\dot{z} = f(z)$ has $(f\bar{f})^{-1}$ as an integrating factor (see [2]). Let X and Y be two differentiable vector fields in $U \in \mathbb{R}^2$ such that $\vartheta(x, y) = \det(X, Y)$ has no zeros in U. Then $[X, Y] = \lambda X$ if and only if ϑ^{-1} is an integrating factor of X. Applying this result to our case (λ=0), we obtain that the inverse of det(f,$i\bar{f}$) where we define $f = u + iv$ and $if = -v + iu$, then

$$\det(f, if) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = f\bar{f},$$

is an integrating factor for equation $\dot{z} = f(z)$. □

From Proposition 5.2.(a) we know that equation (1.1) admits an integrating factor outside the zeros or essential singularities of f, where equation (1.1) may have singular points whose local phase portrait does not preserve the area. Since such points are 'far' from periodic orbits, we easily observe the non-existence of isolated periodic orbits. We would like to extend this result to polycycles. If a graph has a zero of f, it cannot be monodromic since the local phase portrait of a zero of f has no hyperbolic sectors. Thus, the singularities on the polycycle are either poles or essential singularities. If all were poles the integrating factor would be well defined and we could argue, as in the case of the periodic orbit, that, in the set in which the return map is defined, all the orbits should be periodic. The study of the case of polycycles with some essential singularity on them is more delicate. We prove:

Proposition 5.5. Let Γ be a polycycle of equation (1.1). Then in any half-neighborhood of Γ where the return map is defined, all orbits must be periodic.

Proof. The results of Proposition 5.4 imply that the singularities over Γ cannot be zeros of f. Thus, all singular points in Γ are either poles or essential singularities of f. □

Take a small open half-neighborhood U of Γ in which the return map is defined and there are neither zeros, not poles, not essential singularities of f. On U, instead of considering $\dot{z} = f(z)$, we use Proposition 5.4(c) and we can study the solutions of the Hamiltonian system $\dot{z} = 1/\bar{f(z)}$.

Proof of Theorem 5.1. Using the results of Proposition 5.4 and Proposition 5.5 it is easy to prove part (a) and (b). □

6 Bifurcation diagram of a rational family

In this section we give a topological classification of the phase portrait of the family of rational function of the from

$$\dot{z} = \frac{P(z)}{Q(z)}, \tag{6.1}$$

where P and Q are polynomials in the z variable such that $\deg(P) \leq 2$ and $\deg(Q) \leq 2$ without common factors. Notice that $\frac{P(z)}{Q(z)} = \frac{P(z)}{Q(z)}\frac{\overline{Q(z)}}{\overline{Q(z)}} = \lambda(z)P(z)\overline{Q(z)}$, where $\lambda(z) = |Q(z)|^{-2}$. So under a non-liner change of the time variable the equation (6.1) is of the form $\dot{z} = P(z)\overline{Q(z)}$ outside the zeros of $Q(z)$.

We divide the general result into three propositions depending on the degree of the denominator of system (6.1).

6.1 deg(P)≤2 and deg(Q)=0

First we consider the case of the polynomial Q of degree 0, so that it corresponds to de polynomial case. In the following lemma we show a normal form of these equations.

Lemma 6.1. Let $P(z)$ and $Q(z)$ be two polynomials such that $\deg(P) \leq 2$ and $\deg(Q) = 0$, so we write $Q(z) = d$ with $d \neq 0$. Then the corresponding equation (6.1) is linearly conjugate to one and only one of the following vector fields
(a) $\dot{z} = 1$ if $P(z) = c$,
(b) $\dot{z} = kz$ if $P(z) = c(z - c_0)$, where k=$c/d \in \mathbb{C}$,
(c) $\dot{z} = z(z - \xi)$ if $P(z) = c(z - c_0)(z - c_1)$, where $\xi = c(c - c_0)/d \in \mathbb{C}$.

Proof. (a) If we consider the line change of variables $w = \frac{d}{c}z$ then equation $\dot{z} = c/d$ write as

$$\dot{w} = \frac{d}{c}\dot{z} = \frac{d}{c}\frac{c}{d} = 1,$$

so $\dot{z} = c/d$ and $\dot{z} = 1$ are linearly conjugate through the map $\Phi(z) = \frac{d}{c}z$.

(b) If we consider the change of variables $w = z - c_0$ then equation $\dot{z} = c(z - c_0)/d$ write as

$$\dot{w} = \dot{z} = c(z - c_0)/d = c(w + c_0 - c_0)/d = cw/d,$$

so $\dot{z} = c(z - c_0)/d$ and $\dot{z} = cz/d$ are linearly conjugate through the map $\Phi(z) = \frac{d}{c}z - c_0$.

(c) If we consider the change of variables $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w-\beta}{\alpha}$ and

$$\dot{w} = \alpha\dot{z} = \alpha\frac{c}{d}(z - c_0)(z - c_1) = \alpha\frac{c}{d}(\frac{w - \beta}{\alpha} - c_0)(\frac{w - \beta}{\alpha} - c_1) = \frac{c}{d}(w - \beta - \alpha c_0)(w - \beta - \alpha c_1).$$

This equation equivalent $\dot{z} = z(z - \frac{c(c-c_0)}{d})$ so we get $\begin{cases} \beta = -\alpha c_0 \\ \alpha = \frac{c}{d} \end{cases}$.
Finally we have the function $\Phi(z) = \frac{c}{d}(z - c_0)$. ∎

The following proposition gives the phase portrait of the normal forms given in Lemma 6.1.

Proposition 6.2. The following statements hold:
(a) The phase portrait of equation $\dot{z} = 1$ in the Poincaré disc is topologically equivalent to Figure 6.1.
(b) Let $k \in \mathbb{C}$. The phase portrait of equation $\dot{z} = kz$ in the Poincaré disc is topologically equivalent to
(b1) Figure 6.2(1), if $\Im(k) = 0$.
(b2) Figure 6.2(2), if $\Re(k) = 0$.
(b3) Figure 6.2(3), if $\Re(k)\Im(k) \neq 0$.
(c) Let $\xi \in \mathbb{C}$. The hase portrait of equation $\dot{z} = z(z-\xi)$ in the Poincaré disc is topologically equivalent to
(c1) Figure 6.3(1), if $\xi \neq 0, \quad \Re(\xi) = 0$.
(c2) Figure 6.3(2), if $\xi \neq 0, \quad \Re(\xi) \neq 0$.
(c3) Figure 6.3(3), if $\xi = 0$.

Proof. (a) From Corollary 4.2 (a), the phase portrait of $\dot{z} = 1$ near infinity is given figure 4.1. Since these are no finite singular values Poincaré-Bendixson Theorem implies that the global phase portrait should be given by Figure 6.1

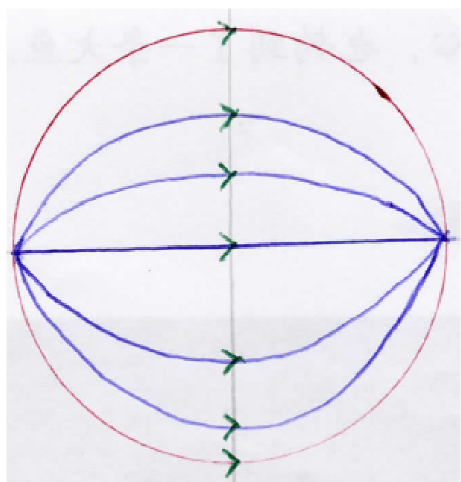

Figure 6.1: case (a)

(b) The unique finite singular point of equation $\dot{z} = kz$, $k \neq 0$ is $z = 0$. From Corollary 3.8 (a) we know that if $\Re(k) = 0$ (respectively $\Re(k) > 0$, $\Re(k) < 0$) then $z = 0$ is a center (respectively an unstable focus, a stable focus).

From Corollary 4.2(b) the phase portrait near infinity is given by Figure 4.2(b1), Figure 4.2(b2) and Figure 4.2(b3) depending on $k = \alpha, k = i\beta$ or $k = \alpha + i\beta$.

Then consequently, the phase portrait in the Poincaré disc is given by Figure 6.2 if $\Im(k) = 0$, if $\Re(k) = 0$ and if $\Re(k)\Im(k) \neq 0$.

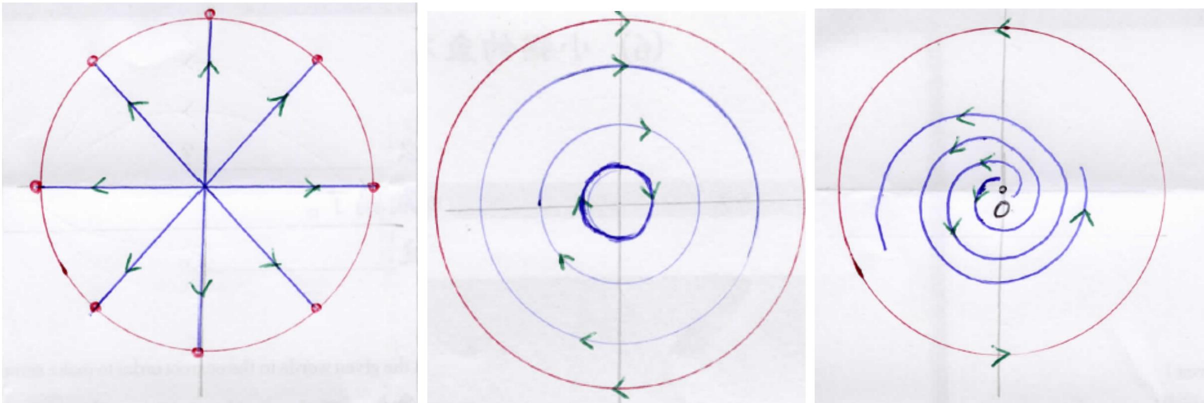

Figure 6.2: These figures correspond to proposition 6.2 (b): (1)$\Im(k) = 0$, (2)$\Re(k) = 0$ and (3)$\Re(k)\Im(k) \neq 0$.

(c) From Corollary 4.2(c) the phase portrait near infinity is given by Figure 4.3.

To get the global phase portrait we need to split the proof into two cases.

If $\xi \neq 0$ these are two finite singular point given by $z = 0$ and $z = \xi$. An easy exercise shows that the derivative at these points are $-\xi$ and ξ, respectively. Hence from Corollary 3.8(a) their local phase portrait is a center or a focus depending on $\Re(\xi) = 0$ or $\Re(\xi) \neq 0$.

Easy topological considerations (and Poincaré-Bendixson Theorem) show that the global phase portrait in the Poincaré disc is given by Figure 6.3(a) if $\Re(\xi) = 0$ and Figure 6.3(b) if $\Re(\xi) \neq 0$.

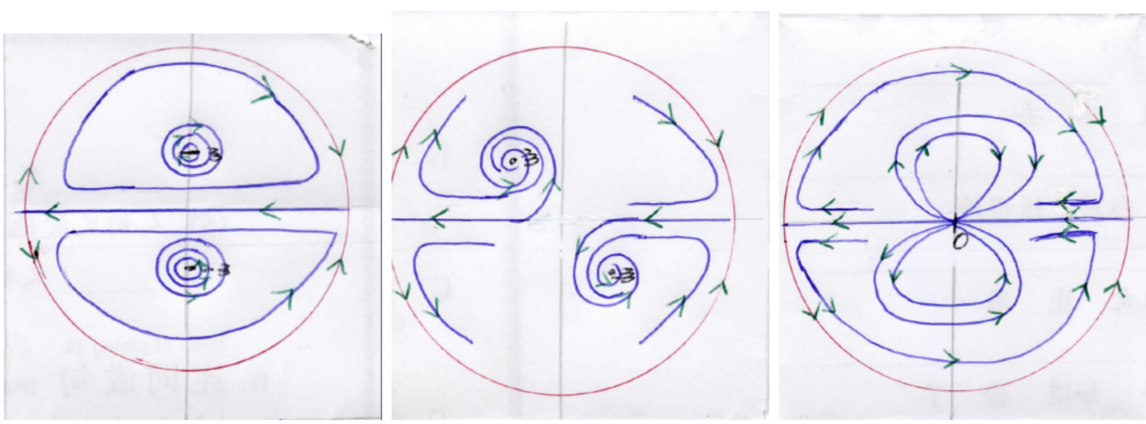

Figure 6.3: These figures correspond to proposition 6.2(c): (1) if $\xi \neq 0$ and $\Re(L_\xi) = \Re(\xi) = 0$, (2) if $\xi \neq 0$ and $\Re(L_\xi) = \Re(\xi) \neq 0$ and (3) if ξ=0.

□

6.2 deg(P)≤2 and deg(Q)=1

Next we consider the case in which Q(z) is a polynomial of degree 1. In the following lemma 6.3 we obtain a normal form for these equations.

Lemma 6.3. Let $P(z)$ and $Q(z)$ be two polynomials such that $\deg(P) \leq 2$ and $\deg(Q) = 1$, so we write $Q(z) = d(z - d_0)$. Then the corresponding equation (6.1) is conformally conjugated to
(a) $\dot{z} = \frac{1}{z}$ if $P(z) = c$,
(b) $\dot{z} = \frac{z-\xi}{z}$ if $P(z) = c(z - c_0)$, where $\xi = \frac{d(c_0-d_0)}{c} \in \mathbb{C}$,
(c) $\dot{z} = \frac{k(z-\xi)(z-1)}{z}$ if $P(z) = c(z - c_0)(z - c_1)$, where $k = \frac{c}{d} \in \mathbb{C}$ and $\xi = \frac{(c_1-d_0)}{(c_0-d_0)} \in \mathbb{C}$.

Proof. (a) To prove that $\dot{z} = \frac{P(z)}{Q(z)} = \frac{c}{d(z-d_0)}$ and $\dot{z} = 1/z$ are conformally conjugated, we must find a conformal map $\Phi(z)$.

If we consider the change of variables $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w+\beta}{\alpha}$ then

$$\dot{w} = \alpha \dot{z} = \alpha \frac{c}{d(z - d_0)} = \alpha^2 \frac{c}{d(w - \beta - d_0)},$$

is conformally conjugated to $\dot{z} = 1/z$ then

$$\begin{cases} \alpha^2 c/d = 1 \\ \beta + \alpha d_0 = 0 \end{cases}$$

and then we get $\alpha = \sqrt{\frac{d}{c}}$, $\beta = -d_0\sqrt{\frac{d}{c}}$ and finally

$$\Phi(z) = \sqrt{\frac{d}{c}}(z - d_0).$$

(b) To prove that $\dot{z} = \frac{z-\xi}{z}$ is conformally conjugated to $\dot{z} = \frac{c(z-c_0)}{d(z-d_0)}$ we apply the change of variable $w = \alpha z + \beta$.

Suppose $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w+\beta}{\alpha}$. If

$$\dot{w} = \alpha \frac{c(z - c_0)}{d(z - d_0)} = \alpha \frac{c}{d} \frac{\frac{w-\beta}{\alpha} - c_0}{\frac{w-\beta}{\alpha} - d_0} = \alpha \frac{c}{d} \frac{w - \beta - \alpha c_0}{w - \beta - \alpha d_0}$$

is conformally conjugated to $\dot{z} = \frac{z-\xi}{z}$ then

$$\begin{cases} \beta + \alpha c_0 = \xi \\ \beta + \alpha d_0 = 0 \ , \\ \alpha \frac{c}{d} = 1 \end{cases}$$

then we get $\alpha = \frac{d}{c}, \beta = -\frac{dd_0}{c}$ and

$$\Phi(z) = \frac{d}{c} z - \frac{dd_0}{c} = \frac{d}{c}(z - d_0).$$

(c) To prove that $\dot{z} = \frac{c(z-c_0)(z-c_1)}{d(z-d_0)}$ and $\dot{z} = \frac{k(z-\xi)(z-1)}{z}$ are conformally conjugated, we apply the change of variable $w = \alpha z + \beta$.

Suppose $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w-\beta}{\alpha}$. Then

$$\dot{w} = \alpha \frac{c(z-c_0)(z-c_1)}{d(z-d_0)} = \alpha \frac{c}{d} \frac{(\frac{w-\beta}{\alpha} - c_0)(\frac{w-\beta}{\alpha} - c_1)}{\frac{w-\beta}{\alpha} - d_0} =$$

$$\alpha \frac{c}{d} \frac{\frac{1}{a^2}(w-\beta-\alpha c_0)(w-\beta-\alpha c_1)}{\frac{1}{a}(w-\beta-\alpha d_0)} = \alpha \frac{c}{d} \frac{(w-\beta-\alpha c_0)(w-\beta-\alpha c_1)}{(w-\beta-\alpha d_0)}.$$

If the equation is conformally conjugated to $\dot{z} = \frac{k(z-\xi)(z-1)}{z}$, then

$$\begin{cases} \beta + \alpha c_0 = 1 \\ \beta + \alpha d_0 = 0 \end{cases} \Rightarrow \begin{cases} \beta = \frac{-d_0}{c_0 - d_0} \\ \alpha = \frac{1}{c_0 - d_0} \end{cases}.$$

and

$$\Phi(z) = \frac{1}{c_0 - d_0} z - \frac{-d_0}{c_0 - d_0} = \frac{1}{c_0 - d_0}(z - d_0)$$

and we know that $\dot{w} = \frac{k(z-\xi)(z-1)}{z}$ then $\xi = \beta + \alpha c_1 = \frac{c_1 - d_0}{c_0 - d_0}$. □

The following proposition provides the phase portrait of the normal forms given in Lemma 6.3.

Proposition 6.4. The following statements hold:
(a) The phase portrait of equation $\dot{z} = \frac{1}{z}$ in the Poincaré disc is topologically equivalent to Figure 6.4.
(b) The phase portrait of equation $\dot{z} = \frac{z-\xi}{z}$ in the Poincaré disc is topologically equivalent to
(b1) Figure 6.5(1), if $\Re(L_\xi) = 0$,
(b2) Figure 6.5(2), if $\Re(L_\xi) \neq 0$,
where $L_\xi = \frac{1}{\xi}$ is the linear part at the singular point $z = \xi$.
(c) Let $\xi \in \mathbb{C}$. The phase portrait of equation $\dot{z} = k\frac{(z-1)(z-\xi)}{z}$ in the Poincaré disc is topologically equivalent to
(c1) Figure 6.6(1), if $\Re(k) = 0, \Re(L_1) = \Re(L_\xi) = 0$ *and* $\Re(\xi) < 0$,
(c2) Figure 6.6(2), if $\Re(k) = 0, \Re(L_1) = \Re(L_\xi) = 0$ *and* $\Re(\xi) > 0$,
(c3) Figure 6.6(3), if $\Re(k) = 0, \Re(L_1) = \Re(L_\xi) \neq 0$,
(c4) Figure 6.6(4), if $\Re(k) = 0, \xi = 1$,
(c5) Figure 6.7(1), if $\Re(k)\Im(f) \neq 0, \Re(L_1)\Re(L_\xi) = 0$,
(c6) Figure 6.7(2), if $\Re(k)\Im(f) \neq 0, \Re(L_1)\Re(L_\xi) < 0$,
(c7) Figure 6.7(3), if $\Re(k)\Im(f) \neq 0, \Re(L_1)\Re(L_\xi) > 0$,
(c8) Figure 6.7(4), if $\Re(k)\Im(f) \neq 0, \xi = 1$,
(c9) Figure 6.8(1), if $\Im(k) = 0, \Re(L_1)\Re(L_\xi) = 0$,
(c10) Figure 6.8(2), if $\Im(k) = 0, \Re(L_1)\Re(L_\xi) < 0$,
(c11) Figure 6.8(3), if $\Im(k) = 0, \Re(L_1)\Re(L_\xi) > 0$,
(c12) Figure 6.8(4), if $\Im(k) = 0, \xi = 1$,
where $L_1 = k(1-\xi)$ and $L_\xi = k\frac{\xi-1}{\xi}$ are the linear part at the singular point $z = 1$ and z=ξ, respectively.

Proof. **Proof of (a)**
In the case, there are no singular points and the origin is a pole of order one, from Corollary 3.8 (c), has local phase portrait in a punctured neighborhood of z=0 is pole that is topologically equivalent to a hyperbolic saddle. On the other hand, the phase portrait of equation (6.1) in a neighborhood of infinity corresponds to $\dot{z} = \frac{1}{z}$. We use the Poincaré Compactidifcation,
$\dot{z} = \dot{x} + i\dot{y} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$ and we can say that

$$\begin{cases} \dot{x} = \frac{x}{x^2+y^2} \\ \dot{y} = \frac{-y}{x^2+y^2}. \end{cases}$$

We multiple the vector field by $x^2 + y^2$ and we get a equivalent polynomial vector field,

$$\begin{cases} \dot{x} = x = P(x, y) \\ \dot{y} = -y = Q(x, y) \end{cases}$$

In this case d=1, and we compute the singular points in the charts U_1 and U_2:

$$\begin{cases} F(u) \equiv Q_d(1, u) - uP_d(1, u) = -u - u = -2u \\ G(u) \equiv P_d(u, 1) - uQ_d(u, 1) = u - u \cdot (-1) = 2u \end{cases} \Rightarrow \begin{cases} \text{if} \quad F(u) = 0 \Rightarrow -2u = 0 \Rightarrow u = 0 \\ \text{and} \quad G(u) = 0 \Rightarrow 2u = 0 \Rightarrow u = 0. \end{cases}$$

Now compute the singular point $u = 0$ in U_1 and U_2:

$$\begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} = A,$$

$$\begin{pmatrix} G'(u) & P_{d-1}(u, 1) - uQ_{d-1}(u, 1) \\ 0 & -Q_d(u, 1) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = B.$$

Then, we have 4 singular points in the infinity two stable nodes and two unstable nodes

Easily, the topological phase portrait in the Poincaré disc is given by Figure 6.4.

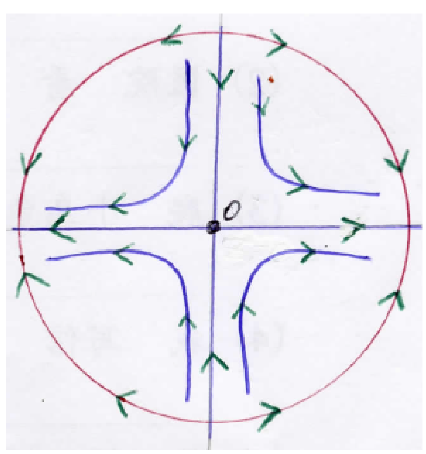

Figure 6.4: case (a): $\dot{z} = \frac{1}{z}$

Proof of (b)
First, we see that the origin is a pole of order one and from Corollary 3.8 (c) the local phase portrait is topologically equivalent to a hyperbolic saddle. The point z=ξ is a singular point with a linear part given by

$$L_\xi = (\dot{z})' = \frac{\xi}{z^2} \mid_{z=\xi} = \frac{1}{\xi}.$$

Hence, from Corollary 4.2 (a) we know that it is a center if $\Re(L_\xi) = 0$, or it is a stable or unstable focus if $L_\xi \neq 0$.

Moreover, from Corollary 4.2 (d), the phase portrait of equation (6.1) in a neighborhood of infinity corresponds to $\dot{z} = 1 + \frac{\xi}{z}$.

Taking all of this into account, we can easily check that the only topological phase portraits in the Poincaré disc are either Figure 6.5 if $\Re(L_\xi) = 0$ and if $\Re(L_\xi) \neq 0$

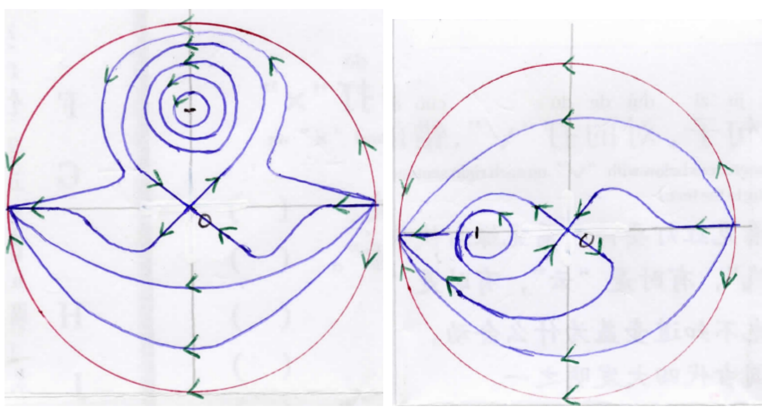

Figure 6.5: These figures correspond to Proposition 6.4(b) : (1) $\Re(L_\xi) = 0$ and (2) $\Re(L_\xi) \neq 0$.

Proof of (c)
In this case, the origin is a pole of order one and from Corollary 3.8(c) the local phase portrait is topologically equivalent to a hyperbolic saddle.

If $\xi \neq 1$, then $z = 1$ and $z = \xi$ are singular points with a linear part given by

$$L_1 = (\dot{z})' = \frac{1}{z^2}[k(z^2 - \xi)] \mid_{z=1} = k(1 - \xi)$$

and

$$L_\xi = (\dot{z})' = \frac{1}{z^2}[k(z^2 - \xi)] \mid_{z=\xi} = k\frac{(\xi - 1)}{\xi}.$$

Moreover, we have that $L_1 + \xi L_\xi = 0$.
If $\xi = 1$, there is a unique singular point and from Corollary 3.8 (b) the local phase portrait is given by the union of two elliptic sectors.

From Corollary 4.2(b) the phase portrait of equation (6.1) in a neighborhood of infinity corresponds to $\dot{z} = kz$. Since

$$\dot{z} = \frac{k(z-1)(z-\xi)}{z} = \frac{k(z^2 - \xi z - z + \xi)}{z} = k(z + \xi - 1 + \frac{1}{z}),$$

when z tends to infinity, $\dot{z} = kz$. The behavior of equation $\dot{z} = kz$ depends on $\Re(k) = 0, \Re(k)\Im(k) \neq 0$, or $\Im(k) = 0$, respectively. We study such cases independently

In what follows $\xi = \xi^R + i\xi^I \in \mathbb{C}, r = \xi^2 = \xi\bar{\xi}$ and $k = \alpha + i\beta \in \mathbb{C}, \alpha, \beta \neq 0$.

Since the proof is quite long we divide it into three cases.

$$\boxed{Case \quad \Re(k) = 0}$$

It is easy to see that

$$\Re(L_1) = \beta\xi^I,$$
$$\Re(L_\xi) = -\beta\xi^I/r.$$

Since if $\Re(k) = 0$, We can write $k = i\beta$ and as before we get

$$L_1 = k(1-\xi) = i\beta(\xi^R + i\xi^I) = i\beta\xi^R - \beta\xi^I \Rightarrow \Re(L_1) = -\beta\xi^I$$

and

$$L_\xi = k\frac{(\xi - 1)}{\xi} = i\beta\frac{(\xi^R + i\xi^I - 1)}{\xi^R + i\xi^I} = \frac{(i\beta\xi^R - \beta\xi^I - i\beta)(\xi^R - i\xi^I)}{(\xi^R + i\xi^I)(\xi^R - i\xi^I)} = \frac{1}{r}(i\beta - i\beta\xi^I - \xi^I\beta)$$

$$\Rightarrow \Re(L_\xi) = -\beta\xi^I/r.$$

If $\boxed{\Re(L_1) = \Re(L_\xi) = 0} \Rightarrow \begin{cases} -\beta\xi^I = 0 & \Rightarrow \xi^I = 0 \\ -\beta\xi^I/r = 0 & \Rightarrow \xi^I = 0 \end{cases}$

then the two centers are located on the real line (on the axis X). The phase portrait of equation (6.1) in the Poincaré disc (the point at infinity corresponds to the equation $\dot{z} = kz$ from Corollary 4.2(b)) is therefore topologically equivalent to Figure 6.6 depending on whether $\xi < 0$ or $\xi > 0$.

If $\boxed{\Re(L_1)\Re(L_\xi) < 0}$ from Corollary 3.8(a) we have one stable and one unstable singular point. The phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.6.

Finally, $\boxed{\xi = 1}$, there is a unique singular point and from Corollary 3.8(b) the local phase portrait is given by the union of two elliptic sectors, the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.6.

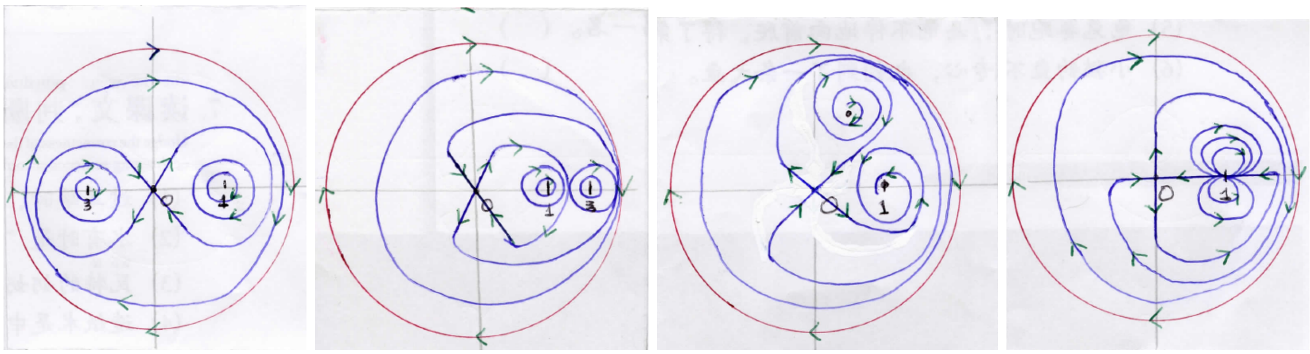

Figure 6.6: These figures correspond to Proposition 6.4(c): if $\xi \neq 0$,then $\Re(L_1) = \Re(L_\xi) = 0$ depending (1) $\xi < 0$ or (2) $\xi > 0$ and (3) $\Re(L_1)\Re(L_\xi) < 0$. If (4) $\xi = 1$.

$\boxed{Case \quad \Re(k)\Im(k) \neq 0}$

Some computations show that

$$\Re(L_1) = \alpha(1 - \xi^R) + \beta\xi^I,$$
$$\Re(L_\xi) = (\alpha(\xi^R(\xi^R - 1) + (\xi^I)^2) - \beta\xi^I)/r.$$

Since we know that $\Re(k)\Im(k) \neq 0 \Rightarrow k = \alpha + i\beta$ then

$$\begin{aligned} L_1 = k(1-\xi) = (\alpha + i\beta)(1 - \xi^R - i\xi^I) = (\alpha + i\beta) - (\alpha + i\beta)\xi^R - (\alpha + i\beta)i\xi^I \\ = \alpha - \alpha\xi^R + \beta\xi^I + i(\alpha\xi^I - \beta - \beta\xi^R) \Rightarrow \Re(L_1) = \alpha(1 - \xi^R) + \beta\xi^I \end{aligned}$$

and

$$L_\xi = k\frac{(\xi - 1)}{\xi} = (\alpha + i\beta)\frac{(\xi^R + i\xi^I - 1)}{\xi^R + i\xi^I} = (\alpha + i\beta)\frac{(\xi^R + i\xi^I - 1)(\xi^R - i\xi^I)}{(\xi^R + i\xi^I)(\xi^R - i\xi^I)} =$$
$$= \alpha + i\beta - \frac{1}{r}(\alpha\xi^R + \alpha i\xi^I + i\beta\xi^R + \beta\xi^I) \Rightarrow \Re(L_\xi) = (\alpha(\xi^R(\xi^R - 1) + (\xi^I)^2) - \beta\xi^I)/r.$$

The only solution to the equation $\Re(L_1) = \Re(L_\xi) = 0$ is $\xi^R = 1$ and $\xi^I = 0$. So two centers are impossible.

Let $\xi \neq 1$.

If $\boxed{\Re(L_1)\Re(L_\xi) = 0}$ (one center and one stable or unstable singular point).

Let us see $z = 1$ and $z = \xi$ which one is a centre and which one is a stable (unstable) singular point.

If $z = 1$ is center, then

$$\Re(L_1) = \alpha(1 - \xi^R) + \beta\xi^I \Rightarrow \xi^R = 1 \text{ and } \xi^I = 0 \text{ impossible.}$$

So $z = 1$ is stable or unstable.

And if $z = \xi$ is center, then

$$\Re(L_\xi) = (\alpha(\xi^R(\xi^R - 1) + (\xi^I)^2) - \beta\xi^I)/r \Rightarrow \xi^R = 1 \text{ or } \xi^R = 0 \text{ and } \xi^I = \beta \text{ or } \xi^I = 0.$$

So we can get $\xi = 1 + i\beta,\ 1,\ i\beta\ or\ 0$, then we can say that $z = \xi$ is one center, then the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to 6.7

If $\boxed{\Re(L_1)\Re(L_\xi) < 0}$ (one stable and one unstable singular point) the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.7.

If $\boxed{\Re(L_1)\Re(L_\xi) > 0}$ (two stables or two unstables singular points) the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.7.

Let $\boxed{\xi = 1}$, there is a unique singular point and from Corollary 3.8(b) the local phase portrait is given by the union of two elliptic sectors, the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.7.

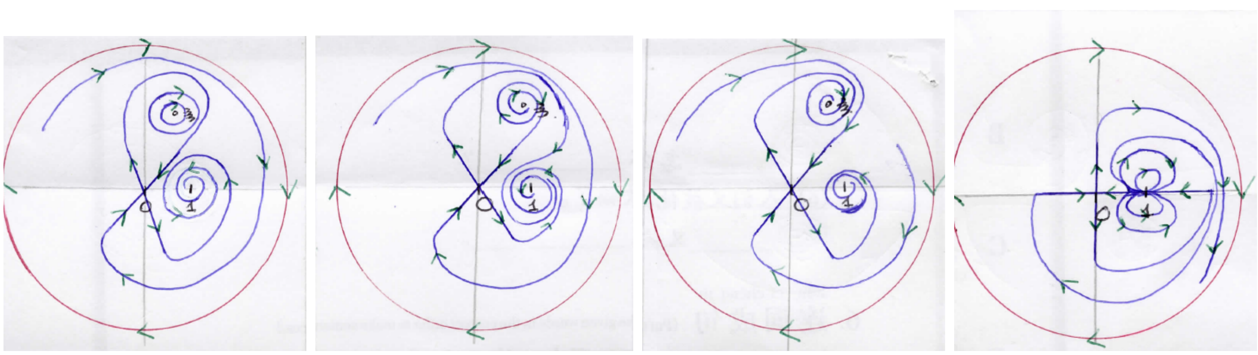

Figure 6.7: These figures correspond to Proposition 6.4(c): If $\xi \neq 1$, (1) $\Re(L_1)\Re(L_\xi) = 0$, (2) $\Re(L_1)\Re(L_\xi) < 0$ and (3)$\Re(L_1)\Re(L_\xi) > 0$. If (4) $\xi = 1$.

$\boxed{Case \quad \Im(k) = 0}$

Some computations show that

$$\Re(L_1) = \alpha(1 - \xi^R),$$
$$\Re(L_\xi) = \alpha(\xi^R(\xi^R - 1) + (\xi^I)^2)/r.$$

Since that we know that $\Im(k) = 0$ then we can write $k = \alpha$ and as before we calculate L_1 and L_ξ

$$L_1 = k(1-\xi) = \alpha(\xi^R + i\xi^I) = \alpha - \alpha\xi^R - i\alpha\xi^I \Rightarrow \Re(L_1) = \alpha(1-\xi^R)$$

and

$$L_\xi = k\frac{(\xi-1)}{\xi} = \alpha\frac{(\xi^R + i\xi^I - 1)}{\xi^R + i\xi^I} = \frac{(\alpha\xi^R - i\alpha\xi^I - \alpha)(\xi^R - i\xi^I)}{(\xi^R + i\xi^I)(\xi^R - i\xi^I)} = \alpha - \frac{1}{r}\alpha\xi^R + i\xi^I\alpha$$
$$\Rightarrow \Re(L_\xi) = \alpha(\xi^R(\xi^R - 1) + (\xi^I)^2)/r.$$

Let $\xi \neq 1$.

If $\boxed{\Re(L_1)\Re(L_\xi) = 0}$ (one center and one stable or unstable focus), to see which one is the center.

If $z = 1$ is center, then $\alpha(1-\xi^R) = 0 \Rightarrow \xi^R = 0$ or if $z = \xi$ is the center, then

$$\alpha(\xi^R(\xi^R - 1) + (\xi^I)^2)/r = 0 \Rightarrow y = \begin{cases} \xi^R(\xi^R - 1) = 0 & \Rightarrow \xi^R = 0 \; or \; \xi^R = 1 \\ (\xi^I)^2 = 0 & \Rightarrow \xi^I = 0 \end{cases}$$

the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.8

If $\boxed{\Re(L_1)\Re(L_\xi) < 0}$ (one stable and one unstable singular point) the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.8.

If $\boxed{\Re(L_1)\Re(L_\xi) > 0}$ (two stables or two unstable singular points) the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.8.

Let $\boxed{\xi = 1}$ there is a unique singular point and from Corollary 3.8(b) the local phase portrait is given by the union of two elliptic sectors, the phase portrait of equation (6.1) in the Poincaré disc is topologically equivalent to Figure 6.8.

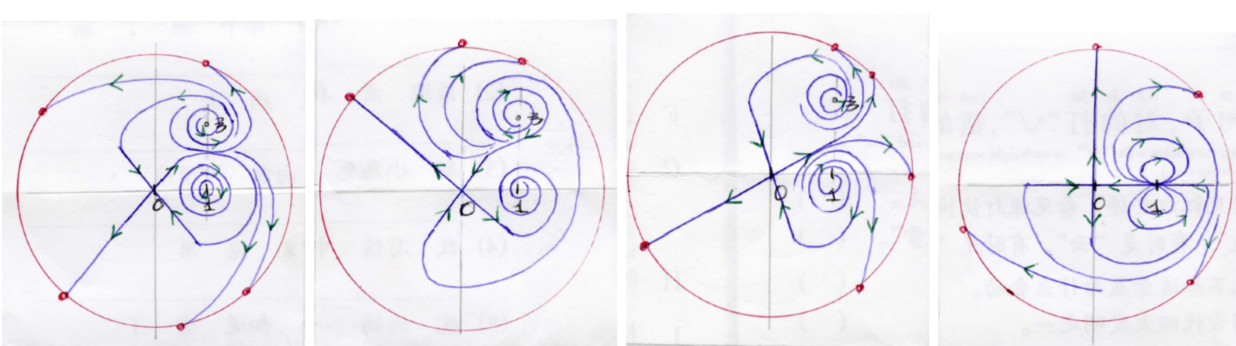

Figure 6.8: These figures correspond to Proposition 6.4(c) : If $\xi \neq 1$, (1) $\Re(L_1)\Re(L_\xi) = 0$, (2) $\Re(L_1)\Re(L_\xi) < 0$ and (3) $\Re(L_1)\Re(L_\xi) > 0$. If (4) $\xi = 1$.

□

6.3 Yaoyao Lemma: deg(P)≤2 and deg(Q)=2

Now we have one new lemma called yaoyao:

Lemma 6.5. Let P(z) and Q(z) be two polynomials such that $\deg(P) \leq 1$ and $\deg(Q) = 2$, we defined $Q(z) = c(z - c_0)(z - c_1)$, then the corresponding equation (6.1) is conformally conjugated to
(a) $\dot{z} = \frac{1}{z(z-\xi)}$ if P(z)=d, and $\xi = \frac{c(c_1-c_0)}{d} \in \mathbb{C}$.
(b) $\dot{z} = \frac{z}{k(z-\xi)(z-1)}$ if P(z)=d(z-d_0), where $\xi = \frac{c_1-d_0}{c_0-d_0} \in \mathbb{C}$.

Proof. (a) There exists a function Φ such that $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w-\beta}{\alpha}$
$\dot{w} = \alpha\dot{z} = \alpha\frac{d}{c}\frac{1}{(z-c_0)(z-c_1)} = \alpha\frac{d}{c}\frac{1}{(\frac{w-\beta}{\alpha}-c_0)(\frac{w-\beta}{\alpha}-c_1)} = \alpha\frac{d}{c}\frac{1}{(w-\beta-\alpha c_0)(w-\beta-\alpha c_1)}$ is equivalent to $\dot{z} = \frac{1}{(z-\xi)}$, so we can see that

$$\begin{cases} \beta + \alpha c_0 = 0 \\ \beta + \alpha c_1 = \xi \\ \alpha\frac{c}{d} = 1 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha c_0 & \Rightarrow \beta = -\frac{c}{d}c_0 \\ \beta = -\alpha c_1 + \frac{c(c_1-c_0)}{d} & \\ \alpha = \frac{d}{c} & \end{cases},$$

thus $\Phi(z) = \frac{c}{d}(z - c_0) \Rightarrow \dot{z} = \frac{d}{c(z-c_0)(z-c_1)}$ is conformally conjugated to $\dot{z} = \frac{1}{(z-\xi)}$

(b) We check that equations $\dot{z} = \frac{d(z-d_0)}{c(z-c_0)(z-c_1)}$ and $\dot{z} = \frac{z}{k(z-\xi)(z-1)}$ are conformally conjugated.
Suppose $\Phi(z) = \alpha z + \beta = w \Rightarrow z = \frac{w-\beta}{\alpha}$

$$\dot{w} = \alpha\frac{d(z-d_0)}{c(z-c_0)(z-c_1)} = \alpha\frac{c}{d}\frac{\frac{w-\beta}{\alpha} - d_0}{(\frac{w-\beta}{\alpha} - c_0)(\frac{w-\beta}{\alpha} - c_1)} =$$

$$= \alpha\frac{c}{d}\frac{\frac{1}{a}(w-\beta-\alpha d_0)}{\frac{1}{a^2}(w-\beta-\alpha c_0)(w-\beta-\alpha c_1)} = \alpha^2\frac{c}{d}\frac{(w-\beta-\alpha d_0)}{(w-\beta-\alpha c_0)(w-\beta-\alpha c_1)}.$$

If the equation is conformally conjugated to $\dot{z} = \frac{k(z-\xi)(z-1)}{z}$, then

$$\begin{cases} \beta + \alpha c_0 = 1 \\ \beta + \alpha d_0 = 0 \end{cases} \Rightarrow \begin{cases} \beta = \frac{-d_0}{c_0-d_0} \\ \alpha = \frac{1}{c_0-d_0} \end{cases}$$

and

$$\Phi(z) = \frac{1}{c_0-d_0}z - \frac{-d_0}{c_0-d_0} = \frac{1}{c_0-d_0}(z-d_0)$$

and we know that $\dot{w} = \frac{z}{k(z-\xi)(z-1)}$ then $\xi = \beta + \alpha c_1 = \frac{-d_0}{c_0-d_0} = \frac{c_1-d_0}{c_0-d_0}$ and $\frac{1}{k} = \frac{a^2 d}{c} = \frac{d}{(c_0-d_0)^2 c} \Rightarrow k = \frac{(c_0-d_0)^2 c}{d}$. □

The following proposition provides the phase portrait of the normal forms given in Yaoyao Lemma.

Proposition 6.6. The following statement hold:
(a) Let $\xi \in \mathbb{C}$. The phase portrait of equation $\dot{z} = \frac{1}{z(z-\xi)}$ in the Poincaré disc is topologically equivalent to
(a1) Figure 6.9(1), if $\xi \neq 0$.
(a2) Figure 6.9(2), if $\xi = 0$.
(b) Let $\xi, k \in \mathbb{C}$. The phase portrait of equation$\dot{z} = \frac{z}{k(z-\xi)(z-1)}$ in the Poincaré disc is topologically equivalent to
(b1) Figure 6.10(1), if $\xi \neq 1 \quad \Re(L_0) = 0$.
(b2) Figure 6.10(2), if $\xi \neq 1 \quad \Im(L_0) = 0$.
(b3) Figure 6.10(3), if $\xi \neq 1 \quad \Re(L_0)\Im(L_0) \neq 0$.
(b4) Figure 6.11, if $\xi = 1$.

Proof. (a) In this case, there are no singular points and the origin ($z = 0$) and $z = \xi$ are poles of order one which, from Corollary 3.8 (c), has local phase portrait that is topologically equivalent to a hyperbolic saddle.

In the second case $\xi = 0$, we only have a point hyperbolic saddle.

On the other hand, the phase portrait of equation (6.1) in a neighborhood of infinity corresponds to $\dot{z} = \frac{1}{z^2}$. We use the Poincaré Compactification, $\dot{z} = \dot{x} + i\dot{y} = \frac{1}{(x+iy)^2} = \frac{(x-iy)^2}{(x^2+y^2)^2}$ and we can write that

$$\begin{cases} \dot{x} = \frac{x^2-y^2}{(x^2+y^2)^2} \\ \dot{y} = \frac{-2xy}{(x^2+y^2)^2} \end{cases}$$

We multiple the vector field by $(x^2 + y^2)^2$ and we get a equivalent polynomial vector field,

$$\begin{cases} \dot{x} = x^2 - y^2 = P(x, y) \\ \dot{y} = -2xy = Q(x, y) \end{cases}$$

In this case $d = 2$, and we compute the singular points in the charts U_1 and U_2

$$\begin{cases} F(u) \equiv Q_d(1, u) - uP_d(1, u) = -2u - u \cdot (1 - u^2) = u(u^2 - 3) \\ G(u) \equiv P_d(u, 1) - uQ_d(u, 1) = u^2 - 1 - u \cdot (-2u) = 3u^2 - 1 \end{cases}$$

$$\begin{cases} \text{if} \quad F(u) = 0 \Rightarrow -3u + u^2 = 0 \Rightarrow u = 0 \quad \text{or} \quad u = \pm\sqrt[2]{3} \\ \text{and} \quad G(u) = 0 \Rightarrow 3u^2 - 1 = 0 \Rightarrow u = \pm\sqrt[2]{\frac{1}{3}} \end{cases} \Rightarrow \begin{cases} F'(u) = 3u^2 + 3 \\ G'(u) = 6u \end{cases}$$

when $u = 0$ in U_1

$$\begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} = A,$$

is a singular point at infinity which is topologically equivalent to a stable node.
when $u = \pm\sqrt[2]{3}$ in the U_1, the first point $u = \sqrt[2]{3}$

$$\begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = B,$$

is a singular point at infinity that is topologically equivalent to an unstable node.

The second point $u = -\sqrt[2]{3}$

$$\begin{pmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = C,$$

is a singular point at infinity which is topologically equivalent to an unstable node.

Now let us study the points $u = \pm\sqrt[2]{\frac{1}{3}}$ in U_2.

When $u = \sqrt[2]{\dfrac{1}{3}}$, $\begin{pmatrix} G'(u) & P_{d-1}(u,1) - uQ_{d-1}(u,1) \\ 0 & -Q_d(u,1) \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix} = B$ and analogously for $u = \sqrt[2]{3}$ in U_1.

Then, we have 6 singular points at infinity, three that are topologically equivalent to an unstable node and the other three that are topologically equivalent to a stable node.

Easily, if $\boxed{\xi \neq 0}$ the topological phase portrait in the Poincaré disc is shown in Figure 6.9(1).

And if $\boxed{\xi = 0}$, the equation is $\dot{z} = \frac{1}{z^2}$ the topological phase portrait in the Poincaré disc is shown in Figure 6.9(2).

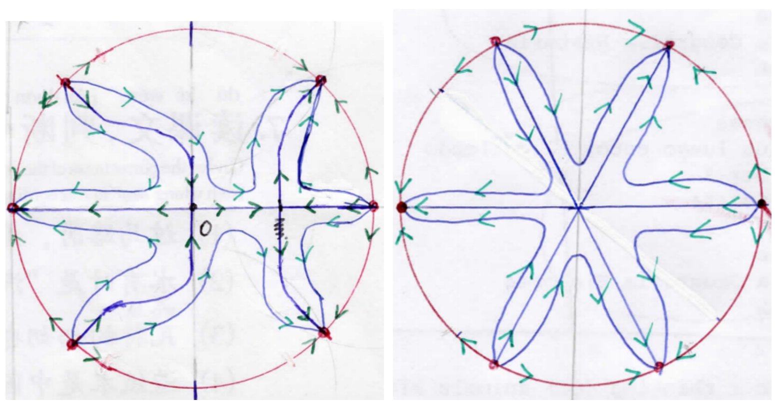

Figure 6.9: These figures correspond to Proposition 6.6(a): (1) $\xi \neq 0$ and (2) $\xi = 0$.

(b) In the Yaoyao Lemma we know that the equation is conformally conjugated to $\dot{z} = \frac{z}{k(z-\xi)(z-1)}$, so we have one zero (singular point) $z = 0$ and two poles $z = \xi$ and $z = 1$.

From Theorem 3.1, we know that if $deg(P) = 1 = n$ and $deg(Q) = 2 = m$, the phase portrait (6.1) is conformally conjugated to the one of (a) $\dot{z} = (\frac{1}{z})^{m-n} + c(\frac{1}{z})^{2(m-n)+1}$, if $n \leq m+1$, then we have in this case that the phase portrait (6.1) conformally conjugated to the one of

$$\dot{z} = (\frac{1}{z})^{2-1} + c(\frac{1}{z})^{2(2-1)+1} = (\frac{1}{z}) + c(\frac{1}{z})^3$$

then from Corollary 4.2, the phase portrait of equation (6.1) in a neighborhood of infinity corresponds to the one of $\dot{z} = \frac{1}{z} + cz^3$.

Moreover, there are either two poles at z=1 and z=ξ (case $\xi \neq 1$), or just one pole at z=1 (case $\xi = 1$).We have one singular point at $z = 0$, the derivative of the vector field at z, is

$$L_z = (\dot{z})' = \frac{k(z-\xi)(z-1) - zk(2z-1-\xi)}{(k(z-\xi)(z-1))^2} = \frac{\xi - z^2}{(k(z-\xi)(z-1))^2},$$

and if z=0,

$$L_0 = \frac{1}{k\xi}.$$

Hence, from Corollary 3.8(a), the phase portrait in the Poincaré sphere is topologically equivalent to either Figure 6.10 depending on whether $\Re(k) = 0$, $\Re(k)\Im(k) \neq 0$ and $\Re(k) \neq 0$, respectively.

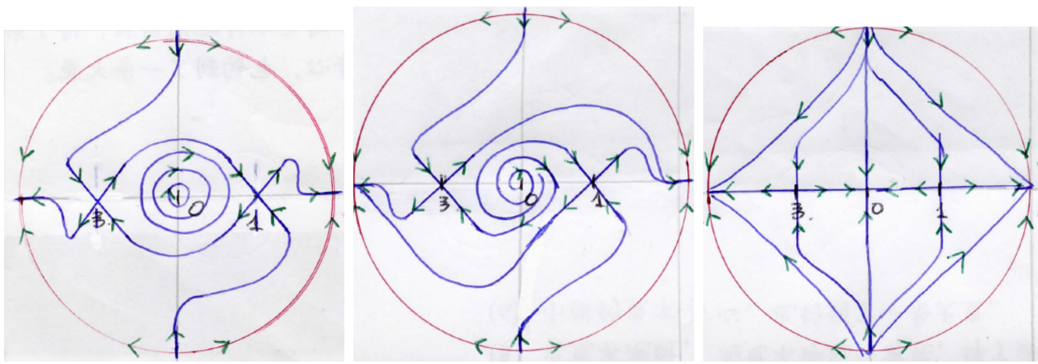

Figure 6.10: These figures correspond to Proposition 6.6(b): If $\xi \neq 1$ depending (1) $\Re(k) = 0$, (2) $\Re(k)\Im(k) \neq 0$ and (3) $\Re(k) \neq 0$.

If $\xi = 1$, then the equation is $\dot{z} = \frac{z}{k(z-1)^2}$. We have a pole $z = 1$ of the f of order 2 and a singular point $z = 0$, from Corollary 3.8(c) we know that $z = 1$ is the union of 6 hyperbolic sectors and from Corollary 3.8(a) $z = 0$ is a singular point depending on k, so the phase portrait in the Poincaré sphere is topologically equivalent to Figure 6.11

□

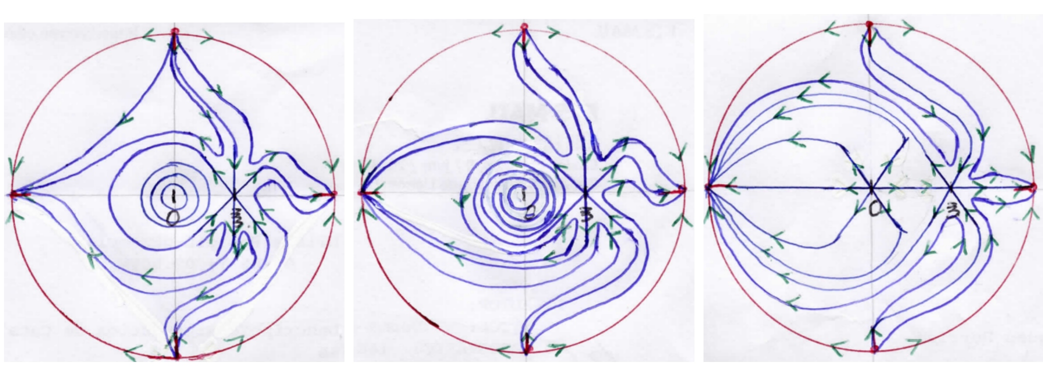

Figure 6.11: These figures correspond to Proposition 6.6(b): If $\xi = 1$ depending $\Re(k) = 0$, $\Re(k)\Im(k) \neq 0$ and $\Re(k) \neq 0$

7 Perturbation

Finally, the last chapter of this work is devoted to study a family of non holomorphic perturbations of the differential equation $\dot{z} = f(z)$ where $f(z) = iz + z^2$.

Namely, we consider the family of perturbations

$$\dot{z} = f(z) + \epsilon R_m(z, \bar{z}).$$

It is established in [10] that for certain values of the parameters of the perturbation the perturbed system presents a limit cycle (or two) associated to one (or two) of its centers before perturbation. We take advantage of [10] to focus the study of the perturbation to those parameters for which we know that the corresponding system has a limit cycle and we compute numerically where it is located.

First we begin by studying the equation $f(z) = iz + z^2$, and we calculate its singular points :

$$\dot{z} = \dot{x} + i\dot{y} = i(x+iy) + (x+iy)^2 = ix - y + x^2 - y^2 + 2ixy \Rightarrow$$

$$\begin{cases} \dot{x} = -y + x^2 - y^2 = P(x,y) \\ \dot{y} = x + 2xy = Q(x,y). \end{cases}$$

If $\dot{y} = 0 \Rightarrow x(1+2y) = 0 \Rightarrow \begin{cases} x = 0 & \text{or} \\ 1 + 2y = 0 & \Rightarrow y = -\frac{1}{2}, \end{cases}$

then

$$\begin{cases} x = 0 \Rightarrow \dot{x} = -y - y^2 = -y(1+y) = 0 \Rightarrow \begin{cases} & y = 0 \\ \text{or} & y = -1 \end{cases} \\ y = -\frac{1}{2} \Rightarrow \dot{x} = \frac{1}{2} + x^2 + \frac{1}{4} \end{cases}$$

so we have two singular points (0,0), (0,-1) and a invariant line $y = -\frac{1}{2}$.

We perform the translation $(\hat{x}, \hat{y}) \to (\hat{x}, \hat{y} + \frac{1}{2}) = (x, y)$, or in complex variables

$$z + i\frac{1}{2} = w,$$

and we obtain

$$\dot{w} = \dot{z} = i(w - i\frac{1}{2}) + (w - i\frac{1}{2})^2 = iw + \frac{1}{2} + w^2 - \frac{1}{4} - iw = w^2 + \frac{1}{4},$$

so the translated equation has the two singular points $(0, \frac{1}{2})$, $(0, -\frac{1}{2})$ and the invariant line $y = 0$.

Next let us see the character of the two singular points:

$$D_{(x,y)}X = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\Rightarrow D_{(x,y)}X(0, \frac{1}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D_{(x,y)}X(0, -\frac{1}{2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so two points $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$ are centers.

Now let us look for the singular points at infinity:

We use the Poincaré Compactidifcation, as explained before

$$\dot{w} = w^2 + \frac{1}{4} = (x+iy)^2 + \frac{1}{4} = x^2 - y^2 + 2ixy + \frac{1}{4}$$

$$\begin{cases} \dot{x} = \frac{1}{4} + x^2 - y^2 = P(x,y) \\ \dot{y} = 2xy = Q(x,y) \end{cases}.$$

In this case $d = 2$ and we compute the singular points in the charts U_1 and U_2

$$\begin{cases} F(u) \equiv Q_d(1,u) - uP_d(1,u) = 2u - u \cdot (1-u^2) = 2u - u + u^3 \\ G(u) \equiv P_d(u,1) - uQ_d(u,1) = u^2 - 1 - u \cdot 2u = u^2 - 1 - 2u^2 \end{cases}$$

$$\begin{cases} \text{if} & F(u) = 0 \Rightarrow u = 0 \\ \text{and} & G(u) \neq 0 \end{cases}$$

and we take u=0 in U_1

$$A(u) = \begin{pmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{pmatrix} \Rightarrow A(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since matrix A(0) is diagonal then the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, hence the singular points at infinity are saddle points.

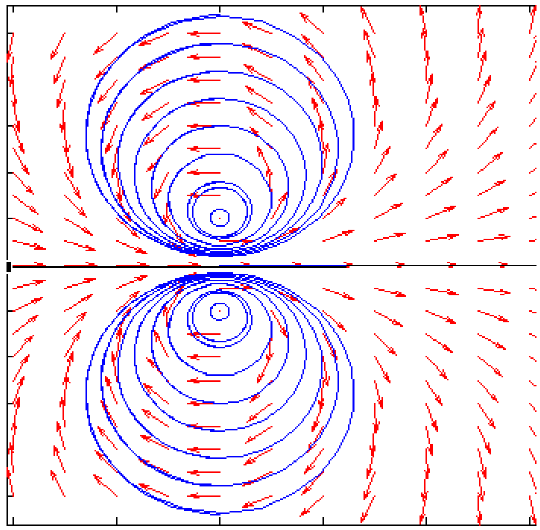

Figure 7.1: Phase portrait of $\dot{w} = w^2 + \frac{1}{4}$ in $\mathbb{C}$

Now we study the perturbed equation

$$\dot{w} = -\frac{1}{4} - w^2 + a_1 w^2 + a_2 w\bar{w} + a_3 \bar{w}^2 + a_4 w + a_5 \bar{w} + a_6$$

with $a_n = \alpha_n + i\beta_n, \alpha_n, \beta_n \in \mathbb{R}$ and $n = 1, 2, 3, 4, 5, 6$. We write it as an equation:

$$\dot{z} = \dot{x} + i\dot{y} = -\frac{1}{4} - (x+iy)^2 + a_1(x+iy)^2 + a_2(x+iy)(x-iy) + a_3(x-iy)^2 + a_4(x+iy) + a_5(x-iy) + a_6$$

$$= -\frac{1}{4} - (x^2 - y^2 + 2ixy) + a_1(x^2 - y^2 + 2ixy) + a_2(x^2 + y^2) + a_3(x^2 - y^2 - 2ixy) + a_4(x+iy) + a_5(x-iy) + a_6$$

$$= (-1 + a_1 + a_2 + a_3)x^2 + (1 - a_1 + a_2 - a_3)y^2(-2i + a_1 2i - a_3 2i)xy + (a_4 + a_5)x + (a_4 i - a_5 i)y + a_6 - \frac{1}{4}$$

$$= -\frac{1}{4} - x^2 + y^2 - 2ixy + (\alpha_1 + i\beta_1)x^2 - (\alpha_1 + i\beta_1)y^2 + (\alpha_1 + i\beta_1)2ixy + (\alpha_2 + i\beta_2)x^2 + (\alpha_2 + i\beta_2)y^2 +$$

$$+(\alpha_3 + i\beta_3)x^2 - (\alpha_3 + i\beta_3)y^2 - (\alpha_3 + i\beta_3)2ixy + (\alpha_4 + i\beta_4)x + (\alpha_4 + i\beta_4)iy + (\alpha_5 + i\beta_5)x - (\alpha_5 + i\beta_5)iy + (\alpha_6 + i\beta_6)$$

$$= -\frac{1}{4} - x^2 + y^2 - 2ixy + \alpha_1 x^2 + i\beta_1 x^2 - \alpha_1 y^2 - i\beta_1 y^2 + 2ixy\alpha_1 - \beta_1 2xy + \alpha_2 x^2 + i\beta_2 x^2 + \alpha_2 y^2 + i\beta_2 y^2 +$$

$$+\alpha_3 x^2 + i\beta_3 x^2 - \alpha_3 y^2 - i\beta_3 y^2 - \alpha_3 2ixy + \beta_3 2xy + \alpha_4 x + i\beta_4 x + \alpha_4 iy - \beta_4 y + \alpha_5 x + i\beta_5 x - \alpha_5 iy + \beta_5 y + \alpha_6 + i\beta_6 ..$$

Then

$$\Re(z) = -\frac{1}{4} - x^2 + y^2 + \alpha_1 x^2 - \alpha_1 y^2 - \beta_1 2xy + \alpha_2 x^2 + \alpha_2 y^2 + \alpha_3 x^2 - \alpha_3 y^2 + \beta_3 2xy + \alpha_4 x - \beta_4 y + \alpha_5 x + \beta_5 y + \alpha_6,$$

$$\Im(z) = -2ixy + i\beta_1 x^2 - i\beta_1 y^2 + 2ixy\alpha_1 + i\beta_2 x^2 + i\beta_2 y^2 + i\beta_3 x^2 - i\beta_3 y^2 - \alpha_3 2ixy + i\beta_4 x + \alpha_4 iy + i\beta_5 x - \alpha_5 iy + i\beta_6.$$

so we get:

$$\begin{cases} \dot{x} = -\frac{1}{4} + (\alpha_1 + \alpha_2 + \alpha_3 - 1)x^2 + (-\alpha_1 + \alpha_2 - \alpha_3 + 1)y^2 + (-\beta_1 + \beta_3)2xy + (\alpha_4 + \alpha_5)x + (-\beta_4 + \beta_5)y + \alpha_6 \\ \dot{y} = (\beta_1 + \beta_2 + \beta_3)x^2 + (-\beta_1 + \beta_2 - \beta_3)y^2 + (\alpha_1 - \alpha_3 - 1)2xy + (\beta_4 + \beta_5)x + (\alpha_4 - \alpha_5)y + \beta_6 \end{cases}.$$

We introduce $A_1 = \alpha_1 + \alpha_3 + \alpha_2,\ A_2 = \alpha_1 + \alpha_2 - \alpha_3,\ A_3 = -\beta_1 + \beta_3,\ A_4 = \alpha_4 + \alpha_5,\ A_5 = -\beta_4 + \beta_5,\ A_6 = \alpha_6$ and hence

$$\dot{x} = -\frac{1}{4} + (A_1 - 1)x^2 + (A_2 + 1)y^2 + 2A_3 xy + A_4 x + A_5 y + A_6,$$

and also $B_1 = \beta_1 + \beta_2 + \beta_3,\ B_2 = -\beta_1 + \beta_2 - \beta_3,\ B_3 = \alpha_1 - \alpha_3,\ B_4 = \beta_4 + \beta_5,\ B_5 = \alpha_4 - \alpha_5,\ B_6 = \beta_6$ so that

$$\dot{y} = B_1 x^2 + B_2 y^2 + 2(B_3 + 1)xy + B_4 x + B_5 y + B_6.$$

Then we have

$$\begin{cases} \dot{x} = -\frac{1}{4} + (A_1 - 1)x^2 + (A_2 + 1)y^2 + 2A_3 xy + A_4 x + A_5 y + A_6 \\ \dot{y} = B_1 x^2 + B_2 y^2 + 2(B_3 + 1)xy + B_4 x + B_5 y + B_6 \end{cases}.$$

We define a vector $\vec{a} = (A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4, B_5, B_6)$.

If X=(P,Q) we have that

$$X(0, \frac{1}{2}, \vec{0}) = 0 \tag{7.1}$$

and

$$X(0, -\frac{1}{2}, \vec{0}) = 0. \tag{7.2}$$

Then the fixed points of the perturbation are (x_p, y_p) and (x_q, y_q), of the form

$$\begin{cases} (x_p, y_p) & = (0, \frac{1}{2}) + (r_1, r_2) \\ (x_q, y_q) & = (0, -\frac{1}{2}) + (r_3, r_4) \end{cases},$$

where $(r_1, r_2), (r_3, r_4)$ small depending on $\vec{a}$.

Let us calculate the first order terms of $(r_1, r_2), (r_3, r_4)$ as a function of $\vec{a}$.

We derivate implicitly $X(x(\vec{a}), y(\vec{a}), \vec{a}) = 0$

$$D_{(x,y)}X(0, \frac{1}{2}, \vec{0}) \begin{pmatrix} D_{\vec{a}}x(\vec{0}) \\ D_{\vec{a}}y(\vec{0}) \end{pmatrix} + D_{\vec{a}}X(0, \frac{1}{2}, \vec{0}) = 0,$$

then we get that

$$\begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} = -D_{(x,y)}X(0, \frac{1}{2}, \vec{0})^{-1} * D_{\vec{a}}X(0, \frac{1}{2}, \vec{0}).$$

We compute

$$D_{(x,y)}X = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} (A_1 - 1)2x + 2A_3 y + A_4 & (A_2 + 1)2y + 2A_3 x + A_5 \\ 2B_1 x + 2(B_3 + 1)x + B_4 & 2B_2 y + 2(B_3 + 1)x + B_5 \end{pmatrix}$$

then $D_{(x,y)}X(0, \frac{1}{2}, \vec{0}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and now

$$D_{\vec{a}}X = \begin{pmatrix} x^2 & y^2 & 2xy & x & y & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & y^2 & 2xy & x & y & 1 \end{pmatrix}$$

and

$$D_{\vec{a}}X(0, \frac{1}{2}, \vec{0}) = \begin{pmatrix} 0 & 1/4 & 0 & 0 & 0 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

then we get

$$\begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} = -D_{(x,y)}X(0, \frac{1}{2}, \vec{0})^{-1} * D_{\vec{a}}X(0, \frac{1}{2}, \vec{0})$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1/4 & 0 & 0 & 0 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

So

$$(r_1, r_2) = \begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} * \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ B_5 \\ B_6 \end{pmatrix} + O(\|\vec{a}\|^2) = \begin{pmatrix} \frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6 \\ -\frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6 \end{pmatrix} + O(\|\vec{a}\|^2)$$

Analogously, we get

$$(r_3, r_4) = \begin{pmatrix} -\frac{1}{4}B_2 + \frac{1}{2}B_5 - B_6 \\ \frac{1}{4}A_2 - \frac{1}{2}A_5 + A_6 \end{pmatrix} + O(\|\vec{a}\|^2).$$

Finally we have the fixed points of the perturbed system

$$(x_p, y_p) = (\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6, \frac{1}{2} - \frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6) + O(\|\vec{a}\|^2)$$

and

$$(x_q, y_q) = (-\frac{1}{4}B_2 + \frac{1}{2}B_5 - B_6, -\frac{1}{2} + \frac{1}{4}A_2 - \frac{1}{2}A_5 + A_6) + O(\|\vec{a}\|^2)$$

Now let us see the character of both points:

The classification of equilibrium points is determined by the eigenvalues λ_1, λ_2 of the matrix $D_{(x,y)}X(x_p, y_p)$ If we write

$$D_{(x,y)}X(x_p, y_p) = \begin{pmatrix} a & b+1 \\ c-1 & d \end{pmatrix}$$

we have

$$a = (A_1 - 1)2(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + 2A_3(\frac{1}{2} - \frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6) + A_4,$$
$$b + 1 = (A_2 + 1)2(\frac{1}{2} - \frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6) + 2A_3(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + A_5,$$
$$c - 1 = 2B_1(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + 2(B_3 + 1)(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + B_4,$$
$$d = 2B_2(\frac{1}{2} - \frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6)) + 2(B_3 + 1)(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + B_5.$$

Is easy to find the eigenvalues λ_1, λ_2 of the matrix $DX(x_p, y_p)$. The numbers λ_1, λ_2 can be found by solving the characteristic equation

$$\lambda^2 - (a + d)\lambda + [ad - (b + 1)(c - 1)] = 0,$$

and we obtain

$$\lambda = \frac{(a + d) \pm \sqrt{(a + b)^2 - 4[ad - (b + 1)(c - 1)]}}{2}.$$

We know that $\lambda_1 = i + \xi$ and if $\vec{a} = 0$, $\xi = 0$, and both singular point are centers. But if $\vec{a} \neq 0$ in general $\xi \neq 0$ and both singular points are focus.

Let us compute the real part of λ_1, λ_2 when $\vec{a} \neq 0$. We have that if $\vec{a}$ is small

$$2\Re(\lambda_1) = a + d = 2(A_1 - 1)(\frac{1}{4}B_2 + \frac{1}{2}B_5 + B_6) + 2A_3(\frac{1}{2} - \frac{1}{4}A_2 - \frac{1}{2}A_5 - A_6) + A_4 +$$

$$+2B_2(\frac{1}{2}-\frac{1}{4}A_2-\frac{1}{2}A_5-A_6))+2(B_3+1)(\frac{1}{4}B_2+\frac{1}{2}B_5+B_6)+B_5.$$

Keeping only the first order terms: $a+d=A_3+A_4+B_2+B_5$, so we have obtained that, if $A_3+A_4+B_2+B_5<0$ the singular point (x_p,y_p) is a stable focus and if $A_3+A_4+B_2+B_5>0$ it is an unstable focus.

In the same way we prove that the eigenvalue λ_1 corresponding to (x_q,y_q) satisfies $2\Re(\lambda_2)=a+d=-A_3+A_4-B_2+B_5$.

Now we study the existence of limit cycle for the perturbed equation

$$\dot{w}=\frac{1}{4}+w^2+\epsilon R(w,\bar{w}).$$

Following [10], let $\dot{z}=f(z)$ be an holomorphic differential equation having a center at p, and consider the following perturbation $\dot{z}=f(z)+\epsilon R(z,\bar{z})$. We give an integral expression, similar to an Abelian integral, whose zeros control the limit cycles that bifurcate from the periodic orbits of the period annulus of p. This expression is given in terms of the liberalizing map of $\dot{z}=f(z)$ at p. The result is applied to control the simultaneous bifurcation of limit cycles from the two period annuli of $\dot{z}=iz+z^2$, after a polynomial perturbation.

The first result is following:

Theorem 7.1. *Consider the differential equation*

$$\dot{z}=f(z)+\epsilon R(z,\bar{z}), \tag{7.3}$$

where $\dot{z}=f(z)$ has a center at p. Let ϕ be a liberalizing change of variable of $\dot{z}=f(z)$ at p. Then the simple zeros $\hat{c}\in(0,c_0)$ of the function

$$I(c)=I^p(c)=-\Im(\int_{\gamma_c:=\{w\bar{w}=c\}}\overline{\phi'(z)R(z,\bar{z})dw}), \tag{7.4}$$

give rise, for ϵ small enough, to limit cycles of (6.3) that tend when ϵ goes t zero to $\phi^{-1}(\gamma_{\hat{c}})$. Here c_0 is given by the image by ϕ of the boundary of the period annulus of p.

As an application of the above result we study the system

$$\dot{z}=iz+z^2+\epsilon R_m(z,\bar{z}), \tag{7.5}$$

where ϵ is real and small and $R_m(z,\bar{z})$ is any polynomial of degree less or equal than m, that is ,

$$R_m(z,\bar{z})=\sum_{l=0}^{m}\sum_{k=0}^{l}\bar{a}_{k,l}z^{l-k}\bar{z}^k, \tag{7.6}$$

where $a_{k,l} \in \mathbb{C}$ are free parameters.

Then from Theorem 7.1 we concluded that the number of limit cycles after perturbation of Eq.(7.5) with $m = 2$ is at most 1 in each period annulus.

With a lemma in [10] we can see that if $m = 2$, then

$$I_2(c) = 2\pi c\{2\Im(a_{0,0}) + \Re(a_{0,1}) + [-\Re(a_{1,1}) + \Im(a_{1,2})]c\}. \tag{7.7}$$

c is the zero of Eq.(7.7) $\Rightarrow I_2(c) = 0 \Rightarrow$

$$c = \frac{2\Im(a_{0,0}) + \Re(a_{0,1})}{\Re(a_{1,1}) - \Im(a_{1,2})}.$$

We choose $\Im(a_{0,0}) = \Im(a_{1,2}) = 0$ and $\Re(a_{0,1}) = \alpha$ and $\Re(a_{1,1}) = \beta$ with $\alpha, \beta \neq 0$ and $\alpha, \beta \in \mathbb{R}$.

Since $\hat{c}$ has to verify, $\hat{c} \in (0, 1)$ we take $\alpha = 0.05, \beta = 0.1$ and $\epsilon = 1$

Since

$$R_2(z, \bar{z}) = \sum_{l=0}^{2} \sum_{k=0}^{l} \bar{a}_{k,l} z^{l-k} \bar{z}^k = \bar{a}_{0,0} + \bar{a}_{0,1} z + \bar{a}_{1,1} \bar{z} + \bar{a}_{0,2} z^2 + \bar{a}_{2,1} z\bar{z} + \bar{a}_{2,2} \bar{z}^2.$$

we can write that

$$\dot{z} = iz + z^2 + 0.05z + 0.1\bar{z}. \tag{7.8}$$

As before we make the translation $z = w - \frac{1}{2}i$, then

$\dot{w} = i(w - \frac{1}{2}i) + (w - \frac{1}{2}i)^2 + 0.05(w - \frac{1}{2}i) + 0.1(\bar{w} + \frac{1}{2}i) = w^2 + \frac{1}{4} + 0.05w - 0.025i + 0.1\bar{w} + 0.05i,$

$\dot{w} = (x + iy)^2 + \frac{1}{4} + 0.05(x + iy) - 0.025i + 0.1(x - iy) + 0.05i,$

therefore we work with the system

$$\begin{cases} \dot{x} = \frac{1}{4} + x^2 - y^2 + 0.15x = P(x, y) \\ \dot{y} = 2xy - 0.05y + 0.025 = Q(x, y) \end{cases}.$$

We define the vector $\vec{a} = (\alpha, \beta)$.

If X=(P,Q) and we can say that

$$X(0, \frac{1}{2}, \vec{0}) = 0. \tag{7.9}$$

Then $(x_p, y_p) = (0, \frac{1}{2})$.

$$D_{(x,y)}X(x_p, y_p) = \begin{pmatrix} 0.15 & -1 \\ 1 & -0.05 \end{pmatrix},$$

we know that the eigenvalue λ is the solution of the equation $\lambda^2 + 0.1\lambda + 0.9925 = 0$ then $\lambda = \frac{-0.1 \pm \sqrt{-3.96}}{2}$ and the part real is negative, so the point singular (x_p, y_p) is an attracting focus.

By the equation (6.8) we can say that, for $x(\vec{0}) \approx 0$, $y(\vec{0}) \approx -1/2$,

$$X(x(\vec{a}), y(\vec{a}), \vec{a}) = D_{(x,y)}X(0, -\frac{1}{2}, \vec{0}) \begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} + D_{\vec{a}}X(0, -\frac{1}{2}, \vec{0}) = 0$$

then we get that

$$\begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} = -D_{(x,y)}X(0, -\frac{1}{2}, \vec{0})^{-1} * D_{\vec{a}}X(0, -\frac{1}{2}, \vec{0}).$$

$$D_{(x,y)}X = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x + 0.15 & -2y \\ 2y & 2x + 0.05 \end{pmatrix} \Rightarrow D_{(x,y)}X(0, -\frac{1}{2}, \vec{0}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

And now

$$D_{\vec{a}}X = \begin{pmatrix} x & x \\ y - 0.5 & -y + 0.5 \end{pmatrix} \Rightarrow D_{\vec{a}}X(0, -\frac{1}{2}, \vec{0}) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

then we get

$$\begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} = -D_{(x,y)}X(0, -\frac{1}{2}, \vec{0})^{-1} * D_{\vec{a}}X(0, -\frac{1}{2}, \vec{0}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

So

$$(r_3, r_4) = \begin{pmatrix} D_{\vec{a}}x(0) \\ D_{\vec{a}}y(0) \end{pmatrix} * \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha + \beta \\ 0 \end{pmatrix}.$$

Then we get $(r_3, r_4) = (\alpha - \beta, 0)$ and $(x_q, y_q) = (-\alpha + \beta, -\frac{1}{2}) = (0.05, -0.5)$.

$$D_{(x,y)}X(x_q, y_q) = \begin{pmatrix} 0.05 & 1 \\ -1 & -0.15 \end{pmatrix},$$

we know that the eigenvalue λ is the solution of the equation $\lambda^2 - 0.1\lambda + 0.9925 = 0$ then $\lambda = \frac{0.1 \pm \sqrt{-3.96}}{2}$ and the part real is positive, so the singular point (x_q, y_q) is a repelling focus.

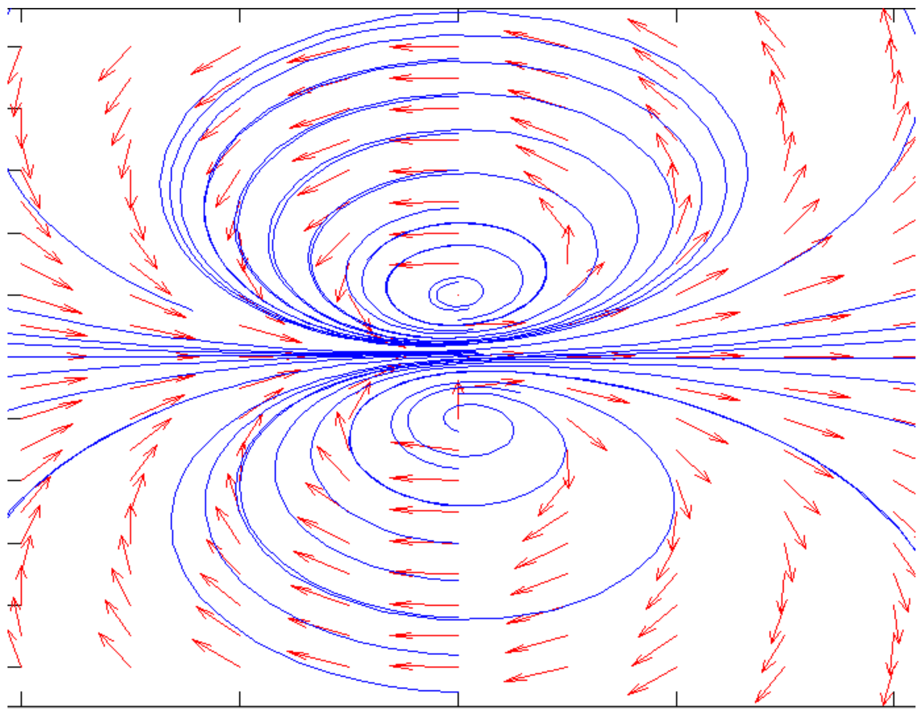

Figure 7.2: $\dot{w} = w^2 + \frac{1}{4} + \epsilon R(w, \bar{w})$ en $\mathbb{C}$

Let us see the existence of the limit cycle after perturbation from the period annuli of the two centers.

As we said at the beginning of this chapter, in [10] the authors study the parameter values for which equations (7.3) have different configurations of simultaneous limit cycles coming from the period annuli of the two centers of the unperturbed system $\dot{z} = iz + z^2$.

Taking into account the precious computations of this section we obtain that the natural transversal segment for computing the Poincaré map are $\Sigma = i\mathbb{R}_+$ for the center located at (0,1/2) and $\Sigma = i\mathbb{R}_- + 0.05$ for the center located at (0,-1/2).

The limit cycles should correspond to the fixed point of the Poincaré map (different of the two perturbed fixed points near $z = 1/2i$ and $z = -1/2i$).

In numerical analysis, the Runge–Kutta methods are an important family of implicit and explicit iterative methods, which are used in temporal discrimination for the approximation of solutions of ordinary differential equations.

To do all the computation we use MATLAB. The idea is as follows.

In our case, the differential equation is given by

$$\begin{cases} \dot{x} = \frac{1}{4} + x^2 - y^2 + 0.15x \\ \dot{y} = 2xy - 0.05y + 0.025 \end{cases}.$$

Then we can use the function runge-kutta4 to compute points with different initial points. And we get two points very close to the $x = 0$ for y positive (or $x = 0.05$ for y negative).

By the definition of the limit cycle, we can see that if we have a limit cycle as in Figure 7.3, we have an initial point (x_0, y_0) and another point on the same line (x_0, y_1), then $y_1 > y_0$ later another line with initial point (x_0, y_2) and (x_0, y_3) then $y_2 < y_3$, then we can say a limit cycle exists.

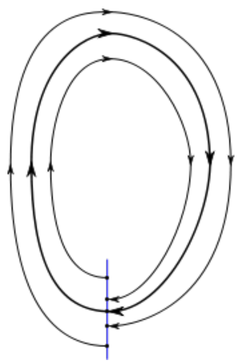

Figure 7.3: Limit cycle

For example, we apply the Runge-Kutta 4 function and initial point (0,0.1) and stepsize $h = 0.05$

$$[x \quad y] = rk4(0, 0.1, 0.05),$$

and we integrate until the x variable passes from positive to negative. We get the point $(x(i), y(i)) = (0.0047, 0.0914)$, then we use the Taylor formula to compute a new stepsize h1:

$$x(i+1) = 0 \approx x(i) * f'(x(i), y(i)) * h1 \Rightarrow h1 = -\frac{x(i)}{P(x(i), y(i))} = -0.0193.$$

Then with the new stepsize and we apply the 'rk4'function one more time, we have another point $x_2 = 6.3082e - 06$ which is closer to $x = 0$, and we go on and we get $x_3 = 1.1080e - 11$ which is the very small.

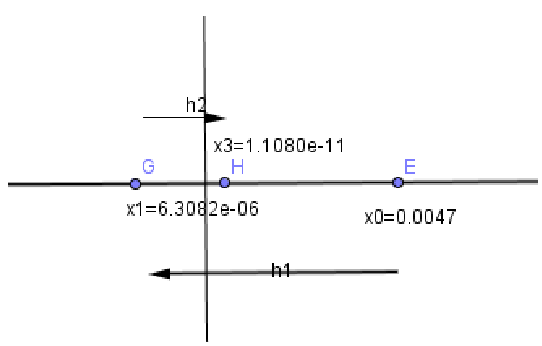

Figure 7.4:

With this formula, we can get all of the points with different initial points in the axis $y \in (0, 0.5)$ in the upper semiplane (and $y \in (-0.52, 0)$ in the lower semiplane).

We take $\Sigma = \{z \in \mathbb{C}, z = ti, t \in (0, 1/2)\}$, and a partion of it and for each point in this partion we compute numerically its image by the Poincaré map $\Pi : \Sigma \to \Sigma$. This computation uses Runge-Kutta 4 and it needs to adapt the stepsize to obtain the precise image of the Poincaré map for each initial condition.

Doing this for each initial conditions gives a list of pairs $(t_i, \pi(t_i))$, $i = 1, 2, ...$ Finally we draw the numerical plot of the resultanting function.

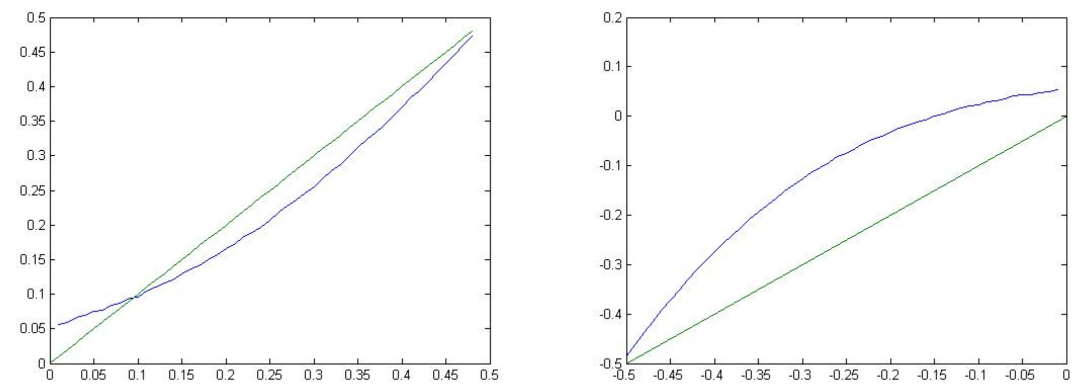

Figure 7.5:

And where is the limit cycle? It is easy to get, because we have all of the points, we use the Secant Method, we get the limit cycle at $y = 0.091$ also we get that there are not a limit cycle in the lower semiplane, let us prove that.

In [10] we have another expression of $I_2^{-i/2}(c)$ which coincides with the expression of $I_2^{-i/2}(c)$ given as before where each $a_{k,l}$ is substituted by the corresponding $b_{k,l}$, being,

$$b_{k,l} = (-1)^{l-k} \sum_{p=l}^{m} \sum_{q=k}^{p-(l-k)} a_{q,p} i^{p-l} (-1)^q \binom{q}{k} \binom{p-q}{l-k}.$$

In this case m=2, as we can see:

If $k = 0, l = 0$, then the formula $b_{0,0}$ is

$$b_{0,0} = (-1)^{0-0} \sum_{p=0}^{2} \sum_{q=0}^{p-0} a_{q,p} i^{p-0} (-1)^q \binom{q}{0} \binom{p-q}{0} = \sum_{p=0}^{2} \sum_{q=0}^{p} a_{q,p} i^p (-1)^q$$

$p=0, q=0 \Rightarrow a_{0,0}(-1)^0 = a_{0,0}$,

$p=1, q=0 \Rightarrow a_{0,1}i^1(-1)^0 = a_{0,1}i$,

$p=1, q=1 \Rightarrow a_{1,1}i^1(-1)^1 = a_{1,1}(-i)$,

$p=2, q=0 \Rightarrow a_{0,2}i^2(-1)^0 = a_{0,2}(-1)$,

$p=2, q=1 \Rightarrow a_{1,2}i^2(-1)^1 = a_{1,2}(-1)(-1)$,

$p=2, q=2 \Rightarrow a_{2,2}i^2(-1)^2 = a_{2,2}(-1)$.

Then

$$b_{0,0} = a_{0,0} + a_{0,1}i - a_{1,1}i - a_{0,2} + a_{1,2} - a_{2,2}.$$

As before, if $k=0, l=1$

$$b_{0,1} = (-1)^{1-0} \textstyle\sum_{p=1}^{2} \sum_{q=0}^{p-(1-0)} a_{q,p} i^{p-1}(-1)^q \binom{q}{0}\binom{p-q}{1} = (-1)^1 \sum_{p=1}^{2}\sum_{q=0}^{p-1} a_{q,p} i^{p-1}(-1)^q (p-q)$$

$p=1, q=0 \Rightarrow a_{0,1}i^{1-1}(-1)^0(p-0) = -a_{0,1}$,

$p=2, q=0 \Rightarrow a_{0,2}i^{2-1}(-1)^0(2-0) = -a_{0,2}2i$,

$p=2, q=1 \Rightarrow a_{1,2}i^{2-1}(-1)^1(2-1) = a_{1,2}i$.

Then

$$b_{0,1} = -a_{0,1} - 2ia_{0,2} + a_{1,2}i.$$

If $k=1, l=1$

$$b_{1,1} = (-1)^{1-1} \textstyle\sum_{p=1}^{2} \sum_{q=1}^{p-(1-1)} a_{q,p} i^{p-1}(-1)^q \binom{q}{1}\binom{p-q}{1-1} = \sum_{p=1}^{2}\sum_{q=1}^{p} -a_{q,p} i^{p-1}(-1)^q q$$

$p=1, q=1 \Rightarrow -a_{1,1}i^{1-1}1 = -a_{1,1}$,

$p=2, q=1 \Rightarrow -a_{1,2}i^{2-1}1 = -a_{1,2}i$,

$p=2, q=2 \Rightarrow a_{2,2}i^{2-1}2 = 2a_{2,2}i$.

$$b_{1,1} = -a_{1,1} - a_{1,2}i + 2a_{2,2}i.$$

If $k = 1, l = 2$

$b_{1,2} = (-1)^{2-1} \sum_{p=2}^{2} \sum_{q=1}^{p-(2-1)} a_{q,p} i^{p-2} (-1)^q \binom{q}{1} \binom{p-q}{2-1} = (-1) \sum_{p=2}^{2} \sum_{q=1}^{p-1} a_{q,p} i^{p-2} (-1)^q q(p-q).$

$p = 2, q = 1 \Rightarrow -a_{1,2} i^{2-2} (-1)^1 1(2-1) = a_{1,2}$, then

$$b_{1,2} = a_{1,2}.$$

As before, we know that,

$$I_2^{i/2}(c) = 2\pi c\{2\Im(a_{0,0}) + \Re(a_{0,1}) + [-\Re(a_{1,1}) + \Im(a_{1,2})]c\},$$
$$I_2^{-i/2}(c) = 2\pi c\{2\Im(b_{0,0}) + \Re(b_{0,1}) + [-\Re(b_{1,1}) + \Im(b_{1,2})]c\}.$$

Then

$$b_{0,0} = a_{0,0} + a_{0,1}i - a_{1,1}i - a_{0,2} + a_{1,2} - a_{2,2} =$$

$= \Re(a_{0,0}) + i\Im(a_{0,0}) + \Re(a_{0,1})i - \Im(a_{0,1}) - \Re(a_{1,1})i + \Im(a_{1,1}) - \Re(a_{0,2}) - i\Im(a_{0,2}) + \Re(a_{1,2}) + i\Im(a_{1,2}) - \Re(a_{2,2}) - i\Im(a_{2,2}).$

And as before, we know that,

$$b_{0,1} = -a_{0,1} - 2ia_{0,2} + a_{1,2}i = -\Re(a_{0,1}) - i\Im(a_{0,1}) - 2i\Re(a_{0,2}) + 2\Im(a_{0,2}) + \Re(a_{1,2})i - \Im(a_{1,2}).$$

$$b_{1,1} = -a_{1,1} - a_{1,2}i + 2a_{2,2}i = -\Re(a_{1,1}) - i\Im(a_{1,1}) - \Re(a_{1,2})i + \Im(a_{1,2}) + 2\Re(a_{2,2})i - 2\Im(a_{2,2}).$$

$$b_{1,2} = a_{1,2} = \Re(a_{1,2}) + i\Im(a_{1,2}).$$

As before $\Im(a_{0,0}) = \Im(a_{1,2}) = 0 \Rightarrow$

$b_{0,0} = \Re(a_{0,0}) + \Re(a_{0,1})i - \Im(a_{0,1}) - \Re(a_{1,1})i + \Im(a_{1,1}) - \Re(a_{0,2}) - i\Im(a_{0,2}) + \Re(a_{1,2}) - \Re(a_{2,2}) - i\Im(a_{2,2}),$

$b_{0,1} = -\Re(a_{0,1}) - i\Im(a_{0,1}) - 2i\Re(a_{0,2}) + 2\Im(a_{0,2}) + \Re(a_{1,2})i,$

$b_{1,1} = -\Re(a_{1,1}) - i\Im(a_{1,1}) - \Re(a_{1,2})i + 2\Re(a_{2,2})i - 2\Im(a_{2,2}),$

$b_{1,2} = \Re(a_{1,2}).$

$\Im(b_{0,0}) = \Re(a_{0,1}) - \Re(a_{1,1}) - \Im(a_{0,2}) - \Im(a_{2,2}),$

$\Re(b_{0,1}) = -\Re(a_{0,1}) + 2\Im(a_{0,2}),$

$\Re(b_{1,1}) = -\Re(a_{1,1}) - 2\Im(a_{2,2}),$

$\Im(b_{1,2}) = 0.$

For $2\Im(b_{0,0}) + \Re(b_{0,1}) = \Re(a_{0,1}) - 2\Re(a_{1,1}) + 2\Im(a_{2,1}) - 2\Im(a_{2,2})$ and
$-\Re(b_{1,1}) + \Im(b_{1,2}) = \Re(a_{1,1}) - \Im(a_{2,1}) - 2\Im(a_{2,2})$, then

$$c_{-i/2} = \frac{2\Im(b_{0,0}) + \Re(b_{0,1})}{\Re(b_{1,1}) - \Im(b_{1,2})} = \frac{\Re(a_{0,1}) - 2\Re(a_{1,1}) - 2\Im(a_{2,2})}{-\Re(a_{1,1}) - 2\Im(a_{2,2})}.$$

As before $\Re(a_{0,1}) = 0.05, \Re(a_{1,1}) = 0.1$ and in my case $a_{2,2} = 0$, then

$c_{-i/2} = \frac{-0.15}{-0.1} > 1 \Rightarrow$do not have cycle limit!

Definition 7.2. Given Eq.(6.5) we will say that the configuration of limit cycles (u, v), $u \geq 0$, $v \geq 0$, is realizable if, for ϵ small enough, exactly u (respectively v) limit cycles bifurcate from the periodic orbits surrounding $z = 0$ (respectively $z = -i$).

And we have a Theorem with the main result in [10].

Theorem 7.3. *For Eq.(6.5) the following configurations of limit cycles are realizable:*
(a) (u, v) with $0 \leq u \leq 1$ and $0 \leq v \leq 1$, if m=1, 2 or 3.
(b) (u, v) with $0 \leq u \leq m - 2$ and $0 \leq v \leq m - 2$, if $4 \leq m \leq 8$.
(a) (u, v) with $0 \leq u \leq m - 2$ and $0 \leq v \leq m - 2$ and $u + v \leq m + 4$, if $m > 8$.

In this case Eq. is $\dot{w} = \frac{1}{4} + w^2 + \epsilon R(w, \bar{w})$, for ϵ small enough, exactly u (respectively v) limit cycles bifurcate from the periodic orbits surrounding $z = i/2$ (respectively $z = -i/2$) and $m = 2$ and by Theorem 6.3 (a) we can see the configuration of limit cycles (u, v) with $0 \leq u \leq 1$ and $0 \leq v \leq 1$, it means the configuration of limit cycles are $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. As before we provide a case of $(1, 0)$. Now let us see another three cases:

As before we know that if we have the simple zeros $c \in (0, 1)$ of the function I_2^p with $p = i/2, -i/2$ where

$$I_2^{i/2}(c) = 2\pi c\{2\Im(a_{0,0}) + \Re(a_{0,1}) + [-\Re(a_{1,1}) + \Im(a_{1,2})]c\}$$

and

$$I_2^{-i/2}(c) = 2\pi c\{2\Im(b_{0,0}) + \Re(b_{0,1}) + [-\Re(b_{1,1}) + \Im(b_{1,2})]c\}.$$

If $I_2^{i/2}(c) = 0$ (respectively $I_2^{-i/2}(c) = 0$) we can prove that the formula of c are:

$$c_{i/2} = \frac{2\Im(a_{0,0}) + \Re(a_{0,1})}{\Re(a_{1,1}) - \Im(a_{1,2})} \quad \text{and} \quad c_{-i/2} = \frac{2\Im(b_{0,0}) + \Re(b_{0,1})}{\Re(b_{1,1}) - \Im(b_{1,2})}.$$

We can write $c_{-i/2}$ where each $a_{k,l}$ is substituted by the corresponding $b_{k,l}$, as before we did ,

$$c_{-i/2} = \frac{2\Im(a_{0,0}) + 2\Re(a_{0,1}) - 2\Re(a_{1,1}) - 2\Im(a_{0,2}) + 2\Im(a_{1,2}) - 2\Im(a_{2,2}) - \Re(a_{0,1}) + 2\Im(a_{0,2}) - \Im(a_{1,2})}{-\Re(a_{1,1}) + \Im(a_{1,2}) - 2\Im(a_{2,2}) - \Im(a_{1,2})} =$$

$$= \frac{2\Im(a_{0,0}) + \Re(a_{0,1}) - 2\Re(a_{1,1}) + \Im(a_{1,2}) - 2\Im(a_{2,2})}{-\Re(a_{1,1}) + \Im(a_{1,2}) - 2\Im(a_{2,2})}.$$

Now with those notations we are able to find values of parameters for which we have configurations of $(0,0)$, $(0,1)$ and $(1,1)$ (we already have obtained an example for the (1,0)).

It means if $c_{i/2} \in (0,1)$ (respectively $I_2^{-i/2}(c) \in (0,1)$), $u = 1$(respectively $v = 1$),and if not, $u = 0$ (respectively $v = 0$).

For example, we make the computations for the case (0,0)

If $\Re(a_{0,1}) = \alpha = 0.05$, $\Re(a_{1,1}) = \beta = -0.1 \Rightarrow c_{i/2} < 0 \Rightarrow u = 0$ and we can compute the conditions above and we get,

$$c_{-i/2} = \frac{-2\Re(a_{1,1}) - 2\Im(a_{0,2}) + \Im(a_{1,2}) - 2\Im(a_{2,2}) + 2\Im(a_{0,2})}{-\Re(a_{1,1}) - 2\Im(a_{2,2})} = 2.5 \Rightarrow c_{-i/2} > 1,$$

so $v = 0 \Rightarrow (u,v) = (0,0)$

Finally, with for the case (1,0) we know that if $\Re(a_{0,1}) = \alpha = 0.05$, and $\Re(a_{1,1}) = \beta = 0.1 \Rightarrow c_{i/2} = 0.5 \Rightarrow u = 1$.

And with the equation $c_{-i/2} = (-0.15 - 2\Im(a_{2,2}))/(0.1 + 2\Im(a_{2,2})) \in (0,1)$ if we let $c_{-i/2} = 0.5$ after compute we get $\Im(a_{2,2}) = 0.067$.

This gives an example for the case (1,1).

8 Bibliography

References

[1] Freddy Dumortier, Jaume Llibre, Joan C.Artés (2000): Qualitative Theory of Planar Differential Systems.

[2] Benzinger, H.E (1991): Plane autonomous systems with rational vector fields. *Trans. of the AMS* **326**, 465-484.

[3] Walcher, S. (2000): Plane polynomial vector fields with prescribed invariant curves. *Proceedings of the Royal Society of Edinburgh* **130**, 633-649.

[4] Brickman, K.A (1997): Conformal equivalence of analytic flows. *J. of Differential Equations* **25**, 310-324.

[5] Gutierrez, C. and Sotomayor J. (1986): Principal lines on surfaces immersed with constant mean curvature. *Trans.AMS* **293**, 751-766.

[6] Olver, P.J. (1995): Equivalence, invariants and symmetries. *Cambridge University Press.*

[7] Jarque X. and Llibre J. (2001): Polynomial foliations of $\mathbb{R}^2$. *Pacific Journal of Mathematics* **197**, 53-72.

[8] Hajek, O. (1996): Notes on meromorphic dynamical systems I. *Mathematical Journal* **91**, 14-27.

[9] Sverdlove, R. (1978): Vector fields defined by complex function. *J. of Differential Equations* **34**, 427-439.

[10] Antonio Garijo, Armengol Gasull, Xavier Jarque (2008): Simultaneous bifurcation of limit cycles from two nests of periodic orbits. *J. Math. Anal. Appl.* **341**, 813-824.