Treball final de màster

MÀSTER DE MATEMÀTICA AVANÇADA

Facultat de Matemàtiques
Universitat de Barcelona

Hodge Theory on Compact Kähler Manifolds

Autor: Victòria Gras Andreu

Director:	**Dr. Ricardo García López**
Realitzat a:	**Departament de Àlgebra i Geometria**

Barcelona,	**June 30, 2014**

Contents

Introduction

The purpose of this master thesis is to study the Hodge Decomposition Theorem for compact Kähler manifolds.

Hodge theory, named after W.V.D. Hodge, is a branch of mathematics belonging to both algebraic topology and differential geometry that enables us to find topological information about a smooth or complex manifold from the study of differential forms and differential operators on these manifolds. Namely, we can find the singular cohomology groups or deduce properties of them with new tools derived from the Hodge theory.

It was first developed in the 1930s as an extension of the de Rham cohomology. Recall that the de Rham cohomology gives an isomorphism between the singular cohomology of a smooth manifold and the de Rham cohomology, which is given by the study of the differential forms on that manifold. Hodge theory includes this particular case and extends the results to more general types of manifolds.

The Theorem we are going to study, the Hodge Decomposition Theorem on compact Kähler manifolds, gives a decomposition of the singular kth cohomology group of a manifold of a specific type called Kähler. Even though the condition of being Kähler may seem very restrictive, there exists lots of examples of manifolds satisfying this condition. For example, we can think of Kähler manifolds as submanifolds of the complex projective space.

To prove this result, we are going to follow diagram 1. Let us explain a little bit the meaning of the diagram. First of all, using the de Rham Theorem, we can find an isomorphism between singular cohomology and the de Rham cohomology groups. Also, using the Dolbeault Theorem, we know that there exists an isomorphism between sheaf cohomology and the Dolbeault cohomology groups. Then, using elliptic operator theory, we will see that there exists an isomorphism between the de Rham cohomology groups and the group of harmonic operators with respect to the operator d and an isomorphism between the Dolbeault cohomology groups and the group of harmonic operators with respect to a new operator denoted by $\bar{\partial}$. Finally, since the groups of harmonic forms decompose as it is seen in the diagram and using the previous isomorphisms, we get the desired decomposition.

In order to follow these steps, the master thesis is divided in four chapters. One can find a brief description at the beginning of each chapter on what it contains. Let us summarize the content of each of them. The first chapter is devoted to present the definition of the principal objects we are going to work with. Concepts like complex manifold, vector bundle, almost complex manifold and the ∂, $\bar{\partial}$-operators are explained and illustrated with examples. In the second chapter, we will give an introduction of sheaf theory and the cohomology groups associated to sheaf complexes. We will see, without proving them, that there are isomorphisms between these cohomology groups and other types of cohomology

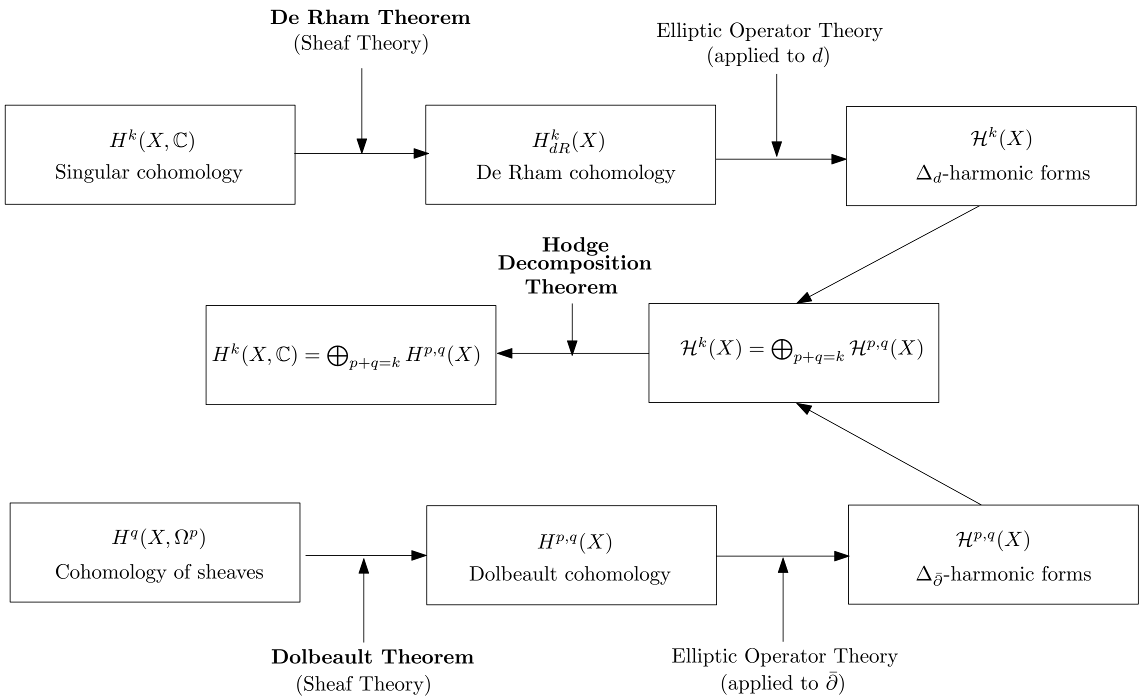

Figure 1: Diagram of the proof of the Hodge Decomposition Theorem.

groups, for instance, singular cohomology and the de Rham cohomology. The third chapter consists of an introduction to elliptic operator theory. The most remarkable result of this chapter is the last theorem, which relates the group of harmonic forms and cohomology groups associated to an elliptic complex, which it turns out that, in our interests, such elliptic complexes are the de Rham complex and the Dolbeault complex. Finally, the last chapter proves the Hodge Decomposition Theorem for compact Kähler manifolds. We will see that a Kähler manifold is a complex manfiold with an hermitian metric satisfying that to the imaginary part of the metric we can associate a 2-form of type $(1,1)$ that is d-closed, for d the exterior derivative.

Finally, I would like to thank Dr. Ricardo García, my thesis advisor, for the opportunity and trust placed in me together with his help and support. Last but not least, I would like to express my special thanks to my family for their advice, support and patience throughout these months.

Chapter 1

Complex and Almost Complex Manifolds

In this chapter we are going to recall and introduce the basic definitions that will appear along this master thesis. Concepts such as complex manifold, vector bundle, section and almost complex structure are the most important objects that we are going to work with.

Throughout this chapter, we are mostly going to follow [WE].

1.1 Manifolds

Let us begin with some notations. Let K denote either $\mathbb{R}$ or $\mathbb{C}$, which as usual denote the field of real and complex numbers respectively. Let D be an open set of K^n. Then,

- For $K = \mathbb{R}$, we denote $\mathcal{E}(D)$ the set of functions $f : D \to \mathbb{R}$ indefinitely differentiable on D.
- For $K = \mathbb{C}$, we denote $\mathcal{O}(D)$ the set of holomorphic functions $f : D \to \mathbb{C}$ on D.

We will denote $\mathcal{S}$ one of these two K-valued families of functions defined on open sets of K^n, we let $\mathcal{S}(D)$ denote the set of functions of $\mathcal{S}$ defined on an open set $D \subset K^n$.

Definition 1.1.1. An *$\mathcal{S}$-structure*, $\mathcal{S}_M$, on a topological manifold M is a family of K-valued continuous functions defined on every open sets of M such that

1. For every $p \in M$, there exists an open neighborhood U of p and a homeomorphism $h : U \to U'$, where U' is open in K^n, such that for any open set $V \subset U$
$$f : V \to K \in \mathcal{S}_M \text{ if and only if } f \circ h^{-1} \in \mathcal{S}(h(V)). \tag{1.1.1}$$
2. If $f : U \to K$, where $U = \cup_i U_i$ and U_i open in M, then $f \in \mathcal{S}_M$ if and only if $f|_{U_i} \in \mathcal{S}_M$ for every i.

The dimension of the topological manifold coincides with n if $K = \mathbb{R}$ and is $2n$ if $K = \mathbb{C}$. In any case, n is called the *K-dimension* of M and we denote it $\dim_K M = n$.

A manifold with an $\mathcal{S}$-structure is called *$\mathcal{S}$-manifold* and is denoted by $(M, \mathcal{S}_M)$. The elements of $\mathcal{S}_M$ are called *$\mathcal{S}$-functions* on M. An open subset $U \subset M$ and a homeomorphism $h : U \to U' \subset K^n$ as in (1.1.1) is called an *$\mathcal{S}$-coordinate system*.

For the previous two classes of functions we have

- $\mathcal{S} = \mathcal{E}$: *differentiable* or *smooth manifolds*, and the functions in $\mathcal{E}_M$ are called $\mathcal{C}^\infty$ or *smooth functions* on open subsets of M.
- $\mathcal{S} = \mathcal{O}$: *complex-analytic* or *complex manifolds*, and the functions in $\mathcal{O}_M$ are called *holomorphic* on open subsets of M.

Definition 1.1.2. An $\mathcal{S}$-*morphism* $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map, $F : M \to N$, such that

$$f \in \mathcal{S}_N \text{ implies that } f \circ F \in \mathcal{S}_M.$$

An $\mathcal{S}$-*isomorphism* is an $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ such that $F : M \to N$ is a homeomorphism, and

$$F^{-1} : (N, \mathcal{S}_N) \to (M, \mathcal{S}_M) \text{ is an } \mathcal{S}\text{-morphism.}$$

Note that if on an $\mathcal{S}$-manifold $(M, \mathcal{S}_M)$ we have two coordinate systems $h_1 : U_1 \to K^n$ and $h_2 : U_2 \to K^n$ such that $U_1 \cap U_2 \neq \varnothing$, then $h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$ is an $\mathcal{S}$-isomorphism on open subsets of $(K^n, \mathcal{S}_{K^n})$. Clearly, let $f \in \mathcal{S}(h_2(U_1 \cap U_2))$ or equivalently $(f \circ h_2) \circ h_2^{-1} \in \mathcal{S}(h_2(U_1 \cap U_2))$ then, by definition of $\mathcal{S}_M$, it implies that $f \circ h_2 : U_2 \cap U_1 \to K \in \mathcal{S}_M$. Again, by definition of $\mathcal{S}$-structure, $f \circ h_2 \in \mathcal{S}_M$ if and only if $(f \circ h_2) \circ h_1^{-1} \in \mathcal{S}(h_1(U_1 \cap U_2))$. Hence, $f \circ (h_2 \circ h_1^{-1}) \in \mathcal{S}(h_1(U_1 \cap U_2))$ as we wanted. Analogously we can see that $h_1 \circ h_2^{-1}$ is an $\mathcal{S}$-morphism.

If we have an open covering $\{U_i\}_{i \in I}$ of a topological manifold M, and a family of homeomorphisms $\{h_i : U_i \to U_i' \subset K^n\}_{i \in I}$, such that they satisfy the previous condition, then this defines an $\mathcal{S}$-structure on M by setting

$$\mathcal{S}_M = \{f : U \to K \text{ continuous, } U \subset M \text{ open, and } f \circ h^{-1} \in \mathcal{S}(h_i(U \cap U_i)) \text{ for all } i \in I\}.$$

The collection of $\{(U_i, h_i)\}_{i \in I}$ is called an *atlas* for $(M, \mathcal{S}_M)$.

Let N be an arbitrary subset of an $\mathcal{S}$-manifold M, then an $\mathcal{S}$-*function on* N is defined to be the restriction to N of an $\mathcal{S}$-function defined in some open set containing N. We denote $\mathcal{S}_M|_N$ the set of all functions defined on relatively open subsets of N which are restrictions of $\mathcal{S}$-functions on the open subsets of M.

Definition 1.1.3. Let N be a closed subset of a $\mathcal{S}$-manifold M, then N is called a $\mathcal{S}$-*submanifold* of M if for each point $x_0 \in N$, there is a coordinate system $h : U \to U' \subset K^n$, where $x_0 \in U$, with the property that $h|_{U \cap N}$ is mapped onto $U' \cap K^k$, where $0 \leq k \leq n$. Here $K^k \subset K^n$ is the standard embedding of the linear subspace K^k into K^n.

An $\mathcal{S}$-submanifold of an $\mathcal{S}$-manifold M is itself an $\mathcal{S}$-manifold with the $\mathcal{S}$-structure given by $\mathcal{S}_M|_N$.

Definition 1.1.4. We say that an $\mathcal{S}$-morphism

$$f : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$$

of two $\mathcal{S}$-manifolds is an $\mathcal{S}$-*embedding* if f is an $\mathcal{S}$-isomorphism onto an $\mathcal{S}$-submanifold of $(N, \mathcal{S}_N)$.

Examples 1.1.5. **a) Euclidean space**: For K^n being either $\mathbb{R}^n$ or $\mathbb{C}^n$, taking as an atlas the chart $U = K^n$ and $h = id$, K^n becomes a smooth or a complex manifold respectively.

b) Projective space: The set of one-dimensional subspaces of a finite dimensional vector space V over K is called *projective space* of V and is denoted $P(V)$. $P_n(K) := P(K^{n+1})$ for $K = \mathbb{R}$ or $\mathbb{C}$ are smooth and complex manifolds respectively.

c) Linear submanifolds: Let $H = \{[z_0, ..., z_n] \in P_n(\mathbb{C}) : a_0 z_0 + ... + a_n z_n = 0\}$, where $(a_0, ..., a_n) \in \mathbb{C}^{n+1} \backslash \{0\}$. Then H is called a *projective hyperplane* and is a submanifold of $P_n(\mathbb{C})$ of dimension $n - 1$.

It can also be proved that an algebraic variety, i.e. $\Sigma = \{[z_0, ..., z_n] \in P_n(\mathbb{C}) : f_1(z_0, ..., z_n) = ... = f_n(z_0, ..., z_n) = 0, f_i \in \mathbb{C}[Z_0, ..., Z_n]\}$, is also a submanifold of $\mathbb{P}_n(\mathbb{C})$.

1.2 Vector Bundles

The study of vector bundles on manifolds has been motivated by the desire of linearize nonlinear problems in geometry. The most important example of vector bundle is the tangent bundle of a differentiable or complex manifold; it linearizes the corresponding manifold. In this section we are going to give the definition of vector bundles and, in particular, of tangent bundle. Moreover, we will also recall what is a section over a vector bundle.

Definition 1.2.1. Let $\pi : E \to M$ be a map between two sets E and M. Then, given $p \in M$, we call $\pi^{-1}(p)$ the *fiber* of π at p.

Definition 1.2.2. Given $\pi : E \to M$, $\pi' : E \to M$, we say that $h : E \to E'$ is *fiber-preserving* if the diagram

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

is commutative, i.e. $\pi = \pi' \circ h$.

Definition 1.2.3. A continuous map $\pi : E \to M$ of one Hausdorff space onto another is called a *K-vector bundle of rank r* if the following conditions are satisfied:

- $E_p := \pi^{-1}(p)$, for $p \in M$, is a K-vector space of dimension r.
- For every $p \in M$ there is a neighborhood U of p in M and a homeomorphism

$$h : \pi^{-1}(U) \to U \times K^r \text{ such that } h(E_p) \subset \{p\} \times K^r,$$

and h^p, defined by the composition

$$h^p : E_p \overset{h}{\to} \{p\} \times K^r \overset{proj.}{\to} K^r,$$

is a K-vector space isomorphism. We call the pair (U, h) a *local trivialization.*

For a K-vector bundle $\pi : E \to M$, E is called the *total space* and M is called the *base space.*

For any two local trivializations (U_α, h_α) and (U_β, h_β) we have the map

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times K^r \to (U_\alpha \cap U_\beta) \times K^r.$$

Hence, for every $p \in M$ we have a linear map

$$h^p_\alpha \circ (h^p_\beta)^{-1} : K^r \to K^r.$$

Thus, we can define

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, K),$$

where $g_{\alpha\beta}(p) = h^p_\alpha \circ (h^p_\beta)^{-1}$, that is, $g_{\alpha\beta}(p)$ is the matrix that corresponds to the linear map $h^p_\alpha \circ (h^p_\beta)^{-1}$.

The functions $g_{\alpha\beta}$ are called *transition functions* of the K-vector bundle $\pi : E \to M$ (with respect to the two local trivializations).

The transition functions $g_{\alpha\beta}$ satisfy the compatibility conditions:

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = I_r, \text{ on } U_\alpha \cap U_\beta \cap U_\gamma,$$

and

$$g_{\alpha\alpha} = I_r, \text{ on } U_\alpha,$$

where the product is a matrix product and I_r is the identity matrix of rank r.

Definition 1.2.4. A K-vector bundle $\pi : E \to M$ is said to be an $\mathcal{S}$*-bundle* if E and M are $\mathcal{S}$-manifolds, π is an $\mathcal{S}$-morphism, and the local trivializations are $\mathcal{S}$-isomorphisms.

Example 1.2.5. (Trivial bundle): Let M be an $\mathcal{S}$-manifold. Then,

$$\pi : M \times K^n \to M,$$

where π is the natural projection, is an $\mathcal{S}$-bundle called a *trivial bundle*.

Example 1.2.6. (The infinite Möbius strip): Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $E = \dfrac{[0,1] \times \mathbb{R}}{(0,x) \sim (1,-x)}$ the Möbius strip. We can define the continuous map

$$\begin{array}{rccc} \pi : & E & \to & \mathbb{S}^1, \\ & (t,x) & \mapsto & e^{2\pi i t}. \end{array}$$

Clearly, for every $p = e^{2\pi i t} \in \mathbb{S}^1$ for some $t \in [0,1]$, $E_p = \pi^{-1}(p) = \{[(t,x)] : x \in \mathbb{R}\} \cong \mathbb{R}$. Thus, for every $p \in \mathbb{S}^1$, E_p is an $\mathbb{R}$-vector space of dimension 1.

Let $U_1 = \mathbb{S}^1 \backslash \{1\}$ and $U_2 = \mathbb{S}^1 \backslash \{-1\}$. These two open sets form an open cover of $\mathbb{S}^1$.

Note that $\pi^{-1}(U_1) = (0,1) \times \mathbb{R}$ and $\pi^{-1}(U_2) = \dfrac{([0,1/2) \cup (1/2,1]) \times \mathbb{R}}{(0,x) \sim (1,-x)}$.

Let us define the mappings

$$\begin{array}{rccc} h_1 : & \pi^{-1}(U_1) & \to & U_1 \times \mathbb{R} \\ & (t,x) & \mapsto & (e^{2\pi i t}, x), \end{array}$$

and also

$$\begin{array}{rccc} h_2 : & \pi^{-1}(U_2) & \to & U_2 \times \mathbb{R} \\ & (t,x) & \mapsto & \begin{cases} (e^{2\pi i t} + 1, -x), & 0 < t < 1/2, \\ (e^{2\pi i t}, x), & 1/2 < t < 1, \end{cases} \end{array}$$

which is clearly well defined since $h_2([(0,x)]) = (1,-x) = h_2([(1,-x)])$. It can be also checked that these two mappings are homeomorphisms and for every $p \in U_i$, $h_i(E_p) \subset \{p\} \times \mathbb{R}$.

Moreover, defining h_i^p as in definition 1.2.3, we have

$$\begin{array}{rccc} h_1^p: & E_p & \to & \mathbb{R}, \\ & [(t,x)] & \mapsto & x, \end{array} \qquad \begin{array}{rccl} h_2^p: & E_p & \to & \mathbb{R}, \\ & [(t,x)] & \mapsto & \begin{cases} -x, & t \in [0,1/2), \\ x, & t \in (1/2,1], \end{cases} \end{array}$$

which are $\mathbb{R}$-vector space isomorphisms.

Thus, $E \to \mathbb{S}^1$ is a $\mathcal{E}$-bundle over $\mathbb{S}^1$.

Making the composition of these mappings we can find the transition functions. Note that we can easily see that

$$\begin{array}{rccc} g_{12}: & \mathbb{S}^1\backslash\{-1,1\} & \to & GL(1,\mathbb{R}) \\ & e^{2\pi it} & \mapsto & \begin{cases} 1, & t \in (0,1/2), \\ -1, & t \in (1/2,1). \end{cases} \end{array}$$

Example 1.2.7. (Tangent bundle): Let M be a smooth manifold. We want to construct a vector bundle over M whose fiber at each point is the linearization of the manifold M. Let $p \in M$ and f and g be $\mathcal{C}^\infty$ functions near p. Then, we say that f and g are equivalent if they coincide at a neighborhood of p. The set of equivalence classes form an algebra over $\mathbb{R}$ and it is denoted by $\mathcal{E}_{M,p}$. Each equivalence class is called *germ* of a $\mathcal{C}^\infty$ function at p.

A *derivation* of $\mathcal{E}_{M,p}$ is a vector space homomorphism $D : \mathcal{E}_{M,p} \to \mathbb{R}$ such that for f and g germs at $p \in M$, $D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g)$. The *tangent space* to M at p is the vector space of all derivations of the algebra $\mathcal{E}_{M,p}$, which is denoted by $T_p(M)$. Since M is a smooth manifold, we can find a diffeomorphism $h : U \to U'$, where U is a neighborhood of p and $U' \subset \mathbb{R}^n$ open. Letting $h^* f(x) = f \circ h(x)$, h has the property that, for $V \subset U'$,

$$h^* : \mathcal{E}_{\mathbb{R}^n}(V) \to \mathcal{E}_M(h^{-1}(V))$$

is an algebra isomorphism. Clearly, for $f : V \subset \mathbb{R}^n \to \mathbb{R}$, V open and f smooth, then, $h^* f = f \circ h \in \mathcal{E}_M(h^{-1}(V))$ since $h^* f \circ h^{-1} = f \in \mathcal{S}_{\mathbb{R}^n}(h(h^{-1}(V)))$. Besides, h^* is a homomorphism of algebras for the properties of the functions in $\mathcal{E}_{\mathbb{R}^n}(V)$. And since h is a diffeomorphism, we can define

$$(h^{-1})^* f(x) = f \circ h^{-1}(x) : \mathcal{E}_M(V) \to \mathcal{E}_{\mathbb{R}^n}(h(V)).$$

As before, it is clear that is a homomorphism of algebras.

It follows that h^* induces an algebra isomorphism on germs, i.e.,

$$h^* : \mathcal{E}_{\mathbb{R}^n,h(p)} \xrightarrow{\cong} \mathcal{E}_{M,p},$$

and hence induces an isomorphism on derivations

$$h_* : T_p(M) \xrightarrow{\cong} T_{h(p)}(\mathbb{R}^n).$$

It is easy to verify that

- $\partial/\partial x_j$ are derivations of $\mathcal{E}_{\mathbb{R}^n,h(p)}$, $j = 1,...,n$ and that
- $\{\partial/\partial x_1,...,\partial/\partial x_n\}$ is a basis for $T_{h(p)}(\mathbb{R}^n)$,

and thus $T_p(M)$ is an n-dimensional vector space over $\mathbb{R}$ for each point $p \in M$ (the derivations are the classical directional derivatives evaluated at the point $h(p)$). Suppose that $f : M \to N$ is a differentiable mapping of differentiable manifolds. Then there is a natural map

$$d_p f : T_p(M) \to T_{f(p)}(N)$$

defined by the diagram

$$\begin{tikzcd} \mathcal{E}_{M,p} \arrow[dr, "D_p"'] & & \mathcal{E}_{N,f(p)} \arrow[ll, "f^*"'] \arrow[dl, "D_p \circ f^* = d_p f(D_p)"] \\ & \mathbb{R} & \end{tikzcd}$$

for $D_p \in T_p(M)$. The mapping $d_p f$ is linear mapping and can be expressed as a matrix of first derivatives with respect to local coordinates. The coefficients of such matrix representation will be $\mathcal{C}^\infty$ functions of the local coordinates. We call $d_p f$ the *differential mapping* or the *Jacobian* of the differentiable map f. It represents a first order approximation at p to the differentiable map f.

Let us now construct the tangent bundle to M:

$$T(M) = \bigcup_{p \in M} \{p\} \times T_p(M),$$

where this union is disjoint. We also define the map

$$\begin{array}{rccc} \pi : & T(M) & \to & M \\ & (p, v_p) & \mapsto & p \end{array}$$

In order to make $T(M)$ a vector bundle, let $\{(U_\alpha, h_\alpha)\}$ be an atlas for M, let $T(U_\alpha) = \pi^{-1}(U_\alpha)$ and define

$$\psi_\alpha : T(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

as follows: let $v \in \{p\} \times T_p(M) \subset T(U_\alpha)$. Then, $d_p h_\alpha(v) \in T_{h_\alpha(p)}(\mathbb{R}^n)$. Thus,

$$d_p h_\alpha(v) = \sum_{j=1}^n \xi_j(p) \left. \frac{\partial}{\partial x_j} \right|_{h_\alpha(p)},$$

where $\xi_j \in \mathcal{E}_M(U_\alpha)$. Set

$$\psi_\alpha(v) = (p, \xi_1(p), ..., \xi_n(p)) \in U_\alpha \times \mathbb{R}^n.$$

Clearly, ψ_α is bijective and also fiber-preserving since the following diagram commutes

$$\begin{tikzcd} T(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

where π_1 is the projection of the first coordinates. Also,

$$\psi_\alpha^p : T_p(M) \overset{\psi_\alpha}{\to} \{p\} \times \mathbb{R}^n \overset{\pi_1}{\to} \mathbb{R}^n$$

is an $\mathbb{R}$-linear isomorphism. We can define transition functions as follows

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$$

where

$$g_{\alpha\beta}(p) = \psi^p_\alpha \circ (\psi^p_\beta)^{-1} : \mathbb{R}^n \to \mathbb{R}^n.$$

The coefficients of the matrices $\{g_{\alpha\beta}\}$ are in $\mathcal{C}^\infty(U_\alpha \cap U_\beta)$, since $g_{\alpha\beta}$ is a matrix representation for the composition $dh_\alpha \circ dh_\beta^{-1}$ with respect to the basis $\{\partial/\partial x_1, ..., \partial/\partial x_n\}$ at $T_{h_\beta(p)}(\mathbb{R}^n)$ and $T_{h_\alpha(p)}(\mathbb{R}^n)$, and that the tangent maps are smooth functions of local coordinates. Thus, $\{(U_\alpha, h_\alpha)\}$ is the trivialization we wanted.

Finally, we have to give $T(M)$ a topology so that $T(M)$ becomes a smooth manifold. In order to do so, it is enough to require that $U \subset T(M)$ is open if and only if $\psi_\alpha(U \cap T(U_\alpha))$ is open in $U_\alpha \times \mathbb{R}^n$ for every α. It is clearly well defined since

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is a diffeomorphism for any α and β such that $U_\alpha \cap U_\beta \neq \varnothing$ (since $\psi_\alpha \circ \psi_\beta^{-1} = id \times g_{\alpha\beta}$). Because transition functions are diffeomorphisms, this defines a differentiable structure on $T(M)$ so that the projection π and the local trivializations ψ_α are differentiable maps.

For example, the tangent bundle of $\mathbb{S}^1$ and $\mathbb{S}^3$ is the trivial bundle but this is false for $\mathbb{S}^2$ the hairy ball theorem.

Example 1.2.8. (Holomorphic tangent bundle to a complex manifold): Let $M = (M, \mathcal{O}_M)$ be a complex manifold of complex dimension n. We will say that if f and g are equivalent if they are defined and holomorphic near p and they coincide on some neighborhood of p. Let $\mathcal{O}_{M,p}$ be the set of equivalence classes, which is $\mathbb{C}$-algebra. We define exactly as before the set $T_p(M)$ of derivations of the algebra $\mathcal{O}_{M,p}$. The complex partial derivatives $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$ form a basis over $\mathbb{C}$ for the vector space $T_p(\mathbb{C}^n)$. As before, taking the union of these tangent spaces,

$$T(M) = \bigcup_{p \in M} \{p\} \times T_p(M),$$

defines a holomorphic vector bundle over M with

$$\begin{array}{rccc} \pi : & T(M) & \to & M, \\ & (p, v_p) & \mapsto & p, \end{array}$$

whose fibers are all isomorphic to $\mathbb{C}^n$.

Definition 1.2.9. Let E and F be $\mathcal{S}$-bundles over M; i.e., $\pi_E : E \to M$ and $\pi_F : F \to M$. Then a *homomorphism of $\mathcal{S}$-bundles*,

$$f : E \to F,$$

is an $\mathcal{S}$-morphism of the total spaces which preserves fibers and is K-linear on each fiber.

An $\mathcal{S}$-bundle isomorphism is an $\mathcal{S}$-bundle homomorphism which is an $\mathcal{S}$-isomorphism on the total spaces and a K-vector space isomorphism on the fibers. Two $\mathcal{S}$-bundles are *equivalent* if there is some $\mathcal{S}$-bundle isomorphism between them.

Definition 1.2.10. An *$\mathcal{S}$-bundle morphism* between two $\mathcal{S}$-bundles $\pi_E : E \to X$ and $\pi_F : F \to Y$ is an $\mathcal{S}$-morphism $f : E \to F$ which takes fibers of E isomorphically (as vector spaces) onto fibers in F. An $\mathcal{S}$-bundle morphism $f : E \to F$ induces an $\mathcal{S}$-morphism $\bar{f}(\pi_E(e)) = \pi_F(f(e))$; in other words, the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Proposition 1.2.11. *Given an $\mathcal{S}$-morphism $f : X \to Y$ and an $\mathcal{S}$-bundle $\pi : E \to Y$, then there exists an $\mathcal{S}$-bundle morphism g such that the following diagram commutes:*

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Moreover, E' is unique up to equivalence. We call E' the pullback of E by f and denote it by f^*E.

Proof. We will prove the existence of E'. Let

$$E' = \{(x, e) \in X \times E : f(x) = \pi(e)\}.$$

Then we can consider

$$\begin{array}{rcl} g : E' & \to & E \\ (x,e) & \mapsto & e \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi' : E' & \to & X \\ (x,e) & \mapsto & x. \end{array}$$

Note that dor every $x \in X$, we can give $E'_x = \{x\} \times E_{f(x)}$ structure of K-vector space induced by $E_{f(x)}$.

Let (U, h) be a local trivialization for E, that is h is a homeomorphism

$$\pi^{-1}(U) \xrightarrow{h} U \times K^n,$$

then we can define for every $h' = id \times (pr \circ h)$, where $pr : U \times K^n \to K^n$ denotes the natural projection. Thus,

$$(\pi')^{-1}(f^{-1}(U)) \to f^{-1}(U) \times K^n$$

is a local trivialization of E' and hence, E' is a K-vector bundle of the same rank as E. □

Definition 1.2.12. An *$\mathcal{S}$-section* of an $\mathcal{S}$-bundle $\pi : E \to M$ is an $\mathcal{S}$-morphism $s : E \to M$ such that

$$\pi \circ s = id_M,$$

where id_M is the identity on M.

We will denote the $\mathcal{S}$-sections of E over M as $\mathcal{S}(M, E)$ and $\mathcal{S}(U, E) = \mathcal{S}(U, E|_U)$, where $E|_U = \pi^{-1}(U)$.

Example 1.2.13. • The sections of the trivial bundle $M \times \mathbb{R}$ over a differentiable manifold M can be identified with $\mathcal{E}(M)$, that is, the global real-valued functions on M. This is clear since an $\mathcal{E}$-section of $M \times \mathbb{R}$ over M is a map $s : M \to M \times \mathbb{R}$ of the form $s = id_M \times f$, for $f : M \to \mathbb{R}$ smooth.

- The term *zero section* is used to refer to the section $0 : M \to E$ given by $0(p) = 0 \in E_p$, that is, it maps every element p of M to the zero element of the vector space E_p.

Let M be a differentiable manifold and let $T(M) \to M$ be its tangent bundle. We would like to consider new differentiable vector bundles over M derived from $T(M)$. We have:

(a) The *cotangent bundle*, $T^*(M)$, whose fiber at $p \in M$, $T_p^*(M)$, is the $\mathbb{R}$-linear dual to $T_p(M)$.

(b) The *exterior algebra bundles*, $\bigwedge^p T(M)$, $\bigwedge^p T^*(M)$, whose fiber at $p \in M$ is the wedge product (of degree k) of the vector spaces $T_p(M)$ and $T_p^*(M)$, respectively, and

$$\bigwedge T(M) = \bigoplus_{i=0}^{k} \bigwedge^{i} T(M), \qquad \bigwedge T^*(M) = \bigoplus_{i=0}^{k} \bigwedge^{i} T^*(M).$$

We define $\mathcal{E}^k(U) := \mathcal{E}(U, \bigwedge^k T^*(M))$, the $\mathcal{C}^\infty$ differential forms of degree k on the open set $U \subset M$. We can define the exterior derivative

$$d : \mathcal{E}^k(U) \to \mathcal{E}^{k+1}(U).$$

To define the exterior derivative we consider $U \subset \mathbb{R}^n$ and $\{\partial/\partial x_1, ..., \partial/\partial x^n\}$ a basis for $T_p(\mathbb{R}^n)$ at $p \in U$. Let $\{dx_1, ..., dx_n\}$ be a dual basis for $T_p^*(\mathbb{R}^n)$. Then the maps

$$dx_j : U \to T^*(\mathbb{R}^n)|_U$$

given by

$$dx_j(p) = dx_j|_p$$

form a basis for $\mathcal{E}(U, T^*(\mathbb{R}^n)) = \mathcal{E}^1(U)$. Moreover, $\{dx_I = dx_{i_1} \wedge ... \wedge dx_{i_p}\}$, where $I = (i_1, ..., i_p)$ and $1 \le i_1 < i_2 < ... < i_p \le n$, form a basis for $\mathcal{E}^p(U)$. We define $d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$ as follows:

Case $p = 0$: Suppose that $f \in \mathcal{E}^0(U) = \mathcal{E}(U)$. We put

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \in \mathcal{E}^1(U).$$

Case $p > 0$: Suppose that $f \in \mathcal{E}^p(U)$. We will have

$$f = \sum_{|I|=p}{}' f_I dx_I,$$

where $f_I \in \mathcal{E}(U)$, $I = (i_1, ..., i_p)$, $|I|$ = the number of indices, and $\sum'$ means that the sum is taken over strictly increasing indices. Then we put

$$df = \sum_{|I|=p}{}' df_I \wedge dx_I = \sum_{|I|=p}{}' \sum_{j=1}^{n} \frac{\partial f_I}{dx_j} dx_j \wedge dx_I.$$

Suppose now that (U,h) is a coordinate system on a differentiable manifold M. We have that $T(M)|_U \to T(\mathbb{R}^n)|_{h(U)}$; hence, $\mathcal{E}^p(U) \leftarrow \mathcal{E}^p(h(U))$, and the mapping

$$d : \mathcal{E}^p(h(U)) \to \mathcal{E}^{p+1}(h(U))$$

defined above induces a mapping (also denoted by d)

$$d : \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U).$$

This defines the exterior derivative d locally on M and, using the chain rule, it can be seen that the definition is independent of the choice of local coordinates. It follows that the exterior derivative is well defined globally on the manifold M.

1.3 Almost Complex manifolds

In the last section of the first chapter, we want to define new operators that will describe the complex structure of a complex manifold.

Definition 1.3.1. Let V be a real vector space. We say that an $\mathbb{R}$-linear isomorphism $J : V \to V$ is a *complex structure* on V if $J^2 = -Id$.

Let us see that for such a real vector space V, we can define a complex structure on V in such a way that it becomes a complex vector space. In order to do so, we just need to define the multiplication by a complex number. Defining the multiplication as follows,

$$(a + ib)v := av + bJv, \qquad a, b \in \mathbb{R}, \quad v \in V, \quad i = \sqrt{-1}.$$

V becomes a complex vector space.

As we know, given a complex vector space V, we can consider it as a real vector space of double dimension. Moreover, we can define a complex structure on the underlying real vector space of V taking J to be multiplication by i.

Example 1.3.2. Let $\mathbb{C}^n$ be the Euclidean space of n-tuples of complex numbers $\{z_1, ..., z_n\}$, with $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}, j = 1, ..., n$. We can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ taking coordinates $\{x_1, y_1, ..., x_n, y_n\}$. We can consider the complex structure in $\mathbb{R}^{2n}$ given by multiplication by i. This complex structure is called the *standard complex structure* in $\mathbb{R}^{2n}$. In this case, $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is

$$J(x_1, y_1, ..., x_n, y_n) = (-y_1, x_1, ..., -y_n, x_n) = \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix},$$

which clearly satisfies that $J^2 = -Id$.

Note that for every $A \in GL(2n, \mathbb{R})$, $A^{-1}JA$ is again a complex structure for $\mathbb{R}^{2n}$ (in fact, isomorphic to the one given by J).

Another example of complex structure is the following:

Let X be a complex manifold and let $T_x(X)$ be the complex tangent space to X at x. Since we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and holomorphic maps are differentiable, we can think of X as a smooth manifold of dimension $2n$. Let us call X_0 the smooth manifold induced by X and $T_x(X_0)$ the real tangent space to X_0 at x.

Proposition 1.3.3. *$T_x(X_0)$ is canonically isomorphic with the underlying real vector space of $T_x(X)$ and therefore, $T_x(X)$ defines a complex structure J_x on $T_x(X_0)$.*

Proof. Let (U,h) be a holomorphic coordinate system near x, that is, $x \in U \subset X$ is open and $h = (h_1,...,h_n) : U \to U'$ is an homeomorphism with $U' \subset \mathbb{C}^n$ open. As we have previously mentioned, this map induces a smooth coordinate system for X_0 near x by defining

$$\begin{array}{rccl} \tilde{h}: & U & \to & \mathbb{R}^{2n} \\ & x & \mapsto & \tilde{h}(x) = (\mathrm{Re}h_1(x), \mathrm{Im}h_1(x), ..., \mathrm{Re}h_n(x), \mathrm{Im}h_n(x)). \end{array}$$

Hence, in order to check that $T_x(X_0)$ is isomorphic to the real vector space induced by $T_x(X)$, it is enough to check that $T_{\tilde{h}(x)}(\mathbb{R}^{2n})$ is isomorphic to the real vector space induced by $T_{h(x)}(\mathbb{C}^n)$. We can assume, without loss of generality, that $h(x) = 0$ and therefore, $\tilde{h}(x) = 0$. Since there exists a canonical isomorphism between $T_0(\mathbb{C}^n)$ and $\mathbb{C}^n$ and between $T_0(\mathbb{R}^{2n})$ and $\mathbb{R}^{2n}$ and there is also an isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ we have

$$\begin{array}{ccc} T_0(\mathbb{C}^n) & \cong & \mathbb{C}^n \\ \wr\Vert & & \wr\Vert \\ T_0(\mathbb{R}^{2n}) & \cong & \mathbb{R}^{2n} \end{array}$$

The complex structure defined on $\mathbb{R}^{2n}$ in the previous example induces a complex structure on $T_0(\mathbb{C}^n)$ and therefore in $T_x(X_0)$, as we wanted.

Finally, let us see that this complex structure is independent on the choice of the coordinate system. Let (V,g) be a coordinate system near x and let $f = g \circ h^{-1} : h(U \cap V) \to g(U \cap V)$ be the biholomorphic change of coordinates. If we denote $u = (u_1,...,u_n)$ the real part of f and $v = (v_1,...,v_n)$ the imaginary part of f, then $\tilde{f} = (u_1, v_1, ..., u_n, v_n)$ is the corresponding differentiable change of coordinates. We will assume that $f(0) = 0$ to simplify notations.

Let J be the standard complex structure in $\mathbb{C}^n$. We have to see that J commutes with the Jacobian of $\tilde{f}$ since we want to see that the following diagram commutes

$$\begin{tikzcd} T_0(h(U\cap V)) \arrow[r, "J"] \arrow[d, "d_0\tilde{f}"'] & T_0(h(U\cap V)) \arrow[d, "d_0\tilde{f}"] \\ T_0(g(U\cap V)) \arrow[r, "J"] & T_0(g(U\cap V)) \end{tikzcd}$$

The Jacobian of $\tilde{f}$ is of the form

$$A = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial v_n} & \frac{\partial u_1}{\partial y_n} \\ -\frac{\partial u_1}{\partial y_1} & \frac{\partial v_1}{\partial y_1} & \cdots & -\frac{\partial u_1}{\partial y_n} & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial v_n}{\partial y_1} & \frac{\partial u_n}{\partial y_1} & \cdots & \frac{\partial v_n}{\partial y_n} & \frac{\partial u_n}{\partial y_n} \\ -\frac{\partial u_n}{\partial y_1} & \frac{\partial v_n}{\partial y_1} & \cdots & -\frac{\partial u_n}{\partial y_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix},$$

where the last equality holds by the Cauchy-Riemann equations. It is easy to verify that $AJ = JA$ and therefore, J induces the same complex structure for each choice of local holomorphic coordinates at x. $\square$

Let V be a real vector space with a complex structure J. We can consider the complexification of V, $V_c = V \otimes_{\mathbb{R}} \mathbb{C}$. We can extend J as follows: $\tilde{J} = J \otimes id$. Clearly, $\tilde{J}^2 = -id$. Hence, $\tilde{J}$ has two possible eigenvalues $i, -i$. Let $v \in V$ and $\alpha \in \mathbb{C}$, then

$$\tilde{J}(v \otimes \alpha - i\tilde{J}(v \otimes \alpha)) = i(v \otimes \alpha - i\tilde{J}(v \otimes \alpha)), \quad \tilde{J}(v \otimes \alpha + i\tilde{J}(v \otimes \alpha)) = -i(v \otimes \alpha + i\tilde{J}(v \otimes \alpha)).$$

Thus, $i, -i$ are eigenvalues of $\tilde{J}$. Let us denote $V^{1,0}$ and $V^{0,1}$ the eigenspace corresponding to the eigenvalue i and $-i$ respectively. Then we have that

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

Let us define the conjugation on the complexification of V as $\overline{v \otimes \alpha} = v \otimes \bar{\alpha}$ for $v \in V$ and $\alpha \in \mathbb{C}$. Notice that conjugation is only $\mathbb{R}$-linear. Clearly, if $v \otimes \alpha \in V^{0,1}$, then

$$\begin{aligned}\tilde{J}(v \otimes \alpha) &= -\, iv \otimes \alpha = \overline{iv \otimes \bar{\alpha}},\\ \tilde{J}(v \otimes \alpha) &= J(v) \otimes \alpha = \overline{J(v) \otimes \bar{\alpha}} = \overline{\tilde{J}(v \otimes \bar{\alpha})} = \overline{\tilde{J}(\overline{v \otimes \alpha})}.\end{aligned}$$

Hence, $\tilde{J}(\overline{v \otimes \alpha}) = i\overline{v \otimes \alpha}$, that is, $\overline{v \otimes \alpha} \in V^{1,0}$. Thus, $V^{1,0} \cong V^{0,1}$ as real vector spaces. We can also see that $V^{1,0} \cong \overline{V^{0,1}}$ defining $\overline{V^{0,1}}$ as $\bar{\alpha} \cdot v$, for $\alpha \cdot v \in V^{0,1}$.

Let us now consider the exterior algebras $\bigwedge V_c, \bigwedge V^{1,0}, \bigwedge V^{0,1}$ and

$$\bigwedge^{p,q} V_c = \langle u \wedge v \in \bigwedge V_c : u \in \bigwedge^{p} V^{1,0}, v \in \bigwedge^{q} V^{0,1} \rangle.$$

The exterior algebra of V_c can be decomposed as

$$\bigwedge V_c = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \bigwedge^{p,q} V.$$

Definition 1.3.4. Let X be a smooth manifold of dimension $2n$ and $J : T(X) \to T(X)$ a differentiable vector bundle isomorphism such that $J_x : T_x(X) \to T_x(X)$ is a complex structure for $T_x(X)$.Then, J is called an *almost complex structure* for the smooth manifold X. If X is equipped with an almost complex structure J, then (X, J) is called an *almost complex manifold*.

Proposition 1.3.5. *A complex manifold X induces an almost complex structure on its underlying differentiable manifold.*

Proof. By proposition 1.3.3, for each $x \in X$, there is a complex structure induced on $T_x(X_0)$, where X_0 is, as before, the real smooth manifold corresponding to X. What remains to see is that

$$J_x : T_x(X_0) \to T_x(X_0), \quad x \in X_0,$$

is a smooth mapping with respect to the parameter x. Since $T(X_0)$ is a smooth vector bundle, for every $x \in X_0$, there exists a diffeomorphism $\tilde{h}$ and a neighborhood $\tilde{U}$ of x such that

$$\tilde{h} : T(\tilde{U}) \to \tilde{U} \times \mathbb{R}^{2n}$$

Thus, it is enough to show that the induced $J : \tilde{U} \times \mathbb{R}^{2n} \to \tilde{U} \times \mathbb{R}^{2n}$ is a diffeomorphism.

In order to do so, let us take a holomorphic coordinate system (U, h) of X. Let $z_1, ..., z_n$ be coordinates on $h(U)$ and $(\xi_1, \eta_1, ..., \xi_n, \eta_n)$ be the attached coordinates in $\mathbb{R}^{2n}$, that is, $\xi_i = \mathrm{Re} z_i$ and $\eta_i = \mathrm{Im} z_i$. Then

$$\begin{array}{ccc} U \times \mathbb{R}^{2n} & \xrightarrow{J = id \times \tilde{J}} & U \times \mathbb{R}^{2n} \\ \downarrow{\scriptstyle h\times id} & & \downarrow{\scriptstyle h\times id} \\ h(U) \times \mathbb{R}^{2n} & \xrightarrow{id \times \tilde{J}} & h(U) \times \mathbb{R}^{2n} \end{array}$$

where, $\tilde{J}(\xi_1, \eta_1, ..., \xi_n, \eta_n) = (-\eta_1, \xi_1, ..., -\eta_n, \xi_n)$. Hence, J is differentiable.

□

Let X be a smooth manifold. Using what we have seen before, let $T(X)_c$ and $T^*(X)_c$ be the complexification of $T(X)$ and $T^*(X)$ respectively. Let us denote

$$\mathcal{E}^r(X)_c = \mathcal{E}(X, \bigwedge^r T^*(X)_c).$$

These are the *complex-valued differential forms of total degree* r on X. When there is no possibility of confusion, we will forget about the subscript c. In local smooth coordinates, $\varphi \in \mathcal{E}^r(X)$ if and only if φ can be expressed in a coordinate neighborhood by

$$\varphi(x) = \sum_{|I|=r}' \varphi_I(x) dx_I,$$

where we use the same notation as in the previous section and $\varphi_I(x)$ is a complex valued smooth function. We can extend the exterior derivative d defined previously by complex linearity to act on complex-valued differential forms and we have

$$\mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \cdots$$

with the property that $d^2 = 0$.

Suppose that (X, J) is an almost complex manifold. Then, we can apply the linear algebra above to $T(X)_c$. The map J extends to a $\mathbb{C}$-linear bundle isomorphism on $T(X)_c$ and has (fiberwise) eigenvalues $\pm i$. Let $T(X)^{1,0}$ and $T(X)^{0,1}$ be the bundle of $(\pm i)$-eigenspaces respectively for J. We will denote the conjugation defined as before by $Q : T(X)_c \to T(X)_c$.

There is a $\mathbb{C}$-linear isomorphism

$$T(X)_J \cong T(X)^{1,0},$$

where $T(X)_J$ is the $\mathbb{C}$-bundle built from $T(X)$ by means of J.

Let $T^*(X)^{1,0}, T^*(X)^{0,1}$ be the $\mathbb{C}$-dual bundles of $T(X)^{1,0}$ and $T(X)^{0,1}$, respectively. Consider the exterior algebra bundles $\bigwedge T^*(X)_c, \bigwedge T^*(X)^{1,0}$ and $\bigwedge T^*(X)^{0,1}$. Let $\bigwedge^{p,q} T^*(X)$ be the bundle whose fiber is $\bigwedge^{p,q} T_x^*(X)$. Its sections are the *complex-valued differential forms of type* (p,q) *on* X, which we denote by

$$\mathcal{E}^{p,q}(X) = \mathcal{E}(X, \bigwedge^{p,q} T^*(X)).$$

Moreover,

$$\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X).$$

Definition 1.3.6. Let $E \to X$ be an $\mathcal{S}$-bundle of rank r and let U be an open subset of X. A *frame for E over U* is a set of r $\mathcal{S}$-sections $\{s_1, ..., s_r\}, s_j \in \mathcal{S}(U, E)$, such that $\{s_1(x), ..., s_r(x)\}$ is a basis of E_x for every $x \in U$.

Proposition 1.3.7. *Every $\mathcal{S}$-bundle $\pi : E \to X$ admits a frame in some neighborhood of any given point in the base space.*

Proof. For being $E \to X$ and $\mathcal{S}$-bundle, for any point $x \in X$ there exists a local trivialization (U, h) so that

$$h : E|_U \to U \times K^r,$$

where $E|_U \coloneqq \pi^{-1}(U)$. Thus, for each section $s_j : U \to E$

$$\begin{tikzcd}
E|_U \arrow[rr, "h"] & & U \times K^r \\
& U \arrow[ul, "s_j"] \arrow[ur, "h \circ s_j"'] &
\end{tikzcd}$$

and therefore we have an isomorphism

$$h_* : \mathcal{S}(U, E|_U) \to \mathcal{S}(U, U \times K^r).$$

As we have seen, we can identify $\mathcal{S}(U, U \times K^r)$ with the $\mathcal{S}$-mappings $U \to K^r$.

Let us take

$$\begin{array}{rccl} \tilde{s}_j : & U & \to & U \times K^r, \\ & x & \mapsto & (x, e_j), \end{array}$$

where, as usual, $e_1 = (1, 0, ..., 0), ..., e_r = (0, ..., 0, 1)$. Clearly, $\{\tilde{s}_1(x), ..., \tilde{s}_r(x)\}$ form a basis for $U \times K^r$. Thus, since h_* is an isomorphism on fibers, it carries this basis to a basis and, therefore, $\{h_*^{-1}(\tilde{s}_1(x)), ..., h_*^{-1}(\tilde{s}_r(x))\}$ forms a frame for $E|_U$.

Hence, every $\mathcal{S}$-bundle admits a frame on a neighborhood of any point. □

In what follows, we will define two new operators that will act on the tangent space of an almost complex manifold.

Let (X, J) be an almost complex manifold and let $\{w_1, ..., w_n\}$ be a local frame for $T^*(X)^{1,0}$. Since conjugation is an isomorphism from $T^*(X)^{1,0}$ to $T^*(X)^{0,1}$, then $\{\bar{w}_1, ..., \bar{w}_n\}$ will be a local frame for $T^*(X)^{0,1}$. Therefore, a local frame for $\bigwedge^{p,q} T^*(X)$ is given by $\{w^I \wedge \bar{w}^J\}, |I| = p, |J| = q$, with I and J strictly increasing. Hence, any section $s \in \mathcal{E}^{p,q}(X)$ can be written in a neighborhood of any point U as

$$s = \sideset{}{'}\sum_{|I|=p, |J|=q} a_{IJ} w^I \wedge \bar{w}^J, \quad a_{IJ} \in \mathcal{E}^0(U).$$

Applying d,

$$ds = \sideset{}{'}\sum_{|I|=p, |J|=q} da_{IJ} \wedge w^I \wedge \bar{w}^J + a_{IJ} d(w^I \wedge \bar{w}^J\},$$

where the second term does not necessarily vanish, since $w_j(x)$ may not be a constant function of the local coordinates in the base space.

Let $\pi_{p,q}$ denote the natural projection operators

$$\pi_{p,q} : \mathcal{E}^r(X) \to \mathcal{E}^{p,q}(X), \quad \text{where } p + q = r.$$

In general, we have

$$d : \mathcal{E}^{p,q}(X) \subset \mathcal{E}^{r}(X) \to \mathcal{E}^{p+q+1}(X) = \bigoplus_{l+s=p+q+1} \mathcal{E}^{l,s}(X).$$

We define

$$\begin{array}{rccc} \partial : & \mathcal{E}^{p,q}(X) & \to & \mathcal{E}^{p+1,q}(X), \\ & s & \mapsto & \partial s = (\pi_{p+1,q} \circ d)(s), \end{array} \qquad \begin{array}{rccc} \bar\partial : & \mathcal{E}^{p,q}(X) & \to & \mathcal{E}^{p,q+1}(X), \\ & s & \mapsto & \bar\partial s = (\pi_{p,q+1} \circ d)(s). \end{array}$$

Finally, we extend them to all $\mathcal{E}^*(X) = \bigoplus_{r=0}^{\dim X} \mathcal{E}^r(X)$.

Proposition 1.3.8.

$$Q\bar\partial(Qf) = \partial f, \ \textit{for } f \in \mathcal{E}^*(X).$$

Proof. Let $f \in \mathcal{E}^r(X)$. As we have seen, at any neighborhood U of any point $x \in X$ we have a frame for E over U. Hence, as we have already disscused, on U, f will be of the form $f = \sum_{s+l=r} \sum'_{|I|=s,|J|=l} a_{IJ} w^I \wedge \bar w^J$ for $a_{IJ} \in \mathcal{E}^0(X)$.

Now, note that $Q\pi_{p,q} f = \pi_{q,p} Q f$:

$$\begin{aligned} Q\pi_{p,q} f =& Q\left(\sum'_{|I|=p,|J|=q} a_{IJ} w^I \wedge \bar w^J \right) = \sum'_{|I|=p,|J|=q} \bar a_{IJ} \bar w^I \wedge w^J, \\ \pi_{q,p} Q f =& \pi_{q,p}\left(\sum_{s+l=r} \sum'_{|I|=s,|J|=l} \bar a_{IJ} \bar w^I \wedge w^J \right) = \sum'_{|I|=p,|J|=q} \bar a_{IJ} \bar w^I \wedge w^J. \end{aligned}$$

And note also that $Q(df) = d(Qf)$ by linearity.

Therefore, for every $f \in \mathcal{E}^r(X)$ and $p + q = r$ we have that

$$Q\bar\partial(Qf) = Q(\pi_{p,q+1} \circ d(Qf)) = Q(\pi_{p,q+1} \circ Q(df)) = QQ\pi_{q+1,p}(df) = \partial f.$$

□

In general, we know that $d^2 = 0$ but it is not the case that $\bar\partial^2 = 0$. However, by the previous proposition we have that $\partial^2 = 0$ if and only if $\bar\partial^2 = 0$.

Note that we can decompose d as

$$d = \sum_{l+s=p+q+1} \pi_{l,s} \circ d = \partial + \bar\partial + \cdots.$$

If $d = \partial + \bar\partial$ we have that $0 = d^2 = \partial^2 + \partial\bar\partial + \bar\partial\partial + \bar\partial^2$ and since they are operators of different type in $\mathcal{E}^{p+q+2}(X)$, we obtain

$$\partial^2 = \partial\bar\partial + \bar\partial\partial = \bar\partial^2 = 0.$$

Definition 1.3.9. We say that an almost complex structure is *integrable* if $d = \partial + \bar\partial$.

Theorem 1.3.10. *The induced almost complex structure on a complex manifold is integrable.*

Proof. Let X be a complex manifold and let (X_0, J) be the underlying differentiable manifold with the induced almost complex structure J. As we know, $T(X)$ is $\mathbb{C}$-linear isomorphic to $T(X_0)_J$ and we have also seen that $T(X_0)_J$ is isomorphic to $T(X_0)^{1,0}$, therefore, $T(X) \cong T(X_0)^{1,0}$. And analogously for the dual bundles: $T^*(X) \cong T^*(X_0)^{1,0}$.

Let $\{dz_1, ..., dz_n\}$ be a local frame for $T^*(X)$, where $(z_1, ..., z_n)$ are local coordinates and $\{dz_1, ..., dz_n\}$ is the dual of $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$. Then, by the isomorphism above, we have a local frame for $T(X)^{1,0}$. We set

$$\begin{aligned}\frac{\partial}{\partial z_j} =& \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad j = 1, ..., n,\\ \frac{\partial}{\partial \bar{z}_j} =& \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad j = 1, ..., n,\end{aligned}$$

where $\{\partial/\partial x_1, ..., \partial/\partial x_n, \partial/\partial y_1, ..., \partial/\partial y_n\}$ is a local frame for $T(X_0)_c$ and $\{\partial/\partial z_1, ..., \partial/\partial z_n\}$ is a local frame for $T(X)$. Note that $\partial/\partial z_j$ is the complex derivative of a holomorphic function, and thus these derivatives form a local frame for $T(X)$. From the above relationships it follows that

$$\begin{aligned}dz_j =& dx_j + idy_j,\\ d\bar{z}_j =& dx_j - idy_j, \quad j = 1, ..., n.\end{aligned}$$

which gives

$$\begin{aligned}dx_j =& \frac{1}{2}(dz_j + d\bar{z}_j),\\ dy_j =& \frac{1}{2i}(dz_j - d\bar{z}_j), \quad j = 1, ..., n.\end{aligned}$$

Thus, we can write $s \in \mathcal{E}^{p,q}(X)$ locally as

$$s = \sum_{|I|=p, |J|=q}{}' a_{IJ} dz^I \wedge d\bar{z}^J.$$

We have

$$\begin{aligned}ds =& \sum_{j=1}^{n} \sum_{I,J}{}' \left(\frac{\partial a_{IJ}}{\partial x_j} dx_j + \frac{\partial a_{IJ}}{\partial y_j} dy_j\right) \wedge dz^I \wedge d\bar{z}^J\\ =& \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial z_j} dz_j \wedge dz^I \wedge d\bar{z}^J + \sum_{j=1}^{n} \sum_{I,J}{}' \frac{\partial a_{IJ}}{\partial \bar{z}_j} d\bar{z}_J \wedge dz^J \wedge d\bar{z}^J.\end{aligned}$$

The first term is of type $(p+1, q)$ and the second of type $(p, q+1)$ and so

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \qquad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j,$$

and hence $d = \partial + \bar{\partial}$.

□

Remark 1.3.11. Not every almost complex structure is integrable. For example, the 6-dimensional sphere, $\mathbb{S}^6$ admits an almost complex structure that is not integrable. However, it is still an open question whether there exists an almost complex structure in $\mathbb{S}^6$ that is integrable.

Chapter 2

Sheaf theory

In this chapter we are going to use what is called sheaf theory in order to define a new type of cohomology groups. They are related to other types of cohomology groups such as the singular or the de Rham cohomology groups. The reason why we use sheaf theory is that it provides a way to compute the singular or the de Rham cohomology of a manifold (which is a global property) from local information.

Throughout this chapter we are also going to follow [WE]. Since we are going to provide an introduction of this theory, we will not prove all results stated in this chapter. For more complete information about this topic, the reader is referred to [WE], [BI] and [GO].

2.1 Sheaves and étalé spaces

First of all, we will begin with the definition of presheaf and sheaf and some examples. We will also associate a sheaf to any presheaf.

Definition 2.1.1. We say that $\mathcal{F}$ is a *presheaf* over a topological space X if

(a) it is an assignment to each nonempty open set $U \subset X$ of a set $\mathcal{F}(U)$,

(b) there exists a collection of mappings (called *restriction homomorphisms*)

$$r^U_V : \mathcal{F}(U) \to \mathcal{F}(V),$$

where $V \subset U \subset X$ satisfying

  (a) $r^U_U = Id$ on $\mathcal{F}(U)$,
  (b) for every $U \supset V \supset W$, $r^U_W = r^V_W \circ r^U_V$.

Remark 2.1.2. We can also define a presheaf as a contravariant functor from a category that has as objects the open subsets of X and as morphisms the inclusions.

Definition 2.1.3. Let $\mathcal{F}, \mathcal{G}$ be presheaves over X. A *morphism of presheaves* $h : \mathcal{F} \to \mathcal{G}$ is a collection of mappings $h_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every $V \subset U \subset X$ open such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "h_U"] \arrow[d, "r^U_V"'] & \mathcal{G}(U) \arrow[d, "r^U_V"] \\ \mathcal{F}(V) \arrow[r, "h_V"] & \mathcal{G}(V) \end{tikzcd}$$

Definition 2.1.4. A *sheaf* $\mathcal{F}$ is a presheaf such that for every $\{U_i\}_{i\in I}$, $U_i \subset X$ open with $U = \bigcup_{i\in I} U_i$ then $\mathcal{F}$ satisfies

1. if $s, t \in \mathcal{F}(U)$ and $r^U_{U_i}(s) = r^U_{U_i}(t)$ for every i we have that $s = t$.
2. if $s_i \in \mathcal{F}(U_i)$ and if for every i, j such that $U_i \cap U_j \neq \varnothing$ we have that

$$r^{U_j}_{U_i\cap U_j}(s_i) = r^{U_j}_{U_i\cap U_j}(s_j)$$

then there exists $s \in \mathcal{F}(U)$ such that

$$r^U_{U_i}(s) = s_i, \text{ for every } i \in I.$$

A *morphism of sheaves* are morphisms of the underlying presheaves.

Note that a sheaf has the property that we can patch local data to obtain global data, a property that, in general, a presheaf does not need to have.

Examples 2.1.5. (i) Let X, Y be topological spaces. Then we consider the presheaf $\mathcal{C}_{X,Y}$ over X defined by

$$\mathcal{C}_{X,Y}(U) := \{f : U \to Y, f \text{ continuous}, U \subset X \text{ open}\}.$$

The restriction functions are defined for every $f \in \mathcal{C}_{X,Y}(U)$ as $r^U_V(f) := f_{|V}$ for every $V \subset U$ open.

It can be seen that defined in this way it is also a sheaf.

(ii) Let X be an $\mathcal{S}$-manifold. Let us define the presheaf

$$\mathcal{S}_X(U) := \mathcal{S}(U),$$

which is the set of $\mathcal{S}$-functions on U. This is a subsheaf of $\mathcal{C}_X := \mathcal{C}_{X,K}$ with $K = \mathbb{R}$ or $\mathbb{C}$. It can also be proved that it is in fact a sheaf.

(iii) Let X be a topological space and G an abelian group. We define the constant sheaf as the sheaf that maps each $U \subset X$ open and connected to G.

(iv) Let $E \to X$ be an $\mathcal{S}$-bundle. We can define the *sheaf of $\mathcal{S}$-sections of the vector bundle E* denoted by $\mathcal{S}(E)$ as $U \mapsto \mathcal{S}(E)(U) = \mathcal{S}(U, E)$ for $U \subset X$ open together with the natural restrictions. It is a subsheaf of $\mathcal{C}_{X,E}$.

Definition 2.1.6. An *étalé space* over a topological space X is a topological space Y together with a continuous surjective local homeomorphism $\pi : Y \to X$.

Definition 2.1.7. A *section of an étalé space* $\pi : Y \to X$ over an open set $U \subset X$ is a continuous map $f : U \to Y$ such that $\pi \circ f = Id_U$.

Let $\mathcal{F}$ be a presheaf over X and let

$$\mathcal{F}_x := \varinjlim_{x\in U} \mathcal{F}(U)$$

be the direct limit of the sets $\mathcal{F}(U)$ with respect to the restriction maps $\{r^U_V\}$ of $\mathcal{F}$. We call $\mathcal{F}_x$ the *stalk* of $\mathcal{F}$ over x.

Let

$$\tilde{\mathcal{F}} = \bigcup_{x \in X} \mathcal{F}_x,$$

and $\pi : \tilde{F} \to X$ be the natural projection taking points in $\mathcal{F}_x$ to x. Giving the appropiate topology to $\tilde{F}$, it becomes an étalé space.

Definition 2.1.8. Let $\mathcal{F}$ be a presheaf over a topological space X and let $\tilde{\mathcal{F}}$ be the associated étalé space to $\mathcal{F}$. We say that the *sheaf generated by* $\mathcal{F}$ is the sheaf of sections of $\tilde{\mathcal{F}}$ and is denoted by $\bar{\mathcal{F}}$.

Theorem 2.1.9. *If $\mathcal{F}$ is a sheaf, then there exists a sheaf isomorphism between $\mathcal{F}$ and $\bar{\mathcal{F}}$.*

Definition 2.1.10. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups over a space X with $\mathcal{G}$ a subsheaf of $\mathcal{F}$. The *quotient sheaf* of $\mathcal{F}$ by $\mathcal{G}$ is the sheaf generated by the presheaf $U \to \mathcal{F}(U)/\mathcal{G}(U)$. It is denoted by $\mathcal{F}/\mathcal{G}$.

2.2 Resolution of sheaves

In order to describe the cohomology of sheaves we need first some tools of category theory.

Definition 2.2.1. If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are sheaves of abelian groups over X and

$$\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C}$$

is a sequence of sheaf morphisms, then this sequence is *exact at* $\mathcal{B}$ if the induced sequence on stalks

$$\mathcal{A}_x \xrightarrow{g_x} \mathcal{B}_x \xrightarrow{h_x} \mathcal{C}_x$$

is exact for all $x \in X$. A *short exact sequence* is a sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0,$$

which is exact at $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, where 0 denotes the (constant) zero sheaf.

Example 2.2.2. Let $\mathcal{A}$ be a subsheaf of $\mathcal{B}$. Then

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{q} \mathcal{B}/\mathcal{A} \longrightarrow 0$$

is an exact sequence of sheaves, where i is the natural inclusion and q is the natural quotient mapping.

Let us use also the following terminology. Let $\mathcal{F}^n$, $n \in \mathbb{N}$ be sheaves. A *graded sheaf* is a family of sheaves indexed by integers. A *sequence of sheaves* is a graded sheaf with sheaf mappings $\mathcal{F}^n \to \mathcal{F}^{n+1}$ for every n. A *differential sheaf* is a sequence of sheaves where the composite of any pair of mappings is zero. A *resolution* of a sheaf $\mathcal{F}$ is an exact sequence of sheaves of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \longrightarrow \mathcal{F}^m \longrightarrow \cdots,$$

which we also denote by

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^*.$$

Examples 2.2.3. (i) Let X be a differentiable manifold of real dimension m and let $\mathcal{E}_X^p$ be the sheaf of real-valued differential forms of degree p. Then there is a resolution of the constant sheaf $\mathbb{R}$ given by

$$0 \longrightarrow \mathbb{R} \xrightarrow{i} \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}_X^m \longrightarrow 0,$$

where i is the natural inclusion and d is the exterior derivative defined in the first chapter. Since $d^2 = 0$, it is clear that the above is a differential sheaf. By the Poincaré lemma, this sequence is exact.

(ii) Let X be a topological manifold and G an abelian group. Using the standard notation of homological algebra, let $S^p(U,G)$ be the group of singular cochains in U with coefficients in G, δ the coboundary operator and $\mathcal{S}^p(G)$ the sheaf over X generated by the presheaf $U \to S^p(U,G)$, with the induced differential mapping $\mathcal{S}^p(G) \xrightarrow{\delta} \mathcal{S}^{p+1}(G)$. Then the sequence

$$0 \longrightarrow G \longrightarrow \mathcal{S}^0(G) \xrightarrow{\delta} \mathcal{S}^1(G) \xrightarrow{\delta} \mathcal{S}^2(G) \longrightarrow \cdots \longrightarrow \mathcal{S}^m(G) \longrightarrow \cdots$$

is a resolution of the constant sheaf G.

(iii) Let X be a differentiable manifold and let us consider $S^p_\infty(U,G)$ the group of $\mathcal{C}^\infty$ chains on X, i.e. linear combinations of maps $f : \Delta^p \to U$, where f is a $\mathcal{C}^\infty$ mapping defined in a neighborhood of the standard p-simplex Δ^p. Let $S^p_\infty(G)$ be the sheaf defined analogously as $S^p(G)$. As in the previous example, we have a resolution by differentiable cochains with coefficients in G

$$0 \longrightarrow G \longrightarrow \mathcal{S}^0_\infty(G) \longrightarrow \mathcal{S}^1_\infty(G) \longrightarrow \cdots \longrightarrow \mathcal{S}^m_\infty(G) \longrightarrow \cdots.$$

(iv) Let X be a complex manifold of complex dimension n, let $\mathcal{E}^{p,q}$ be the sheaf of (p,q) forms on X. We define the sheaf of *holomorphic differential forms of type* $(p,0)$ (or of *degree* p) Ω^p as the kernel sheaf of the mapping $\mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1}$. In local coordinates, $\varphi \in \Omega^p(U)$ if and only if

$$\varphi = \sum_{|I|=p}{}' \varphi_I dz^I, \quad \varphi_I \in \mathcal{O}(U).$$

Then we have a resolution of the sheaf Ω^p for each $p \geq 0$ fixed,

$$0 \longrightarrow \Omega^p \xrightarrow{i} \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n} \longrightarrow 0. \tag{2.2.1}$$

By the Dolbeault's lemma, this sequence is exact.

Definition 2.2.4. Let $\mathcal{L}^*$ and $\mathcal{M}^*$ be differential sheaves. We say that $f : \mathcal{L}^* \to \mathcal{M}^*$ is a *homomorphism* if it is a sequence of homomorphisms $f_i : \mathcal{L}^i \to \mathcal{M}^i$ which commutes with the differentials of both differential sheaves.

$$\begin{tikzcd}
\mathcal{L}^i \arrow[r, "f_i"] \arrow[d, "d_{\mathcal{L}}"'] & \mathcal{M}^i \arrow[d, "d_{\mathcal{M}}"] \\
\mathcal{L}^{i+1} \arrow[r, "f_{i+1}"] & \mathcal{M}^{i+1}
\end{tikzcd}$$

A *homomorphism of resolution* of sheaves is a homomorphism of the underlying differential sheaves.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}^* \arrow[d] \\ 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{B}^* \end{tikzcd}$$

Example 2.2.5. Let X be a smooth manifold and let

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^*$$
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^*_\infty(\mathbb{R})$$

be the resolutions of $\mathbb{R}$ already explained. Then, there is a natural homomorphism of differential sheaves

$$I : \mathcal{E}^* \longrightarrow \mathcal{S}^*_\infty(\mathbb{R})$$

defined by integration over chains

$$I_U : \mathcal{E}^*(U) \longrightarrow \mathcal{S}^*_\infty(U, \mathbb{R})$$
$$I_U(\varphi)(c) = \int_c \varphi,$$

where c is a $\mathcal{C}^\infty$ chain. We have

$$\begin{tikzcd} & & \mathcal{E}^* \arrow[dd, "I"] \\ 0 \arrow[r] & \mathbb{R} \arrow[ur, "i"] \arrow[dr, "i"'] & \\ & & \mathcal{S}^*_\infty(\mathbb{R}). \end{tikzcd}$$

By Stokes' theorem it follows that the mapping I commutes with the differentials, and therefore is a homomorphism of resolutions.

2.3 Cohomology of sheaves

In this section we will construct the cohomology with coefficients in a given sheaf. We will see how this cohomology is useful to prove results like the de Rham and the Dolbeault theorem.

First of all note that given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0,$$

the induced sequence

$$0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{B}(X) \longrightarrow \mathcal{C}(X) \longrightarrow 0$$

is exact at $\mathcal{A}(X)$ and $\mathcal{B}(X)$ but not necessarily at $\mathcal{C}(X)$.

Cohomology gives a measure to the amount of inexactness of the sequence at $\mathcal{C}(X)$.

Let $\mathcal{S}$ be a given sheaf and let $\tilde{\mathcal{S}} \xrightarrow{\pi} X$ be the étalé space associated to $\mathcal{S}$. We define the *sheaf of discontinuous sections of* $\mathcal{S}$ *over* X as follows

$$\mathcal{C}^0(\mathcal{S})(U) := \{f : U \to \tilde{\mathcal{S}} : \pi \circ f = 1_U\}.$$

Note that $\mathcal{S} \subset \mathcal{C}^0(\mathcal{S})$.

Now let $\mathcal{F}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{S})/\mathcal{S}$ and define $\mathcal{C}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{S}))$. By induction we define

$$\mathcal{F}^i(\mathcal{S}) = \mathcal{C}^{i-1}(\mathcal{S})/\mathcal{F}^{i-1}(\mathcal{S})$$

and

$$\mathcal{C}^i(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^i(\mathcal{S})).$$

We then have the following short exact sequences of sheaves:

$$\begin{gathered} 0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^0(\mathcal{S}) \longrightarrow \mathcal{F}^1(\mathcal{S}) \longrightarrow 0 \\ 0 \longrightarrow \mathcal{F}^i(\mathcal{S}) \longrightarrow \mathcal{C}^i(\mathcal{S}) \longrightarrow \mathcal{F}^{i+1}(\mathcal{S}) \longrightarrow 0. \end{gathered}$$

By splicing these two short exact sequences together we obtain the long exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^0(\mathcal{S}) \longrightarrow \mathcal{C}^1(\mathcal{S}) \longrightarrow \mathcal{C}^2(\mathcal{S}) \longrightarrow \cdots,$$

which we call the *canonical resolution of* $\mathcal{S}$.

Now let us define the cohomology groups of a space with coefficients in a given sheaf. Suppose that $\mathcal{S}$ is a sheaf over a space X and consider the canonical resolution described above. Taking global sections, it induces the cochain complex

$$0 \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{C}^0(\mathcal{S})(X) \longrightarrow \mathcal{C}^1(\mathcal{S})(X) \longrightarrow \cdots \longrightarrow \mathcal{C}^q(\mathcal{S})(X) \longrightarrow \cdots.$$

This sequence is exact at $\mathcal{C}^0(\mathcal{S})(X)$. Let

$$C^*(X, \mathcal{S}) := \mathcal{C}^*(\mathcal{S})(X).$$

We write the previous cochain complex as

$$0 \longrightarrow \mathcal{S}(X) \longrightarrow C^*(X, \mathcal{S}).$$

Definition 2.3.1. Let $\mathcal{S}$ be a sheaf over a space X and let

$$H^q(X, \mathcal{S}) := H^q(C^*(X, \mathcal{S})),$$

where $H^q(C^*(X, \mathcal{S}))$ is the qth cohomology of the cochain complex $C^*(X, \mathcal{S})$; i.e.,

$$H^q(C^*) = \frac{\operatorname{Ker}(C^q \to C^{q+1})}{\operatorname{Im}(C^{q-1} \to C^q)}, \quad \text{where } C^{-1} = 0.$$

The abelian groups $H^q(X, \mathcal{S})$ are defined for $q \geq 0$ and are called the *sheaf cohomology groups of the space* X *of degree* q *and with coefficients in* $\mathcal{S}$.

It can be proved that for "nice" X and $\mathcal{S}$ the constant sheaf, $H^q(X, \mathcal{S})$ is canonically isomorphic to the singular cohomology of X. In particular, for smooth manifolds this result holds.

Definition 2.3.2. A resolution of a sheaf $\mathcal{S}$ over a space X

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$$

is called *acyclic* if $H^q(X, \mathcal{A}^p) = 0$ for all $q > 0$ and $p \geq 0$.

Theorem 2.3.3. *Let $\mathcal{S}$ be a sheaf over a space X and let*

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$$

be a resolution of $\mathcal{S}$. Then there is a natural homomorphism

$$\gamma^p : H^p(\mathcal{A}^*(X)) \to H^p(X, \mathcal{S}),$$

where $H^p(\mathcal{A}^*(X))$ is the pth derived group of the cochain complex $\mathcal{A}^*(X)$. Moreover, if

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$$

is acyclic, γ^p is an isomorphism.

Corollary 2.3.4. *Suppose that*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, "f"] & \mathcal{A}^* \arrow[d, "g"] \\ 0 \arrow[r] & \mathcal{J} \arrow[r] & \mathcal{B}^* \end{tikzcd}$$

is a homomorphism of resolutions of sheaves. Then there is an induced homomorphism

$$H^p(\mathcal{A}^*(X)) \xrightarrow{g_p} H^p(\mathcal{B}^*(X)),$$

which is an isomorphism if f is an isomorphism of sheaves and the resolutions are both acyclic.

As a consequence we have the following theorems.

Theorem 2.3.5. ***(de Rham)****: Let X be a differentiable manifold. Then the natural mapping*

$$I : H^p(\mathcal{E}^*(X)) \to H^p(\mathcal{S}^*_\infty(X, \mathbb{R}))$$

induced by integration of differential forms over $\mathcal{C}^\infty$ singular chains with real coefficients is an isomorphism.

In order to prove this theorem, we should apply corollary 2.3.4 once we see that both resolutions are acyclic. To prove that they are acyclic it is enough to see that both resolutions are what is called *soft* (which we will not define in this project).

As we have already noted, $H^p(\mathcal{S}^*_\infty(X, \mathbb{R})) \cong H^p(X, \mathbb{R})$, where the last group is the singular cohomology group.

Theorem 2.3.6. ***(Dolbeault)****: Let X be a complex manifold. Then*

$$H^q(X, \Omega^p) \cong \frac{Ker(\mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X))}{Im(\mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X))}.$$

As in the de Rham theorem, to prove this result one shows that the resolution (2.2.1) is soft and then apply theorem 2.3.3.

Chapter 3

Elliptic Operator Theory

In this chapter we will provide information about differential operators required to prove the Hodge Decomposition Theorem.

In order to prove the main result of this chapter, Theorem 3.2.10, it is necessary to use numerous results from Functional Analysis that involve working with Sobolev spaces and develop the theory of Pseudodifferential Operators. Since the amount of work is considerable, as in the previous chapter, we will focus on giving the basic definitions and results with examples to clarify them but without proving such results. Thus, the reader is invited to look at [WE], the book we are going to follow, if he or she is interested in this topic.

In what follows, unless specified otherwise, X will denote a smooth manifold of dimension n and $E \to X$, $F \to X$ will be differentiable $\mathbb{C}$-vector bundles over X.

3.1 Differential Operators

In the first section of this chapter, we will define a Hermitian metric on X, the concept of differential operator and symbol of a differential operator with properties of them that we will use in the next chapter.

Definition 3.1.1. A *Hermitian metric* $\langle , \rangle$ on E is an assignment of a Hermitian inner product $\langle , \rangle_x$ to each fiber E_x of E such that, for any open set $U \subset X$ and $\xi, \eta \in \mathcal{E}(U, E)$, the function

$$\begin{array}{rccc} \langle \xi, \eta \rangle : & U & \to & \mathbb{C} \\ & x & \mapsto & \langle \xi(x), \eta(x) \rangle_x \end{array}$$

is smooth.

A *Hermitian vector bundle* $E \to X$ is a differentiable $\mathbb{C}$-vector bundle over X equipped with a Hermitian metric.

In order to find local representations for the Hermitian metric, we are going to use frames. Recall that if $E \to X$ is a differentiable bundle of rank r and U an open subset of X, a set of r sections $\mathcal{E}(U, E)$ is a frame if it forms a basis for E_x for every $x \in U$. We have seen in the first chapter that every differentiable bundle E admits a frame at a neighborhood of any given point in X. Let r be the rank of the vector bundle E and $e = \{e_1, ..., e_r\}$ be a frame at $x \in X$ for some $U \subset X$ neighborhood of x. Set

$$h(e)_{i,j} = \langle e_i, e_j \rangle,$$

and $h(e) = (h(e)_{i,j})_{i,j}$ the $r \times r$ matrix of the $\mathcal{C}^\infty$-functions defined above. The matrix $h(e)$ will be a positive definite Hermitian symmetric matrix by the properties of $\langle , \rangle$ and will be a local representation of the Hermitian metric $\langle , \rangle$ with respect to the frame e.

Let $\xi, \eta \in \mathcal{E}(U, E)$. In this frame, they can be expressed as

$$\xi(e) = \sum_{i=1}^{r} \xi(e)_i e_i, \qquad \eta(e) = \sum_{i=1}^{r} \eta(e)_i e_i.$$

Therefore,

$$\langle \xi, \eta \rangle = \langle \sum_{i=1}^{r} \xi(e)_i e_i, \sum_{j=1}^{r} \eta(e)_j e_j \rangle = \sum_{i,j=1}^{r} \overline{\eta(e)_j} h(e)_{i,j} \xi(e)_i.$$

Given any two frames e, e' of a neighborhood over U, there exists a change of frames $g : U \to \mathrm{GL}(r, \mathbb{C})$ differentiable such that $e' = ge$. It is easy to see that $h(e') = h(ge) = \bar{g}^t h(e) g$. Thus, all local results involving frames are independent of the choice of frame. In the next chapter, this observation will be very useful.

Theorem 3.1.2. *Every differentiable $\mathbb{C}$-vector bundle $E \to X$ admits a Hermitian metric.*

Thus, since in this chapter we are working with E and F differentiable $\mathbb{C}$-vector bundles, they can also be regarded as Hermitian vector bundles.

For simplicity, from now on we will also assume that X is compact and with a strictly positive smooth measure μ. That is, $d\mu$ is a volume element which can be expressed in local coordinates $(x_1, ..., x_n)$ by

$$d\mu = \rho(x) dx = \rho(x) dx_1 \cdots dx_n.$$

Note that for every Hermitian vector bundle E, we can define an inner product $(,)$ on $\mathcal{E}(X, E)$ by setting

$$(\xi, \eta) = \int_X \langle \xi(x), \eta(x) \rangle_x d\mu, \tag{3.1.1}$$

where $\langle , \rangle$ is a Hermitian metric on E.

Definition 3.1.3. Let

$$L : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F), \qquad S : \mathcal{E}(X, F) \longrightarrow \mathcal{E}(X, E)$$

be $\mathbb{C}$-linear maps. Then S is called an *adjoint* of L if

$$(Lf, g) = (f, Sg)$$

for all $f \in \mathcal{E}(X, E)$, $g \in \mathcal{E}(X, F)$.

It can be seen that if the adjoint of an operator L exists, it is unique. It is usually denoted by L^*.

The following definition is the generalization of a differential operator in $\mathbb{R}^n$ to any manifold X as before. Roughly speaking, a differential operator on a smooth manifold X will be an operator that in any local coordinates is a linear partial differential operator in $\mathbb{R}^n$.

Definition 3.1.4. Let $L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$ be a $\mathbb{C}$-linear map. We say that L is a *differential operator* if for any choice of local coordinates on X and local trivializations on E and F there exists a linear partial differential operator $\tilde{L}$ such that the diagram for such a trivialization commutes:

$$\begin{array}{ccc} [\mathcal{E}(U)]^p & \xrightarrow{\tilde{L}} & [\mathcal{E}(U)]^q \\ \wr\Vert & & \wr\Vert \\ \mathcal{E}(U, U \times \mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U \times \mathbb{C}^q) \\ \cup & & \cup \\ \mathcal{E}(X,E)|_U & \xrightarrow{L} & \mathcal{E}(X,F)|_U. \end{array}$$

that is, for $f = (f_1, ..., f_p) \in [\mathcal{E}(U)]^p$

$$\tilde{L}(f)_i = \sum_{j=1, |\alpha| \le k}^{p} a_\alpha^{ij}(x) D^\alpha f_j, \quad i = 1, ..., q,$$

where $a_{i,j}^\alpha \in \mathcal{C}^\infty$, $\alpha = (\alpha_1, ..., \alpha_n)$ and $D^\alpha = (-i)^{|\alpha|}(\partial/\partial x_1)^{\alpha_1} ... (\partial/\partial x_n)^{\alpha_n}$.

Definition 3.1.5. We say that a differential operator has *order k* if there are no derivatives of order greater than or equal to $k+1$ in the local representation.

A priori this definition depends on the local representation but it can be seen that it is independent.

We will denote $\mathrm{Diff}_k(E,F)$ the vector space of all differential operators of order k mapping $\mathcal{E}(X,E)$ to $\mathcal{E}(X,F)$.

Proposition 3.1.6. *If $L \in \mathrm{Diff}_k(E,F)$, then there exists an adjoint L^* and moreover, $L^* \in \mathrm{Diff}_k(F,E)$.*

Now we are going to define the symbol of an operator.

Let $T'(X)$ denote $T^*(X)$ without the zero section and let $T'(X) \xrightarrow{\pi} X$ denote the projection mapping. We set, for any $k \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{Smbl}_k(E,F) :=& \{\sigma \in \mathrm{Hom}(\pi^* E, \pi^* F) : \sigma(x, \rho v) \\ =& \rho^k \sigma(x,v), (x,y) \in T'(X), \rho > 0\}. \end{aligned}$$

We are going to define a map

$$\sigma_k : \mathrm{Diff}_k(E,F) \longrightarrow \mathrm{Smbl}_k(E,F),$$

where $\sigma_k(L)$ will be called the *k-symbol* of the differential operator L. To define it, note that $\sigma_k(L)(x,v)$ has to be a linear mapping from E_x to F_x, where $(x,v) \in T'(X)$. Therefore let $(x,v) \in T'(X)$ and $e \in E_x$ be given. Find $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ such that $d_x g = v$ and $f(x) = e$. Then we define the linear mapping

$$\begin{aligned} \sigma_k(L)(x,v) :& E_x \longrightarrow F_x, \\ \sigma_k(L)(x,v)e =& L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x). \end{aligned}$$

which then defines an element of $\mathrm{Smbl}_k(E,F)$ independent of the choices made. We call $\sigma_k(L)$ the *k-symbol* of L.

Proposition 3.1.7. *Let E, F, G be $\mathbb{C}$-differentiable vector bundles over X. Let $L_1 \in \mathrm{Diff}_{k_1}(E, F)$ and $L_2 \in \mathrm{Diff}_{k_2}(F, G)$. Then $L_2 \circ L_1 \in \mathrm{Diff}_{k_1+k_2}(E, G)$ and*

$$\sigma_{k_1+k_2}(L_2 \circ L_1) = \sigma_{k_2}(L_2)\sigma_{k_1}(L_1),$$

where the product on the right-hand side is product of linear mappings.

In order to see more explicitly what is the symbol of an operator, let us see some examples.

Example 3.1.8. Let us consider the differential operator $L : [\mathcal{E}(\mathbb{R}^n)]^p \to [\mathcal{E}(\mathbb{R}^n)]^q$ such that $L = \sum_{|\alpha|\le k} a_\alpha D^\alpha$, with $D^\alpha = (-i)^{|\alpha|}(\frac{\partial}{\partial x_1})^{\alpha_1}...(\frac{\partial}{\partial \alpha_n})^{\alpha_n}$. Clearly, $L \in \mathrm{Diff}_k(\mathbb{R}^n \times \mathbb{C}^p, \mathbb{R}^n \times \mathbb{C}^q)$. Let $(x, v) \in T'(\mathbb{R}^n)$, $e \in \mathbb{C}^p$, $g \in \mathcal{E}(\mathbb{R}^n)$ such that $v = d_x g = \sum_{j=1}^n \xi_j dx_j$; i.e. $\frac{\partial g}{\partial x_j}(x) = \xi_j$, and $f \in \mathcal{E}(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{C}^p)$ such that $f(x) = e$. Then

$$\sigma_k(L)(x, v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) = \frac{i^k}{k!}\sum_{|\alpha|\le k} a_\alpha D^\alpha((g - g(x))^k f)(x).$$

Note that all terms of order strictly less than k vanish and we are left with

$$\sigma_k(L)(x, v)e = \frac{i^k}{k!}\sum_{|\alpha|=k} a_\alpha k!(D^\alpha g)(x)e = \sum_{|\alpha|=k} a_\alpha \xi^\alpha e,$$

where $\xi^\alpha = \xi_1^{\alpha_1}...\xi_n^{\alpha_n}$.

Thus, if L is a differential operator, in any local representation, L is of the previous form

$$L = p(x, D) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha.$$

Replacing the differentials by new variables $\xi = (\xi_1, ..., \xi_n)$ we can consider the polynomial

$$p(x, \xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1}...\xi_n^{\alpha_n}$. The symbol of L is the highest degree component of this polynomial, that is $\sum_{|\alpha|=k} a_\alpha \xi^\alpha$.

Example 3.1.9. Let us denote $T^* = T^*(X) \otimes \mathbb{C}$ and, as usual d the exterior derivative of differential forms. Consider the de Rham complex

$$\mathcal{E}(X, \overset{0}{\bigwedge} T^*) \xrightarrow{d} \mathcal{E}(X, \overset{1}{\bigwedge} T^*) \xrightarrow{d} \cdots \longrightarrow \mathcal{E}(X, \overset{n}{\bigwedge} T^*) \longrightarrow 0. \tag{3.1.2}$$

We want to compute the associated 1-symbol mappings

$$\bigwedge^0 T_x^* \xrightarrow{\sigma_1(d)(x,v)} \bigwedge^1 T_x^* \xrightarrow{\sigma_1(d)(x,v)} \bigwedge^2 T_x^* \longrightarrow \cdots.$$

Let $f, g, (x, v), e$ be as before. Then

$$\begin{aligned}\sigma_1(d)(x, v)e =& d\left(i(g - g(x))f\right)(x) = id((g - g(x))f)(x)\\ =& id_x(g - g(x)) \wedge f(x) + i(g - g(x))(x)d_x f = id_x g \wedge e = iv \wedge e.\end{aligned}$$

Example 3.1.10. Consider the Dolbeault complex on a complex manifold X,

$$\mathcal{E}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}(X) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,n}(X) \longrightarrow 0, \tag{3.1.3}$$

and the associated symbol sequence

$$\cdots \longrightarrow \bigwedge^{p,q-1} T_x^*(X) \xrightarrow{\sigma_1(\bar{\partial})(x,v)} \bigwedge^{p,q} T_x^*(X) \xrightarrow{\sigma_1(\bar{\partial})(x,v)} \bigwedge^{p,q+1} T_x^*(X) \longrightarrow \cdots,$$

where $\bigwedge^{p,q} T^*(X)$ are defined in chapter 1. Let $v \in T_x^*(X)$, then $v = v^{1,0} + v^{0,1}$ with notations as in the first chapter. Proceeding as before we can see that

$$\sigma_1(\bar{\partial})(x,v)(e) = iv^{0,1} \wedge e.$$

Proposition 3.1.11. *Let $L \in \mathrm{Diff}_k(E,F)$, then $\sigma_k(L^*) = \sigma_k(L)^*$, where $\sigma_k(L)^*$ is the adjoint of the linear map*

$$\sigma_k(L)(x,v) : E_x \to F_x.$$

3.2 Elliptic Operators and Complexes

In this section we will define what are the so-called elliptic operators between two vector bundles. Once we have defined them, we will introduce what is an elliptic complex, that generalizes the concept of an elliptic operator between two vector bundles to a sequence of a finite number of elliptic operators between vector bundles. Once we have these definitions, we will introduce the cohomology groups associated to this complex and state one of the main important theorems of this master theses, Theorem 3.2.10.

Definition 3.2.1. Let $s \in \mathrm{Smbl}_k(E,F)$. Then s is said to be *elliptic* if and only if for every $(x,v) \in T'(X)$, the linear map

$$s(x,v) : E_x \to F_x,$$

is an isomorphism.

Definition 3.2.2. Let $L \in \mathrm{Diff}_k(E,F)$. Then L is said to be *elliptic* (*of order k*) if and only if $\sigma_k(L)$ is an elliptic symbol.

Example 3.2.3. Let us consider $X = \mathbb{R}^n$, $E = F = \mathbb{R}^n \times \mathbb{C}^p$. We have already seen that the symbol of a differential operator $L = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$ is $\sigma_k(L)(x,v) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$, with notations as above. Thus, $\sigma_k(L)$ is elliptic if and only if for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n \backslash \{0\}$, $\sigma_k(L)(x,v) \ne 0$.

The reason why we use elliptic operators is because for this type of operators there exists what is called a *parametrix* or *pseudoinvers*. A parametrix of a differential operator is the "invers in a weak sense" of an operator. With "weak sense" we mean that, if I_E and I_F denote the identity operators in $\mathrm{Diff}(E)$ and $\mathrm{Diff}(F)$ respectively and $L \in \mathrm{Diff}(E,F)$, then $\tilde{L}$ is a parametrix if

$$\begin{aligned} L \circ \tilde{L} - I_F &: \mathcal{E}(X,F) \to \mathcal{E}(X,F), \\ \tilde{L} \circ L - I_E &: \mathcal{E}(X,E) \to \mathcal{E}(X,E), \end{aligned}$$

are $\mathbb{C}$-linear mappings that admit a continuous extension to the completion of $\mathcal{E}(X,E)$ and $\mathcal{E}(X,F)$ respectively (where the norm is the Sobolev norm induced by the Hermitian metric and the volume element).

Now that we know what are elliptic operators, let us extend this definition to a finite number of vector bundles. Let $E_0, E_1, ..., E_N$ be a sequence of differentiable vector bundles over $\mathbb{C}$ defined over a compact smooth manifold X. Suppose that there is a sequence of differential operators, of some fixed order k, $L_0, L_1, ..., L_{N-1}$ mapping as in the following sequence

$$\mathcal{E}(X,E_0) \xrightarrow{L_0} \mathcal{E}(X,E_1) \xrightarrow{L_1} \mathcal{E}(X,E_2) \longrightarrow \cdots \xrightarrow{L_{N-1}} \mathcal{E}(X,E_N). \tag{3.2.1}$$

Let $\sigma(L_j)$ be the k-symbol of the operator L_j. Let us consider the associated symbol sequence

$$0 \longrightarrow \pi^*E_0 \xrightarrow{\sigma(L_0)} \pi^*E_1 \xrightarrow{\sigma(L_1)} \pi^*E_2 \longrightarrow \cdots \xrightarrow{\sigma(L_{N-1})} \pi^*E_N \longrightarrow 0. \tag{3.2.2}$$

As usual, we say that the sequence of operators and vector bundles as in (3.2.1) is a *complex* if $L_j \circ L_{j-1} = 0, j = 1, ..., N-1$. We will denote the complex (3.2.1) by $E_\bullet$.

Definition 3.2.4. A complex $E_\bullet$ is called an *elliptic complex* if the associated symbol sequence (3.2.2) is exact.

Definition 3.2.5. Let $E_\bullet$ be a complex as (3.2.1) which is also elliptic. We define the *cohomology groups* of the complex $E_\bullet$ for $q = 0, ..., N$ as

$$H^q(E_\bullet) = \frac{\mathrm{Ker}(L_q : \mathcal{E}(X,E_q) \longrightarrow \mathcal{E}(X,E_{q+1}))}{\mathrm{Im}(L_{q-1} : \mathcal{E}(X,E_{q-1}) \longrightarrow \mathcal{E}(X,E_q))}$$

where $L_{-1} = L_N = E_{-1} = E_{N+1} = 0$.

Example 3.2.6. The de Rham complex 3.1.2 with the exterior derivative is an elliptic complex. The cohomology of groups defined before is exactly the de Rham cohomology.

The Dolbeault complex 3.1.3 with the operator $\bar{\partial}$ is an elliptic complex. The corresponding complex changing $\bar{\partial}$ with ∂ is also an elliptic complex.

As we have seen at the beginning of this chapter, we can equip each vector bundle E_j in $E_\bullet$ with a Hermitian metric and an inner product like (3.1.1).

Since we are considering that L_j with $j = 0, ..., N-1$ are differential operators, by Proposition 3.1.6, we know that there exists a unique adjoint operator $L_j^* : \mathcal{E}(X,E_{j+1}) \to \mathcal{E}(X,E_j)$ which is also a differential operator. Taking this into account, we can define a new operator as follows.

Definition 3.2.7. The *Laplacian operator* of the elliptic complex $E_\bullet$ are the differential operator:

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \mathcal{E}(X,E_j) \to \mathcal{E}(X,E_j), \quad j = 0, 1, ..., N.$$

Proposition 3.2.8. *If $E_\bullet$ is an elliptic complex, Δ_j, $j = 0, ..., N$, are well-defined self-adjoint elliptic operators of order $2k$.*

Proposition 3.2.9. *Let $E_\bullet$ be an elliptic complex as in* (3.2.1) *and $\xi \in \mathcal{E}(X,E_j)$ for every $j = 0, ..., N-1$. Then $\Delta_j \xi = 0$ if and only if $L_j \xi = L_j^* \xi = L_{j-1} \xi = L_{j-1}^* \xi = 0$.*

In the following chapter we are going to see the proof of the two previous propositions in the case of $L = d, \partial, \bar{\partial}$ on the previous elliptic complexes.

Let us introduce some notation that will be useful in what follows. Let $L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)$ be a differential operator. Then

$$\mathcal{H}_L(E, F) = \{\xi \in \mathcal{E}(X, E) : L\xi = 0\} = \operatorname{Ker}(L : \mathcal{E}(X, E) \to \mathcal{E}(X, F)).$$

If E is a elliptic complex as before and L is the Laplacian operator, the elements of $\mathcal{H}_{\Delta_j}(E_j)$ are called *harmonic.*

One important result that can be proved using methods of functional analysis, is that $\mathcal{H}_L(E, F)$ is finite dimensional.

Theorem 3.2.10. *Let E be an elliptic complex equipped with an inner product and Δ_j be the Laplacian operator of E. Then, there is a canonical isomorphism*

$$\mathcal{H}_{\Delta_j}(E_j) \cong H^j(E).$$

Example 3.2.11. Let X be a compact differentiable manifold X with a Riemannian metric (which induces an inner product on $\bigwedge^p T^*(X)$ for each p). Then de Rham complex on X with the exterior derivative is an elliptic complex with an inner product.

We have already seen that

$$H^r(X, \mathbb{C}) \cong H^r(\mathcal{E}^*(X)),$$

where the first group is the singular cohomology and the second is the rth derived group of the de Rham complex. Clearly, $H^r(\mathcal{E}^*(X)) = H^r(\wedge^* T^*(X) \otimes \mathbb{C})$, where the second group is the cohomology group of the complex defined above. We denote the Laplacian by $\Delta = \Delta_d = dd^* + d^*d$. Let

$$\mathcal{H}^r(X) := \mathcal{H}_\Delta\left(\bigwedge^r T^*(X)\right)$$

be the vector space of Δ-*harmonic r-forms* on X. By Theorem 3.2.10 we obtain

$$H^r(X, \mathbb{C}) \cong \mathcal{H}^r(X).$$

This means that for each cohomology class $c \in H^r(X, \mathbb{C})$ there exist a unique harmonic form φ representing this class c which is d-close by Proposition 3.2.9.

Example 3.2.12. Let X be a compact complex manifold of complex dimension n, and consider the complex

$$\cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+2}(X) \xrightarrow{\bar{\partial}} \cdots,$$

for a fixed p, $0 \le p \le n$. By Dolbeault's theorem,

$$H^q(X, \Omega^p) \cong H^q\left(\bigwedge^{p,*} T^*(X)\right).$$

Equipping $\bigwedge^{p,q} T^*(X)$ with a Hermitian metric for each $0 \le p, q \le n$, the complex above becomes an elliptic complex with an inner product (parametrized by the integer p). Then we can also define the Laplacian

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

and

$$\mathcal{H}^{p,q}(X) := \mathcal{H}_{\Delta_{\bar{\partial}}}\left(\bigwedge^{p,q} T^*(X)\right).$$

Similar to the de Rham situation, we have the following canonical isomorphism

$$H^q(X, \Omega^p) \cong \mathcal{H}^{p,q}(X).$$

Chapter 4

The Hodge Decomposition Theorem

In the last chapter of this master thesis we are going to prove the Hodge Decomposition Theorem for compact Kähler manifolds. First of all, we will define what are Kähler manifolds and we will see important properties of them. Using the contraction and the Hodge $\star$-operator defined in the second section, we are going to compute the adjoint operator for several differential operators such as d, ∂ and $\bar{\partial}$ to derive some interesting properties that will allow us to prove some results of the previous chapter for these operators, the so-called Kähler identities and, finally, the Hodge Decomposition Theorem.

We will follow mostly [BH] and [VO]. For another approach of the proof that uses representations of $\mathfrak{sl}(2,\mathbb{C})$, the reader is invited to look at [WE].

4.1 Kähler Manifolds

In order to introduce the definition of a Kähler manifold, we need first to see a one-to-one correspondence between a Hermitian metric and a positive-definite $(1,1)$-form on a complex manifold X.

Let V be a complex vector space of dimension n, and let $V_{\mathbb{R}}$ be the underlying real vector space of dimension $2n$. Recall that the multiplication by i induces a complex structure on $V_{\mathbb{R}}$. Furthermore this complex structure induces a canonical decomposition of the complexified vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ in $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

In the same way, let $W_{\mathbb{R}} := \operatorname{Hom}(V_{\mathbb{R}}, \mathbb{R})$ be the dual space and $W_{\mathbb{C}} := W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, then we get a decomposition

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear forms. Let now $W^{1,1} = W^{1,0} \wedge W^{0,1} \subset \bigwedge^2 W_{\mathbb{C}}$ be the 2-forms of type $(1,1)$, and set

$$W_{\mathbb{R}}^{1,1} := W^{1,1} \cap \bigwedge^2 W_{\mathbb{R}},$$

where $\bigwedge^2 W_{\mathbb{R}} \subset \bigwedge^2 W_{\mathbb{C}}$ are the 2-forms with real coefficients.

Lemma 4.1.1. *Let V be a complex vector space of dimension n, and let J be the induced complex structure on the underlying real vector space $V_{\mathbb{R}}$. There is a natural isomorphism* *between the Hermitian forms on V and $W_{\mathbb{R}}^{1,1}$ given by*

$$h \mapsto \omega = -\mathrm{Im} h,$$

and

$$\omega \mapsto h : V \times V \to \mathbb{C},$$

where for every $u, v \in V$, $h(u,v) = \omega(u, J(v)) - i\omega(u,v)$.

Proof. Let

$$h = \mathrm{Re} h + i \mathrm{Im} h,$$

where Re and Im are real bilinear forms acting on V. Then for all u, v in V

$$\mathrm{Re} h(u,v) + i\mathrm{Im} h(u,v) = h(u,v) = \overline{h(v,u)} = \mathrm{Re} h(u,v) - i\mathrm{Im} h(v,u),$$

so $\omega = -\mathrm{Im} h$ is an alternating real form. In other words $\omega \in \bigwedge^2 W_{\mathbb{R}}$.

Let us see that it is of type $(1,1)$. First note that by contruction $W^{1,1} = (V^{1,1})^*$ where

$$\bigwedge^2 V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

so $\omega \in W^{1,1}$ if and only if the natural extension of ω by $\mathbb{C}$-bilinearity vanishes on every couple of vectors (u,v) of type $V^{1,0}$ or $V^{0,1}$. Since $\omega = \bar{\omega}$ and $V^{1,0} = \overline{V^{0,1}}$, it is sufficient to check the first case. As we have seen in previous chapters, a generating system of $V^{1,0}$ is given by $v - iJ(v)$ for $v \in V$. For such vectors we have

$$\omega(u - iJ(u), v - iJ(v)) = \omega(u,v) - \omega(J(u), J(v)) - i(\omega(u, J(v)) + \omega(J(u), v)). \quad (4.1.1)$$

Since J is defined as multiplication by i we have

$$h(J(u), J(v)) = ih(u, J(v)) = -i^2 h(u,v) = h(u,v)$$

and

$$h(u, J(v)) = -ih(u,v) = -h(J(u), v),$$

so we have

$$\omega(u,v) = \omega(J(u), J(v)) \text{ and } \omega(u, J(v)) = -\omega(J(u), v).$$

and substituting these expressions into (4.1.1) we get that the natural extension of ω vanishes on $V^{1,0}$ as we wanted.

Conversely, let $\omega \in W_{\mathbb{R}}^{1,1}$ and define h as before. Then, for every $u, v \in V$,

$$h(u,v) = \omega(u, J(v)) - i\omega(u,v) = -\omega(J(u), v) + i\omega(v,u) = \omega(v, J(u)) + i\omega(v,u) = \overline{h(v,u)}.$$

Thus, h is a Hermitian metric.

□

Definition 4.1.2. We say that a real form of type $(1,1)$ is positive if the corresponding Hermitian form is positive definite.

Let $\{z_1, ..., z_n\}$ be a basis of $\bigwedge^{1,0} V$ and $\{\bar{z}_1, ..., \bar{z}_n\}$ a basis of $\bigwedge^{0,1} V$. Then any Hermitian form on V can be written as

$$h = \sum_{1 \le j,k \le n} h_{j,k} z_j \otimes \bar{z}_k,$$

where $(h_{j,k})_{j,k}$ is the corresponding Hermitian matrix. Note that

$$\omega = -\mathrm{im} h = -\frac{1}{2i}(h - \bar{h}) = \frac{i}{2}(h - \bar{h}).$$

Therefore, the corresponding $(1,1)$-form is exactly

$$\begin{aligned} \omega =& \frac{i}{2} \sum_{1 \le j,k \le n} (h_{j,k} z_j \otimes \bar{z}_k - h_{k,j} \bar{z}_j \otimes z_k) = \frac{i}{2} \sum_{1 \le j,k \le n} h_{j,k} (z_j \otimes \bar{z}_k - \bar{z}_k \otimes z_j) \\ =& \frac{i}{2} \sum_{1 \le j,k \le n} h_{j,k} z_j \wedge \overline{z_k}. \end{aligned}$$

Now, let us apply the previous theory to the case we are interested in: a Hermitian complex manifold X with an Hermitian metric h. Recall that, by definition, a Hermitian metric on a complex manifold is a $\mathcal{C}^\infty$ collection of Hermitian metrices h_x on each tangent space $T_x(X)$. By Lemma 4.1.1, every Hermitian metric h_x has an associated $(1,1)$-form ω_x. We denote ω the collection of such $(1,1)$-forms.

Definition 4.1.3. Let X be a Hermitian complex manifold and ω the corresponding $(1,1)$-form. We say that ω is a *Kähler form* or *Kähler metric* if it is closed, that is

$$d\omega = 0.$$

We say that a complex manifold is a *Kähler manifold* if it admits a Kähler form.

Example 4.1.4. Let $X = \mathbb{C}^n$ and consider the standard metric $h = \sum_{j=1}^n dz_j \otimes d\bar{z}_j$. The corresponding real $(1,1)$-form is

$$\omega = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j.$$

Clearly it defines a Kähler metric since it has constant coefficients.

Example 4.1.5. In $\mathbb{P}_n(\mathbb{C})$ we can also find a Kähler metric, which is called the Fubini-study metric. This metric is constructed using the following idea: a metric measures the distance between two elements (in this case lines in $\mathbb{C}^{n+1}$ through the origin). It is possible to define a complex angle invariant under multiplication by nonzero constants between two complex lines using the standard hermitian metric on $\mathbb{C}^{n+1}$ and thus it descends to give a hermitian metric on $\mathbb{P}^n$. The Kähler metric is given by

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log(|z|^2),$$

where $z = (z_0, ..., z_n)$ is a coordinate on $\mathbb{C}^{n+1}$. It can be seen that it is invariant under multiplication by a nonzero constant and that $d\omega_{FS} = 0$.

Proposition 4.1.6. *Let $i : Y \hookrightarrow X$ be a submanifold of a Kähler manifold. Then Y is a Kähler manifold. More precisely if ω is a Kähler form on X, then $i^*\omega$ is a Kähler form on Y.*

Proof. Clearly, $i^*\omega$ is a hermitian metric on Y (where $i^*\omega$ is the pullback of ω by i) since

$$d(i^*\omega) = i^*(d\omega) = 0,$$

and therefore Y is Kähler. □

Therefore, any submanifold of the projective space is a Kähler manifold.

Although there are lots of examples of Kähler manifolds, being Kähler is a very restrictive property. For example, it is known that, in dimension 2, a compact complex surface is Kähler if and only if its first Betti number is even.

In what follows, we are going to see that a Kähler metric admits a local representation that will allow us to simplify computations in the next sections. Let (X, ω) be a Kähler manifold and fix a point $x \in X$. Let $z_1, ..., z_n$ be local holomorphic coordinates around x, then we have

$$\omega = \frac{i}{2} \sum_{1 \le j,k \le n} h_{j,k} dz_j \wedge d\bar{z}_k,$$

where the $h_{j,k}$ are differentiable functions. In these coordinates,

$$d\omega = \frac{i}{2} \sum_{1 \le j,k,l \le n} \frac{\partial h_{j,k}}{\partial z_l} dz_l \wedge dz_j \wedge d\bar{z}_k.$$

Note that, since $dz_l \wedge dz_j \wedge d\bar{z}_k = -dz_j \wedge dz_l \wedge d\bar{z}_k$, we have that $d\omega = 0$ if

$$\frac{\partial h_{j,k}}{\partial z_l} = \frac{\partial h_{l,k}}{\partial z_j}, \qquad \text{for all } 1 \le j,k,l \le n. \tag{4.1.2}$$

Theorem 4.1.7. *Let (X, ω) be a Kähler manifold. Then locally we can choose holomorphic coordinates $\zeta_1, ..., \zeta_n$ such that $h_{j,k} = \delta_{j,k} + O(|\zeta|^2)$.*

Proof. Starting with any choice of local coordinates $z_1, ..., z_n$ around 0, it is clear that we can make a linear change of coordinates such that $dz_1, ..., dz_n$ induce a basis of $T_0^*(X)$ that is orthonormal with respect to ω (Gram-Schmidt method). That is, $h_{jk}(0) = \delta_{j,k}$ or in other words, $h_{j,k}(z) = \delta_{j,k} + O(|z|)$. The Taylor expansion to the first order is then

$$h_{j,k} = \delta_{j,k} + \sum_{l=1}^{n} (a_{jkl} z_l + a'_{jkl} \bar{z}_l) + O(|z|^2), \tag{4.1.3}$$

where

$$a_{jkl} = \frac{\partial h_{j,k}}{\partial z_l}(0), \qquad a'_{jkl} = \frac{\partial h_{j,k}}{\partial \bar{z}_l}(0).$$

Since $\omega = \bar{\omega}$, we have $\overline{h_{k,j}} = h_{j,k}$, and then

$$\delta_{j,k} + \sum_{l=1}^{n} (a_{jkl} z_l + a'_{jkl} \bar{z}_l) + O(|z|^2) = h_{j,k} = \overline{h_{k,j}} = \delta_{j,k} + \sum_{l=1}^{n} (\overline{a'_{kjl}} z_l + \overline{a_{kjl}} \bar{z}_l) + O(|z|^2).$$

In particular

$$\overline{a_{kjl}} = a'_{jkl}. \tag{4.1.4}$$

Furthermore we have by (4.1.7),

$$a_{jkl} = a_{lkj}. \tag{4.1.5}$$

Set now

$$\zeta_k = z_k + \frac{1}{2}\sum_{1\le j,k\le n} a_{jkl} z_j z_l \qquad \text{for all } k = 1,\dots,n.$$

By the Inverse Function Theorem, ζ_k defines a local holomorphic coordinate system and

$$d\zeta_k = dz_k + \frac{1}{2}\sum_{1\le j,l\le n} a_{jkl}(z_j dz_l + z_l dz_j) = dz_k + \frac{1}{2}\sum_{1\le j,l\le n}(a_{jkl} + a_{lkj}) z_l dz_j,$$

which by (4.1.5) equals

$$d\zeta_k = dz_k + \sum_{1\le j,l\le n} a_{jkl} z_l dz_j.$$

Therefore

$$i\sum_{1\le k\le n} d\zeta_k \wedge d\bar{\zeta}_k = i\sum_{1\le k\le n} dz_k \wedge d\bar{z}_k + i\sum_{1\le j,k,l\le n} \overline{a_{jkl}}\bar{z}_l dz_k \wedge d\bar{z}_j + a_{jkl} z_l dz_j \wedge d\bar{z}_k + O(|z|^2).$$

Now (4.1.4) implies

$$\sum_{1\le j,k,l\le n} \overline{a_{jkl}}\bar{z}_l dz_k \wedge d\bar{z}_j = \sum_{1\le j,k,l\le n} \overline{a_{kjl}}\bar{z}_l dz_j \wedge d\bar{z}_k = \sum_{1\le j,k,l\le n} a'_{jkl}\bar{z}_l dz_j \wedge d\bar{z}_k.$$

Thus,

$$i\sum_{1\le k\le n} d\zeta_k \wedge d\bar{\zeta}_k = i\sum_{1\le j,k,l\le n}\left(\delta_{j,k} + \sum_{1\le l\le n} a_{jkl} z_l + a'_{jkl}\bar{z}_l\right) dz_j \wedge \bar{z}_k + O(|z|^2).$$

Comparing with (4.1.3), we see that $\sum_{1\le j,k\le n}\delta_{j,k} d\zeta_j \wedge d\bar{\zeta}_k = \omega + O(|z|^2)$ as we wanted.

□

4.2 The Contraction and Hodge *-operators

In this section, we are going to introduce two linear operators that will be very useful in the following sections.

As we did in other cases, in order to define operators at the tangent bundle we will first define them in a vector space and in the following sections we will apply these results to our case.

4.2.1 Contraction of a vector field

Definition 4.2.1. Let V be a real vector space of dimension n and let $v \in V$ and $\alpha \in \bigwedge^p V$. We define the *contraction* $v \lrcorner \alpha \in \bigwedge^{p-1} V$ by

$$(v \lrcorner \alpha)(v_1,\dots,v_{p-1}) = \alpha(v, v_1,\dots,v_{p-1}),$$

for every $v_1,\dots,v_{p-1} \in V$.

We can extend it by complex linearity.

It is also easy to see that if $e_1, ..., e_n$ is a basis of V and $\epsilon_1, ..., \epsilon_n$ is its dual basis, then

$$e_j \lrcorner (\epsilon_{i_1} \wedge ... \wedge \epsilon_{i_k}) = \begin{cases} 0, & \text{if } j \neq \{i_1, ..., i_k\}, \\ (-1)^{l-1} \epsilon_{i_1} \wedge ... \wedge \widehat{\epsilon_{i_l}} \wedge ... \wedge \epsilon_{i_k} & \text{if } j = i_l. \end{cases}$$

Using this formula it can be deduced that for every $v \in V$, $\alpha \bigwedge^p V$, $\beta \in \bigwedge^q V$

$$v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta). \tag{4.2.1}$$

4.2.2 The Hodge *-operator

Let V be a Euclidean space, i.e. a real finite-dimensional vector space of dimension n with an inner product $\langle , \rangle$. Without loss of generality we can take $\langle , \rangle$ the standard scalar product in $\mathbb{R}^n$. Let us consider $\{e_1, ..., e_n\}$ a orthonormal basis of V. As in previous chapters, we denote the exterior algebra of V as

$$\bigwedge V = \bigoplus_{p=1}^{n} \bigwedge^p V.$$

For every $1 \leq p \leq n$, a basis of $\bigwedge^p V$ is $\{e_{i_1} \wedge ... \wedge e_{i_p}, 1 \leq i_1 < i_2 < ... < i_p \leq n\}$. Note that we can extend the inner product of V to $\bigwedge^p V$ so that the previous set is an orthonormal basis of $\bigwedge^p V$.

Let us also choose an orientation on V, that is, an ordering of a basis such as $\{e_1, ..., e_n\}$ up to an even permutation. This is equivalent to a choice of sign for the n-form $e_1 \wedge ... \wedge e_n$.

Definition 4.2.2. Let V be as before, a Euclidean space of dimension n with a fixed orientation, and $\{e_1, ..., e_n\}$ a orthonormal basis of V. The *Hodge $*$-operator* is the mapping that acts as follows

$$\begin{array}{cccc} * : & \bigwedge^p V & \longrightarrow & \bigwedge^{n-p} V \\ & e_{i_1} \wedge ... \wedge e_{i_p} & \longmapsto & \epsilon e_{j_1} \wedge ... \wedge e_{j_{n-p}}, \end{array}$$

where $\{j_1, ..., j_{n-p}\}$ is the complement of $\{i_1, ..., i_p\}$ in $\{1, ..., n\}$ and

$$\epsilon = \begin{cases} 1, & \text{if } \{i_1, ..., i_p, j_1, ..., j_{n-p}\} \text{ is an even permutation of } \{1, ..., n\}, \\ -1, & \text{otherwise.} \end{cases}$$

To simplify notation, set vol $= e_1 \wedge ... \wedge e_n$. Then, $*(e_{i_1} \wedge ... \wedge e_{i_p})$ is defined so that $(e_{i_1} \wedge ... \wedge e_{i_p}) \wedge *(e_{i_1} \wedge ... \wedge e_{i_p}) = \text{vol}$.

Proposition 4.2.3. *Let $*$ denote the Hodge $*$-operator extended by linearity to all of $\bigwedge^p V$. For every $\alpha, \beta \in \bigwedge^p V$ we have the equality*

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}.$$

Proof. Let

$$\alpha = \sum_{|I|=p}{}' a_I e_I, \qquad \beta = \sum_{|J|=p}{}' b_J e_J,$$

for $a_I, b_J \in \mathbb{R}$.

Then

$$\alpha \wedge *\beta = \left({\sum_{|I|=p}}' a_I e_I \right) \wedge * \left({\sum_{|J|=p}}' b_J e_J \right) = {\sum_{|I|=p,|J|=p}}' a_I b_J e_I \wedge *e_J,$$

Note that $e_I \wedge *e_J \neq 0$ if and only if $I = J$, since otherwise, some indexes will be repeated. Thus,

$$\alpha \wedge *\beta = {\sum_{|I|=p}}' a_I b_I e_I \wedge *e_I = {\sum_{|I|=p}}' a_I b_I \mathrm{vol} = \langle \alpha, \beta \rangle \mathrm{vol}.$$

□

Note that the Hodge $*$-operator is independent of the choice of the orthonormal basis on V and only depends on the inner product and the orientation.

Proposition 4.2.4. *With notations as above, the composition $**$ is equal to multiplication by $(-1)^{p(n-p)}$.*

Proof. By definition, $*e_I$ is a $(n-p)$-form that satisfies

$$e_I \wedge *e_I = \mathrm{vol}.$$

On the other hand, recall that, by the properties of the wedge product, if $\alpha \in \bigwedge^i V$ and $\beta \in \bigwedge^j V$, then

$$\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha.$$

Therefore,

$$e_I \wedge *e_I = (-1)^{p(n-p)} (*e_I) \wedge e_I = (-1)^{p(n-p)} \epsilon \mathrm{vol},$$

where $\epsilon = \pm 1$ depending on the sign of the permutation one has to apply to $*e_I$ and e_I to get vol (if the permutation is even +1, if odd, −1). Splicing the two equalities we get $\epsilon = (-1)^{p(n-p)}$.

Note also that $*e_I \wedge e_I = (-1)^{p(n-p)}\mathrm{vol}$ implies that

$$*e_I \wedge (-1)^{p(n-p)} e_I = \mathrm{vol}.$$

That is, $* * e_I = (-1)^{p(n-p)} e_I$ by definition of the Hodge $*$-operator, since $* * e_I$ is the p-form that satisfies that

$$*e_I \wedge (* * e_I) = \mathrm{vol}.$$

□

Corollary 4.2.5. *If n is even, $**$ is multiplication by $(-1)^p$.*

Once we have defined the Hodge $*$-operator for Euclidean spaces, we have to generalize it for complex-valued p-forms in order to be able to apply it to p-forms of the complexified tangent space of a smooth manifold.

Let us consider a complex vector space V of dimension n with $\{e_1, ..., e_n\}$ a basis of V and the p-forms of V, $\bigwedge^p V$. Assume that we have a Hermitian inner product $\langle , \rangle$ on V. For every $\alpha, \beta \in \bigwedge^p V$, that is, $\alpha = \sum'_{|I|=p} a_I e_I$, $\beta = \sum'_{|J|=p} b_J e_J$ with $a_I, b_J \in \mathbb{C}$, the Hermitian inner product has the form

$$\langle \alpha, \beta \rangle = {\sum_{|I|=p}}' a_I \bar{b}_I.$$

Note that in the case $a_I, b_I \in \mathbb{R}$, the inner product is a Euclidean inner product as in the previous case.

Extending the Hodge $*$-operator by complex linearity, we can see that

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \mathrm{vol}.$$

Thus, if V is a real vector space of dimension n, we can apply the previous results to the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V, and we have that the Hodge $*$-operator

$$* : \bigwedge^{p} V \longrightarrow \bigwedge^{2n-p}$$

works as before.

Now, we are going to show how the Hodge $*$-operator acts on forms of type (p,q). Set $\{e_1, ..., e_n\}$ a basis of $\bigwedge^{1,0} V_{\mathbb{C}}$ and $\{\bar{e}_1, ..., \bar{e}_n\}$ a basis of $\bigwedge^{0,1} V_{\mathbb{C}}$. Let

$$u = \sum_{|J|=p, |K|=q} u_{J,K} e_J \wedge \bar{e}_K, \qquad v = \sum_{|J|=p, |K|=q} v_{J,K} e_J \wedge \bar{e}_K$$

be (p,q)-forms. Then, $\langle u, v \rangle$ is given by

$$\langle u, v \rangle = \sum_{|J|=p, |K|=q} u_{J,K} \overline{v_{J,K}}.$$

Since by definition $u \wedge *\bar{v} = \langle u, v \rangle \mathrm{vol}$, this shows that the Hodge $*$-operator gives $\mathbb{C}$-linear operator

$$* : \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{n-q,n-p}(X),$$

since $*\bar{v}$ is of type $(n-p, n-q)$.

Moreover, this also shows that the decomposition

$$\bigwedge^{k} V = \bigoplus_{p+q=k} \bigwedge^{p,q} V$$

is orthogonal with respect to the inner product $(,)$ defined as

$$(\alpha, \beta) = \int_V \langle \alpha, \beta \rangle \mathrm{vol} = \int_V \alpha \wedge *\bar{\beta}.$$

Indeed if v is of type (p', q') with $p' + q' = p + q$ then $u \wedge *\bar{v}$ is of type $(n - p' + p, n - q' + q)$, so it is zero unless $p = p', q = q'$, since otherwise either $n - p' + q$ or $n - q' + q$ will be strictly greater than n.

4.3 Differential and Hodge *-operators

Once we have seen the definition of the Hodge $*$-operator in vector spaces, let us apply the previous results to the case we are interested in, Hermitian compact complex manifolds.

Let X be a compact complex manifold of dimension n endowed with a Hermitian metric. As before, we will denote the volume element of this metric as vol.

Proposition 4.3.1. *Let X be a compact complex manifold of dimension n. For all $\alpha \in \mathcal{E}^{p,q}(X)$ and $\beta \in \mathcal{E}^{p,q+1}(X)$, we have*

$$\bar{\partial}^* := - * \partial * .$$

And for all $\alpha \in \mathcal{E}^{p,q}(X)$ and $\beta \in \mathcal{E}^{p+1,q}(X)$, we have

$$\partial^* := - * \bar{\partial} * .$$

Proof. We show the first statement, the proof of the second is analogous. By definition

$$(\bar{\partial}\alpha, \beta) = \int_X \bar{\partial}\alpha \wedge *\bar{\beta}.$$

Recall that $d = \partial + \bar{\partial}$, because X is a complex manifold. Hence, since $\alpha \wedge *\bar{\beta} \in \mathcal{E}^{n,n-1}(X)$, we have

$$d(\alpha \wedge *\bar{\beta}) = \bar{\partial}(\alpha \wedge *\bar{\beta}) = \bar{\partial}\alpha \wedge *\bar{\beta} + (-1)^{p+q}\alpha \wedge \bar{\partial} * \bar{\beta}.$$

Since X is a manifold without boundary and $\alpha \wedge *\beta \in \mathcal{E}^{n,n-1}(X)$, by Stokes' theorem

$$\int_X d(\alpha \wedge *\beta) = 0.$$

Thus,

$$\int_X \bar{\partial}\alpha \wedge *\bar{\beta} = - \int_X (-1)^{p+q}\alpha \wedge \bar{\partial} * \bar{\beta}.$$

Using that $*$ is a real operator and Proposition 4.2.4 we have

$$\bar{\partial} * \bar{\beta} = \overline{\partial * \beta} = (-1)^{(2n-p-q)(p+q)} * \overline{*\partial * \beta} = (-1)^{p+q} * \overline{*\partial * \beta}.$$

Therefore

$$(\bar{\partial}\alpha, \beta) = - \int_X (-1)^{p+q}\alpha \wedge \bar{\partial} * \bar{\beta} = - \int_X \alpha \wedge * \overline{*\partial * \beta} = (\alpha, - * \partial * \beta),$$

as we wanted. □

Let us now use this result to give a local expression of the formal adjoint in terms of contractions by vector fields. Let $U \subset \mathbb{C}^n$ be an open set and endow $T(U)$ with the standard hermitian metric $\sum_{j=1}^n dz_j \otimes d\bar{z}_j$ where $z_1, ..., z_n$ are the linear coordinates on $\mathbb{C}^n$. Thus $\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}$ is a holomorphic frame for $T(U)$. Let α be a form with compact support of type (p,q) given in these local coordinates by $\alpha = \sum_{|J|=p,|K|=q} \alpha_{J,K} dz_J \wedge d\bar{z}_K$. To simplify notations, since all operators are linear, we can take simply $\alpha = \alpha_{J,K} dz_J \wedge d\bar{z}_K$ with $dz_J = dz_{j_1} \wedge ... \wedge dz_{j_p}$ and $d\bar{z}_K = d\bar{z}_{k_1} \wedge ... \wedge d\bar{z}_{k_q}$. Then,

$$*\bar{\alpha} = \bar{\alpha}_{J,K} * (d\bar{z}_J \wedge dz_K) = \epsilon\bar{\alpha}_{J,K} dz_{J^c} \wedge d\bar{z}_{K^c},$$

where J^c and K^c denote the complement of J and K respectively in $\{1, ..., n\}$ and $\epsilon = \pm 1$ depending on the sign of the corresponding permutation.

$$\bar{\partial} * \bar{\alpha} = \epsilon \sum_{l=1}^{n} \frac{\partial \bar{\alpha}_{J,K}}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_{J^c} \wedge d\bar{z}_{K^c} = \epsilon \sum_{k_l=1}^{q} \frac{\partial \bar{\alpha}_{J,K}}{\partial \bar{z}_{k_l}} d\bar{z}_{k_l} \wedge dz_{J^c} \wedge d\bar{z}_{K^c}.$$

Again,

$$*\overline{\bar\partial * \bar\alpha} = \epsilon\epsilon' \sum_{k_l=1}^{q} \overline{\frac{\partial \bar\alpha_{J,K}}{\partial \bar z_{k_l}}} dz_J \wedge d\bar z_{K\setminus k_l} = \epsilon\epsilon' \sum_{k_l=1}^{q} \frac{\partial \alpha_{J,K}}{\partial z_{k_l}} dz_J \wedge d\bar z_{K\setminus k_l}.$$

Now, note that

$$\begin{aligned}(d\bar z_{k_l} \wedge dz_{J^c} \wedge d\bar z_{K^c}) \wedge (dz_J \wedge d\bar z_{K\setminus k_l}) =&(-1)^{(n-p)+(n-q)+p+(l-1)}(dz_{J^c} \wedge d\bar z_{K^c}) \wedge (dz_J \wedge d\bar z_K)\\ =&(-1)^{q+l-1}(-1)^{p+q}\epsilon * \overline{dz_J \wedge d\bar z_K} \wedge *\overline{*\overline{dz_J \wedge d\bar z_K}}\\ =&(-1)^{p+l-1}\epsilon \text{vol}.\end{aligned}$$

Thus, $\epsilon' = (-1)^{p+l-1}\epsilon$ which implies that

$$\partial^*\alpha = -(-1)^{p+l-1} \sum_{k_l=1}^{q} \frac{\partial \alpha_{J,K}}{\partial z_{k_l}} dz_J \wedge d\bar z_{K\setminus k_l} = -\sum_{l=1}^{n} \frac{\partial \alpha_{J,K}}{\partial z_{k_l}} \frac{\partial}{\partial \bar z_l} \lrcorner\, dz_J \wedge d\bar z_K.$$

To simplify notation, we will denote $\frac{\partial \alpha}{\partial z_l} = \frac{\partial \alpha_{J,K}}{\partial z_l} dz_J \wedge d\bar{z_K}$, where α is defined as before. Therefore, we have that

$$\partial^*\alpha = -\sum_{l=1}^{n} \frac{\partial}{\partial \bar z_l} \lrcorner \frac{\partial \alpha}{\partial z_l}. \tag{4.3.1}$$

Lemma 4.3.2. *Let X be a compact complex manifold, then we have*

$$\begin{aligned}(\alpha, \Delta_d\beta) =&(d\alpha, d\beta) + (d^*\alpha, d^*\beta),\\ (\alpha, \Delta_\partial\beta) =&(\partial\alpha, \partial\beta) + (\partial^*\alpha, \partial^*\beta),\\ (\alpha, \Delta_{\bar\partial}\beta) =&(\bar\partial\alpha, \bar\partial\beta) + (\bar\partial^*\alpha, \bar\partial^*\beta).\end{aligned}$$

Proof. They hold by definition of the adjoint. For example, let us see the first equality

$$(\alpha, \Delta_d\beta) = (\alpha, (dd^* + d^*d)\beta) = (\alpha, dd^*\beta) + (\alpha, d^*d\beta) = (d\alpha, d\beta) + (d^*\alpha, d^*\beta).$$

□

Lemma 4.3.3. *Let X be a complex compact manifold. A form α is harmonic (resp. Δ_∂-harmonic, resp. $\Delta_{\bar\partial}$-harmonic) if and only if it is $d-$ and d^*-closed (resp. $\partial-$ and ∂^*-closed, resp. $\bar\partial-$ and $\bar\partial^*$-closed).*

Proof. Recall that $(\Delta_d\alpha, \Delta_d\alpha) \geq 0$ with equality if and only if $\Delta_d\alpha = 0$. By the previous Lemma,

$$(\alpha, \Delta_d\alpha) = (d\alpha, d\alpha) + (d^*\alpha, d^*\alpha).$$

Thus, α is harmonic if and only if $(d\alpha, d\alpha) = (d^*\alpha, d^*\alpha) = 0$, i.e. $d\alpha = d^*\alpha = 0$.

Analogously with ∂ and $\bar\partial$.

□

4.4 Hodge Decomposition Theorem

Finally, we are going to prove the Hodge Decomposition Theorem on compact Kähler manifolds. In order to do so, we are going to introduce a new operator that will play an important role in the Kähler identities. These identities are used in the proof of the main result of this theorem.

Let (X,ω) be a Kähler manifold, and denote by vol the volume form of the corresponding Hermitian metric. The exterior product with the Kähler form defines a differential operator

$$\begin{array}{rccc} L: & \mathcal{E}^k(X)_c & \to & \mathcal{E}^{k+2}(X)_c, \\ & \alpha & \mapsto & \omega\wedge\alpha, \end{array}$$

of degree zero. This operator is called *Lefschetz operator.*

Proposition 4.4.1. *We denote by*

$$\Lambda : \mathcal{E}^{k+2}(X)_c \to \mathcal{E}^k(X)_c$$

the formal adjoint of L. Then,

$$\Lambda\beta = (-1)^k(*L*)\beta$$

for every k-form β.

Proof. Note that since ω is a real differential form, it is sufficient to show the claim for $\beta \in \mathcal{E}^{k+2}(X)$.

Then we have

$$\langle L\alpha,\beta\rangle\text{vol} = L\alpha\wedge *\beta = (\omega\wedge\alpha)\wedge *\beta = (-1)^{2k}\alpha\wedge\omega\wedge *\beta = \alpha\wedge\omega\wedge *\beta.$$

Note that $*\beta \in \mathcal{E}^{2n-k-2}$. Thus, $\omega\wedge *\beta \in \mathcal{E}^{2n-k}$. By Proposition 4.2.4, we have that

$$\omega\wedge *\beta = (-1)^{(2n-k)k} * *(\omega\wedge *\beta) = *((-1)^k * (\omega\wedge *\beta) = *((-1)^k * L * \beta).$$

Therefore,

$$\langle L\alpha,\beta\rangle\text{vol} = \alpha\wedge *((-1)^k * L * \beta) = \langle\alpha,((-1)^k * L*)\beta\rangle\text{vol},$$

as we wanted.

□

Note furthermore that since ω is real form, we have

$$\bar{L}(\alpha) = \alpha\wedge\bar{\omega} = \alpha\wedge\omega = L(\alpha),$$

so L and therefore Λ are invariant under conjugation.

If A, B are two differential operators of degree a and b respectively, their Lie bracket is a differential operator of degree $a+b$ defined by

$$[A,B] := AB - (-1)^{ab}BA.$$

Lemma 4.4.2. *Let $U \subset \mathbb{C}^n$ be an open set endowed with the constant Kähler metric*

$$\omega = \frac{i}{2}\sum_{1\le j\le n} dz_j\wedge d\bar{z}_j.$$

Then we have

$$[\bar{\partial}^*, L] = i\partial.$$

Proof. Let $u \in \mathcal{E}^l(U)$ be a l-form, then by (4.3.1)

$$\bar{\partial}^* u = -\sum_{1\le k\le n} \frac{\partial}{\partial \bar{z}_l} \lrcorner \frac{\partial u}{\partial z_k}.$$

Therefore we have

$$[\bar{\partial}^*, L]u = -\sum_{1\le k\le n} \frac{\partial}{\partial \bar{z}_k} \lrcorner \frac{\partial}{\partial z_k}(\omega \wedge u) + \omega \wedge \sum_{1\le k\le n} \frac{\partial}{\partial \bar{z}_k} \lrcorner \frac{\partial u}{\partial z_k}.$$

Since the coeffitiens of ω are constant, we have $\frac{\partial}{\partial z_k}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_k}$. Furthermore by (4.2.1)

$$\frac{\partial}{\partial \bar{z}_k} \lrcorner \left(\omega \wedge \frac{\partial u}{\partial z_k}\right) = \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner \omega\right) \wedge \frac{\partial u}{\partial z_k} + \omega \wedge \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner \frac{\partial u}{\partial z_k}\right)$$

thus we get

$$[\bar{\partial}^*, L]u = -\sum_{1\le k\le n} \left(\frac{\partial}{\partial \bar{z}_k} \lrcorner \omega\right) \wedge \frac{\partial u}{\partial z_k}.$$

Moreover, $\frac{\partial}{\partial \bar{z}_k} \lrcorner (\omega) = -i dz_k$, so

$$[\bar{\partial}^*, L]u = i \sum_{1\le k\le n} dz_k \wedge \frac{\partial u}{\partial z_k} = i\partial u.$$

□

Proposition 4.4.3. *Let (X, ω) be a Kähler manifold. Then we have*

$$[\bar{\partial}^*, L] = i\partial \tag{4.4.1}$$
$$[\partial^*, L] = -i\bar{\partial} \tag{4.4.2}$$
$$[\Lambda, \bar{\partial}] = -i\partial^* \tag{4.4.3}$$
$$[\Lambda, \partial] = i\bar{\partial}^*. \tag{4.4.4}$$

Proof. Note that the second equation can be derived from the first one by conjugation:

$$-i\bar{\partial} = \overline{i\partial} = \overline{[\bar{\partial}^*, L]} = [\partial^*, \bar{L}] = [\partial^*, L].$$

Analogously the third and the fourth. Furthermore the third relation follows from the first by the formal adjoint property: let u, v be (p,q)-forms, then

$$\begin{aligned}([\Lambda, \bar{\partial}]u, v) &= (\Lambda\bar{\partial}u, v) + (\bar{\partial}\Lambda u, v) = (u, \bar{\partial}^* L v) + (u, L\bar{\partial}^* v)\\ &= (u, [\bar{\partial}^*, L]v) = (u, i\partial v) = (-i\partial^* u, v).\end{aligned}$$

Thus we are left to show the first relation. Note now that the local expressions of the differential operators only use the coefficients of the metric up to first order: indeed the operator L uses the metric only up to order zero and $\bar{\partial}^* = -\ast\partial\ast$ shows that we only use the metric and its first derivatives. By lemma 4.1.7, we can choose holomorphic coordinates such that

$$h = Id + O(|z|^2),$$

thus it is sufficient to consider the situation of an open set in $\mathbb{C}^n$ endowed with the standard Kähler metric. By Lemma 4.4.2 the first equality holds and therefore we are done. □

Theorem 4.4.4. *Let (X,ω) be a Kähler manifold, and let $\Delta_d, \Delta_\partial$, and $\Delta_{\bar\partial}$ be the Laplacians associated to the operators d, ∂ and $\bar\partial$. Then we have*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}.$$

In particular a k-form is harmonic if and only if it is Δ_∂-harmonic if and only if it is $\Delta_{\bar\partial}$-harmonic.

Proof. We will show the first equality, the proof of the second one is analogous. We have $d = \partial + \bar\partial$, so

$$\Delta_d = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial).$$

Since $\bar\partial^* = -i[\Lambda, \partial] = -i\Lambda\partial + i\partial\Lambda$ by formula (4.4.4) and $\partial^2 = 0$, we have

$$(\partial + \bar\partial)(\partial^* + \bar\partial^*) = \partial\partial^* - i\partial\Lambda\partial + \bar\partial\partial^* - i\bar\partial\Lambda\partial + i\bar\partial\partial\Lambda$$

and

$$(\partial^* + \bar\partial^*)(\partial + \bar\partial) = \partial^*\partial + i\partial\Lambda\partial + \partial^*\bar\partial + i\partial\Lambda\bar\partial - i\Lambda\partial\bar\partial.$$

By (4.4.3) we have $\partial^* = i[\Lambda, \bar\partial] = i\Lambda\bar\partial - i\bar\partial\Lambda$, which implies

$$\partial^*\bar\partial = -i\bar\partial\Lambda\bar\partial = -\bar\partial\partial^*. \tag{4.4.5}$$

Thus we get

$$\Delta_d = \partial\partial^* - i\bar\partial\Lambda\partial + i\bar\partial\partial\Lambda + \partial^*\partial + i\partial\Lambda\bar\partial - i\Lambda\partial\bar\partial = \Delta_\partial - i\Lambda\partial\bar\partial - i\bar\partial\Lambda\partial + i\partial\Lambda\bar\partial + i\bar\partial\partial\Lambda.$$

Since $\partial\bar\partial = -\bar\partial\partial$, we have

$$-i\Lambda\partial\bar\partial - i\bar\partial\Lambda\partial + i\partial\Lambda\bar\partial + i\bar\partial\partial\Lambda = i(\Lambda\bar\partial - \bar\partial\Lambda)\partial + i\partial(\Lambda\bar\partial - \bar\partial\Lambda),$$

so another application of the Kähler identity (4.4.3) gives

$$i(\Lambda\bar\partial - \bar\partial\Lambda)\partial + i\partial(\Lambda\bar\partial - \bar\partial\Lambda) = \Delta_\partial.$$

□

Corollary 4.4.5. *Let (X,ω) be a Kähler manifold, and let α be a form of type (p,q). Then $\Delta_d\alpha$ has type (p,q).*

Proof. Obvious, since $\Delta_d\alpha = \frac{1}{2}\Delta_\partial\alpha$ has type (p,q). □

Theorem 4.4.6. *Let (X,ω) be a Kähler manifold, and let $\alpha = \sum_{k=p+q}\alpha^{p,q}$ be the decomposition of a k-form in its components of type (p,q). Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic for all p,q. In particular we have*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k}\mathcal{H}^{p,q}(X)$$

where $\mathcal{H}^{p,q}(X)$ is the space of harmonic forms of type (p,q). Furthermore we have

$$\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)} \qquad \forall p,q \in \mathbb{N}.$$

Proof. By the preceding corollary,

$$\Delta_d \alpha = \sum_{k=p+q} \Delta_d \alpha^{p,q}$$

is the decomposition of $\Delta_d \alpha$ in forms of type (p, q). Clearly, it is zero if and only if all components are zero.

If β is a harmonic form of type (p, q), then $\bar{\beta}$ has type (q, p). By hypothesis $\Delta_\partial \beta = 0$, so

$$\overline{\Delta_\partial \bar{\beta}} = \overline{\Delta_\partial} \beta = \Delta_{\bar{\partial}} \beta = \Delta_\partial \beta = 0,$$

hence $\Delta_\partial \bar{\beta} = 0$ if and only if $\Delta_\partial \beta = 0$ as we wanted. □

Lemma 4.4.7. *The symbol of the differential operator Δ_d is*

$$\sigma_2(\Delta_d)(x, \theta) = -\|\theta\|^2 Id.$$

In particular, the operator Δ_d is elliptic.

Proof. Recall that, in the last chapter we saw that $\sigma_1(d)(x, \theta)e = i\theta \wedge e$. By Proposition 3.1.7, it only remains to compute $\sigma_1(d^*)$. But by Proposition 3.1.11, $\sigma_1(d^*) = (\sigma_1(d))^*$.

Let $\alpha \in \mathcal{E}^k(X)$, $\beta \in \mathcal{E}^{k+1}(X)$ and $(x, \theta) \in T'(X)$. Then,

$$\begin{aligned}
(\theta \wedge \alpha, \beta) = &\int_X \theta \wedge \alpha \wedge *\beta = (-1)^k \int_X \alpha \wedge \theta \wedge *\beta)) \\
=&(-1)^k(-1)^{k(2n-k)} \int_X \alpha \wedge *(*(\theta \wedge *\beta)) = \int_X \alpha \wedge *(v \lrcorner \beta) \\
=&(\alpha, v \lrcorner \beta),
\end{aligned}$$

where $\theta \in T_x^*(X)$ is the 1-form associated to $v \in T_x(X)$ with respect to the metric $(,)$, that is, v is such that $\theta(v) = (\theta, \theta) = \|\theta\|^2$.

Thus, $(\sigma_1(d)(x, v))^* e = iv \lrcorner e$. Moreover, we have

$$\begin{aligned}
\sigma_2(\Delta_d)(x, v)e =&(\sigma_1(d)(x, v)(\sigma_1(d)(x, v))^* + (\sigma_1(d)(x, v))^* \sigma_1(d)(x, v))e \\
=& - (\theta \wedge (v \lrcorner e) + v \lrcorner (\theta \wedge e)) = -\|\theta\|^2 e,
\end{aligned}$$

where the last equality holds by (4.2.1). □

Corollary 4.4.8. *The symbol of the laplacian operator Δ_∂, $\Delta_{\bar{\partial}}$ is $\sigma_2(\Delta_\partial)(x, \theta) = \sigma_2(\Delta_{\bar{\partial}})(x, \theta) = -\frac{1}{2}\|\theta\|^2 Id$.*

This shows that we can apply the theory of elliptic operators to $\Delta_d, \Delta_\partial$ and $\Delta_{\bar{\partial}}$.

If X is a compact Kähler manifold, we can combine Theorem 4.4.6 and Theorem 3.2.10 so we get

Theorem 4.4.9. ***(Hodge decomposition theorem)*** *Let X be a compact Kähler manifold. Then we have the Hodge decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X),$$

and the Hodge duality

$$H^{q,p}(X) = \overline{H^{p,q}(X)}.$$

Note that a priori, we only have an isomorphism $H^k(X,\mathbb{C}) \cong \bigoplus_{k=p+q} H^{p,q}(X)$. It can be seen that the isomorphism is canonical, which justifies the equality.

The decomposition of the cohomology groups of M can be represented in what is called the *Hodge diamond*. It goes as follows:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
H^{n,0}\,H^{n-1,1} & & \cdots & \cdots & \cdots & & H^{1,n-1}\,H^{0,n} \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

Note that this diagram has several symmetries:

- By the Hodge Decomposition Theorem, $H^{q,p} = \overline{H^{p,q}}$.
- The Hodge $*$-operator induces an isomorphism between $H^{p,q}$ and $H^{n-p,n-q}$.

Conclusions

As a conclusion to this master thesis, I would like to emphasize, first of all, the huge amount of results in several branches of mathematics I had to skip. From category theory to functional analysis, in order to prove in all detail the Hodge Decomposition Theorem, it is required to have a broad knowledge in those fields. Even though I did not had the time to study more carefully the missing results, seeing how some of them apply here has been very enriching.

Furthermore, I wanted to provide some examples of how to use this important Theorem but I ran out of time. It is well known that the Hodge Decomposition Theorem on compact Kähler manifolds has lots of applications. For instance, it can be used to determine that a smooth manifold cannot be a complex manifold. If $\mathbb{S}^1 \times \mathbb{S}^3$ were a complex manifold, by the Hodge decomposition theorem, the first cohomology group would have even dimension, but this is not the case.

Finally, I also wanted to prove the functoriality or naturality of the isomorphism between the group of harmonic sections and the de Rham or Dolbeault cohomology groups.

Bibliography

[BI] I. Birger, *Cohomology of Sheaves,* Universitext, Springer, Cop., Berlin, 1986.

[BH] O. Biquard and A. Höring, *Kähler Geometry and Hodge Theory,* Universtié Nice Sophia Antipolis, France,
`http://math.unice.fr/~hoering/hodge/hodge.pdf`.

[GO] R. Gonzales, *Sheaf Cohomology,* Western University, Canada,
`http://www.math.uwo.ca/~rgonzal3/atp.pdf`.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry,* Pure and Applied Mathematics, Inc., Wiley, New York, NY, 1978.

[MA] G. Magnússon, *Being Kählet,* University of Iceland, Iceland, 2010.

[SA] C. Sabbah, *Théorie de Hodge et Théorème de Lefschetz "Difficile",* Notes de cours, Strasbourg, 2000, Bordeaux, 2001,
`http://www.math.polytechnique.fr/~sabbah/hodge-str.pdf`.

[VO] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I,* Cambridge studies in advanced mathematics, **76, 77**, Cambridge University Press, Cambridge, 2002.

[WE] R. O. Wells, *Differential Analysis on Complex Manifolds,* Graduate texts in mathematics,**65**, Springer-Verlag, Cop., New York, NY, 2008.