DEFORMATION OF GABOR SYSTEMS

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Abstract. We introduce a new notion for the deformation of Gabor systems. Such deformations are in general nonlinear and, in particular, include the standard jitter error and linear deformations of phase space. With this new notion we prove a strong deformation result for Gabor frames and Gabor Riesz sequences that covers the known perturbation and deformation results. Our proof of the deformation theorem requires a new characterization of Gabor frames and Gabor Riesz sequences. It is in the style of Beurling's characterization of sets of sampling for bandlimited functions and extends significantly the known characterization of Gabor frames "without inequalities" from lattices to non-uniform sets.

1. Introduction

The question of robustness of a basis or frame is a fundamental problem in functional analysis and in many concrete applications. It has its historical origin in the work of Paley and Wiener [40] who studied the perturbation of Fourier bases and was subsequently investigated in many contexts in complex analysis and harmonic analysis. Particularly fruitful was the study of the robustness of structured function systems, such as reproducing kernels, sets of sampling in a space of analytic functions, wavelets, or Gabor systems. In this paper we take a new look at the stability of Gabor frames and Gabor Riesz sequences with respect to general deformations of phase space.

To be explicit, let us denote the time-frequency shift of a function $g \in L^2(\mathbb{R}^d)$ along $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$ by

$$\pi(z)g(t) = e^{2\pi i \xi \cdot t} g(t - x) \,.$$

For a fixed non-zero function $g \in L^2(\mathbb{R}^d)$, usually called a "window function", and $\Lambda \subseteq \mathbb{R}^{2d}$, a Gabor system is a structured function system of the form

$$\mathcal{G}(g, \Lambda) = \left\{ \pi(\lambda) g := e^{2\pi i \xi \cdot} g(\cdot - x) : \lambda = (x, \xi) \in \Lambda \right\} \,.$$

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The index set Λ is a discrete subset of the phase space $\mathbb{R}^{2d}$ and λ indicates the localization of a time-frequency shift $\pi(\lambda)g$ in phase space.

The Gabor system $\mathcal{G}(g,\Lambda)$ is called a *frame* (a Gabor frame), if

$$A \le \|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|_2^2, \qquad f \in L^2(\mathbb{R}^d),$$

for some constants $0 < A \le B < \infty$. In this case every function $f \in L^2(\mathbb{R}^d)$ possesses an expansion $f = \sum_\lambda c_\lambda \pi(\lambda)g$, for some coefficient sequence $c \in \ell^2(\Lambda)$ such that $\|f\|_2 \asymp \|c\|_2$. The Gabor system $\mathcal{G}(g,\Lambda)$ is called a *Riesz sequence* (or Riesz basis for its span), if $\|\sum_\lambda c_\lambda \pi(\lambda)g\|_2 \asymp \|c\|_2$ for all $c \in \ell^2(\Lambda)$.

For meaningful statements about Gabor frames it is usually assumed that

$$\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g\rangle| \, dz < \infty.$$

This condition describes the modulation space $M^1(\mathbb{R}^d)$, also known as the Feichtinger algebra. Every Schwartz function satisfies this condition.

In this paper we study the stability of the spanning properties of $\mathcal{G}(g,\Lambda)$ with respect to a set $\Lambda \subseteq \mathbb{R}^{2d}$. If Λ' is "close enough" to Λ, then we expect $\mathcal{G}(g,\Lambda')$ to possess the same spanning properties. In this context we distinguish perturbations and deformations. Whereas a perturbation is local and Λ' is obtained by slightly moving every $\lambda \in \Lambda$, a deformation is a global transformation of $\mathbb{R}^{2d}$. The existing literature is rich in perturbation results, but not much is known about deformations of Gabor frames.

(a) *Perturbation or jitter error:* The jitter describes small pointwise perturbations of Λ. For every Gabor frame $\mathcal{G}(g,\Lambda)$ with $g \in M^1(\mathbb{R}^d)$ there exists a maximal jitter $\epsilon > 0$ with the following property: if $\sup_{\lambda\in\Lambda} \inf_{\lambda'\in\Lambda'} |\lambda - \lambda'| < \epsilon$ and $\sup_{\lambda'\in\Lambda'} \inf_{\lambda\in\Lambda} |\lambda - \lambda'| < \epsilon$, then $\mathcal{G}(g,\Lambda')$ is also a frame. See [19, 23] for a general result in coorbit theory, the recent paper [21], and Christensen's book on frames [10] for more details and references. Conceptually the jitter error is easy to understand, because the frame operator is continuous in the operator norm with respect to the jitter error. The proof techniques go back to Paley and Wiener [40] and amount to norm estimates for the frame operator.

(b) *Linear deformations:* The fundamental deformation result is due to Feichtinger and Kaiblinger [20]. Let $g \in M^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice, and assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. Then there exists $\epsilon > 0$ with the following property: if A is a $2d \times 2d$-matrix with $\|A - \mathrm{I}\| < \epsilon$ (in some given matrix norm), then $\mathcal{G}(g, A\Lambda)$ is again a frame. Only recently, this result was generalized to non-uniform Gabor frames [3]. The proof for the case of a lattice [20] was based on the duality theory of Gabor frames, the proof for non-uniform Gabor frames in [3] relies on the stability under chirps of the Sjöstrand symbol class for pseudodifferential operators, but this technique does not seem to adapt to nonlinear deformations. Compared to perturbations, (linear) deformations of Gabor frames are much more difficult to understand, because the frame operator no longer depends (norm-) continuously on Λ and a deformation may change the density of Λ (which may affect significantly the spanning properties of $\mathcal{G}(g,\Lambda)$).

Perhaps the main difficulty is to find a suitable notion for deformations that preserves Gabor frames. Except for linear deformations and some preliminary observations in [12, 15] this question is simply unexplored. In this paper we introduce a general concept of deformation, which we call *Lipschitz* deformations. Lipschitz deformations include both the jitter error and linear deformations as a special case. The precise definition is somewhat technical and will be given in Section 6. For simplicity we formulate a representative special case of our main result.

Theorem 1.1. *Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$. Let $T_n : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ for $n \in \mathbb{N}$ be a sequence of differentiable maps with Jacobian DT_n. Assume that*

$$\sup_{z\in\mathbb{R}^{2d}} |DT_n(z) - I| \longrightarrow 0 \qquad \text{as } n \to \infty\,. \tag{1}$$

Then the following holds.

(a) *If $\mathcal{G}(g,\Lambda)$ is a frame, then $\mathcal{G}(g,T_n(\Lambda))$ is a frame for all sufficiently large n.*
(b) *If $\mathcal{G}(g,\Lambda)$ is a Riesz sequence, then $\mathcal{G}(g,T_n(\Lambda))$ is a Riesz sequence for all sufficiently large n.*

We would like to emphasize that Theorem 1.1 is quite general. It deals with *non-uniform* Gabor frames (not just lattices) under *nonlinear* deformations. In particular, Theorem 1.1 implies the main results of [20, 3]. The counterpart for deformations of Gabor Riesz sequences (item (b)) is new even for linear deformations.

Condition (1) roughly states that the mutual distances between the points of Λ are preserved locally under the deformation T_n. Our main insight was that the frame property of a deformed Gabor system $\mathcal{G}(g,T_n(\Lambda))$ does not depend so much on the position or velocity of the sequences $(T_n(\lambda))_{n\in\mathbb{N}}$ for $\lambda \in \Lambda$, but on the relative distances $|T_n(\lambda)-T_n(\lambda')|$ for $\lambda, \lambda' \in \Lambda$. For an illustration see Example 7.4.

As an application of Theorem 1.1, we derive a *non-uniform Balian-Low theorem* (BLT). For this, we recall that the lower Beurling density of a set $\Lambda \subseteq \mathbb{R}^{2d}$, which is given by

$$D^-(\Lambda) = \lim_{R\to\infty} \min_{z\in\mathbb{R}^{2d}} \frac{\#\,\Lambda \cap B_R(z)}{\mathrm{vol}(B_R(0))}\,,$$

and likewise the upper Beurling density $D^+(\Lambda)$ (where the minimum is replaced by a supremum). The fundamental density theorem of Ramanathan and Steger [31] asserts that if $\mathcal{G}(g,\Lambda)$ is a frame then $D^-(\Lambda) \geq 1$. Analogously, if $\mathcal{G}(g,\Lambda)$ is a Riesz sequence, then $D^+(\Lambda) \leq 1$ [5]. The so-called Balian-Low theorem (BLT) is a stronger version of the density theorem and asserts that for "nice" windows g the inequalities in the density theorem are strict. For the case when $g \in M^1(\mathbb{R}^d)$ and Λ is a lattice, the Balian-Low theorem is a consequence of [20]. A Balian-Low theorem for non-uniform Gabor frames was open for a long time and was proved only recently by Ascensi, Feichtinger, and Kaiblinger [3]. The corresponding statement for Gabor Riesz sequences was open and is settled here as an application of our deformation theorem. We refer to Heil's detailed survey [27] of the numerous contributions to the density theorem for Gabor frames after [31] and to [13] for the Balian-Low theorem.

As an immediate consequence of Theorem 1.1 we obtain the following version of the Balian-Low theorem for non-uniform Gabor systems.

Corollary 1.2 (Non-uniform Balian-Low Theorem)**.** *Assume that* $g \in M^1(\mathbb{R}^d)$.
(a) If $\mathcal{G}(g,\Lambda)$ *is a frame for* $L^2(\mathbb{R}^d)$*, then* $D^-(\Lambda) > 1$.
(b) If $\mathcal{G}(g,\Lambda)$ *is a Riesz sequence in* $L^2(\mathbb{R}^d)$*, then* $D^+(\Lambda) < 1$.

Proof. We only prove the new statement (b), part (a) is similar [3]. Assume $\mathcal{G}(g,\Lambda)$ is a Riesz sequence, but that $D^+(\Lambda) = 1$. Let $\alpha_n > 1$ such that $\lim_{n\to\infty} \alpha = 1$ and set $T_n z = \alpha_n z$. Then the sequence T_n satisfies condition (1). On the one hand, we have $D^+(\alpha_n \Lambda) = \alpha_n^{2d} > 1$, and on the other hand, Theorem 1.1 implies that $\mathcal{G}(g,\alpha_n\Lambda)$ is a Riesz sequence for n large enough. This is a contradiction to the density theorem, and thus the assumption $D^+(\Lambda) = 1$ cannot hold. ∎

The proof of Theorem 1.1 does not come easily and is technical. It combines methods from the theory of localized frames [22, 25], the stability of operators on ℓ^p-spaces [1, 36] and weak limit techniques in the style of Beurling [7]. We say that $\Gamma \subseteq \mathbb{R}^{2d}$ is a weak limit of translates of $\Lambda \subseteq \mathbb{R}^{2d}$, if there exists a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{2d}$, such that $\Lambda + z_n \to \Gamma$ uniformly on compact sets. See Section 4 for the precise definition and more details on weak limits.

We will prove the following characterization of non-uniform Gabor frames "without inequalities".

Theorem 1.3. *Assume that* $g \in M^1(\mathbb{R}^d)$ *and* $\Lambda \subseteq \mathbb{R}^{2d}$*. Then* $\mathcal{G}(g,\Lambda)$ *is a frame for* $L^2(\mathbb{R}^d)$*, if and only if for every weak limit* Γ *of* Λ *the map* $f \to \big(\langle f, \pi(\gamma) g\rangle\big)_{\gamma\in\Gamma}$ *is one-to-one on* $(M^1(\mathbb{R}^d))^*$.

The full statement with five equivalent conditions characterizing a non-uniform Gabor frame will be given in Section 5, Theorem 5.1. An analogous characterization of Gabor Riesz sequences with weak limits is stated in Theorem 5.4.

For the special case when Λ is a lattice, the above characterization of Gabor frames without inequalities was already proved in [26]. In the lattice case, the Gabor system $\mathcal{G}(g,\Lambda)$ possesses additional invariance properties that facilitate the application of methods from operator algebras. The generalization of [26] to non-uniform Gabor systems was rather surprising for us and demands completely different methods.

To make Theorem 1.3 more plausible, we make the analogy with Beurling's results on balayage in Paley-Wiener space. Beurling [7] characterized the stability of sampling in the Paley-Wiener space of bandlimited functions $\{f \in L^\infty(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subseteq S\}$ for a compact spectrum $S \subseteq \mathbb{R}^d$ in terms of sets of uniqueness for this space. It is well-known that the frame property of a Gabor system $\mathcal{G}(g,\Lambda)$ is equivalent to a sampling theorem for an associated transform. Precisely, let $z \in \mathbb{R}^{2d} \to V_g f(z) = \langle f, \pi(z) g\rangle$ be the short-time Fourier transform, for fixed non-zero $g \in M^1(\mathbb{R}^d)$ and $f \in (M^1(\mathbb{R}^d))^*$. Then $\mathcal{G}(g,\Lambda)$ is a frame, if and only if Λ is a set of sampling for the short-time Fourier transform on $(M^1)^*$.

In this light, Theorem 1.3 is the precise analog of Beurling's theorem for bandlimited functions.

One may therefore try to adapt Beurling's methods to Gabor frames and the sampling of short-time Fourier transforms. Beurling's ideas have been used for many sampling problems in complex analysis following the pioneering work of Seip on the Fock space [32],[33] and the Bergman space [34], see also [8] for a survey. A remarkable fact in Theorem 1.3 is the absence of a complex structure (except when g is a Gaussian). This explains why we have to use the machinery of localized frames and the stability of operators in our proof. We mention that Beurling's ideas have been transferred to a few other contexts outside complex analysis, such as sampling theorems with spherical harmonics in the sphere [28], or, more generally, with eigenvectors of the Laplace operator in Riemannian manifolds [30].

This article is organized as follows: In Section 2 we collect the main definitions from time-frequency analysis. In Section 3 we discuss time-frequency molecules and their ℓ^p-stability. Section 4 is devoted to the details of Beurling's notion of weak convergence of sets. In Section 5 we state and prove the full characterization of non-uniform Gabor frames and Riesz sequences without inequalities. In Section 6 we introduce the general concept of a Lipschitz deformation of a set and prove the main properties. Finally, in Section 7 we state and prove the main result of this paper, the general deformation result. The appendix provides the technical proof of the stability of time-frequency molecules.

2. Preliminaries

2.1. **General notation.** Throughout the article, $|x| := (|x_1|^2 + \ldots + |x_d|^2)^{1/2}$ denotes the Euclidean norm, and $B_r(x)$ denotes the Euclidean ball. Given two functions $f, g : X \to [0, \infty)$, we say that $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$, for all $x \in X$. We say that $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

2.2. **Sets of points.** A set $\Lambda \subseteq \mathbb{R}^d$ is called *relatively separated* if

$$\operatorname{rel}(\Lambda) := \sup\{\#(\Lambda \cap B_1(x)) : x \in \mathbb{R}^d\} < \infty. \tag{2}$$

It is called *separated* if

$$\operatorname{sep}(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda\} > 0. \tag{3}$$

We say that Λ is δ-separated if $\operatorname{sep}(\Lambda) \geq \delta$. A separated set is relatively separated and

$$\operatorname{rel}(\Lambda) \lesssim \operatorname{sep}(\Lambda)^{-d}, \qquad \Lambda \subseteq \mathbb{R}^d. \tag{4}$$

Relatively separated sets are finite unions of separated sets.

The *hole* of a set $\Lambda \subseteq \mathbb{R}^d$ is defined as

$$\rho(\Lambda) := \sup_{x \in \mathbb{R}^d} \inf_{\lambda \in \Lambda} |x - \lambda|. \tag{5}$$

A sequence Λ is called *relatively dense* if $\rho(\Lambda) < \infty$. Equivalently, Λ is relatively dense if there exists $R > 0$ such that

$$\mathbb{R}^d = \bigcup_{\lambda \in \Lambda} B_R(\lambda).$$

In terms of the Beurling densities defined in the Introduction, a set Λ is relatively separated if and only if $D^+(\Lambda) < \infty$ and it is relatively dense if and only if $D^-(\Lambda) > 0$.

2.3. **Amalgam spaces.** The *amalgam space* $W(L^\infty, L^1)(\mathbb{R}^d)$ consists of all functions $f \in L^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{W(L^\infty,L^1)} := \int_{\mathbb{R}^d} \|f\|_{L^\infty(B_1(x))} dx \asymp \sum_{k\in\mathbb{Z}^d} \|f\|_{L^\infty([0,1]^d+k)} < \infty.$$

The subspace of $W(L^\infty, L^1)(\mathbb{R}^d)$ consisting of continuous functions is denoted by $W(C_0, L^1)(\mathbb{R}^d)$. This space will be used as a convenient collection of test functions. We will repeatedly use the following sampling inequality: *Assume that $f \in W(L^\infty, L^1)(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is relatively separated, then*

$$\sum_{\lambda\in\Lambda} |f(\lambda)| \lesssim \operatorname{rel}(\Lambda) \, \|f\|_{W(L^\infty,L^1)} . \tag{6}$$

The dual space of $W(C_0, L^1)(\mathbb{R}^d)$ will be denoted $W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$. It consists of all the complex-valued Borel measures $\mu : \mathcal{B}(\mathbb{R}^d) \to \mathbb{C}$ such that

$$\|\mu\|_{W(\mathcal{M},L^\infty)} := \sup_{x\in\mathbb{R}^d} \|\mu\|_{B_1(x)} = \sup_{x\in\mathbb{R}^d} |\mu| \, (B_1(x)) < \infty.$$

For the general theory of Wiener amalgam spaces we refer to [17].

2.4. **Time-frequency analysis.** The *time-frequency shifts* of a function $f : \mathbb{R}^d \to \mathbb{C}$ are

$$\pi(z)f(t) := e^{2\pi i\xi t} f(t-x), \qquad z = (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, t \in \mathbb{R}^d.$$

These operators satisfy the commutation relations

$$\pi(x,\xi)\pi(x',\xi') = e^{-2\pi i\xi' x}\pi(x+x', \xi+\xi'), \qquad (x,\xi),(x',\xi') \in \mathbb{R}^d \times \mathbb{R}^d. \tag{7}$$

Given a non-zero Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, the *short-time Fourier transform* of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is defined as

$$V_g f(z) := \langle f, \pi(z)g\rangle , \qquad z \in \mathbb{R}^{2d}. \tag{8}$$

For $\|g\|_2 = 1$ the short-time Fourier transform is an isometry:

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)}, \qquad f \in L^2(\mathbb{R}^d). \tag{9}$$

The commutation rule (7) implies the covariance property of the short-time Fourier transform:

$$V_g(\pi(x,\xi))f(x',\xi') = e^{-2\pi i(x-x')(\xi-\xi')} V_g f(x'-x, \xi'-\xi), \qquad (x,\xi),(x',\xi') \in \mathbb{R}^d \times \mathbb{R}^d.$$

In particular,

$$|V_g \pi(z) f| = |V_g f(\cdot - z)| , \qquad z \in \mathbb{R}^{2d}. \tag{10}$$

We then define the *modulation spaces* as follows: fix a non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ and let

$$M^p(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, V_g f \in L^p(\mathbb{R}^{2d}) \right\}, \qquad 1 \le p \le \infty, \tag{11}$$

endowed with the norm $\|f\|_{M^p} := \|V_g f\|_{L^p}$. Different choices of non-zero windows $g \in \mathcal{S}(\mathbb{R}^d)$ yield the same space with equivalent norms, see [18]. The space $M^1(\mathbb{R}^d)$, known as the Feichtinger algebra, plays a central role. It can also be characterized as

$$M^1(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \,:\, V_f f \in L^1(\mathbb{R}^{2d}) \right\}.$$

The modulation space $M^0(\mathbb{R}^d)$ is defined as the closure of the Schwartz-class with respect to the norm $\|\cdot\|_{M^\infty}$. Then $M^0(\mathbb{R}^d)$ is a closed subspace of $M^\infty(\mathbb{R}^d)$ and can also be characterized as

$$M^0(\mathbb{R}^d) = \left\{ f \in M^\infty(\mathbb{R}^d) \,:\, V_g f \in C_0(\mathbb{R}^{2d}) \right\}.$$

The duality of modulation spaces is similar to sequence spaces; we have $M^0(\mathbb{R}^d)^* = M^1(\mathbb{R}^d)$ and $M^1(\mathbb{R}^d)^* = M^\infty(\mathbb{R}^d)$ with respect to the duality $\langle f, h\rangle := \langle V_g f, V_g h\rangle$.

In this article we consider a fixed a function $g \in M^1(\mathbb{R}^d)$ and will be mostly concerned with $M^1(\mathbb{R}^d)$, its dual space $M^\infty(\mathbb{R}^d)$, and $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. The weak* topology in $M^\infty(\mathbb{R}^d)$ will be denoted by $\sigma(M^\infty, M^1)$ and the weak* topology on $M^1(\mathbb{R}^d)$ by $\sigma(M^1, M^0)$. Hence, a sequence $\{f_k : k \geq 1\} \subseteq M^\infty(\mathbb{R}^d)$ converges to $f \in M^\infty(\mathbb{R}^d)$ in $\sigma(M^\infty, M^1)$ if and only if for every $h \in M^1(\mathbb{R}^d)$: $\langle f_k, h\rangle \longrightarrow \langle f, h\rangle$.

We mention the following facts that will be used repeatedly (see for example [19, Theorem 4.1] and [24, Proposition 12.1.11]).

Lemma 2.1. *Let $g \in M^1(\mathbb{R}^d)$ be nonzero. Then the following hold true.*

(a) *If $f \in M^1(\mathbb{R}^d)$, then $V_g f \in W(C_0, L^1)(\mathbb{R}^{2d})$.*

(b) *Let $\{f_k : k \geq 1\} \subseteq M^\infty(\mathbb{R}^d)$ be a bounded sequence and $f \in M^\infty(\mathbb{R}^d)$. Then $f_k \longrightarrow f$ in $\sigma(M^\infty, M^1)$ if and only if $V_g f_k \longrightarrow V_g f$ uniformly on compact sets.*

(c) *Let $\{f_k : k \geq 1\} \subseteq M^1(\mathbb{R}^d)$ be a bounded sequence and $f \in M^1(\mathbb{R}^d)$. Then $f_k \longrightarrow f$ in $\sigma(M^1, M^0)$ if and only if $V_g f_k \longrightarrow V_g f$ uniformly on compact sets.*

In particular, if $f_n \to f$ in $\sigma(M^\infty, M^1)$ and $z_n \to z \in \mathbb{R}^{2d}$, then $V_g f_n(z_n) \to V_g f(z)$.

2.5. **Analysis and synthesis maps.** Given $g \in M^1(\mathbb{R}^d)$ and a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$, consider the *analysis operator* and the *synthesis operator* that are formally defined as

$$C_{g,\Lambda} f := \left(\langle f, \pi(\lambda) g\rangle\right)_{\lambda\in\Lambda}, \qquad f \in M^\infty(\mathbb{R}^d),$$
$$C^*_{g,\Lambda} c := \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g, \qquad c \in \ell^\infty(\Lambda).$$

These maps are bounded between M^p and ℓ^p spaces [24, Cor. 12.1.12] with estimates

$$\|C_{g,\Lambda} f\|_{\ell^p} \lesssim \operatorname{rel}(\Lambda) \|g\|_{M^1} \|f\|_{M^p},$$
$$\|C^*_{g,\Lambda} c\|_{M^p} \lesssim \operatorname{rel}(\Lambda) \|g\|_{M^1} \|c\|_{\ell^p}.$$

The implicit constants in the last estimates are independent of $p \in [1, \infty]$.

For $z = (x, \xi) \in \mathbb{R}^{2d}$, the *twisted shift* is the operator $\kappa(z) : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda + z)$ given by

$$(\kappa(z) c)_{\lambda+z} := e^{-2\pi i x \lambda_2} c_\lambda, \qquad \lambda = (\lambda_1, \lambda_2) \in \Lambda.$$

As a consequence of the commutation relations (7), the analysis and synthesis operators satisfy the covariance property

$$\pi(z)C^*_{g,\Lambda} = C^*_{g,\Lambda+z}\kappa(z) \text{ and } e^{2\pi i x\xi}C_{g,\Lambda}\pi(-z) = e^{-2\pi i x\xi}\kappa(-z)C_{g,\Lambda+z} \tag{12}$$

for $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$.

A Gabor system $\mathcal{G}(g, \Lambda)$ is a *frame* if and only if $C_{g,\Lambda} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ is bounded below, and $\mathcal{G}(g, \Lambda)$ is a *Riesz sequence* if and only if $C^*_{g,\Lambda} : \ell^2(\Lambda) \to L^2(\mathbb{R}^d)$ is bounded below. As the following lemma shows, each of these conditions implies a restriction of the geometry of the set Λ.

Lemma 2.2. *Let $g \in L^2(\mathbb{R}^d)$ and let $\Lambda \subseteq \mathbb{R}^{2d}$ a set. Then the following holds.*

(a) *If $\mathcal{G}(g, \Lambda)$ is a frame, then Λ is relatively separated and relatively dense.*
(b) *If $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, then Λ is separated.*

Proof. For part (a) see for example [9, Theorem 1.1]. For part (b), suppose that Λ is not separated. Then there exist two sequences $\{\lambda_n : n \geq 1\}, \{\gamma_n : n \geq 1\} \subseteq \Lambda$ with $\lambda_n \neq \gamma_n$ such that $|\lambda_n - \gamma_n| \longrightarrow 0$. Hence we derive the following contradiction: $\sqrt{2} = \|\delta_{\lambda_n} - \delta_{\gamma_n}\|_{\ell^2(\Lambda)} \asymp \|\pi(\lambda_n)g - \pi(\gamma_n)g\|_{L^2(\mathbb{R}^d)} \longrightarrow 0$. ■

We extend the previous terminology to other values of $p \in [1, \infty]$. We say that $\mathcal{G}(g, \Lambda)$ is a p-frame for $M^p(\mathbb{R}^d)$ if $C_{g,\Lambda} : M^p(\mathbb{R}^d) \to \ell^p(\Lambda)$ is bounded below, and that $\mathcal{G}(g, \Lambda)$ is a p-Riesz sequence within $M^p(\mathbb{R}^d)$ if $C^*_{g,\Lambda} : \ell^p(\Lambda) \to M^p(\mathbb{R}^d)$ is bounded below. Since boundedness below and left invertibility are different properties outside the context of Hilbert spaces, there are other reasonable definitions of frames and Riesz sequences for M^p. This is largely immaterial for Gabor frames with $g \in M^1$, since the theory of localized frames asserts that when such a system is a frame for L^2, then it is a frame for all M^p and moreover the operator $C_{g,\Lambda} : M^p \to \ell^p$ is left invertible [25, 22, 4, 5]. Similar statements apply to Riesz sequences.

3. Stability of time-frequency molecules

We say that $\{f_\lambda : \lambda \in \Lambda\} \subseteq L^2(\mathbb{R}^d)$ is a set of time-frequency molecules if $\Lambda \subseteq \mathbb{R}^{2d}$ is a relatively separated set and there exists a non-zero $g \in M^1(\mathbb{R}^d)$ and an envelope function $\Phi \in W(L^\infty, L^1)(\mathbb{R}^{2d})$ such that

$$|V_g f_\lambda(z)| \leq \Phi(z - \lambda), \qquad z \in \mathbb{R}^d, \lambda \in \Lambda. \tag{13}$$

If (13) holds for some $g \in M^1(\mathbb{R}^d)$, then it holds for all $g \in M^1(\mathbb{R}^d)$ (with an envelope depending on g).

Remark 3.1. Every Gabor system $\mathcal{G}(g, \Lambda)$ with window $g \in M^1(\mathbb{R}^d)$ and a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$ is a set of time-frequency molecules. In this case the envelope can be chosen to be $\Phi = |V_g g|$, which belongs to $W(L^\infty, L^1)(\mathbb{R}^{2d})$ by Lemma 2.1.

The following stability result will be one of our main technical tools.

Theorem 3.2. *Let $\{f_\lambda : \lambda \in \Lambda\}$ be a set of time-frequency molecules. Then the following holds.*

(a) *Assume that*

$$\|f\|_{M^p} \asymp \|(\langle f, f_\lambda\rangle)_{\lambda\in\Lambda}\|_p, \qquad \forall\, f \in M^p(\mathbb{R}^d), \tag{14}$$

holds for some $1 \le p \le \infty$. Then (14) holds for all $1 \le p \le \infty$. In other words, if $\{f_\lambda : \lambda \in \Lambda\}$ is a p-frame for $M^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$, then it is a p-frame for $M^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$.

(b) *Assume that*

$$\|\sum_{\lambda\in\Lambda} c_\lambda f_\lambda\|_{M^p} \asymp \|c\|_p, \qquad c \in \ell^p(\Lambda), \tag{15}$$

holds for some $1 \le p \le \infty$. Then (15) holds for all $1 \le p \le \infty$.

The result is similar in spirit to other results in the literature [37, 1, 35, 39, 38], but none of these is directly applicable to our setting. We postpone the proof of Theorem 3.2 to the appendix, so as not to interrupt the natural flow of the article. As in the cited references, the proof elaborates on Sjöstrand's Wiener-type lemma [36].

As a special case of Theorem 3.2 we record the following corollary.

Corollary 3.3. *Let $g \in M^1(\mathbb{R}^d)$ and let $\Lambda \subseteq \mathbb{R}^{2d}$ be a relatively separated set. Then the following holds.*

(a) *If $\mathcal{G}(g,\Lambda)$ if a p-frame for $M^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$, then it is a p-frame for $M^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$.*

(b) *If $\mathcal{G}(g,\Lambda)$ if a p-Riesz sequence within $M^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$, then it p-Riesz sequence within $M^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$.*

The space $M^1(\mathbb{R}^d)$ is the largest space of windows for which the corollary holds. Under a stronger condition on g, statement (a) was already derived in [1], the general case was left open.

4. Weak convergence

4.1. **Convergence of sets.** The Hausdorff distance between two sets $X, Y \subseteq \mathbb{R}^d$ is defined as

$$d_H(X,Y) := \inf\{\varepsilon > 0 : X \subseteq Y + B_\varepsilon(0), Y \subseteq X + B_\varepsilon(0)\}.$$

Note that $d_H(X,Y) = 0$ if and only if $\overline{X} = \overline{Y}$.

Let $\Lambda \subseteq \mathbb{R}^d$ be a set. A sequence $\{\Lambda_n : n \ge 1\}$ of subsets of $\mathbb{R}^d$ *converges weakly* to Λ, in short $\Lambda_n \xrightarrow{w} \Lambda$, if

$$d_H\big((\Lambda_n \cap \bar{B}_R(z)) \cup \partial\bar{B}_R(z)), (\Lambda \cap \bar{B}_R(z)) \cup \partial\bar{B}_R(z))\big) \to 0, \qquad \forall z \in \mathbb{R}^d, R > 0. \tag{16}$$

(To understand the role of the boundary of the ball in the definition, consider the following example in dimension $d = 1$: $\Lambda_n := \{1 + 1/n\}$, $\Lambda := \{1\}$ and $B_R(z) = [0,1]$.)

The following lemma provides an alternative description of weak convergence.

Lemma 4.1. *Let $\Lambda \subseteq \mathbb{R}^d$ and $\Lambda_n \subseteq \mathbb{R}^d, n \geq 1$ be sets. Then $\Lambda_n \xrightarrow{w} \Lambda$ if and only if for every $R > 0$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\Lambda \cap B_R(0) \subseteq \Lambda_n + B_\varepsilon(0) \quad \text{and} \quad \Lambda_n \cap B_R(0) \subseteq \Lambda + B_\varepsilon(0).$$

The following consequence of Lemma 4.1 is often useful to identify weak limits.

Lemma 4.2. *Let $\Lambda_n \xrightarrow{w} \Lambda$ and $\Gamma_n \xrightarrow{w} \Gamma$. Suppose that for every $R > 0$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\Lambda_n \cap B_R(0) \subseteq \Gamma_n + B_\varepsilon(0).$$

Then $\overline{\Lambda} \subseteq \overline{\Gamma}$.

The notion of weak convergence will be a technical tool in the proofs of deformation results.

4.2. **Measures and compactness.** In this section we explain how the weak convergence of sets can be understood by the convergence of some associated measures. First we note the following semicontinuity property, that follows directly from Lemma 4.1.

Lemma 4.3. *Let $\{\mu_n : n \geq 1\} \subset W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ be a sequence of measures that converges to a measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ in the $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))$ topology. Suppose that $\operatorname{supp}(\mu_n) \subseteq \Lambda_n$ and that $\Lambda_n \xrightarrow{w} \Lambda$. Then $\operatorname{supp}(\mu) \subseteq \overline{\Lambda}$.*

The example $\mu_n = \frac{1}{n}\delta$, $\mu = 0$ shows that in Lemma 4.3 the inclusions cannot in general be improved to equalities. Such improvement is however possible for certain classes of measures. A Borel measure μ is called *natural-valued* if $\mu(E)$ is a non-negative integer for all Borel sets E. For these measures the following holds.

Lemma 4.4. *Let $\{\mu_n : n \geq 1\} \subset W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ be a sequence of natural-valued measures that converges to a measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ in the $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))$ topology. Then $\operatorname{supp}(\mu_n) \xrightarrow{w} \operatorname{supp}(\mu)$.*

The proof of Lemma 4.4 is elementary and therefore we skip it. Lemma 4.4 is useful to deduce properties of weak converge of sets from properties of convergence of measures, as we now show. For a set $\Lambda \subseteq \mathbb{R}^d$, let us consider the natural-valued measure

$$\sqcup\!\sqcup_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda. \tag{17}$$

One can readily verify that Λ is relatively separated if and only if $\sqcup\!\sqcup_\Lambda \in W(\mathcal{M}, L^\infty)(\mathbb{R}^d)$ and moreover,

$$\|\sqcup\!\sqcup_\Lambda\|_{W(\mathcal{M}, L^\infty)} \asymp \operatorname{rel}(\Lambda). \tag{18}$$

For sequences of sets $\{\Lambda_n : n \geq 1\}$ with uniform separation, i.e.

$$\inf_n \operatorname{sep}(\Lambda_n) = \inf\{|\lambda - \lambda'| : \lambda \neq \lambda', \lambda, \lambda' \in \Lambda_n, n \geq 1\} > 0,$$

the convergence $\Lambda_n \xrightarrow{w} \Lambda$ is equivalent to the convergence $\sqcup\!\sqcup_{\Lambda_n} \to \sqcup\!\sqcup_{\Lambda}$ in $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))$. For sequences without uniform separation the situation is slightly more technical because of possible multiplicities in the limit set.

Lemma 4.5. *Let $\{\Lambda_n : n \geq 1\}$ be a sequence of relatively separated sets in $\mathbb{R}^d$. Then the following hold.*

(a) *If $\sqcup\!\sqcup_{\Lambda_n} \longrightarrow \mu$ in $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))$ for some measure $\mu \in W(\mathcal{M}, L^\infty)$, then $\sup_n \mathrm{rel}(\Lambda_n) < \infty$ and $\Lambda_n \xrightarrow{w} \Lambda := \mathrm{supp}(\mu)$.*

(b) *If $\limsup_n \mathrm{rel}(\Lambda_n) < \infty$, then there exists a subsequence $\{\Lambda_{n_k} : k \geq 1\}$ that converges weakly to a relatively separated set.*

(c) *If $\limsup_n \mathrm{rel}(\Lambda_n) < \infty$ and $\Lambda_n \xrightarrow{w} \Lambda$, for some set $\Lambda \subseteq \mathbb{R}^d$, then Λ is relatively separated (and in particular closed).*

The lemma follows easily from Lemma 4.4, (18) and the weak*-compactness of the ball of $W(\mathcal{M}, L^\infty)$, and hence we do not prove it. We remark that the limiting measure μ in the lemma is not necessarily $\sqcup\!\sqcup_{\Lambda}$. For example, if $d = 1$ and $\Lambda_n := \{0, 1/n, 1, 1 + 1/n, 1 - 1/n\}$, then $\sqcup\!\sqcup_{\Lambda_n} \longrightarrow 2\delta_0 + 3\delta_1$. In this case we can interpret μ as representing a set with multiplicities.

The following lemma provides a version of (18) for linear combinations of point measures.

Lemma 4.6. *Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set and consider a measure*

$$\mu := \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda$$

with coefficients $c_\lambda \in \mathbb{C}$. Then

$$\begin{aligned} &\|\mu\| = |\mu|(\mathbb{R}^d) = \|c\|_1, \\ &\|c\|_\infty \leq \|\mu\|_{W(\mathcal{M}, L^\infty)} \lesssim \mathrm{rel}(\Lambda)\|c\|_\infty. \end{aligned}$$

Proof. The identity $|\mu|(\mathbb{R}^d) = \|c\|_1$ is elementary. The estimate for $\|\mu\|_{W(\mathcal{M}, L^\infty)}$ follows from the fact that, for all $\lambda \in \Lambda$, $|c_\lambda|\,\delta_\lambda \leq |\mu| \leq \|c\|_\infty \sqcup\!\sqcup_\Lambda$, where $\sqcup\!\sqcup_\Lambda$ is defined by (17). ■

5. Gabor Frames and Gabor Riesz Sequences without Inequalities

As a first step towards the main results, we characterize frames and Riesz bases in terms of uniqueness properties for certain limit sequences. The corresponding results for lattices have been derived by different methods in [26]. For the proofs we combine Theorem 3.2 with Beurling's methods [7, p.351-365].

For a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$, let $W(\Lambda)$ be the set of weak limits of the translated sets $\Lambda + z, z \in \mathbb{R}^{2d}$, i.e., $\Gamma \in W(\Lambda)$ if there exists a sequence $\{z_n : n \in \mathbb{N}\}$ such that $\Lambda + z_n \xrightarrow{w} \Gamma$. It is easy to see that then Γ is always relatively separated. When Λ is a lattice, i.e., $\Lambda = A\mathbb{Z}^{2d}$ for an invertible real-valued $2d \times 2d$-matrix A, then $W(\Lambda)$ consists only of translates of Λ.

Throughout this section we use repeatedly the following special case of Lemma 4.5(b,c): given a relatively separated set $\Lambda \subseteq \mathbb{R}^{2d}$ and any sequence of points $\{z_n : n \geq 1\} \subseteq \mathbb{R}^{2d}$, there is a subsequence $\{z_{n_k} : k \geq 1\}$ and a relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$ such that $\Lambda + z_{n_k} \xrightarrow{w} \Gamma$.

5.1. **Characterization of frames.** In this section we characterize the frame property of Gabor systems in terms of the sets in $W(\Lambda)$.

Theorem 5.1. *Assume that $g \in M^1(\mathbb{R}^d)$ and that $\Lambda \subseteq \mathbb{R}^{2d}$ is relatively separated. Then the following are equivalent.*

(i) *$\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.*
(ii) *$\mathcal{G}(g, \Lambda)$ is a p-frame for $M^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$ (for all $p \in [1, \infty]$).*
(iii) *$\mathcal{G}(g, \Lambda)$ is a ∞-frame for $M^\infty(\mathbb{R}^d)$.*
(iv) *$C^*_{g,\Lambda}$ is surjective from $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$.*
(v) *$C_{g,\Gamma}$ is bounded below on $M^\infty(\mathbb{R}^d)$ for every weak limit $\Gamma \in W(\Lambda)$.*
(vi) *$C_{g,\Gamma}$ is one-to-one on $M^\infty(\mathbb{R}^d)$ for every weak limit $\Gamma \in W(\Lambda)$.*

Remark 5.2. 1. When Λ is a lattice, then $W(\Lambda)$ consists only of translates of Λ. In this case, Theorem 5.1 reduces to main result in [26].
2. In the more general context of sampling measures, the implication $(v) \Rightarrow (i)$ was recently shown by Ascensi [2] with completely different methods.

Proof. The equivalence of (i), (ii) and (iii) follows immediately from Corollary 3.3.

In the sequel we will use several times the following version of the closed range theorem [11, p. 166]: Let $T : X \to Y$ be a bounded operator between two Banach spaces X and Y. Then T is onto Y, if and only if $T^* : Y^* \to X^*$ is one-to-one on Y^* and has closed range in X^*, if and only if T^* is bounded below.

Conditions (iii) and (iv) are equivalent by applying the closed range theorem to the synthesis operator $C^*_{g,\Lambda}$ on $\ell^1(\Lambda)$.

For the remaining equivalences we adapt Beurling's methods.

(iv) $\Rightarrow$ (v). Consider a convergent sequence of translates $\Lambda - z_n \xrightarrow{w} \Gamma$. Since $C^*_{g,\Lambda}$ maps $\ell^1(\Lambda)$ onto $M^1(\mathbb{R}^d)$, because of (12) and the open mapping theorem, the synthesis operators $C^*_{g,\Lambda - z_n}$ are also onto $M^1(\mathbb{R}^d)$ with bounds on preimages independent of n. Thus for every $f \in M^1(\mathbb{R}^d)$ there exist sequences $\{c^n_\lambda\}_{\lambda \in \Lambda - z_n}$ with $\|c^n\|_1 \lesssim 1$ such that

$$f = \sum_{\lambda \in \Lambda - z_n} c^n_\lambda \pi(\lambda) g,$$

with convergence in $M^1(\mathbb{R}^d)$.

Consider the measures $\mu_n := \sum_{\lambda \in \Lambda - z_n} c^n_\lambda \delta_\lambda$. Note that $\|\mu_n\| = \|c^n\|_1 \lesssim 1$. By passing to a subsequence we may assume that $\mu_n \longrightarrow \mu$ in $\sigma(\mathcal{M}, C_0)$, for some measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$.

By assumption $\operatorname{supp}(\mu_n) \subseteq \Lambda - z_n$, $\Lambda - z_n \xrightarrow{w} \Gamma$, and Γ is relatively separated and thus closed. It follows from Lemma 4.3 that $\operatorname{supp}(\mu) \subseteq \Gamma$. Hence,

$$\mu = \sum_{\lambda \in \Gamma} c_\lambda \delta_\lambda$$

for some sequence c. In addition, $\|c\|_1 = \|\mu\| \le \liminf_n \|\mu_n\| \lesssim 1$. Let $f' := \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g$. This is well-defined in $M^1(\mathbb{R}^d)$, because $c \in \ell^1(\Gamma)$. Let $z \in \mathbb{R}^{2d}$. Since by Lemma 2.1 $V_g\pi(z)g \in W(C_0, L^1)(\mathbb{R}^{2d}) \subseteq C_0(\mathbb{R}^{2d})$ we can compute

$$\begin{aligned}\langle f, \pi(z)g\rangle &= \sum_{\lambda\in\Lambda - z_n} c^n_\lambda \overline{V_g\pi(z)g(\lambda)} \\ &= \int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu_n \longrightarrow \int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu = \langle f', \pi(z)g\rangle\,.\end{aligned}$$

(Here, the interchange of summation and integration is justified because c and c^n are summable.) Hence $f = f'$ and thus $C^*_{g,\Gamma} : \ell^1(\Gamma) \to M^1(\mathbb{R}^d)$ is surjective. By duality $C_{g,\Gamma}$ is one-to-one from $M^\infty(\mathbb{R}^d)$ to $\ell^\infty(\Gamma)$ and has closed range, whence $C_{g,\Gamma}$ is bounded below on $M^\infty(\infty)$.

(v) $\Rightarrow$ **(vi)** is clear.

(vi) $\Rightarrow$ **(iii).** Suppose $\mathcal{G}(g, \Lambda)$ is not an ∞-frame for $M^\infty(\mathbb{R}^d)$. Then there exists a sequence of functions $\{f_n : n \ge 1\} \subset M^\infty(\mathbb{R}^d)$ such that $\|V_g f_n\|_\infty = 1$ and $\sup_{\lambda\in\Lambda} |V_g f_n(\lambda)| \longrightarrow 0$. Let $z_n \in \mathbb{R}^{2d}$ be such that $|V_g f_n(z_n)| \ge 1/2$ and consider $h_n := \pi(-z_n) f_n$. By passing to a subsequence we may assume that $h_n \longrightarrow h$ in $\sigma(M^\infty, M^1)$ for some $h \in M^\infty(\mathbb{R}^d)$, and that $\Lambda - z_n \xrightarrow{w} \Gamma$ for some relatively separated Γ. Since $|V_g h_n(0)| = |V_g f_n(z_n)| \ge 1/2$ by (10), it follows that $h \ne 0$. Given $\gamma \in \Gamma$, there exists a sequence $\{\lambda_n : n \ge 1\} \subseteq \Lambda$ such that $\lambda_n - z_n \longrightarrow \gamma$. Since, by Lemma 2.1, $V_g h_n \longrightarrow V_g h$ uniformly on compact sets, we can use (10) to obtain that

$$|V_g h(\gamma)| = \lim_n |V_g h_n(\lambda_n - z_n)| = \lim_n |V_g f_n(\lambda_n)| = 0.$$

As $\gamma \in \Gamma$ is arbitrary, this contradicts (vi). ■

Although Theorem 5.1 seems to be purely qualitative, it can be used to derive quantitative estimates for Gabor frames. We fix a non-zero window g in $M^1(\mathbb{R}^d)$ and assume that $\|g\|_2 = 1$. We measure the modulation space norms with respect to this window by $\|f\|_{M^p} = \|V_g f\|_p$ and observe that the isometry property of the short-time Fourier transform extends to $M^\infty(\mathbb{R}^d)$ as follows: if $f \in M^\infty(\mathbb{R}^d)$ and $h \in M^1(\mathbb{R}^d)$, then

$$\langle f, h\rangle = \int_{\mathbb{R}^{2d}} V_g f(z) \overline{V_g h(z)}\, dz = \langle V_g f, V_g h\rangle. \tag{19}$$

For $\delta > 0$, we define the M^1-modulus of continuity of g as

$$\omega_\delta(g)_{M^1} = \sup_{\substack{z,w\in\mathbb{R}^{2d}\\ |z-w|\le\delta}} \|\pi(z)g - \pi(w)g\|_{M^1} = \sup_{\substack{z,w\in\mathbb{R}^{2d}\\ |z-w|\le\delta}} \|V_g(\pi(z)g - \pi(w)g)\|_{L^1}. \tag{20}$$

It is easy to verify that $\lim_{\delta\to 0+} \omega_\delta(g)_{M^1} = 0$, because time-frequency shifts are continuous on $M^1(\mathbb{R}^d)$.

Then we deduce the following quantitative conditions for Gabor frames from Theorem 5.1.

Corollary 5.3. *For $g \in M^1(\mathbb{R}^d)$ with $\|g\|_2 = 1$ choose $\delta > 0$ so that $\omega_\delta(g)_{M^1} < 1$. If $\Lambda \subseteq \mathbb{R}^{2d}$ is relatively separated and $\rho(\Lambda) < \delta$, then $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.*

Proof. We argue by contradiction and assume that $\mathcal{G}(g,\Lambda)$ is not a frame. By condition (vi) of Theorem 5.1 there exists a weak limit $\Gamma \in W(\Lambda)$ and $f \in M^\infty(\mathbb{R}^d)$, such that $V_g f\big|_\Gamma = 0$. Since $\rho(\Lambda) < \delta$, we also have $\rho(\Gamma) \le \delta$. By normalizing, we may assume that $\|f\|_{M^\infty} = \|V_g f\|_\infty = 1$. For $0 < \epsilon < 1 - \omega_\delta(g)_{M^1}$ we find $z \in \mathbb{R}^{2d}$ such that $|V_g f(z)| = |\langle f, \pi(z) g\rangle| > 1 - \epsilon$. Since $\rho(\Gamma) \le \delta$, there is a $\gamma \in \Gamma$ such that $|z - \gamma| \le \delta$. Consequently, since $V_g f\big|_\Gamma = 0$, we find that

$$\begin{aligned}
1 - \epsilon &< |\langle f, \pi(z) g\rangle - \langle f, \pi(\gamma) g\rangle| = |\langle f, \pi(z) g - \pi(\gamma) g\rangle| \\
&= |\langle V_g f, V_g(\pi(z) g - \pi(\gamma) g)\rangle| \\
&\le \|V_g f\|_\infty \, \|V_g(\pi(z) g - \pi(\gamma) g)\|_1 \\
&= \|f\|_{M^\infty} \, \|\pi(z) g - \pi(\gamma) g\|_{M^1} \\
&\le \omega_\delta(g)_{M^1} \,.
\end{aligned}$$

Since we have chosen $1 - \epsilon > \omega_\delta(g)_{M^1}$, we have arrived at a contradiction. Thus $\mathcal{G}(g,\Lambda)$ is a frame. ∎

This theorem is analogous to Beurling's famous sampling theorem for multivariate bandlimited functions [6]. The proof is in the style of [29].

5.2. **Characterization of Riesz sequences.** We now derive analogous results for Riesz sequences. The existence of a result of this type for interpolating sequences in the context of bandlimited functions had been conjectured by Beurling [7, Problem 3, p. 359]. In our context this can established because (i) and (ii) are equivalent in Theorem 5.4 below. The analogous statement for bandlimited functions is not true.

Theorem 5.4. *Assume that $g \in M^1(\mathbb{R}^d)$ and that $\Lambda \subseteq \mathbb{R}^{2d}$ is separated. Then the following are equivalent.*

(i) *$\mathcal{G}(g,\Lambda)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$.*
(ii) *$\mathcal{G}(g,\Lambda)$ is a p-Riesz sequence in $M^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$ (for all $p \in [1,\infty]$).*
(iii) *$\mathcal{G}(g,\Lambda)$ is a ∞-Riesz sequence in $M^\infty(\mathbb{R}^d)$, i.e., $C^*_{g,\Lambda} : \ell^\infty(\Lambda) \to M^\infty(\mathbb{R}^d)$ is bounded below.*
(iv) *$C_{g,\Lambda} : M^1 \to \ell^1(\Lambda)$ is surjective.*
(v) *$C^*_{g,\Gamma} : \ell^\infty(\Gamma) \to M^\infty(\mathbb{R}^d)$ is bounded below for every weak limit $\Gamma \in W(\Lambda)$.*
(vi) *$C^*_{g,\Gamma} : \ell^\infty(\Gamma) \to M^\infty(\mathbb{R}^d)$ is one-to-one for every weak limit $\Gamma \in W(\Lambda)$.*

Remark 5.5. Note that we are assuming that Λ is separated. This is necessarily the case if $\mathcal{G}(g,\Lambda)$ is a Riesz sequence (Lemma 2.2), but needs to be assumed in some of the other conditions.

Before proving Theorem 5.4, we prove the following continuity property of $C^*_{g,\Lambda}$ with respect to Λ.

Lemma 5.6. *Let $g \in M^1(\mathbb{R}^d), g \neq 0$ and let $\{\Lambda_n : n \geq 1\}$ be a sequence of uniformly separated subsets of $\mathbb{R}^{2d}$, i.e.,*

$$\inf_n \operatorname{sep}(\Lambda_n) = \delta > 0\,. \tag{21}$$

For every $n \in \mathbb{N}$, let $c^n \in \ell^\infty(\Lambda_n)$ be such that $\|c^n\|_\infty = 1$ and suppose that

$$\sum_{\lambda\in\Lambda_n} c^n_\lambda \pi(\lambda) g \longrightarrow 0 \text{ in } M^\infty(\mathbb{R}^d), \qquad \text{as } n \longrightarrow \infty.$$

Then there exist a subsequence $(n_k) \subset \mathbb{N}$, points $\lambda_{n_k} \in \Lambda_{n_k}$, a separated set $\Gamma \subseteq \mathbb{R}^{2d}$ and a non-zero sequence $c \in \ell^\infty(\Gamma)$ such that

$$\Lambda_{n_k} - \lambda_{n_k} \xrightarrow{w} \Gamma, \qquad \text{as } k \longrightarrow \infty$$

$$\text{and} \qquad \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g = 0.$$

Proof. Combining the hypothesis (21) and observation (4), we also have the uniform relative separation

$$\sup_n \operatorname{rel}(\Lambda_n) < \infty. \tag{22}$$

Since $\|c^n\|_\infty = 1$ for every $n \geq 1$, we may choose $\lambda_n \in \Lambda_n$ be such that $\left|c^n_{\lambda_n}\right| \geq 1/2$. Let $\theta_{\lambda,n} \in \mathbb{C}$ such that

$$\theta_{\lambda,n}\pi(\lambda - \lambda_n) = \pi(-\lambda_n)\pi(\lambda),$$

and consider the measures $\mu_n := \sum_{\lambda\in\Lambda_n} \theta_{\lambda,n} c^n_\lambda \delta_{\lambda-\lambda_n}$. Then by Lemma 4.6, $\|\mu_n\|_{W(\mathcal{M},L^\infty)} \lesssim \operatorname{rel}(\Lambda_n - \lambda_n)\|c^n\|_\infty = \operatorname{rel}(\Lambda_n)\|c^n\|_\infty \lesssim 1$. Using (22) and Lemma 4.5, we may pass to a subsequence such that (i) $\Lambda_n - \lambda_n \xrightarrow{w} \Gamma$ for some relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$ and (ii) $\mu_n \longrightarrow \mu$ in $\sigma(W(\mathcal{M}, L^\infty), W(C_0, L^1))(\mathbb{R}^{2d})$ for some measure $\mu \in W(\mathcal{M}, L^\infty)(\mathbb{R}^{2d})$. The uniform separation condition in (21) implies that Γ is also separated.

Since $\operatorname{supp}(\mu_n) \subseteq \Lambda_n - \lambda_n$ it follows from Lemma 4.3 that $\operatorname{supp}(\mu) \subseteq \overline{\Gamma} = \Gamma$. Hence,

$$\mu = \sum_{\lambda\in\Gamma} c_\lambda \delta_\lambda,$$

for some sequence of complex numbers c, and, by Lemma 4.6, $\|c\|_\infty \leq \|\mu\|_{W(\mathcal{M},L^\infty)} < \infty$.

From (21) it follows that for all $n \in \mathbb{N}$, $B_{\delta/2}(\lambda_n) \cap \Lambda_n = \{\lambda_n\}$. Let $\varphi \in C(\mathbb{R}^{2d})$ be real-valued, supported on $B_{\delta/2}(0)$ and such that $\varphi(0) = 1$. Then

$$\left|\int_{\mathbb{R}^{2d}} \varphi\, d\mu\right| = \lim_n \left|\int_{\mathbb{R}^{2d}} \varphi\, d\mu_n\right| = \lim_n \left|c^n_{\lambda_n}\right| \geq 1/2.$$

Hence $\mu \neq 0$ and therefore $c \neq 0$.

Finally, we show that the short-time Fourier transform of $\sum_\lambda c_\lambda \pi(\lambda)g$ is zero. Let $z \in \mathbb{R}^{2d}$ be arbitrary and recall that by Lemma 2.1 $V_g\pi(z)g \in W(C_0, L^1)(\mathbb{R}^{2d})$. Now we estimate

$$\begin{aligned}
\left|\left\langle \sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda)g, \pi(z)g \right\rangle\right| &= \left|\sum_{\lambda\in\Gamma} c_\lambda \overline{V_g\pi(z)g}(\lambda)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu\right| = \lim_n \left|\int_{\mathbb{R}^{2d}} \overline{V_g\pi(z)g}\, d\mu_n\right| \\
&= \lim_n \left|\left\langle \sum_{\lambda\in\Lambda_n} \theta_{\lambda,n} c^n_\lambda \pi(\lambda-\lambda_n)g, \pi(z)g \right\rangle\right| \\
&\le \lim_n \Big\| \sum_{\lambda\in\Lambda_n} \theta_{\lambda,n} c^n_\lambda \pi(\lambda-\lambda_n)g \Big\|_{M^\infty} \big\|g\big\|_{M^1} \\
&= \lim_n \Big\| \pi(-\lambda_n) \sum_{\lambda\in\Lambda_n} c^n_\lambda \pi(\lambda)g \Big\|_{M^\infty} \big\|g\big\|_{M^1} \\
&= \lim_n \Big\| \sum_{\lambda\in\Lambda_n} c^n_\lambda \pi(\lambda)g \Big\|_{M^\infty} \big\|g\big\|_{M^1} = 0.
\end{aligned}$$

We have shown that $V_g(\sum_{\lambda\in\Gamma} c_\lambda\pi(\lambda)g) \equiv 0$ and thus $\sum_{\lambda\in\Gamma} c_\lambda\pi(\lambda)g \equiv 0$, as desired. ■

Proof of Theorem 5.4. The equivalence of (i), (ii), and (iii) follows from Corollary 3.3(b), and the equivalence of (iii) and (iv) follows by duality.

(iv) ⇒ (v). Assume (iv) and consider a sequence $\Lambda - z_n \xrightarrow{w} \Gamma$. Let $\lambda \in \Gamma$ be arbitrary and let $\{\lambda_n : n \in \mathbb{N}\} \subseteq \Lambda$ be a sequence such that $\lambda_n - z_n \longrightarrow \lambda$. By the open map theorem, every sequence $c \in \ell^1(\Lambda)$ with $\|c\|_1 = 1$ has a preimage $c = C^*_{g,\Lambda}(f)$ with $\|f\|_{M^1} \lesssim 1$. With the covariance property (12) we deduce that there exist $f_n \in M^1(\mathbb{R}^d)$, such that $c = C^*_{g,\Lambda-z_n}(f_n)$ and $\|f_n\|_{M^1} \lesssim 1$.

In particular, for each $n \in \mathbb{N}$ there exists an interpolating function $h_n \in M^1(\mathbb{R}^d)$ such that $\|V_g h_n\|_1 \lesssim 1$, $V_g h_n(\lambda_n - z_n) = 1$ and $V_g h_n \equiv 0$ on $\Lambda - z_n \setminus \{\lambda_n - z_n\}$. By passing to a subsequence we may assume that $h_n \longrightarrow h$ in $\sigma(M^1, M^0)$. It follows that $\|h\|_{M^1} \lesssim 1$. Since $V_g h_n \longrightarrow V_g h$ uniformly on compact sets by Lemma 2.1, we obtain that

$$V_g h(\lambda) = \lim_n V_g h_n(\lambda_n - z_n) = 1.$$

Similarly, given $\gamma \in \Gamma \setminus \{\lambda\}$, there exists a sequence $\{\gamma_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that $\gamma_n - z_n \longrightarrow \gamma$. Since $\lambda \neq \gamma$, for $n \gg 0$ we have that $\gamma_n \neq \lambda_n$ and consequently $V_g h_n(\gamma_n - z_n) = 0$. It follows that $V_g h(\gamma) = 0$.

Hence, we have shown that for each $\lambda \in \Gamma$ there exists an interpolating function $h_\lambda \in M^1(\mathbb{R}^d)$ such that $\|h_\lambda\|_{M^1} \lesssim 1$, $V_g h_\lambda(\lambda) = 1$ and $V_g h_\lambda \equiv 0$ on $\Gamma \setminus \{\lambda\}$. Given an arbitrary sequence $c \in \ell^1(\Gamma)$ we consider

$$f := \sum_{\lambda\in\Gamma} c_\lambda h_\lambda.$$

It follows that $f \in M^1(\mathbb{R}^d)$ and that $C_{g,\Gamma} f = c$. Hence, $C_{g,\Gamma}$ is onto $\ell^1(\Gamma)$, and therefore $C^*_{g,\Gamma}$ is bounded below.

(v) $\Rightarrow$ **(vi)** is clear.

(vi) $\Rightarrow$ **(iii)**. Suppose that (iii) does not hold. Then there exists a sequence $\{c^n : n \in \mathbb{N}\} \subseteq \ell^\infty(\Lambda)$ such that $\|c^n\|_\infty = 1$ and

$$\Big\|\sum_{\lambda\in\Lambda} c^n_\lambda \pi(\lambda) g\Big\|_{M^\infty} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

We now apply Lemma 5.6 with $\Lambda_n := \Lambda$ and obtain a set $\Gamma \in W(\Lambda)$ and a non-zero sequence $c \in \ell^\infty(\Gamma)$ such that $\sum_{\lambda\in\Gamma} c_\lambda \pi(\lambda) g = C^*_{g,\Gamma}(c) = 0$. This contradicts (vi). ■

6. Deformation of sets and Lipschitz convergence

The characterizations of Theorem 5.1 suggest that Gabor frames are invariant under "weak deformations" of Λ. One might expect that if $\mathcal{G}(g,\Lambda)$ is a frame and Λ' is close to Λ in the weak sense, then $\mathcal{G}(g,\Lambda')$ is also a frame. This view is too simplistic. Just choose $\Lambda_n = \Lambda \cap B_n(0)$, then $\Lambda_n \xrightarrow{w} \Lambda$, but Λ_n is a finite set and thus $\mathcal{G}(g,\Lambda_n)$ is never a frame. For a deformation result we need to introduce a finer notion of convergence.

Let $\Lambda \subseteq \mathbb{R}^d$ be a (countable) set. We consider a sequence $\{\Lambda_n : n \geq 1\}$ of subsets of $\mathbb{R}^d$ produced in the following way. For each $n \geq 1$, let $\tau_n : \Lambda \to \mathbb{R}^d$ be a map and let $\Lambda_n := \tau_n(\Lambda) = \{\tau_n(\lambda) : \lambda \in \Lambda\}$. We assume that $\tau_n(\lambda) \longrightarrow \lambda$, as $n \longrightarrow \infty$, for all $\lambda \in \Lambda$. The sequence of sets $\{\Lambda_n : n \geq 1\}$ together with the maps $\{\tau_n : n \geq 1\}$ is called a *deformation* of Λ. We think of each sequence of points $\{\tau_n(\lambda) : n \geq 1\}$ as a (discrete) path moving towards the endpoint λ.

We will often say that $\{\Lambda_n : n \geq 1\}$ is a deformation of Λ, with the understanding that a sequence of underlying maps $\{\tau_n : n \geq 1\}$ is also given.

Definition 6.1. *(a) A deformation $\{\Lambda_n : n \geq 1\}$ of Λ is called* Lipschitz*, denoted by $\Lambda_n \xrightarrow{Lip} \Lambda$, if the following two conditions hold:*

(L1) *Given $R > 0$,*

$$\sup_{\substack{\lambda,\lambda'\in\Lambda \\ |\lambda-\lambda'|\leq R}} |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| \to 0, \quad \textit{as } n \longrightarrow \infty.$$

(L2) *Given $R > 0$, there exists $R' > 0$ and $n_0 \in \mathbb{N}$ such that if $|\tau_n(\lambda) - \tau_n(\lambda')| \leq R$ for* some *$n \geq n_0$ and some $\lambda, \lambda' \in \Lambda$, then $|\lambda - \lambda'| \leq R'$.*

Condition (L1) means that $\tau_n(\lambda) - \tau_n(\lambda') \longrightarrow \lambda - \lambda'$ uniformly in $|\lambda - \lambda'|$. In particular, by fixing λ', we see that Lipschitz convergence implies the weak convergence $\Lambda_n \xrightarrow{w} \Lambda$. Furthermore, if $\{\Lambda_n : n \geq 1\}$ is Lipschitz convergent to Λ, then so is every subsequence $\{\Lambda_{n_k} : k \geq 1\}$.

Example 6.2. *Jitter error:* Let $\Lambda \subseteq \mathbb{R}^d$ be relatively separated and let $\{\Lambda_n : n \geq 1\}$ be a deformation of Λ. If $\sup_\lambda |\tau_n(\lambda) - \lambda| \longrightarrow 0$, as $n \longrightarrow \infty$, then $\Lambda_n \xrightarrow{Lip} \Lambda$.

Example 6.3. *Linear deformations:* Let $\Lambda = A\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$, with A an invertible $2d \times 2d$ matrix, $\Lambda_n = A_n\mathbb{Z}^{2d}$ for a sequence of invertible $2d \times 2d$-matrices and assume that $\lim A_n = A$. Then $\Lambda_n \xrightarrow{Lip} \Lambda$. In this case conditions (L1) and (L2) are easily checked by taking $\tau_n = A_n A^{-1}$.

The third class of examples contains differentiable, nonlinear deformations.

Lemma 6.4. *Let $p \in (d, \infty]$. For each $n \in \mathbb{N}$, let $T_n = (T_n^1, \dots, T_n^d) : \mathbb{R}^d \to \mathbb{R}^d$ be a map such that each coordinate function $T_n^k : \mathbb{R}^d \to \mathbb{R}$ is continuous, locally integrable and has a weak derivative in $L^p_{loc}(\mathbb{R}^d)$. Assume that*

$$\begin{aligned} &T_n(0) = 0, \\ &|DT_n - I| \longrightarrow 0 \quad \text{in } L^p(\mathbb{R}^d). \end{aligned}$$

(Here, DT_n is the Jacobian matrix consisting of the partial derivatives of T_n and the second condition means that each entry of the matrix $DT_n - I$ tends to 0 in L^p.)

Let $\Lambda \subseteq \mathbb{R}^d$ be a relatively separated set and consider the deformation $\Lambda_n := T_n(\Lambda)$ (i.e $\tau_n := T_n\big|_{\Lambda}$). Then Λ_n is Lipschitz convergent to Λ.

Remark 6.5. In particular, the hypothesis of Lemma 6.4 is satisfied by every sequence of differentiable maps $T_n : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\begin{aligned} &T_n(0) = 0, \\ &\sup_{z \in \mathbb{R}^{2d}} |DT_n(z) - I| \longrightarrow 0. \end{aligned}$$

Proof of Lemma 6.4. Let $\alpha := 1 - \frac{d}{p} \in (0, 1]$. We use the following Sobolev embedding known as Morrey's inequality (see for example [16, Chapter 4, Theorem 3]). If $f : \mathbb{R}^d \to \mathbb{R}$ is locally integrable and possesses a weak derivative in $L^p(\mathbb{R}^d)$, then f is α-Hölder continuous (after being redefined on a set of measure zero). If $x, y \in \mathbb{R}^d$, then

$$|f(x) - f(y)| \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)} |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d.$$

Applying Morrey's inequality to each coordinate function of $T_n - I$ we obtain that there is a constant $C > 0$ such that

$$|(T_n x - T_n y) - (x - y)| = |(T_n - I)x - (T_n - I)y| \le C\|DT_n - I\|_{L^p(\mathbb{R}^d)} |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d.$$

Let $\epsilon_n = C\|DT_n - I\|_{L^p(\mathbb{R}^d)}$, where $\|DT_n - I\|_{L^p(\mathbb{R}^d)}$ is the L^p-norm of $|DT_n(\cdot) - I|$. Then $\epsilon_n \to 0$ by assumption and

$$|(T_n x - T_n y) - (x - y)| \le \varepsilon_n |x - y|^\alpha, \qquad x, y \in \mathbb{R}^d. \tag{23}$$

Choose $x = \lambda$ and $y = 0$, then $T_n(\lambda) \longrightarrow \lambda$ for all $\lambda \in \Lambda$ (since $T_n(0) = 0$). Hence Λ_n is a deformation of Λ.

If $\lambda, \lambda' \in \Lambda$ and $|\lambda - \lambda'| \le R$, then (23) implies that

$$|(T_n\lambda - T_n\lambda') - (\lambda - \lambda')| \le \varepsilon_n R^\alpha.$$

Thus condition (L1) is satisfied.

For condition (L2), choose n_0 such that $\varepsilon_n \leq 1/2$ for $n \geq n_0$. If $|\lambda - \lambda'| \leq 1$, there is nothing to show (choose $R' \geq 1$). If $|\lambda - \lambda'| \geq 1$ and $|T_n\lambda - T_n\lambda'| \leq R$ for some $n \geq n_0$, $\lambda, \lambda' \in \Lambda$, then by (23) we obtain

$$|(T_n\lambda - T_n\lambda') - (\lambda - \lambda')| \leq 1/2\, |\lambda - \lambda'|^\alpha \leq 1/2\, |\lambda - \lambda'| .$$

This implies that

$$|\lambda - \lambda'| \leq 2\, |(T_n\lambda - T_n\lambda')| , \text{ for all } n \geq n_0. \tag{24}$$

Since $|T_n\lambda - T_n\lambda'| \leq R$, we conclude that $|\lambda - \lambda'| \leq 2R$, and we may actually choose $R' = \max(1, 2R)$ in condition (L2). ■

Remark 6.6. Property (L1) can be proved under slightly weaker conditions on the convergence of $DT_n - I$. In fact, we need (23) to hold only for $|x - y| \leq R$. Thus it suffices to assume locally uniform convergence in L^p, i.e.,

$$\sup_{y \in \mathbb{R}^{2d}} \int_{B_R(y)} |DT_n(x) - I|^p \, dx \to 0$$

for all $R > 0$.

We prove some technical lemmas about Lipschitz convergence.

Lemma 6.7. *Let $\{\Lambda_n : n \geq 1\}$ be a deformation of a relatively separated set $\Lambda \subseteq \mathbb{R}^d$. Then the following hold.*

(a) *If Λ_n is Lipschitz convergent to Λ and* $\operatorname{sep}(\Lambda) > 0$*, then* $\liminf_n \operatorname{sep}(\Lambda) > 0$*.*
(b) *If Λ_n is Lipschitz convergent to Λ, then* $\limsup_n \operatorname{rel}(\Lambda_n) < \infty$*.*
(c) *If Λ_n is Lipschitz convergent to Λ and $\rho(\Lambda) < \infty$, then* $\limsup_n \rho(\Lambda_n) < \infty$*.*

Proof. (a) By assumption $\delta := \operatorname{sep}(\Lambda) > 0$. Using (L2), let $n_0 \in \mathbb{N}$ and $R' > 0$ be such that if $|\tau_n(\lambda) - \tau_n(\lambda')| \leq \delta/2$ for some $\lambda, \lambda' \in \Lambda$ and $n \geq n_0$, then $|\lambda - \lambda'| \leq R'$. By (L1), choose $n_1 \geq n_0$ such that for $n \geq n_1$

$$\sup_{|\lambda - \lambda'| \leq R'} |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| < \delta/2.$$

Claim. $\operatorname{sep}(\Lambda_n) \geq \delta/2$ for $n \geq n_1$.

If the claim is not true, then for some $n \geq n_0$ there exist two distinct points $\lambda, \lambda' \in \Lambda$ such that $|\tau_n(\lambda) - \tau_n(\lambda')| \leq \delta/2$. Then $|\lambda - \lambda'| \leq R'$ and consequently

$$|\lambda - \lambda'| \leq |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| + |\tau_n(\lambda) - \tau_n(\lambda')| < \delta,$$

contradicting the fact that Λ is δ-separated.

(b) Since Λ is relatively separated we can split it into finitely many separated sets $\Lambda = \Lambda^1 \cup \ldots \cup \Lambda^L$ with $sep(\Lambda^k) > 0$.

Consider the sets defined by restricting the deformation τ_n to each Λ^k

$$\Lambda_n^k := \left\{\tau_n(\lambda) : \lambda \in \Lambda^k\right\} .$$

As proved above in (a), there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\operatorname{sep}(\Lambda_n^k) \geq \delta$ for all $n \geq n_0$ and $1 \leq k \leq L$. Therefore, using (4),

$$\operatorname{rel}(\Lambda_n) \leq \sum_{k=1}^{L} \operatorname{rel}(\Lambda_n^k) \lesssim L\delta^{-d}, \qquad n \geq n_0,$$

and the conclusion follows.

(c) By (b) we may assume that each Λ_n is relatively separated. Assume that $\rho(\Lambda) < \infty$. Then there exists $r > 0$ such that every cube $Q_r(z) := z + [-r, r]^d$ intersects Λ. By (L1), there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\sup_{\substack{\lambda,\lambda' \in \Lambda \\ |\lambda - \lambda'|_\infty \leq 6r}} \left|(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')\right|_\infty \leq r. \tag{25}$$

Let $R := 8r$ and $n \geq n_0$. We will show that every cube $Q_R(z)$ intersects Λ_n. This will give a uniform upper bound for $\rho(\Lambda_n)$. Suppose on the contrary that some cube $Q_R(z)$ does not meet Λ_n and consider a larger radius $R' \geq R$ such that Λ_n intersects the boundary but not the interior of $Q_{R'}(z)$. (This is possible because Λ_n is relatively separated and therefore closed.) Hence, there exists $\lambda \in \Lambda$ such that $|\tau_n(\lambda) - z|_\infty = R'$. Let us write

$$\begin{aligned}
&(z - \tau_n(\lambda))_k = \delta_k c_k, \qquad k = 1, \dots, d, \\
&\delta_k \in \{-1, 1\}, \qquad k = 1, \dots, d, \\
&0 \leq c_k \leq R', \qquad k = 1, \dots, d\,,
\end{aligned}$$

and $c_k = R'$ for some k. We now argue that we can select a point $\gamma \in \Lambda$ such that

$$(\lambda - \gamma)_k = -\delta_k c'_k, \qquad k = 1, \dots, d, \tag{26}$$

with coordinates

$$2r \leq c'_k \leq 6r, \qquad k = 1, \dots, d. \tag{27}$$

Using the fact that Λ intersects each of the cubes $\left\{Q_r(2rj) : j \in \mathbb{Z}^d\right\}$, we first select an index $j \in \mathbb{Z}^d$ such that $\lambda \in Q_r(2rj)$. Second, we define a new index $j' \in \mathbb{Z}^d$ by $j'_k = j_k + 2\delta_k$ for $k = 1, \dots, d$. We finally select a point $\gamma \in \Lambda \cap Q_r(2rj')$. This procedure guarantees that (26) and (27) hold true. See Figure 1.

Since by (26) and (27) $|\lambda - \gamma|_\infty \leq 6r$, we can use (25) to obtain

$$(\tau_n(\lambda) - \tau_n(\gamma))_k = -\delta_k c''_k, \qquad k = 1, \dots, d,$$

with coordinates

$$r \leq c''_k \leq 7r, \qquad k = 1, \dots, d.$$

We write $(z - \tau_n(\gamma))_k = (z - \tau_n(\lambda))_k + (\tau_n(\lambda) - \tau_n(\gamma))_k = \delta_k(c_k - c''_k)$ and note that $-7r \leq c_k - c''_k \leq R' - r$. Hence,

$$|z - \tau_n(\gamma)|_\infty \leq \max\{R' - r, 7r\} = R' - r,$$

since $7r = R - r \leq R' - r$. This shows that $Q_{R'-r}(z)$ intersects Λ_n, contradicting the choice of R'. ■

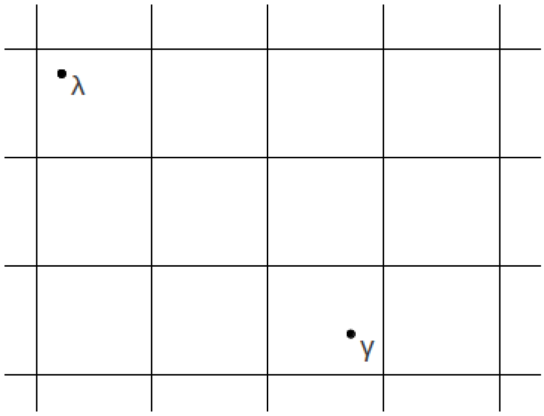

FIGURE 1.

The following lemma relates Lipschitz convergence to the weak-limit techniques.

Lemma 6.8. *Let $\Lambda \subseteq \mathbb{R}^d$ be relatively separated and let $\{\Lambda_n : n \geq 1\}$ be a Lipschitz deformation of Λ. Then the following holds.*

(a) *Let $\Gamma \subseteq \mathbb{R}^d$ and $\{\lambda_n : n \geq 1\} \subseteq \Lambda$ some sequence in Λ. If $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{w} \Gamma$, then $\Gamma \in W(\Lambda)$.*

(b) *Suppose that Λ is relatively dense and $\{z_n : n \geq 1\} \subseteq \mathbb{R}^d$ is an arbitrary sequence. If $\Lambda_n - z_n \xrightarrow{w} \Gamma$, then $\Gamma \in W(\Lambda)$.*

Proof. (a) We first note that Γ is relatively separated. Indeed, since by Lemma 6.7

$$\limsup_{n\to\infty} \operatorname{rel}(\Lambda_n - \tau_n(\lambda_n)) = \limsup_{n\to\infty} \operatorname{rel}(\Lambda_n) < \infty.$$

Lemma 4.5(c) implies that Γ is relatively separated (and in particular closed).

By extracting a subsequence, we may assume that $\Lambda - \lambda_n \xrightarrow{w} \Gamma'$ for some relatively separated set $\Gamma' \in W(\Lambda)$. We will show that $\Gamma' = \Gamma$ and consequently $\Gamma \in W(\Lambda)$.

Let $R > 0$ and $0 < \varepsilon \leq 1$ be given. By (L1), there exists $n_0 \in \mathbb{N}$ such that

$$\lambda, \lambda' \in \Lambda, |\lambda - \lambda'| \leq R, n \geq n_0 \Longrightarrow |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| \leq \varepsilon\,. \tag{28}$$

If $n \geq n_0$ and $z \in (\Lambda - \lambda_n) \cap B_R(0)$, then there exists $\lambda \in \Lambda$ such that $z = \lambda - \lambda_n$ and $|\lambda - \lambda_n| \leq R$. Consequently (28) implies that

$$|(\tau_n(\lambda) - \tau_n(\lambda_n)) - z| = |(\tau_n(\lambda) - \tau_n(\lambda_n)) - (\lambda - \lambda_n)| \leq \varepsilon.$$

This shows that

$$(\Lambda - \lambda_n) \cap B_R(0) \subseteq (\Lambda_n - \tau_n(\lambda_n)) + B_\varepsilon(0) \quad \text{for } n \geq n_0\,. \tag{29}$$

Since $\Lambda - \lambda_n \xrightarrow{w} \Gamma'$ and $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{w} \Gamma$, it follows from (29) and Lemma 4.2 that $\Gamma' \subseteq \overline{\Gamma} = \Gamma$.

For the reverse inclusion, let again $R > 0$ and $0 < \varepsilon \le 1$. Let $R' > 0$ and $n_0 \in \mathbb{N}$ be the numbers associated with R in (L2). Using (L1), choose $n_1 \ge n_0$ such that

$$\lambda, \lambda' \in \Lambda, |\lambda - \lambda'| \le R', n \ge n_1 \implies |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| \le \varepsilon \,. \tag{30}$$

If $n \ge n_1$ and $z \in (\Lambda_n - \tau_n(\lambda_n)) \cap B_R(0)$, then $z = \tau_n(\lambda) - \tau_n(\lambda_n)$ for some $\lambda \in \Lambda$ and $|\tau_n(\lambda) - \tau_n(\lambda_n)| \le R$. Condition (L2) now implies that $|\lambda - \lambda_n| \le R'$ and therefore, using (30) with $\lambda' = \lambda_n$, we get

$$|z - (\lambda - \lambda_n)| = |(\tau_n(\lambda) - \tau_n(\lambda_n)) - (\lambda - \lambda_n)| \le \varepsilon.$$

Hence we have proved that

$$(\Lambda_n - \tau_n(\lambda_n)) \cap B_R(0) \subseteq (\Lambda - \lambda_n) + B_\varepsilon(0), \text{ for } n \ge n_1 \,.$$

Since $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{w} \Gamma$ and $\Lambda - \lambda_n \xrightarrow{w} \Gamma'$, Lemma 4.2 implies that $\Gamma \subseteq \overline{\Gamma'} = \Gamma'$. In conclusion $\Gamma' = \Gamma \in W(\Lambda)$, as desired.

(b) Since $\rho(\Lambda) < \infty$, Lemma 6.7(c) implies that $\limsup_n \rho(\Lambda_n) < \infty$. By omitting finitely many n, there exists $L > 0$ such that $\Lambda_n + B_L(0) = \mathbb{R}^d$ for all $n \in \mathbb{N}$. This implies the existence of a sequence $\{\lambda_n : n \ge 1\} \subseteq \Lambda$ such that $|z_n - \tau_n(\lambda_n)| \le L$. By passing to a subsequence we may assume that $z_n - \tau_n(\lambda_n) \longrightarrow z_0$, for some $z_0 \in \mathbb{R}^d$.

Since $\Lambda_n - z_n \xrightarrow{w} \Gamma$ and $z_n - \tau_n(\lambda_n) \longrightarrow z_0$, it follows that $\Lambda_n - \tau_n(\lambda_n) \xrightarrow{w} \Gamma + z_0$. By (a), we deduce that $\Gamma + z_0 \in W(\Lambda)$ and thus $\Gamma \in W(\Lambda)$, as desired. ■

7. Deformation of Gabor systems

We now prove the main results on the deformation of Gabor systems. The proofs combine the characterization of non-uniform Gabor frames and Riesz sequences without inequalities and the fine details of Lipschitz convergence.

First we formulate the stability of Gabor frames under a class of nonlinear deformations.

Theorem 7.1. *Let $g \in M^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$ and assume that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. If Λ_n is Lipschitz convergent to Λ, then $\mathcal{G}(g, \Lambda_n)$ is a frame for all sufficiently large n.*

Theorem 1.1(a) of the Introduction now follows by combining Theorem 7.1 and Lemma 6.4. Note that in Theorem 1.1 we may assume without loss of generality that $T_n(0) = 0$, because the deformation problem is invariant under translations.

Proof. Suppose that $\mathcal{G}(g, \Lambda)$ is a frame. According to Lemma 2.2 Λ is relatively separated and relatively dense. Now suppose that the conclusion does not hold. By passing to a subsequence we may assume that $\mathcal{G}(g, \Lambda_n)$ fails to be a frame for all $n \in \mathbb{N}$.

By Theorem 5.1 every $\mathcal{G}(g, \Lambda_n)$ also fails to be a ∞-frame for $M^\infty(\mathbb{R}^d)$. It follows that for every $n \in \mathbb{N}$ there exist $f_n \in M^\infty(\mathbb{R}^d)$ such that $\|V_g f_n\|_\infty = 1$ and

$$\|C_{g,\Lambda_n}(f_n)\|_{\ell^\infty(\Lambda_n)} = \sup_{\lambda \in \Lambda} |V_g f_n(\tau_n(\lambda))| \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

For each $n \in \mathbb{N}$, let $z_n \in \mathbb{R}^{2d}$ be such that $|V_g f_n(z_n)| \ge 1/2$ and let us consider $h_n := \pi(-z_n) f_n$. By passing to a subsequence we may assume that $h_n \to h$ in $\sigma(M^\infty, M^1)$ for some function $h \in M^\infty$. Since $|V_g h_n(0)| = |V_g f_n(z_n)| \geq 1/2$, it follows that $|V_g h(0)| \geq 1/2$ and the weak*-limit h is not zero.

In addition, by Lemma 6.7

$$\limsup_{n\to\infty} \mathrm{rel}(\Lambda_n - z_n) = \limsup_{n\to\infty} \mathrm{rel}(\Lambda_n) < \infty.$$

Hence, using Lemma 4.5 and passing to a further subsequence, we may assume that $\Lambda_n - z_n \xrightarrow{w} \Gamma$, for some relatively separated set $\Gamma \subseteq \mathbb{R}^{2d}$. Since Λ is relatively dense, Lemma 6.8 guarantees that $\Gamma \in W(\Lambda)$.

Let $\gamma \in \Gamma$ be arbitrary. Since $\Lambda_n - z_n \xrightarrow{w} \Gamma$, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ such that $\tau_n(\lambda_n) - z_n \to \gamma$. By Lemma 2.1, the fact that $h_n \to h$ in $\sigma(M^\infty, M^1)$ implies that $V_g h_n \to V_g h$ uniformly on compact sets. Consequently, by (10),

$$|V_g h(\gamma)| = \lim_n |V_g h_n(\tau_n(\lambda_n) - z_n)| = \lim_n |V_g f_n(\tau_n(\lambda_n))| = 0.$$

Hence, $h \not\equiv 0$ and $V_g h \equiv 0$ on $\Gamma \in W(\Lambda)$. According to Theorem 5.1(vi), $\mathcal{G}(g, \Lambda)$ is not a frame, thus contradicting the initial assumption. ■

The corresponding deformation result for Gabor Riesz sequences reads as follows.

Theorem 7.2. *Let $g \in M^1(\mathbb{R}^d)$, $\Lambda \subseteq \mathbb{R}^{2d}$ and assume that $\mathcal{G}(g, \Lambda)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$.*

If Λ_n is Lipschitz convergent to Λ, then $\mathcal{G}(g, \Lambda_n)$ is a Riesz sequence for all sufficiently large n.

Proof. Assume that $\mathcal{G}(g, \Lambda)$ is a Riesz sequence. Lemma 2.2 implies that Λ is separated. With Lemma 6.7 we may extract a subsequence such each Λ_n is separated with a uniform separation constant, i.e.,

$$\inf_{n\geq 1} \mathrm{sep}(\Lambda_n) > 0. \tag{31}$$

We argue by contradiction and assume that the conclusion does not hold. By passing to a further subsequence, we may assume that $\mathcal{G}(g, \Lambda_n)$ fails to be a Riesz sequence for all $n \in \mathbb{N}$. As a consequence of Theorem 5.4(iii), there exist sequences $c^n \in \ell^\infty(\Lambda_n)$ such that $\|c^n\|_\infty = 1$ and

$$\|C^*_{g,\Lambda_n}(c^n)\|_{M^\infty} = \Big\| \sum_{\lambda\in\Lambda} c^n_{\tau_n(\lambda)} \pi(\tau_n(\lambda)) g \Big\|_{M^\infty} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{32}$$

Thus g, Λ_n, c^n satisfy the assumptions of Lemma 5.6. The conclusion of Lemma 5.6 yields a subsequence (n_k), a separated set $\Gamma \subseteq \mathbb{R}^{2d}$, a non-zero sequence $c \in \ell^\infty(\Gamma)$, and a sequence of points $\{\lambda_{n_k} : k \geq 1\} \subseteq \Lambda$ such that

$$\Lambda_{n_k} - \tau_{n_k}(\lambda_{n_k}) \xrightarrow{w} \Gamma.$$

and

$$\sum_{\gamma\in\Gamma} c_\gamma \pi(\gamma) g = C^*_{g,\Gamma}(c) = 0\,.$$

By Lemma 6.8, we conclude that $\Gamma \in W(\Lambda)$. According to condition (vi) of Theorem 5.4, $\mathcal{G}(g, \Lambda)$ is not a Riesz sequence, which is a contradiction. ■

Remark 7.3. *Uniformity of the bounds:* Under the assumptions of Theorem 7.1 it follows that there exists $n_0 \in \mathbb{N}$ and constants $A, B > 0$ such that for $n \geq n_0$,

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\tau_n(\lambda))g\rangle|^2 \leq B\|f\|_2^2, \qquad f \in L^2(\mathbb{R}^d).$$

The uniformity of the upper bound follows from the fact that $g \in M^1(\mathbb{R}^d)$ and $\sup_n \operatorname{rel}(\Lambda_n) < \infty$ (cf. Section 2.5). For the lower bound, note that the proof of Theorem 7.1 shows that there exists a constant $A' > 0$ such that, for $n \geq n_0$,

$$A'\|f\|_{M^\infty} \leq \sup_{\lambda \in \Lambda} |\langle f, \pi(\tau_n(\lambda))g\rangle|, \qquad f \in M^\infty(\mathbb{R}^d).$$

This property implies a uniform L^2-bound as is made explicit in Remarks 8.2 and 8.3.

To show why local preservation of differences is related to the stability of Gabor frames, let us consider the following example.

Example 7.4. From [3] or from Theorem 7.1 we know that if $g \in M^1(\mathbb{R}^d)$ and $\mathcal{G}(g, \Lambda)$ is a frame, then $\mathcal{G}(g, (1 + 1/n)\Lambda)$ is also a frame for sufficiently large n. For every n we now construct a deformation of the form $\tau_n(\lambda) := \alpha_{\lambda,n}\lambda$, where $\alpha_{\lambda,n}$ is either 1 or $(1+1/n)$ with roughly half of the multipliers equal to 1. Since only a subset of Λ is moved, we would think that this deformation is "smaller" than the full dilation $\lambda \to (1 + \frac{1}{n})\lambda$. and thus should preserve the spanning properties of the corresponding Gabor system. Surprisingly, this is completely false. We now indicate how the coefficients $\alpha_{\lambda,n}$ need to be chosen. Let $\mathbb{R}^{2d} = \bigcup_{l=0}^\infty B_l$ be a partition of $\mathbb{R}^{2d}$ into the annuli

$$B_l = \{z \in \mathbb{R}^{2d} : (1 + \tfrac{1}{n})^l \leq |z| < (1 + \tfrac{1}{n})^{l+1}\},$$

and define

$$\alpha_{\lambda,n} = \begin{cases} 1 & \text{if } \lambda \in B_{2l+1}, \\ 1 + \frac{1}{n} & \text{if } \lambda \in B_{2l}. \end{cases}$$

Since $(1 + \frac{1}{n})B_l = B_{l+1}$, the deformed set $\Lambda_n = \tau_n(\Lambda) = \{\alpha_{\lambda,n}\lambda : \lambda \in \Lambda\}$ is contained in $\bigcup_{l=0}^\infty B_{2l+1}$ and thus contains arbitrarily large holes. So $\rho(\Lambda_n) = \infty$ and $D^-(\Lambda_n) = 0$. Consequently the corresponding Gabor system $\mathcal{G}(g, \Lambda_n)$ cannot be a frame. See Figure 2 for a plot of this deformation in dimension 1.

8. Appendix

We finally prove Theorem 3.2. Both the statement and the proof are modelled on Sjöstrand's treatment of Wiener's lemma for convolution-dominated matrices [36]. Several stability results are built on his techniques [37, 1, 35, 39, 38]. The following proposition exploits the flexibility of Sjöstrand's methods to transfer lower bounds for a matrix from one value of p to all others, under the assumption that the entries of the matrix decay away from a collection of lines.

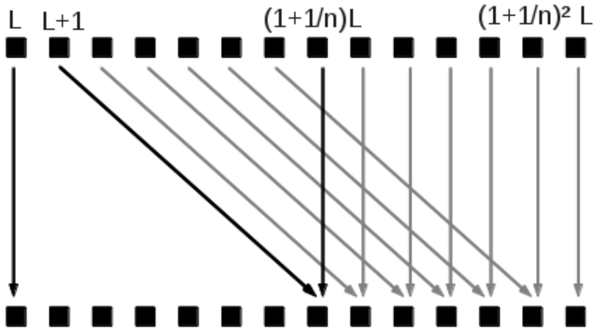

FIGURE 2. A deformation "dominated" by the dilation $\lambda \to (1 + 1/n)\lambda$.

8.1. **A variation of Sjöstrand's Wiener-type lemma.** Let G be the group $G := \{-1, 1\}^d$ with coordinatewise multiplication, and let $\sigma \in G$ act on $\mathbb{R}^d$ by $\sigma x := (\sigma_1 x_1, \dots, \sigma_d x_d)$. The group inverse of $\sigma \in G$ is $\sigma^{-1} = (-\sigma_1, \dots, -\sigma_d)$ and the orbit of $x \in \mathbb{R}^d$ by G is $G \cdot x := \{\sigma x : \sigma \in G\}$. We note that the cardinality of $G \cdot x$ depends on the number of non-zero coordinates of $x \in \mathbb{R}^d$. Consequently, $\mathbb{Z}^d = \bigcup_{k \in \mathbb{N}_0^d} G \cdot k$ is a disjoint union.

Proposition 8.1. *Let Λ and Γ be relatively separated subsets of $\mathbb{R}^d$. Let $A \in \mathbb{C}^{\Lambda \times \Gamma}$ be a matrix such that*

$$|A_{\lambda,\gamma}| \le \sum_{\sigma \in G} \Theta(\lambda - \sigma\gamma) \qquad \lambda \in \Lambda, \gamma \in \Gamma, \qquad \textit{for some } \Theta \in W(L^\infty, L^1)(\mathbb{R}^d).$$

Assume that there exist a $p \in [1, \infty]$ and $C_0 > 0$, such that

$$\|Ac\|_p \ge C_0 \|c\|_p \qquad \textit{for all } c \in \ell^p(\Gamma)\,. \tag{33}$$

Then there exists a constant $C > 0$ independent of q such that, for all $q \in [1, \infty]$

$$\|Ac\|_q \ge C \|c\|_q \qquad \textit{for all } c \in \ell^q(\Gamma)\,. \tag{34}$$

In other words, if A is bounded below on some ℓ^p, then A is bounded below on ℓ^p for all $p \in [1, \infty]$.

Proof. By considering $\sum_{y \in G \cdot x} \Theta(y)$ we may assume that Θ is G-invariant, i.e. $\Theta(x) = \Theta(\sigma x)$ for all $\sigma \in G$.

Step 1. *Construction of a partition of unity.* Let $\psi \in C^\infty(\mathbb{R}^d)$ be G-invariant and such that $0 \le \psi \le 1$, $\operatorname{supp}(\psi) \subseteq B_2(0)$ and

$$\sum_{k \in \mathbb{Z}^d} \psi(\cdot - k) \equiv 1.$$

For $\varepsilon > 0$, define $\psi_k^\varepsilon(x) := \psi(\varepsilon x - k)$, $I := \mathbb{N}_0^d$, and

$$\varphi_k^\varepsilon := \sum_{j \in G \cdot k} \psi_j^\varepsilon.$$

Since $\mathbb{Z}^d = \bigcup_{k\in I} G\cdot k$ is a disjoint union, it follows that

$$\sum_{k\in I} \varphi_k^\varepsilon = \sum_{k\in I}\sum_{j\in G\cdot k} \psi_k^\varepsilon \equiv 1\,.$$

Thus $\{\varphi_k^\varepsilon : k\in I\}$ generates a partition of unity, and it easy to see that it has the following additional properties:

- $\Phi^\varepsilon := \sum_{k\in I} (\varphi_k^\varepsilon)^2 \asymp 1$,
- $0 \le \varphi_k^\varepsilon \le 1$,
- $|\varphi_k^\varepsilon(x) - \varphi_k^\varepsilon(y)| \lesssim \varepsilon\,|x-y|\,$,
- $\varphi_k^\varepsilon(x) = \varphi_k^\varepsilon(\sigma x)$ for all $\sigma\in G$.

Combining the last three properties, we obtain that

$$|\varphi_k^\varepsilon(x) - \varphi_k^\varepsilon(y)| \lesssim \min\{1, \varepsilon d(x, G\cdot y)\}, \tag{35}$$

where $d(x,E) := \inf\{|x-y| : y\in E\}$.

Step 2. *Commutators.* For a matrix $B = (B_{\lambda,\gamma})_{\lambda\in\Lambda,\gamma\in\Gamma} \in \mathbb{C}^{\Lambda\times\Gamma}$ we denote the Schur norm by

$$\|B\|_{\mathrm{Schur}(\Gamma\to\Lambda)} := \max\left\{\sup_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda}|B_{\lambda,\gamma}|\,, \sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}|B_{\lambda,\gamma}|\,\right\}.$$

Let us assume that A is bounded below on $\ell^p(\Gamma)$. After multiplying B with a constant, we may assume that

$$\|c\|_p \le \|Ac\|_p, \qquad c\in\ell^p(\Gamma).$$

For given $\varepsilon > 0$ and $k\in I$ let $\varphi_k^\varepsilon c := \varphi_k^\varepsilon\big|_\Gamma \cdot c$ denote the multiplication operator by φ_k^ϵ and $[A,\varphi_k^\varepsilon] = A\varphi_k^\epsilon - \varphi_k^\epsilon A$ the commutator with A. Now let us estimate

$$\begin{aligned}\|\varphi_k^\varepsilon c\|_p &\le \|A\varphi_k^\varepsilon c\|_p \le \|\varphi_k^\varepsilon Ac\|_p + \|[A,\varphi_k^\varepsilon]c\|_p\\ &\le \|\varphi_k^\varepsilon Ac\|_p + \sum_{j\in I}\|[A,\varphi_k^\varepsilon]\varphi_j^\varepsilon(\Phi^\varepsilon)^{-1}\varphi_j^\varepsilon c\|_p\\ &\le \|\varphi_k^\varepsilon Ac\|_p + K\sum_{j\in I} V_{j,k}^\varepsilon \|\varphi_j^\varepsilon c\|_p, \end{aligned}\tag{36}$$

where $K = \max_x \Phi^\epsilon(x)^{-1}$ and

$$V_{j,k}^\varepsilon := \|[A,\varphi_k^\varepsilon]\varphi_j^\varepsilon\|_{\mathrm{Schur}(\Gamma\to\Lambda)}, \qquad j,k\in\ I. \tag{37}$$

The goal of the following steps is to estimate the Schur norm of the matrix V^ϵ with entries $V_{j,k}^\epsilon$ and to show that $\|V^\epsilon\|_{\mathrm{Schur}(I\to I)} \to 0$ for $\epsilon\to 0+$.

Step 3. *Convergence of the entries $V_{j,k}^\epsilon$*. We show that

$$\sup_{j,k\in I} V_{j,k}^\varepsilon \longrightarrow 0,\ \text{ as } \varepsilon\longrightarrow 0^+. \tag{38}$$

We first note that the matrix entries of $[A,\varphi_k^\varepsilon]\varphi_j^\varepsilon$, for $j,k\in I$, are

$$([A,\varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\gamma} = -A_{\lambda,\gamma}\varphi_j^\varepsilon(\gamma)(\varphi_k^\varepsilon(\lambda) - \varphi_k^\varepsilon(\gamma)), \qquad \gamma\in\Gamma, \lambda\in\Lambda.$$

Using (35) and the hypothesis on A, we bound the entries of the commutator by

$$\begin{aligned}\left|([A,\varphi^\varepsilon_k]\varphi^\varepsilon_j)_{\lambda,\gamma}\right| &\lesssim \sum_{\sigma\in G}\Theta(\lambda-\sigma\gamma)\min\{1,\varepsilon d(\lambda,G\cdot\gamma)\}\\ &\leq \sum_{\sigma\in G}\Theta(\lambda-\sigma\gamma)\min\{1,\varepsilon\left|\lambda-\sigma\gamma\right|\}.\end{aligned}$$

Hence, if we define $\Theta^\varepsilon(x) := \Theta(x)\min\{1,\varepsilon\left|x\right|\}$, then by (6)

$$V^\varepsilon_{j,k} = \|[A,\varphi^\varepsilon_k]\varphi^\varepsilon_j\|_{\mathrm{Schur}(\Gamma\to\Lambda)} \lesssim \max\{\mathrm{rel}(\Lambda),\mathrm{rel}(\Gamma)\}\|\Theta^\varepsilon\|_{W(L^\infty,L^1)}.$$

Since $\Theta\in W(L^\infty,L^1)$, it follows that $\|\Theta^\varepsilon\|_{W(L^\infty,L^1)}\longrightarrow 0$, as $\varepsilon\longrightarrow 0^+$. This proves (38).

Step 4. *Refined estimates for V^ϵ_{jk}.* For $s\in\mathbb{Z}^d$ let us define

$$\triangle^\varepsilon(s) := \sum_{t\in\mathbb{Z}^d:\left|\varepsilon t-s\right|_\infty\leq 5}\ \sup_{z\in[0,1]^d+t}\left|\Theta(z)\right|\,. \tag{39}$$

Claim: If $|j-k|>4$ and $\varepsilon\leq 1$, then

$$V^\varepsilon_{j,k}\lesssim\sum_{s\in G\cdot j-G\cdot k}\triangle^\varepsilon(s)\,.$$

If $|k-j|>4$, then $\varphi^\varepsilon_j(\gamma)\varphi^\varepsilon_k(\gamma)=0$. Indeed, if this were not the case, then $\varphi^\varepsilon_j(\gamma)\neq 0$ and $\varphi^\varepsilon_k(\gamma)\neq 0$. Consequently, $|\varepsilon\gamma-\sigma j|\leq 2$ and $|\varepsilon\gamma-\tau k|\leq 2$ for some $\sigma,\tau\in G$. Hence, $d(k,G\cdot j)\leq|k-\tau^{-1}\sigma j|=|\tau k-\sigma j|\leq 4$. Since $k,j\in I$, this implies that $|k-j|\leq 4$, contradicting the assumption.

As a consequency, for $|k-j|>4$, the matrix entries of $[A,\varphi^\varepsilon_k]\varphi^\varepsilon_j$ simplify to

$$([A,\varphi^\varepsilon_k]\varphi^\varepsilon_j)_{\lambda,\gamma} = -A_{\lambda,\gamma}\varphi^\varepsilon_j(\gamma)\varphi^\varepsilon_k(\lambda),\qquad \gamma\in\Gamma,\lambda\in\Lambda\,.$$

Hence, for $|k-j|>4$ we have the estimate

$$\left|([A,\varphi^\varepsilon_k]\varphi^\varepsilon_j)_{\lambda,\gamma}\right|\leq\sum_{\sigma\in G}\Theta(\lambda-\sigma\gamma)\varphi^\varepsilon_j(\gamma)\varphi^\varepsilon_k(\lambda)=\sum_{\sigma\in G}\Theta(\lambda-\sigma\gamma)\varphi^\varepsilon_j(\sigma\gamma)\varphi^\varepsilon_k(\lambda).$$

Consequently, for $|k-j|>4$ we have

$$\begin{aligned}\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\left|([A,\varphi^\varepsilon_k]\varphi^\varepsilon_j)_{\lambda,\gamma}\right| &\leq \sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\sum_{\sigma\in G}\Theta(\lambda-\sigma\gamma)\varphi^\varepsilon_j(\sigma\gamma)\varphi^\varepsilon_k(\lambda)\\ &\lesssim \sup_{\lambda\in\Lambda}\sum_{\gamma\in G\cdot\Gamma}\Theta(\lambda-\gamma)\varphi^\varepsilon_j(\gamma)\varphi^\varepsilon_k(\lambda).\end{aligned}$$

If $\varphi^\varepsilon_j(\gamma)\varphi^\varepsilon_k(\lambda)\neq 0$, then $|\varepsilon\gamma-\sigma j|\leq 2$ and $|\varepsilon\lambda-\tau k|\leq 2$ for some $\sigma,\tau\in G$. The triangle inequality implies that

$$d(\varepsilon(\lambda-\gamma),G\cdot j-G\cdot k)\leq 4. \tag{40}$$

Hence, we further estimate

$$\sup_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma}\left|([A,\varphi^\varepsilon_k]\varphi^\varepsilon_j)_{\lambda,\gamma}\right|\lesssim\sup_{\lambda\in\Lambda}\sum_{s\in G\cdot j-G\cdot k}\ \sum_{\substack{\gamma\in G\cdot\Gamma,\\ |\varepsilon(\lambda-\gamma)-s|\leq 4}}\Theta(\gamma-\lambda). \tag{41}$$

For fixed $s \in G \cdot j - G \cdot k$ and $\varepsilon \leq 1$ we bound the inner sum in (41) by

$$\begin{aligned}\sum_{\gamma \in G\cdot\Gamma : |\varepsilon(\lambda-\gamma)-s| \leq 4} \Theta(\gamma - \lambda) &\leq \sum_{t \in \mathbb{Z}^d} \sum_{\substack{\gamma \in G\cdot\Gamma : |\varepsilon(\lambda-\gamma)-s| \leq 4 \\ (\lambda-\gamma) \in [0,1]^d + t}} \Theta(\gamma - \lambda) \\ &\lesssim \operatorname{rel}(\lambda - G \cdot \Gamma) \sum_{t \in \mathbb{Z}^d : |\varepsilon t - s|_\infty \leq 5} \sup_{z \in [0,1]^d + t} |\Theta(z)| \\ &\lesssim \operatorname{rel}(\Gamma) \sum_{t \in \mathbb{Z}^d : |\varepsilon t - s|_\infty \leq 5} \sup_{z \in [0,1]^d + t} |\Theta(z)| \\ &\lesssim \triangle^\varepsilon(s).\end{aligned}$$

Substituting this bound in (41), we obtain

$$\sup_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \left| ([A, \varphi_k^\varepsilon] \varphi_j^\varepsilon)_{\lambda,\gamma} \right| \lesssim \sum_{s \in G\cdot j - G \cdot k} \triangle^\varepsilon(s).$$

Inverting the roles of λ and γ we obtain a similar estimate, and the combination yields

$$V_{j,k}^\varepsilon \lesssim \sum_{s \in G\cdot j - G\cdot k} \triangle^\varepsilon(s), \qquad \text{for } |j-k| > 4 \text{ and } \varepsilon \leq 1, \tag{42}$$

as claimed.

Step 5. *Schur norm of V^ϵ.* Let us show that $\|V^\epsilon\|_{\mathrm{Schur}(I\to I)} \to 0$, i.e.,

$$\sup_{k \in I} \sum_{j \in I} V_{j,k}^\varepsilon, \quad \sup_{j \in I} \sum_{k \in I} V_{j,k}^\varepsilon \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0^+. \tag{43}$$

We only treat the first limit; the second limit is analogous. Using the definition of $\triangle^\varepsilon$ from (39) and the fact $\Theta \in W(L^\infty, L^1)$, we obtain that

(44)

$$\sum_{s \in \mathbb{Z}^d, |s| > 6\sqrt{d}} \triangle^\varepsilon(s) \leq \sum_{s \in \mathbb{Z}^d, |s|_\infty > 6} \triangle^\varepsilon(s) \lesssim \sum_{t \in \mathbb{Z}^d, |t|_\infty > 1/\varepsilon} \sup_{z \in [0,1]^d + t} |\Theta(z)| \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0^+.$$

Fix $k \in I$ and use (42) to estimate

$$\sum_{j : |j-k| > 6\sqrt{d}} V_{j,k}^\varepsilon \lesssim \sum_{j : |j-k| > 6\sqrt{d}} \sum_{s \in G\cdot j - G\cdot k} \triangle^\varepsilon(s) \leq \sum_{\sigma,\tau \in G} \sum_{j : |j-k| > 6\sqrt{d}} \triangle^\varepsilon(\sigma j - \tau k).$$

If $|j-k| > 6\sqrt{d}$ and $j, k \in I$, then also $|s| = |\sigma j - \tau k| > 6\sqrt{d}$ for all $\sigma, \tau \in G$. Hence we obtain the bound

$$\sum_{j \in I : |j-k| > 6\sqrt{d}} V_{j,k}^\varepsilon \lesssim \sum_{\sigma,\tau \in G} \sum_{\substack{s \in \mathbb{Z}^d \\ |s| > 6\sqrt{d}}} \triangle^\varepsilon(s) \lesssim \sum_{\substack{s \in \mathbb{Z}^d \\ |s| > 6\sqrt{d}}} \triangle^\varepsilon(s).$$

For the sum over $\{j : |j-k| \leq 6\sqrt{d}\}$ we use the bound

$$\sum_{j : |j-k| \leq 6\sqrt{d}} V_{j,k}^\varepsilon \leq \#\{j : |j-k| \leq 6\sqrt{d}\} \sup_{s,t} V_{s,t}^\varepsilon \lesssim \sup_{s,t} V_{s,t}^\varepsilon.$$

Hence,

$$\sum_{j\in I} V^{\varepsilon}_{j,k} \lesssim \sup_{s,t} V^{\varepsilon}_{s,t} + \sum_{|s|>6\sqrt{d}} \triangle^{\varepsilon}(s),$$

which tends to 0 uniformly in k as $\varepsilon \to 0^+$ by (38) and (44).

Step 6. *The stability estimate.* According to the previous step we may choose $\varepsilon > 0$ such that

$$\|V^{\epsilon} a\|_q \le \frac{1}{2K}\|a\|_q, \qquad a \in \ell^q(I)$$

uniformly for all $q \in [1,\infty]$. Using this bound in (36), we obtain that

$$\left(\sum_{k\in I}\|\varphi^{\varepsilon}_k c\|_p^q\right)^{1/q} \le \left(\sum_{k\in I}\|\varphi^{\varepsilon}_k Ac\|_p^q\right)^{1/q} + 1/2\left(\sum_{k\in I}\|\varphi^{\varepsilon}_k c\|_p^q\right)^{1/q},$$

with the usual modification for $q=\infty$. Hence,

$$\left(\sum_{k\in I}\|\varphi^{\varepsilon}_k c\|_p^q\right)^{1/q} \le 2\left(\sum_{k\in I}\|\varphi^{\varepsilon}_k Ac\|_p^q\right)^{1/q}. \tag{45}$$

Step 7. *Comparison of ℓ^p-norms.* Let us show that for every $1 \le q \le \infty$,

$$\left(\sum_{k\in I}\|\varphi^{\varepsilon}_k a\|_p^q\right)^{1/q} \asymp \|a\|_q, \qquad a \in \ell^q(\Gamma), \tag{46}$$

with constants independent of p, q, and the usual modification for $q=\infty$.

First note that for fixed $\varepsilon > 0$

$$N := \sup_{k\in I} \#\,\mathrm{supp}(\varphi^{\varepsilon}_k\big|_{\Gamma}) = \sup_{k\in I} \#\big\{\, \gamma \in \Gamma \,:\, \varphi^{\varepsilon}_k(\gamma) \ne 0 \,\big\} < \infty.$$

Then for $q \in [1,\infty]$ we have

$$\|\varphi^{\varepsilon}_k a\|_p \le \|\varphi^{\varepsilon}_k a\|_1 \le N\|\varphi^{\varepsilon}_k a\|_\infty \le N\|\varphi^{\varepsilon}_k a\|_q, \qquad a \in \ell^\infty(\Gamma),$$

and similarly

$$\|\varphi^{\varepsilon}_k a\|_q \le N\|\varphi^{\varepsilon}_k a\|_p, \qquad a \in \ell^\infty(\Gamma).$$

As a consequence,

$$\Big(\sum_{k\in I}\|\varphi^{\varepsilon}_k a\|_p^q\Big)^{1/q} \asymp \Big(\sum_{k\in I}\|\varphi^{\varepsilon}_k a\|_q^q\Big)^{1/q}, \qquad a \in \ell^q(\Gamma). \tag{47}$$

with constants independent of p and q and with the usual modification for $q=\infty$.

Next note that

$$\eta := \sup_{\varepsilon>0}\sup_{x\in\mathbb{R}^d} \#\big\{\, k \in I \,:\, \varphi^{\varepsilon}_k(x) \ne 0 \,\big\} = \sup_{\varepsilon>0}\sup_{x\in\mathbb{R}^d} \#\{k \in I \cap B_2(\varepsilon x)\} < \infty. \tag{48}$$

because $\mathrm{supp}(\psi) \subseteq B_2(0)$. So we obtain the following simple bound for all $x \in \mathbb{R}^d$:

$$1 = \sum_{k \in I} \varphi_k^\varepsilon(x) = \sum_{k \in I : \varphi_k^\varepsilon(x) \neq 0} \varphi_k^\varepsilon(x) \leq \eta \sup_{k \in I} \varphi_k^\varepsilon(x).$$

Therefore, for all $x \in \mathbb{R}^d$,

$$\frac{1}{\eta} \leq \sup_{k \in I} \varphi_k^\varepsilon(x) \leq \Big(\sum_{k \in I} (\varphi_k^\varepsilon(x))^q \Big)^{1/q} \leq \sum_{k \in I} \varphi_k^\varepsilon(x) = 1\,. \tag{49}$$

If $q < \infty$ and $a \in \ell^q(\Gamma)$, then

$$\frac{1}{\eta^q} \sum_{\gamma \in \Gamma} |a_\gamma|^q \leq \sum_{\gamma \in \Gamma} \sum_{k \in I} (\varphi_k^\varepsilon(\gamma))^q \, |a_\gamma|^q \leq \sum_{\gamma \in \Gamma} |a_\gamma|^q\,,$$

which implies that

$$\Big(\sum_{k \in I} \|\varphi_k^\varepsilon a\|_q^q \Big)^{1/q} = \Big(\sum_{\gamma \in \Gamma} \sum_{k \in I} (\varphi_k^\varepsilon(\gamma))^q \, |a_\gamma|^q \Big)^{1/q} \asymp \|a\|_q, \tag{50}$$

with constants independent of q. The corresponding statement for $q = \infty$ follows similarly. Finally, the combination of (47) and (50) yields (46).

Step 8. We finally combine the norm equivalence (46) with (45) and deduce that for all $1 \leq q \leq \infty$

$$\|c\|_q \lesssim \|Ac\|_q,$$

with a constant independent of q. This completes the proof. ■

Remark 8.2. We emphasize that the lower bound guaranteed by Proposition 8.1 is uniform for all p. The constant depends only on the decay properties of the envelope Θ, the lower bound for the given value of p, and on upper bounds for the relative separation of the index sets.

8.2. **Wilson bases.** A Wilson basis associated with a window $g \in L^2(\mathbb{R}^d)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ of the form $\mathcal{W}(g) = \big\{ g_\gamma : \gamma = (\gamma_1, \gamma_2) \in \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d \big\}$ with

$$g_\gamma = \sum_{\sigma \in \{-1,1\}^d} \alpha_\gamma \pi(\gamma_1, \sigma\gamma_2) g, \tag{51}$$

where $\alpha_\gamma \in \mathbb{C}$ and, as before, $\sigma x = (\sigma_1 x_1, \dots, \sigma_d x_d)$, $x \in \mathbb{R}^d$.

There exist Wilson bases associated to functions g in the Schwartz class [14] (see also [24, Chapters 8.5 and 12.3]). In this case $\mathcal{W}(g)$ is a p-Riesz sequence and a p-frame for $M^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$. This means that the associated coefficient operator $C_{\mathcal{W}}$ defined by $C_{\mathcal{W}} = (\langle f, g_\gamma \rangle)_\gamma$ is an isomorphism from $M^p(\mathbb{R}^d)$ onto $\ell^p(\Gamma)$ for every $p \in [1, \infty]$ and that the synthesis operator $C_{\mathcal{W}}^* c = C_{\mathcal{W}}^{-1} c = \sum_\gamma c_\gamma g_\gamma$ is an isomorphism from $\ell^p(\Gamma)$ onto $M^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$. (For $p = \infty$ the series converge in the weak* topology).

8.3. **Proof of Theorem 3.2.**

Proof. Let $\Gamma := \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d$ and let $\mathcal{W}(g) = \{g_\gamma : \gamma \in \Gamma\}$ be a Wilson basis with $g \in M^1(\mathbb{R}^d)$.

(a) We assume that $\{f_\lambda : \lambda \in \Lambda\}$ is a set of time-frequency molecules with associated coefficient operator S, $Sf := (\langle f, f_\lambda \rangle)_{\lambda\in\Lambda}$.

We need to show that if S is bounded below on $M^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then it is bounded below for all $p \in [1, \infty]$. Since the synthesis operator $C^*_{\mathcal{W}}$ associated with the Wilson basis $\mathcal{W}(g)$ is an isomorphism for all $1 \le p \le \infty$, S is bounded below on $M^p(\mathbb{R}^d)$, if and only if $SC^*_{\mathcal{W}}$ is bounded below on $\ell^p(\Lambda)$. Thus it suffices to show that $SC^*_{\mathcal{W}}$ is bounded below for some $p \in [1, \infty]$ and then apply Proposition 8.1. The operator $SC^*_{\mathcal{W}}$ is represented by the matrix A with entries

$$A_{\lambda,\gamma} := \langle g_\gamma, f_\lambda \rangle, \qquad \lambda \in \Lambda, \gamma \in \Gamma.$$

In order to apply Proposition 8.1 we provide an adequate envelope. Let Φ be the function from (13), $\Phi^\vee(z) := \Phi(-z)$ and $\Theta := \Phi^\vee * |V_g g|$. Then $\Theta \in W(L^\infty, L^1)(\mathbb{R}^{2d})$ by Lemma 2.1. Using (51) and the time-frequency localization of $\{f_\lambda : \lambda \in \Lambda\}$ and of g we estimate

$$\begin{aligned}
|A_{\lambda,\gamma}| &= |\langle f_\lambda, g_\gamma \rangle| = |\langle V_g f_\lambda, V_g g_\gamma \rangle| \\
&\le \int_{\mathbb{R}^{2d}} |\Phi(z - \lambda)|\, |V_g g_\gamma(z)|\, dz \\
&\lesssim \sum_{\sigma\in\{-1,1\}^d} \int_{\mathbb{R}^{2d}} |\Phi(z - \lambda)|\, |V_g g(z - (\gamma_1, \sigma\gamma_2))|\, dz \\
&= \sum_{\sigma\in\{-1,1\}^d} \Theta(\lambda - (\gamma_1, \sigma\gamma_2)) \le \sum_{\sigma\in\{-1,1\}^{2d}} \Theta(\lambda - \sigma\gamma).
\end{aligned}$$

Hence, the desired conclusion follows from Proposition 8.1.

(b) Here we assume that $\{f_\lambda : \lambda \in \Lambda\}$ is a set of time-frequency molecules such that the associated synthesis operator $S^*c = \sum_{\lambda\in\Lambda} c_\lambda f_\lambda$ is bounded below on some $\ell^p(\Lambda)$. We must show that S^* is bounded below for all $p \in [1, \infty]$. Since $C_{\mathcal{W}}$ is an isomorphism on $M^p(\mathbb{R}^d)$, this is equivalent to saying the operator $C_{\mathcal{W}} S^*$ is bounded below on some (hence all) $\ell^p(\Lambda)$.

The operator $C_{\mathcal{W}} S^*$ is represented by the matrix $B = A^*$ with entries

$$B_{\gamma,\lambda} := \langle g_\gamma, f_\lambda \rangle, \qquad \gamma \in \Gamma, \lambda \in \Lambda,$$

and satisfies

$$|B_{\gamma,\lambda}| \le \sum_{\sigma\in\{-1,1\}^{2d}} \Theta(\lambda - \sigma\gamma), \qquad \gamma \in \Gamma, \lambda \in \Lambda.$$

To apply Proposition 8.1, we consider the symmetric envelope $\Theta^*(x) = \sum_{y\in G\cdot x} \Theta(y), x \in \mathbb{R}^{2d}$. Then $\Theta^* \in W(L^\infty, L^1)(\mathbb{R}^{2d})$, $\Theta^*(\sigma x) = \Theta(x)$, and

$$|B_{\gamma,\lambda}| \le \sum_{\sigma\in\{-1,1\}^{2d}} \Theta^*(\lambda - \sigma\gamma) = \sum_{\sigma\in\{-1,1\}^{2d}} \Theta^*(\gamma - \sigma\lambda).$$

This shows that we can apply Proposition 8.1 and the proof is complete. ∎

Remark 8.3. As in Remark 8.2, we note that the norm bounds for all p in Theorem 3.2 depend on the envelope Θ, on upper bounds for $\mathrm{rel}(\Lambda)$ and frame or Riesz basis bounds for a particular value of p.

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