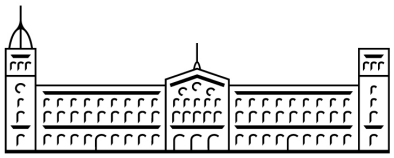

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Breakdown of tori in symplectic maps

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Summary

Most of physical phenomena can be explained in terms of Hamiltonian systems. These continuous dynamical systems can be related with symplectic maps. Under certain hypothesis one can see that these maps present some invariant tori. Then, it is really interesting to understand how these tori behave. One of the most important properties these tori present is that they persist under small perturbation of our initial systems but that for higher perturbations they are going to break down. These perturbations are usually related with equations depending on a parameter, K. For $2D$ symplectic maps, renormalization techniques allow to understand the mechanisms concerning the destruction of invariant circles. Rotation numbers of these circles play a key role in the analysis of their breakdown. Throughout this work we will show some of the most important tools to deal with these invariant circles.

First we present KAM Theory which gives some numerical conditions on the rotation number which will ensure the persistence of the invariant tori under small perturbation of our systems. These conditions will be related with the kind of irrationality of the considered rotation number and are called Diophantine conditions. KAM Theory is developed for several different systems, as for example, circle maps, Hamiltonian systems and symplectic maps. We will prove their corresponding KAM Theorems by using Arnold's technique, consisting in doing some Fourier analysis with zero mean functions.

Later we will do some numerical analysis. This analysis is based in computing the frequencies (rotation numbers) of our curves. We will consider two different methods to approximate these rotation numbers. The first one is a classical method based in projecting our curve, and the second one is Luque-Villanueva's Method which considers an unfolded version of our signal and then uses some extrapolation methods. In both cases if the method does not converge we can conclude that the orbit is not an invariant circle. Doing this analysis for several possible initial points, and several possible values of our parameter K, we can approximate the critical value K_C for which all invariant tori disappear.

We will continue by presenting an obstruction method to explain why these invariant circles break down. The method consists in looking for some heteroclinic tangencies of hyperbolic periodic points of our system. If they appear we are not going to be able to have invariant circles with rotation number between the periods of our points. This fact will explain why they break down and also gives an approximation value of the critical parameter too.

Finally we will present renormalisation techniques. These techniques are based in constructing a renormalisation operator, that is, doing some scale changes in our system. The renormalisation operator depends on the rotation number of the invariant circles we want to study at any moment. The scale changes can be viewed as zooming up our system and then we can forget about some non-relevant phenomena which are far away of our interesting rotation curve. To present this operators we concentrate on noble rotation numbers, in particular in the golden mean $\gamma = \frac{1+\sqrt{5}}{2}$. There exist two fixed points of the renormalisation operator. The first one presents an invariant circle with the corresponding rotation number and is going to be attracting. So if we consider a map close to it (with small perturbation) it is going to tend to it under iteration and that can be viewed as having also an invariant curve with that rotation number. The second fixed point is going to present a non-smooth invariant curve with the interesting rotation number and a 1-parameter universal unstable manifold. This fact gives us the information related with when this curves are going to break down.

CHAPTER 1

Introduction and first definitions

First we will set our work space and show some examples. To do so we introduce the kind of dynamical systems we are going to work with, and show some real applications of them.

We consider some continuous dynamical systems (Hamiltonian systems) and also some discrete ones (symplectic maps). As working with discrete dynamical systems is usually easier, we concentrate on the symplectic maps. We are able to do so, because it is possible to pass from a Hamiltonian system to a symplectic map in a very simple way, just by using Poincaré maps.

The reason of studying this kind of systems is given in Section 1.2, by showing some real problems that can be modelled by a Hamiltonian system or by a symplectic map.

1.1 First definitions

The first kind of systems we are going to consider are Hamiltonian systems:

Definition 1.1.1. A *Hamiltonian system with N degrees of freedom* is given by the scalar map $H(p_1, \ldots, p_N, q_1, \ldots, q_N)$ and the differential equations:

$$\begin{cases} \dfrac{dp_i}{dt} &= -\dfrac{\partial H}{dq_i} \\ \dfrac{dq_i}{dt} &= \dfrac{\partial H}{dp_i} \end{cases} \tag{1.1}$$

We suppose that H is at least $\mathcal{C}^2$ to have that our differential equations are at least $\mathcal{C}^1$ and be able to apply the common theorems of existence and uniqueness of solutions. Components q_i are called *configuration coordinates* and p_i *canonical momenta*.

Remark 1.1.2. During the work we consider analytic systems.

Notation. We denote by $\mathbf{p} = (p_1, \ldots, p_N)$ and by $\mathbf{q} = (q_1, \ldots, q_N)$.

Definition 1.1.3. The scalar function from Definition 1.1.1 is called *Hamiltonian equation* or *energy equation*. Generally it represents the energy of our real system and $H^{-1}(h)$, $h \in \mathbb{R}$ are the level sets.

Remark 1.1.4. Hamiltonian systems preserve energy. That is, assume we fix $h \in \mathbb{R}$, a possible value of the energy equation. We select $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{q}}$ such that $H(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}) = h$ and we integrate our equations from 0 to $t = t_f$ obtaining $\widehat{\mathbf{p}_f}$ and $\widehat{\mathbf{q}_f}$ then, one has $H(\widehat{\mathbf{p}_f}, \widehat{\mathbf{q}_f}) = h$.

Example 1.1.5. One of the most famous Hamiltonian systems is the Hénon-Heiles System, which is given by:

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) + q_1^2 q_2 - \frac{q_2^3}{3}$$

Remarks 1.1.6.

1. All Hamiltonian systems with 1 degree of freedom are integrable and have $H(\mathbf{p}, \mathbf{q})$ as its first integral.

2. If $H(\mathbf{p}, \mathbf{q}) = H(\mathbf{p})$ the Hamiltonian system is also integrable.

Remark 1.1.7. The role of variables $\mathbf{p}$ and $\mathbf{q}$ has no mathematical meaning, it just play a role on physical questions. Therefore Part 2 of Remark 1.1.6 can also be written in terms of $\mathbf{q}$.

This physical meaning is explained in the following example: If we deal with Newtonian equations under the action of a potential. Then q_i represent the component of the position of the point mass (or masses) and p_i represents the component of the linear momenta, $p_i = m_j \dot{q}_i$, if q_i denotes one of the components of the particle with mass m_j.

Sometimes we can see that $\mathbf{p}$ will represent the canonical momenta coordinates and that $\mathbf{q}$ the configuration ones. In that case we just have to take care with the minus sign when constructing the differential equations.

Once we have spoken about the considered continuous dynamical systems, we present the discrete ones, symplectic maps. To do so first we should present the symplectic forms.

Definition 1.1.8. Let M be a manifold, and consider τ a 2-form. Then we say that τ is:

- *closed* if it has zero differential, that is, $d\tau = 0$.
- *non-degenerate* if for all $p \in M$, if there exists $X \in T_pM$ such that $\tau(X, Y) = 0\ \forall Y \in T_pM$ then $X = 0$.

- *symplectic* if it is closed and non-degenerate.

Remark 1.1.9. By the definition of 2-form we have also the skew-symmetry property and therefore M must have even dimension.

Now we are able to define a symplectic map:

Definition 1.1.10. We say that a diffeomorphism, T, is a *symplectic map* if it preserves a symplectic form.

Remark 1.1.11. If the preserved symplectic form is $\tau = dp \wedge dq$ then 2-dimensional symplectic maps are area preserving. Otherwise they preserve "some kind of area".

Example 1.1.12. We consider the *standard map*, which is a 1-parameter family defined on $\mathbb{S}^1 \times \mathbb{R}$. The lift of the family is given by:

$$T_K(x, y) = (x + y + K \sin(2\pi x), y + K \sin(2\pi x))$$

This map is a symplectic one, and can also be seen as a map from $\mathbb{T}^2$ to itself.

As working with maps it is easier than working with continuous systems, we need to relate Hamiltonian systems with symplectic maps. This can be done in a very simple way:

Proposition 1.1.13. *If we have a Hamiltonian system with N degrees of freedom we can consider a transversal section, Σ, to the associated Hamiltonian vector field, as defined by (1.1), and its corresponding Poincaré map, $\mathcal{P}$. If we also fix a level of energy, that is, we take points such that $H(\boldsymbol{p}, \boldsymbol{q}) = h$ for a fixed value of h, then $\mathcal{P}$ for this level of energy is a symplectic map.*

Remark 1.1.14. Using Proposition 1.1.13 we pass from a Hamiltonian with N degrees of freedom (that is $2N$ variables) to a $(2N-2)$ dimensional symplectic map.

We lose one of the variable by fixing the transversal section, Σ, and the other one by the energy conservation of Hamiltonian systems. For example, if Σ is defined as $\Sigma = \{(\mathbf{p}, \mathbf{q}) | q_N = 0\}$ (assuming transversality in Σ or in a subset of it) then p_N should be taken in such a way that $H(\mathbf{p}, \mathbf{q}) = h$.

Remark 1.1.15. If we have $N = 1$, since everything is integrable, the study of these systems is very simple and therefore we are not interested in that case.

We are going to consider always Hamiltonians with 2 degrees of freedom or 2 dimensional symplectic maps. The reason is that we want present some techniques that are available in this lower dimension but not extended to higher ones. This problem remains still open when, for example, $N = 3$. It is planned to consider this topic in a future work.

Integrability and compactness make us think in having some circular movements, therefore we want to define invariant circles. This fact is given by Liouville-Arnold's Theorem, see Appendix 26 from Arnold-Avez [3].

Definition 1.1.16. Given a symplectic map, T, in $\mathbb{S}^1 \times \mathbb{R}$. We say that an orbit, $\mathcal{C}$, is *an invariant rotating circle*, if it is a closed curve dividing the cylinder in two invariant sets and it verifies that $T\mathcal{C} = \mathcal{C}$.

1.2 Motivation and examples.

Studying this kind of systems and maps is particularly interesting since many of different physical phenomena can be expressed in terms of a Hamiltonian equation that determines its corresponding energy. Therefore, studying the behaviour of these equations would lead us to a complete understanding of how the models of these physical systems behave.

Example 1.2.1 (Charged particle)**.** Consider a particle with mass m and charge Q_1 moving in a bound orbit around a fixed charge Q_2 in the plane perpendicular to a constant magnetic field B. If we consider polar coordinates (r, θ), then we have that:

$$H(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2m} + \frac{(p_\theta - \frac{1}{2}Q_1 B r^2)^2}{2mr^2} + \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}$$

where p_r and p_θ represent the momenta, gives us the Hamiltonian system that leads the movement of the charged particle.

Example 1.2.2 (The restricted three body problem)**.** Let us consider that we have two punctual masses, we are going to called them primaries, moving in a circular orbit around their common centre of mass according to the Kepler laws. These 2 punctual masses can be for example a star and a planet around it (e.g., Sun - Jupiter system). The restricted three body problem describes the motion of a new particle under the gravitational attraction of the primaries. We assume that this third particle has negligible mass respect to the primaries and hence it does not affect their motion.

We take units in order to have gravitational constant equal to one, period for the primaries equal to 2π and that the sum of the masses of the primaries is equal to 1. Having this consideration, the distance between the primaries is also going to be 1.

We denote by $\mu \in (0, \frac{1}{2}]$ the mass of the smallest primary (for example $\mu = 0.953771180363 \times 10^{-3}$ corresponds to the Sun-Jupiter system).

Finally we use a rotating coordinate system with origin at the centre of mass of the primaries, making them to be fixed on the x-axis. Consider that the x-axis points forwards the biggest primary, that the z-axis points to the direction of angular motion of the primaries with respect to the centre of masses, and that the y-axis is defined in order to obtain an orthogonal and positive-oriented system of reference. In this system the primary mass μ is at the point $(\mu - 1, 0, 0)$, and the other one at the point $(\mu, 0, 0)$. We define the following momenta, $p_x = \dot{x} - y$, $p_y = \dot{y} + x$ and $p_z = \dot{z}$, and then we have that the equations of motion can be written has a Hamiltonian system with the following energy function:

$$H(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2} + y p_x - x p_y - \frac{1-\mu}{r_S} - \frac{\mu}{r_J},$$

where:

$$r_S^2 = (x - \mu)^2 + y^2 + z^2, \qquad r_J^2 = (x - \mu + 1)^2 + y^2 + z^2.$$

To show that the existence of invariant tori in these Hamiltonian equations, and therefore in the symplectic maps, is interesting, we present the following example related with particle accelerators.

Example 1.2.3. The main idea on how these particle accelerators work is the following. We have some particles rotating in a huge torus. This rotation is due to some external forces, generally they are electro-magnetic. The target is that one of this particles collides with a different one. This fact will make some new particles to appear, and these are the interesting particles that physicists want to study. This movement can be modelled by a Hamiltonian [15].

We can imagine for example, that we have our particle rotation in the tube, and we put another particle, stopped in a position such that to have a collision between the particles, the first one has to describe an invariant circle. It becomes really interesting then to know when this invariant circle will disappear under perturbation, to have some control about how to control the behaviour of the magnets.

As we can pass always from a continuous dynamical system to a discrete one (via Poincaré map), to study the behaviour of this Hamiltonians, it will be enough to study their corresponding associated symplectic maps.

CHAPTER 2

KAM Theory

In this chapter we present KAM Theory. This theory due to Kolmogorov, Arnold and Moser is very useful to assure the existence of invariant tori under small perturbations of our system. It gives us some conditions on the rotation number and the "size" of the perturbation to ensure the persistence of these invariant tori. There are different approaches to this problem depending on the kind of mappings one considers. We are going to introduce and prove the theorem for mappings in the circle and finally state the KAM Theorem for both Hamiltonian systems and symplectic maps.

2.1 KAM Theorem for circle maps

2.1.1 Introduction

Let Π_ρ be the band given by $|\text{Im } z| < \rho$. Let f be a holomorphic bounded function on this band, we can define:

$$\|f\|_\rho := \sup_{z \in \Pi_\rho} |f(z)|$$

Definition 2.1.1. Let $\mu > 0$ be an irrational number, $K > 0$ and $\sigma \geq 0$. We will say that μ is *Diophantine of type* (K, σ) (or that it *verifies the Diophantine condition of type* (K, σ)) if for any $u \in \mathbb{N}$ and any $v \in \mathbb{N} \backslash \{0\}$ we have that:

$$\left| \mu - \frac{u}{v} \right| \geq \frac{K}{|v|^{2+\sigma}}$$

Remark 2.1.2. If we admit $u \in \mathbb{Z}$ then we can extend this definition to any irrational number, μ.

Theorem 2.1.3 (KAM Theorem for circle maps)**.** *Assume that there exists $\varepsilon > 0$, which only depends on ρ, K, σ, such that, if f is a 2π-periodic analytic function, that it is real over the real axis and verifies $\|f\|_\rho < \varepsilon$. If the map:*

$$y \overset{F}{\mapsto} y + 2\pi\mu + f(y)$$

defines a diffeomorphism of the circle with number of rotation, μ, Diophantine of type (K, σ). Then this diffeomorphism is analytically conjugate to a rotation of angle $2\pi\mu$.

What this Theorem says is that if we are close to a rotation with rotation number Diophantine, then our map is also conjugated to a rotation with same rotation number.

Remark 2.1.4. Note that the main hypothesis of the Theorem is that the rotation number is Diophantine, and that the perturbation is small. In case the map F is not as in the statement, but has rotation number μ we can always modify it and write it in that form.

2.1.2 Proof of Theorem 2.1.3

Introduction

First we introduce our problem. If we call U the rotation of angle $2\pi\mu$ and H the diffeomorphism we are looking for. Then H should conjugate the rotation U with F, and therefore the following diagram should be conmutative:

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{F} & \mathbb{S}^1 \\ \uparrow{\scriptstyle H} & & \uparrow{\scriptstyle H} \\ \mathbb{S}^1 & \xrightarrow{U} & \mathbb{S}^1 \end{array}$$

that is, we need $H \circ U = F \circ H$.

So writing H as $H(z) = z + h(z)$, where $h(z + 2\pi) = h(z)$, we will obtain the following functional equation:

$$h(z + 2\pi\mu) - h(z) = f(z + h(z))$$

By hypothesis we have that f is "small", making that F has a small difference from a rotation. It is natural to suppose that h is of the same order of smallness than f. Hence we have that $f(z+h(z))$ differs of $f(z)$ on a quantity which is less than the order of f (and therefore less than ε). Then we can obtain our first approximation of h using the following equation:

$$h(z + 2\pi\mu) - h(z) = f(z) \tag{2.1}$$

Definition 2.1.5. Equation (2.1) is called *homological equation.*

Solution in the class of formal Fourier Series

We can express the function f and the unknown h as their Fourier expressions, so:

$$f(z) = \sum_{k\in\mathbb{Z}} f_k e^{ikz} \qquad h(z) = \sum_{k\in\mathbb{Z}} h_k e^{ikz}$$

So identifying coefficients in (2.1) we obtain that:

$$h_k = \frac{f_k}{e^{2\pi i\mu k} - 1}$$

To have a solution we need that the numerators vanish at the same time as the denominators (as we work with Diophantine rotation numbers this fact will never happen), in particular we need that $f_0 = 0$. If we suppose $f_0 = 0$ and μ irrational then there is going to exist a solution to the homological equation (2.1) in the class of formal Fourier series. In order to obtain the "real" solution we need to study the convergence of this series. Before studying this convergence we have to solve the small divisors problem.

Small divisors problem

Taking μ to be irrational we assure that the denominators do not vanish, but some of them may be very close to zero. In fact, each number admits rational approximations, $\frac{u}{v}$, with an error:

$$\left|\mu - \frac{u}{v}\right| < \delta$$

for any δ and for v large enough (density of irrational numbers). Then for $k = q$ the expression $e^{2\pi ik\mu} - 1$ would be very close to zero. We can establish that those small divisors are going to be bounded by below from a power of k almost surely.

Lemma 2.1.6. *Let $\sigma > 0$. Then for almost every real number μ, there exists $K = K(\mu, \sigma) > 0$ such that:*

$$\left|\mu - \frac{p}{q}\right| \geq \frac{K}{|q|^{2+\sigma}} \tag{2.2}$$

for p, q non zero entire numbers.

Proof. We work only over the interval $[0, 1]$, then it can be generalized. Let's consider the numbers in $[0, 1]$ such that the inequality in (2.2) its not verified (for some p, q, K, σ fixed). These numbers make a set of length less than or equal to $\frac{2K}{q^{2+\sigma}}$. If we consider the union of these sets for every p (keeping the other constants fixed) we obtain that it has length $\frac{2K}{q^{1+\sigma}}$. Summarizing we have that over q we have a set of measure less than or equal to CK where $C = 2\sum q^{-(1+\sigma)} < \infty$. Then the set of numbers $\mu \in [0, 1]$ such that there is not any K as in the statement of the lemma is covered by sets of measure as close to zero as we want. Hence it has measure zero. □

Remark 2.1.7. Any Diophantine number of type (K, σ) verifies that the small divisor admits the following lower bound:

$$\left|e^{2\pi ik\mu} - 1\right| \geq \frac{2\pi K}{|k|^{1+\sigma}} \qquad |k| > 0.$$

Combining Lemma 2.1.6 and 2.1.7 we solve the problem of small denominators. Lemma 2.1.6 assures that almost every real number is Diophantine and Remark 2.1.7 gives us the lower bound.

Convergence of the Fourier series

Once we have solved the small denominators problem, we want those Fourier series to be convergent. To study this convergence we need to show some properties about the Fourier coefficients of this analytic functions.

Lemma 2.1.8. *If f is a 2π-periodic function, which is analytic over the band, Π_ρ, continuous in the closure of the band and having $\|f\|_\rho \leq M$. Then the Fourier coefficients decrease as a geometric progression. In particular:*

$$|f_k| \leq Me^{-|k|\rho}$$

Proof. We know that:

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(z)e^{-ikz}dz$$

Let $k > 0$. Doing the change of variable $x = z - i\rho$ and using that our function f is 2π-periodic. Then:

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(z - i\rho)e^{-ikz-k\rho}dx \Rightarrow |f_k| \leq Me^{-k\rho}$$

If $k < 0$ we have to consider the change $x = z + i\rho$. □

Lemma 2.1.9. *If $|f_k| \leq Me^{-|k|\rho}$ then the function $f(z) = \sum f_k e^{ikz}$ is analytic over the band Π_ρ and moreover:*

$$\|f\|_{\rho-\delta} \leq \frac{4M}{\delta} \qquad \text{if } \delta < \rho \ \&\ \delta < 1$$

Proof. Using Lemma 2.1.8, we have that:

$$\begin{aligned}
\|f\|_{\rho-\delta} &\leq \sup_{z\in\Pi_{\rho-\delta}} \sum_{k\in\mathbb{Z}} |f_k||e^{ikz}| \leq \sup_{z\in\Pi_{\rho-\delta}} M\sum_{k\in\mathbb{Z}} e^{-|k|\rho}|e^{ikz} \\
&\leq \sup_{z\in\Pi_{\rho-\delta}} M\sum_{k\in\mathbb{Z}} e^{-|k|}e^{|k|(\rho-\delta)} = M\sum_{k\in\mathbb{Z}} e^{-|k|\delta} \\
&\leq \frac{2M}{1-e^{-\delta}} \leq \frac{4M}{\delta}
\end{aligned}$$

□

Remark 2.1.10. Lemma 2.1.8 is still valid if we work with functions in n-variables. On the other hand the bound in Lemma 2.1.9 has to be changed to $\frac{CM}{\delta^n}$ where $C = C(n)$ is independent of δ and f.

Solution for zero mean functions

Now we are ready to solve the problem for 2π-periodic analytic functions having mean value zero, i.e., $f_0 = 0$. This is shown in the following Lemma:

Lemma 2.1.11. *If f is 2π-periodic and has zero mean, then for almost every number μ the homological equation has a 2π-periodic analytic solution (which is real if f is real). There exists a constant $\nu = \nu(K, \sigma) > 0$ such that if μ is Diophantine of type (K, σ), then $\forall \delta > 0$, less than ρ and $\rho < 1/2$ one has that:*

$$\|h\|_{\rho-\delta} \leq \|f\|_\rho \delta^{-\nu}$$

Remark 2.1.12. Going form f to h does not modify the properties of the function. It is just a ν-uple derivation. (It can be proved that $\|\frac{d^\nu f}{dz^\nu}\|_{\rho-\delta} \leq C\|f\|_\rho \delta^{-\nu}$. This is due to the Cauchy majorization of the Taylor coefficients). If we neglect the alteration of the function due to a ν-uple derivation, we can say that the solution to the homological equation, h, is of the same order of smallness than its second member f.

Proof of Lemma 2.1.11. The proof of this Lemma is divided in 4 steps:

1. By Lemma 2.1.8 we have that, if $\|f\|_\rho \leq M$ then:
$$|f_k| \leq M e^{-|k|\rho}$$

2. Using Lemmas 2.1.8 and 2.1.6, μ Diophantine of the type (K, σ) implies that:
$$|h_k| \leq \frac{2M e^{-|k|\rho} |k|^{1+\sigma}}{K}$$

3. The function $w(x) = x^m e^{-\alpha x}$ if $x > 0$ has a maximum in $x = \frac{m}{\alpha}$. Then:
$$x^m e^{-\alpha x} \leq C\alpha^{-m} \quad \text{where } C = \left(\frac{m}{\alpha}\right)^m \quad \forall \alpha > 0 \;\; \forall x > 0$$
Hence, $\forall \alpha > 0$, taking $m = 1 + \sigma$:
$$|k|^{1+\sigma} e^{-\alpha|k|} \leq C\alpha^{-(1+\sigma)}$$

4. Finally, we have, using points 2 and 3 that:
$$|h_k| \leq M e^{-|k|(\rho-\alpha)} 2CK^{-1}\alpha^{-m}$$
Therefore using Lemma 2.1.9, we have:
$$\begin{aligned} \|h\|_{\rho-\delta} &\leq \frac{4}{\delta} M e^{-|k|(\rho-\alpha)} 2CK^{-1}\alpha^{-m} \\ &= \frac{8C}{K\alpha^m} \frac{e^{-|k|(\rho-\alpha)}}{\delta} \\ &\leq \frac{8C}{K\alpha^m(\delta-\alpha)} M = DM \end{aligned}$$
So taking $\alpha = \delta/2$ we have that $D \leq \delta^{-\nu}$ if ν is big enough, since $\delta < 1/2$.

□

General case: iterated approximations

We want to extend this fact to any function f. To do so we are going to use Arnold's method, which is based on Newton's Method. The key point is that we transform our general function to a function with zero mean, just by substracting is mean value. Then we can apply the tools explained above obtaining a small error.

We start solving the homological equation (2.1) taking as the function on the right hand side: $\tilde{f} = f - f_0$ where f_0 is the mean value of our general function f. We can do this since we are going to prove in Part 4 of the proof of Lemma 2.1.13 that this mean value has order two and we can take it with the residue. This $\tilde{f}$ is going to have zero mean and therefore we can take h^0 to be the solution of the homological equation. We define $H_0 : z \mapsto z + h^0(z)$, and construct:

$$F_1 = H_0^{-1} \circ F \circ H_0$$

we denote as f^1 the map defined by $F_1(z) = z + 2\pi\mu + f^1(z)$.

What we have done is to introduce a new coordinate over the circle, z_1 (where $z = H_0(z_1)$), and we express F in terms of this new coordinate. Then, the map, $z_1 \mapsto F(z_1)$, obtained differs of a rotation of angle $2\pi\mu$ a residue f^1.

Our next approximation is built in the same way, but using F_1 in the place of F. We construct then h^1 and the transformation H_1 that takes F_1 into $F_2 = H_1^{-1} \circ F_1 \circ H_1$.

If we iterate this method, we obtain a sequence of changes, F_n. We are going to consider the change $\mathcal{H}_n = H_0 \circ H_1 \circ \ldots \circ H_{n-1}$ and we haven then that:

$$F_n = \mathcal{H}_n^{-1} \circ F \circ \mathcal{H}_n$$

We will prove that this sequence is going to converge when μ verifies the Diophantine condition of type (K, σ) and if $\|f\|_\rho$ is sufficiently small. The limit change $\mathcal{H}$ will take our initial map to $\mathcal{H}^{-1} \circ F \circ \mathcal{H} = \lim F_n$ which is a rotation of angle $2\pi\mu$.

Estimation of the residue after one iteration

We are interested now in the behaviour of the residues in order to make this method convergent. We do the study for the first step, but if we work for the n-th step everything would be analogous.

Lemma 2.1.13. *There exist constants $\chi, \lambda > 0$ depending only in K and σ, such that $\forall \delta \in (0, \rho)$, with $\rho < \frac{1}{2}$, we have:*

$$\|f\|_\rho \leq \delta^\chi \Rightarrow \|f^1\|_{\rho-\delta} \leq \|f\|_\rho^2 \delta^{-\lambda}$$

Remark 2.1.14. This means that the residue, f^1, that appears after the first iteration (first change of variables) is of second order with respect to the initial residue, f, of the rotation (the modification close to a derivation of the type λ-upla of the function). So we have that in the scheme of the iterated approximations, the

error committed in each step is of the order of the square of the one made in the previous step. So at the end of n iterations we will have an error of the order of ε^{2^n} if ε is sufficiently small. We have also some terms in $\delta^{-\lambda}$ but they are absorbed by the ε terms, and so they are not relevant.

This characteristic of convergence of the Tangents Newton's Method allows us to avoid the influence of the small divisors that appears at each step, that is, the influence of the modifying factor, $\delta^{-\lambda}$. This method to deal with the small divisors is due to A.Kolmogorov (1954).

Proof of Lemma 2.1.13. We are going to divide this proof in five steps:

1. Let $\Omega \subset \mathbb{C}^n$ convex domain, $g : \Omega \to \mathbb{C}^n$ a differentiable map such that:

$$\|g'\| = \sup_{x\in\Omega} |g'(x)| < 1$$

The map H, where $G(x) \mapsto x + g(x)$ is a diffeomorphism from Ω to $G\Omega$.

Proof. Indeed, the eigenvalues of $G(x)$ are different from zero, then G is a local diffeomorphism. We have also that g is a contraction since $|g(x)| < q < 1$ and since Ω is convex, therefore we have that G is bijective. Considering all this facts we have that G is a diffeomorphism from Ω to $G\Omega$. □

2. If the constant χ is big enough then the map F is analytic in the band $\Pi_{\rho-\delta}$.

Proof. Let $\|f\|_\rho \leq M = \delta^\chi$. Then we have $|f_0| \leq M$ and $\|\tilde{f}\|_\rho \leq 2M$ (we have used Lemma 2.1.8 and the definition of $\tilde{f}$). Using Lemma 2.1.11:

$$\|h^0\|_{\rho-\alpha} \leq 2M\alpha^{-\nu}$$

So:

$$\left\|\frac{dh^0}{dz}\right\|_{\rho-2\alpha} \leq 2M\alpha^{-(\nu+1)}$$

Let now, $\alpha = \frac{\delta}{8}$. If χ is big enough then the previous inequalities told us that:

$$\|f\|_\rho < \alpha, \qquad \|h^0\|_{\rho-\alpha} < \alpha, \qquad \left\|\frac{dh^0}{dz}\right\|_{\rho-2\alpha} < \alpha$$

Using Lemmas 2.1.8 and 2.1.9 we have that H_0 is a diffeomorphism over the band $\Pi_{\rho-2\alpha}$ and the image contains the band $\Pi_{\rho-3\alpha}$. We have then that $H_0\Pi_{\rho-\delta} \subset \Pi_{\rho-\delta+\alpha}$ and $F \circ H_0\Pi_{\rho-\delta} \subset \Pi_{\rho-\delta+2\alpha} \subset \Pi_{\rho-3\alpha}$.Therefore the diffeomorphism H_0^{-1} is deffined over $F \circ H_0\Pi_{\rho-\delta}$. And then the map $F_1 = H_0^{-1} \circ F \circ H_0$ is analytic in $\Pi_{\rho-\delta}$ and it is a diffeomorphism. □

3. We are going to evaluate the residue, f^1, now.

   The commutative diagram that gives us f^1 tells us:

   $$x + 2\pi\mu + f^1(z) + h^0(z + 2\pi\mu + f^1(z)) = z + h^0(z) + 2\pi\mu + f(z + h^0(z))$$

   or in terms of the homological equation:

   $$f^1(z) = [f(z + h^0(z)) - f(z)] - [h^0(z + 2\pi\mu + f^1(z)) - h^0(z + 2\pi\mu)] + f_0$$

   Using the Mean Theorem and Cauchy's Inequality we can bound the first bracket by:

   $$\|f(z + h^0(z)) - f(z)\|_{\rho-\delta} \leq \frac{M}{\delta}\|h^0\|_{\rho-\delta} \leq M^2\delta^{-u}$$

   where the constant u does only depend on ν, that is, it depends only on K and σ.

   In an analogously way we can bound the second bracket by:

   $$\begin{aligned} \|h^0(z + 2\pi\mu + f^1(z)) - h^0(z + 2\pi\mu)\| &\leq 2M\alpha^{-(\nu+4)}\|f^1\|_{\rho-\delta} \\ &\leq M\delta^{-u_1}\|f^1\|_{\rho-\delta} \end{aligned}$$

   Therefore

   $$\|f^1\|_{\rho-\delta}(1 - M\delta^{-u_1}) \leq |f_0| + M^2\delta^{-u}$$

4. Let's evaluate now $|f_0|$ using that the rotation number of F and in fact of F^1 is equal to $2\pi\mu$. We have that f^1 is zero in a real point z_0. Therefore, using the formula for $f^1(z)$, we have that in z_0:

   $$f_0 = f(z_0) - f(z_0 + h^0(z_0))$$

   and then, using the same bounds as in point 3:

   $$|f_0| \leq M^2\delta^{-u}$$

5. Using points 3 and 4 we obtain that:

   $$\|f^1\|_{\rho-\delta} \leq 4M^2\delta^{-u}$$

□

Convergence of the method

Now that we have estimated the residue made at each step, we are able to study de convergence of the sequence. We are going to do it in 4 steps:

1. We study the map F_n, constructed in the n-th step, over the band of radius ρ_n. We have that the radius decreases at each step: $\rho_0 = \rho$ and $\rho_n = \rho_{n-1} - \delta_{n-1}$.

   We can take the sequence, δ_n, in the following way:

   $$\delta_n = \delta_{n-1}^{3/2} \qquad \text{with } \delta_0 < \frac{1}{2}$$

   Then for δ_0 small enough we have:

   $$\sum_n \delta_n < \frac{\delta}{2}$$

2. Take a sequence of numbers, M_n, such that:

   $$M_n = \delta_n^N$$

   where N, that should be big enough and only depending on K and σ, is chosen in advanced. We can have that:

   $$M_n = M_{n-1}^{3/2}$$

3. If we suppose that $\|f\|_\rho \leq M_0$ then we have that $\|f^n\|_{\rho_n} \leq M_n$.

   Proof. We have seen in the Proof of Lemma 2.1.11 that if $N > \chi$, then:

   $$\|f^1\|_{\rho_1} \leq M_0^2 \delta_0^{-\lambda} = \delta_0^{2N-\lambda}$$

   and

   $$\delta_0^{2N-\lambda} < \delta_0^{3N/2} = \delta_1^N \qquad \text{if } N > 2\lambda$$

   If we take N bigger than 2λ and than χ, we have then:

   $$\|f^1\|_{\rho_1} \leq \delta_1^N = M_1$$

   Doing the same we can do the step from f^{n-1} to f^n and then we are done. □

4. Let's prove now the convergence of the composition of maps $\mathcal{H}_n = H_0 \circ \ldots \circ H_{n-1}$ in the band $\Pi_{\rho/2}$.

   The diffeomorphism H_0 is analytic over Π_{ρ_1} and verifies:

   $$\|h^0\|_{\rho_1} \leq \delta_0 \qquad \left\| \frac{dh^0}{dz} \right\|_{\rho_1} \leq \delta_0$$

as we have seen also in the proof of Lemma 2.1.11.

Analogously we can see that for H_{n-1}:

$$\|h^{n-1}\|_{\rho_n} \leq \delta_{n-1} \qquad \left\|\frac{dh^{n-1}}{dz}\right\|_{\rho_n} \leq \delta_{n-1}$$

Then $\mathcal{H}_n$ is analytic over Π_{ρ_n} and has derivative bounded from above and from below by

$$C = \prod_k (1+\delta_k) \qquad \& \qquad c = \prod_k (1-\delta_k)$$

From here we have that $\mathcal{H}_n$ is a diffeomorphism in Π_{ρ_n} and that the sequence $\mathcal{H}_n$ is convergent in $\Pi_{\rho/2}$. Indeed:

$$\|\mathcal{H}_n - \mathcal{H}_{n+1}\| \leq C\|h^n\|_{\rho/2} \leq C\delta_n$$

Let $\mathcal{H}$ be the limit of $\mathcal{H}_n$. Then going to the limit in $F \circ \mathcal{H}_n = \mathcal{H}_n \circ F_n$ we have that $F \circ \mathcal{H} = \mathcal{H} \circ U$ where U is the rotation of angle $2\pi\mu$.

So we have proved the result for a general function f verifying the hypothesis of Theorem 2.1.3.

2.2 KAM Theorem for Hamiltonian systems

As we are interested on working with Hamiltonian flows and symplectic maps we introduce KAM Theorem for those kind of maps. To study this theory for Hamiltonian systems, first of all we have to consider some integrable systems.

2.2.1 Some integral systems

Consider the phase space $\Omega = \mathcal{B}^n \times \mathbb{T}^n$, where $\mathcal{B}^n$ is a bounded domain. Imagine that $\mathbf{p} = (p_1, \ldots, p_n)$ are the coordinates on $\mathcal{B}^n$ and that $\mathbf{q} = (q_1, \ldots, q_n) \mod 2\pi$ are the coordinates on $\mathbb{T}^n$. We consider the Hamiltonian system, having as energy equation:

$$H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p})$$

Then we have that the equations of the Hamiltonian system are given by:

$$\begin{cases} \dot{\mathbf{p}} &= 0 \\ \dot{\mathbf{q}} &= \omega_0(\mathbf{p}) \end{cases} \tag{2.3}$$

where $\omega_0(\mathbf{p}) = \dfrac{\partial H_0}{\partial \mathbf{p}}$.

These equations describe a quasi-periodic movement with frequency $\omega(\mathbf{p})$ over the invariant tori $\mathbf{p} = \text{ctant}$. The condition to have a change of frequency from a torus to another is the following one:

$$\det\left(\frac{\partial^2 H_0}{d\mathbf{p}^2}\right) = \det\left(\frac{\partial \omega_0}{\partial \mathbf{p}}\right) \neq 0. \tag{2.4}$$

Therefore for each neighbourhood of the torus there exist some invariant tori where the frequencies are independent over the rationals and the orbits are dense (Jacobi's Theorem see Appendix 1 from Arnold - Avez [3]). On the other hand we have that there exist also some other tori with frequencies satisfying a resonance condition, that is $\langle k, \omega \rangle = 0$ for $k \in \mathbb{Z}^n \backslash \{0\}$, this set of tori has Lebesgue measure zero.

2.2.2 Systems close to an integrable system

Suppose now that we perturb the Hamiltonian of our integrable system in a small way. That is, we define a new energy equation:

$$H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p}) + H_1(\mathbf{p}, \mathbf{q})$$

where H_1 is 2π-periodic on $\mathbf{q}$ and is small enough, $\|H_1(\mathbf{p}, \mathbf{q})\| \ll 1$. So in this case our system is given by the following equations:

$$\begin{cases} \dot{\mathbf{p}} &= -\dfrac{\partial H_1}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} &= \omega_0(\mathbf{p}) + \dfrac{\partial H_1}{\partial \mathbf{p}} \end{cases} \tag{2.5}$$

Notation. We consider ω^* a non-resonant vector of frequencies, that is $\langle k, \omega \rangle \neq 0$ $\forall k \in \mathbb{Z}^n$. We denote by $T_0(\omega^*)$ the invariant torus of the non-perturbed system (2.3) having ω^* as its vector of frequencies.

On the other hand $T(\omega^*)$ represents a torus for the perturbed system (2.5) having ω^* as its vector of frequencies.

Having these considerations we can write the following Theorem:

Theorem 2.2.1 (KAM Theorem for Hamiltonian Systems)**.** *There exists $\varepsilon > 0$ such that if $\|H_1\| < \varepsilon$ and our system verifies (2.4), then for almost every ω^* there exists an invariant torus, $T(\omega^*)$, for the perturbed system (2.5), and it is going to be a neighbour of $T_0(\omega^*)$.*

Remark 2.2.2. Condition (2.4) plays an important role during the proof of this Theorem. We need it in order recover the Diophantine frequencies if they are changed by the map obtained from the solution of the homological equation.

The proof of this Theorem uses the same ideas as the proof of KAM Theorem for circle maps, and hence we are not going to show it, for more information one can have a look at *Chapter 21: Proof of a Theorem of A.N. Kolomogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian* of Arnold in [2].

This theorem assures that under a small perturbation and assuming the "torsion" condition holds, then almost every invariant torus will persist.

Remark 2.2.3. The condition for the persistence of the torus is going to be a Diophantine one.

2.3 KAM Theorem for symplectic maps

Let's consider symplectic maps defined from the cylinder to itself. Suppose that the variable x represents the angle coordinate.

Definition 2.3.1. Let $T : (x, y) \to (x', y')$ be a symplectic map from the cylinder to itself. If T is differentiable we say that T is a *twist map (with twist to the right)* if there exists $K > 0$ such that:

$$\left.\frac{dx'}{dy}\right|_x \geq K. \tag{2.6}$$

Twist condition (2.6) means that x' is a monotonically increasing function of y.

Remark 2.3.2. Analogously we can define twist maps with twist to the left. When referring to twist maps, we just need that $\left.\frac{dx'}{dy}\right|_x$ has constant sign.

Remark 2.3.3. In general we have that T^2 is not a twist map.

Example 2.3.4. We consider the standard map defined in Example 1.1.12 Therefore:

$$\left.\frac{dx'}{dy}\right|_x = 1 > 0$$

so it verifies the twist condition for any value of the parameter K.

As in the case of Hamiltonian systems we can have integrable maps. In this case we have are going to be able to write our map with some angle coordinates θ verifying:

$$\begin{cases} \dot{\theta} &= \theta + \Omega(I) \\ \dot{I} &= I \end{cases}$$

Remark 2.3.5. In this case our twist condition (2.6) is defined in the following way:

$$\frac{d\Omega}{dI} \geq K > 0 \qquad \left(\text{or } \frac{d\Omega}{dI} \leq \hat{K} < 0\right) \tag{2.7}$$

As in the Hamiltonian case, for this integrable systems everything is very easy to study and there appear some invariant quasi-periodic tori, in fact, most of the phase space is filled up by tori. We are interested in how these tori behave under small perturbation. In this case to define small we are going to take care about the j derivatives of the perturbation.

Definition 2.3.6. We define the *j-th norm of f* as;

$$\|f(x, y)\|_j = \sum_{m+n \leq j} \left|\frac{\partial^{m+n} f}{\partial x^m \partial y^n}\right|$$

We consider then the following perturbed system:

$$\begin{cases} \dot{\theta} &= \theta + \Omega(I) + g(I,\theta) \\ \dot{I} &= I + f(I,\theta) \end{cases} \tag{2.8}$$

We need to suppose that the average of $f(I,\theta)$ is zero in order to have invariant circles. Once we have introduced this perturbed system we have the following KAM Theorem:

Theorem 2.3.7 (KAM Theorem for Symplectic maps)**.** *If $\Omega(y)$ verifies the twist condition (2.7) and is j times differentiable, then there exists $\varepsilon > 0$ such that all the perturbed system of the form (2.8) with $\|f\|_j + \|g\|_j < \varepsilon$ has rotational invariant circles for all frequencies that satisfies a Diophantine condition with:*

$$1 < \sigma < (j-1)/2$$

Again the proof is based in the same tricks as the one for circle maps, and the theorem gives us conditions under which this invariant circles persist under small perturbation.

CHAPTER 3

Frequency Analysis

To detect the existence of invariant circles in our symplectic maps we can do a numerical study. This study is going to be based in the computation of orbits corresponding to possible invariant circles and then to look numerically if they are indeed invariant circles or not.

The procedure is very simple. What we have to do is to take a grid in all the possible initial conditions and possible values of the parameter of our equation. Then compute the orbit starting at any possible initial condition and any possible parameter value on the grid. And finally see if it is an invariant circle or not.

In order to see if each orbit corresponds to an invariant circle or not, we can proceed in several different ways. In general, we are interested in computing the possible rotation number of each curve. During this work we concentrate in two possible methods:

1. Classical Method.

2. Luque-Villanueva's Method.

During this work we are going to present both algorithms and finally use them to present an example.

Remark 3.0.8. It can be seen that Luque-Villanueva's Method require less CPU time and hence is more efficient. As the target of the work is not to compare them we won't prove this fact.

Remark 3.0.9. Both methods can be used for maps in any dimension.

3.1 Classical Method

Let F be a general map:

$$F : \omega \subset \mathbb{R}^n \to \mathbb{R}^n$$

having an invariant curve, $\mathcal{C}$. This method be based in the computation of some "angles". We can have two different options, and then we proceed in a different way to compute these "angles".

3.1.1 Existence of a 1-1 projection

In the first case we suppose that there exists a projection, $\mathcal{T}$, of the invariant circle, $\mathcal{C}$, into a $2D$ plane, where we are going to be able to take polar coordinates in a 1-1 way and not repeating arguments.

Our curve is given by some iterations of the map, and then for any of our points of the orbit, p_n, we can obtain an "angle" or argument, α_n. We are suppose that $\alpha_0 = 0$. We are able to do this simple assumption just rotating the axes in the $2D$ plane. Therefore we have that $\alpha_0 = 0$ and that $\alpha_j \in [0, 2\pi)$ for any possible j.

Suppose that the iterates in $\mathcal{TC}$ move in the positive direction (otherwise we can use symmetry).

Then we associate a new parameter, n_j, to all of our points. This parameter is computed in the following way:

1. We set $n_0 = 0$.
2. If we have $\alpha_j > \alpha_{j-1}$ then we take $n_j = n_{j-1}$, and otherwise we take $n_j = n_{j-1} + 1$.

Remark 3.1.1. If the orbit of the initial point is not a periodic orbit we can suppose that all the α's are different.

Therefore after M iterations we obtain two lists of data, one with the α's and one with the n's. Once we have this values, we order the values of the α's in an increasing way. So we obtain:

$$\alpha_0 < \alpha_{j_1} < \alpha_{j_2} < \ldots < \alpha_{j_k} < \ldots < \alpha_{j_M}$$

Then if we call ρ the rotation number of $\mathcal{C}$ and we take two consecutive k indexes, we have two different possibilities:

- If $j_{k+1} > j_k$ then we have $\rho > \dfrac{n_{j_{k+1}} - n_{j_k}}{j_{k+1} - j_k}$.
- If $j_{k+1} < j_k$ then we have $\rho < \dfrac{n_{j_{k+1}} - n_{j_k}}{j_{k+1} - j_k}$.

In both cases we obtain a bound of the rotation number, so iterating we can obtain a good approximation of its value.

3.1.2 Non-existence of a 1-1 projection

Now we have to consider the possibility that the 1-1 projection, $\mathcal{T}$, does not exist (see Figure 3.1). This does not allow us to use the same method to compute the angles and we have to use a new method.

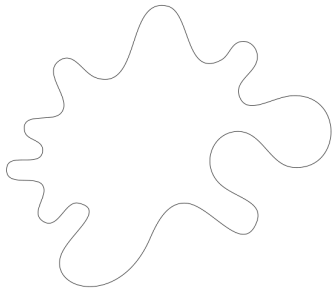

Figure 3.1: Non-projectable curve.

What we have to do then is to produce some artificial "angles". To do so we need to assume that our curve is smooth (for example of class $\mathcal{C}^2$).

First of all we take the initial point and give it index 0. Now we look for the closest point to this point zero. Imagine it has index k, we call $v_{0,k}$ the vector with origin at the point with index 0 and end at the point with index k. To select the next point we look for the closest point to p_k which is different form p_0 and also verifies that the inner product between $v_{0,k}$ and $v_{k,i}$ is positive, where $v_{k,i}$ is the vector linking p_k with this new point.

Remark 3.1.2. We make the assumption of the inner product to take points going in the same direction.

We have ordered the points along the curve, and we can take the following "angles":

- For p_0 we take $\alpha_0 = 0$
- For p_k (the closest point to p_0) we take $\alpha_k = \dfrac{2\pi}{M}$.
- For p_i (the closest to p_k verifying all the conditions needed) we take $\alpha_i = 2\dfrac{2\pi}{M}$.
- ...

Now we have constructed these artificial "angles" we can compute the n's in the same way as we did when we have that projection and we can do just the same things to obtain the approximation of the rotation number.

Remark 3.1.3. This can be applied to arbitrary systems, not only to symplectic or Hamiltonian. For instance, it can be applied to a Poincaré section of a $2D$ torus in a discretization of a PDE with a large number of nodes.

3.1.3 Algorithm

Once we have covered all the possibilities we are ready to introduce the algorithm in charge of compute the rotation number of the invariant circle $\mathcal{C}$. The algorithm is the following:

- We fix some initial bounds for ρ: $\rho_{\text{sup}} = 0$ and $\rho_{\text{inf}} = 1$.
- We compute all the α's and n's and sort them.
- Loop: $k = 0, \ldots, M-1$.

  We compare j_{k+1} and j_k:

  If $j_{k+1} > j_k$ redefine $\rho_{\text{sup}} = \min\left\{\rho_{\text{inf}}, \dfrac{n_{j_{k+1}} - n_{j_k}}{j_{k+1} - j_k}\right\}$.

  Otherwise if $j_{k+1} < j_k$ redefine $\rho_{\text{inf}} = \max\left\{\rho_{\text{sup}}, \dfrac{n_{j_{k+1}} - n_{j_k}}{j_{k+1} - j_k}\right\}$.

We stop if at any moment we have:

$$|\rho_{\text{sup}} - \rho_{\text{inf}}| < \text{tol}$$

for any fixed tolerance.

We have that ρ_{sup} represents the supremum of the lower bounds of ρ and ρ_{inf} the infimum of the upper bounds. Therefore we should have $\rho_{\text{sup}} < \rho_{\text{inf}}$. If at any moment this does not happen that means that we are not in an invariant circle. We have to take care if the difference is to small it can be due to roundoff errors and therefore we can keep our approximation.

Remark 3.1.4. This algorithm gives us the rotation number of an invariant circle in case we really have it, but it is also useful to see if they really exist since the method would not be convergent or would fail ($\rho_{\text{sup}} > \rho_{\text{inf}}$ at any moment) in case we don't have this invariant circle.

3.2 Luque-Villanueva's Method

We present now a different method to do frequency analysis using averaging-extrapolation methods. This tactic is due to Alejandro Luque (Universitat de Barcelona) and Jordi Villanueva (Universitat Politècnica de Catalunya) [6]. The target is to compute numerically the basic frequencies of quasi-periodic signals. The method can be viewed as a tool to refine a rough approximation of the frequency, so first of all we have to compute the first approximation. Although we are going to present it for general signals we are interested in using it at the orbits we obtain.

A refinement of the results given by a Discrete Fourier Transform was introduced by J.Laskar at the end of the 80's to detect chaotic motion in the solar system. At the beginning of the 90's it was improved by a Newton-like method by Gómez,

Jorba, Masdemont and Simó to be applied to space missions design. Later on, Jordi Villanueva and Tere Seara present some improvements to these methods. Finally, we get the method we present here.

First we give a formal definition for quasi-periodic signals.

Definition 3.2.1. Let $\mathbb{T}^r = \mathbb{R}^r/(2\pi\mathbb{Z})^r$ be the r-dimensional standard torus. We say that a complex sequence $\{z_n\}_{n\in\mathbb{Z}}$ is a *quasi-periodic signal with frequency vector* $\omega \in \mathbb{R}^r$, if there exists $\gamma : \mathbb{T}^r \to \mathbb{C}$ such that $z_n = \gamma(n\omega)$, $\forall n \in \mathbb{Z}$.

If we consider the Fourier expansion of γ:

$$\gamma(\theta) = \sum_{k\in\mathbb{Z}^r} \hat{\gamma}_k e^{i\langle k,\theta\rangle} \qquad \text{where} \qquad \hat{\gamma}_k = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} \gamma(\theta) e^{-i\langle k,\theta\rangle}\, d\theta$$

then we have:

$$z_n = \gamma(n\omega) = \sum_{k\in\mathbb{Z}^r} \hat{\gamma}_k e^{in\langle k,\omega\rangle}$$

We continue by presenting a definition of vector of basic frequencies of a signal.

Definition 3.2.2. A vector $\omega \in \mathbb{R}^r$ is called *vector of basic frequencies of the signal* if it verifies non-resonance conditions, that is:

$$\langle k, \omega\rangle + 2\pi m \neq 0 \tag{3.1}$$

for any $k \in \mathbb{Z}^r\backslash\{0\}$ and $m \in \mathbb{Z}$.

Remark 3.2.3. In case (3.1) is not verified we can rewrite our signal in terms of a new frequency vector, having less number of frequencies.

3.2.1 Unfolding of the signal to isolate a frequency

Let's describe how to unfold a quasi-periodic signal. The target is to build a new signal having the signal we are looking for as a frequency around the origin.

We start presenting first a first order scheme and then a second order one, finally we show the general case having order p.

In order to present this method we suppose certain Diophantine conditions:

Definition 3.2.4. We say that the vector $\omega \in \mathbb{R}^r$ verifies the *Diophantine condition* if:

$$\left|e^{i\langle k,\omega\rangle} - 1\right| \geq \frac{C}{\|k\|_1^{\tau+r}} \qquad \forall k \in \mathbb{Z}^r\backslash\{0\} \tag{3.2}$$

for some $C, \tau \geq 0$, where $\|\cdot\|_1$ represents the usual 1-norm.

First order unfolding

Definition 3.2.5. Let $\{z_n\}_{n\in\mathbb{Z}}$ be a complex-valued quasi-periodic signal. Suppose this signal is analytic and that the frequency vector $\omega \in \mathbb{R}^r$ verifies the Diophantine condition (3.2). For $\omega_0 \in \mathbb{R}$ and $L \in \mathbb{N}$, we define the following iterates:

$$z_n^{(L,\omega_0)} = \frac{1}{L}\sum_{m=n}^{L+n-1} z_m e^{i(n-m)\omega_0} \tag{3.3}$$

Remark 3.2.6. Analyticity is fundamental in this approaching. This contrasts with the "topological" method we have presented before.

One can see that for almost every choice of ω_0, the new sequence $\{z_n^{(L,\omega_0)}\}_{n\in\mathbb{Z}}$ is a quasi-periodic signal with the same frequency vector as the original one.

Moreover, if we represent our new signal (3.3) as its Fourier expansion:

$$z_n^{(L,\omega_0)} = \gamma^{(L,\omega_0)}(n\omega) = \sum_{k\in\mathbb{Z}^r} \hat{\gamma}_k^{(L,\omega_0)} e^{in\langle k,\omega\rangle}$$

Then we have the following expression for the new coefficients:

$$\hat{\gamma}_k^{(L,\omega_0)} = \frac{\hat{\gamma}_k}{L}\sum_{m=n}^{L+n-1} e^{i(m-n)(\langle k,\omega\rangle-\omega_0)} = \frac{\hat{\gamma}_k\left(1-e^{iL(\langle k,\omega\rangle-\omega_0)}\right)}{L\left(1-e^{i(\langle k,\omega\rangle-\omega_0)}\right)}$$

If we suppose that ω_0 is any of the frequencies, for example the chosen one, we call it ω_1, and omitting the dependence on ω_0 we can write this new signal as:

$$z_n^{(L)} = \sum_{k\in\mathbb{Z}^r} \hat{\gamma}_k^{(L)} e^{in\langle k,\omega\rangle}$$

where:

$$\hat{\gamma}_{e_1}^{(L)} = \hat{\gamma_{e_1}}, \qquad \hat{\gamma}_k^{(L)} = \frac{\hat{\gamma}_k\left(1-e^{iL(\langle k,\omega\rangle-\omega_1)}\right)}{L\left(1-e^{i(\langle k,\omega\rangle-\omega_1)}\right)} \text{ for } k \neq e_1$$

Using the fact that the signal is analytic and ω verifies the Diophantine condition (3.2) we have that if $k \neq e_1$:

$$\hat{\gamma}_k^{(L)} = \mathcal{O}(e^{-\rho\|k\|_1}\|k-e_1\|_1^{\tau}/L)$$

Hence for large values of L, this new "unfolded signal", $\{z_n^{(L)}\}_{n\in\mathbb{Z}}$, becomes a quasi-periodic perturbation of a planar rotation of angle ω_1.

Second order unfolding

We want to accelerate this unfolding process. In order to do so we perform higher order averages and Richardson extrapolation. First of all we introduce the second order averages.

Definition 3.2.7. Under the same notations and hypotheses of Definition 3.2.5, we introduce the sequences:

$$\mathcal{Z}_n^{(L,\omega_0,2)} = \sum_{l=1}^{L} \sum_{m=n}^{l+n-1} z_m e^{-im\omega_0}$$

and:

$$\widetilde{\mathcal{Z}}_n^{(L,\omega_0,2)} = \left(\frac{2}{L(L+1)} \mathcal{Z}_n^{(L,\omega_0,2)} \right) e^{in\omega_0}$$

Finally, we define the following iterates:

$$z_n^{(2L,\omega_0,2)} = 2\widetilde{\mathcal{Z}}_n^{(2L,\omega_0,2)} - \widetilde{\mathcal{Z}}_n^{(L,\omega_0,2)}$$

As it happened for the first order unfolding, the new signal $\{z_n^{(2L,\omega_0,2)}\}_{n\in\mathbb{Z}}$ is quasi-periodic with frequency vector ω.

Doing the analogous analysis as for the first order unfolding we can see that if ω_0 is close enough to a particular frequency ω_1 and $L|\omega_0 - \omega_1|$ is sufficiently small, then, this new signal is a perturbation of order $\mathcal{O}(1/L^2)$ of a rotation of angle ω_1 and therefore we have earned some regularity.

Unfolding of order p

Finally we extend Definitions 3.2.5 and 3.2.7 for any order $p \in \mathbb{N}$, by introducing higher order averages.

Definition 3.2.8. Under the same notations and hypotheses of Definition 3.2.5, we introduce the following recursive sums:

$$\mathcal{Z}_n^{(L,\omega_0,1)} = \sum_{m=n}^{L+n-1} z_m e^{-im\omega_0}, \qquad \mathcal{Z}_n^{(L,\omega_0,p)} = \sum_{l=1}^{L} \mathcal{Z}_n^{(l,\omega_0,p-1)}, \ p \geq 2$$

and the averaged sums of order p:

$$\widetilde{\mathcal{Z}}_n^{(L,\omega_0,p)} = \left[\begin{pmatrix} N+p-1 \\ p \end{pmatrix}^{-1} \mathcal{Z}_n^{(L,\omega_0,p)} \right] e^{in\omega_0}$$

If we fix values for $p \geq 1$ and $q \geq p$ we define the new sequence:

$$z_n^{(2^q,\omega_0,p)} = \sum_{j=0}^{p-1} c_j^{(p-1)} \widetilde{\mathcal{Z}}_n^{(L_j,\omega_0,p)}$$

where $L_j = 2^{q-p+j+1}$ and the (extrapolation) coefficients $c_j^{(m)}$ are given by :

$$c_j^{(m)} = (-1)^{m-j} \frac{2^{j(j+1)/2}}{\delta(j)\delta(m-j)} \tag{3.4}$$

where:

$$\delta(0) = 1 \text{ and } \delta(n) = (2^n - 1)(2^{n-1} - 1)\dots(2^1 - 1) \text{ if } n \in \mathbb{N}$$

As in the previous cases, $\{z_n^{(2^q,\omega_0,p)}\}_{n\in\mathbb{Z}}$ is a quasi-periodic signal with frequency vector ω for almost every ω_0.

Remark 3.2.9. We select L_j as powers of two to be able to give an explicit simple formula for the extrapolation coefficients.

This new sequence gives us an unfolding of order p of $\{z_n\}_{n\in\mathbb{Z}}$. With the previous arguments we can see that taking ω_0 close enough to a particular frequency ω_1 and having that $L|\omega_0-\omega_1|$ is sufficiently small, then, this new signal is a perturbation of order $\mathcal{O}(1/L^p)$ of a rotation of angle ω_1 and therefore we have earned some regularity.

3.2.2 Computation of frequencies

At this moment we are going to sketch how to compute Diophantine rotation numbers of our signals.

We are going to work with quasi-periodic signals on $\mathbb{T}$ whose lift to $\mathbb{R}$ takes the following form:

$$x_n = n\omega_1 + \Phi(\omega\theta) = n\omega_1 + \sum_{k\in\mathbb{Z}^r} \hat{\Phi}_k e^{i\langle k,\omega\rangle}, \qquad n\in\mathbb{N} \tag{3.5}$$

where $\Phi : \mathbb{T}^r \to \mathbb{R}$. Therefore we have that ω_1 is the frequency that leads the rotation, and our objective.

Using that $\omega_1 = \lim_{n\to\infty}(x_n - x_0)/n$ to compute the frequency requires a huge number of iterates since we expect a convergence rate of $\mathcal{O}(1/n)$. We want to accelerate this process introducing a suitable recursive and averaged sums.

Definition 3.2.10. Given a sequence $\{x_n\}_{n\geq 1}$, we introduce the following recursive sums:

$$S_N^{(0)} = x_N, \qquad S_N^{(p)} = \sum_{n=1}^{N} S_n^{(p-1)}, \text{ for } N,p\geq 1,$$

and the following averaged sums of order p:

$$\widetilde{S}_N^{(p)} = \begin{pmatrix} N+p \\ p+1 \end{pmatrix} S_N^{(p)}$$

Proposition 3.2.11. *Let us consider a real-valued signal $\{x_n\}_{n\in\mathbb{N}}$ of the form (3.5). We assume that Φ is analytic and that the vector $\omega\in\mathbb{R}^n$ verifies the Diophantine conditions (3.2). Then we have the following expression for the averaged sums of Definition 3.2.10:*

$$\widetilde{S}_N^{(p)} = \omega_1 + \sum_{l=1}^{p} \frac{\widetilde{A}_l^{(p)}(n\omega)}{N^l} + \widetilde{E}^{(N,p)}(n\omega), \qquad N\geq 1,$$

where $\{\widetilde{A}_l^{(p)}(n\omega)\}$ and $\widetilde{E}^{(N,p)}(n\omega)$ are analytic and quasi-periodic expressions with $\widetilde{E}^{(N,p)} = \mathcal{O}(1/N^{p+1})$. Specifically, $|\widetilde{E}^{(N,p)}| \leq C_p/N^{p+1}$, for some $C_p > 0$.

Furthermore, we consider fixed integer values $q \geq p \geq 1$ and we define the following extrapolation operator of order p, which involves $N = 2^q$ consecutive iterates of $\{x_n\}_{n\in\mathbb{N}}$:

$$\Theta^{(p,q)} = \sum_{j=0}^{p} c_j^{(p)} \widetilde{S}_{N_j}^{(p)}$$

where $N_j = 2^{q-p+j}$ and the coefficients $c_j^{(p)}$ are given by (3.4). Then, we have:

$$\omega_1 = \Theta^{(p,q)} + e(p,q)$$

where $e(p,q) = \mathcal{O}(1/N^{p+1}) = \mathcal{O}(2^{-(p+1)q})$.

Using this proposition we obtain a good approximation of the leading frequency of our signal. For further information about the scheme, and the proof of this Proposition, we recommend to read Luque-Villanueva [6].

3.2.3 First approximation of the frequency

We have seen that our method works properly if we have an initial approximation of the frequency we are interested in. If we don't have any a priori information about it we can use the following method to obtain a good starting approximation. This starting approximation is based in the Discrete Fourier Transform (DFT) of the signal.

Let $\{z_n\}$ be a quasi-periodic signal. If we have a sample of L points we can compute its DFT as a function of ω_0 given by:

$$\omega_0 \mapsto \frac{1}{L}\sum_{m=0}^{L-1} z_m e^{-im\omega_0}$$

It is clear that this expression coincide with $z_0^{(L,\omega_0)}$ from the unfolded signal of order 1. We are going to denote by $Z^{(L,1)}(\omega_0) = |z_0^{(L,\omega_0)}|$ its modulus.

Imagine first that we have $\{z_n\}$ a quasi-periodic signal that only has one Fourier harmonic, say it $z_n = e^{in\omega_1}$. Then we have that:

$$Z_0^{(L,1)}(\omega_0) = \frac{1}{L}\left|\frac{\sin(L(\omega_1-\omega_0)/2)}{\sin((\omega_1-\omega_0)/2)}\right| \tag{3.6}$$

has a local maximum at $\omega_0 = \omega_1$.

If we have now a generic signal, $\{z_n\}$, then the effect of the other harmonics on $Z^{(L,1)}(\omega_0)$ can be viewed as a perturbation of (3.6) around $\omega_0 = \omega_1$. This leads to a displacement of the maximum of $Z_0^{(L,1)}(\omega_0)$, that allows us to compute the dominant frequency ω_1 of $\{z_n\}$ with accuracy of $\mathcal{O}(1/L)$.

Remark 3.2.12. Sometimes is useful to use some kind of filters, for example Hanning filters, to decrease the effect of the other harmonics having closed frequencies.

We can use higher order unfoldings in order to have better accuracy. If we consider a sample of $2L$ points of our signal we can introduce:

$$Z^{(2L,2)}(\omega_0) = |z_0^{(2L,\omega_0,2)}|$$

where $z_0^{(2L,\omega_0,2)}$ is the zero iterate of the second order unfolded signal. In this case we are also going to reach a local maximum at $\omega_0 = \omega_1$. We have also that around the maximum the peaks are going to be softer and therefore this is going to magnify the maximum. In this case we have a rate of convergence of $\mathcal{O}(1/L^2)$.

We can also extend this method to higher order, using the unfolded signal of higher order. For example, we can extend it to order p by considering:

$$Z^{(2^q,p)}(\omega_0) = |z_0^{(2^q,\omega_0,p)}|$$

3.3 An example: The standard map

Let's consider now the standard map defined in Definition 1.1.12. We are interested in computing some of the different invariant rotating circles that appear for some values of the parameter K.

We have an approximation for the critical parameter, this approximation due to Greene verifies that:

$$2\pi K_C \approx 0.9716...$$

Some ways of approximating this parameter will be shown in the following chapters, for example the obstruction method or renormalisation techniques.

The existence of invariant rotating circles for $K = 0.06$ can be viewed in Figure 3.2. These invariant rotating circles are identified with lines crossing from the left part of the square to the right one. However, in Figure 3.3 these circles do not appear. The reason is that this value of K exceed Greene's critical value, and therefore no invariant rotational circles could live there.

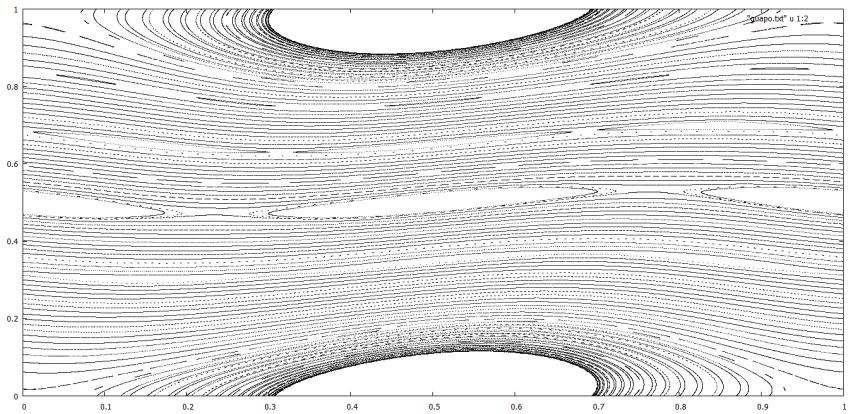

Figure 3.2: Orbits of the standard map for $K = 0.06$.

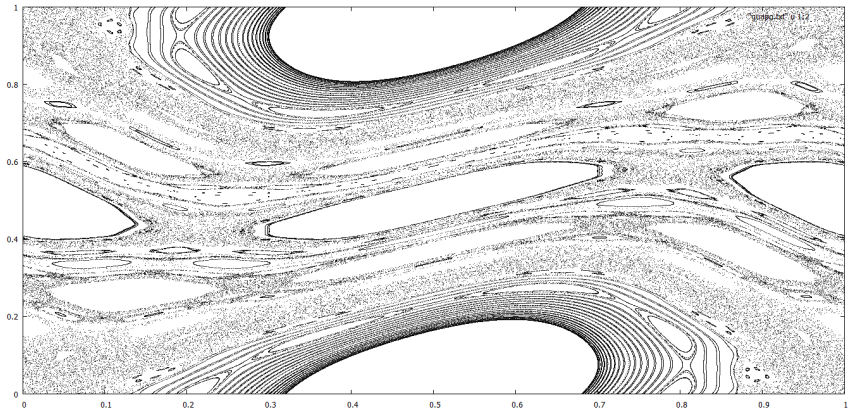

Figure 3.3: Orbits of the standard map for $K = 0.16$.

Next step in our work is to compute where we can find these invariant rotating circles as a function of the initial position and the value of the parameter. To do so, we fix a value of the x-parameter, for example $x = 0$, and compute the possible periodic orbits for $y_0 \in [0.5, 0.8]$. We select this interval because here we have the existence of a golden curve. We follow these steps:

1. Take a grid in the possible values of the parameter from 0.08 to 0.16, and in the possible initial condition y_0 from 0.5 to 0.6. We take for example step 10^{-4} in both directions.

2. We compute the orbit with initial condition $(0, y_0)$ and parameter value K_0, for the possible pairs (y_0, K_0) on the grid.

3. We unfold the signal by taking $L = 1024$ and $p = 2$.

4. We approximate the frequency starting by $p = 4$ and $q = 4$. We iterate for different values of q until $q = 13$. In case two different iterates differ less than a tolerance (for example, 10^{-10}) we stop iterating and store the point.

In Figure 3.4 we have the results obtained. Notice that using Luque-Villanueva's Method we also localize the circular orbits around the islands. These orbits are not the desired ones. We can avoid them by non-storing the ones having rotation number that can be considered rational. To exclude them we consider the following procedure: Once we have the rotation number we have a look at if it is rational or not. This computation is done in the following way. We compute its convergents. If at any point the denominator is very big, for example greater than 10^5, we consider the point rational. Also if it is big, but not so big, for example 10^3 we compute the error between our rotation number and the corresponding convergent. If it is less than 10^{-10} then we say that the rotation number is rational too. We do this for the 10 first convergents.

We obtain also the rotation number of the points in the peaks closed to $y = 0.65$ and an approximation of the critical value of the parameter. The rotation number

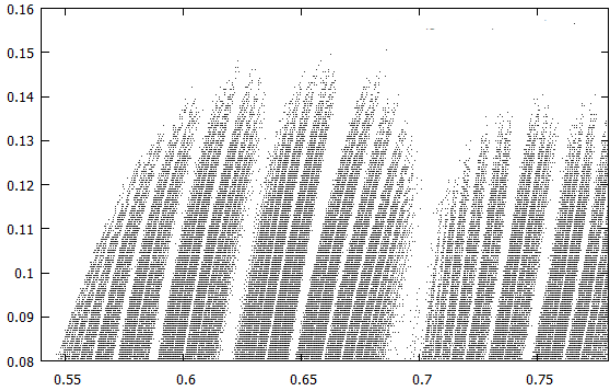

Figure 3.4: Invariant circles with initial point $(0, y_0)$ for values of $y_0 \in [0.5, 0.8]$.

we obtain 1.6178450768819137 which is a good approximation of the golden mean, $\gamma = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$. The approximation last parameter value is $K = 0.1535$ and therefore we have $2\pi K = 0.96446894465$ which is closed to Greene's critical value.

Remark 3.3.1. While computing the rotation numbers we have to compute 4 "huge" sums while obtaining the S_N's. In order to avoid that huge computational time we develop the expressions of the sums. We have obtained a much simpler formula which just involves 2 different sums. This fact saves a lot of memory.

The same method can be use to find the curves around the island for higher values of the parameter. In particular, Narcís Miguel, Carles Simó and Arturo Vieiro [10], obtained Figure 3.5. They consider points surrounding the accelerator mode island one see at the point $(1/4, 0)$ for $K = 1$.

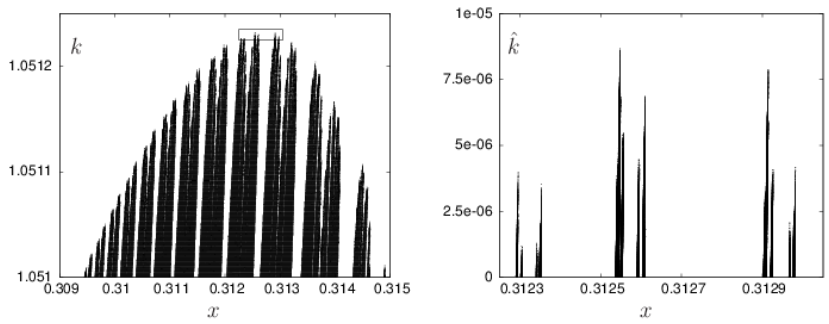

Figure 3.5: Initial conditions in the (x, K) plane taken on $y = 0$ for which there exist rotational invariant curves surronding the accelerator mode island. Right plot is a magnification of the box in the left one. $\hat{K} = K - 1.051225$.

They have obtained also that the rotation number of the last two peaks, near $x = 0.3125$ and 0.3129, are given by $[4, 14, 1, 1, 1, \ldots]$ and $[4, 13, 1, 1, 1, \ldots]$, respectively.

An analogous representation close to different resonances can be seen in Figure 6 of [10].

CHAPTER 4

Obstruction Method

To explain why the invariant circles disappear and to approximate the critical value of the parameter we are going to introduce an obstruction method presented by Arturo Olvera and Carles Simó [11].

We explain the method following our main example, the standard map. Finally we explain how to use this method numerically in order to compute the critical value of the parameter up to which some invariant circles persist.

4.1 Birkhoff's Theorem

Before introducing the method we have to prove Birkhoff's Theorem. To state the theorem we have to present the concept of end preserving map:

Definition 4.1.1. We say that a 2 dimensional symplectic map, $T(x, y)$, defined on $\mathbb{S}^1 \times \mathbb{R}$ is *end preserving* if points with arbitrary large component y are mapped into similar points. That is, if we map to points we are "close" they remain "close".

Now we are able to state the Theorem.

Theorem 4.1.2 (Birkhoff's Theorem)**.** *Let T be a $\mathcal{C}^1$ two dimensional symplectic twist map on the cylinder. Let T be end preserving and U be an open invariant set homeomorphic to the cylinder verifying that there exist $a < b$ such that:*

$$\{(x, y)|y < a\} \subset U \subset \{(x, y)|y < b\}$$

Then we have that the boundary of U is going to be the graph, $\{(x, f(x))\}$, of some continuous function f.

The proof of Birkhoff's Theorem is based in the concept of accessible points.

Definition 4.1.3. Let $\gamma(t) = (x(t), y(t))$ be a curve embedded on U such that $\lim_{t\to-\infty} = -\infty$. We define the *deviation of* γ as the angle, δ, between a tangent vector to γ and the vertical direction. For points such that $y(t) > y(t')\ \forall t' < t$ we choose $\delta \in [-\pi/2, \pi/2]$, otherwise we take it to make the deviation continuous.

Definition 4.1.4. A curve γ^R is *tilted to the right* if $\delta \leq 0$ everywhere. Analogously we can define *left-tilting curves* that we are going to denote by γ^L.

Definition 4.1.5. We say that a point z_0 is *right accessible* if there exists γ^R tilted to the right, $\gamma^R(t) \subset U$, such that $\gamma^R(t_0) = z_0$ for some $t_0 \in \mathbb{R}$.

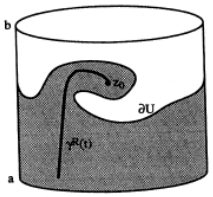

Figure 4.1: Right accessible point.

Now we are able to prove Birkhoff's Theorem:

Proof of Birkhoff's Theorem 4.1.2. We have that any right-tilting curve, γ^R, is mapped onto another such curve by T.

Indeed, we know that any vector v at z is mapped to Mv where M represents the linearisation of T at point z. In particular, by twist condition, we have that the vertical direction is mapped into a tilting-right vector with $\delta \in [-\pi, 0]$. Since T preserves orientation, we have that the angle δ' between Mv and the tangent to $T(\gamma^R)$ at $T(Z)$ has to be in range $[-\pi, 0]$. Finally the deviation of $T(\gamma^R)$ has to be the sum of those two angles so it has to be tilted to the right.

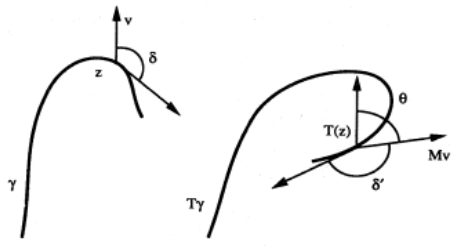

Figure 4.2: Iterate of a curve that tilts to the right and which also tilts to the right.

Now we can take A^R and A^L the subsets of U which are right and left accessible respectively. We have that ∂A^R is made of parts of ∂U with some vertical segments (see Figure 4.3). Also we have that by what we have just proved $T(A^R) \subset A^R$.

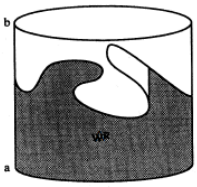

Figure 4.3: Right accessible region.

Similarly we have the same property with ∂A^L and that $T^{-1}(A^L) \subset A^L$.

In fact, as T is symplectic we have that $A^R = U$.

Indeed, imagine this is false, then there exists a lobe, $\mathcal{L}$. Under iteration we have that vertical direction start tilting to the right, therefore some part of $\mathcal{L}$ is mapped on A^R. Consider now the circle $y = y_0$ far below ∂U. Since ∂U is contained between $y = a$ and $y = b$ the area above y_0 is finite. Furthermore, we have that this region is going to be mapped onto a region with the same area. This is a contradiction since the flux crossing $y = y_0$ has to be zero.

Analogously one can see that $A^L = U$.

So each point of U is vertically accessible and hence we can find the continuous function describing ∂U.

□

Corollary 4.1.6. *All invariant rotating curves are graphs of continuous functions.*

Corollary 4.1.7. *This function f is in fact Lipschitz.*

4.2 Obstruction to invariant curves by heteroclinic tangencies

To describe the method to find this upper bound for the critical value of the parameter, K_c, such that an invariant rotating curve (IRC) disappears, we consider the standard map family (see Example 1.1.12).

We remind that the standard map is the 1-parameter family of maps on the cylinder, $\mathbb{S}^1 \times \mathbb{R}$ or in $\mathbb{T}^2$, f_k, such that its lift on $\mathbb{R}^2$ is given by:

$$F_K(x, y) = (x + y + K \sin(2\pi x), y + K \sin(2\pi x))$$

By Birkhoff's Theorem 4.1.2 we have that the IRC can always be identified with the graph of a Lipschitz function, $y = \Phi(x)$, that separates the cylinder in two invariant sets . Any of these curves have rotation number (RN) defined as:

$$\omega = \lim_{n\to\infty} \frac{\Pi_1 F_K^n(x, \Phi(x)) - x}{n}$$

where Π_1 represents the projection of the first coordinate.

We know that every periodic orbit has a rational rotation number $\frac{p}{q}$, where for any (x, y) in the orbit, we have:

$$\Pi_1 F^q(x, y) = x + p.$$

The points on this periodic orbits are all formed by hyperbolic, elliptic or parabolic points. It is easy to prove that when $K = 0$ we are under an integrable system and that the orbits are either IRC (in case $\omega \in \mathbb{R}\backslash\mathbb{Q}$) or parabolic periodic points ($\omega \in \mathbb{Q}$).

If we consider $[\omega_0, \omega_1]$, where ω_0 and ω_1 are related with some orbits in the cylinder, then we have that any $\omega \in [\omega_0, \omega_1]$ has an invariant set between those orbits associated to it. In the case that this ω is rational we have that this set contains periodic orbits with rotation number ω and different points such as, for example, the stable manifold of the hyperbolic periodic points.

Considering all this things we can obtain the following results.

Lemma 4.2.1. *Let x_0 and x_1 be two hyperbolic periodic points of F_K, with rotation number ω_0 and ω_1 respectively. If the unstable manifold of x_0 and the stable manifold of x_1 have non-empty intersection then there is not an IRC with rotation number $\omega \in (\omega_0, \omega_1)$.*

Proof. Take $\omega_C \in (\omega_0, \omega_1)$ and let $\Phi_C(x)$ be the Lipschitz function such that its graph (on the lift) gives us the invariant circle with rotation number ω_C. Let P_h be one of heteroclinic points. And let (x_0, P_h, x_1) denote the continuous path joining x_0 with x_1 passing along P_h and formed by the union of $W^u_{x_0}$ and $W^s_{x_1}$.

For a given function Φ we are going to denote its graph by $G\Phi$.

By preservation of the measure under our map we should have homoclinic points for the point x_0. And then we should have the existence of some homoclinic connections (separatrices) or not. In case we have them, we are going to define Φ_{x_0} such that $G\Phi_{x_0}$ is the union of the separatrices. In case they do not exist we take $G\Phi_{x_0}$ the concatenation of segments of $W^u_{x_0}$ and $W^s_{x_1}$. In any of the cases we obtain a closed curve encircling the cylinder. Finally we do the same process with x_1 and construct $G\Phi_{x_1}$,

Once we have construct this graphs we have that:

$$\begin{cases} G\Phi_C \cap (x_0, P_h, x_1) &= \emptyset \\ G\Phi_C \cap G\Phi_{x_0} &= \emptyset \\ G\Phi_C \cap G\Phi_{x_1} &= \emptyset \end{cases}$$

Therefore we have that $\Phi_C(x) - \Phi_{x_0}(x)$ and $\Phi_C(x) - \Phi_{x_1}(x)$ do not change their sign.

We have that, since $\omega_1 > \omega_C$, we can obtain from $\Pi_1 F_K(x, \Phi_{x_1}(x)) - \Pi_1 F_K(x, \Phi_{x_C}(x)) = \Phi_{x_1}(x) - \Phi_{x_C}(x)$ and then it follows that: $\Phi_{x_1}(x) - \Phi_{x_C}(x) > 0$.

Analogously we can see that $\Phi_{x_C}(x) - \Phi_{x_0}(x) > 0$. Hence we have that $G\Phi_C$ lies between $G\Phi_{x_0}$ and $G\Phi_{x_1}$, so (x_0, P_h, x_1) should have non-empty intersection with $G\Phi_C$. This is an absurdity and hence we have that such Φ_C does not exist and therefore there are no IRC with rotation number ω_C.

□

Remark 4.2.2. Lemma 4.2.1 is a general result where we need to suppose twist condition.

Lemma 4.2.3. *Under the same hypothesis than in Lemma 4.2.1, let x_2 and x_3 also hyperbolic periodic points with rotation number ω_2 and ω_3 verifying that $\omega_0 < \omega_2 < \omega_3 < \omega_1$. If x_0 and x_1 have heteroclinic points for the parameter value K_0 then x_2 and x_3 also have heteroclinic points.*

Proof. By Lemma 4.2.1 we have that $G\Phi_{x_0}$ and $G\Phi_{x_1}$ define an instability domain, that means that there are not IRC there. We can construct, as in the previous Lemma, continuous functions Φ_{x_2} and Φ_{x_3}, then for any x we have:

$$\Phi_{x_0}(x) < \Phi_{x_2}(x) < \Phi_{x_3}(x) < \Phi_{x_1}(x)$$

So $G\Phi_{x_2}$ and $G\Phi_{x_3}$ define also an unstable domain.

Let $\varepsilon > 0$ and consider $U^{\varepsilon}_{x_2}$ the disk of radius ε centred at x_2. Then we have that:

$$\forall \varepsilon > 0 \quad W^u_{x_2} \cap U_\varepsilon \neq \emptyset.$$

Let $V = \cup_{n\in\mathbb{Z}} F^n_K(U^{\varepsilon}_{x_2})$. We have then that $\Phi_{x_3} \cap V \neq \emptyset$. Since this is true for all $\varepsilon > 0$ then we have:

$$\Phi_{x_3} \cap W^u_{x_2} \neq \emptyset$$

Hence,

$$W^u_{x_2} \cap W^s_{x_3} \neq \emptyset \quad \text{that is } x_2 \text{ and } x_3 \text{ have heteroclinic points}$$

□

Corollary 4.2.4. *Under the same hypothesis than in Lemma 4.2.3 we also have:*

$$W^u x_0 \cap W^s x_2 \neq \emptyset \quad \text{and} \quad W^s x_1 \cap W^u x_2 \neq \emptyset$$

Lemma 4.2.5. *Let ω_n be a sequence of rational numbers such that:*

1. *$\omega_n = p_n/q_n$ where $p_n, q_n \in \mathbb{N}$.*
2. *$\lim_{n\to\infty} \omega_n = \omega$.*
3. *$|\omega - \omega_{n+1}| < |\omega - \omega_n|$.*
4. *$(\omega - \omega_{n+1})(\omega - \omega_n) < 0$.*

Let x_n and x_{n+1} be hyperbolic periodic points with rotation number ω_n, ω_{n+1} and suppose that they don't have heteroclinic points for $K < K_n$ and that they have such heteroclinic points for $K > K_n$. Then if $m > n$ we have that $K_m \leq K_n$.

Proof. Using Lemma 4.2.3 we have that for every $K_1 \geq K \geq K_n$ by our hypothesis, x_n and x_{m+1} have heteroclinic points. Therefore we have $K_m \leq K_n$. □

Corollary 4.2.6. *The sequence K_n is decreasing.*

Once we have proved this results we are able to obtain this upper bound for the critical parameter. To do so we have to define the continued fraction expansion of a number.

Definition 4.2.7. Given a number ω we say that $[a_0, a_1, \ldots]$ is its *continued fraction expansion* if:

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

Remark 4.2.8. ω is rational if and only if $\omega = [a_0, \ldots, a_n]$.

Definition 4.2.9. We say that ω is *noble* if there exists $n_0 \in \mathbb{N}$ such that $a_N = 1$ for every $N \geq n_0$. The most noble number is the *golden mean*, $\gamma = \frac{1+\sqrt{5}}{2}$.

Once we have defined the continued fraction expansion, we can explain the method. In order to find the critical value for the IRC with rotation number ω, we consider its continued fraction expansion:

$$\omega = [a_1, a_2, a_3, \ldots]$$

And we are going to denote as:

$$\omega_n = [a_1, \ldots, a_n]$$

the *truncated expansion*. Note that $\omega_n \in \mathbb{Q}$.

We look then for the "first" heteroclinic tangency (therefore the tangency that happens for K_n and there are no other heteroclinic points for lower parameters), of the hyperbolic periodic points with rotation number ω_n and ω_{n+1}.

Then from Lemma 4.2.5 and Corollary 4.2.6 we obtain a decreasing sequence $\{K_n\}$ such that its limit is going to be the upper bound of K_c. We have then than there is not any IRC with rotation number ω for $K_\infty = \lim_{n\to\infty} K_n$ and there are those curves for $0 < K < K_\infty$.

Although it is not possible to compute this "first" heteroclinic tangency of the two hyperbolic periodic points since in that case we should extend infinitely their invariant manifolds. We can fix a particular tangency, for example in some selected domain of the phase space, that gives us a unique value of this parameter, k_n.

Definition 4.2.10. Let x_0 and x_1 hyperbolic periodic points and $W^u_{x_0}$ and $W^s_{x_1}$ its invariant manifolds expanded in parametric form, that is:

$$W^u_{x_0} = (\psi_0(s), \eta_0(s))$$
$$W^s_{x_1} = (\psi_1(t), \eta_1(t))$$

for $s, t \in \mathbb{R}^+ \cup \{0\}$. such that $(\psi_0(0), \eta_0(0)) = x_0$ and$(\psi_1(0), \eta_1(0)) = x_1$. We call then *heteroclinic tangency on the "first tonge"* a tangency which happens for values of t and s such that:

1. $(\psi_0(s_0), \eta_0(s_0)) = (\psi_1(t_0), \eta_1(t_0))$.
2. $\left\langle \left(\frac{d\psi_0}{ds}(s_0), \frac{d\eta_0}{ds}(s_0) \right), \left(\frac{d\eta_1}{dt}(t_0), -\frac{d\psi_1}{dt}(t_0) \right) \right\rangle = 0.$

Then we take k_n the minimal value for which we find this kind of tangency.

4.3 Numerical method

The numerical method to compute the heteroclinic tangencies is divided in 5 steps.

4.3.1 Search of the hyperbolic periodic points

First we have to find the hyperbolic periodic points having rotation number ω_n and ω_{n+1} respectively.

Let $\hat{\omega} = \frac{\hat{p}}{\hat{q}}$ be one of the rotation numbers and let K be the parameter. We look for fixed points of $F^{\hat{q}}$ and once we have them, we compute the periodic orbits and select the ones with the required rotation number.

In the case of the standard map, one can see that every periodic orbit has two points belonging to the symmetry lines of the map ($x = 0$, $x = 1/2$, $x = y/2$ and $x = (y+1)/2$).

4.3.2 Expansion of the invariant manifolds of the points

We are interested in obtaining a piece of the invariant manifold of a hyperbolic periodic point. To do so for a hyperbolic periodic point with high period by iteration of points of its linear approximation has a huge computational cost.

We consider a better approximation, that is, a degree-m polynomial approximation of the invariant manifold, $y = \Phi(x)$. We are going to approximate it by:

$$y = \Phi_m(x) = y_0 + \sum_{j=1}^{m} a_j (x - x_0)^j$$

where (x_0, y_0) is the hyperbolic periodic point.

In order to obtain this representation we apply the definition of invariant set:

$$\Pi_2 f^q(x, \Phi(x)) = \Phi(\Pi_1 f^q(x, \Phi(x))) \tag{4.1}$$

Now we can represent f^q by its Taylor series of order m, so:

$$\Pi_r f^q_m(x, y) = \sum_{i+j \le m} \left(\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial x^i \partial y^j} \left(\Pi_r f^q(x_0, y_0)\right) (x - x_0)^i (y - y_0)^j \right) \tag{4.2}$$

where $r = 1, 2$. This derivatives can be obtained in a recurrent way or better by considering compositions and using "automatic differentiation".

If we replace Φ and f^q by their approximations then we can write (4.1) as a polynomial on x.

Identifying terms we will obtain the a_j's as function of the previous coefficients.

There is also an alternative way to compute it: using parametrization.

4.3.3 Numerical continuation of the invariant manifolds

We fix an interval on x inside the invariant manifold. This interval is defined as follows:

$$[\Pi_1 f^{\pm q}(x_0 + \varepsilon, \Phi(x_0 + \varepsilon)), x_0 + \varepsilon]$$

where we take $+$ in order to work with the stable manifold and $-$ to work with the unstable one. If we iterate this points we can continue the invariant manifold until the sign of curvature changes, that is, we arrived at the "first tongue" of the invariant manifold. Once we have done so we can interpolate this manifold (for example using a cubic spline to get a smooth curve).

4.3.4 Determine the distance between manifolds

Once we have computed the invariant manifolds of the hyperbolic periodic points with rotation numbers ω_n and ω_{n+1} , we have to compute the distance between them. To do so we take intervals of the variable y where the "first tongue" tangencies of $W^u_{x_n}$ and $W^s_{x_{n+1}}$ are defined.

After that we look for the minimum of the distance between those two intervals and store it.

4.3.5 Computation of k_n. Loop

For each value of K we obtain a value for the distance between the manifolds. This can be viewed as a function from $\mathbb{R}$ to $\mathbb{R}$. A zero of this function is identified with an heteroclinic tangency of "first tongue". So we can determine numerically the zero and save it as k_n.

If we repeat this process for each pair of the ω_n's we obtain the required sequence. Repeating the process a sufficient number of times we are going to obtain a good approximation of the critical value.

Remark 4.3.1. The key fact of this method is that since $\dim(W^u_x) = \dim(W^s_x) = 1$ for a hyperbolic fixed point x in the cylinder, we have that in dimension two, if we have heteroclinic intersection, we are going to isolate a region of the cylinder between parts of the invariant manifolds of the points having this heteroclinic intersection, and therefore non IRC can live inside.

When we try to extend this fact to higher dimension, for example to 4 dimensional symplectic maps, we have that, if x is a hyperbolic periodic point (or complex saddle), then $\dim(W^u_x) = \dim(W^s_x) = 2$ therefore when we have an heteroclinic intersection, we do not isolate any region and hence IRC or 2D tori can still appear there.

CHAPTER 5

Renormalisation

In this chapter we will show the key techniques to deal with the problem of invariant circles, renormalisation. To do so, we are introduce a renormalisation operator which is relevant to invariant circles. We will restrict our work to the case of noble rotation numbers, but the analysis can be done in an analogous way for other irrational rotation numbers. This renormalisation operator presents two important fixed points, one is called simple and the other one critical. The study of this fixed points will give us a new point of view for KAM Theory.

The critical fixed point corresponds to maps having a non-smooth noble curve. We have that the unstable manifold of this fixed point gives us a universal one-parameter family describing the break down of the invariant circles we are studying. Therefore this approach does not only determine that under small perturbation these invariant circles persist but also gives an approximation of the break down moment.

The main idea of this renormalisation operator is to make some scale changes in order to zoom up and start working with an easier system which has the same behaviour than the original one.

First we will introduce this operator and try to motivate it. Later we will study its fixed points in order to understand when they are attracting or not and so, when we will know if our system will present invariant circles or not.

We consider only some particular maps. We denote by A the class of area preserving, end-preserving twist maps of a cylinder having zero Calabi invariant is the interesting set. Then, we have to define the Calabi invariant.

Definition 5.0.2. Consider F an area preserving map. Let U be a set containing all the points below some level z_1 and no points above a different level z_2 (this is a set in the same conditions as in Birkhoff's Theorem). Then we define the *Calabi invariant* as the difference of the areas of U and $F(U)$. This definition does not depend on the set U since F is area preserving.

5.1 Introduction

We introduce an operator which is going to be well adapted to questions involving rotations numbers. However, we will need to generalize most of the concepts from periodic maps to commuting pairs of maps. This operator can be viewed as defined in commuting pairs of maps or in actions.

5.1.1 Action representation

We start by defining the action representation. We have that twist maps can come from the return map of a Lagrangian system. This return map can be given in a variational form. Consider the section defined by the angular coordinate,

$$\varphi = \varphi_0 \mod 1.$$

Definition 5.1.1. For a given energy we can define the *action, $\tau(x, x')$, from position x to x' on the previous section* to be the value:

$$\tau(x, x') = \text{sta} \int L(x, \dot{x}, \varphi, \dot{\varphi}),$$

for the orbit from (x, φ_0) to $(x', \varphi_0 + 1)$ having that energy. By sta we mean the stationary value, which we assume locally unique.

Then we have that:

$$\text{sta} \int_{(x_0,\varphi_0)}^{(x_n,\varphi_0+n)} L(x, \dot{x}, \varphi, \dot{\varphi}) = \text{sta} \sum_{i=0}^{n-1} \tau(x_i, x_{i+1})$$

and therefore the orbits of the discrete time system are given by sequences x_n for which the total action is stationary for all finite segments.

Notation. Talking about actions, the subscript $i = 1, 2$ will represent the derivative with respect the i^{th} argument.

We can arise symplectic maps from discrete time Lagrangian systems. Given an action $\tau(x, x')$, verifying that $\tau_{12}(x, x')$ is of constant sign, then the following relations:

$$\begin{aligned} y' &= \tau_2(x, x') \\ y &= -\tau_1(x, x') \end{aligned}$$

can be inverted in order to construct a map $T(x, y) = (x', y')$. This map is area preserving, in fact:

$$DT = \begin{vmatrix} -\dfrac{\tau_{11}}{\tau_{12}} & -\dfrac{1}{\tau_{12}} \\ \tau_{21} - \dfrac{\tau_{22}\tau_{11}}{\tau_{12}} & -\dfrac{\tau_{22}}{\tau_{12}} \end{vmatrix} = 1$$

Moreover, as we have that τ_{12} has constant sign then this map verifies the twist condition.

Conversely, we have that any area preserving map satisfying the twist condition can be generated by an action function, τ, at least locally (Spivak [14]).

Definition 5.1.2. We call this action, $\tau(x, x')$, the *generating function*.

In a given coordinate system this generating function is unique up to addition of constants. So the existence of generating functions with τ_{12} of constant sign gives us the representation of an area preserving twist map.

Suppose that the composition of two twist maps, T and U, is also a twist map. If T and U have generating functions τ, ν, respectively, then the composition TU has the following generating function:

$$\nu \oplus \tau(x, x'') = \nu(x, x') + \tau(x', x''),$$

where $x'(x, x'')$ is such that:

$$0 = \nu_2(x, x') + \tau_1(x', x''). \tag{5.1}$$

Definition 5.1.3. We have that if we work on the cylinder then $\tau(\theta + 1, \theta' + 1)$ generates the same map as $\tau(\theta, \theta')$, so they must differ only in a constant, C. We call C the *Calabi invariant*. This definition is equivalent to Definition 5.0.2

5.1.2 Motivation

We have seen that the proof of the persistence of invariant circles in our KAM Theorems, depends on the rotation number of the circle, ω. In particular it depends on its type of irrationality (i.e., if they verify some Diophantine condition or not). We are going to define an operator which is based on the continued fraction expansion of ω.

We suppose, without loss of generality, that we consider the orbit at the origin, $\mathbf{0}$. We have then that if this orbit has rotation number ω then:

$$\Pi_1 F^q B^p(\mathbf{0}) = q\omega - p + o(q) \quad \text{as} \quad q \to \infty, \tag{5.2}$$

where B is the *back rotation map* and it is given by:

$$B(\theta, z) = (\theta - 1, z).$$

If it belongs to a circle on which our map F is topologically conjugate to a rotation map, then we have even a stronger fact

$$\Pi_1 F^{q_n} B^{p_n}(\mathbf{0}) \to 0 \quad \text{as} \quad q_n\omega - p_n \to 0.$$

Therefore, one thinks in a sequence of maps of the form:

$$C_n F^{q_n} B^{p_n} C_n^{-1} \tag{5.3}$$

where C_n represents a coordinate change expanding the scale in the θ direction. In order to conserve some nice properties of the system it would be helpful to rescale also the z direction.

We want to select a good choice for p_n and q_n, making $q_n\omega - p_n$ particulary small. This can be done by taking the convergents of ω.

Definition 5.1.4. If we represent $\omega = [m_0, m_1, \ldots]$ by its continued fraction expansion, we define the *convergents of* ω as:

$$\frac{p_n}{q_n} = [m_0, \ldots, m_n].$$

They verify that:

$$\begin{cases} p_i &= m_i p_{i-1} + p_{i-2} \\ q_i &= m_i q_{i-1} + q_{i-2} \end{cases}$$

for $i \geq 0$. Where:

$$\begin{cases} p_{-2} = 0 & p_{-1} = 1 \\ q_{-2} = 1 & q_{-1} = 0 \end{cases}$$

Recall that these convergents are exactly the ones we have picked in Chapter 3.

For this choice of p_n and q_n we can generate the sequence (5.3) in a very easy way.

Definition 5.1.5. For each integer m, we define the *renormalisation operator*, R_m, acting on pairs of maps in the following way:

$$R_m(U, T) = (CTC^{-1}, CT^m UC^{-1}).$$

So by induction we have that these renormalisation operators verify:

$$R_{m_n} \cdots R_{m_0}(U, T) = (C_n U^{q_{n-1}} T^{p_{n-1}} C_n^{-1}, C_n U^{q_n} T^{p_n} B_n^{-1}),$$

where C_n represents the composition of the corresponding change of coordinates. So we can generate (5.3) by consecutive applications of the corresponding R_{m_i} on the pair (F, B), taking as m_i elements in the continued fraction expansion of ω.

5.1.3 Some comments on commuting pairs

As our operator is defined in pairs of maps, we want to extend some of the definitions we have for the case of periodic maps. To present these definitions, we denote by $\mathbf{x} = (x, y)$ the components of our map. We take these general coordinates just to show that this can be done for general maps. Then we have the following definitions.

Definition 5.1.6. Given a point $\mathbf{x}$ we define *the orbit of* $\mathbf{x}$ *under(U,T)* as the set:

$$\{U^q T^p \mathbf{x} | p, q \in \mathbb{Z}\}.$$

Definition 5.1.7. We say that a point $\mathbf{x}$ is periodic if $\exists (p, q) \in \mathbb{Z}^2 \backslash \{0\}$ such that:

$$U^q T^p \mathbf{x} = \mathbf{x}.$$

We will say that it has *type* (p, q) if they are the smallest possible integers verifying the condition.

Definition 5.1.8. We define an *invariant curve* as a curve from $-\infty$ to ∞ which is invariant under both U and T.

The next concept we are interested in defining, is the concept of rotation number. To do so we use relationship (5.2) (dividing both sides by q), and that it tends to zero for some p_n and q_n if and only if $p_n/q_n \to \omega$. So now we are able to generalize the concept by allowing $\omega = \infty$. This can be done, for example considering the rotation numberas an element of the real projective line, $\mathbb{R}P$.

Definition 5.1.9. We say that $\mathbf{x}$ has *rotation number* $\omega \in \mathbb{R}P$ *under* (U,T) if for all sequences p_n, $q_n \in \mathbb{Z}$ such that $r_n = \max(|p_n|, |q_n|) \to \infty$ then:

$$\frac{\Pi_1 U^{q_n} T^{p_n} \mathbf{x}}{r_n} \to 0 \quad \text{if and only if} \quad \frac{p_n}{q_n} \to \omega.$$

Definition 5.1.10. We say that an invariant curve with rotation number μ/η is *smooth* if there is a sufficiently differentiable coordinate function $\varphi(\mathbf{x})$ on the curve, such that:

$$\varphi U^q T^p \varphi^{-1}(t) = t + q\mu - p\eta$$

In order to extend the concept of twist condition we consider a very restrictive fact, probably even more than necessary. We suppose that both U and T have action generating functions, that is, both verify the twist condition. Notice that this is not true for the back rotation map, B.

Definition 5.1.11. When we have that U and T commute and have generating functions ν and τ respectively, we have that then TU and UT have generating functions that only differ on a constant. This constant is called *Calabi invariant*, $C_{\nu,\tau}$.

This is not a generalization of the periodic case definition since B does not necessarily have a generating function. However, this concept is essentially the same.

We denote by $Ext(A)$ the extension of the class A to commuting pairs.

5.1.4 Renormalisation and rotation numbers

Now that we have extended those concepts we are ready to state some properties that the renormalisation operator fulfils. These properties are crucial in the study of the possible invariant circles. We have then that:

Proposition 5.1.12. *The following statements are true:*

1. $\boldsymbol{x}$ *has rotation number* ω *under the pair* (U,T) *if and only if* $C\boldsymbol{x}$ *has rotation number* $\widehat{\omega}$ *under* $R_m(U,T)$ *where* $\omega = m + \frac{1}{\widehat{\omega}}$.

2. $\boldsymbol{x}$ *has rotation number* $\omega = [m_0, m_1, \ldots]$ *if and only if* $C_{n-1} \cdots C_0 \boldsymbol{x}$ *has rotation number* $\omega_n = [m_n, \ldots]$ *under* $R_{m_{n-1}} \cdots R_{m_0}(U,T)$, *where the* C_j*'s are the corresponding coordinate changes.*

3. *(U,T) has an invariant circle with rotation number ω if and only if $R_{m_{n-1}} \cdots R_{m_0}(U,T)$ has an invariant circle with rotation number ω_n.*

In particular we have the following corollary:

Corollary 5.1.13. *A periodic map F has an invariant circle of rotation number ω_0 if and only if $R_{m_{n-1}} \cdots R_{m_0}(F,B)$ has an invariant circle of rotation number ω_n.*

If we restrict C to be a linear diagonal change of coordinates, that is, $B(x,y) = (\alpha x, \beta y)$, then R_m will induce a renormalisation on action generating functions. It is given by:

$$\begin{cases} \nu'(x,x') &= \alpha\beta\tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right) \\ \tau'(x,x') &= \alpha\beta\nu \oplus \tau \oplus \cdots \oplus \tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right) \end{cases}$$

This would imply that the Calabi invariant is going to be renormalised as follows:

$$C_{\nu',\tau'} = -\alpha\beta C_{\nu,\tau}.$$

5.2 Fixed points analysis

From now on we concentrate just in noble rotation numbers. The analysis we are going to do can be extended to any quadratic irrational number, that is, $\omega^* = [r_0, \ldots, r_k, (s_1, \ldots, s_{k^*})^\infty]$. To do the extension we just correct the first terms until we arrive at the periodic part of the continued fraction expansion and then study the composition of the operators related with the s_i part of the continued fraction expansion. It is important to say that quadratic irrational numbers have zero measure.

Moreover, we consider the case of golden invariant circles, that is, invariant circles having the golden mean, γ as rotation number. In this case the renormalisation operator is going to present two fixed points. We are going to explain what happens with both of them. Before we present general properties of both of them and also some invariant spaces under renormalisation that would be helpful to construct some eigenvectors.

As we are concentrated in the golden case, our renormalisation operator has the following form:

$$N_1(U,T) = (CTC^{-1}, CTUC^{-1}) =: (U',T').$$

Another assumption we do is that C is in diagonal form.

5.2.1 Stability

If we obtain a fixed point, the first thing one wants to do is to study its stability. Therefore, we have to study the derivative of the operator. In this case we have that this derivative is given by the following expressions:

$$DN_1 : \begin{cases} \delta U' &= DC_{TC^{-1}}\delta TC^{-1} \\ \delta T' &= DC_{TUC^{-1}}\delta TUC^{-1} + DC_{TUC^{-1}}DT_{UC^{-1}}\delta UC^{-1} \end{cases}$$

where δT and δU represent the variation of T and U. We ignore the contributions of the variation of C, since we can see that they would produce the following correction on $\delta U'$:

$$\delta U' = \delta C\, C^{-1} U' - DU' \delta C\, C^{-1}$$

where $\delta C^{-1} = -DC^{-1}\delta C\, C^{-1}$. This only contributes on the direction of the coordinate changes and therefore will have not essentially effect. Analogously, we see the same fact for $\delta T'$.

What we have to do now is to take this derivative in our fixed point and study its eigenvalues and eigenvectors in order to see the behaviour of our fixed point.

Sometimes, for example while working with area preserving maps, it is useful to use the action representation. In this case, and if C is linear and diagonal, $C(x,y) = (\alpha x, \beta y)$, we have the following renormalisation over pairs of actions, (ν, τ).

$$N_1 : \begin{cases} \nu'(x,x') &= \alpha\beta\tau\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right) \\ \tau'(x,x') &= \alpha\beta\left[\nu\left(\dfrac{x}{\alpha}, \bar{x}\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right)\right) + \tau\left(\bar{x}\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right), \dfrac{x'}{\alpha}\right)\right] \end{cases} \tag{5.4}$$

where $\bar{x}$ is chosen to make the sum stationary as in (5.1).

Therefore we have that the derivative is given by:

$$DN_1 : \begin{cases} \delta\nu'(x,x') &= \alpha\beta\delta\tau\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right) \\ \delta\tau'(x,x') &= \alpha\beta\left[\delta\nu\left(\dfrac{x}{\alpha}, \bar{x}\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right)\right) + \delta\tau\left(\bar{x}\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right), \dfrac{x'}{\alpha}\right)\right] \end{cases}$$

By stationarity the variations in $\bar{x}$ do not contribute to the system. As we have done for maps, we avoid contributions from variations on C, in particular in α and β. They would be the following:

$$\begin{aligned} \delta\nu'(x,x') &= \delta(\alpha\beta)\tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right) + \alpha\beta\left(x\tau_1\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right) + x'\tau_2\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right)\right)\delta\left(\frac{1}{\alpha}\right) \\ &= \frac{\delta(\alpha\beta)}{\alpha\beta}\nu'(x,x') + \alpha\delta\left(\frac{1}{\alpha}\right)\left(x\nu_1'(x,x') + \nu_2'(x,x')\right) \end{aligned}$$

analogously we construct $\delta\tau'$ replacing ν' by τ'.

5.2.2 Invariant sets

We proceed now to present some invariant sets under renormalisation that would be useful in order to compute the required eigenvectors of our map.

To present these invariant sets we have to consider the concept of symmetry.

Definition 5.2.1. We say that T is *symmetric* if $(TS)^2(x,y) = (x,y)$ where $S(x,y) = (-x,y)$.

We have then that:

Remark 5.2.2. For U and T symmetric, we have UT symmetric if and only if U and T commute.

Proof. If U and T are symmetric we have then that:

$$(SUT)^2 = SUTSUT = U^{-1}SST^{-1}UT = U^{-1}T^{-1}UT$$

so we are done. □

We have then that the following spaces remain invariant under renormalisation:

1. Commuting pairs.
2. Area preserving pairs.
3. Commuting area preserving pairs with zero Calabi invariant (what we have called $Ext(A)$).
4. Symmetric commuting pairs.
5. Coordinate transforms of any fixed point.
6. The intersection of the previous items.

5.3 Simple fixed point

Let's concentrate now in the simple fixed point that this renormalisation operator presents. It is identified with an integral twist map having a golden invariant circle. Then we study its stability in the space of maps. We decompose the spectrum of this fixed point in order to see that it is essentially attracting. This would imply that noble circles are going to persist under small perturbations of our maps. This fact can be viewed as a different way of proving KAM Theorem for these special Diophantine numbers.

This simple fixed point for N_1 is given by:

$$T(x,y) = (x+y+1, y) \qquad U(x,y) = \left(x + \frac{y}{\gamma} - \gamma, y\right) \qquad C(x,y) = (-\gamma x, -\gamma^2 y).$$

In terms of actions the fixed point is given by:

$$\tau(x,x') = \frac{1}{2}(x'-x-1)^2 \qquad \nu(x,x') = \frac{\gamma}{2}(x'-x+\gamma)^2 \qquad \bar{x}(x,x') = \frac{x}{\gamma} + \frac{x'}{\gamma^2} - 1 - \frac{1}{\gamma^2},$$

where $\alpha = -\gamma$ and $\beta = -\gamma^2$, in the change of variables.

Before entering in more details we show some properties about the golden mean, that are really useful in order to simplify computations.

Remark 5.3.1. The golden mean γ verifies the following relations:

$$\begin{cases} \gamma^2 &= \gamma+1 \\ \gamma^3 &= 2\gamma+1 \\ \gamma^4 &= 3\gamma+2 \\ \gamma^5 &= 5\gamma+3 \\ \gamma^{-1} &= \gamma-1 \\ \gamma^{-2} &= 2-\gamma \\ \gamma^{-3} &= 2\gamma-3 \end{cases}$$

5.3.1 Study of the derivative

Now we are going to consider the derivative of N_1 at our fixed point.

$$DN_1 : \begin{cases} \Pi_1\delta U'(x,y) &= -\gamma\Pi_1\delta T\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right) \\ \Pi_1\delta T'(x,y) &= -\gamma\Pi_1\delta U\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right) - \gamma\Pi_1\delta T\left(-\frac{x}{\gamma}-\frac{y}{\gamma^3}-\gamma,-\frac{y}{\gamma}^2\right) \\ & \quad -\gamma\Pi_2\delta U\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right) \\ \Pi_2\delta U'(x,y) &= -\gamma^2\Pi_2\delta T\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right) \\ \Pi_2\delta T'(x,y) &= -\gamma^2\Pi_2\delta U\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right) - \gamma^2\Pi_2\delta T\left(-\frac{x}{\gamma}-\frac{y}{\gamma^3}-\gamma,-\frac{y}{\gamma^2}\right) \end{cases}$$

As we have said in advanced we ignore the effects of variation of C with (U,T) which are only in the direction of the coordinate changes.

We consider the following order on monomials:

$$0 < y < x < y^2 < yx < x^2 < \cdots$$

Definition 5.3.2. We say that a polynomial is *of rank* p,q if its larger monomial with respect to this order is $x^p y^q$.

This derivative has the property that at the simple fixed point it never increases the rank of a polynomial perturbation. Hence we can write DN_1 into Jordan normal form on the space of polynomial perturbations, with polynomial eigenvectors or generalised eigenvectors.

If we expand in power series:

$$\Pi_1\delta U(x,y) = \sum_{p,q} a_{pq}x^p y^q \qquad \Pi_1\delta T(x,y) = \sum_{p,q} b_{pq}x^p y^q$$

$$\Pi_2\delta U(x,y) = \sum_{p,q} c_{pq}x^p y^q \qquad \Pi_2\delta T(x,y) = \sum_{p,q} d_{pq}x^p y^q,$$

and order the coefficients by rank we can see that DN_1 is block upper triangular, with 4×4 blocks on the diagonal having the following form:

$$(-\gamma)^{-p}(-\gamma^2)^{-q}\begin{pmatrix} 0 & -\gamma & 0 & 0 \\ -\gamma & -\gamma & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma^2 \\ 0 & 0 & -\gamma^2 & -\gamma^2 \end{pmatrix}$$

for polynomials of rank p, q. Each block has as the following eigenvectors:

$$A_{p,q} = \begin{pmatrix} \frac{1}{2\gamma} \\ \frac{\gamma}{2} \\ 1 \\ \gamma \end{pmatrix} \quad B_{p,q} = \begin{pmatrix} 1 \\ \gamma \\ 0 \\ 0 \end{pmatrix} \quad C_{p,q} = \begin{pmatrix} \gamma \\ -\gamma \\ \gamma \\ 1 \end{pmatrix} \quad D_{p,q} = \begin{pmatrix} \gamma \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

with corresponding respective eigenvalues:

$$-\gamma^3(-\gamma)^{-p}(-\gamma^2)^{-q} \qquad -\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q} \qquad \gamma(-\gamma)^{-p}(-\gamma^2)^{-q} \qquad (-\gamma)^{-p}(-\gamma^2)^{-q}.$$

Now we have these eigenvectors and their corresponding eigenvalues we can determine some lower rank coefficients to construct eigenvectors of the derivative, DN_1. We have to notice that these coefficients may not be unique. We take these eigenvectors related with the invariants sets we have shown in Section 5.2.2.

If we consider the renormalisation operator in terms of actions it is easy to diagonalise DN_1 just by doing the following change of variables:

$$\begin{aligned} \sigma &= x' + x \\ \delta &= x' - x \end{aligned}$$

and taking:

$$t(\sigma, \delta) = \tau(x, x') \qquad u(\sigma, \delta) = \nu(x, x')$$

hence the fixed point is written as:

$$t(\sigma, \delta) = \frac{1}{2}(\delta - 1)^2 \qquad u(\sigma, \delta) = \frac{\gamma}{2}(\delta + \gamma)^2 \qquad \bar{x}(\sigma, \delta) = \frac{\sigma}{2} - \frac{\delta}{2\gamma^3} - 1 - \frac{1}{\gamma^2}$$

In this new variables, the composition $u \oplus t$ is given by:

$$u \oplus t(\sigma, \delta) = u\left(\bar{x} + \frac{\sigma - \delta}{2}, \bar{x} - \frac{\sigma - \delta}{2}\right) + t\left(\frac{\sigma + \delta}{2} + \bar{x}, \frac{\sigma + \delta}{2} - \bar{x}\right).$$

Finally we have that the derivative of the renormalisation at the fixed point is given by:

$$DN_1 : \begin{cases} \delta u'(\sigma, \delta) &= \gamma^3 \delta t\left(-\frac{\sigma}{\gamma}, -\frac{\delta}{\gamma}\right) \\ \delta t'(\sigma, \delta) &= \gamma^3\left(\delta u\left(-\frac{\sigma}{\gamma} + \frac{\delta}{\gamma^2} + \frac{1}{\gamma} + \frac{1}{\gamma^3}, \frac{\delta}{\gamma^3} + \frac{1}{\gamma} + \frac{1}{\gamma^3}\right) + \right. \\ & \left. \quad + \delta u\left(-\frac{\sigma}{\gamma} - \frac{\delta}{\gamma^3} + \frac{1}{\gamma} + \frac{1}{\gamma^3}, -\frac{\delta}{\gamma^2} - \frac{1}{\gamma} - \frac{1}{\gamma^3}\right)\right) \end{cases}$$

Proceeding in the same way as for maps, if we consider the following order by ranks $1 < \delta < \sigma < \delta^2 < \sigma\delta < \sigma^2 < \ldots$ and the power expressions of δu and δt:

$$\delta u(\sigma,\delta) = \sum l_{pq}\sigma^p\delta^q \qquad \delta t(\sigma,\delta) = \sum \tilde{l}_{pq}\sigma^p\delta^q,$$

we obtain DN_1 as a block upper triangular matrix, with 2×2 diagonal blocks:

$$\gamma^3(-\gamma)^{-p}\left|\begin{array}{cc} 0 & (-\gamma)^{-q} \\ (-\gamma^3)^{-q} & (-\gamma^2)^{-q}\end{array}\right|$$

having the following eigenvectors:

$$I_{p,q} = \begin{pmatrix} \gamma^q \\ \gamma \end{pmatrix} \qquad II_{p,q} = \begin{pmatrix} \gamma^q \\ -\dfrac{1}{\gamma} \end{pmatrix},$$

and their corresponding eigenvalues:

$$\gamma^4(-\gamma)^{-p}(-\gamma^2)^{-q} \qquad -\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q}.$$

We want to relate now the eigenvectors we have obtained for both cases. Then we have to compare perturbations in actions with perturbations on maps. We have that a perturbation (δu,δt) of our simple fixed point in actions, gives the following perturbation in maps:

$$\begin{cases} \Pi_1\delta U(x,y) &= \dfrac{1}{\gamma}(u_1-u_2)\left(2x-\dfrac{y}{\gamma}-\gamma,\dfrac{y}{\gamma}-\gamma\right) \\ \Pi_1\delta T(x,y) &= (t_1-t_2)(2x+y+1,y+1) \\ \Pi_2\delta U(x,y) &= 2u_1\left(2x+\dfrac{y}{\gamma}-\gamma,\dfrac{y}{\gamma}-\gamma\right) \\ \Pi_2\delta T(x,y) &= 2t_1(2x+y+1,y+1) \end{cases}$$

Hence, we have that, when $q \neq 0$, a $\sigma^p\delta^q$ perturbation in actions gives us a x^py^{q-1} one in maps. In case $q = 0$ and $p \neq 0$ we have a perturbation of the form x^{p-1}. If both of them are zero we obtain a zero perturbation. Therefore, we obtain the following correspondence relationship:

$$I_{p,q} = B_{p,q-1} \quad q \neq 0 \qquad\qquad II_{p,q} = D_{p,q-1} \quad q \neq 0$$

$$I_{p,0} = A_{p-1,0} \quad p \neq 0 \qquad\qquad II_{p,0} = C_{p-1,0} \quad p \neq 0$$

and $I_{0,0}$, $II_{0,0}$ generate the zero perturbation. All these equalities are in terms of perturbations.

5.3.2 Decomposition of the spectrum

Now we are interested in decomposing the spectrum of DN_1 in order to have a clear idea of what happens with the fixed point. We present the following eigenvectors.

1. Coordinate changes:

   We start considering the coordinate change eigenvectors. These eigenvectors present the following possible forms:

   - $\delta\sigma(x,y) = (x^p y^q, 0)$ which generates:

$$\begin{cases} \Pi_1\delta U(x,y) &= -\left(x + \dfrac{y}{\gamma} - \gamma\right)^p y^q + x^p y^q \\ \Pi_1\delta T(x,y) &= -(x+y+1)^p y^q + x^p y^q \\ \Pi_2\delta U(x,y) &= 0 \\ \Pi_2\delta T(x,y) &= 0 \end{cases}$$

     whose maximal rank terms give a $B_{p-1,q+1}$ eigenvector when $p \neq 0$ and 0 when $p = 0$.

   - $\delta\sigma(x,y) = (0, x^p y^q)$ which generates:

$$\begin{cases} \Pi_1\delta U(x,y) &= \dfrac{1}{\gamma} x^p y^q \\ \Pi_1\delta T(x,y) &= x^p y^q \\ \Pi_2\delta U(x,y) &= -\left(x + \dfrac{y}{\gamma} - \gamma\right)^p y^q + x^p y^q \\ \Pi_2\delta T(x,y) &= -(x+y+1)^p y^q + x^p y^q \end{cases}$$

     whose maximal rank terms give a $B_{p,q}$ eigenvector.

   - If we consider the following combinations where the maximal rank terms cancel: $\delta\sigma(x,y) = (x^{p+2}y^{q-2}, (p+2)x^{p+1}y^{q-1})$ where $q \geq 2$ we obtain then an $A_{p,q}$ eigenvector.
   - We can also consider $\delta\sigma(x,y) = (xy^q, y^{q+1})$, that gives us a $D_{0,q}$ eigenvector.

   So we can take $A_{p,q}$ with $q \geq 2$ and $D_{0,q}$ to be those coordinate changes. To have a good choice for $B_{p,q}$ we can take the area preserving coordinate changes:

$$\begin{cases} \delta\sigma(x,y) &= (qx^{p+1}y^{q-1}, -(p+1)x^p y^q) & q \geq 1 \\ \delta\sigma(x,y) &= (0, x^p) & q = 0 \end{cases}$$

Remark 5.3.3. The coordinate change $(y^q, 0)$ does not affect the map at the fixed point since y is invariant. This implies that the eigenvector generated by it is going to vanish, and therefore it is not interesting.

By same arguments we can deduce that $D_{0,q}$ are area preserving.

2. Area preservation:

   Let's study now which of the eigenvectors are area preserving. We have that:

$$\det D(T+\delta T) = \det \begin{pmatrix} 1+\partial_x \Pi_1 \delta T & 1+\partial_y \Pi_1 \delta T \\ \partial_x \Pi_2 \delta T & 1+\partial_y \Pi_2 \delta T \end{pmatrix}$$

   So we have that this perturbation δT is area preserving if and only if it verifies the following condition:

$$\partial_x \Pi_1 \delta T + \partial_y \Pi_2 \delta T - \partial_x \Pi_2 \delta T = \mathcal{O}((\delta T)^2)$$

   In the same way, if we perturb U we have δU is area preserving if and only if:

$$\partial_x \Pi_1 \delta U + \partial_y \Pi_2 \delta U - \frac{1}{\gamma} \partial_x \Pi_2 \delta U = \mathcal{O}((\delta U)^2)$$

   These properties applied to the term of maximal rank of the eigenvectors give us that $A_{p,q}$ and $C_{p,q}$ with $q \neq 0$ are non-area preserving eigenvectors.

   If we count area preserving dimension using the action representations, we obtain that there are $2(d+1)$ dimensions for actions of degree d. At the simple fixed point a perturbation of degree d in actions gives us a perturbation of degree $d-1$ on maps. So we have obtained $2(d+2)$ area preserving dimensions. The total number of dimensions of degree d is $4(d+1)$, so there are $2d$ non-area preserving dimensions. The eigenvectors we have already present span all of them, therefore we will concentrate in the area preserving ones.

3. Commutation:

   Let's study what happens with commutation. We have that the perturbation $(\delta U, \delta T)$ preserves commutation if and only if

$$\delta U\, T + DU_T \delta T\, U - DT_U \delta U = 0.$$

   At our simple fixed point we have the following x component:

$$\Pi_1 \delta U(x+y+1, y) + \Pi_1 \delta T(x,y) + \frac{1}{\gamma} \Pi_2 \delta T \left(x + \frac{y}{\gamma} - \gamma, y \right) - \Pi_1 \delta U(x,y) - \Pi_2 \delta U(x,y)$$

   and y component:

$$\Pi_2 \delta U(x+y+1, y) + \Pi_2 \delta T(x,y) - \Pi_2 \delta T \left(x + \frac{y}{\gamma} - \gamma, y \right) - \Pi_2 \delta U(x,y)$$

   for the commutation expression above. We call these components x-commutators and y-commutators respectively.

We are going to show that the eigenvectors of the form $D_{p,q}$ where $p \neq 0$ are non-commuting. Let's evaluate the next rank. We have that the $D_{p,q}$ eigenvectors have as eigenvalues $(-\gamma)^{-p}(-\gamma^2)^{-q}$. Therefore its next coefficients $c_{p-1,q+q}$ and $d_{p-1,q+1}$ are going to be determined by:

$$\begin{pmatrix} 0 & -\gamma & 0 & 0 \\ -\gamma & -\gamma & 0 & 0 \\ 0 & 0 & 0 & -\gamma \\ 0 & 0 & -\gamma & -\gamma \end{pmatrix} \begin{pmatrix} c \\ d \\ \gamma \\ -1 \end{pmatrix} = \begin{pmatrix} c \\ d \\ \gamma \\ -1 \end{pmatrix}$$

Hence we have that:

$$c = -\gamma d$$

where d is arbitrary. We can avoid arbitrariness by requiring $D_{p,q}$ to be area preserving. In that case:

$$d = \frac{p}{q+1}$$

We have that the x-commutator has a term in $x^{p-1}y^{q+1}$ of the form:

$$\gamma p + \frac{d}{\gamma} + \frac{p}{\gamma} - c \neq 0 \quad \text{for} \quad d \neq -p$$

Therefore, $D_{p,q}$ with $p \neq 0$ are non-commuting.

Next we want to show that the $A_{p,0}$, with $p \geq 1$ are non-commuting too. To do so we can check that in terms of actions there exists the following eigenvector in terms of maximal degree $p+1$:

$$\begin{aligned} \delta\tau(x,x') &= \sum_{i+j=p+1} x^i (x')^j = \frac{x^{p+2} - x^{p+2}}{x - x'} \\ \delta\nu(x,x') &= \frac{1}{\gamma}\delta\tau(x,x') \end{aligned}$$

with eigenvalue $-\gamma^3(-\gamma)^{-p}$ for degree. That expression gives an $A_{p,0}$ eigenvector. We have then that as there is only one eigenvalue $-\gamma^3(-\gamma)^{-p}$ for degree p and hence, $A_{p,0}$ is uniquely determined to terms of maximal degree. The terms of maximal degree for the y-components are:

$$\begin{aligned} \Pi_2 \delta T(x,y) &= f(x, x+y) \\ \Pi_2 \delta U(x,y) &= \frac{1}{\gamma} f\left(x, x + \frac{y}{\gamma}\right) \end{aligned}$$

where:

$$f(x,x') = (p+2)\frac{x^{p+1} - (x')^{p+1}}{x - x'}$$

so, correct to rank $(p-1,0)$ the y-commutator is:

$$\frac{1}{\gamma} f(x+y+1, x+\gamma y+1) + f(x, x+y) - f\left(x + \frac{y}{\gamma} - \gamma, x + \gamma y - y\right) - \frac{1}{\gamma} f\left(x, x + \frac{y}{\gamma}\right)$$

$$= (p+2)\sum_{i+j=p}\frac{1}{\gamma}(x+y+1)^i(x+\gamma y+1)^j - \left(x+\frac{y}{\gamma}-\gamma\right)^i(x+\gamma y-\gamma)^j - \frac{1}{\gamma}x^i\left(x+\frac{y}{\gamma}\right)^j$$

Evaluating the coefficient x^{p-1} with $p \neq 0$ in the y-commutator we obtain:

$$p(p+2)\left(\frac{1}{\gamma}+\gamma\right) \neq 0$$

so we have no commutation.

In a similar way we can see that $A_{0,0}$ has non-zero Calabi invariant and it is commuting.

4. Symmetry:

Definition 5.3.4. A perturbation δT preserves symmetry if and only if $(S(T+\delta T))^2(x,y) = (x,y)$.

This fact can be expressed in the following way:

$$S\,\delta T\,ST + D(STS)\delta T = 0$$

which is translated as:

$$\begin{array}{rcl} -\Pi_1\delta T(-x-y-1,y) + \Pi_1\delta T(x,y) - \Pi_2\delta T(x,y) & = & 0 \\ \Pi_2\delta T(-x-y-1,y) + \Pi_2\delta T(x,y) & = & 0 \end{array}$$

and the analogous terms for δU. Hence, considering the terms on maximum rank, we can see that the following type of eigenvectors do not have symmetry: $A_{p,q}$ when p is even, $B_{p,q}$ when q is odd, $C_{p,q}$ for any p and $D_{p,q}$ for p odd.

We have that any coordinate change $\delta\sigma$ is symmetric if $\Pi_1\delta\sigma$ is odd in x and $\Pi_2\delta\sigma$ is even in x. Therefore, $A_{p,q}$ with $q \geq 2$ and p odd, $B_{p,q}$ with p even and $D_{0,q}$ preserve symmetry.

We have to discard also the non-commuting eigenvectors that cannot be symmetric.

Finally we have to see what happens with $A_{p,1}$ where p is odd. We have that to maximum degree it is given by:

$$\begin{array}{rcl} \Pi_1\delta T(x,y) & = & -(x+y)^{p+2}y^{-1} + x^{p+2}y^{-1} + (p+2)x^{p+1} \\ \Pi_2\delta T(x,y) & = & -(p+1)(x+y)^{p+1} + (p+2)x^{p+1} \end{array}$$

Even if it appears y^{-1}, this is well defined since those terms are going to be cancelled. This is just the only possible form for $A_{p,1}$ since with less rank the eigenvalues are going to be different. Let us consider then the terms of degree p.

We have a degree of freedom in choosing these x^p terms since $C_{p,0}$ also has the same eigenvalue. We can add always a multiple of $C_{p,0}$ so we have:

$$\begin{array}{rcl} \Pi_1\delta T(x,y) & = & -a\gamma x^p \\ \Pi_2\delta T(x,y) & = & -ax^p \end{array}$$

This gives us that $A_{p,0}$ is non-symmetric except if $a = 0$ by the x-symmetry of the third definition of symmetry.

We evaluate now $\Pi_2\delta T$ to rank $x^{p-1}y$, we have then:

$$\begin{aligned}\gamma(-\gamma)^p\Pi_2\delta U(x,y) &= -\gamma^2\Pi_2\delta T\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right)\\ \gamma(-\gamma)^{-p}\Pi_2\delta T(x,y) &= -\gamma^2\Pi_2\delta U_y\left(-\frac{x}{\gamma},-\frac{y}{\gamma^2}\right)-\gamma^2\Pi_2\delta T\left(-\frac{x}{\gamma}-\frac{y}{\gamma^3}-\gamma,-\frac{y}{\gamma^2}\right)\end{aligned}$$

Then we obtain:

$$\gamma^2(-\gamma)^{-2p}\Pi_2\delta T(x,y)=\gamma^4\Pi_2\delta T\left(\frac{x}{\gamma^2},\frac{y}{\gamma^3}\right)-\gamma^3(-\gamma)^{-p}\Pi_2\delta T\left(\frac{x}{\gamma}-\frac{y}{\gamma^3}-\gamma,-\frac{y^2}{\gamma}\right)$$

Then trying the following solution:

$$\Pi_2 T(x,y)=(p+2)\left(-(x+y)^{p+1}+x^{p+1}+bx^{p-1}y\right)+l.r.t$$

We have

$$b=-\gamma p(p+1)$$

And then this is not symmetric by y-symmetry property in the third definition of symmetry.

Summarizing all the previous analysis we can construct Table 5.1 where we have all the information related with this eigenvectors.

Classification				Eigenvector	Eigenvalue	Eigenvalues with modulus greater or equal 1
AP	Co	CC	Sy	$B_{p,q}$, p even	$-\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q}$	$-\gamma^2,\ 1,\ -1$
				$D_{0,q}$	$(-\gamma^2)^{-q}$	1
			NSy	$B_{p,q}$, p odd	-$\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q}$	γ
		CI	NSy	$A_{0,0}$	$-\gamma^3$	$-\gamma^3$
	NCo		NSy	$D_{p,q},\ p\geq 1$	$(-\gamma)^{-1}(-\gamma^2)^{-q}$	
				$C_{p,0}$	$\gamma(-\gamma)^{-p}$	$\gamma,-1$
NAP				$A_{p,0}$	$-\gamma^3(-\gamma)^{-p}$	$\gamma^2,\ -\gamma,\ 1$
	Co	CC	Sy	$A_{p,q},\ q\geq 2,\ p$ odd	$-\gamma^3(-\gamma)^{-p}(-\gamma^2)^{-q}$	
			NSy	$A_{p,q},\ q\geq 2,\ p$ even	$-\gamma^3(-\gamma)^{-p}(-\gamma^2)^{-q}$	
		NCC	NSy	$A_{p,1}$	$\gamma(-\gamma)^{-p}$	$\gamma,\ -1$
	NCo		NSy	$C_{p,q},\ q\geq 1$	$\gamma(-\gamma)^{-p}(-\gamma^2)^{-q}$	

Table 5.1: Decomposition of the spectrum of DN_1 at the simple point. AP (Area preservation), Co (commutativity), Sy (symmetry), CC (coordinate change), CI (non-zero Calabi invariant), and N means non.

5.3.3 Compactness of DN_1 at the simple fixed point

We consider the following norm:

$$\|(\delta U, \delta T)\| = w_1\|\Pi_1\delta U\| + w_2\|\Pi_1\delta T\| + w_3\|\Pi_2\delta U\| + w_4\|\Pi_4\delta T\|,$$

where the w_i are some weights.

Theorem 5.3.5. *The operator DN_1 is compact.*

Proof. It is enough to show that N_1 is analyticity improving in a neighbourhood of our simple fixed point, on some suitable domains. Suppose now that T is analytic on $|x| \leq X$ and $|y| \leq Y$, for X and Y verifying $X > \gamma^3 + \frac{Y}{\gamma}$.

Suppose also that U is analytic on $|x| \leq X_2$ and $|y| \leq Y_2$ where $\frac{X}{\gamma} < X_2 < \gamma X$ and $\frac{Y}{\gamma^2} < Y_2 < \gamma^2 Y$.

We have then that if (U', T') is going to be analytic on a larger domain by the construction of our renormalisation operator. So we have analyticity improving and therefore DN_1 is compact. □

Theorem 5.3.6. *The error when truncating DN_1 at degree d goes to zero as d goes to ∞. Moreover, $\|DN_1 - DN_1^{(d)}\| \leq C\lambda^d$ where:*

$$\lambda = \max\left(\frac{X}{\gamma X_2}, \frac{X_2}{\gamma X}, \frac{Y}{\gamma Y_2}, \frac{Y_2}{\gamma Y}, \frac{\frac{X}{\gamma} + \frac{Y}{\gamma^3} + \gamma}{X}\right) < 1,$$

where C depends on the weights of our norm.

Proof. We know that the norm of a bounded linear operator is given by:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

We have then that if we take vectors of degree d or lower then $DN_1 - DN_1^{(d)}$ has zero action. On the other hand, for monomials $x^p y^q$ of higher degree it has the same action as DN_1 and hence:

$$\begin{cases} \|\Pi_1\delta U'\| & \leq & \gamma\left(\frac{X_2}{\gamma X}\right)^p\left(\frac{Y_2}{\gamma Y}\right)^q\|\Pi_1\delta T\| \\ \|\Pi_1\delta T'\| & \leq & \gamma\left(\frac{X}{\gamma X_2}\right)^p\|\Pi_1\delta U\| + \gamma\left(\frac{\frac{X}{\gamma} + \frac{Y}{\gamma^3} + \gamma}{X}\right)^p\gamma^{-2q}\|\Pi_1\delta T\| \\ & & + \left(\frac{X}{\gamma X_2}\right)^p\left(\frac{Y}{\gamma^2 Y_2}\right)^q\|\Pi_2\delta U\| \\ \|\Pi_2\delta U'\| & \leq & \gamma\left(\frac{X_2}{\gamma X}\right)^p\left(\frac{Y_2}{\gamma Y}\right)^q\|\Pi_2\delta T\| \\ \|\Pi_2\delta T'\| & \leq & \gamma^2\left(\frac{X}{\gamma X_2}\right)^p\|\Pi_2\delta U\| + \gamma^2\left(\frac{\frac{X}{\gamma} + \frac{Y}{\gamma^3} + \gamma}{X}\right)^p\gamma^{-2q}\|\Pi_2\delta T\| \end{cases}$$

With these estimates and the definition of the norm we are done.

□

Having those two lasts results we are able to use standard results in functional analysis to compute all the spectrum of DN_1. In this case we have that Table 5.1 is complete, apart from a component having eigenvalue 0.

5.3.4 Meaning of the simple fixed point

If we consider the class $Ext(A)$ we have seen that all polynomial directions from the simple fixed point are coordinates changes. This would imply that the simple fixed point is attracting, even faster than exponentially, in a neighbourhood. This was what we have previously seen using KAM Theory.

We can also see that the fixed point is attracting in the space of symmetric commuting pairs.

We had said that $N_1(U,T)$ has a golden curve if and only if (U,T) does. This result can be improved in such a way that having $N_1^n(U,T)$ converging to a pair having golden curve implies that (U,T) has this golden curve too.

5.4 Critical fixed point

There exists also a critical fixed point for our renormalisation operator. This point presents essentially one unstable direction, and it would correspond to a map with a non-smooth noble circle.

5.4.1 Introduction

Considering any quadratic irrational map, one can notice that the behaviour of the maps seems to be independent of the choice of the map. They seem to have the same repeated pattern. In particular, for nobles, we have Figure 5.1. We have to consider a small enough scale and also some appropriate coordinates (we will see them later). We have that the smaller box repeats the whole picture on a smaller scale and turned over. Asymptotically, we have that the self-similarity is exact and has the following scaling factors:

$$\alpha = -1.4148360 \text{ in } Y \qquad \beta = -3.0668882 \text{ in } X$$

Let's show how to find the critical cases in one parameter families of reversible maps and also how to compute these scaling factors. To do so, first of all we have to compute the convergents p_n/q_n of our noble rotation number, and the dominant half-line where there is a point of non-negative residue for each co-prime pair (p,q). After that, we compute the parameter values P_n for which we have a periodic point of type (p_n,q_n) living on the dominant half-line has some given residue, for example we can take it equal 1. In general this sequence of values of the parameter P_n converges and gives us the value for a critical case.

Some numerical examples were made by Robert Mackay. Taking as rotation number $\gamma^{-2} = [0,2,1,1,\ldots]$, he obtained numerically that for the quadratic map,

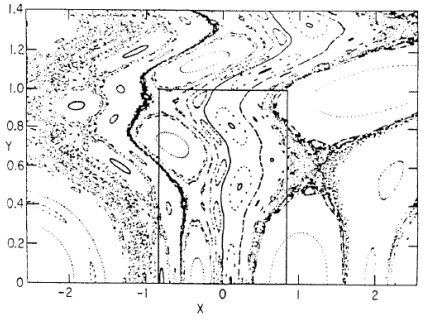

Figure 5.1: Orbits for the universal map F^*.

$T_{\text{quad}}(x, y) = (p - y - x^2, x)$, this sequence, P_n seems to converge. Notice that this quadratic map is conjugated to one of the Hénon area preserving maps. The results he obtained can be seen in Table 4.4.1.1 of [8]. One can also see that the ratio of convergence tends to the following value:

$$\delta = 1.6280$$

Once we have this approximation, we want to know how to approximate those scaling factors. Considering the symmetry coordinates (X, Y) we look for the positions X_n on the dominant half-line of the periodic points of type (p_n, q_n), and also to their convergence rate. We have then that this position will have ratio of convergence converging to the value of β. This fact can be shown in Table 4.4.1.2 of [8]. Analogously, if we consider the Y-coordinates we will obtain the value of α. There is also a different way to compute this value α. If we evaluate the following differentials:

$$DF^{q_n} = \begin{vmatrix} A_n & B_n \\ C_n & D_n \end{vmatrix}$$

at the dominant periodic points. Then our scaling will verify:

$$C_n \sim c \left(\frac{\beta}{\alpha}\right)^n,$$

so we can take the value of α from there.

5.4.2 Computation of the critical fixed point

Doing some numerical analysis we can see that it seems as if the values of the parameters converge. This fact plus Section 5.4.1 suggests the existence of another fixed point of N_1, the critical one. We expect this fixed point to be symmetric. Therefore, we can make a little change in the renormalisation over actions. This change would consist in taking $\oplus$ in the opposite order. This order change is equivalent to reflect (x, x') onto $(-x', -x)$.

We have to look for nice domains where we can expand our equation. To do so we study (5.4) as a second order equation:

$$\tau''(x,x') = \alpha'\beta'\tau'\left(\frac{x}{\alpha'}, \bar{x}\left(\frac{x}{\alpha'}, \frac{x'}{\alpha'}\right)\right) + \alpha'\beta'\alpha\beta\tau\left(\frac{1}{\alpha'}\bar{x}\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right), \frac{x'}{\alpha'\alpha}\right), \tag{5.5}$$

where α' and β' are the values of α and β for τ'. To have τ'' defined at (x,x') it is necessary to have:

$$\tau' \text{ defined at } \left(\frac{x}{\alpha'}, \bar{x}\left(\frac{x}{\alpha'}, \frac{x'}{\alpha'}\right)\right)$$

and

$$\tau \text{ defined at } \left(\frac{1}{\alpha'}\bar{x}\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right), \frac{x'}{\alpha'\alpha}\right)$$

Now, following Mackay's work we can obtain a good domain where we are able to evaluate τ''. His idea was to chose an arbitrary starting point, for example $(0,0)$, and look for the points which are required to evaluate τ''. These points are called *preimages*. If one repeats this process several times one can obtain Figure 5.2. It is important to remark that each time we repeat the process the number of points increases. It seems that this process has an arc as a limit set. We have that any domain must contain at least this arc. It can be proved that this corresponds with a segment of the golden curve.

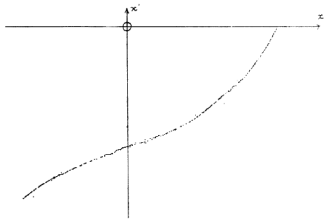

Figure 5.2: Essential points in the domain of N_1.

The next step is to look for a suitable parallelogram domain. This can be done by writting:

$$\tau(x,x') = t(\xi,\xi'),$$

where:

$$\begin{aligned} x &= c + a_{11}\xi + a_{12}\xi' \\ x' &= c' + a_{21}\xi + a_{22}\xi' \end{aligned} \tag{5.6}$$

and then look for $t(\xi,\xi')$ analytic on the product of units disks, that is, when $|\xi|, |\xi^2| \leq 1$. This will correspond to a parallelogram on the (x,x') space. We restrict to parallelograms because the corresponding coordinate change is simple to implement, it is only affine.

Then one can try several parallelogram which are closed to be centred on the arc. We repeat the process until one finds that the preimages are strictly inside. This fact is not sufficient, because we are looking for a complex domain. So one has

to introduce a quantity called *analyticity safety factor*. This quantity is an upper bound of the amount of preimages remaining inside the domain. We do it by taking estimates of the form:

$$\left|\sum a_n z^n\right| \leq \sum |a_n| \quad \text{when } |z| \leq 1.$$

In our particular case, this job is done in the following way. Given a constant $\hat{c}$, and a matrix A as in (5.6) we can compute the coefficients X_{ij} of:

$$\bar{x}\left(\frac{\hat{c} + A \cdot \vec{\xi}}{\alpha}\right) = \sum_{i,j} X_{ij} \xi^i (\xi')^j,$$

then we define:

$$X^+ = \sum_{i+j \geq 2} |X_{ij}|.$$

Then we can define for $i = 1, 2$:

$$\begin{cases} f_i &= \left|A_{i1}^{-1} c \left(\frac{1}{\alpha} - 1\right) + A_{i2}^{-1}(X_{00} - c')\right| + \left|\frac{A_{11}^{-1} A_{11}}{\alpha} + A_{i2}^{-1} X_{10}\right| + \left|\frac{A_{i1}^{-1} A_{12}}{\alpha} + A_{i2}^{-1} X_{01}\right| + |A_{i2}^{-1} X^+| \\ g_i &= \left|\frac{A_{i1}^{-1}}{\alpha}(X_{00} - c) + A_{i2}^{-1} c' \left(\frac{1}{\alpha^2} - 1\right)\right| + \left|\frac{A_{i1}^{-1} X_{10}}{\alpha} + \frac{A_{i2}^{-1} A_{21}}{\alpha^2}\right| + \left|\frac{A_{i1}^{-1} X_{01}}{\alpha} + \frac{A_{i2}^{-1} A_{22}}{\alpha^2}\right| + \left|\frac{A_{i1}^{-1} X^+}{\alpha}\right| \end{cases}$$

We have then that the analyticity safety factor is $\max(f_1, f_2, g_1, g_2)$.

By using some Monte Carlo routines Mackay has minimized the safety factor over all the possible choices of parallelogram domains. We consider Monte Carlo's routines since there exists a lot of local minimums and our function is non-differentiable there. A different option will be to look for $f_1 = f_2 = g_1 = g_2$ which has 6 unknowns but only 3 equations. The reason of choosing a parallelogram is that other possible domains will give us lots of problems to compute them. This work has been done in 1982, nowadays methods have improved a lot and we are able tod o this estimations in a simpler way and with better domains.

Then the best parallelogram obtained can be viewed in Figure 5.3. It is given by:

$$|x - c| \leq r \qquad |x' - c'| \leq r',$$

where

$$\begin{array}{rclrcl} c &=& 0.18319019 & r &=& 0.86715977 \\ c' &=& -0.63829713 & r' &=& 0.725794827 \end{array}$$

and safety factor of 0.93252.

Therefore we can represent $\tau(x, x')$ by:

$$t(\xi, \xi') = \tau(c + r\xi, c' + r\xi'),$$

and expand $t(\xi, \xi')$ in the product of the unit disks.

Given a domain for τ we can take its rescaled version for ν. Even this can be done as we can do the rescaling exactly, it would be interesting to take some slightly

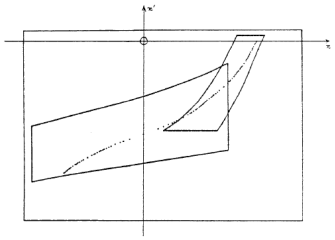

Figure 5.3: Optimal domain for N_1 and its preimages.

smaller domains. Hence τ analytic on its domain would imply that ν' analytic on slightly more than the domain for ν.

At this moment one has to discuss the choice of α and β. We determine α in the following way:

$$\tau_1\left(0, \frac{1}{\alpha}\right) = 0.$$

This value is unique since we have the twist condition. Finally β is given by:

$$\frac{\alpha}{\beta} = \tau_{12}\left(0, \frac{1}{\alpha}\right).$$

This choice of the parameters forces:

$$\nu_1'(0,1) = 0 \qquad \nu_{12}'(0,1) = 1.$$

Once we have the domain it is time to implement Newton's Method on a truncated version of N_1, we truncate all operations at some degree in the expansion in these domains. Notice that in order to evaluate $N_1(\nu, \tau)$ we have to do two more Newton steps, one to determine α and the other one to determine $\bar{x}(x, x')$. Suppose that the domain is centred at $(0, 0)$, otherwise just do the corresponding affine changes of coordinates.

Given $\nu(x, x')$ and $\tau(x, x')$ as polynomials of degree d, we start by evaluating α. As:

$$\tau_1\left(0, \frac{1}{\alpha}\right) = 0.$$

We obtain it by iterating the following Newton step:

$$\frac{1}{\alpha} \to \frac{1}{\alpha} - \frac{\tau_1\left(0, \frac{1}{\alpha}\right)}{\tau_{12}\left(0, \frac{1}{\alpha}\right)}.$$

Now we can compute

$$\beta = \frac{\alpha}{\tau_{12}\left(0, \frac{1}{\alpha}\right)}.$$

Hence set

$$\nu'(x, x') = \alpha\beta\tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right).$$

The next step consists in finding $z(x,x')$ of degree d such that:

$$\tau_2\left(\frac{x}{\alpha}, z(x,x')\right) + \nu_1\left(z(x,x'), \frac{x'}{\alpha}\right) = 0,$$

by iterating the following Newton step:

$$z(x,x') \to z(x,x') - \frac{\tau_2\left(\frac{x}{\alpha}, z(x,x')\right) + \nu_1\left(z(x,x'), \frac{x'}{\alpha}\right)}{\tau_{22}\left(\frac{x}{\alpha}, z(x,x')\right) + \nu_{11}\left(z(x,x'), \frac{x'}{\alpha}\right)},$$

where the quantities are truncated at degree d. Dividing power series is as easy as multiplying them, for some ideas one can see Knuth [5]. Notice that to find the fixed point it would be enough to find $z(x,x')$ to degree $d/2$, evaluate the Newton step to degree $d/2$, since the composition will will be exact to degree d. Observe that this z plays the role of $\bar{x}$.

Now we can evaluate:

$$\tau'(x,x') = \alpha\beta\left(\tau\left(\frac{x}{\alpha}, z(x,x')\right) + \nu\left(z(x,x'), \frac{x'}{\alpha}\right)\right)$$

up to order d.

Later we compute the derivative of the renormalisation operator:

$$\begin{aligned}
\delta\nu'(x,x') &= \alpha\beta\delta\tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right)\frac{\delta(\alpha\beta)}{\alpha\beta}\nu'(x,x') + \alpha\delta\left(\frac{1}{\alpha}\right)\left(x\nu_1'(x,x') + x'\nu_2'(x,x')\right) \\
\delta\tau'(x,x') &= \alpha\beta\left(\delta\tau\left(\frac{x}{\alpha}, z(x,x')\right) + \delta\nu\left(z(x,x'), \frac{x'}{\alpha}\right)\right) + \frac{\delta(\alpha\beta)}{\alpha\beta}\tau'(x,x') \\
&\quad + \alpha\delta\left(\frac{1}{\alpha}\right)\left(x\tau_1'(x,x') + x'\tau_2'(x,x')\right)
\end{aligned}$$

We have that the correction to the scaling α, $\delta\left(\frac{1}{\alpha}\right)$, due to the change $\delta\tau$, is given by:

$$\delta\tau_1\left(0, \frac{1}{\alpha}\right) + \tau_{12}\left(0, \frac{1}{\alpha}\right)\delta\left(\frac{1}{\alpha}\right) = 0.$$

Similarly we have that, $\delta(\alpha\beta)$ is given by:

$$\frac{\delta(\alpha\beta)}{\alpha\beta} = -\frac{\beta}{\alpha}\delta\left(\frac{\alpha}{\beta}\right) - 2\alpha\delta\left(\frac{1}{\alpha}\right),$$

where

$$\delta\left(\frac{\alpha}{\beta}\right) = \delta\tau_{12}\left(0, \frac{1}{\alpha}\right) + \tau_{122}\left(0, \frac{1}{\alpha}\right)\delta\left(\frac{1}{\alpha}\right).$$

Finally, perform a Newton step:

$$(\nu,\tau) \to (\nu',\tau') - \left(I + (DN_1 - I)^{-1}\right)\cdot((\nu',\tau') - (\nu,\tau))$$

At degree 14 Mackay obtained reasonable results, he obtained a fixed point where:

$$\alpha = -1.418 \qquad \beta = -3.055$$

which are very closed to the expected values. At the same time one can evaluate the spectrum of DN_1. He shows and interprets some of them (the ones having modulus greater than 0.8) in Table 4.4.2.1 of [8], for degree 16. The problem of this approximation appears at higher order, when there start appearing more than one eigenvalue outside the unit circle.

To solve this problem, as we expect the fixed point to be symmetric we are just interested in obtaining its spectrum in the space of symmetric commuting pairs. We can reduce also the domain by evaluating the second term in (5.5) at its reflection. This is equivalent to consider the modified renormalisation operator:

$$N_{1s} : \begin{cases} \nu'(x,x') &= \alpha\beta\tau\left(-\dfrac{x'}{\alpha}, -\dfrac{x}{\alpha}\right) \\ \tau'(x,x') &= \alpha\beta\,\tau \oplus \nu\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right) \end{cases}$$

Doing the same procedures as before, one can obtain that the optimal domain is again a rectangle (see Figure 5.4), having:

$$\begin{array}{rclrcl} c &=& 0.050707985 & r &=& 0.502060282 \\ c' &=& -0.655406307 & r' &=& 0.309680205 \end{array}$$

and safety factor of 0.87918. In this case we have to take care and take the rescaled and reflected ν.

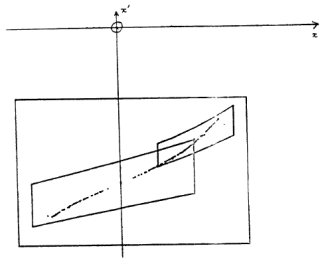

Figure 5.4: Optimal domain for N_{1s} and its preimages.

Now we proceed by Newton iteration as we did for N_1. This approximation gives better results. Some good approximations of the coefficients of the fixed point are presented in Table 4.4.2.4 of [8].

One can also appreciate the biggest eigenvalues in Table 4.4.2.6 of [8].

Therefore its unstable manifold is a universal one parameter family, which will explain the behaviour of the system and when the invariant circles break down.

Conclusions

The problem of the existence and persistence of invariant tori in Hamiltonian systems (symplectic maps) is well understood for Hamiltonian systems having 2 degrees of freedom ($2D$ symplectic maps). Throughout this work we have shown the most important tools and techniques to deal with this problem. We have seen that it can be approached in several ways. Both analytical and numerical techniques are presented and useful to explain when the invariant circles persist, break down etc. Using a geometrical approach we have been able to explain why these invariant curves break down.

We get that, considering renormalisation theory, one can understand what happens with almost every possible invariant circle. Studying the renormalisation operator leads us to the knowledge of how any of the considered systems behaves.

The problem of considering invariant tori in higher dimension remains open. The study done throughout this work helps us to understand why some of these methods can not be extended into higher dimension, for example the obstruction one. Having a good knowledge of what happens in dimension 2 may help us in future works to try to deal with higher dimension problems.

Bibliography

[1] V. I. Arnold. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*. "Mir", Moscow, 1984. Translated from the Russian by Djilali Embarek, Reprint of the 1980 edition.

[2] V. I. Arnold. *Vladimir I. Arnold—collected works. Vol. I.* Springer-Verlag, Berlin, 2009. Representations of functions, celestial mechanics and KAM theory, 1957–1965.

[3] V. I. Arnold and A. Avez. *Problèmes ergodiques de la mécanique classique*. Monographies Internationales de Mathématiques Modernes, No. 9. Gauthier-Villars, Éditeur, Paris, 1967.

[4] Rafael de la Llave. A tutorial on KAM theory.

[5] Donald E. Knuth. *The art of computer programming. Vol. 2: Seminumerical algorithms*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1969.

[6] A. Luque and J. Villanueva. Quasi-periodic frequency analysis using averaging-extrapolation methods. *SIAM J. Appl. Dyn. Syst.*, 13:1–46, 2014.

[7] R. S. MacKay. A renormalisation approach to invariant circles in area-preserving maps. *Phys. D*, 7(1-3):283–300, 1983. Order in chaos (Los Alamos, N.M., 1982).

[8] R. S. MacKay. *Renormalisation in area-preserving maps*, volume 6 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[9] J. D. Meiss. Symplectic maps, variational principles, and transport. *Rev. Modern Phys.*, 64(3):795–848, 1992.

[10] N. Miguel, C. Simó, and A. Vieiro. On the effect of islands in the diffusive properties of the standard map, for large parameter values.

[11] Arturo Olvera and Carles Simó. An obstruction method for the destruction of invariant curves. *Phys. D*, 26(1-3):181–192, 1987.

[12] Jürgen Pöschel. A lecture on the classical KAM theorem. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 707–732. Amer. Math. Soc., Providence, RI, 2001.

[13] C. Simó. *Quadratic Area Preserving Maps in* $\mathbb{R}^2$. Universitat de Barcelona, 2013.

[14] Michael Spivak. *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*. W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[15] R. Tomás. Direct measurement of resonance driving terms in the super proton synchrotron (sps) of cern using beam position monitors. 2003.