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Topological Quantum Field Theories: Towards The Cobordism Hypothesis

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Contents

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Introduction

Topological quantum field theories (TQFTs) have been a past century attempt to axiomatize *quantum field theories* from physics. While the underlying theory for quantum mechanics had been fully developed in terms of Hilbert spaces and operator theory, the analytic basis of quantum field theories remained unsettled, and several approaches are still nowadays being considered. Surprisingly or not, the introduction of these new theories was received with high interest not only by physicists but also by the mathematical community. TQFTs became a recurrent field of study in mathematics mainly because of the interest they had from a topological standpoint. While TQFTs became less popular over the time in physics, the mathematical approach has increasingly attracted the attention of researchers because of its natural drift towards the homotopy theory of higher categories, showing some outstanding results such as the *Cobordism Hypothesis*, formulated by John Baez and James Dolan [2] and recently proved by Jacob Lurie [8].

The original definition of TQFTs was first given by Michael Atiyah's [1] in 1988 as a generalization to category theory of group representations. A TQFT was defined as a functor from the category of cobordisms (smooth manifolds with boundary and additional structure), to the category of vector spaces (originally Atiyah formulated the definition in terms of Λ-modules for a ring Λ). The definition, as Atiyah himself stated, was inspired in the previous work done by Edward Witten on super-symmetry [17] and Graeme Segal on conformal theory. A successful understanding of TQFTs in low dimensions was rapidly achieved, and several theories in dimensions ≤ 4 were developed. Baez and Dolan [2], foreseeing a near future, suggested in 1995 that a more complex theory was behind the classical formulation, and although their work lacked formality, it delimited essential guidelines of study.

The mathematical importance of TQFTs has not passed unnoticed, and at least four Fields Medals have been given to this date to mathematicians for research related to TQFTs: Simon Donaldson, Vaughan Jones, Edward Witten and Maxim Kontsevich.

Motivation and goals

Our goal in this thesis is to study the properties of TQFTs and their implication to the development of a theory of higher categories and the formulation of the Cobordism Hypothesis. We will try to build enough theoretical background, always following the historical perspective, to produce a self-consistent work. Having set this as the main goal, we assume that the difficulty and novelty of the topic will eventually bring us beyond our limits, but we will put all our efforts in being precise, formal and omitting the least possible steps. The background acquired during the undergraduate years will serve as a basis, and it will be complemented with other notions belonging to more advanced fields in mathematics. Being still undecided about the near future, this project will also serve as a first introduction to a current field of research in mathematics.

The structure of the thesis will be as natural as possible: in the first section we will introduce basic concepts. In the second section we will start by unraveling the classical definition of TQFTs, studying their essential properties and their classification in dimensions 1 and 2. This low-dimensional cases will be used as essential models to be compared at the end of the work. This path, however, will eventually bring us to a dead end: while classification of TQFTs in lower dimensions is rather simple, their behavior in higher dimensions is much more intrincate. This fact will force us to go around the problem by defining a more complex mathematical structure, a higher category, to model an enhanced notion of TQFTs, the *extended* TQFTs. Due to the complexity of defining higher categories, in the third section we will momentarily forget about TQFT and instead we will focus on the understanding of (∞, n)-categories, and in particular, the model of Segal n-categories. In the fourth section we will define extended TQFTs and describe the fundamental (∞, n)-category on which extended TQFTs are based, finally being able to formulate and explain the Cobordism Hypothesis. We will finally remark its importance on the classification of extended TQFTs.

Chapter 1

Preliminary Concepts

In this first section we list the basic material needed throughout this thesis. *Topological quantum field theories* are formulated in terms of category theory, geometry and topology. More details can be found, for example, in [10] and [15].

Recall that a *category*, which we will generally denote by $\mathscr{C}$, is a mathematical structure with objects and morphisms satisfying certain axioms: morphisms can be composed, the composition is associative and each object has an identity morphism. Examples of categories include the category of sets and functions, the category of topological spaces and continuous functions, the category of groups and group homomorphisms, etc. *Functors* between categories preserve identities and composition of morphisms. A typical example is the fundamental group in topology, assigning groups to spaces with base points and group homomorphisms to continuous functions.

A *monoidal category* is a category $\mathscr{C}$ equipped with a bifunctor $\otimes\colon \mathscr{C}\times\mathscr{C} \to \mathscr{C}$, usually called *tensor product*, and a *unit* object $\mathbf{1}_{\mathscr{C}}$, such that associativity and unitarity hold up to natural isomorphisms subject to coherence conditions [10]. If, in addition, for each two objects X, Y there is an invertible natural *twist* map $\tau_{X,Y}\colon X\otimes Y \to Y\otimes X$, then $\mathscr{C}$ is a *symmetric* monoidal category. Vector spaces over a field k and linear maps between them form a symmetric monoidal category (taking $\otimes$ as the usual tensor product over k), which will be denoted by $\mathbf{Vect}_{\otimes}(k)$. In this case, the unit is the ground field k, since for every vector space we have $V\otimes k \cong V$. A *monoidal functor* is a functor between monoidal categories preserving tensor products and units up to natural isomorphisms.

We also need some basic topology notions. A *manifold* of dimension n with boundary is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of the *half-space* $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The set of points in a manifold M that have a neighborhood homeomorphic to $\mathbb{R}^n$ is called *interior* of M and denoted by $\mathrm{Int}(M)$. The *boundary* of a manifold is defined as $\partial M = M \setminus \mathrm{Int}(M)$. A manifold can be equipped with a smooth structure, and it can have an orientation. Given a smooth manifold M, an *orientation* on M is a choice of sign on each basis of the tangent space T_xM at each point such that the differentials of the coordinate changes acting on T_xM have positive determinant. If a manifold admits an orientation then we say that it is orientable. The boundary of an orientable manifold M is also orientable in a canonical way. For embedded manifolds, the induced orientation on ∂M is such that the normal vector pointing outwards plus a positive basis of $T_x\partial M$ yields the given orientation of M. We will denote by $\overline{M}$ the oriented manifold M with the opposite orientation.

Remark. Throughout this thesis we will use the word *closed* manifold to refer to a compact manifold without boundary. Unless specified, the manifolds we consider in the following sections will be smooth and orientable. We will generally draw manifolds of dimension 2 or 1, but most of the conceptual ideas behind the images hold for all dimensions.

1.1 Cobordisms

The fundamental category in which topological quantum field theories are formulated is the category of oriented cobordisms. To say it quickly: manifolds in physics represent space, while cobordisms, having an extra dimension, represent space-time.

Definition 1.1. Given two closed oriented manifolds Σ and Σ' of dimension $n-1$, an *oriented cobordism* from Σ to Σ' is a smooth oriented manifold M of dimension n with boundary and with an orientation-preserving diffeomorphism $\partial M \cong \overline{\Sigma} \amalg \Sigma'$, where $\amalg$ denotes disjoint union. Remember that $\overline{\Sigma}$ denotes the manifold Σ with the opposite orientation. We will say that M is a cobordism from the *incoming boundary* Σ towards the *outgoing boundary* Σ'. Both incoming and outgoing boundaries of a cobordism can have several connected components.

We will represent cobordisms with pictures where the incoming boundary is situated at the left of the image and the outgoing boundary at the right. Figure 1.1 is an example of a cobordism from the circle S^1 to $S^1 \amalg S^1$.

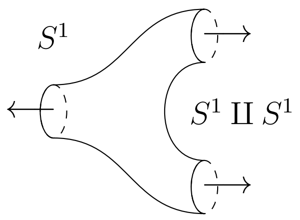


Figure 1.1: Example of a cobordism

Equivalence of cobordisms

We can define the following equivalence relation in the set of smooth orientable manifolds with boundary $\overline{\Sigma} \amalg \Sigma'$:

Definition 1.2. Given two cobordisms M and N from Σ to Σ', we say they are *equivalent* if there exists an orientation-preserving diffeomorphism $\psi\colon M \to N$ rendering the following diagram commutative:

$$\begin{tikzcd}
& M \arrow[dd, "\psi"', "\cong"] & \\
\overline{\Sigma} \arrow[ur] \arrow[dr] & & \Sigma' \arrow[ul] \arrow[dl] \\
& N &
\end{tikzcd}$$

Composing cobordisms

Composition of cobordisms is given by identifying the outgoing boundary of the first one with the incoming boundary of the second one. Thus, given two cobordisms M_1 and M_2 with respective boundaries $\overline{\Sigma}_0 \amalg \Sigma_1$ and $\overline{\Sigma}_1 \amalg \Sigma_2$, we define their composition as $M = M_1 \amalg_{\Sigma_1} M_2$. Here there is an aspect that needs further care. While there is no problem in defining $M = M_1 \amalg_{\Sigma_1} M_2$ as a topological manifold, there is no canonical choice of a smooth structure on it. However, M admits a smooth structure which is unique up to a diffeomorphism fixing Σ_0, Σ_1 and Σ_2, and the embeddings $M_1 \hookrightarrow M$ and $M_2 \hookrightarrow M$ are diffeomorphisms onto their images; see [11, Theorem 1.4].

The category of cobordisms

With the above definitions we can build a category, taking manifolds of a fixed dimension as objects and cobordisms between them as morphisms. We will denote this category by $\mathbf{Cob}_{\mathrm{II}}(n)$, and we define it in the following way:

- Objects are closed oriented manifolds of dimension $(n-1)$.
- For every two objects Σ and Σ' of dimension $(n-1)$, the morphism set $\mathrm{Hom}_{\mathbf{Cob}_{\mathrm{II}}(n)}(\Sigma, \Sigma')$ is the set of equivalence classes of cobordisms from Σ to Σ' in the sense of Definition 1.2.
- The composition law is given by gluing cobordisms along their common boundaries.
- For every object Σ, the identity morphism is given by the cylinder $[0,1] \times \Sigma$.

This category admits a symmetric monoidal structure where the tensor product is the disjoint union and the unit object is the empty set. We next illustrate the composition law in $\mathbf{Cob}_{\mathrm{II}}(n)$, as well as associativity, identities, monoidal structure and symmetry.

- Associativity of composition: Since morphisms are diffeomorphism classes of cobordisms, we just need to see that the different outcomes of gluing representatives of given equivalence classes are diffeomorphic. Figure 1.2 is an example of a composition of three cobordisms:

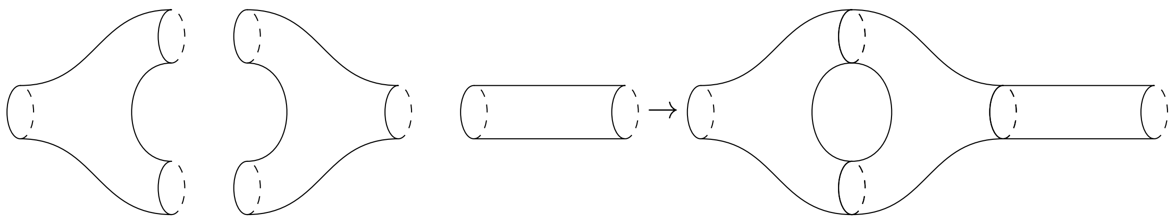


Figure 1.2: Associativity of the composition law

- Identity cobordism: $[0,1] \times \Sigma$ with the normal vector pointing outwards has incoming boundary $\{0\} \times \overline{\Sigma}$ and outgoing boundary $\{1\} \times \Sigma$, hence it is the identity cobordism on Σ (Figure 1.3) and satisfies the desired axioms. If we choose $[0,1] \times \Sigma$ to have the opposite orientation, we obtain the identity cobordism on $\overline{\Sigma}$.

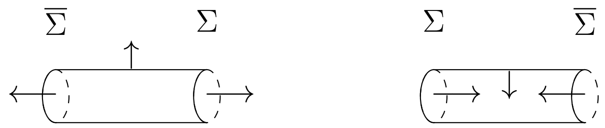


Figure 1.3: Identity cobordisms on Σ and $\overline{\Sigma}$

The composition of identities with another cobordism M is diffeomorphic to M:

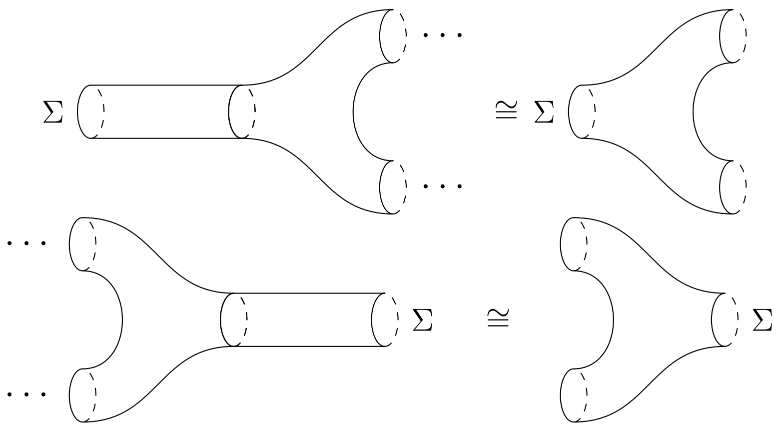


Figure 1.4: Left and right identities

- Monoidal structure: The monoidal structure is given by the disjoint union $\amalg$ of manifolds and by the empty set, viewed as an $(n-1)$-dimensional manifold, acting as the unit $\mathbf{1}_{\amalg}$. Disjoint unions are indeed associative, and for any manifold Σ we have $\Sigma \amalg \emptyset = \Sigma$.
- Symmetry: The symmetry of the disjoint union is given by the *twist* cobordism. The composition of two twists is the identity cobordism. In Figure 1.5 we can visualize a twist between two manifolds Σ and Σ'. Keep in mind that these cobordisms are not embedded in any common space, so there is no real intersection between them:

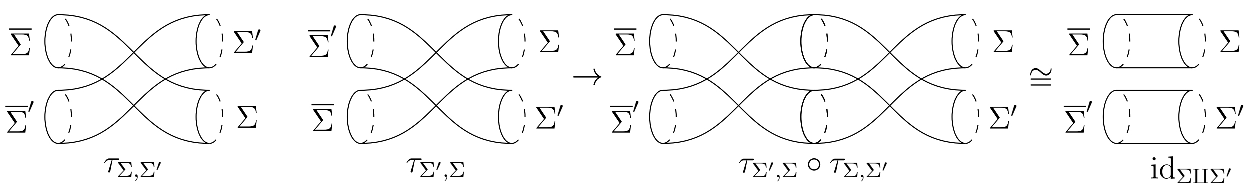


Figure 1.5: Twist cobordism

Chapter 2

Topological Quantum Field Theories

2.1 Atiyah's definition

The following definition is due to Atiyah [1].

Definition 2.1. A *topological quantum field theory* (TQFT) of dimension n is a monoidal functor

$$Z\colon \mathbf{Cob}_{\amalg}(n) \to \mathbf{Vect}_{\otimes}(k),$$

where k is a field.

According to this definition, a TQFT of dimension n will assign to each closed oriented $(n-1)$-dimensional manifold Σ a k-vector space $Z(\Sigma)$, and to each cobordism M between two closed oriented manifolds Σ and Σ' a linear map $Z(M)\colon Z(\Sigma) \to Z(\Sigma')$.

In this chapter we will review some general properties of topological quantum field theories. There are essential properties that are satisfied regardless of the dimension of the given theory.

- As a consequence of Z being monoidal, it is going to map the unit with respect to the disjoint union, namely the empty set $\emptyset$, to the unit with respect to the tensor product, namely the ground field k. Therefore, the image of any n-dimensional closed manifold viewed as a cobordism between $\emptyset$ and itself must be a linear map from k to itself, thus an element of k. In this sense, we can say that Z assigns a number to every closed oriented manifold of dimension n.

- $Z(\Sigma \amalg \Sigma') \cong Z(\Sigma) \otimes Z(\Sigma')$. Disjoint unions of manifolds or cobordisms map into tensor products of vector spaces or linear maps respectively.

2.2 TQFTs and duality

Given an oriented closed manifold Σ of any dimension, we can construct a cobordism by multiplying it by the unit interval: $N = [0,1] \times \Sigma$. This cobordism can have different interpretations depending on the choice of incoming and outgoing boundaries. The boundary of this cobordism is $\overline{\Sigma} \amalg \Sigma$ and it can seen in the four following ways, each of them resulting in a different image under Z:

- The identity cobordism $N\colon \Sigma \to \Sigma$, $N = \mathrm{id}_\Sigma$. Then $Z(N)\colon Z(\Sigma) \to Z(\Sigma)$ is the identity linear map on $Z(\Sigma)$.
- A cobordism $N\colon \emptyset \to \overline{\Sigma} \amalg \Sigma$. We will call it *coevaluation map* and denote it by coev_Σ. Then $Z(\mathrm{coev}_\Sigma)\colon k \cong Z(\emptyset) \to Z(\Sigma) \otimes Z(\overline{\Sigma})$.
- The identity cobordism $N\colon \overline{\Sigma} \to \overline{\Sigma}$, $N = \mathrm{id}_{\overline{\Sigma}}$. Then $Z(N)\colon Z(\overline{\Sigma}) \to Z(\overline{\Sigma})$ is the identity linear map on $Z(\overline{\Sigma})$.
- A cobordism $N\colon \Sigma \amalg \overline{\Sigma} \to \emptyset$. We will call it *evaluation map* and denote it by ev_Σ. Then $Z(\mathrm{ev}_\Sigma)\colon Z(\Sigma) \otimes Z(\overline{\Sigma}) \to Z(\emptyset) \cong k$ is a canonical pairing between $Z(\Sigma)$ and $Z(\overline{\Sigma})$.

The four possibilities above can be visualized in the following figure:

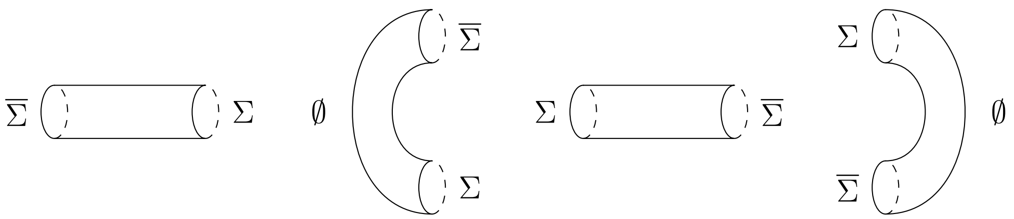


Figure 2.1: The basic cobordisms

This simple interpretation of $N = [0,1] \times \Sigma$ results especially interesting. A fascinating aspect comes from the fact that the pairing $Z(\mathrm{ev}_\Sigma)$ is non-degenerate, and therefore induces an isomorphism between $Z(\overline{\Sigma})$ and $Z(\Sigma)^V$, the dual vector space of $Z(\Sigma)$.

Definition 2.2. A pairing between two k-vector spaces V and W (that is, a linear map $\gamma\colon V \otimes W \to k$) is *non-degenerate* if there is a linear map $\beta\colon k \to W \otimes V$, called *copairing*, such that the following compositions are the identity maps in V and W respectively:

$$V \xrightarrow{\mathrm{id}_V \otimes \beta} V \otimes W \otimes V \xrightarrow{\gamma \otimes \mathrm{id}_V} V$$

$$W \xrightarrow{\beta \otimes \mathrm{id}_W} W \otimes V \otimes W \xrightarrow{\mathrm{id}_W \otimes \gamma} W$$

Lemma 2.1. *A non-degenerate pairing between two vector spaces V and W induces isomorphisms $V^V \cong W$ and $W^V \cong V$. Moreover, V and W have finite dimension.*

Proof. Let γ be the pairing and β the copairing. Let $\sum_{i=1}^n w_i \otimes v_i$ be the image of 1_k by β. Then the image of a vector $x \in V$ through the above composition is

$$x \xrightarrow{\mathrm{id}_V \otimes \beta} \sum_{i=1}^n x \otimes w_i \otimes v_i \xrightarrow{\gamma \otimes \mathrm{id}_V} \sum_{i=1}^n \gamma(x, w_i) v_i.$$

If this composition is the identity map for all $x \in V$, then v_i span all of V, therefore V has finite dimension. A similar argument holds for W.

Let now $v \in V$, $w \in W$ and $\gamma(v, w)$ be the image under γ of $v \otimes w$. Then for every fixed w, we have a form $\gamma(-, w)\colon V \to k$, and for every fixed v a form $\gamma(v, -)\colon W \to k$. They induce the following maps: $\gamma^+\colon W \to V^V$ such that $\gamma^+(w) = \gamma(-, w)$ and $\gamma^-\colon V \to W^V$ such that $\gamma^-(v) = \gamma(v, -)$. We can see that γ^+ is an injective map. Imagine we have an element $x \in V$ such that $\gamma(x, w) = 0$ for all $w \in W$. Then, by the same composition before, we have

$$x = \sum_{i}^{n} \gamma(x, w_i) v_i = 0.$$

A similar argument holds for γ^-. The injectivity of γ^+ and γ^- proves that V and W have the same dimension, and thus γ^+ and γ^- are automatically isomorphisms. □

Corollary 2.1. *Given an oriented manifold Σ, if $\overline{\Sigma}$ denotes the same manifold with the opposite orientation, then $Z(\overline{\Sigma}) \cong Z(\Sigma)^V$, where the superscript V denotes the dual vector space.*

Proof. The cylinder $I \times \Sigma$ has boundary $\Sigma \amalg \overline{\Sigma}$ and yields us a pairing $Z(\mathrm{ev}_\Sigma)$ and a copairing $Z(\mathrm{coev}_\Sigma)$ between $Z(\Sigma)$ and $Z(\overline{\Sigma})$. We have to see that the following compositions correspond to the identity map:

$$Z(\Sigma) \xrightarrow{\mathrm{id}_{Z(\Sigma)} \otimes Z(\mathrm{coev}_\Sigma)} Z(\Sigma) \otimes Z(\overline{\Sigma}) \otimes Z(\Sigma) \xrightarrow{Z(\mathrm{ev}_\Sigma) \otimes \mathrm{id}_{Z(\Sigma)}} Z(\Sigma)$$

$$Z(\overline{\Sigma}) \xrightarrow{Z(\mathrm{coev}_\Sigma) \otimes \mathrm{id}_{Z(\overline{\Sigma})}} Z(\overline{\Sigma}) \otimes Z(\Sigma) \otimes Z(\overline{\Sigma}) \xrightarrow{\mathrm{id}_{Z(\overline{\Sigma})} \otimes Z(\mathrm{ev}_\Sigma)} Z(\overline{\Sigma})$$

The argument is depicted in Figure 2.2 for the first composite, and the second one is analogous.

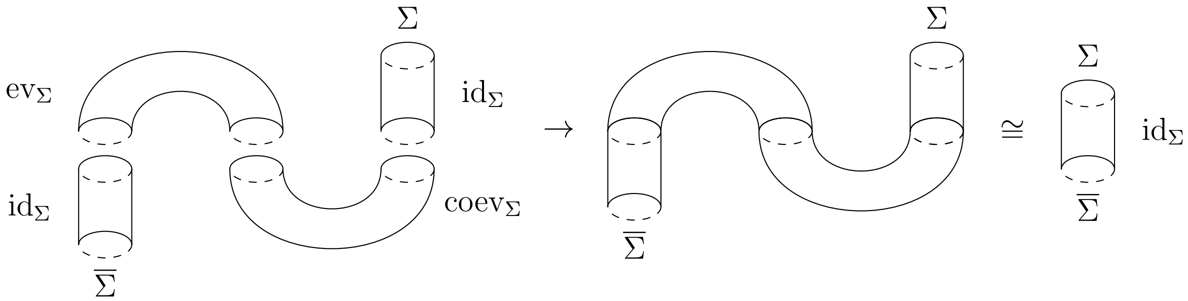


Figure 2.2: Snake decomposition of a cylinder

The fact that the above cobordisms are diffeomorphic to the identity cobordism implies that the image under Z has to be the identity map between vector spaces. This shows that $Z(\mathrm{ev}_\Sigma)$ and $Z(\mathrm{coev}_\Sigma)$ make a non-degenerate pairing. □

Corollary 2.2. *If Σ is a manifold of dimension $n-1$ and Z is an n-dimensional TQFT, then the vector space $Z(\Sigma)$ has finite dimension.*

Proof. This is also a consequence of the fact that the pairing between $Z(\Sigma)$ and $Z(\overline{\Sigma})$ is non-degenerate. □

Remark. We have seen that we can think of $Z(\overline{\Sigma})$ as the dual vector space of $Z(\Sigma)$. In this way, ev_Σ and coev_Σ can be interpreted as follows in terms of linear maps. Recall that, for a vector space A, there is a canonical isomorphism between $A \otimes A^V$ and $\mathrm{End}(A)$.

- $Z(\mathrm{ev}_\Sigma)$ is evaluation of a form on a vector $(\lambda, \nu) \mapsto \lambda(\nu)$. Under the isomorphism above this becomes the trace map $\mathrm{tr}\colon \mathrm{End}(Z(\Sigma)) \to k$.
- $Z(\mathrm{coev}_\Sigma)$, in the same way, is multiplication by the identity map: $x \mapsto x \cdot \mathrm{id}_{Z(\Sigma)} \in \mathrm{End}(Z(\Sigma))$.

2.3 Classification of TQFTs

In this section we will focus on further properties of a monoidal functor Z as in the previous section. Classifying a TQFT means understanding how the assignments of vector spaces and linear maps to manifolds and cobordisms have to be. The fact that

Z is a monoidal functor implies that it preserves the structure and suggests that a full understanding of $\mathbf{Cob}_{\amalg}(n)$ is necessary in order to determine these assignments. We need to map cobordisms to linear maps, so we will start trying to simplify cobordisms by cutting them down and composing them again after.

2.3.1 Decomposing cobordisms

Imagine we want to compute the value of Z on an arbitrary given cobordism M of dimension n, with boundary ∂M. We can regard M as a cobordism from the empty set to ∂M, so $Z(M)$ will be a map $Z(M)\colon k \to Z(\partial M)$, thus an element of the vector space $Z(\partial M)$. But we have more information, namely the fact that Z is a functor allows us to compute $Z(M)$ by cutting M into smaller parts, as a composition of smaller cobordisms. Imagine that we have an $(n-1)$-dimensional manifold Σ that breaks M into two pieces M_1 and M_2. Our new boundary is going to be $\partial M \amalg \Sigma \amalg \overline{\Sigma}$. If we evaluate Z on this new boundary we get the following pairing:

$$Z(\partial M) \otimes Z(\Sigma) \otimes Z(\overline{\Sigma}) = Z(\partial M) \otimes Z(\Sigma) \otimes Z(\Sigma)^V \xrightarrow{\mathrm{id}_{Z(\partial M)} \otimes Z(\mathrm{ev}_\Sigma)} Z(\partial M).$$

If we consider now the linear maps $Z(M_1)\colon k \to Z(\partial M_1)$ and $Z(M_2)\colon k \to Z(\partial M_2)$, then we can view $Z(M)$ as the composite of $Z(M_1) \otimes Z(M_2)$ with the above pairing. Hence, $Z(M)$ is determined as an element of $Z(\partial M)$ by $Z(M_1)$ and $Z(M_2)$. Cutting down cobordisms generates additional boundaries, but the essence is that we have a canonical way to cancel disjoint unions of a manifold and its opposite. In Figure 2.3 we decompose a cobordism by cutting through a manifold Σ into two *pairs of pants*, which we will study later.

The following question arises: Is there any finite set of *generators* of $\mathbf{Cob}_{\amalg}(n)$? Or similarly: Is there any finite set of *easy* cobordisms from which we can construct any other cobordism in some chosen dimension? If the answer is yes then an n-dimensional topological quantum field theory is going to be completely determined by its image on the set of generators of $\mathbf{Cob}_{\amalg}(n)$, and the rest of the theory will be automatically obtained by composing the little pieces. In the next section we will classify TQFTs in dimension 1 and 2, and this will be used as a motivation to try to find a generalization for the classification of TQFTs in higher dimensions. We will now finish with the following proposition:

Proposition 2.1. *Let Z be a topological quantum field theory of dimension n. Let Σ be a closed oriented manifold of dimension $n-1$ and $A = Z(\Sigma)$ its image. Then*

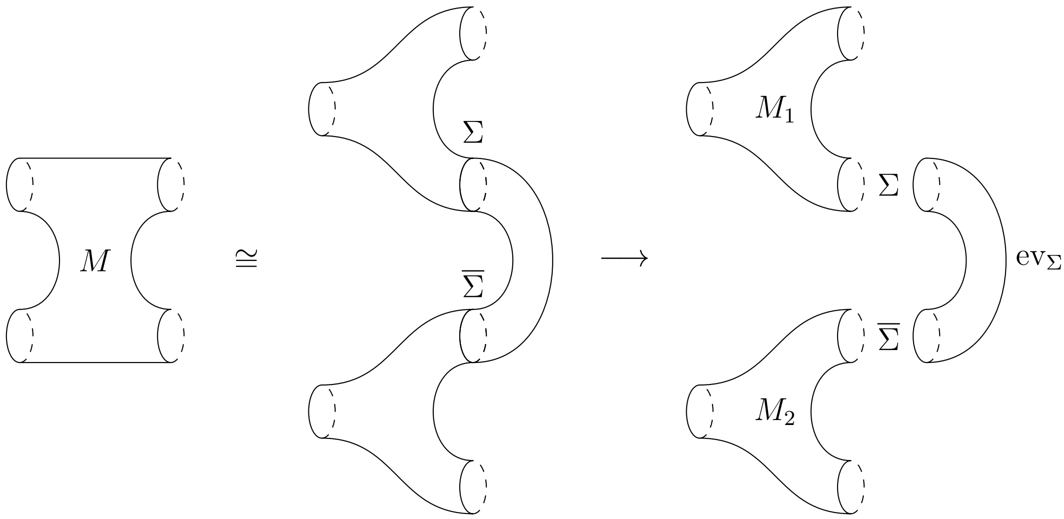


Figure 2.3: Decomposing cobordisms

$Z(S^1 \times \Sigma) = \dim(A)$.

Proof. We can cut along two copies of Σ decomposing the cobordism as shown in the picture:

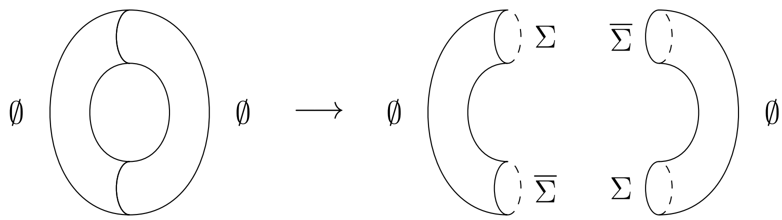


Figure 2.4: The O decomposition

This corresponds to the composition of cobordisms $\mathrm{ev}_\Sigma \circ \mathrm{coev}_\Sigma$. Now, applying Z we obtain the following composition of linear maps:

$$Z(S^1 \times \Sigma) : k \xrightarrow{Z(\mathrm{coev}_\Sigma)} A \otimes A^V \xrightarrow{Z(\mathrm{ev}_\Sigma)} k,$$

and we have seen previously that this corresponds to the trace $\mathrm{tr}(\mathrm{id}_A) = \dim(A)$. $\square$

2.3.2 TQFTs in dimension 1

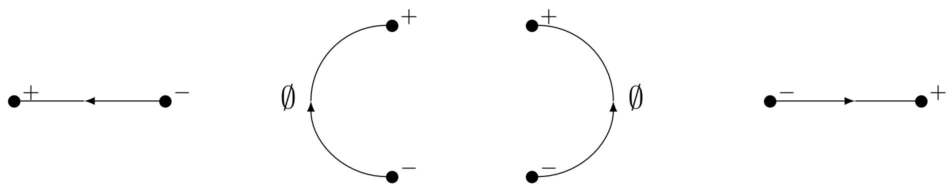
Although it might seem easy, the classification of TQFTs in dimension 1 already reveals clues that we will later compare. A TQFT in dimension 1 is a monoidal functor $Z\colon \mathbf{Cob}_{\amalg}(1) \to \mathbf{Vect}_{\otimes}(k)$, where k is a field. The category $\mathbf{Cob}_{\amalg}(1)$ has compact 0-dimensional manifolds as objects, that is, finite sets of points. The tangent space to a single point is the trivial vector space $\{0\}$ which has the empty set as basis. An orientation is a choice of sign of this basis. So an object of $\mathbf{Cob}_{\amalg}(1)$ will be a finite disjoint union of points with positive orientation $\bullet^+$ and points with negative orientation $\bullet^-$. There are only two diffeomorphism classes of oriented 1-manifolds: the circle S^1 and the interval $I = [0, 1]$.

The functor Z will assign a vector space to each of the objects above. Let us start supposing that $Z(\bullet^+) = A$ is a certain vector space of dimension d (not related with the dimension of Z). Then, as previously seen, $Z(\bullet^-) = A^V$, the dual space of A. Thus the value of Z on objects of $\mathbf{Cob}_{\amalg}(1)$ is:

$$Z(\bullet^+ \amalg \cdots \amalg \bullet^+ \amalg \bullet^- \amalg \cdots \amalg \bullet^-) = A \otimes \cdots \otimes A \otimes A^V \otimes \cdots \otimes A^V.$$

The functor Z also needs to assign linear maps to cobordisms. As said above, there are two different classes of cobordisms:

- The interval $I \times \bullet^+ = I$ can be read as a cobordism in four different ways, as we discussed previously: as the identity cobordism from $\bullet^+$ to itself, as the identity cobordism from $\bullet^-$ to itself, as a cobordism from $\emptyset$ to $\bullet^+ \amalg \bullet^-$ or as a cobordism from $\bullet^+ \amalg \bullet^-$ to $\emptyset$. Then $Z(I)$ is determined by the identity maps on $Z(M)$ and its dual and by the pairing and co-pairing between them. If $A = Z(\bullet^+)$, these are the pictures of the cobordisms and their corresponding linear maps:



$A^V \xrightarrow{\mathrm{id}_{A^V}} A^V \qquad k \xrightarrow{Z(\mathrm{coev})} A \otimes A^V \qquad A \otimes A^V \xrightarrow{Z(\mathrm{ev})} k \qquad A \xrightarrow{\mathrm{id}_A} A$

Figure 2.5: Basic cobordisms in dimension 1

- The circle S^1. It has no boundary so it can only be regarded as a cobordism from $\emptyset$ to itself. It is the reduction of the composition of ev_Σ and coev_Σ to dimension 1, discussed in the previous section, thus $Z(S^1) = \mathrm{tr}(\mathrm{id}_A) = \dim(A)$.

At this point we can state and prove a first elementary version of what is known by the Cobordism Hypothesis.

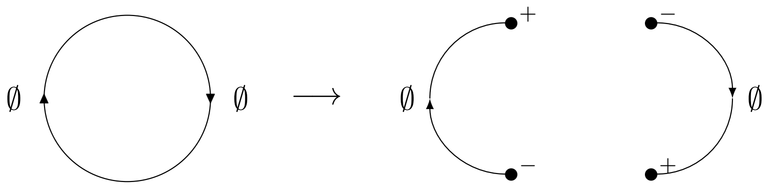


Figure 2.6: The O decomposition in dimension 1

Theorem 2.1. *If A is a finite-dimensional vector space, then there is a unique topological quantum field theory Z of dimension 1 such that $Z(\bullet^+) = A$.*

Proof. Set $Z(\bullet^-) = A^V$. To evaluate Z on cobordisms it is enough to see that any two opposed boundary points will cancel out, so the value of Z on any 1-manifold is determined by its values on elementary cobordisms. □

In fact, if choosing a finite-dimensional vector space is essentially the same thing as choosing a 1-dimensional TQFT, we can state the following theorem, where $\mathrm{Fun}^{\otimes}$ denotes the set of monoidal functors between the given categories.

Theorem 2.2. *Let $\mathbf{Vect}_{\otimes}^{<\infty}(k)$ denote the class of finite-dimensional vector spaces. Then evaluation on a positively oriented point sets up a bijective correspondence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Cob}_{\amalg}(1),\mathbf{Vect}_{\otimes}(k)) \longleftrightarrow \mathbf{Vect}_{\otimes}^{<\infty}(k).$$

2.3.3 TQFTs in dimension 2

We will now describe a similar classification in dimension 2. Recall that a 2-dimensional TQFT is a monoidal functor $Z\colon \mathbf{Cob}_{\amalg}(2) \to \mathbf{Vect}_{\otimes}(k)$. Therefore it assigns vector spaces to closed oriented manifolds of dimension 1, and linear maps to cobordisms between them. There is only one closed manifold of dimension 1, namely the circle S^1. Notice that there is an orientation-preserving diffeomorphism between $\overline{S}1$ and S^1 (a reflection). This fact is essential because it means that we only have to consider finite disjoint unions of circles as objects in the category $\mathbf{Cob}_{\amalg}(2)$. Let again $A = Z(S^1)$. Evaluation of Z on the objects of $\mathbf{Cob}_{\amalg}(2)$ is completely determined by $Z(S^1)$ in the following way:

$$Z\left(\coprod_{i=1}^{n} S^1\right) = \bigotimes_{i=1}^{n} A.$$

We have assigned a vector space A to every object, but we still need to assign linear maps to cobordisms between disjoint unions of circles. These cobordisms will endow A with additional algebraic structure. There are more cobordisms to consider than in the case of dimension 1:

- The disk D^2:
  - As a cobordism from $\emptyset$ to S^1, it yields a map $Z(D^2)\colon k \cong Z(\emptyset) \longrightarrow A$. Let $\mathbf{1}_A$ be the image of 1 under this map.
  - As a cobordism from S^1 to $\emptyset$, it yields a map $Z(D^2)\colon A \longrightarrow Z(\emptyset) \cong k$, which will be called *trace* and denoted by tr.
- Given a cobordism B from $S^1 \amalg S^1$ to S^1, $Z(B)$ is a linear map

  $$Z(B)\colon A \otimes A \to A.$$

  This map is a bilinear multiplication m on A. It has the property of being associative, commutative and having a unit, as can be seen in Figures 2.7, 2.8 and 2.9.

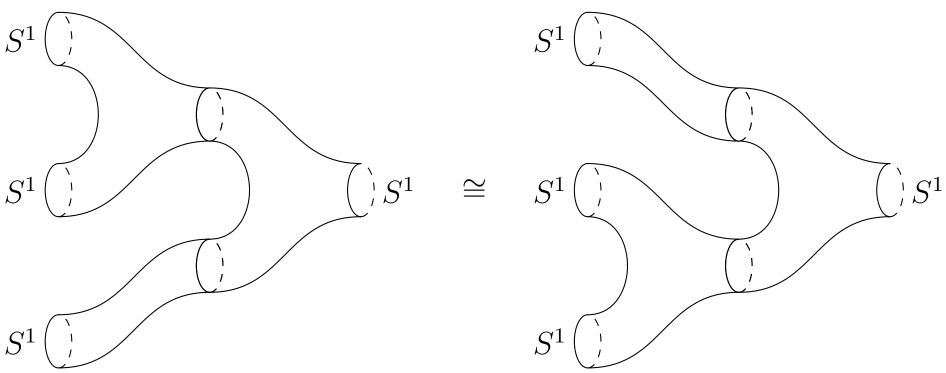


Figure 2.7: Associativity

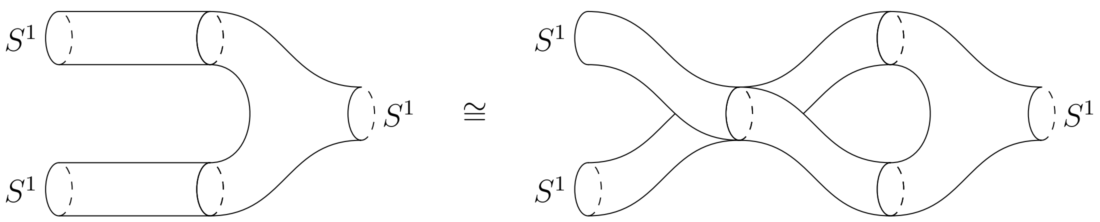


Figure 2.8: Commutativity

At this point we can see that the multiplication m and the trace tr are composable, and the composition is the evaluation map ev_{S^1}; see Figure 2.10. There is, therefore, a canonical pairing between A and itself corresponding to the composition $A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{tr}} k$.

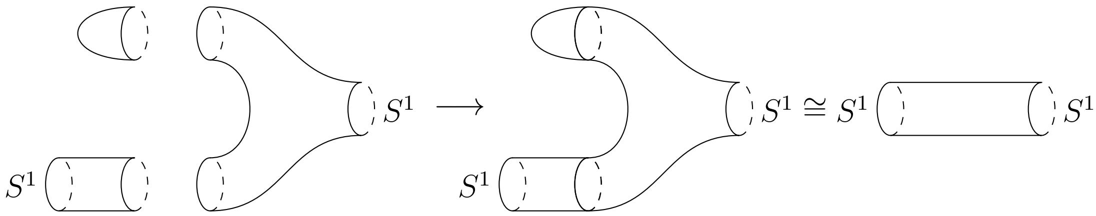


Figure 2.9: Unit

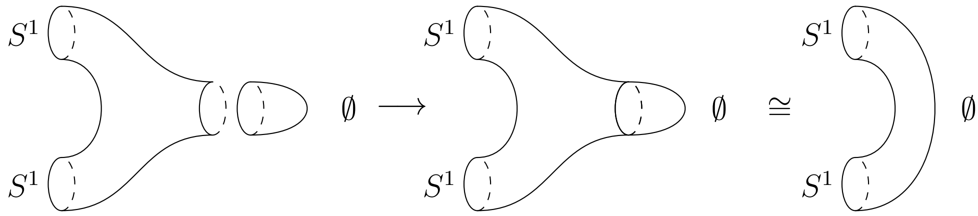


Figure 2.10: Composition of tr and m

Frobenius algebras

An algebra over a field k is a k-vector space over k together with two linear maps $\mu\colon A \otimes A \to A$ and $\eta\colon k \to A$ such that μ is associative and η acts as both right and left identities. A *commutative Frobenius algebra* over k is a finite-dimensional k-algebra A equipped with an associative and commutative non-degenerate pairing.

Proposition 2.2. *A monoidal functor $Z\colon \mathbf{Cob}_{\text{II}}(2) \to \mathbf{Vect}_{\otimes}(k)$ induces a structure of commutative Frobenius k-algebra on A.*

Proof. The algebra structure is given by the pair of pants and the disk. We have already checked associativity and commutativity. The existence of a pairing is given by the composition of the pair of pants with the disk, and its non-degeneracy has already been discussed in 2.1. □

What we have proved is that a 2-dimensional topological quantum field theory Z endows $Z(S^1)$ with a structure of a commutative Frobenius algebra. In fact, the converse statement is also true: given a commutative Frobenius algebra A we can construct a 2-dimensional topological quantum field theory Z such that $Z(S^1) = A$. This is a consequence of the fact that every cobordism in $\mathbf{Cob}_{\text{II}}(2)$ can be decomposed into pairs of pants, cylinders and disks. The proof requires the use of Morse theory [6]. Proving that the value of Z is independent of the choice of decomposition of a given cobordism is beyond our scope; details can be found in [7]. It is however important to highlight that as a consequence of this fact we can now give the second elementary version of the Cobordism Hypothesis:

Theorem 2.3. *If A is a commutative Frobenius k-algebra, then there is a unique topological quantum field theory of dimension 2 such that $Z(S^1) = A$.*

Or, in other words:

Theorem 2.4. *Let $\mathbf{Vect}_{\otimes}^{\mathrm{Frob}}(k)$ denote the class of Frobenius k-algebras. Then evaluation on the circle S^1 sets up a bijective correspondence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Cob}_{\mathrm{II}}(2), \mathbf{Vect}_{\otimes}(k)) \longleftrightarrow \mathbf{Vect}_{\otimes}^{\mathrm{Frob}}(k).$$

2.3.4 Passage to higher dimensions

Although low-dimensional topological quantum field theories can be easily classified, the complexity increases when we move to higher dimensions. The main reason is that, while the topology of curves and surfaces is very well understood, higher-dimensional manifolds are far less easy. The cutting method for cobordisms becomes much more intricate. Some approaches, however, have been totally or partially developed for dimensions ≤ 5.

Example. Picturing how higher-dimensional cobordisms look like is rather hard, but we can think of examples of cobordisms of dimension 3 between two manifolds of dimension 2. For instance, Figure 2.11 depicts a cobordism from a torus to a sphere.

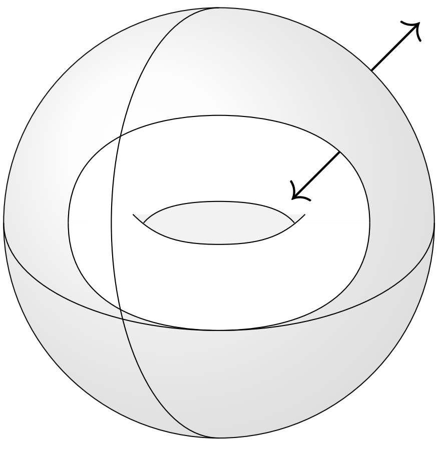


Figure 2.11: A cobordism in dimension 3

2.4 Quantum Field Theory

We have given a mathematical description of topological quantum field theories without discussing the physics background. We would like to take a break during this short section to exemplify some key aspects of the theory that come from the physical perspective, and encode some essential properties that motivated the definition and usage. This section does not follow the general guideline of the thesis, but it might be interesting as a motivation.

There are several aspects that we expect that a *quantum field theory* will have, emerging only from the topology of space-time. In TQFTs closed oriented manifolds represent space, and cobordisms represent space-time.

Quantum Mechanics

A 1-dimensional topological quantum field theory does not involve any spatial dimension, so we expect it to capture the essence of quantum mechanics. In quantum mechanics, the state of a particle is given by a vector in a Hilbert space. A property of it, an *observable*, is given by a unit operator on the given Hilbert space. The hamiltonian operator $\mathscr{H}$, which encodes information about the energy of the system, determines the evolution of the system over time through the unitary operator

$$U_t = e^{-it\mathscr{H}/\hbar}.$$

Time-evolution operators behave like cobordisms in $\mathbf{Cob}_{\mathrm{II}}(1)$ (Figure 2.12). Given τ_1 and τ_2, the time-evolution operator satisfies:

$$U_{\tau_1+\tau_2} = e^{-i(\tau_1+\tau_2)\mathscr{H}/\hbar} = e^{-i\tau_1\mathscr{H}/\hbar}e^{-i\tau_2\mathscr{H}/\hbar} = U_{\tau_2} \circ U_{\tau_1}.$$

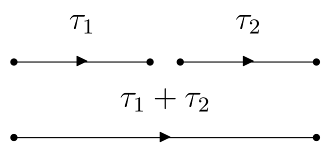


Figure 2.12: Composition of cobordisms in $\mathbf{Cob}_{\mathrm{II}}(1)$

The comparison also allows the existence of exotic events, such as creation/annihilation of a pair of particles (Figure 2.13). Another property of quantum mechanics that is also reflected in TQFTs is encoded in the monoidal structure. The space of states of the

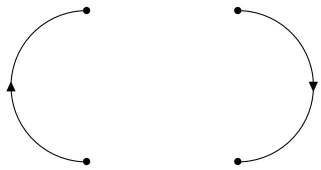


Figure 2.13: Creation/annihilation

composition of several quantum subsystems is given by the tensor product of Hilbert spaces. In classical mechanics, the configuration space is given by the cartesian product of the configuration spaces of subsystems, implying that the properties of particles are independent. In quantum mechanics this is not true, and perhaps the best known example is *Heisenberg's uncertainty principle.* In quantum mechanics, *Kets* and *Bras*

Quantum Mechanics	**Category Theory**
Space of quantum states	Object in a category
Linear operator	Morphism in a category
Time-evolution operator	Functor
Entanglement	Monoidal structure
Bra/Ket	Duality

Table 2.1: Comparison between quantum mechanics and TQFTs

represent respectively states on a Hilbert space and states on its dual space. We have seen in 2.1 that duality is also encoded in the formalism of TQFTs.

The path integral

The *path integral* shows a key functorial behavior. In classical mechanics, the evolution of a system is characterized by the *Lagrangian* $\mathscr{L}$. A given system evolves following the path which minimizes the *action* S, defined as the integral

$$S = \int_{t_1}^{t_2} \mathscr{L} \mathrm{d}t.$$

The *path integral* is the corresponding generalization to quantum field theory of the above concept. In quantum mechanics, the system does not follow a single path. Its behavior, instead, depends on all permitted paths and the value of their respective actions. An example of path integral can be the following:

$$Z(M) = \int \mathcal{D}\mathcal{A}\, e^{\mathrm{i}\int_{\mathrm{M}} \mathcal{S}[\mathrm{A}]/\hbar}.$$

Here M is an arbitrary closed 3-dimensional manifold and $\mathcal{S}[A]$ is the *Chern–Simons* action [18]. The fact that this action is topological implies that $Z(M)$ is a topological invariant of M. Furthermore, the value of $Z(M)$ can be obtained from the value of Z in a decomposition of M into subdomains, such as cobordisms. This property is related with *locality*, in the sense that global invariants are determined by local data.

Feynman diagrams

Feynman diagrams have correspondences in terms of cobordisms. In Figure 2.14 we show some examples.

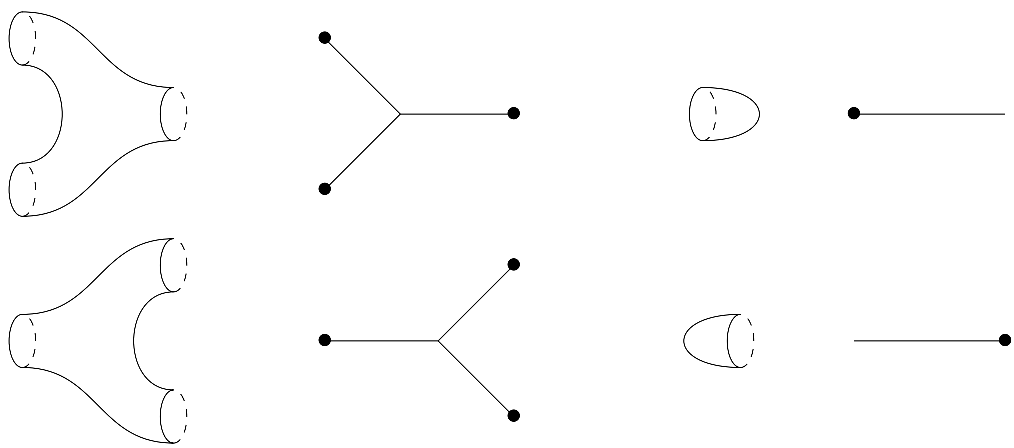


Figure 2.14: Feynman diagrams and 2-dimensional cobordisms

Chapter 3

Extended Topological Quantum Field Theories

3.1 Motivation for extending TQFTs

3.1.1 Mathematical motivation

We have seen in Section 2.3 a classification of TQFTs in low dimensions. While it is rather simple to classify them for dimensions 1 and 2, already in dimension 3 the complexity increases greatly. Decomposing manifolds by cutting in codimension 1 is no longer effective, since no finite set of generators for $\mathbf{Cob}_{\mathrm{II}}(n)$ can be found for dimensions $n \geq 3$. Thus it is necessary to use other methods. A triangulation, for example, would allow us to write a given manifold as the union of a finite number of simplices. Unfortunately it would not completely help, since the cutting and gluing of these pieces would involve the usage of manifolds of lower dimensions with *corners*, which do not belong to our approach. In fact, Morse theory also uses manifolds with corners surrounding critical points in higher dimensions.

Thus, an essential fact that would make the classification potentially possible is being able to cut in all codimensions. Why not defining a richer structure than $\mathbf{Cob}_{\mathrm{II}}(n)$ involving manifolds of all dimensions $\leq n$, not only with boundaries but with corners of any dimension? This, indeed, is the solution that has been historically adopted and which we will discuss at the end of the section. Cobordism pieces will no longer look like tubes, but they will be more similar to Figure 3.1.

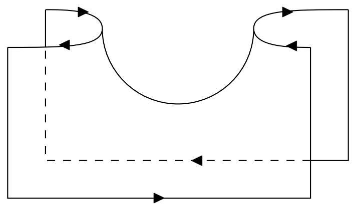


Figure 3.1: Example of a 2-manifold with corners

This change of perspective could make us think that the new theories have little to do with the ones that we have described so far. This is in fact false, and we will see at the end that the motivational examples in low dimensions are crucial, and can be seen as simple models of the same idea. Remember that TQFTs in dimension 1 and 2 are completely determined by the assignment of a vector space with a certain structure to the image of a single object. In dimension 1 this happened by assigning a finite-dimensional vector space to the positively oriented point $\bullet^+$, and in dimension 2 by assigning a commutative Frobenius algebra to the circle S^1. Thus Theorems 2.2 and 2.4 motivate the Cobordism Hypothesis.

3.1.2 Physical motivation

Although we have taken a purely mathematical point of view and our main motivation in extending TQFTs is to be able to classify them, *extended* TQFTs also have a physical interpretation. Ordinary TQFTs are *local* in the sense that they are functorial, as we were able to cut every given cobordism into smaller ones recursively and use the resulting pieces to recover the value on the initially given cobordism. This cutting, however, only involved submanifolds of codimension 1. In other words, we were only able to cut along *time* sections. In *extended* TQFTs, we are going to be able to cut along time but also through spatial dimensions, thus it will be possible to localize properties down to a single point.

3.2 Higher categories

In Definition 1.1 we defined a cobordism as a morphism between two closed manifolds (where *closed* means compact without boundary). Although cobordisms are manifolds, and one could be tempted to imagine 2-cobordisms between cobordisms, this is not well defined, since cobordisms can have a non-empty boundary, so they do not satisfy the

$$
\begin{tikzcd}
X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & Y
\end{tikzcd}
$$

Figure 3.2: Diagram of a 2-morphism α

condition of being closed. Therefore we would like to broaden the original definition in order to be able to, starting from a point, recursively define cobordisms between any type of manifolds with corners until some fixed dimension. We will then have points, cobordisms, cobordisms between cobordisms, etc.

Here is where the concept of higher category appears. A higher category (more precisely, an n-category) is a mathematical structure that contains objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, up to n-morphisms satisfying some coherence laws (Figure 3.2). An n-category with all morphisms invertible is called an *n-groupoid*.

If $n = \infty$, then we talk about an ∞-category, and if all k-morphisms are invertible (in a weaker sense) for $k > n$ we say that it is an (∞, n)-category. The latest has proven to be the most successful model to lead to the goals of this thesis and the one we will use.

The first approach to higher category theory involves *strict* n-categories, which are simple to define [12], and whose k-morphism compositions are associative at all levels. It has been seen, however, that strictness is not suitable for the construction of useful mathematical structures for many different purposes [14, §2.7]. Instead, what is useful is to have a mathematical structure which comprises objects, 1-morphisms, 2-morphisms, and so on, where the composition of such morphisms is not associative but associative up to invertible higher morphisms.

Basic notions

Definition 3.1. Let Δ denote the *simplex category*, whose objects are finite ordered sets $[n] = \{0, 1, \ldots, n\}$ and whose morphisms are non-decreasing maps between them.

Here is an example of a morphism in Δ:

$$\begin{tikzcd}[row sep=tiny]
f\colon [3] \arrow[rr] & & {[2]} \\
0 \arrow[rr] & & 0 \\
1 \arrow[rr] & & 1 \\
2 \arrow[rru] & & 2 \\
3 \arrow[rru] & &
\end{tikzcd}$$

There are exactly $n+1$ injective maps δ_i in $\mathrm{Hom}_\Delta([n-1],[n])$ and $n+1$ surjective maps σ_i in $\mathrm{Hom}_\Delta([n+1],[n])$. To prove this claim, observe that an injective morphism in $\mathrm{Hom}_\Delta([n-1],[n])$ will miss a single element in $[n]$. If we choose j to be this element, then there is a single non-decreasing morphism from $[n-1]$ to $[n]$ satisfying this condition. There are in total $n+1$ possible choices of j in $[n] = \{0,\ldots,n\}$, hence there are $n+1$ maps δ_i. A surjective morphism in $\mathrm{Hom}_\Delta([n+1],[n])$ will target twice a single element in $[n]$. The fact that the morphism is non-decreasing leaves us again with a single option. There are $n+1$ choices of elements in $[n] = \{0,\ldots,n\}$, so there exactly $n+1$ morphisms σ_i.

For a category $\mathscr{C}$, we denote by $\mathscr{C}^{\mathrm{op}}$ the category with the same objects and with morphisms exchanging target and source:

- $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$;
- $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X)$.

Definition 3.2. If $\mathscr{C}$ is a category, a *simplicial object* in $\mathscr{C}$ is a functor $X\colon \Delta^{\mathrm{op}} \to \mathscr{C}$. We denote $X_n = X([n])$.

A simplicial object has two particularly important families of morphisms, namely the images under X of the above maps δ_i and σ_i. The *face* morphisms d_i are the images under X of the maps δ_i, and the *degeneracy* morphisms s_i are the images under X of the maps σ_i. We will denote by d_i^j the image of the map $\delta\colon [j] \to [j+1]$ that misses i, and by s_i^j the image of the map $\sigma\colon [j] \to [j-1]$ that targets i twice. A simplicial object can be depicted by a diagram like the following one (recall that X is a contravariant functor):

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[l, shift right=2] \arrow[l, shift right=4] \arrow[r, shift left=1] \arrow[r, shift right=1] & X_2 \arrow[l, shift right=3] \arrow[l, shift right=5] \arrow[l, shift right=7] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots \arrow[l, shift right=4] \arrow[l, shift right=6] \arrow[l, shift right=8] \arrow[l, shift right=10]
\end{tikzcd}$$

Because every order-preserving map in Δ can be decomposed into face and degeneracy maps [5], a simplicial object will be completely determined by the objects X_n and the face and degeneracy morphisms. These morphisms are related by the *simplicial identities* inherited from Δ, which can be found in [5]. If we let $\mathscr{C}$ be the category *Set* of sets, then a simplicial set is a functor $X\colon \Delta^{\mathrm{op}} \to Set$. Simplicial sets, in turn, form a category whose objects are simplicial sets and whose morphisms are natural transformations between them. We will denote this category by *SSet*. Simplicial sets are a combinatorial model of topological spaces.

Enriching

The concept of enriching categories is crucial for our approach. In a category $\mathscr{C}$, between two objects X, Y we have a set of morphisms $\mathrm{Hom}_{\mathscr{C}}(X,Y)$. In other words, categories are canonically enriched over the category *Set* of sets. Given a monoidal category $\mathscr{D}$, we say that a category $\mathscr{C}$ is *enriched* over $\mathscr{D}$ if for every two objects X, Y in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is an object of $\mathscr{D}$ and composition of morphisms is a map $\mathrm{Hom}_{\mathscr{C}}(X,Y) \otimes \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$ in $\mathscr{D}$ subject to compatibility conditions. A category is called *small* if its objects form a set (i.e., not a proper class).

Definition 3.3. A 2-*category* is a category enriched over the category **Cat** of small categories and functors between them.

Hence, a 2-category admits morphisms between morphisms, which are called 2-*morphisms*. Note that **Cat** is itself a 2-category where 2-morphisms are natural transformations of functors. Using this language, one can define recursively an *n-category* as a category enriched over the category of small $(n-1)$-categories.

Topological spaces, simplicial sets and $(\infty, 0)$-categories

Definition 3.4. Let X, Y be topological spaces or simplicial sets, and let $f\colon X \to Y$ be a map. Then f is a *weak homotopy equivalence* if it induces a bijection of connected components and isomorphisms in all homotopy groups:

$$\pi_i(X,x) \xrightarrow{\cong} \pi_i(Y,f(x)), \forall x \in X, \forall i \geq 1.$$

The homotopy category of topological spaces is obtained by formally inverting weak homotopy equivalences, and similarly with simplicial sets. In fact, the homotopy category of topological spaces is equivalent to the homotopy category of simplicial sets [10].

Given a topological space (or a simplicial set) X, we associate to it a structure denoted by $\Pi_{\leq\infty}X$ consisting of the following items:

- The objects of $\Pi_{\leq\infty}X$ are the points of X.
- Given a pair of objects, a 1-morphism between them is a path in X between these.
- Given a pair of objects and a pair of 1-morphisms between them, a 2-morphism between them is a homotopy with fixed endpoints.
- Given a pair of objects, a pair of 1-morphisms between them and a pair of 2-morphisms between them, a 3-morphism is a homotopy between the two given homotopies with fixed boundaries.
- And so on.

Note that composition of paths or homotopies is not strictly associative, since it requires reparametrization. Similarly, inverses of paths or homotopies are given by a reversal of parameter $t \mapsto 1 - t$, although such inverses are not strict, but up to homotopy. In fact, k-morphisms in $\Pi_{\leq\infty}X$ are invertible for all k, yet only up to higher morphisms. A structure with this property is called an *∞-groupoid*.

Any two homotopy equivalent topological spaces have equivalent ∞-groupoids, if we define equivalence between ∞-groupoids in an appropriate way. Moreover, each ∞-groupoid comes from some space. Lurie's approach to make these claims precise is to define (∞, n)-categories recursively, as detailed in the next subsection, starting with the convention that an $(\infty, 1)$-category is a *weak Kan complex* (i.e., a simplicial set satisfying the weak Kan condition [5]). Then an $(\infty, 0)$-category, that is, an ∞-groupoid, is an $(\infty, 1)$-category where all 1-morphisms are invertible, which corresponds to a (full) Kan complex. Since the homotopy category of Kan complexes is equivalent to the homotopy category of simplicial sets, this choice of models provides a rigorous formulation and proof of *Grothendieck's Homotopy Hypothesis*: The homotopy category of ∞-groupoids is equivalent to the homotopy category of topological spaces.

3.2.1 Segal categories

Segal categories are based on earlier work of Graeme Segal [13]. They will be used here to provide a precise definition of ∞-categories. There are other models for the homotopy theory of ∞-categories, such as complete Segal spaces, which were used by

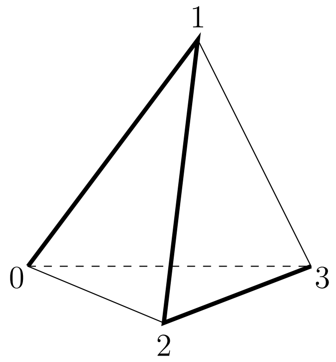


Figure 3.3: Principal edges

Lurie in [8].

Definition 3.5. A *Segal category* is a simplicial object A in the category of simplicial sets,

$$A\colon \Delta^{\mathrm{op}} \longrightarrow SSet,$$

such that

- The simplicial set A_0 is discrete.
- For $m \geq 1$, the following *Segal maps* are weak equivalences of simplicial sets:

$$A_m \xrightarrow{\prod p_i^*} A_1 \times_{A_0} \cdots \times_{A_0} A_1,$$

where p_i^* corresponds to the image under A of the following map in Δ:

$$\begin{aligned} p_i\colon [1] &\longrightarrow [n] \\ 0 &\longrightarrow i-1 \\ 1 &\longrightarrow i \end{aligned}$$

The Segal maps are morphisms of simplicial sets whose components are given by the principal edges of $[m]$. The morphisms $A_1 \to A_0$ in the fiber product $A_1 \times_{A_0} A_1$ are the images of the two inclusions $[0] \to [1]$, so an element of $A_1 \times_{A_0} A_1$ corresponds to two composable elements in A_1. For example, if $m = 3$, then the principal edges correspond to the thick edges in the tetrahedron in Figure 3.3.

The set A_0 is the *set of objects* of the Segal category A. Between any two objects X, Y in A_0 we now have a simplicial set of morphisms, which will be called *mapping space* and denoted by $\mathrm{Map}_A(X, Y)$. This mapping space is defined by the following pullback diagram:

$$\begin{tikzcd}
\mathrm{Map}_A(X,Y) \arrow[r] \arrow[d] & A_1 \arrow[d, "d_0^1 \times d_1^1"] \\
* \arrow[r, "{(X,Y)}"'] & A_0 \times A_0
\end{tikzcd}$$

The composition law in Segal categories is given by the Segal maps in A_2. Given two morphisms, i.e., two elements in A_1, the following weak equivalence allows us to draw the dotted arrow:

$$\begin{tikzcd}
A_2 \arrow[r, "p_1^* \times p_2^*"] \arrow[d, "d_1^2"'] & A_1 \times_{A_0} A_1 \arrow[dl, dashed] \\
A_1 &
\end{tikzcd}$$

In this case, p_1^* and p_2^* correspond to the faces d_2^2 and d_0^2 respectively. For a pair of composable morphisms f, g in $A_1 \times_{A_0} A_1$, the fact that $p_1^* \times p_2^*$ is a weak equivalence implies that there is an element $\alpha \in A_2$ such that $d_1^2(\alpha) = g \circ f$:

$$\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "g \circ f"'] \arrow[rr, phantom, shift left=3, "\alpha"] & & \bullet
\end{tikzcd}$$

Figure 3.4: Composition of 1-morphisms

Associativity is encoded in higher levels. In fact, the following diagram and Figure 3.3 provide the associativity for 1-morphisms:

$$\begin{tikzcd}
A_3 \arrow[r, "p_1^* \times p_2^* \times p_3^*"] \arrow[d, "d_1^2 \circ d_1^3 / d_1^2 \circ d_2^3"'] & A_1 \times_{A_0} A_1 \times_{A_0} A_1 \arrow[dl, dashed] \\
A_1 &
\end{tikzcd}$$

Definition 3.6. An $(\infty,1)$-*category* is a Segal category.

Definition 3.7. Every $(\infty,1)$-category A has an underlying category, called the *homotopy category*. It is denoted it by $\mathrm{h}A$ and defined in the following way:

- The objects of $\mathrm{h}A$ are the objects of A, that is the elements of A_0.
- The morphisms of $\mathrm{h}A$ are elements of $\pi_0(A_1)$.

Once having the definition, we need to take care of several aspects before iterating the process. We need a notion of good transformation between $(\infty,1)$-categories and a notion of weak equivalence between them. For the first, we just consider the category whose objects are functors $\Delta^{\mathrm{op}} \to SSet$, and whose morphisms are natural transformations between them. After discarding all functors that do not satisfy the Segal conditions we obtain a category with $(\infty,1)$-categories as objects and functors between them, which will be called $(\infty,1)$-*functors*. We denote this category by $(\infty,1)$-**Cat**.

Definition 3.8. Given two $(\infty,1)$-categories A and B, an $(\infty,1)$-functor

$$F\colon A \longrightarrow B$$

is a *weak equivalence* if it satisfies the following conditions:

- It is *fully faithful*: For all $X, Y \in A_0$, the induced map of $(\infty,0)$-categories

  $$\mathrm{Map}_{A_1}(X,Y) \longrightarrow \mathrm{Map}_{B_1}(FX,FY)$$

  is a weak equivalence.
- It is *essentially surjective*: The induced functor of homotopy categories

  $$[F]\colon \mathrm{h}A \longrightarrow \mathrm{h}B$$

  is an equivalence of categories.

Recall that a functor between ordinary categories is an *equivalence* if it is fully faithful (i.e., bijective on all morphism sets) and essentially surjective (i.e., bijective on isomorphism classes of objects).

Iterating the process: Segal n-categories

Having defined $(\infty,1)$-categories and a notion of weak equivalence between them, we now carry out the inductive step to recursively define (∞,n)-categories.

Lets us suppose that we have defined the concept of $(\infty, n-1)$-category and a notion of weak equivalence between $(\infty, n-1)$-categories, and let $(\infty, n-1)$-**Cat** stand for the category of $(\infty, n-1)$-categories.

Definition 3.9. An (∞, n)-*category* is a simplicial object in the category of $(\infty, n-1)$-categories,

$$A\colon \Delta^{\mathrm{op}} \longrightarrow (\infty, n-1)\text{-}\mathbf{Cat},$$

such that

- The $(\infty, n-1)$-category A_0 is discrete.
- For $m \geq 1$, the following Segal maps are weak equivalences of $(n-1)$-categories:

$$A_m \xrightarrow{\Pi p_i^*} A_1 \times_{A_0} \cdots \times_{A_0} A_1.$$

The underlying homotopy category $\mathrm{h}A$ of an (∞, n)-category A is defined as before:

- The objects of $\mathrm{h}A$ are the objects of A, that is the elements of A_0.
- The morphisms of $\mathrm{h}A$ are isomorphism classes of 1-morphisms in A.

We next define functors between (∞, n)-categories. An (∞, n)-*functor* is a natural transformation between functors $\Delta^{\mathrm{op}} \to (\infty, n-1)$-**Cat** satisfying the Segal conditions.

Finally, we generalize the notion of weak equivalence:

Definition 3.10. Given two (∞, n)-categories A and B, an (∞, n)-functor

$$F\colon A \longrightarrow B$$

is a *weak equivalence* if it satisfies the following conditions:

- It is *fully faithful*: For all $X, Y \in A_0$ the induced functor

  $$\mathrm{Map}_{A_1}(X, Y) \longrightarrow \mathrm{Map}_{B_1}(FX, FY)$$

  is a weak equivalence of $(\infty, n-1)$-categories.
- It is *essentially surjective*: The induced functor of homotopy categories

  $$[F]\colon \mathrm{h}A \longrightarrow \mathrm{h}B$$

  is an equivalence of categories.

We have given all the guidelines to iterate the process for recursively obtaining (∞, n)-categories. At this point, we would like to highlight the following fact: in an (∞, n)-category, between any two objects there is an $(\infty, n-1)$-category of morphisms.

Monoidal structure

An ordinary monoidal category can be viewed as *weak* 2-category with a single object. For (∞, n)-categories, we generalize this idea and define the property of being monoidal in the following way:

Definition 3.11. A *monoidal* (∞, n)*-category* is an $(\infty, n+1)$-category with a single isomorphism class of objects.

3.2.2 Duality and adjunction

We have observed several times in this thesis that values of TQFTs are vector spaces with restrictions and additional structure. In fact, the finite dimension condition is a consequence of the duality imposed by the functoriality of a TQFT. This notion of duality can be extended naturally for higher categories, and not only for objects, but also for k-morphisms for any k.

Dualizable objects

If $\mathscr{C}_\otimes$ is a symmetric monoidal category, we say that an object X in $\mathscr{C}$ has a *dual* if there exists another object Y in $\mathscr{C}$ and two morphisms $u\colon \mathbf{1}_\mathscr{C} \to Y \otimes X$ and $v\colon X \otimes Y \to \mathbf{1}_\mathscr{C}$ such that the following compositions are equal to the identity maps of X and Y respectively:

$$X \cong X \otimes \mathbf{1}_\mathscr{C} \xrightarrow{\mathrm{id}_X \otimes u} X \otimes Y \otimes X \xrightarrow{v \otimes \mathrm{id}_X} \mathbf{1}_\mathscr{C} \otimes X \cong X$$
$$Y \cong Y \otimes \mathbf{1}_\mathscr{C} \xrightarrow{u \otimes \mathrm{id}_Y} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}_Y \otimes v} \mathbf{1}_\mathscr{C} \otimes Y \cong Y$$

Definition 3.12. Let $A_\otimes$ be a symmetric monoidal (∞, n)-category. We say that an object X is *dualizable* if it admits a dual when regarded as an object of the homotopy category $\mathrm{h}A_\otimes$.

Adjoints

The concept of adjunction in higher categories is closely related to the classical adjunction between functors, so it is worth reminding it. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ a pair of functors. The functor F is said to be a *left* *adjoint* for G if for all $X \in \mathrm{Ob}(\mathscr{C})$ and for all $Y \in \mathrm{Ob}(\mathscr{D})$ there is a natural bijection $\Phi_{X,Y}$ between the following sets:

$$\mathrm{Hom}_{\mathscr{D}}(FX, Y) \xleftrightarrow{\Phi_{X,Y}} \mathrm{Hom}_{\mathscr{C}}(X, GY).$$

A pair of adjoint functors naturally define a unit map and a counit map, so an equivalent definition (very similar to the duality condition) of an adjunction can be given [10]: Given two categories $\mathscr{C}$, $\mathscr{D}$ and a pair of functors $F\colon \mathscr{C} \to \mathscr{D}$, $G\colon \mathscr{D} \to \mathscr{C}$, then F is a left adjoint for G if there exist two natural transformations $u\colon \mathrm{id}_{\mathscr{C}} \to G \circ F$ and $v\colon F \circ G \to \mathrm{id}_{\mathscr{D}}$ such that the following composites are the identities of F and G repectively:

$$F = F \circ id_{\mathscr{C}} \xrightarrow{\mathrm{id}_F \times u} F \circ G \circ F \xrightarrow{v \times \mathrm{id}_F} \mathrm{id}_{\mathscr{D}} \circ F = F$$
$$G = G \circ id_{\mathscr{D}} \xrightarrow{u \times \mathrm{id}_G} G \circ F \circ G \xrightarrow{\mathrm{id}_G \times v} \mathrm{id}_{\mathscr{C}} \circ G = G$$

The definition of an adjunction will be first given for 2-categories and recursively defined for (∞, n)-categories afterwards.

Definition 3.13. Let $\mathscr{C}$ be a 2-category, X and Y two objects in $\mathscr{C}$ and $f\colon X \to Y$ and $g\colon Y \to X$ 1-morphisms. Then f is a *left adjoint* for g if there exist 2-morphisms $u\colon \mathrm{id}_X \Rightarrow g \circ f$ and $v\colon \mathrm{id}_Y \Rightarrow f \circ g$ such that the following compositions are equal to the identity 2-morphisms. The operation $\times$ in 2-morphisms has to be interpreted as in Figure 3.5:

$$f = f \circ \mathrm{id}_X \xRightarrow{\mathrm{id}_f \times u} f \circ g \circ f \xRightarrow{v \times \mathrm{id}_f} \mathrm{id}_X \circ f = f$$
$$g = \mathrm{id}_Y \circ g \xRightarrow{u \times \mathrm{id}_g} g \circ f \circ g \xRightarrow{\mathrm{id}_g \times v} g \circ \mathrm{id}_Y = g$$

$$\begin{tikzcd}
X \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[dr, bend right, "f"'] \arrow[rr, phantom, "\Downarrow u"] & & X \arrow[r, "f"] & Y \\
& Y \arrow[ur, bend right, "g"'] & &
\end{tikzcd}$$

Figure 3.5: Diagram of $\mathrm{id}_f \times u$

Definition 3.14. Given an (∞, n)-category A, we define the *homotopy* 2*-category* $\mathrm{h}_2 A$ as follows:

- The objects of h_2A are the objects of A.
- The 1-morphisms of h_2A are the 1-morphisms of A.
- Given a pair of objects X, Y in A and a pair of 1-morphisms $f, g \colon X \to Y$, a 2-morphisms from f to g in h_2A is an isomorphism class of 2-morphisms from f to g in A.

Definition 3.15. We will say that a 1-morphism in an (∞, n)-category A is *adjointable* if it is part of a pair of adjoints in the homotopy 2-category h_2A.

Definition 3.16. Given an (∞, n)-category A and a pair of k-morphisms, we say they are *adjointable* if they are adjointable as $(k-1)$-morphisms in the $(\infty, n-1)$-category of 1-morphisms in A.

Lemma 3.1. *Let $f \colon X \to Y$ and $g \colon Y \to X$ be n-morphisms in an (∞, n)-category. If they are adjointable, then they are invertible.*

Proof. Let u and v be the unit and counit of the adjunction. By the definition of (∞, n)-category they are invertible, so they yield isomorphisms

$$f \circ g \cong id_Y \qquad\qquad g \circ f \cong id_X,$$

showing that f and g are inverse to each other up to isomorphism. □

3.3 The (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$

The last step is to introduce the (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$. Omitting extra structure on the manifolds we can intuitively think of $\mathbf{Bord}_n$ as the following:

- The objects of $\mathbf{Bord}_n$ are manifolds of dimension 0.
- The 1-morphisms of $\mathbf{Bord}_n$ are cobordisms between manifolds of dimension 0.
- The 2-morphisms of $\mathbf{Bord}_n$ are cobordisms between cobordisms between manifolds of dimension 0, i.e., manifolds of dimension 2 with corners.
- $\cdots$
- The n-morphisms of $\mathbf{Bord}_n$ are cobordisms between $(n-1)$-morphisms, i.e., manifolds of dimension n with corners.
- The $(n+1)$-morphisms of $\mathbf{Bord}_n$ are diffeomorphisms.

- The $(n+2)$-morphisms of $\mathbf{Bord}_n$ are isotopies between diffeomorphisms.
- $\cdots$

In ordinary TQFTs, orientation was a tool to determine the direction of cobordisms as morphisms. This is generalized over $\mathbf{Bord}_n$ by considering manifolds with *framings*.

Definition 3.17. Let M be a manifold of dimension n and TM its tangent bundle. A *framing* on M is a trivialization of TM, that is, an isomorphism with the trivial bundle $\mathbb{R}^n$. More generally, for $m \geq n$, we define an *m-framing* as a trivialization of the bundle $TM \oplus \mathbb{R}^{n-m}$.

In order to state the Cobordism Hypothesis as in Lurie's work [8], we need to describe an (∞, n)-category which is a model for $\mathbf{Bord}_n^{\mathrm{fr}}$. Although Lurie did it by means of *complete Segal spaces*, we present it here using Segal categories as models for simplicity. While the proof of the Cobordism Hypothesis in [8] seems to require the full machinery of complete Segal spaces, the homotopy category of these is equivalent to the homotopy category of Segal categories, and therefore the latter suffice in order to give a precise statement, which is indeed our aim.

Definition 3.18. We define $\mathbf{Bord}_n^{\mathrm{fr}}$ as the Segal n-category obtained by discretizing the zero space $(P_n)_{0,\ldots,0}$ of the following n-fold simplicial space P_n: For every n-tuple of nonnegative integers $k_1, \ldots, k_n$ and every finite-dimensional vector space V over $\mathbb{R}$, let

$$(P_n^V)_{k_1,\ldots,k_n} = \{(M, \{t_0^1 \leq \cdots \leq t_{k_1}^1\}, \ldots, \{t_0^n \leq \cdots \leq t_{k_n}^n\})\},$$

where M is an n-dimensional framed manifold properly embedded into $V \times \mathbb{R}^n$ and, for every subset $S \subseteq \{1, \ldots, n\}$ and every collection $\{0 \leq r_i \leq k_i\}_{i \in S}$, the composite map $M \to \mathbb{R}^n \to \mathbb{R}^S$ does not have $(r_i^i)_{i \in S}$ as a critical value. Then P_n is defined by passing to the direct limit the family P_n^V as V ranges over all finite-dimensional subspaces of Euclidean spaces.

For example, if $n = 1$, then P_1^V is a simplicial space such that, for each $k \geq 0$, the space $(P_1^V)_k$ consists of pairs $(M, \{t_0 \leq \cdots \leq t_k\})$ where M is a 1-manifold properly embedded into $V \times \mathbb{R}$ which is cut into $k+2$ pieces at the slices $V \times \{t_i\}$. For bigger values of n, the condition on the critical values is imposed in order to ensure that gluing is made along manifolds.

Discretization of a space amounts to replacing each connected component by a single point; a more precise definition is given in [3, §6], together with a proof that the homotopy categories of complete Segal spaces and Segal categories are equivalent. Furthermore, it is shown in [8] that $\mathbf{Bord}_n^{\mathrm{fr}}$ is a symmetric monoidal (∞, n)-category.

3.4 The Cobordism Hypothesis

Now we can define the notion of an extended TQFT in a precise way.

Definition 3.19. Given an (∞, n)-category $\mathscr{C}$, an *extended topological quantum field theory* of dimension n with values in $\mathscr{C}$ is a symmetric monoidal (∞, n)-functor

$$Z\colon \mathbf{Bord}_n^{\mathrm{fr}} \longrightarrow \mathscr{C}.$$

The following remarks are contained in [8, §2.3, §2.4].

Lemma 3.2. *Let $\mathscr{C}$ be a symmetric monoidal (∞, n)-category. There is an (∞, n)-category $\mathscr{C}^{\mathrm{fd}}$ and an (∞, n)-functor $i\colon \mathscr{C}^{\mathrm{fd}} \longrightarrow \mathscr{C}$ such that:*

- *Every object in $\mathscr{C}^{\mathrm{fd}}$ is fully dualizable and every k-morphism, for $1 \leq k \leq n-1$, is adjointable.*
- *$i\colon \mathscr{C}^{\mathrm{fd}} \longrightarrow \mathscr{C}$ is universal with respect to this condition.*

Proof. (Sketch) Take as $\mathscr{C}^{\mathrm{fd}}$ the smallest Segal category containing the subcategory of $\mathscr{C}$ resulting from discarding the non-adjointable morphisms.

One proves similarly:

Lemma 3.3. *Let $\mathscr{D}$ be an (∞, n)-category. There is an $(\infty, 0)$-category $\mathscr{D}^{\sim}$ and an (∞, n)-functor $j\colon \mathscr{D}^{\sim} \longrightarrow \mathscr{D}$ such that:*

- *Every k-morphism, $k > 0$, in $\mathscr{D}^{\sim}$ is invertible.*
- *$j\colon \mathscr{D}^{\sim} \longrightarrow \mathscr{D}$ is universal with respect to this condition.*

The construction of this $(\infty, 0)$-category $\mathscr{D}^{\sim}$ can be achieved by removing all non-invertible morphisms in $\mathscr{D}$.

Lemma 3.4. *Let $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathscr{C})$ denote the (∞, n)-category of monoidal functors from $\mathbf{Bord}_n^{\mathrm{fr}}$ to $\mathscr{C}$. Then k-morphisms in $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathscr{C})$ are invertible for all $k \geq 1$. Therefore $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathscr{C})$ is an $(\infty, 0)$-category.*

Proof. (Sketch) This is shown, as outlined in [8, 2.4.7], by reversing the orientation of framings on manifolds in order to provide a homotopy inverse for each natural transformation $Z \to Z'$ of extended TQFTs.

These definitions are the last steps that we need to finalize our work.

Cobordism Hypothesis. *Let $\mathscr{C}$ be a symmetric monoidal (∞, n)-category. The evaluation functor $Z \to Z(\bullet^+)$ induces a weak equivalence of spaces*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathscr{C}) \to (\mathscr{C}^{\mathrm{fd}})^{\sim}.$$

This is a precise formulation of the original statement written down by Baez and Dolan in [2]. With this terminology and machinery, a proof was given by Lurie in [8].

Further work

We would like to conclude this thesis by discussing some possible lines of future work. Here, we have chosen a particular approach towards the Cobordism Hypothesis. Although our description is complete, there are several aspects, such as the definition of complete Segal spaces, that require a deeper understanding. Moreover, the formulation of the Cobordism Hypothesis is not unique. Several variants can be found, for example, in [4] and [8]. Finally, there are applications of the Cobordism Hypothesis in topology, algebra and representation theory, that can be explored further.

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