Optimal Sobolev Embeddings in Spaces with Mixed Norm

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CERTIFICA:

Que la presente memoria ha sido realizada, bajo su dirección, por Nadia F. Clavero y que constituye su tesis para aspirar al grado de Doctora en Matemáticas.

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Resumen

Nuestro principal objetivo en este trabajo es estudiar las estimaciones clásicas de las inclusiones de Sobolev, probadas por Gagliardo [29] y Nirenberg [50], para derivadas de orden superior y espacios más generales. En particular, estamos interesados en describir el dominio y el rango óptimos para estas inclusiones entre los espacios invariantes por reordenamiento (r.i.) y espacios de normas mixtas.

Los primeros trabajos en esta línea de investigación fueron presentados por Sobolev [59] en los años treinta, quien introdujo los espacios que hoy en día reciben su nombre:

$$W^mL^p(I^n) = \Big\{u \in L^p(I^n) : |D^m u| \in L^p(I^n)\Big\}, \quad m \in \mathbb{N}\,,\ 1 \le p \le \infty,$$

donde $D^m u$ denota el vector de todas las derivadas parciales $\partial^\alpha u$, con $0 \le |\alpha| \le m$, $\alpha \in (\mathbb{N} \cup \{0\})^n$ y $|D^m u|$ es su norma Euclídea, y también demostró su teorema clásico (para más detalles véase [1, 45]):

$$W^1L^p(I^n) \hookrightarrow L^{pn/(n-p)}(I^n), \quad 1 \le p < n. \tag{1}$$

En realidad, su método no funcionaba para $p = 1$ y solo a finales de los cincuenta, Gagliardo [29] y Nirenberg [50] probaron este caso excepcional. Sus trabajos, basados en realizar estimaciones en secciones lineales de una función, dieron lugar a introducir los espacios de normas mixtas (véase la Definición 3.1.2):

$$\mathcal{R}(X,Y) = \bigcap_{k=1}^{n} \mathcal{R}_k(X,Y),$$

donde $\mathcal{R}_k(X,Y)$ es el espacio de Benedek-Panzone (véase la Definición 3.1.1) cuya norma viene dada por:

$$\|f\|_{\mathcal{R}_k(X,Y)} = \|\psi_k(f,Y)\|_{X(I^{n-1})}, \quad \psi_k(f,Y)(\widehat{x_k}) = \|f(\widehat{x_k},\cdot)\|_{Y(I)}.$$

Concretamente, Gagliardo [29] y Nirenberg [50] observaron que:

$$W^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty), \tag{2}$$

y luego, utilizando una forma iterada de la desigualdad de Hölder, concluyeron que:

$$W^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(I^n). \tag{3}$$

Más tarde, una serie de resultados sobre espacios de normas mixtas fue presentada por Fournier [28], y posteriormente desarrollada, vía diferentes métodos, por

numerosos autores, tales como Blei y Fournier [10], Milman [47], Barza, Kamińska, Persson y Soria [4], Algervik y Kolyada [2], Kolyada [37, 38, 39] y Kolyada y Soria [40]. Concretamente, la parte central del trabajo de Fournier [28] fue probar que

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n), \tag{4}$$

y luego, teniendo en cuenta la estimación (2), obtuvo la siguiente mejora del teorema clásico de Sobolev (1):

$$W^1L^1(I^n) \hookrightarrow L^{n',1}(I^n). \tag{5}$$

Cabe señalar que esta inclusión fue demostrada con anterioridad por Poornima [55].

Desde que la inclusión (4) fue probada por primera vez, numerosas demostraciones y diferentes extensiones han aparecido en la literatura. En particular, relaciones entre los espacios de norma mixta de espacios de Lorentz fueron estudiadas en [2] donde, por ejemplo, se probó:

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)',1}, L^1), \quad n \geq 2. \tag{6}$$

Recientemente, extensiones de (1) a pares de espacios r.i. arbitrarios han sido exhaustivamente estudiadas por diversos autores, tales como Edmunds, Kerman y Pick [27], Kerman, y Pick [36], Martín, Milman y Pustylnik [44] y Cianchi [19]. Concretamente, Kerman y Pick [36] determinaron condiciones necesarias y suficientes para que las siguientes inclusiones, que involucran espacios r.i. $X(I^n)$ y $Z(I^n)$, se cumplan:

$$W^mZ(I^n) \hookrightarrow X(I^n), \quad m \in \mathbb{N}, \tag{7}$$

obteniendo así una versión unificada de la teoría clásica anteriormente mencionada. Dicha caracterización fue utilizada para estudiar los problemas del dominio y rango óptimos para tales inclusiones (7), dentro de la clase de los espacios r.i. De este modo, extendieron las estimaciones clásicas (5) de Poornima [55] y Peetre [54], y el denominado caso límite o crítico de la inclusión de Sobolev, probada por Hansson [33], Brezis y Wainger [15], y Maz'ya [45], a derivadas de orden superior y, como nueva contribución, demostraron la optimalidad de tales estimaciones en el contexto de los espacios r.i. En particular, estos autores describieron totalmente los mejores rangos posibles en (7), cuando $Z(I^n) = L^p(I^n)$:

(i) si $1 \leq p < m/n$, entonces el espacio de Lorentz $L^{np/(n-mp),p}(I^n)$ es el rango óptimo en la inclusión:

$$W^mL^p(I^n) \hookrightarrow L^{np/(n-mp),p}(I^n). \tag{8}$$

Es decir, si (8) se cumple con $L^{np/(n-mp),p}(I^n)$ reemplazado por otro espacio r.i. , entonces este último debe contener a $L^{np/(n-mp),p}(I^n)$;

(ii) si $p = n/m$, el espacio Lorentz-Zygmund $L^{\infty,n/m;-1}(I^n)$ es el menor rango r.i. que satisface

$$W^mL^{n/m}(I^n) \hookrightarrow L^{\infty,n/m;-1}(I^n); \tag{9}$$

(iii) si $p > n/m$, entonces $L^\infty(I^n)$ es el menor espacio r.i. que verifica

$$W^m L^p(I^n) \hookrightarrow L^\infty(I^n).$$

Todos estos trabajos nos motivan a considerar las estimaciones (2) de orden arbitrario en el marco de los espacios r.i. , así como también, describir los dominios y los rangos óptimos para tales inclusiones entre los espacios r.i. y los espacios de normal mixta. Como consecuencia, nuestros objetivos son:

(i) obtener propiedades funcionales de los espacios de norma mixta;

(ii) estudiar el problema del dominio-rango óptimo para los siguientes tipos de inclusiones de Sobolev:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, Y), \quad m \in \mathbb{N}.$$

A continuación presentamos brevemente los principales contenidos que desarrollaremos en esta tesis doctoral:

Con el fin de llevar a cabo este proyecto como monografía auto-contenida, estudiamos, en el capítulo 2, conceptos y resultados básicos de la teoría de espacios invariantes por reordenamiento que serán utilizados a lo largo de este trabajo. Aquí, entre otras cosas, introducimos los espacios invariantes por reordenamiento (véase la Definición 2.1.6), es decir, espacios funcionales de Banach, donde las estimaciones en norma dependen exclusivamente de los conjuntos de nivel de la función. Además, recordamos los siguientes ejemplos: espacios de Lebesgue, espacios de Orlicz, espacios de Lorentz y sus generalizaciones, tales como espacios de Lorentz-Zygmund.

En este capítulo, también recopilamos algunas definiciones y resultados de la Teoría de Interpolación. En particular, recordamos la definición del K-funcional de Peetre, que será utilizado en diferentes contextos. Entre los resultados clásicos, mencionamos el teorema de Holmstedt, el cual proporciona una expresión explícita para el K-funcional de Peetre asociado a pares de espacios de Lorentz (para más detalles véase el Teorema 2.1.18). Finalmente, definimos los espacios de Sobolev (véase la Definición **??**).

En el capítulo 3, estudiamos propiedades de los espacios de norma mixta, focalizando nuestro interés en algunos resultados de la Teoría de Interpolación y relaciones entre espacios de norma mixta y espacios r.i.

Para tal fin, introducimos los espacios de Benedek-Panzone $\mathcal{R}_k(X, L^\infty)$ y los espacios de norma mixta $\mathcal{R}(X, L^\infty)$. Asimismo, calculamos el K-funcional de Peetre asociado a algunas parejas de espacios de norma mixta. De este modo, encontramos expresiones explícitas para los casos $(\mathcal{R}(X, L^\infty), L^\infty)$ y $(L^p, \mathcal{R}(L^p, X))$ (véase el Teorema 3.1.7 y el Teorema 3.1.11, respectivamente). Como consecuencia, probamos que el espacio de interpolación real

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta,q}, \quad 0 < \theta < 1, \ \leq q \leq \infty,$$

puede obtenerse a partir de $(X, L^\infty)_{\theta,q}$ del siguiente modo (véase el Corolario 3.1.8):

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta,q} = \mathcal{R}((X, L^\infty)_{\theta,q}, L^\infty).$$

Por otro lado, exploramos conexiones entre los espacios de norma mixta. Es importante observar que, para cualquier $k \in \{1, \dots, n\}$,

$$\mathcal{R}_k(X_1, Y_1) \hookrightarrow \mathcal{R}_k(X_2, Y_2) \Leftrightarrow \begin{cases} X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}), \\ Y_1(I) \hookrightarrow Y_2(I), \end{cases} \tag{10}$$

(véase el Lema 3.2.1). Es decir, mirando a cada componente específica, las inclusiones son triviales. Sin embargo hay casos, por ejemplo (6), que prueban que si en (10) reemplazamos los espacios de Benedek-Panzone por los espacios de norma mixta globales, entonces la correspondiente equivalencia ya no es cierta. Este hecho ilustra que los espacios de norma mixta pueden tener una estructura mucho más complicada que los espacios de Benedek-Panzone. Por consiguiente, es natural analizar sus inclusiones en este contexto (véase también [31]).

Motivados por este problema, encontramos condiciones necesarias y suficientes para que se verifiquen los siguientes tipos de inclusiones:

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2), \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty), \quad \mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y).$$

Además, extendemos (6) a espacios r.i. más generales, mejorando, aún más, la estimación de norma mixta probada por Algervik y Kolyada [2]. Concretamente, el Teorema 3.2.14 aporta una descripción del menor espacio rango $\mathcal{R}(X_2, L^1)$ en

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^1),$$

cuando el espacio de norma mixta $\mathcal{R}(X_1, L^\infty)$ es dado.

Asimismo, establecemos una extensión de la estimación clásica (4), probada por Fournier [28], al caso donde los espacios de Lebesgue son reemplazados por espacios r.i. más generales. Concretamente, encontramos condiciones necesarias y suficientes sobre los espacios r.i. $X(I^n)$ y $Z(I^n)$ bajo las cuales se cumple la siguiente inclusión (véase el Teorema 3.3.2):

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n). \tag{11}$$

Una consecuencia general del Teorema 3.3.2 está implícita en el Teorema 3.3.5, que proporciona una caracterización del mayor espacio de la forma $\mathcal{R}(X, L^\infty)$ en (11) correspondiente a un espacio r.i. $Z(I^n)$ previamente fijado. Además, para un espacio de norma mixta dado $\mathcal{R}(X, L^\infty)$, el Teorema 3.3.8 describe el menor espacio r.i. $Z(I^n)$ que satisface (11).

Un operador que será de importancia en este trabajo es el operador de Hardy

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds, \quad f \in \mathcal{M}_+(0,1),\ t > 0, \tag{12}$$

que, junto con sus diversas modificaciones, constituye una herramienta muy útil para estudiar el problema del dominio-rango óptimo en las inclusiones de tipo Sobolev.

Importantes propiedades intrínsecas de este operador, como la acotación y la compacidad en diferentes espacios funcionales, han sido intensamente estudiadas desde el siglo pasado. En particular, numerosos autores, tales como Hardy, Littlewood y Pólya [34], Muckenhoupt [49], Bradley [13], Maz'ya [45] y Sinnamon [57] dieron una completa caracterización de los pesos v para los cuales el operador (12) es acotado sobre espacios de Lebesgue $L^p(v)$. Desde entonces, diferentes extensiones al caso n-dimensional han aparecido en la literatura. Específicamente, Sinnamon [56] consideró el operador de Hardy n-dimensional:

$$P_n f(x) = \int_0^1 f(sx)ds, \quad f \in \mathcal{M}_+(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \tag{13}$$

y caracterizó completamente las desigualdades con peso para este operador.

Motivados por estos resultados, en el capítulo 4, estudiamos los operadores de tipo Hardy, recordando algunos resultados conocidos que serán usados posteriormente. Asimismo describimos para qué pares de parámetros p y q se cumple (véase el Teorema 4.2.2)

$$P_n : \mathcal{R}(L^p, L^\infty) \to L^q(\mathbb{R}^n).$$

Además, encontramos condiciones necesarias y suficientes sobre los espacios r.i. $X(\mathbb{R})$ e $Y(\mathbb{R}^2)$ tales que cumplen (véase el Corolario 4.2.5):

$$P_2 : \mathcal{R}(X, L^\infty) \to Y(\mathbb{R}^2).$$

El capítulo 5 está dedicado a estudiar las inclusiones de la forma

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty), \quad m \in \mathbb{N}, \tag{14}$$

extendiendo la estimación clásica (2). Siguiendo las ideas de [36], establecemos la equivalencia entre (14) y la acotación de un operator de tipo Hardy (véase el Teorema 5.1.5). Esta relación es luego una herramienta clave para determinar el domino óptimo, y el rango óptimo, para (14) entre los espacios r.i. y los espacios de norma mixta.

Después de esta discusión, nuestro análisis se centra en dar construcciones explícitas de tales espacios óptimos. En particular, el Teorema 5.2.1 proporciona una caracterización del menor espacio de la forma $\mathcal{R}(X, L^\infty)$ en (14), cuando el dominio r.i. es dado. Finalmente, para un espacio de norma mixta $\mathcal{R}(X, L^\infty)$ fijado previamente, el Teorema 5.3.2 describe el mayor espacio r.i. $Z(I^n)$ que verifica (14).

Todos estos resultados son luego utilizados para establecer las inclusiones clásicas de Sobolev en este contexto. Resumiendo, obtenemos una completa caracterización de los rangos óptimos de la forma $\mathcal{R}(X, L^\infty)$ en (14) cuando $Z(I^n) = L^p(I^n)$:

(i) si $1 \leq p < m/n$, entonces $\mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty)$ es el menor rango de la forma $\mathcal{R}(X, L^\infty)$ satisfaciendo

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty);$$

(ii) si $p = n/m$, entonces $\mathcal{R}(L^{\infty,n/m;-1}, L^\infty)$ es el espacio de norma mixta óptimo que verifica

$$W^m L^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty); \tag{15}$$

(iii) si $p > n/m$, entonces $\mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n)$ es el menor rango de la forma $\mathcal{R}(X, L^\infty)$ que cumple:

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n).$$

Como consecuencia, recuperamos la estimación (5) y, como nueva contribución, demostramos su optimalidad en el contexto de los espacios de norma mixta.

Como hemos mencionado anteriormente, el problema del rango óptimo para las inclusiones de Sobolev fue estudiado en [36] dentro de la clase de los espacios r.i. Concretamente, para un dominio r.i. fijo $Z(I^n)$, estos autores determinaron el menor espacio rango r.i. $X^{\mathrm{op}}(I^n)$ que satisface (7). Motivados por este problema, comparamos el espacio rango óptimo r.i. con el espacio de norma mixta óptimo. En particular, el Teorema 5.4.3 prueba que la siguiente cadena de inclusiones se cumple:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n),$$

donde $\mathcal{R}(X, L^\infty)$ es el espacio de norma mixta construido en el Teorema 5.2.1. Por consiguiente, probamos que las estimaciones clásicas para el espacio de Sobolev estándar W^1L^p de Poornima [55] y Peetre [54] $(1 \leq p < n)$, y de Hansson [33], Brezis y Wainger [15] y Maz'ya [45] $(p = n)$ pueden mejorarse aún más considerando espacios de norma mixta en los rangos. Así, por ejemplo, concluimos que:

(i) si $1 \leq p < m/n$, entonces

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty) \underset{\neq}{\hookrightarrow} L^{np/(n-mp),p}(I^n);$$

(ii) si $p = n/m$, entonces

$$W^m L^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n/m;-1}(I^n). \tag{16}$$

Aunque la estimación (9) es la mejor posible en cuanto a espacios rangos r.i. se refiere, sin embargo Bastero, Milman y Ruiz [5], e independientemente Malý y Pick [43], demostraron que, si el requisito de que el rango sea un espacio lineal es eliminado, entonces una mejora de (9), con $m = 1$, es todavía posible. Para este fin, introdujeron la clase funcional no lineal definida por

$$L(\infty, n)(I^n) = \left\{ f \in \mathcal{M}(I^n) : \|f\|_{L(\infty,n)(I^n)} = \left(\int_0^1 \left(f^{**}(t) - f^*(t) \right)^n \frac{dt}{t} \right)^{1/n} < \infty \right\},$$

y luego, usando una versión débil de la inclusión Sobolev-Gagliardo-Nirenberg (3) con un argumento de truncación, probado por Mazýa, concluyeron que

$$W_0^1 L^n(I^n) \hookrightarrow L(\infty, n)(I^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n). \tag{17}$$

Motivados por estos trabajos, en el capítulo 6, reformulamos sus resultados en términos de los espacios de norma mixta. Es decir, encontramos un refinamiento de (15), con $m = 1$, en el contexto de los espacios no lineales de la forma $\mathcal{R}(X, L^\infty)$.

Concretamente, introducimos la clase funcional $\mathcal{R}(L(\infty, n), L^\infty)$ y establecemos algunas de sus propiedades básicas (véase el Lema 6.2.6). Así, por ejemplo, demostramos que dicho conjunto no es lineal y, tambíen, deducimos que

$$\mathcal{R}(L(\infty, n), L^\infty)) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Por consiguiente, siguiendo el mismo planteamiento que en [5, 43], obtenemos (véase el Teorema 6.2.7):

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Finalmente, teniendo en cuenta (16), con $m = 1$, y (17), comparamos el espacio no lineal $L(\infty, n)$ con $\mathcal{R}(L^{\infty,n;-1}, L^\infty)$. De hecho, concluimos que dichos espacios no son comparables (véase el Teorema 6.3.2 y el Teorema 6.3.3).

En el capítulo 7, estudiamos inclusiones de Sobolev de la forma:

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1). \tag{18}$$

En particular, obtenemos condiciones necesarias y suficientes para la existencia de dichas estimaciones (véase el Teorema 7.1.6). Para ello, aplicamos un método basado en el estudio de inclusiones entre espacios de Sobolev (construidos a partir de espacios de normas mixtas). Concretamente, en el Teorema 7.1.3 probamos que se cumple

$$W^1 \mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1), \tag{19}$$

si y solo si se satisface la siguiente inclusión de Sobolev

$$W^1 Z(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Esta equivalencia es crucial en la discusión posterior, donde realizamos un exhaustivo estudio del problema del rango óptimo para (18) y (19). Concretamente, fijado un dominio r.i., describimos el mejor rango posible de la forma $\mathcal{R}(X, L^1)$ en (18) (véase el Teorema 7.2.1). Asimismo, dado un espacio dominio $\mathcal{R}(Z, L^1)$, caracterizamos el menor espacio $\mathcal{R}(X, L^1)$ que cumple (19) (véase el Teorema 7.2.2). Como consecuencia, obtenemos nuevas inclusiones, con espacios de llegada de la forma $\mathcal{R}(X, L^1)$ óptimos, para los espacios de Sobolev de primer orden $W^1 L^p$ y $W^1 \mathcal{R}(L^p, L^1)$ (véase el Corolario 7.2.3 y el Corolario 7.2.5):

(i) si $1 \leq p < n-1$, entonces $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ es el menor espacio rango de la forma $\mathcal{R}(X, L^1)$ satisfaciendo

$$W^1 L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1), \tag{20}$$

y

$$W^1 \mathcal{R}(L^p, L^1) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1);$$

(ii) si $p = n-1$, entonces $\mathcal{R}(L^{\infty,n-1;-1}, L^1)$ es el rango óptimo en

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1),$$

y

$$W^1\mathcal{R}(L^{n-1}, L^1) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1);$$

(iii) si $p > n-1$, entonces $\mathcal{R}(L^\infty, L^1)$ es el menor espacio de norma mixta que verifica

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1) \quad \text{y} \quad W^1\mathcal{R}(L^p, L^1) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Posteriormente, resolvemos el problema del dominio óptimo para las inclusiones de la forma (20). Es decir, dado un espacio de norma mixta $\mathcal{R}(X, L^1)$, construimos el mayor dominio r.i. que verifica (18) (véase el Teorema 7.3.2).

Como hemos mencionado anteriormente, los espacios de norma mixta de la forma $\mathcal{R}(X, L^\infty)$ y $\mathcal{R}(X, L^1)$ son espacios funcionales que tienen intersección no vacía (por ejemplo, $L^\infty(I^n)$ está contenido a ambos). Motivados por este hecho, consideramos $Z(I^n) = L^p(I^n)$ y comparamos sus rangos óptimos del tipo $\mathcal{R}(X, L^\infty)$ y $\mathcal{R}(X, L^1)$.

Uno de los primeros resultados, en esta dirección, fue dado por Algervik y Kolyada [2], quienes probaron que la inclusión (20), con $p = 1$, puede mejorarse permitiendo espacios de diferente índole, tales como los espacios de norma mixta de la forma $\mathcal{R}(X, L^\infty)$. Es decir, demostraron que

$$W^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{(n-1)',1}, L^1). \tag{21}$$

Podríamos pensar que (21) debería verificarse para cualquier inclusión de Sobolev de primer orden. Sin embargo, veremos algunos ejemplos que demuestran que tales espacios óptimos no son comparables en general, como ocurre en las desigualdades para espacios de Sobolev W^1L^p (véase el Ejemplo 7.4.2).

Finalmente, buscamos una relación entre el rango óptimo de la forma $\mathcal{R}(X, L^1)$ en (18) y el menor espacio r.i. en (7), cuando $Z(I^n) = L^p(I^n)$. En particular, concluimos que tales espacios tampoco son comparables en general.

Los resultados de los capítulos 3, 5, 6 y 7 se incluyen en [23], [24], [22] y [25], respectivamente.

Chapter 1

Introduction

The main purpose of this work is to study the classical Sobolev-type inequalities due to Gagliardo [29] and Nirenberg [50] for higher order derivatives and more general spaces. In particular, we concentrate on seeking the optimal domains and the optimal ranges for these embeddings between rearrangement-invariant spaces (r.i.) and mixed norm spaces.

The pioneering works in this line of research were given by Sobolev [59] in the 1930s, who introduced the spaces that now bear his name:

$$W^m L^p(I^n) = \Big\{ u \in L^p(I^n) : |D^m u| \in L^p(I^n) \Big\}, \quad m \in \mathbb{N}, 1 \leq p \leq \infty,$$

where $D^m u$ denotes the vector of all partial derivatives $\partial^\alpha u$, with $0 \leq |\alpha| \leq m$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, and $|D^m u|$ is its Euclidean norm and, at the same time, he proved his nowadays classical theorem (for more details see [1, 45]):

$$W^1 L^p(I^n) \hookrightarrow L^{pn/(n-p)}(I^n), \quad 1 \leq p < n. \tag{1.1}$$

Actually, his proof did not apply when $p = 1$, and only at end of 1950s Gagliardo [29] and Nirenberg [50] found a method which worked in that exceptional case. Their ideas, based on the behaviour of linear sections of functions, led to introduce the mixed norm spaces (see Definition 3.1.2):

$$\mathcal{R}(X, Y) = \bigcap_{k=1}^{n} \mathcal{R}_k(X, Y),$$

where $\mathcal{R}_k(X, Y)$ is the Benedek-Panzone space (see Definition 3.1.1) , with norm given by

$$\|f\|_{\mathcal{R}_k(X,Y)} = \|\psi_k(f, Y)\|_{X(I^{n-1})}, \quad \psi_k(f, Y)(\widehat{x_k}) = \|f(\widehat{x_k}, \cdot)\|_{Y(I)}.$$

Specifically, Gagliardo [29] and Nirenberg [50] observed that

$$W^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty), \tag{1.2}$$

and then, using an iterated form of Hölder's inequality, deduced that:

$$W^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(I^n). \tag{1.3}$$

Later, a new approach based on properties of mixed norm spaces was introduced by Fournier [28] and was subsequently developed, via different methods, by various authors, including Blei and Fournier [10], Milman [47], Barza, Kamińska, Persson, and Soria [4], Algervik and Kolyada [2], Kolyada [37, 38, 39], and Kolyada and Soria [40]. To be more precise, the central part of Fournier's work was to prove that

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n), \tag{1.4}$$

and then taking into account (1.2), he obtained the following improvement of (1.1):

$$W^1L^1(I^n) \hookrightarrow L^{n',1}(I^n). \tag{1.5}$$

It should be mentioned that the embedding (1.5) was first proved by Poornima [55].

Since the embedding (1.4) was first proved, many other proofs and different extensions have appeared in the literature. In particular, relations between mixed norm spaces of Lorentz spaces were studied in [2], where it was shown, for instance,

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)',1}, L^1), \quad n \geq 2. \tag{1.6}$$

In recent years, extensions of (1.1) for more general r.i. spaces have been exhaustively studied by various authors, including Edmunds, Kerman, and Pick [27], Kerman and Pick [36], Martín, Milman, and Pustylnik [44] and Cianchi [19]. To be more specific, Kerman and Pick [36] were interested on seeking necessary and sufficient conditions for the following embeddings involving r.i. spaces to hold :

$$W^mZ(I^n) \hookrightarrow X^{\mathrm{op}}(I^n), \quad m \in \mathbb{N}, \tag{1.7}$$

giving a unified version of the classical theory which was previously mentioned. This characterization was then exploited to study the optimal domain-range problems for the embedding (1.7), within the class of r.i. spaces. Thus, for instance, they extended the classical estimates (1.5) by Poornima [55] and Peetre [54] and the so-called limiting or critical case of Sobolev embedding due to Hansson [33], Brezis and Wainger [15] and Maz'ya [45] for higher order derivatives and, as a new contribution, they showed their optimality in the framework of all r.i. spaces. Summarizing, they totally described the best possible targets in (1.7), whenever $Z(I^n) = L^p(I^n)$:

(i) if $1 \leq p < m/n$, then the Lorentz space $L^{np/(n-mp),p}(I^n)$ is the optimal r.i. range space in

$$W^mL^p(I^n) \hookrightarrow L^{np/(n-mp),p}(I^n). \tag{1.8}$$

This should be understood as follows: if an estimate of type (1.8) holds with $L^{np/(n-mp),p}(I^n)$ replaced by some other r.i; then the latter must contain $L^{np/(n-mp),p}(I^n)$;

(ii) if $p = n/m$, the Lorentz-Zygmund space $L^{\infty,n/m;-1}(I^n)$ is the smallest r.i. space satisfying

$$W^mL^{n/m}(I^n) \hookrightarrow L^{\infty,n/m;-1}(I^n); \tag{1.9}$$

(iii) if $p > n/m$, then the range space $L^\infty(I^n)$ is the smallest r.i. space that verifies

$$W^m L^p(I^n) \hookrightarrow L^\infty(I^n).$$

All these works provide us a strong motivation to consider (1.2) of arbitrary order in the setting of r.i. spaces, as well as to describe the optimal domain and the optimal range for this embedding between r.i. spaces and mixed norm spaces. Consequently, our goal in this work is twofold:

(i) to obtain functional properties of mixed norm spaces;

(ii) to study the optimal domain-range problem for the following type of Sobolev embeddings:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X,Y), \quad m \in \mathbb{N}. \tag{1.10}$$

We now briefly describe the main contents of this dissertation:

In order to carry out this project as a self-contained monograph, we study in Chapter 2 the basic background from the theory of rearrangement-invariant spaces that will be needed for the rest of the work. Here, among other things, we introduce rearrangement-invariant spaces (see Definition 2.1.6), i.e., a Banach function space in which the norm of any function depends only on the measure of its level sets. Moreover, we recall basic examples of such spaces: Lebesgue spaces, Orlicz spaces, Lorentz spaces and their generalizations, including Lorentz-Zygmund spaces.

In this chapter we also collect some definitions and results from the Theory of Interpolation. In particular, we introduce the Peetre K-functional , which will be used in different contexts. Furthermore, among the classical results, we mention Holmstedt's theorem which gives explicit expressions for the Peetre K-functional associated to couples of Lorentz spaces (for more details see Theorem 2.1.18). Finally, we recall the definition of the Sobolev spaces (see Definition **??**).

In Chapter 3, we study properties of mixed norm spaces, focusing on some results from Interpolation Theory and relations between mixed norm spaces and r.i. spaces.

In particular, we introduce Benedek-Panzone spaces $\mathcal{R}_k(X, L^\infty)$ and mixed norm spaces $\mathcal{R}(X, L^\infty)$. Moreover, we compute the Peetre K-functional associated to certain couples of mixed norm spaces. In this way, we find explicit expressions for the cases $(\mathcal{R}(X, L^\infty), L^\infty)$ and $(L^p, \mathcal{R}(L^p, X))$ (see Theorem 3.1.7 and Theorem 3.1.11 resp.). Consequently, we show that real interpolation spaces

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta,q}, \quad 0 < \theta < 1, \ \leq q \leq \infty,$$

can in fact be obtained from $(X, L^\infty)_{\theta,q}$ as follows (see Corollary 3.1.8):

$$(\mathcal{R}(X, L^\infty), L^\infty)_{\theta,q} = \mathcal{R}((X, L^\infty)_{\theta,q}, L^\infty).$$

On the other hand, we explore connections between mixed norm spaces. It is important to observe that, for any $k \in \{1, \dots, n\}$,

$$\mathcal{R}_k(X_1, Y_1) \hookrightarrow \mathcal{R}_k(X_2, Y_2) \Leftrightarrow \begin{cases} X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}), \\ Y_1(I) \hookrightarrow Y_2(I), \end{cases} \tag{1.11}$$

(see Lemma 3.2.1). That is, looking at each specific component, the embeddings are trivial. However, there are examples, for instance (1.6), showing that if in (1.11) we replace Benedek-Panzone spaces by the global mixed norm spaces, then the corresponding equivalence is not longer true. This fact illustrates that mixed norm spaces may have a much more complicated structure than Benedek-Panzone spaces. Therefore, it is natural to analyze their embeddings in this setting (see also [31]).

Motivated by this problem we find necessary and sufficient conditions for the existence of the following types of embeddings:

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2), \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty), \quad \mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y).$$

In addition, we further improve the mixed norm estimate (1.6) due Algervik and Kolyada [2] to more general r.i. spaces. To be more specific, Theorem 3.2.14 provides a description of the smallest range of the form $\mathcal{R}(X_2, L^1)$ in

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^1),$$

once the mixed norm space $\mathcal{R}(X_1, L^\infty)$ is given.

Furthermore, we establish an extension of the classical estimate (1.4) due to Fournier [28] to the case where the Lebesgue spaces are replaced by more general r.i. spaces. More specifically, we find necessary and sufficient conditions on the r.i. spaces $X(I^n)$ and $Z(I^n)$ under which the following estimate is fulfilled (see Theorem 3.3.2):

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n). \tag{1.12}$$

A general consequence of Theorem 3.3.2 is contained in Theorem 3.3.5, which provides a characterization of the largest space of the form $\mathcal{R}(X, L^\infty)$ in (1.12), once the r.i. space $Z(I^n)$ is given. Moreover, for a fixed mixed norm space $\mathcal{R}(X, L^\infty)$, Theorem 3.3.8 describes the smallest r.i. space $Z(I^n)$ for which (1.12) holds.

Another operator of interest considered in this work is the Hardy operator:

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds, \quad f \in \mathcal{M}_+(0,1),\ t > 0, \tag{1.13}$$

which, together with its many various modifications, constitutes a handy tool for studying the optimal domain-range problem for Sobolev type embeddings.

Important intrinsic properties of this operator, such as boundedness and compactness, on various function spaces, have been intensively studied over almost a century. In particular, various authors, including Hardy, Littlewood, and Pólya [34], Muckenhoupt [49], Bradley [13], Maz'ya [45] and Sinnamon [57], gave a complete characterization of weights for which (1.13) is bounded on weighted Lebesgue spaces. Since then, different extensions to the higher dimensional case have appeared in the literature. To be more precise, Sinnamon [56] considered n-dimensional Hardy operator

$$P_n f(x) = \int_0^1 f(sx)ds, \quad f \in \mathcal{M}_+(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \tag{1.14}$$

and he described the weighted inequalities for such operator.

Motivated by these results, in Chapter 4, we describe for what pairs of parameters p and q the Hardy operator (1.14) is bounded from $\mathcal{R}(L^p, L^\infty)$ to $L^q(\mathbb{R}^n)$ (see Theorem 4.2.2), i.e. ,

$$P_n : \mathcal{R}(L^p, L^\infty) \to L^q(\mathbb{R}^n).$$

Furthermore, we give necessary and sufficient conditions on the r.i. spaces $X(\mathbb{R})$ and $Y(\mathbb{R}^2)$ such that (see Corollary 4.2.5)

$$P_2 : \mathcal{R}(X, L^\infty) \to Y(\mathbb{R}^2).$$

Embeddings of Sobolev spaces into mixed norm spaces play also an important role in this monograph. More specifically, Chapter 5 is devoted to study the Sobolev embedding of the form

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty), \quad m \in \mathbb{N}, \tag{1.15}$$

extending the classical estimate (1.2). Following the ideas of [36], we establish the equivalence between (1.15) and the boundedness of a Hardy type operator (see Theorem 5.1.5). Then, this relation will be a key tool in determining the optimal domain and the optimal range for (1.15) between r.i. spaces and mixed norm spaces.

After this discussion, our analysis focuses on giving explicit constructions of such optimal spaces. In particular, Theorem 5.2.1 provides a characterization of the smallest space of the form $\mathcal{R}(X, L^\infty)$ in (1.15), once the r.i. domain is given. Finally, for a fixed mixed norm space $\mathcal{R}(X, L^\infty)$, Theorem 5.3.2 describes the largest r.i. space $Z(I^n)$ for which (1.15) holds.

All these results are then employed to establish classical Sobolev embeddings in the context of mixed norm spaces. Summarizing, we give a complete characterization of the optimal ranges of the form $\mathcal{R}(X, L^\infty)$ in (1.15) when $Z(I^n) = L^p(I^n)$:

(i) If $1 \leq p < m/n$, then $\mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ satisfying

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty);$$

(ii) If $p = n/m$, then $\mathcal{R}(L^{\infty,n/m;-1}, L^\infty)$ is the optimal mixed norm space that verifies

$$W^m L^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty); \tag{1.16}$$

(iii) If $p > n/m$, then $\mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ which renders the following embedding true:

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n).$$

Consequently, we recover the classical estimate (1.5) and, as a new contribution, we show its sharpness in the framework of mixed norm spaces.

As we have already mentioned, the optimal range problem for the Sobolev embedding was studied in [36] within the class of r.i. spaces. More concretely, for a fixed r.i. domain space $Z(I^n)$ they determined the smallest r.i. range space $X^{\mathrm{op}}(I^n)$ satisfying (1.7). Motivated by this problem, we compare the optimal r.i. range space with the optimal mixed norm space. In particular, Theorem 5.4.3 proves that the following chain of embeddings holds:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n),$$

with $\mathcal{R}(X, L^\infty)$ the mixed norm space constructed in Theorem 5.2.1. Consequently, we prove that the classical estimate for the standard Sobolev space W^1L^p by Poornima [55] and Peetre [54] ($1 \le p < n$), and by Hansson [33], Brezis and Wainger [15] and Maz'ya [45] ($p = n$) can be further strengthened by considering mixed norms of the form $\mathcal{R}(X, L^\infty)$ on the target spaces. Thus, for instance, we conclude that

(i) If $1 \le p < m/n$, then

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty) \underset{\neq}{\hookrightarrow} L^{np/(n-mp),p}(I^n);$$

(ii) If $p = n/m$, then

$$W^m L^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n/m;-1}(I^n). \tag{1.17}$$

Although the estimate (6.5) is the best possible as far as r.i. range spaces are concerned, Bastero, Milman, and Ruiz [5], and independently Malý and Pick [43], proved that if the requirement that the target space should be a linear space is abandoned, then a further improvement of (6.5), with $m = 1$, is still possible. To this end, they introduced the non-linear function class defined by

$$L(\infty, n)(I^n) = \left\{ f \in \mathcal{M}(I^n) : \|f\|_{L(\infty,n)(I^n)} = \left(\int_0^1 \left(f^{**}(t) - f^*(t) \right)^n \frac{dt}{t} \right)^{1/n} < \infty \right\},$$

and then, using a weak version of the Sobolev-Gagliardo-Nirenberg (1.3) embedding together with a truncation argument due to Mazýa, they concluded that

$$W_0^1 L^n(I^n) \hookrightarrow L(\infty, n)(I^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n). \tag{1.18}$$

Motivated by these works, in Chapter 6 we reformulate their results in terms of mixed norm spaces. That is, we find an improvement of (1.16), with $m = 1$, within the class of non-linear spaces of the form $\mathcal{R}(X, L^\infty)$. To be more precise, we introduce the functional class $\mathcal{R}(L(\infty, n), L^\infty)$ and we establish some of its basic properties (see Lemma 6.2.6). Thus, for instance, we show that it is not a linear set and we also deduce that

$$\mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Consequently, following the same approach as in [5, 43], we get (Theorem 6.2.7)

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Moreover, taking into account (1.17), with $m = 1$ and (1.18), we compare the non-linear space $L(\infty, n)$ with $\mathcal{R}(L^{\infty,n;-1}, L^{\infty})$. In fact, we conclude that these spaces are not comparable (see Theorem 6.3.2 and Theorem 6.3.3).

In Chapter 7, we study Sobolev embeddings of the form

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1). \tag{1.19}$$

In particular, we obtain necessary and sufficient conditions for the existence of these estimates (see Theorem 7.1.6). For this, we apply a method, which relies on studying embeddings between Sobolev spaces built upon mixed norm spaces. Specifically, in Theorem 7.1.3 we prove that the embeddings of the form

$$W^1\mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1) \tag{1.20}$$

hold if and only if the following Sobolev type estimates are fulfilled:

$$W^1 Z(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Such relation is then crucial in the forthcoming discussion where an exhaustive study of the optimal range problem for (1.19) and (1.20) is made. To be more precise, we answer the question of finding the best possible target of the form $\mathcal{R}(X, L^1)$ for (1.19) when the domain space is fixed (see Theorem 7.2.1). Furthermore, given a domain space $\mathcal{R}(Z, L^1)$, we characterize the smallest range space $\mathcal{R}(X, L^1)$ for which (1.20) holds (see Theorem 7.2.2). As a consequence, we derive new embeddings, with optimal target spaces of the form $\mathcal{R}(X, L^1)$, for the first order Sobolev spaces W^1L^p and $W^1\mathcal{R}(L^p, L^1)$ (see Corollary 7.2.3 and Corollary 7.2.5):

(i) If $1 \le p < n-1$, then $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ is the smallest range of the form $\mathcal{R}(X, L^1)$ satisfying

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1), \tag{1.21}$$

and

$$W^1\mathcal{R}(L^p, L^1) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1);$$

(ii) If $p = n-1$, then $\mathcal{R}(L^{\infty,n-1;-1}, L^1)$ is the optimal mixed norm space in

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1),$$

and

$$W^1\mathcal{R}(L^{n-1}, L^1) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1);$$

(iii) If $p > n-1$, then $\mathcal{R}(L^{\infty}, L^1)$ is the smallest range of the form $\mathcal{R}(X, L^1)$ for

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{\infty}, L^1) \quad \text{and} \quad W^1\mathcal{R}(L^p, L^1) \hookrightarrow \mathcal{R}(L^{\infty}, L^1).$$

Next, we solve the optimal domain problem for Sobolev embeddings of the form (1.19). Namely, given a mixed norm space $\mathcal{R}(X, L^1)$, we find the largest r.i. domain satisfying (1.19) (see Theorem 7.3.2).

As we have pointed out before, the mixed norm spaces of the forms $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$ are function spaces, having non-trivial intersection (for example, $L^\infty(I^n)$ is contained in both). Motivated by this fact, we consider $Z(I^n) = L^p(I^n)$ and we compare its optimal ranges of the form $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$.

An early result in this direction was obtained by Algervik and Kolyada [2] who proved that the embedding (1.21), with $p = 1$, can be strengthened on allowing spaces of a different nature, such as mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$. Namely, it was shown that:

$$W^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{(n-1)',1}, L^1).$$

It seems natural to expect that this chain of embeddings should hold for any Sobolev type estimate of first order. However, we present (see Example 7.4.2) some examples, including inequalities for the standard Sobolev spaces W^1L^p showing that such mixed norm spaces are not comparable.

Finally, we find a relation between the optimal mixed norm space $\mathcal{R}(X, L^1)$ in (1.19) and the smallest r.i. space in (1.7), when $Z(I^n) = L^p(I^n)$. In particular, we deduce that such spaces are not comparable in general.

The results of Chapters 3, 5, 6, and 7 are included in [23], [24], [22], and [25], respectively.

Chapter 2

Preliminaries

In order to carry out this project as a self-contained monograph, in this chapter we study the main definitions and results needed for the rest of the work.

The sections are organized as follows: in the first and second section, we collect background material on r.i. spaces and Sobolev spaces, together with the definitions and results from Interpolation Theory that will be needed in what follows.

2.1 Rearrangement invariant Banach spaces

In this section, we briefly recall some basic facts from the theory of rearrangement-invariant spaces. For a detailed treatment of this topic, we refer to [8, 30].

Let $n \in \mathbb{N}$, $n \geq 1$ and let $I \subset \mathbb{R}$ be an interval, with $|I| = 1$. We write $\mathcal{M}(I^n)$ for the set of all real-valued measurable functions on I^n. Moreover, $|\cdot|$ denotes the Lebesgue measure . Finally, if A and B are two positive quantities, we say that they are equivalent ($A \approx B$) if there exists a positive constant C such that $C^{-1}B \leq A \leq CB$. The case $A \leq CB$ is denoted by $A \lesssim B$.

Definition 2.1.1. *Let $f \in \mathcal{M}(I^n)$. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by*

$$f^*(t) = \inf\{s \geq 0 : \lambda_f(s) \leq t\}, \quad t \geq 0,$$

where λ_f is the distribution function of f given by

$$\lambda_f(t) = |\{x \in I^n : |f(x)| > t\}|, \quad t \geq 0.$$

We now state some basic properties of f^* (for more details see [8]).

Proposition 2.1.2. *Let f, g , $\{f_j\}_j \in \mathcal{M}(I^n)$ and let α be any scalar. Then,*

(i) f^* *is a non-negative, decreasing and right-continuous function on* $[0, \infty)$;

(ii) $f^*(\lambda_f(t)) \leq t$, *for all* $t \geq 0$ *with* $\lambda_f(t) < \infty$;

(iii) $\lambda_f(f^*(t)) \leq t$, *for any* $t \geq 0$ *with* $f^*(t) < \infty$;

(iv) if $|f| \leq |g|$ *a.e., then* $f^* \leq g^*$;

(v) $(\alpha f)^* = |\alpha| f^*$;

(vi) *if* $|f_j| \uparrow |f|$ *a.e., then* $f_j^* \uparrow f^*$.

The following technical lemma will be used in the forthcoming chapters (for further information see [19]).

Lemma 2.1.3. *If* g *is any radial function on* $\mathbb{R}^n$ *having the form*

$$g(x) = f(\omega_n |x|^n),$$

for some real-valued measurable function $f : [0, \infty) \to \mathbb{R}$, *then* $g^* = f^*$.

As usual, we shall use the notation

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0.$$

Some useful properties of f^{**} are listed bellow (see [8] for further details).

Proposition 2.1.4. *Let* f, g , $\{f_j\}_j \in \mathcal{M}(I^n)$ *and let* α *be any scalar. Then,*

(i) f^{**} *is a non-negative, decreasing and continuous function on* $(0, \infty)$;

(ii) *if* $|f| \le |g|$ *a.e., then* $f^{**} \le g^{**}$;

(iii) $(\alpha f)^* = |\alpha| f^{**}$;

(iv) *if* $|f_j| \uparrow |f|$ *a.e., then* $f_j^{**} \uparrow f^{**}$;

(v) $(f + g)^{**}(t) \le f^{**}(t) + g^{**}(t)$, *for any* $t > 0$.

A basic property of rearrangements is the following well-known result (see [8] for further details).

Theorem 2.1.5 (Hardy-Littlewood inequality)**.** *If* $f, g \in \mathcal{M}(I^n)$, *then*

$$\int_{I^n} |f(x)g(x)| dx \le \int_0^1 f^*(t) g^*(t) dt, \quad f, g \in \mathcal{M}(I^n).$$

We introduce now the rearrangement-invariant (for short, r.i.) spaces which play an important role in this theory.

Definition 2.1.6. *A Banach function space* $X(I^n)$ *is the collection of all* $f \in \mathcal{M}(I^n)$ *for which* $\|f\|_{X(I^n)} < \infty$, *where* $\|\cdot\|_{X(I^n)}$ *satisfies the following properties:*

(A1) $\|\cdot\|_{X(I^n)}$ *is a norm;*

(A2) *if* $0 \le g \le f$ a.e., *then* $\|g\|_{X(I^n)} \le \|f\|_{X(I^n)}$;

(A3) *if* $0 \le f_j \uparrow f$ a.e., *then* $\|f_j\|_{X(I^n)} \uparrow \|f\|_{X(I^n)}$;

(A4) $\|\chi_{I^n}\|_{X(I^n)} < \infty$;

(A5) $\int_{I^n} |f(x)|dx \lesssim \|f\|_{X(I^n)}$.

If, in addition, $\|\cdot\|_{X(I^n)}$ *verifies:*

(A6) if $f^* = g^*$*, then* $\|f\|_{X(I^n)} = \|g\|_{X(I^n)}$,

then we say that $X(I^n)$ *is a rearrangement invariant Banach function space (briefly an r.i. space).*

It is easy to see that $L^1(I^n)$ and $L^\infty(I^n)$ are the largest and the smallest, respectively, r.i. spaces on I^n, in the sense that if $X(I^n)$ is any other r.i. space, then

$$L^\infty(I^n) \hookrightarrow X(I^n) \hookrightarrow L^1(I^n). \tag{2.1}$$

Next let us introduce the associate of an r.i. space.

Definition 2.1.7. *Let* $X(I^n)$ *be an r.i. space. The associate space of* $X(I^n)$ *is defined as the set*

$$X'(I^n) = \left\{ f \in \mathcal{M}(I^n) : \int_{I^n} |f(x)g(x)|dx < \infty, \text{ for any } g \in X(I^n) \right\},$$

equipped with the norm

$$\|f\|_{X'(I^n)} = \sup_{\|g\|_{X(I^n)} \le 1} \int_{I^n} |f(x)g(x)|dx.$$

The following results will be useful in the next chapters (for more details see [8]).

Theorem 2.1.8. *If* $X(I^n)$ *is an r.i. space, then so is its associate* $X'(I^n)$.

Theorem 2.1.9 (Lorentz-Luxemburg Theorem)**.** *Let* $X(I^n)$ *be an r.i. space. Then,*

$$X''(I^n) := (X'(I^n))' = X(I^n).$$

Theorem 2.1.10 (Hölder's inequality)**.** *Let* $X(I^n)$ *be an r.i. space. Then, for* $f \in X(I^n)$ *and* $g \in X'(I^n)$,

$$\int_{I^n} |f(x)g(x)|dx \le \|f\|_{X(I^n)} \|g\|_{X'(I^n)}.$$

Theorem 2.1.11 (Hardy-Littlewood-Pólya Principle)**.** *Let* $X(I^n)$ *be an r.i. space and let* $g \in \mathcal{M}(I^n)$ *and* $f \in X(I^n)$ *such that*

$$\int_0^t g^*(s)ds \le \int_0^t f^*(s)ds, \quad 0 < t < 1.$$

Then $g \in X(I^n)$ *and* $\|g\|_{X(I^n)} \le \|f\|_{X(I^n)}$.

Theorem 2.1.12 (Luxemburg Representation Theorem)**.** *For any r.i. space* $X(I^n)$*, there exists another r.i. space* $\overline{X}$ *such that*

$$f \in X(I^n) \Longleftrightarrow f^* \in \overline{X}(0,1),$$

and in this case $\|f\|_{X(I^n)} = \|f^*\|_{\overline{X}(0,1)}$.

We now introduce the fundamental function of an r.i. space, which plays a significant part in the theory, especially in connection with interpolation (for more details see [8]).

Definition 2.1.13. *Let $X(I^n)$ be an r.i. space. The function $\varphi_X : [0,1) \to [0,\infty)$ given by*

$$\varphi_X(t) = \|\chi_{(0,t)}\|_{\overline{X}(0,1)}, \quad \textit{for } t \in [0,1)$$

is called the fundamental function of $X(I^n)$.

The fundamental function φ_X of any r.i. space $X(I^n)$ is quasiconcave, in the sense that it is non-decreasing on $[0,1)$, $\varphi_X(0) = 0$ and $\varphi_X(s)/s$ is non-increasing on $(0,1)$. Moreover, one has that

$$\varphi_X(t)\varphi_{X'}(t) = t, \quad \text{for any } s \in [0,1). \tag{2.2}$$

The following theorem (see [58] for more details) will be very useful later on.

Theorem 2.1.14. *Let $X(I^n)$ be an r.i. space. Then, the following statements are equivalent:*

(i) $X(I^n) \neq L^\infty(I^n)$;

(ii) $\lim_{t\to 0^+} \varphi_X(t) = 0$.

Next, we introduce the Boyd indices of an r.i. space. But first, we recall that the dilation operator E_t is given by

$$E_t f(s) = \begin{cases} f(s/t), & \text{if } 0 \le s \le \min(1,t), \\ 0, & \text{otherwise,} \end{cases} \quad t > 0, \;\; f \in \mathcal{M}(0,1). \tag{2.3}$$

Let us just mention that the operator E_t is bounded on $\overline{X}(0,1)$, for every r.i. space $X(I^n)$ and for every $t > 0$.

By means of the norm of E_t on $\overline{X}(0,1)$, denoted by $h_X(t)$, we define the lower and upper Boyd indices of $X(I^n)$ as

$$\underline{\alpha}_X = \sup_{0<t<1} \frac{\log h_X(t)}{\log(t)} \quad \text{and} \quad \overline{\alpha}_X = \inf_{1<t<\infty} \frac{\log h_X(t)}{\log(t)}. \tag{2.4}$$

It is easy to see that $0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le 1$.

Now, we recall definitions and basic properties of those function spaces that will be involved in our results.

The Lebesgue spaces $L^p(I^n)$, with $1 \le p \le \infty$, endowed with the standard norm, are the simplest example of r.i. spaces. We shall also work with the Lorentz spaces, defined either for $p = q = 1$ or $p = q = \infty$, or $1 < p < \infty$ and $1 \le q \le \infty$ as

$$L^{p,q}(I^n) = \Big\{ f \in \mathcal{M}(I^n) : \|f\|_{L^{p,q}(I^n)} < \infty \Big\},$$

where

$$\|f\|_{L^{p,q}(I^n)} = \left\|t^{1/p-1/q} f^*(t)\right\|_{L^q(I^n)}. \tag{2.5}$$

and, more generally, with the Lorentz-Zygmund spaces, defined for $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$ as

$$L^{p,q;\alpha}(I^n) = \Big\{ f \in \mathcal{M}(I^n) : \|f\|_{L^{p,q;\alpha}(I^n)} < \infty \Big\},$$

where

$$\|f\|_{L^{p,q;\alpha}(I^n)} = \left\|t^{1/p-1/q}[1+\log(t)]^\alpha f^*(t)\right\|_{L^q(I^n)}. \tag{2.6}$$

Observe that $L^{p,p}(I^n) = L^p(I^n)$ and $L^{p,q;0}(I^n) = L^{p,q}(I^n)$. Let us also mention, for the sake of completeness, that the quantities $\|\cdot\|_{L^{p,q}(I^n)}$ and $\|\cdot\|_{L^{p,q;\alpha}(I^n)}$, given in (2.5) and (2.6), are in general only quasi-norms, since they may fail to satisfy the triangle inequality. They can be turned into equivalent norms replacing u^* by u^{**} in definitions (2.5) and (2.6). However, in the special cases when the weights $t^{1/p-1/q}$ or $t^{1/p-1/q}[1+\log(t)]^\alpha$ are non-increasing (and hence in all the cases involved in this monograph), then $\|\cdot\|_{L^{p,q}(I^n)}$ and $\|\cdot\|_{L^{p,q;\alpha}(I^n)}$ are norms.

Given any Young function $A : [0,\infty) \to [0,\infty)$, namely a convex increasing function vanishing at 0, the Orlicz spaces $L_A(I^n)$ are the r.i. spaces of all measurable function f in I^n such that the Luxembourg norm

$$\|f\|_{L_A(I^n)} = \inf\left\{\lambda > 0 : \int_{I^n} A(|f(x)|/\lambda)dx \le 1\right\} \tag{2.7}$$

is finite. The Orlicz spaces generalize many well-known spaces such as the Lebesgue spaces $L^p(I^n)$ and the Lorentz-Zygmund spaces $L^{p,q;\alpha}(I^n)$. In fact, in view of (2.7), we have that:

(i) if $A(t) = t^p$, then we recover the Lebesgue spaces $L_A(I^n) = L^p(I^n)$;

(ii) if $A(t) = e^{t^\alpha}$, then we obtain $L_A(I^n) = L^{\infty,\infty;-1/\alpha}(I^n)$.

Other relevant families of classical r.i. spaces are the Lorentz spaces Λ_{φ_X} which consist of all $f \in \mathcal{M}(I^n)$ for which the expression:

$$\|f\|_{\Lambda_{\varphi_X}} = \|f\|_{L^\infty(I^n)}\varphi_X(0^+) + \int_0^1 f^*(t)\varphi_X'(t)dt \tag{2.8}$$

is finite. It is well-known [8, Theorem II.5.13] that if $X(I^n)$ is an r.i. space then,

$$\Lambda_{\varphi_X} \hookrightarrow X(I^n).$$

Finally, let us recall some special results from Interpolation Theory, which we shall need in what follows. For further information on this topic see [8, 9, 16].

Definition 2.1.15. *Given a pair of compatible Banach spaces (X_0, X_1) (compatible in the sense that they are continuously embedded into a common Hausdorff topological vector space), their K-functional is defined, for each $f \in X_0 + X_1$, by*

$$K(f,t;X_0,X_1) := \inf_{f=f_0+f_1}(\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \quad t > 0.$$

Definition 2.1.16. *For $0 < \theta < 1$, $1 \leq q < \infty$ or $0 \leq \theta \leq 1$, $q = \infty$, the space $(X_0, X_1)_{\theta,q}$ consists of all $f \in X_0 + X_1$ for which the functional*

$$\|f\|_{\theta,q} = \begin{cases} \left(\int_0^\infty t^{-\theta q - 1}[K(f,t;X_0,X_1)]^q dt \right)^{1/q}, & 0 < \theta < 1\,, 1 \leq q < \infty, \\ \sup_{t>0} t^{-\theta} K(f,t;X_0,X_1), & 0 \leq \theta \leq 1\,, q = \infty \end{cases}$$

is finite.

The fundamental result concerning the K-functional is [8]:

Theorem 2.1.17. *Let (X_0, X_1) and (Y_0, Y_1) be two compatible pairs of Banach spaces and let T be a sublinear operator satisfying*

$$T\, : X_0 \to Y_0, \quad \textit{and} \quad T\, : X_1 \to Y_1.$$

Then, there exists a constant $C > 0$ (depending only on the norms of T between X_0 and Y_0 and between X_1 and Y_1) such that

$$K(Tf,t;Y_0,Y_1) \leq CK(f,Ct;X_0,X_1), \quad \textit{for every } f \in X_0 + X_1 \textit{ and } t > 0.$$

The K-functional for pairs of Lorentz spaces $L^{p,q}(I^n)$ is given, up to equivalence, by the following result [35].

Theorem 2.1.18 (Holmstedt's formulas)**.** *Let $p_0 = q_0 = 1$ or $1 < p_0 < \infty$ and $1 \leq q_0 < \infty$. Then, for any $t > 0$,*

(i) $K(f,t;L^{p_0,q_0},L^\infty) \approx \left(\int_0^{t^{p_0}} \left[s^{1/p_0 - 1/q_0} f^*(s) \right] ds \right)^{1/q_0}$;

(ii) $K(f,t;L^{q_0,\infty},L^\infty) \approx \sup\limits_{0<s<t^{q_0}} s^{1/q_0} f^*(s)$.

2.2 Sobolev spaces

We now recall the definition of the Sobolev spaces and collect some important results that will be necessary for our work. For a complete account we refer to [1, 45, 14]. For a given multi-index $\alpha \in (N \cup \{0\})^n$ we write

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}},$$

for the α^{th} weak derivative of $f \in L^1_{\text{loc}}(I^n)$.

The Sobolev space $W^m Z(I^n)$ is a Banach function space. Moreover, if we denote by

$$D^m f = (\partial^\alpha f)_{0 \leq |\alpha| \leq m},$$

then

$$\|f\|_{W^m Z(I^n)} \approx \||D^m f|\|_{Z(I^n)},$$

with $|D^m f|$ is the Euclidean length of the vector as an element of $\mathbb{R}^N$, where N the number of multiindices $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $0 \le |\alpha| \le m$.

Concerning the K-functional for a couple of Sobolev spaces, we mention the work of DeVore and Scherer [26], who proved that, for every $m \in \mathbb{Z}_+$ and $f \in W^m L^1(I^n)$,

$$K(f, t; W^m L^1, W^m L^\infty) \approx \int_0^t |D^m f|^*(s)ds, \quad t > 0. \tag{2.9}$$

Furthermore, thanks to (2.9), the reiteration theorem [8, Theorem V.2.4] and Holmstedt's formulas, one can also describe the K-functional for any couple of Sobolev spaces $(W^m L^{p_0,q_0}, W^m L^{p_1,q_1})$ (see [8] for more details).

Theorem 2.2.1. *Let $p_0 = q_0 = 1$, or $1 < p_0 < p_1 < \infty$ and $1 \le q_0, q_1 < \infty$, and let $1/\alpha = 1/p_0 - 1/p_1$. Then, for any $t > 0$,*

$$K(f, t; W^m L^{p_0,q_0}, W^m L^{p_1,q_1}) \approx \left(\int_0^{t^\alpha} \left[s^{1/p_0 - 1/q_0} |D^m f|^*(s) \right]^{q_0} ds \right)^{1/q_0} + t \left(\int_{t^\alpha}^1 \left[s^{1/p_1 - 1/q_1} |D^m f|^*(s) \right]^{q_1} ds \right)^{1/q_1}.$$

Chapter 3

Mixed norm spaces

Estimates on mixed norm spaces already appeared in the works of Gagliardo [29] and Nirenberg [50] who proved an endpoint case of the classical Sobolev embeddings. However, a more systematic approach to these spaces was first given explicitly by Fournier [28] and was subsequently developed, via different methods, by various authors, including Blei and Fournier [10], Milman [47], Algervik and Kolyada [2], Kolyada [37, 38, 39], and Kolyada and Soria [40].

Motivated by these works, in this chapter we study the mixed norm spaces, focusing on the properties that we will need throughout this monograph. The results of this chapter are included in [23]

The sections are organized as follows: First, we introduce the mixed norm spaces $\mathcal{R}(X,Y)$ (see Definition 3.1.2). Here, among other things, we find an explicit formula for the Peetre K-functional for $(\mathcal{R}(X,L^\infty),L^\infty)$ and $(L^p,\mathcal{R}(L^\infty,L^p))$.

In the second section, we characterize the following types of embeddings:

$$\mathcal{R}(L^\infty,Y_1)\hookrightarrow\mathcal{R}(X_2,Y_2),\quad \mathcal{R}(X_1,L^\infty)\hookrightarrow\mathcal{R}(X_2,L^\infty),\quad \mathcal{R}(X_1,Y)\hookrightarrow\mathcal{R}(X_2,Y).$$

Moreover, given a mixed norm space $\mathcal{R}(X_1,L^\infty)$ we describe the smallest space of the form $\mathcal{R}(X_2,L^1)$ into which $\mathcal{R}(X_1,L^\infty)$ is continuously embedded (Theorem 3.2.14).

After this discussion we explore connections between mixed norm spaces and r.i. spaces. In particular, in the third section, we improve the classical estimate due to Fournier [28] in the setting of r.i. spaces. Namely, we establish necessary and sufficient conditions on the r.i. spaces $X(I^n)$ and $Z(I^n)$ under which the following estimate is fulfilled (see Theorem 3.3.2):

$$\mathcal{R}(X,L^\infty)\hookrightarrow Z(I^n). \tag{3.1}$$

A general consequence of Theorem 3.3.2 is contained in Theorem 3.3.5, which provides a characterization of the largest space of the form $\mathcal{R}(X,L^\infty)$ in (3.1), once the r.i. space $Z(I^n)$ is given. Furthermore, for a fixed mixed norm space $\mathcal{R}(X,L^\infty)$, Theorem 3.3.8 describes the smallest r.i. space $Z(I^n)$ for which (3.1) holds.

Finally, in the the fourth section, special attention will be given to embeddings of the form

$$Z(I^n)\hookrightarrow\mathcal{R}(X,L^1). \tag{3.2}$$

In particular, we describe the best possible target for (3.2), within the class of mixed norm spaces (see Theorem 3.4.2).

3.1 Definition and some properties

Let $n \in \mathbb{N}$, with $n \geq 2$. Our goal in this section is to present some basic properties of mixed norm spaces. Let $k \in \{1, \dots, n\}$. We write $\widehat{x_k}$ for the point in I^{n-1} obtained from a given vector $x \in I^n$ by removing its kth coordinate. That is,

$$\widehat{x_k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in I^{n-1}.$$

Moreover, for any $f \in \mathcal{M}(I^n)$, we use the notation $f_{\widehat{x_k}}$ for the function obtained from f, with $\widehat{x_k}$ fixed. Observe that, since f is measurable, $f_{\widehat{x_k}}$ is also measurable, for a.e. $\widehat{x_k}$.

For later purposes, let us first recall some geometric properties of the projections. We refer to the book [32] for basic facts on this topic.

Let $E \subset I^n$ be any measurable set and let $\widehat{x_k} \in I^{n-1}$ be fixed. The x_k-section of E is defined as

$$E(\widehat{x_k}) = \left\{ x_k \in I : (\widehat{x_k}, x_k) \in E \right\}.$$

Let us just emphasize that since E is measurable, its x_k-section is also measurable, for a.e. $\widehat{x_k}$. The essential projection of E onto the hyperplane $x_k = 0$ is defined as

$$\Pi_k^* E = \left\{ \widehat{x_k} \in I^{n-1} : |E(\widehat{x_k})| > 0 \right\}.$$

An important result, for our purpose, is the so-called Loomis-Whitney inequality [41, Theorem 1] which says

$$\big|E\big| \leq \prod_{k=1}^{n} \big|\Pi_k E\big|^{1/(n-1)}, \tag{3.3}$$

where $\Pi_k E$ is the orthogonal projection of E into the coordinate hyperplane $x_k = 0$.

An improvement of (3.3) was given in [28, 2], where a version of the Loomis-Whitney inequality with the measures of the essential projections was proved:

$$\big|E\big| \leq \prod_{k=1}^{n} \big|\Pi_k^* E\big|^{1/(n-1)}. \tag{3.4}$$

We now recall the Benedek-Panzone spaces, which were introduced in [6]. For further information on this topic see [17, 11, 12, 4].

Definition 3.1.1. *Let $k \in \{1, \dots, n\}$. Given two r.i. spaces $X(I^{n-1})$ and $Y(I)$, the Benedek-Panzone space $\mathcal{R}_k(X, Y)$ is defined as the collection of all $f \in \mathcal{M}(I^n)$ satisfying*

$$\|f\|_{\mathcal{R}_k(X,Y)} = \big\|\psi_k(f, Y)\big\|_{X(I^{n-1})} < \infty,$$

where $\psi_k(f, Y)(\widehat{x_k}) = \|f(\widehat{x_k}, \cdot)\|_{Y(I)}$.

Buhvalov [17] and Blozinski [11] proved that $\mathcal{R}_k(X, Y)$ is a Banach function space. Moreover Boccuto, Bukhvalov, and Sambucini [12] proved that $\mathcal{R}_k(X, Y)$ is an r.i. space, if and only if $X = Y = L^p$.

Now, we shall give the definition of the mixed norm spaces sometimes also called symmetric mixed norm spaces.

Definition 3.1.2. *Given two r.i. spaces $X(I^{n-1})$ and $Y(I)$, the mixed norm space $\mathcal{R}(X,Y)$ is defined as*

$$\mathcal{R}(X,Y)=\bigcap_{k=1}^{n}\mathcal{R}_k(X,Y).$$

For each $f\in\mathcal{R}(X,Y)$, we set $\|f\|_{\mathcal{R}(X,Y)}=\sum_{k=1}^{n}\|f\|_{\mathcal{R}_k(X,Y)}$.

It is not difficult to verify that $\mathcal{R}(X,Y)$ is a Banach function space. Before going on, for the sake of completeness, let us mention that examples of these spaces are the Lebesgue spaces $\mathcal{R}(L^p,L^p)=L^p(I^n)$, with $1\le p\le\infty$.

Since the pioneering works of Gagliardo [29], Nirenberg [50], and Fournier [28], many useful properties and generalizations of these spaces have been studied, via different methods, by various authors, including Blei and Fournier [10], Milman [47], Algervik and Kolyada [2], Kolyada [37, 38, 39], and Kolyada and Soria [40].

All these works, together with the Gagliardo-Nirenberg embedding, provide us a strong motivation to better understand the mixed norm spaces of the form $\mathcal{R}(X,L^\infty)$. For this, we start with a useful lemma:

Lemma 3.1.3. *Let $f\in\mathcal{M}(I^n)$ and let $E_\alpha=\{x\in I^n : |f(x)|>\alpha\}$, with $\alpha\ge 0$. Then,*

$$\Pi_k^* E_\alpha=\big\{\widehat{x}_k\in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x}_k)>\alpha\big\}.$$

Proof. Let us see that

$$\big\{\widehat{x}_k\in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x}_k)>\alpha\big\}\subset\Pi_k^* E_\alpha. \tag{3.5}$$

In fact, if $\widehat{x}_k\notin\Pi_k^*E_\alpha$, then, by definition of $\Pi_k^*E_\alpha$, we have

$$\big|\big\{x_k\in I : |f(\widehat{x}_k,x_k)|>\alpha\big\}\big|=0.$$

But,

$$\psi_k(f,L^\infty)(\widehat{x}_k)=\inf\big\{s\ge 0 : \big|\big\{x_k\in I : |f(\widehat{x}_k,x_k)|>s\big\}\big|=0\big\},\quad \widehat{x}_k\in I^{n-1},$$

and hence, we get $\psi_k(f,L^\infty)(\widehat{x}_k)\le\alpha$. As a consequence, we have

$$\widehat{x}_k\notin\big\{\widehat{x}_k\in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x}_k)>\alpha\big\}.$$

This proves that (3.5) holds. To complete the proof, it only remains to see that

$$\Pi_k^*E_\alpha\subset\big\{\widehat{x}_k\in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x}_k)>\alpha\big\}.$$

In fact, if $\widehat{x}_k\in\Pi_k^*E_\alpha$, then

$$\big|\big\{x_k\in I : |f(\widehat{x}_k,x_k)|>\alpha\big\}\big|>0.$$

So, $\psi_k(f,L^\infty)(\widehat{x}_k)>\alpha$. Therefore,

$$\widehat{x}_k\in\big\{\widehat{x}_k\in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x}_k)>\alpha\big\}.$$

Thus, the proof is complete. □

As an immediate consequence of inequality (3.4) and Lemma 3.1.3, we have the following inequality, which was previously proved in [28].

Corollary 3.1.4. *Let $f \in \mathcal{M}(I^n)$. Then, for any $t > 0$,*

$$\lambda_f(t) \le \Big(\prod_{k=1}^{n} \lambda_{\psi_k(f,L^\infty)}(t) \Big)^{1/(n-1)}.$$

Proof. We fix any $t > 0$ and we set

$$E_t = \big\{ x \in I^n : |f(x)| > t \big\}.$$

Then, using (3.4) together with Lemma 3.1.3, we get

$$\lambda_f(t) = |E_t| \le \Big(\prod_{k=1}^{n} |\Pi_k^* E_t| \Big)^{1/(n-1)} = \Big(\prod_{k=1}^{n} \lambda_{\psi_k(f,L^\infty)}(t) \Big)^{1/(n-1)},$$

from which the result follows. □

We now give a reformulation of Corollary 3.1.4 in terms of the decreasing rearrangement function that will be useful to get several estimates in the next chapters.

Corollary 3.1.5. *For any $f \in \mathcal{M}(I^n)$, it holds that*

$$f^*(t) \le \sum_{j=1}^{n} \psi_j^*(f, L^\infty)(t^{1/n'}), \quad t > 0. \tag{3.6}$$

Proof. We fix any $f \in \mathcal{M}(I^n)$ and $t > 0$, such that the right-hand side of (3.6) is finite. We also set

$$\alpha_t = \sum_{j=1}^{n} \psi_j^*(f, L^\infty)(t^{1/n'}).$$

Using now Proposition 2.1.2, we have

$$\lambda_{\psi_k(f,L^\infty)}(\alpha_t) \le \lambda_{\psi_k(f,L^\infty)}(\psi_k^*(f, L^\infty)(t^{1/n'})) \le t^{1/n'}, \quad k \in \{1, \dots, n\},$$

and so

$$\prod_{k=1}^{n} \lambda_{\psi_k(f,L^\infty)}(\psi_k^*(f, L^\infty)(t^{1/n'})) \le t^{n-1}. \tag{3.7}$$

So, taking into account Corollary 3.1.4, with t replaced by α_t, and (3.7), we get

$$\lambda_f(\alpha_t) \le \Big(\prod_{k=1}^{n} \lambda_{\psi_k(f,L^\infty)}(\alpha_t) \Big)^{1/(n-1)} \le t.$$

From this, we deduce that

$$\alpha_t \in \inf\{y \ge 0 : \lambda_f(y) \le t\},$$

and hence

$$f^*(t) \le \alpha_t = \sum_{j=1}^{n} \psi_j^*(f, L^\infty)(t^{1/n'}),$$

as we wanted to show. □

Our next goal is to compute (or at least to estimate) the K-functional associated to couples of mixed norm spaces. To be more precise, we shall find explicit expressions for the cases $(\mathcal{R}(X, L^\infty), L^\infty)$ and $(L^p, \mathcal{R}(L^\infty, X))$ (recall that $\mathcal{R}(L^p, L^p) = L^p$). For further information on this topic see [47].

Let us start with a lower bound for the K-functional for the couple of mixed norm spaces $(\mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y))$.

Lemma 3.1.6. *Let $X(I^{n-1})$ and $Y(I)$ be r.i. spaces. Then,*

$$\sum_{k=1}^{n} \left\|\psi_k^*(f, Y)\chi_{(0,t)}\right\|_{\overline{X}(0,1)} \lesssim K\big(f, \varphi_X(t); \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)\big), \quad 0 < t < 1.$$

Proof. We fix $0 < t < 1$ and $k \in \{1, \dots, n\}$. If $f = f_0 + f_1$, with $f_0 \in \mathcal{R}(X, Y)$ and $f_1 \in \mathcal{R}(L^\infty, Y)$, then

$$\psi_k(f, Y)(\widehat{x_k}) \leq \psi_k(f_0, Y)(\widehat{x_k}) + \psi_k(f_1, Y)(\widehat{x_k}), \quad \widehat{x_k} \in I^{n-1}.$$

So, it holds that

$$\psi_k^*(f, Y)(t) \leq \psi_k^*(f_0, Y)(t) + \psi_k^*(f_1, Y)(0) = \psi_k^*(f_0, Y)(t) + \|f_1\|_{\mathcal{R}_k(L^\infty, Y)}.$$

Therefore, we have

$$\begin{aligned}\left\|\psi_k^*(f, Y)\chi_{(0,t)}\right\|_{\overline{X}(0,1)} &\leq \left\|\psi_k^*(f_0, Y)\chi_{(0,t)}\right\|_{\overline{X}(0,1)} + \varphi_X(t)\|f_1\|_{\mathcal{R}_k(L^\infty, Y)} \\ &\leq \|f_0\|_{\mathcal{R}(X,Y)} + \varphi_X(t)\|f_1\|_{\mathcal{R}(L^\infty, Y)}.\end{aligned}$$

Hence, taking the infimum over all decompositions of f of the form $f = f_0 + f_1$, with $f_0 \in \mathcal{R}(X, Y)$ and $f_1 \in \mathcal{R}(L^\infty, Y)$, we obtain

$$\left\|\psi_k^*(f, Y)\chi_{(0,t)}\right\|_{\overline{X}(0,1)} \leq K(f, \varphi_X(t); \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)),$$

for any $k \in \{1, \dots, n\}$ and $0 < t < 1$. Consequently,

$$\sum_{k=1}^{n} \left\|\psi_k^*(f, Y)\chi_{(0,t)}\right\|_{\overline{X}(0,1)} \leq nK(f, \varphi_X(t); \mathcal{R}(X, Y), \mathcal{R}(L^\infty, Y)), \quad 0 < t < 1,$$

from which the result follows. □

Next, let us find an explicit formula for the K-functional for the couple of mixed norm spaces $(\mathcal{R}(X, L^\infty), L^\infty)$.

Theorem 3.1.7. *Let $X(I^{n-1})$ be an r.i. space and let $f \in \mathcal{R}(X, L^\infty) + L^\infty(I^n)$. Then,*

$$K(f, \varphi_X(t); \mathcal{R}(X, L^\infty), L^\infty) \approx \sum_{k=1}^{n} \left\|\psi_k^*(f, L^\infty)\chi_{(0,t)}\right\|_{\overline{X}(0,1)}, \quad 0 < t < 1,$$

where $\varphi_X(t)$ is the fundamental function of $X(I^{n-1})$.

Proof. In view of Lemma 3.1.6, we only need to prove

$$K(f,\varphi_X(t);\mathcal{R}(X,L^\infty),L^\infty) \lesssim \sum_{k=1}^{n} \left\|\psi_k^*(f,L^\infty)\chi_{(0,t)}\right\|_{\overline{X}(0,1)}, \quad 0<t<1.$$

For this, we fix any $0<t<1$. Then, we define

$$\alpha_t = \sum_{j=1}^{n} \psi_j^*(f,L^\infty)(t), \tag{3.8}$$

$$F(x) = \begin{cases} f(x) - \dfrac{\alpha_t f(x)}{|f(x)|}, & \text{if } x \in A_t = \big\{x \in I^n : |f(x)| > \alpha_t\big\}, \\ 0, & \text{otherwise,} \end{cases}$$

and $G = f - F$. Let $k \in \{1,\dots,n\}$ be fixed. Then, for any $\widehat{x_k} \in I^{n-1}$,

$$F_{\widehat{x_k}}(y) = \begin{cases} f_{\widehat{x_k}}(y) - \dfrac{\alpha_t f_{\widehat{x_k}}(y)}{|f_{\widehat{x_k}}(y)|}, & \text{if } y \in A_t(\widehat{x_k}), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$A_t(\widehat{x_k}) = \big\{y \in I : (\widehat{x_k}, y) \in A_t\big\} = \big\{y \in I : |f_{\widehat{x_k}}(y)| > \alpha_t\big\}.$$

Thus, for any $s \geq 0$ and $\widehat{x_k} \in I^{n-1}$,

$$\begin{aligned} \lambda_{F_{\widehat{x_k}}}(s) &= \big|\big\{y \in I : |F_{\widehat{x_k}}(y)| > s\big\}\big| = \big|\big\{y \in A_t(\widehat{x_k}) : \big||f_{\widehat{x_k}}(y)| - \alpha_t\big| > s\big\}\big| \\ &= \big|\big\{y \in I : |f_{\widehat{x_k}}(y)| - \alpha_t > s\big\}\big| = \lambda_{f_{\widehat{x_k}}}(s+\alpha_t). \end{aligned}$$

Now, let us suppose that $\widehat{x_k} \notin \Pi_k^* A_t$. Then, by definition, $\lambda_{f_{\widehat{x_k}}}(\alpha_t) = 0$. As a consequence, we have that if $\widehat{x_k} \notin \Pi_k^* A_t$, then $\lambda_{F_{\widehat{x_k}}}(s) = 0$, for any $s \geq 0$. Therefore, for any $s \geq 0$, it holds that

$$\lambda_{F_{\widehat{x_k}}}(s) = \begin{cases} \lambda_{f_{\widehat{x_k}}}(s+\alpha_t), & \text{if } \widehat{x_k} \in \Pi_k^* A_t, \\ 0, & \text{otherwise.} \end{cases}$$

So, Lemma 3.1.3 implies that, for any $s \geq 0$,

$$\lambda_{F_{\widehat{x_k}}}(s) = \begin{cases} \lambda_{f_{\widehat{x_k}}}(s+\alpha_t), & \text{if } \widehat{x_k} \in \{\widehat{x_k} \in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x_k}) > \alpha_t\}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for any $\widehat{x_k} \in \big\{\widehat{x_k} \in I^{n-1} : \psi_k(f,L^\infty)(\widehat{x_k}) > \alpha_t\big\}$, we get

$$\begin{aligned} \psi_k(F,L^\infty)(\widehat{x_k}) &= \inf\big\{y > 0 : \lambda_{F_{x_k}}(y) = 0\big\} = \inf\{y > 0 : \lambda_{f_{x_k}}(y+\alpha_t) = 0\big\} \\ &= \psi_k(f,L^\infty)(\widehat{x_k}) - \alpha_t. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|F\|_{\mathcal{R}_k(X,L^\infty)} &= \big\|(\psi_k(f,L^\infty)(\widehat{x_k})-\alpha_t)\chi_{\{\widehat{x_k}\in I^{n-1}:\psi_k(f,L^\infty)(\widehat{x_k})>\alpha_t\}}(\widehat{x_k})\big\|_{X(I^{n-1})}\\ &= \big\|(\psi_k^*(f,L^\infty)-\alpha_t)\chi_{(0,\lambda_{\psi_k(f,L^\infty)}(\alpha_t))}\big\|_{\overline{X}(0,1)}.\end{aligned}$$

But, using (3.8), we get

$$\|F\|_{\mathcal{R}_k(X,L^\infty)} \le \big\|\psi_k^*(f,L^\infty)\chi_{(0,t)}\big\|_{\overline{X}(0,1)}, \quad \text{for any } k\in\{1,\dots,n\}.$$

Therefore, using the above inequality, we obtain

$$\begin{aligned}K(f,\varphi_X(t);\mathcal{R}(X,L^\infty),L^\infty) &\le \|F\|_{\mathcal{R}(X,L^\infty)}+\varphi_X(t)\|G\|_{L^\infty(\mathbb{R}^n)}\\ &= \sum_{k=1}^n \|F\|_{\mathcal{R}_k(X,L^\infty)}+\varphi_X(t)\alpha_t\\ &\le \sum_{k=1}^n \big\|\psi_k^*(f,L^\infty)\chi_{(0,t)}\big\|_{\overline{X}(0,1)}+\varphi_X(t)\alpha_t.\end{aligned}$$

But, it holds that

$$\varphi_X(t)\alpha_t \le \sum_{k=1}^n \big\|\psi_k^*(f,L^\infty)\chi_{(0,t)}\big\|_{\overline{X}(0,1)},$$

and hence, we have

$$K(f,\varphi_X(t);\mathcal{R}(X,L^\infty),L^\infty) \le 2\sum_{k=1}^n \big\|\psi_k^*(f,L^\infty)\chi_{(0,t)}\big\|_{\overline{X}(0,1)}, \quad t>0,$$

as we wanted to show. □

As a consequence we obtain the following result. Recall that $(A,B)_{\theta,q}$ stands for the real interpolation space of couple (A,B) (see Definition 2.1.16).

Corollary 3.1.8. *Let $0<\theta<1$ and $1\le q\le\infty$ and let $X(I^{n-1})$ be an r.i. space. Then,*

$$(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q} = \mathcal{R}((X,L^\infty)_{\theta,q},L^\infty),$$

with equivalent norms.

To prove it, we first need to recall a result concerning the K-functional of pairs of r.i. spaces. For further information see [46, 3].

Theorem 3.1.9. *Let $X(I^n)$ be an r.i. space. Then,*

$$K(f,\varphi_X(t);X,L^\infty) \approx \big\|f^*\chi_{(0,t)}\big\|_{\overline{X}(0,1)}, \quad t>0.$$

Proof of Corollary 3.1.8. For the sake of simplicity, we prove this result only when $1 \le q < \infty$. Let $f \in (\mathcal{R}(X, L^\infty), L^\infty)_{\theta,q}$. Then, by a change of variables, we get

$$\begin{aligned}\|f\|^q_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}} &= \int_0^\infty t^{-\theta q-1}\big[K(f,t;\mathcal{R}(X,L^\infty),L^\infty)\big]^q dt \\ &= \int_0^\infty \big(\varphi_X(s)\big)^{-\theta q-1}\big[K(f,\varphi_X(s);\mathcal{R}(X,L^\infty),L^\infty)\big]^q d\varphi_X(s).\end{aligned}$$

Hence, using Theorem 3.1.7, we obtain

$$\begin{aligned}\|f\|^q_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}} &\approx \int_0^\infty \big(\varphi_X(s)\big)^{-\theta q-1}\left[\sum_{k=1}^n \|\psi_k^*(f,L^\infty)\chi_{(0,s)}\|_{\overline{X}(0,1)}\right]^q ds \\ &\ge \int_0^\infty \big(\varphi_X(s)\big)^{-\theta q-1}\|\psi_k^*(f,L^\infty)\chi_{(0,s)}\|^q_{\overline{X}(0,1)} ds,\end{aligned}$$

for any $k \in \{1,\ldots,n\}$. So, Theorem 3.1.9 implies that

$$\begin{aligned}\|f\|^q_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}} &\gtrsim \int_0^\infty \big(\varphi_X(s)\big)^{-\theta q-1}\Big[K(\psi_k(f,L^\infty),\varphi_X(s);X,L^\infty)\Big]^q d\varphi_X(s) \\ &= \big\|\psi_k(f,L^\infty)\big\|^q_{(X,L^\infty)_{\theta,q}} = \|f\|^q_{\mathcal{R}_k((X,L^\infty)_{q,\theta},L^\infty)}.\end{aligned}$$

As a consequence, we get

$$\|f\|_{\mathcal{R}((X,L^\infty)_{\theta,q},L^\infty)} = \sum_{k=1}^n \|f\|_{\mathcal{R}_k((X,L^\infty)_{\theta,q},L^\infty)} \lesssim \|f\|_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}}.$$

Thus, we have seen that the embedding

$$(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q} \hookrightarrow \mathcal{R}((X,L^\infty)_{\theta,q},L^\infty)$$

holds. Hence, to complete the proof, it only remains to see that

$$\mathcal{R}((X,L^\infty)_{\theta,q},L^\infty) \hookrightarrow (\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q},$$

also holds. To do it, we fix any $f \in \mathcal{R}((X,L^\infty)_{\theta,q},L^\infty)$. Then, using Theorem 3.1.7 and the subadditive property of $\|\cdot\|_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}}$, we get

$$\|f\|_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}} \le \sum_{k=1}^n \left(\int_0^\infty \big(\varphi_X(s)\big)^{-\theta q-1}\|\psi_k^*(f,L^\infty)\chi_{(0,s)}\|^q_{\overline{X}(0,1)} d\varphi_X(s)\right)^{1/q}.$$

So, using Theorem 3.1.9, we obtain

$$\|f\|_{(\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q}} \lesssim \sum_{k=1}^n \big\|\psi_k(f,L^\infty)\big\|_{(X,L^\infty)_{\theta,q}} = \|f\|_{\mathcal{R}((X,L^\infty)_{\theta,q},L^\infty)}.$$

From this, we conclude that

$$\mathcal{R}((X,L^\infty)_{\theta,q},L^\infty) \hookrightarrow (\mathcal{R}(X,L^\infty),L^\infty)_{\theta,q},$$

as we wanted to prove. $\square$

Finally, let us describe the K-functional associated to the couple of mixed norm spaces $(L^p, \mathcal{R}(L^\infty, L^p))$. But first, we need to recall a result concerning commutation properties for interpolation spaces [42] (for more details see also [48]).

Theorem 3.1.10. *Let $Y(I^n)$ and $X_i(I^n)$, with $i \in \{1, \dots, n\}$, be Banach function spaces. Then, for any $f \in Y(I^n) + \bigcap_{k=1}^n X_k(I^n)$,*

$$K\Big(f, t; Y, \bigcap_{k=1}^{n} X_k\Big) \approx \sum_{k=1}^{n} K(f, t; Y, X_k), \quad t > 0.$$

Theorem 3.1.11. *Let $1 \le p < \infty$. Then, for any $f \in L^p(I^n) + \mathcal{R}(L^\infty, L^p)$,*

$$K(f, t, L^p; \mathcal{R}(L^\infty, L^p)) \approx \sum_{k=1}^{n} \bigg(\int_0^{t^p} \big(\psi_k^*(f, L^p)(s)\big)^p ds \bigg)^{1/p}, \quad t > 0.$$

Proof. Let any $f \in L^p(I^n) + \mathcal{R}_k(L^\infty, L^p)$. By Lemma 3.1.6 and Theorem 3.1.10, we only need to prove that, for any $k \in \{1, \dots, n\}$,

$$K(f, t; L^p, \mathcal{R}_k(L^\infty, L^p)) \lesssim \bigg(\int_0^{t^p} \big(\psi_k^*(f, L^p)(s)\big)^p ds \bigg)^{1/p}, \quad t > 0.$$

We fix any $k \in \{1, \dots, n\}$ and $t > 0$. Next, we define

$$A_{t,k} = \big\{ \widehat{x}_k \in I^{n-1} : |\psi_k(f, L^p)(\widehat{x}_k)| > \alpha_t \big\}, \quad \text{where } \alpha_t = \psi_k^*(f, L^p)(t^p),$$

$$F(x) = \begin{cases} \dfrac{f(x)\big(\psi_k(f, L^p)(\widehat{x}_k) - \alpha_t\big)}{\psi_k(f, L^p)(\widehat{x}_k)}, & \text{if } (\widehat{x}_k, x_k) \in A_{t,k} \times I, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$G(x) = f(x) - F(x) = \begin{cases} f(x), & \text{if } (\widehat{x}_k, x_k) \in (A_{t,k})^c \times I, \\ \dfrac{\alpha_t f(x)}{\psi_k(f, L^p)(\widehat{x}_k)}, & \text{if } (\widehat{x}_k, x_k) \in A_{t,k} \times I. \end{cases}$$

Then,

$$\psi_k(F, L^p)(\widehat{x}_k) = \begin{cases} \psi_k(f, L^p)(\widehat{x}_k) - \alpha_t, & \text{if } \widehat{x}_k \in A_{t,k}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi_k(G, L^p)(\widehat{x}_k) = \begin{cases} \psi_k(f, L^p)(\widehat{x}_k), & \text{if } \widehat{x}_k \in (A_{t,k})^c, \\ \alpha_t, & \text{if } \widehat{x}_k \in A_{t,k}. \end{cases}$$

Thus, for any $s \ge 0$,

$$\lambda_{\psi_k(F, L^p)}(s) = \lambda_{\psi_k(f, L^p)}(s + \alpha_t),$$

and

$$\begin{aligned}\lambda_{\psi_k(G,L^p)}(s) &= \left|\left\{\widehat{x_k} \in I^{n-1} : |\psi_k(G,L^p)(\widehat{x_k})| > s\right\}\right| \\ &= \left|\left\{\widehat{x_k} \in (A_{t,k})^c : \psi_k(f,L^p)(\widehat{x_k}) > s\right\}\right| + \left|\left\{\widehat{x_k} \in A_{t,k} : \alpha_t > s\right\}\right| \\ &= \left|\left\{\widehat{x_k} \in I^{n-1} : s < \psi_k(f,L^p)(\widehat{x_k}) \le \alpha_t\right\}\right| + \left|\left\{\widehat{x_k} \in A_{t,k} : \alpha_t > s\right\}\right| \\ &= \begin{cases} \lambda_{\psi_k(f,L^p)}(s), & 0 \le s < \alpha_t, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

As a consequence, we have

$$\psi_k^*(F,L^p)(s) = \begin{cases} \psi_k^*(f,L^p)(s) - \alpha_t, & 0 \le s < \lambda_{\psi_k(f,L^p)}(\alpha_t), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi_k^*(G,L^p)(s) = \begin{cases} \alpha_t, & 0 \le s < \lambda_{\psi_k(f,L^p)}(\alpha_t), \\ \psi_k^*(f,L^p)(s), & s > \lambda_{\psi_k(f,L^p)}(\alpha_t). \end{cases}$$

Therefore, using the same ideas as in the proof of Theorem 3.1.7,

$$K(f,t;L^p,\mathcal{R}_k(L^\infty,L^p)) \lesssim \left(\int_0^{t^p} \left(\psi_k^*(f,L^p)(s)\right)^p ds\right)^{1/p},$$

as we wanted to show. □

3.2 Embeddings between mixed norm spaces

Our aim in this section is to characterize certain embeddings between mixed norm spaces. Before that, let us emphasize that relations between mixed norm spaces of Lorentz spaces were studied in [2], where it was shown, for instance,

$$\mathcal{R}(L^1,L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)',1},L^1), \quad n \ge 2. \tag{3.9}$$

Let us start with some preliminary lemmas:

Lemma 3.2.1. *Let $k \in \{1,\dots,n\}$. Let $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$, and $Y_2(I)$ be r.i. spaces. Then,*

$$\mathcal{R}_k(X_1,Y_1) \hookrightarrow \mathcal{R}_k(X_2,Y_2) \Leftrightarrow \begin{cases} X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}), \\ Y_1(I) \hookrightarrow Y_2(I). \end{cases}$$

Proof. To prove the implication "⇒", we just have to apply the hypothesis to the functions

$$g_1(x) = f_1(\widehat{x_k})\chi_{I^n}(x) \quad \text{and} \quad g_2(x) = f_2(x_k)\chi_{I^n}(x),$$

with $f_1 \in X_1(I^{n-1})$ and $f_2 \in Y_1(I)$. The converse follows from Definition 3.1.1. □

It is important to observe that there are examples, for instance (3.9), showing that if in Lemma 3.2.1 we replace the Benedek-Panzone spaces by mixed norm spaces, then the corresponding equivalence is not longer true. However, we always have this result:

Lemma 3.2.2. *Let $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$, and $Y_2(I)$ be r.i. spaces. Then,*

$$X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}) \text{ and } Y_1(I) \hookrightarrow Y_2(I) \Rightarrow \mathcal{R}(X_1, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2).$$

Proof. It is immediate from Definition 3.1.2 and Lemma 3.2.1. □

Lemma 3.2.1 and Lemma 3.2.2 show that it is natural to study when embeddings between mixed norm spaces are true (see also [31]). Motivated by this problem, we shall find necessary and sufficient conditions in the following cases:

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2), \quad \mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty), \quad \mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y).$$

Theorem 3.2.3. *For any r.i. spaces $X_1(I^{n-1})$, $X_2(I^{n-1})$, $Y_1(I)$ and $Y_2(I)$, if the following embedding*

$$\mathcal{R}(X_1, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2)$$

holds, then $Y_1(I) \hookrightarrow Y_2(I)$.

Proof. By Lemma 3.2.2, we may assume, without loss of generality, that the following embedding

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(L^1, Y_2)$$

holds. Also, we shall suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$. Let $r \in \mathbb{R}$ such that $0 < r < \min(a, b)$. Given any function $g \in Y_1(I)$, with $\lambda_g(0) \le 2r/n$, we define

$$f(x) = g^*\Big(2\Big|\sum_{i=1}^n x_i\Big|\Big)\chi_{(-r,r)^n}(x).$$

For any $k \in \{1, \dots, n\}$, we denote

$$\beta_k = \sum_{i=1, i\neq k}^n x_i, \quad \text{whenever } \widehat{x_k} \in (-r, r)^{n-1}.$$

Now, if $s \ge 0$, we have

$$\begin{aligned}
\lambda_{f_{\widehat{x_k}}}(s) &= |\{x \in (-r, r) : g^*(2|x + \beta_k|) > s\}| \\
&= |\{x \in (-r, r) \cap (-r/n - \beta_k, r/n - \beta_k) : g^*(2|x + \beta_k|) > s\}| \\
&\le |\{x \in (-r/n - \beta_k, r/n - \beta_k) : g^*(2|x + \beta_k|) > s\}| \\
&= |\{x \in (-r/n, r/n) : g^*(2|x|) > s\}| = \lambda_g(s).
\end{aligned}$$

Thus,

$$\{s \ge 0 : \lambda_g(s) \le t\} \subseteq \{s \ge 0 : \lambda_{f_{\widehat{x_k}}}(s) \le t\}, \quad \text{for any } t \ge 0,$$

and so $f^*_{\widehat{x_k}} \leq g^*$. Hence, we get

$$\psi_k(f, Y_1)(\widehat{x_k}) \leq \|g^*\|_{\overline{Y}_1(0,1)}, \quad \widehat{x_k} \in (-r, r)^{n-1}.$$

Therefore,

$$\|f\|_{\mathcal{R}_k(L^\infty, Y_1)} \leq \|g^*\|_{\overline{Y}_1(0,1)}, \quad k \in \{1, \dots, n\}.$$

Hence, our assumption on g ensures that $f \in \mathcal{R}(L^\infty, Y_1)$ and

$$\|f\|_{\mathcal{R}(L^\infty, Y_1)} \leq n\|g^*\|_{\overline{Y}_1(0,1)}. \tag{3.10}$$

Thus, using $\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(L^1, Y_2)$ and (3.10), we get

$$\|f\|_{\mathcal{R}(L^1, Y_2)} \lesssim \|g^*\|_{\overline{Y}_1(0,1)}. \tag{3.11}$$

Now, let us compute $\|f\|_{\mathcal{R}(L^1, Y_2)}$. In order to do it, we fix any $k \in \{1, \dots, n\}$ and $\widehat{x_k} \in (0, r/n)^{n-1}$, and set

$$\gamma_k = \sum_{i=1, i\neq k}^{n} x_i.$$

As before, if $s \geq 0$, we have

$$\lambda_{f_{\widehat{x_k}}}(s) = |\{x \in (-r, r) \cap (-r/n - \gamma_k, r/n - \gamma_k) : g^*(2|x + \gamma_k|) > s\}|.$$

But, $0 < \gamma_k < r/n'$, so we obtain

$$\lambda_{f_{\widehat{x_k}}}(s) = |\{x \in (-r/n - \gamma_k, r/n - \gamma_k) : g^*(2|x + \gamma_k|) > s\}| = \lambda_g(s),$$

for any $s \geq 0$. As a consequence, if $\widehat{x_k} \in (0, r/n)^{n-1}$, then $f^*_{\widehat{x_k}} = g^*$. Thus,

$$\begin{aligned}\psi_k(f, Y_2)(\widehat{x_k}) &= \|f(\widehat{x_k}, \cdot)\|_{Y_2(I)}\chi_{(-r,r)^{n-1}}(\widehat{x_k}) \geq \|f(\widehat{x_k}, \cdot)\|_{Y_2(I)}\chi_{(0,r/n)^{n-1}}(\widehat{x_k}) \\ &= \|g^*\|_{\overline{Y}_2(0,1)}\chi_{(0,r/n)^{n-1}}(\widehat{x_k}),\end{aligned}$$

and so

$$\|f\|_{\mathcal{R}(L^1, Y_2)} \gtrsim \|g^*\|_{\overline{Y}_2(0,1)}.$$

Therefore, inequality (3.11) gives us that

$$\|g^*\|_{\overline{Y}_2(0,1)} \lesssim \|g^*\|_{\overline{Y}_1(0,1)}, \tag{3.12}$$

for any $g \in Y(I)$, with $\lambda_g(0) \leq 2r/n$. Now, let us consider a general function $g \in Y_1(I)$. We define

$$g_1(x) = \max[|g(x)| - g^*(2r/n), 0] \operatorname{sgn} g(x),$$

and

$$g_2(x) = \min[(|g(x)|, g^*(2r/n)] \operatorname{sgn} g(x).$$

Since $\lambda_{g_1}(0) \le 2r/n$, the inequality (3.12), with g replaced by g_1, implies that

$$\|g_1\|_{Y_2(I)} \lesssim \|g_1\|_{Y_1(I)}. \tag{3.13}$$

Thus, combining the conditions $g_1 \le g$ a.e. with (3.13), we get

$$\|g_1\|_{Y_2(I)} \lesssim \|g\|_{Y_1(I)}. \tag{3.14}$$

On the other hand, by Hölder's inequality, we obtain

$$\|g_2\|_{Y_2(I)} \le \varphi_{Y_2}(1) g^{**}(2r/n) \lesssim \|g\|_{Y_1(I)}. \tag{3.15}$$

Finally, using (3.14) and (3.15), we get

$$\|g\|_{Y_2(I)} = \|g_1 + g_2\|_{Y_2(I)} \lesssim \|g\|_{Y_1(I)}, \quad f \in Y_1(I),$$

and the proof is complete. □

As a consequence we have the following corollaries:

Corollary 3.2.4. *Let $X_2(I^{n-1})$, $Y_1(I)$ and $Y_2(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2) \Leftrightarrow Y_1(I) \hookrightarrow Y_2(I).$$

Proof. The necessary part of this result follows from Theorem 3.2.3. Now, if $Y_1(I)$ is continuously embedded into $Y_2(I)$, then, using (2.1) together with Lemma 3.2.2, we conclude that

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, Y_2).$$

Thus, the proof is complete. □

Corollary 3.2.5. *Let $X(I^{n-1})$, $Y_1(I)$ and $Y_2(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(X, Y_1) \hookrightarrow \mathcal{R}(X, Y_2) \Leftrightarrow Y_1(I) \hookrightarrow Y_2(I).$$

Proof. The sufficient part of this result is an immediate consequence of Lemma 3.2.2. On the other hand, if the embedding

$$\mathcal{R}(X, Y_1) \hookrightarrow \mathcal{R}(X, Y_2)$$

holds, then, using (2.1), we get

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X, Y_1) \hookrightarrow \mathcal{R}(X, Y_2).$$

Therefore, the result follows by Corollary 3.2.4. □

Corollary 3.2.6. *Let $X_2(I^{n-1})$, and $Y_1(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, L^\infty) \Leftrightarrow Y_1(I) = L^\infty(I).$$

Proof. In view of follows from Lemma 3.2.2, it suffices to prove the necessary part of this result. On the other hand, if the embedding

$$\mathcal{R}(L^\infty, Y_1) \hookrightarrow \mathcal{R}(X_2, L^\infty),$$

then, using Corollary 3.2.4 together with (2.1), we conclude that $Y_1(I) = L^\infty(I)$, as we wanted to show. □

Another consequence of Theorem 3.2.3 is the following result regarding the Lorentz spaces $L^{p,q}$:

Corollary 3.2.7. *Let $1 < p_1, p_3 < \infty$, $1 \le q_1, q_3 \le \infty$ and either $p_2 = q_2 = 1$, $p_2 = q_2 = \infty$ or $1 < p_2 < \infty$ and $1 \le q_2 \le \infty$. Then,*

$$\mathcal{R}(L^\infty, L^{p_1,q_1}) \hookrightarrow \mathcal{R}(L^{p_2,q_2}, L^{p_3,q_3}) \Leftrightarrow \begin{cases} p_3 < p_1\,, 1 \le q_1\,, q_3 \le \infty, \\ p_1 = p_3\,, 1 \le q_1 \le q_3 \le \infty. \end{cases}$$

Proof. It follows from Theorem 3.2.3 and the classical embeddings for Lorentz spaces (see [8]). □

Let us now study embeddings between mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$.

Theorem 3.2.8. *Let $X_1(I^{n-1})$ and $X_2(I^{n-1})$ be r.i. spaces. Then,*

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty) \Leftrightarrow X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}).$$

Proof. In view of Lemma 3.2.2, we only need to prove that if the embedding

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty)$$

holds, then $X_1(I^{n-1}) \hookrightarrow X_2(I^n)$. As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any $f \in X_1(I^{n-1})$, with $\lambda_f(0) \le \omega_{n-1} r^{n-1}$, we define

$$g(x) = \begin{cases} f^*(\omega_{n-1}|x|^{n-1}), & \text{if } x \in B_n(0,r), \\ 0, & \text{otherwise.} \end{cases}$$

We fix any $k \in \{1, \dots, n\}$. Then, it holds that

$$\psi_k(g, L^\infty)(\widehat{x}_k) = \|g(\widehat{x}_k, \cdot)\|_{L^\infty(I)} = f^*(\omega_{n-1}|\widehat{x}_k|^{n-1}), \quad \text{if } \widehat{x}_k \in B_{n-1}(0,r),$$

and $\psi_k(g, L^\infty)(\widehat{x}_k) = 0$ otherwise. So, Lemma 2.1.3 implies that

$$\|g\|_{\mathcal{R}_k(X_1, L^\infty)} = \|\psi_k(g, L^\infty)\|_{X_1(I^{n-1})} = \|f\|_{X_1(I^{n-1})},$$

for any $k \in \{1, \dots, n\}$, Hence, since we are assuming that $f \in X_1(I^{n-1})$, we obtain $g \in \mathcal{R}(X_1, L^\infty)$ and

$$\|g\|_{\mathcal{R}(X_1, L^\infty)} = n\|f\|_{X_1(I^{n-1})}.$$

So, using $\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^\infty)$ and the previous inequality, we get

$$\|g\|_{\mathcal{R}(X_2,L^\infty)} \lesssim \|f\|_{X_1(I^{n-1})}.$$

But, as before,

$$\|g\|_{\mathcal{R}(X_2,L^\infty)} = n\|f\|_{X_2(I^{n-1})},$$

hence, we have

$$\|f\|_{X_2(I^{n-1})} \lesssim \|f\|_{X_1(I^{n-1})}.$$

This proves that if $f \in X_1(I^{n-1})$, with $\lambda_f(0) \le \omega_{n-1} r^{n-1}$, then $f \in X_2(I^{n-1})$. The rest of the proof is essentially the same as in Theorem 3.2.3. □

For a general r.i. space $Y(I)$, we have a similar result assuming some conditions on $X_1(I^{n-1})$.

Theorem 3.2.9. *Let $X_1(I^{n-1})$ be an r.i. space, with $\underline{\alpha}_{X_1} > 0$, and let $X_2(I^{n-1})$ and $Y(I)$ be r.i. spaces. Then,*

$$\mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y) \Leftrightarrow X_1(I^{n-1}) \hookrightarrow X_2(I^{n-1}).$$

Proof. As before, according to Lemma 3.2.2, it suffices to prove the necessary part of this result. Also, by Theorem 3.2.8, we assume that $Y(I) \neq L^\infty(I)$. Let us suppose that the embedding

$$\mathcal{R}(X_1, Y) \hookrightarrow \mathcal{R}(X_2, Y)$$

holds and that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any function $f \in X_1(I^{n-1})$, with $\lambda_f(0) \le \omega_{n-1} r^{n-1}$, we define

$$g(x) = \begin{cases} \displaystyle\int_{\omega_{n-1}|x|^{n-1}}^{\omega_{n-1}r^{n-1}} \frac{f^*(t)}{t\varphi_Y\left(2\left(t/\omega_{n-1}\right)^{1/(n-1)}\right)}\, dt, & \text{if } x \in B_n(0,r), \\ 0, & \text{otherwise.} \end{cases}$$

We fix $k \in \{1, \dots, n\}$ and $\widehat{x_k} \in B_{n-1}(0, r)$. Using now Lemma 2.1.3 together with Theorem 2.1.14 and recalling the definition of Λ_{φ_Y} given in (2.8), we obtain

$$\begin{aligned} \psi_k(g, Y)(\widehat{x_k}) \le \psi_k(g, \Lambda_{\varphi_Y})(\widehat{x_k}) &= \psi_k(g, L^\infty)(\widehat{x_k})\varphi_Y(0^+) + \int_0^1 g^*_{\widehat{x_k}}(t)\varphi_Y'(t)dt \\ &= \int_0^{2((\lambda_f(0)/\omega_{n-1})^{1/(n-1)} - |\widehat{x_k}|)} \varphi_Y'(t)dt \\ &\quad \times \left(\int_{\omega_{n-1}(t/2+|\widehat{x_k}|)^{n-1}}^{\omega_{n-1}r^{n-1}} \frac{f^*(s)}{s\varphi_Y\left(2\left(s/\omega_{n-1}\right)^{1/(n-1)}\right)}\, ds \right). \end{aligned}$$

Then, Fubini's theorem gives

$$\psi_k(g,Y)(\widehat{x_k}) \lesssim \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{\lambda_f(0)} \frac{f^*(s)}{s\varphi_Y(2\,(s/\omega_{n-1})^{1/(n-1)})} \times \left(\int_0^{2(s/\omega_{n-1})^{1/(n-1)}-2|\widehat{x_k}|} \varphi_Y'(t)dt\right)ds$$
$$= \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{\lambda_f(0)} \frac{f^*(s)\varphi_Y\left(2(s/\omega_{n-1})^{1/(n-1)}-2|\widehat{x_k}|\right)}{s\varphi_Y(2\,(s/\omega_{n-1})^{1/(n-1)})}\,ds.$$

Hence, using that φ_Y is an increasing function, we deduce that

$$\psi_k(g,Y)(\widehat{x_k}) \lesssim \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{\lambda_f(0)} f^*(s)\frac{ds}{s}.$$

Since $\underline{\alpha}_{X_1} > 0$, [8, Theorem V.5.15] ensures that the integral operator

$$\int_t^1 f^*(s)\frac{ds}{s}$$

is bounded on $\overline{X}_1(I^{n-1})$ and, as a consequence, we obtain

$$\|g\|_{\mathcal{R}_k(X_1,Y)} = \|\psi_k(g,Y)\|_{X_1(I^{n-1})} \lesssim \left\|\int_t^{\lambda_f(0)} f^*(s)\frac{ds}{s}\right\|_{\overline{X}_1(0,1)} \lesssim \|f\|_{X_1(I^{n-1})},$$

for any $k\in\{1,\dots,n\}$. Hence, our assumption on f gives that $g\in\mathcal{R}(X_1,Y)$ and

$$\|g\|_{\mathcal{R}(X_1,Y)} \lesssim \|f\|_{X_1(I^{n-1})}. \tag{3.16}$$

So, using $\mathcal{R}(X_1,Y)\hookrightarrow\mathcal{R}(X_2,Y)$ and (3.16), we get

$$\|g\|_{\mathcal{R}(X_2,Y)} \lesssim \|f\|_{X_1(I^{n-1})}. \tag{3.17}$$

We next find a lower estimate for $\|g\|_{\mathcal{R}(X_2,Y)}$. In fact, we fix $k\in\{1,\dots,n\}$ and $\widehat{x_k}\in B_{n-1}(0,r/2)$. Then, by Hölder's inequality, we get

$$\frac{1}{\varphi_{Y'}(2|\widehat{x_k}|)}\int_{B_1(0,|\widehat{x_k}|)} g(\widehat{x_k},x_k)dx_k \le \psi_k(g,Y)(\widehat{x_k}). \tag{3.18}$$

On the other hand, by a change of variables, it holds that

$$\int_{B_1(0,|\widehat{x_k}|)} g(\widehat{x_k},x_k)dx_k \approx \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}} t^{1/(n-1)}\frac{dt}{t} \times \left(\int_t^{\omega_{n-1}r^{n-1}} \frac{f^*(s)}{s\varphi_Y(2\,(s/\omega_{n-1})^{1/(n-1)})}ds\right),$$

and so Fubini's theorem and (2.2) give

$$\begin{aligned}\int_{B_1(0,|\widehat{x_k}|)} g(\widehat{x_k}, x_k) &\gtrsim \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}} \frac{f^*(t)}{t\varphi_Y(2\,(t/\omega_{n-1})^{1/(n-1)})}\,dt \\ &\quad \times \left(\int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{t} s^{1/(n-1)-1} ds\right) \\ &\gtrsim \int_{(3/2)^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}}^{2^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}} \frac{f^*(t)\left(t^{1/(n-1)} - \omega_{n-1}^{1/(n-1)}|\widehat{x_k}|\right)}{t\varphi_Y(2\,(t/\omega_{n-1})^{1/(n-1)})} dt \\ &\gtrsim \varphi_{Y'}(2|\widehat{x_k}|) f^*(2^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}).\end{aligned}$$

Hence, using (3.18), we obtain

$$f^*(2^{n-1}\omega_{n-1}|\widehat{x_k}|^{n-1}) \lesssim \psi_k(g, Y)(\widehat{x_k}), \quad \widehat{x_k} \in B_{n-1}(0, r/2),$$

and hence Lemma 2.1.3 gives

$$\|f^*\|_{\overline{X}_2(0,1)} \lesssim \|\chi_{(0,\lambda_f(0)/2)} f^*(2t)\|_{\overline{X}_2(0,1)} \lesssim \|\psi_k(g, Y)\|_{X_2(I^{n-1})}.$$

Thus, using (3.17), we get

$$\|f\|_{X_2(I^{n-1})} \lesssim \|f\|_{X_1(I^{n-1})}.$$

This proves that if $f \in X_1(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1} r^{n-1}$, then $f \in X_2(I^{n-1})$. The general case can be treated as at the end of the proof of Theorem 3.2.3. □

As a consequence of Theorem 3.2.8 and Theorem 3.2.9, we obtain the following result:

Corollary 3.2.10. *Let* $1 < p_1\,, p_3 < \infty$, $1 \leq q_1\,, q_3 \leq \infty$ *and either* $p_2 = q_2 = 1$, $p_2 = q_2 = \infty$ *or* $1 < p_2 < \infty$ *and* $1 \leq q_2 \leq \infty$. *Then*

$$\mathcal{R}(L^{p_1,q_1}, L^{p_2,q_2}) \hookrightarrow \mathcal{R}(L^{p_3,q_3}, L^{p_2,q_2}) \Leftrightarrow \begin{cases} p_3 < p_1\,, 1 \leq q_1\,, q_3 \leq \infty, \\ p_1 = p_3\,, 1 \leq q_1 \leq q_3 \leq \infty. \end{cases}$$

Proof. It follows from Theorem 3.2.8, Theorem 3.2.9 and the classical embeddings for Lorentz spaces (see [8]). □

Now, let us study the embedding

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^1). \tag{3.19}$$

Let us start by analyzing the case $n = 2$. The following result will be useful for our purposes.

Lemma 3.2.11. *Let* $X(I)$ *be an r.i. space. Then,*

$$\mathcal{R}(L^1, X) \hookrightarrow \mathcal{R}(X, L^1).$$

Proof. Let $f \in \mathcal{R}(L^1, X)$ and $k \in \{1,2\}$. Then, using Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned}\|f\|_{\mathcal{R}_k(X,L^1)} &= \sup_{\|g\|_{X'(I)} \leq 1} \int_I \int_I |g(\widehat{x_k}) f(\widehat{x_k}, x_k)| d\widehat{x_k} dx_k \\ &\leq \int_I \psi_k(f, X)(\widehat{x_k}) d\widehat{x_k} = \|f\|_{\mathcal{R}_k(L^1,X)} \leq \|f\|_{\mathcal{R}(L^1,X)}.\end{aligned}$$

That is, $\mathcal{R}(L^1, X) \hookrightarrow \mathcal{R}(X, L^1)$ and the proof is complete. □

Corollary 3.2.12. *For any couple of r.i. spaces $X_1(I)$ and $X_2(I)$, we have*

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(X_2, L^1).$$

Proof. Using Lemma 3.2.2 and Lemma 3.2.11, we get

$$\mathcal{R}(X_1, L^\infty) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^1, X_2) \hookrightarrow \mathcal{R}(X_2, L^1),$$

as we wanted to see. □

Now, let us consider the embedding (3.19), for the case $n \geq 3$. In particular, we shall provide a characterization of the smallest mixed norm space of the form $\mathcal{R}(Y, L^1)$ in (3.19) once the mixed norm space $\mathcal{R}(X_1, L^\infty)$ is given. In order to do it, we begin with a preliminary lemma.

Lemma 3.2.13. *Let $X(I^{n-1})$ be an r.i. space, with $n \geq 3$. Then, the functional defined by*

$$\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} = \|f^*(t^{(n-1)'})\|_{\overline{X}(0,1)}, \quad f \in \mathcal{M}_+(I^{n-1}) \tag{3.20}$$

is an r.i. norm.

Proof. The positivity and homogeneity of $\|\cdot\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}$ are clear. Next, let f and g be measurable functions on I^n. Then,

$$\begin{aligned}\|f+g\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} &= \|(f+g)^*(t^{(n-1)'})\|_{\overline{X}(0,1)} \\ &= \frac{1}{(n-1)'} \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^1 (f+g)^*(t) t^{-1/(n-1)} h^*(t^{1/(n-1)'}) dt.\end{aligned}$$

Thus, using Proposition 2.1.4 together with Hardy-Littlewood-Pólya Principle (see Theorem 2.1.11), we get

$$\begin{aligned}\|f+g\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} &\leq \frac{1}{(n-1)'} \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^1 f^*(t) t^{-1/(n-1)} h^*(t^{1/(n-1)'}) dt \\ &\quad + \frac{1}{(n-1)'} \sup_{\|h\|_{X'(I^{n-1})} \leq 1} \int_0^1 g^*(t) t^{-1/(n-1)} h^*(t^{1/(n-1)'}) dt \\ &= \|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} + \|g\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.\end{aligned}$$

The proof of (A2)-(A4) and (A6) for $\|\cdot\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}$ requires only the corresponding axioms for $\|\cdot\|_{\overline{X}(0,1)}$, hence we shall omit them. Finally, to prove property (A5), we fix any $f \in \mathcal{M}(I^{n-1})$. Then,

$$\|f^*(t^{(n-1)'})\|_{\overline{X}(0,1)} \gtrsim \int_0^1 f^*(t^{(n-1)'})dt \geq \int_0^1 f^*(t)dt.$$

□

Theorem 3.2.14. *Let $n \geq 3$. Let $X(I^{n-1})$ be an r.i. space and let $X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$ be as in (3.20). Then, the embedding*

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^1) \tag{3.21}$$

holds. Moreover, $\mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^1)$ is the smallest space of the form $\mathcal{R}(Y, L^1)$ that verifies (3.21).

Proof. By Lemma 3.2.13, we have that $X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}$. Now, let us see that the embedding (3.21) holds. In fact, if $f \in \mathcal{R}(X, L^\infty)$ then, combining

$$L^\infty(I^n) = \mathcal{R}(L^\infty, L^\infty) \hookrightarrow \mathcal{R}(L^\infty, L^1),$$

with the embedding (3.9), we deduce that

$$K(f, t; \mathcal{R}(L^{(n-1)',1}, L^1), \mathcal{R}(L^\infty, L^1)) \lesssim K(f, Ct; \mathcal{R}(L^1, L^\infty), L^\infty).$$

Hence, using Lemma 3.1.6 and Theorem 3.1.7, we get

$$\int_0^t \psi_j^*(f, L^1)(s^{(n-1)'})ds \lesssim \sum_{k=1}^n \int_0^{Ct} \psi_k^*(f, L^\infty)(s)ds, \quad j \in \{1, \dots, n\}.$$

So, using the Hardy-Littlewood inequality (see Theorem 2.1.5) and the subadditivity of $\|\cdot\|_{\overline{X}(0,1)}$, we get

$$\begin{aligned}\|f\|_{\mathcal{R}_j(X_{\mathcal{R}(X,L^\infty)},L^1)} &= \|\psi_j^*(f, L^1)\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})} = \|\psi_j^*(f, L^1)(s^{(n-1)'})\|_{\overline{X}(0,1)} \\ &\lesssim \sum_{k=1}^n \|\psi_k^*(f, L^\infty)(Ct)\|_{\overline{X}(0,1)} \lesssim \|f\|_{\mathcal{R}(X,L^\infty)},\end{aligned}$$

for any $j \in \{1, \dots, n\}$. Therefore, the embedding (3.21) holds.

Now, let us prove that $\mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^1)$ is the smallest r.i. space satisfying (3.21). That is, let us see that if a mixed norm space $\mathcal{R}(Y, L^1)$ satisfies

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(Y, L^1),$$

then

$$\mathcal{R}(X_{\mathcal{R}(X,L^\infty)}, L^1) \hookrightarrow \mathcal{R}(Y, L^1). \tag{3.22}$$

We assume that $I=(-a,b)$, with $a,b\in\mathbb{R}_+$, and $0<r<\min(a,b)$. Given any function $f\in X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$, with $\lambda_f(0)\le\omega_{n-1}r^{(n-1)}$, we define

$$g(x)=\begin{cases} f^*(\omega_{n-1}|\widehat{x_n}|^{(n-1)}), & \text{if } (\widehat{x_n},x_n)\in B_{n-1}(0,r)\times I,\\ 0, & \text{otherwise.}\end{cases}$$

We fix $k\in\{1,\dots,n-1\}$. Then,

$$\psi_k(g,L^\infty)(\widehat{x_k})=f^*(\omega_{n-1}|\widehat{x_{k,n}}|^{(n-1)}),\quad \text{if } (\widehat{x_n},x_n)\in B_{n-2}(0,r)\times I,$$

and $\psi_k(g,L^\infty)(\widehat{x_k})=0$, otherwise. Thus, Lemma 2.1.3 gives

$$\|g\|_{\mathcal{R}_k(X,L^\infty)}\lesssim\|f^*(t^{(n-1)'})\|_{\overline{X}(0,1)}=\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})},\tag{3.23}$$

for any $k\in\{1,\dots,n-1\}$. On the other hand, it holds that

$$\psi_n(g,L^\infty)(\widehat{x_n})\begin{cases} f^*(\omega_{n-1}|\widehat{x_n}|^{n-1}), & \text{if } \widehat{x_n}\in B_{n-1}(0,r),\\ 0, & \text{otherwise,}\end{cases}$$

and so, using again Lemma 2.1.3, we get

$$\|g\|_{\mathcal{R}_n(X,L^\infty)}=\|f\|_{X(I^{n-1})}\le\|f^*(t^{(n-1)'})\|_{\overline{X}(0,1)}=\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.\tag{3.24}$$

Therefore, by (3.23) and (3.24), we have that

$$\|g\|_{\mathcal{R}(X,L^\infty)}\lesssim\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.\tag{3.25}$$

So, using $\mathcal{R}(X,L^\infty)\hookrightarrow\mathcal{R}(Y,L^1)$ and (3.25), we get

$$\|g\|_{\mathcal{R}(Y,L^1)}\lesssim\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.\tag{3.26}$$

But, as before,

$$\|g\|_{\mathcal{R}(Y,L^1)}\approx\|f\|_{Y(I^{n-1})}$$

and hence, using (3.26), we get

$$\|f\|_{Y(I^{n-1})}\lesssim\|f\|_{X_{\mathcal{R}(X,L^\infty)}(I^{n-1})}.$$

From this, we obtain that any $f\in X_{\mathcal{R}(X,L^\infty)}(I^{n-1})$, with $\lambda_f(0)\le\omega_{n-1}r^{n-1}$, belongs to $Y(I^{n-1})$. The general case can be proved as in the proof of Theorem 3.2.3. Thus, we have

$$X_{\mathcal{R}(X,L^\infty)}(I^{n-1})\hookrightarrow Y(I^{n-1}),$$

and so, Lemma 3.2.2 implies that (3.22) holds, as we wanted to see. □

Remark 3.2.15. Theorem 3.2.14 can be extended to the case $p>1$ as follows: if we define $\|f\|_{\widetilde{X}_p(I^{n-1})}=\|f^*(t^{(p(n-1))'})\|_{\overline{X}(0,1)}$, then with a similar proof one can get

$$\mathcal{R}(X,L^\infty)\hookrightarrow\mathcal{R}(\widetilde{X}_p,L^{p,1}),\tag{3.27}$$

although this embedding is not optimal in general. For example, if $X = L^{p,1}(I^{n-1})$ then $\widetilde{X}_p(I^{n-1}) = L^{p(p(n-1))',1}(I^{n-1})$, but in this case

$$\mathcal{R}(L^{p,1}, L^\infty) \hookrightarrow \mathcal{R}(L^{p(n-1)',\infty}, L^{p,1}), \tag{3.28}$$

and clearly $L^{p(n-1)',\infty}(I^{n-1}) \hookrightarrow L^{p(p(n-1))',1}(I^{n-1})$.

However, if $X = L^1(I^{n-1})$, then (3.27) gives

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(p(n-1))',1}, L^{p,1}),$$

which is indeed optimal (as proved in [2]).

Before going on and for the sake of completeness, let us prove (3.28). But first we need the following result.

Lemma 3.2.16. *Let $\alpha \in \mathbb{R}_+$, $0 < r < 1$ and*

$$F(x,y) = \begin{cases} x^{-\alpha}, & (x,y) \in [0,1] \times [0,r], \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$F^*(t) = \begin{cases} r^\alpha t^{-\alpha}, & 0 \le t < r, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We observe that if $s \ge 0$, then

$$\begin{aligned} \lambda_F(s) &= |\{(x,y) \in [0,1] \times [0,r] : F(x,y) > s\}| = r|\{x \in [0,1] : x^{-\alpha} > s\}| \\ &= \begin{cases} r, & 0 \le s < 1, \\ rs^{-1/\alpha}, & s > 1, \end{cases} \end{aligned}$$

Hence, using the definition of f^*, the result follows. □

Remark 3.2.17. From now on, we denote the rearrangement of a measurable function f with respect to the variables x_k by $\mathcal{R}_k f$. That is, we set

$$\mathcal{R}_k f(\widehat{x}_k, t) = f^*_{\widehat{x}_k}(t).$$

Let us just mention that $\mathcal{R}_k f$ is a measurable function equimeasurable with f. See [11] for some related results.

Proposition 3.2.18. *Let $n \ge 3$ and $1 < p < \infty$. Then,*

$$\mathcal{R}(L^{p,1}, L^\infty) \hookrightarrow \mathcal{R}(L^{p(n-1)',\infty}, L^{p,1}).$$

Proof. It suffices to prove

$$\mathcal{R}(L^{p,1}, L^\infty) \hookrightarrow \mathcal{R}_n(L^{p(n-1)',\infty}, L^{p,1}),$$

since the general case can be proved in the same way. We fix any $f \in \mathcal{R}(L^{p,1}, L^\infty)$. Then, [61, Lemma 3.17] and Remark 3.2.17 imply that

$$\int_0^t \psi_n^*(f, L^{p,1})(v)dv = \sup_{|E|\le t} \int_E \psi_n(f, L^{p,1})(\widehat{x_n})d\widehat{x_n}$$
$$= \sup_{|E|\le t} \int_E \left(\int_0^1 s^{-1/p'} \mathcal{R}_n f(\widehat{x_n}, s)ds \right) d\widehat{x_n}.$$

So, Fubini's theorem and Hardy-Littlewood inequality (see Theorem 2.1.5) give

$$\int_0^t \psi_n^*(f, L^{p,1})(v)dv \le \int_0^1 s^{-1/p'} \left(\int_0^t F_s^*(v)dv \right) ds, \tag{3.29}$$

where $F_s(\widehat{x_n}) = \mathcal{R}_n f(\widehat{x_n}, s)$. Now, we fix any $\ell \in \{1, \ldots, n-1\}$ and $v > 0$. Then, by Corollary 3.1.5 with f replaced by F_s, we get

$$F_s^*(v) \le \sum_{j=1}^{n-1} \psi_j^*(F_s, L^\infty)(v^{1/(n-1)'}).$$

Therefore, using (3.29), we obtain

$$\int_0^t \psi_n^*(f, L^{p,1})(v)dv \le \sum_{j=1}^{n-1} \int_0^1 s^{-1/p'} \left(\int_0^t \psi_j^*(F_s, L^\infty)(v^{1/(n-1)'})dv \right) ds. \tag{3.30}$$

But, if we fix any $\ell \in \big\{1, \ldots, n-1\big\}$, then, for almost all $x \in I^n$, it holds that

$$|f(x)| \le \psi_\ell(f, L^\infty)(\widehat{x_{\ell,n}}, x_n),$$

and, therefore, we deduce that

$$\mathcal{R}_n f(\widehat{x_k}, s) \le \mathcal{R}_n(\psi_\ell(f, L^\infty)(\widehat{x_{\ell,n}}, s).$$

As a consequence, we have that

$$\psi_\ell(F_s, L^\infty)(\widehat{x_{\ell,n}}) \le \mathcal{R}_n \psi_\ell(f, L^\infty)(\widehat{x_{\ell,n}}, t),$$

and so, we conclude that

$$\psi_\ell^*(F_s, L^\infty)(v) \le (\mathcal{R}_n \psi_\ell(f, L^\infty)(\cdot, s))^*(v) = G_\ell(v, s). \tag{3.31}$$

Therefore, combining (3.30) and (3.31), we obtain

$$\int_0^t \psi_n^*(f, L^{p,1})(v)dv \le \sum_{j=1}^{n-1} \int_0^1 s^{-1/p'} \left(\int_0^t \psi_j^*(F_s, L^\infty)(v^{1/(n-1)'})dv \right) ds$$
$$\approx \sum_{j=1}^{n-1} \int_0^1 s^{-1/p'} \left(\int_0^{t^{1/(n-1)'}} v^{1/(n-2)} \psi_j^*(F_s, L^\infty)(v)dv \right) ds$$
$$\le \sum_{j=1}^{n-1} \int_0^1 s^{-1/p'} \left(\int_0^{t^{1/(n-1)'}} v^{1/(n-2)} G_j(v, s)dv \right) ds$$
$$\le t^{1/(n-1)} \sum_{j=1}^{n-1} \int_0^1 s^{-1/p'} \left(\int_0^{t^{1/(n-1)'}} G_j(v, s)dv \right) ds.$$

So, Hardy-Littlewood inequality (see Theorem 2.1.5) and Lemma 3.2.16 imply that

$$\int_0^t \psi_n^*(f, L^{p,1})(v)dv \leq t^{-1/(n-1)'} t^{1/((n-1)'p')} \sum_{j=1}^{n-1} \int_0^{t^{1/(n-1)'}} s^{-1/p'} G_j^*(s)ds. \tag{3.32}$$

We fix any $j \in \{1, \dots, n-1\}$. Using now that $\psi_j(f, L^\infty)$ and $\mathcal{R}_n \psi_j(f, L^\infty)$ are equimeasurable functions (see Remark 3.2.17), we have

$$\begin{aligned}
\lambda_{\psi_j(f,L^\infty)}(v) = \lambda_{\mathcal{R}_n\psi_j(f,L^\infty)}(v) &= \int_0^1 |\{\widehat{x_{\ell,n}} \in I^{n-2} : \mathcal{R}_n\psi_j(f, L^\infty)(\widehat{x_{\ell,n}}, t) > v\}|dt \\
&= \int_0^1 |\{s \in [0,1] : (\mathcal{R}_n\psi_j(f, L^\infty)(\cdot, t))^*(s) > v\}|dt \\
&= |\{(s,t) \in [0,1]^2 : G_j(s,t) > v\}| = \lambda_{G_j}(v).
\end{aligned}$$

Thus,

$$\psi_j^*(f, L^\infty) = G_j^*, \quad j \in \{1, \dots, n-1\}. \tag{3.33}$$

Hence, by (3.32) and (3.33), we get

$$\begin{aligned}
\psi_n^{**}(f, L^{p,1})(t) &\lesssim t^{-1/(n-1)'} t^{1/((n-1)'p')} \sum_{j=1}^{n-1} \int_0^{t^{1/(n-1)'}} s^{-1/p'} \psi_j^*(f, L^\infty)(s)ds \\
&= t^{-1/((n-1)'p)} \sum_{j=1}^{n-1} \int_0^{t^{1/(n-1)'}} s^{-1/p'} \psi_j^*(f, L^\infty)(s)ds \\
&\leq t^{-1/((n-1)'p)} \|f\|_{\mathcal{R}(L^{p,1}, L^\infty)}.
\end{aligned}$$

Therefore,

$$\|f\|_{\mathcal{R}_n(L^{p(n-1)',\infty}, L^{p,1})} = \sup_{0<t<1} t^{1/(p(n-1)')} \psi_n^{**}(f, L^{p,1})(t) \lesssim \|f\|_{\mathcal{R}(L^{p,1}, L^\infty)}.$$

Thus, the proof is complete. □

3.3 Fournier embeddings

Our main goal, in this section, is to study the following embedding

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n). \tag{3.34}$$

In particular, we are interested in the following problems:

(i) Given a mixed norm space $\mathcal{R}(X, L^\infty)$, we would like to find the smallest r.i. range space $Z(I^n)$ satisfying (3.34).

(ii) Now, let us suppose that the range space is given $Z(I^n)$. We would like to find the largest mixed norm space of the form $\mathcal{R}(X, L^\infty)$ for which (3.34) holds.

The main motivation to consider these questions come from the embedding due to Fournier [28], which shows that, if $n \geq 2$,

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n). \tag{3.35}$$

Remark 3.3.1. We observe that by means of Corollary 3.1.4, it is possible to prove (3.35) in a slightly different form. In fact, let $f \in \mathcal{R}(L^1, L^\infty)$. Then, writing $\|f\|_{L^{n',1}(I^n)}$ in terms of the distribution function of f (cf. e. g. [30, Proposition 1.4.9]), and using Corollary 3.1.4, we get

$$\|f\|_{L^{n',1}(I^n)} = n' \int_0^\infty \big(\lambda_f(s)\big)^{1/n'} ds \leq n' \int_0^\infty \prod_{k=1}^n \big(\lambda_{\psi_k(f,L^\infty)}(s)\big)^{1/n} ds.$$

So, the geometric-arithmetic mean inequality implies that

$$\begin{aligned} \|f\|_{L^{n',1}(I^n)} &\leq \frac{1}{(n-1)} \sum_{k=1}^n \int_0^\infty \lambda_{\psi_k(f,L^\infty)}(s) ds = \frac{1}{(n-1)} \sum_{k=1}^n \|\psi_k(f, L^\infty)\|_{L^1(I^{n-1})} \\ &\leq \frac{1}{(n-1)} \sum_{k=1}^n \|f\|_{\mathcal{R}_k(L^1,L^\infty)} = \frac{1}{(n-1)} \|f\|_{\mathcal{R}(L^1,L^\infty)}. \end{aligned}$$

That is, $\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n)$, as we wanted to see.

3.3.1 Necessary and sufficient conditions

Now, our main purpose is to find necessary and sufficient conditions on $X(I^{n-1})$ and $Z(I^n)$ under which we have the embedding (3.34).

Theorem 3.3.2. *Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces. Then, the embedding*

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n) \tag{3.36}$$

holds, if and only if,

$$\big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,1)} \lesssim \|f^*\|_{\overline{X}(0,1)}, \quad f \in X(I^{n-1}). \tag{3.37}$$

Proof. Let us first suppose that (3.36) holds. As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $0 < r < \min(a, b)$. Given any function $f \in X(I^{n-1})$, with $\lambda_f(0) \leq \omega_n^{1/n'} r^{n-1}$, we define

$$g(x) = \begin{cases} f^*(\omega_n^{1/n'} |x|^{n-1}), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

Now, we fix any $k \in \{1, \ldots, n\}$. Then,

$$\psi_k(g, L^\infty)(\widehat{x}_k) = f^*(\omega_n^{1/n'} |\widehat{x}_k|^{n-1}), \quad \text{for any } \widehat{x}_k \in B_{n-1}(0, r),$$

and $\psi_k(g, L^\infty)(\widehat{x}_k) = 0$ otherwise. Thus, using Lemma 2.1.3 and the boundedness of the dilation operator in r.i. spaces, we get

$$\|g\|_{\mathcal{R}_k(X,L^\infty)} = \big\|\psi_k(g, L^\infty)\big\|_{X(I^{n-1})} \lesssim \|f^*\|_{\overline{X}(0,1)}, \quad k \in \{1, \ldots, n\}.$$

So, our assumption on f shows that $g \in \mathcal{R}(X, L^\infty)$ and

$$\|g\|_{\mathcal{R}(X,L^\infty)} \lesssim \|f^*\|_{\overline{X}(0,1)}. \tag{3.38}$$

Thus, using $\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$ and (3.38), we obtain

$$\|g\|_{Z(I^n)} \lesssim \|f^*\|_{\overline{X}(0,1)}. \tag{3.39}$$

But, again by Lemma 2.1.3, it holds that

$$\|g\|_{Z(I^n)} = \left\|f^*(t^{1/n'})\right\|_{\overline{Z}(0,1)},$$

and hence (3.39) gives

$$\left\|f^*(t^{1/n'})\right\|_{\overline{Z}(0,1)} \lesssim \|f\|_{\overline{X}(0,1)}.$$

This proves (3.37), for any function $f \in X(I^{n-1})$, with $\lambda_f(0) \leq \omega_n^{1/n'} r^{n-1}$. The general case can be proved as in the proof of Theorem 3.2.3.

Now, let us suppose that (3.37) holds. We fix any $f \in \mathcal{R}(X, L^\infty)$ and $s \in (0,1)$. Then, by Corollary 3.1.5, we obtain

$$f^*(s) \leq \sum_{j=1}^{n} \psi_j^*(f, L^\infty)(s^{1/n'}).$$

Thus, we have

$$\|f\|_{Z(I^n)} \leq \left\|\sum_{k=1}^{n} \psi_k^*(f, L^\infty)(s^{1/n'})\right\|_{\overline{Z}(0,1)} \leq \sum_{k=1}^{n} \left\|\psi_k^*(f, L^\infty)(s^{1/n'})\right\|_{\overline{Z}(0,1)}.$$

Hence, using (3.37), we get

$$\|f\|_{Z(I^n)} \leq \sum_{k=1}^{n} \left\|\psi_k^*(f, L^\infty)(s^{1/n'})\right\|_{\overline{Z}(0,1)} \lesssim \sum_{k=1}^{n} \left\|\psi_k(f, L^\infty)\right\|_{X(I^{n-1})} = \|f\|_{\mathcal{R}(X,L^\infty)}.$$

That is, the embedding $\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n)$ holds and the proof is complete. □

Now, we shall see when the inclusion (3.36) is strict.

Theorem 3.3.3. *Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces satisfying (3.36). Then,*

$$\mathcal{R}(X, L^\infty) = Z(I^n) \iff \begin{cases} Z(I^n) = L^\infty(I^n); \\ X(I^{n-1}) = L^\infty(I^{n-1}). \end{cases}$$

Proof. We only need to prove the necessary part of this result. For this, we shall see that

$$Z(I^n) \neq L^\infty(I^n) \Longrightarrow \mathcal{R}(X, L^\infty) \neq Z(I^n).$$

As before, we suppose that $I = (-a, b)$, $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any function $g \in Z(I^n)$, but $g \notin L^\infty(I^n)$, we define

$$f(x) = \begin{cases} g^*(2|x_n|), & \text{if } (\widehat{x_n}, x_n) \in I^{n-1} \times (-r, r), \\ 0, & \text{otherwise.} \end{cases}$$

Let us see that $f \in Z(I^n)$ and $f \notin \mathcal{R}(X, L^\infty)$. In fact, by Lemma 2.1.3, we have

$$f^*(t) = \begin{cases} g^*(t), & \text{if } 0 \leq t < \min\big(2r, \lambda_g(0)\big), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, our assumption on g ensures that

$$\|f\|_{Z(I^n)} \leq \|g\|_{Z(I^n)} < \infty.$$

That is, $f \in X(I_n)$. On the other hand, for any $\widehat{x_n} \in I^{n-1}$, it holds that

$$\psi_n(f, L^\infty)(\widehat{x_n}) = \|g\|_{L^\infty(I)}.$$

But, by hypothesis, $g \notin L^\infty(I)$, and hence $f \notin \mathcal{R}_n(X, L^\infty)$. Therefore, f does not belong to $\mathcal{R}(X, L^\infty)$ and so the proof is complete. □

3.3.2 The optimal domain problem

Let $Z(I^n)$ be an r.i. space. Now, we want to find the largest space of the form $\mathcal{R}(X, L^\infty)$ satisfying

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

In order to do this, let us introduce a new space, denoted by $X_{Z,L^\infty}(I^{n-1})$, consisting of those functions $f \in \mathcal{M}(I^{n-1})$ for which the quantity

$$\|f\|_{X_{Z,L^\infty}(I^{n-1})} = \big\|f^{**}(s^{1/n'})\big\|_{\overline{Z}(0,1)} \tag{3.40}$$

is finite. It is not difficult to verify that $X_{Z,L^\infty}(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{X_{Z,L^\infty}(I^{n-1})}$.

The next lemma gives an equivalent expression for the norm $\|\cdot\|_{X_{Z,L^\infty}(I^{n-1})}$.

Lemma 3.3.4. *Let $Z(I^n)$ be an r.i space, with $\overline{\alpha}_Z < 1/n'$. Then,*

$$\|f\|_{X_{Z,L^\infty}(I^{n-1})} \approx \|f^*(t^{1/n'})\|_{\overline{Z}(0,1)}, \quad f \in \mathcal{M}(I^{n-1}).$$

Proof. We follow the ideas given in [27, Theorem 4.4]. We fix $f \in \mathcal{M}(I^{n-1})$. Since $f^* \leq f^{**}$, it will be enough to prove that

$$\|f\|_{X_{Z,L^\infty}(I^{n-1})} \lesssim \big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,|I|^n)}.$$

Using Hölder's inequality, together with an elementary change of variable, we get

$$\begin{aligned}\|f\|_{X_{Z,L^\infty}(I^{n-1})} = \big\|f^{**}(t^{1/n'})\big\|_{\overline{Z}(0,1)} &= \sup_{\|g\|_{Z'(I^n)}\le 1} \int_0^1\int_0^1 g^*(t) f^*(v\, t^{1/n'})dt\, dv \\ &\le \big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,1)} \int_0^1 h_Z(v^{-n'})dv \\ &\approx \big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,1)} \int_0^1 v^{-1/n'-1} h_Z(v)dv.\end{aligned}$$

But $\overline{\alpha}_Z < 1/n'$, hence [8, Lemma III, 5.9] implies that

$$\|f\|_{X_{Z,L^\infty}(I^{n-1})} \lesssim \big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,|I|^n)}.$$

Thus, the proof is complete. □

Theorem 3.3.5. *Let $Z(I^n)$ be an r.i. space, with $\alpha_Z < 1/n'$, and let $X_{Z,L^\infty}(I^{n-1})$ be the r.i. space defined in (3.40) Then, the embedding*

$$\mathcal{R}(X_{Z,L^\infty}, L^\infty) \hookrightarrow Z(I^n) \tag{3.41}$$

holds. Moreover, $\mathcal{R}(X_{Z,L^\infty}, L^\infty)$ is the largest space of the form $\mathcal{R}(X, L^\infty)$ for which the embedding (3.41) holds.

Proof. The embedding (3.41) follows from Theorem 3.3.2. Thus, to complete the proof, it only remains to see that $\mathcal{R}(X_{Z,L^\infty}, L^\infty)$ is the largest domain space of the form $\mathcal{R}(X, L^\infty)$ corresponding to $Z(I^n)$. In fact, we shall see that that if $\mathcal{R}(Y, L^\infty)$ is another mixed norm space such that (3.41) holds with $\mathcal{R}(X_{Z,L^\infty}, L^\infty)$ replaced by $\mathcal{R}(Y, L^\infty)$, then

$$\mathcal{R}(Y, L^\infty) \hookrightarrow \mathcal{R}(X_{Z,L^\infty}, L^\infty).$$

We fix any $f \in \mathcal{R}(Y, L^\infty)$. Then, Theorem 3.3.2 ensures us that

$$\big\|f^*(t^{1/n'})\big\|_{\overline{Z}(0,1)} \lesssim \|f\|_{X(I^{n-1})},$$

and so, using Lemma 3.3.4, we get

$$\|f\|_{X_{Z,L^\infty}(I^{n-1})} \lesssim \|f\|_{X(I^{n-1})}, \qquad f \in X(I^{n-1}).$$

That is, $X(I^{n-1}) \hookrightarrow X_{Z,L^\infty}(I^{n-1})$. Hence, using Theorem 3.2.3, we deduce that

$$\mathcal{R}(X, L^\infty) \hookrightarrow \mathcal{R}(X_{Z,L^\infty}, L^\infty).$$

as we wanted to see. □

Let us see an application of Theorem 3.3.5 to the case of Lorentz spaces.

Corollary 3.3.6. *Let $n' < p_1 < \infty$, and $1 \le q_1 \le \infty$. Then, the mixed norm space $\mathcal{R}(L^{p_1/n', q_1}, L^\infty)$ is the largest space of the form $\mathcal{R}(X, L^\infty)$ satisfying*

$$\mathcal{R}(L^{p_1/n', q_1}, L^\infty) \hookrightarrow L^{p_1,q_1}(I^n).$$

Proof. It follows from Theorem 3.3.5, with $Z(I^n)$ replaced by $L^{p_1,q_1}(I^n)$. □

3.3.3 The optimal range problem

Let $X(I^{n-1})$ be an r.i. space. We would like to describe the smallest r.i. space $Z(I^n)$ satisfying

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n).$$

We begin with a preliminary lemma.

Lemma 3.3.7. *Let $X(I^{n-1})$ be an r.i. space. Then, the functional defined by*

$$\|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} = \left\|f^*(t^{n'})\right\|_{\overline{X}(0,1)}, \quad f \in \mathcal{M}_+(I^n), \tag{3.42}$$

is an r.i. norm.

Proof. It follows using the same argument as in the proof of Lemma 3.2.13. □

Theorem 3.3.8. *Let $X(I^{n-1})$ be an r.i. space and let $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ be as in (3.42). Then, the embedding*

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X,L^\infty)}(I^n)$$

holds. Moreover, $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ is the smallest r.i. space that verifies this embedding.

Proof. Lemma 3.3.7 gives us that $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ is an r.i. space equipped with the norm $\|\cdot\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}$. Now, let us see that the embedding

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X,L^\infty)}(I^n)$$

holds. In fact, let f be any function from $\mathcal{R}(X, L^\infty)$. Then, using Fournier's embedding

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n),$$

we get

$$K(f, t; L^{n',1}, L^\infty) \leq K(f, t; \mathcal{R}(L^1, L^\infty), L^\infty), \quad 0 < t < 1.$$

Thus, Theorem 2.1.18 and Theorem 3.1.7, with $X(I^{n-1})$ replaced by $L^1(I^{n-1})$, imply that

$$\int_0^t f^*(s^{n'})ds \lesssim \sum_{k=1}^{n} \int_0^t \psi_k^*(f, L^\infty)(s)ds, \quad 0 < t < 1.$$

Therefore, using Hardy-Littlewood-Pólya Principle (see Theorem 2.1.11) and the subadditive property of $\|\cdot\|_{\overline{X}(0,1)}$, we get

$$\left\|f^*(s^{n'})\right\|_{\overline{X}(0,1)} \lesssim \sum_{k=1}^{n} \left\|\psi_k^*(f, L^\infty)\right\|_{\overline{X}(0,1)} = \|f\|_{\mathcal{R}(X,L^\infty)}.$$

That is, $\mathcal{R}(X, L^\infty) \hookrightarrow Z_{\mathcal{R}(X,L^\infty)}(I^n)$.

Now, let us see that $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ is the smallest r.i. space for which this embedding holds, i.e., let us see that if an r.i. space $Z(I^n)$ satisfies

$$\mathcal{R}(X, L^\infty) \hookrightarrow Z(I^n), \tag{3.43}$$

then $Z_{\mathcal{R}(X,L^\infty)}(I^n) \hookrightarrow Z(I^n)$. As before, assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $0 < r < \min(a, b)$. Given any function $f \in Z_{\mathcal{R}(X,L^\infty)}(I^n)$, with $\lambda_f(0) \leq \omega_n r^n$, we define

$$g(x) = \begin{cases} f^*(\omega_n |x|^n), & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

Then, applying the same technique as in the proof of Theorem 3.2.3 and using the boundedness of the dilation operator in r.i. spaces, we get

$$\|g\|_{\mathcal{R}(X,L^\infty)} \lesssim \left\| f^*(t^{n'}) \right\|_{\overline{X}(0,1)} = \|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}. \tag{3.44}$$

By hypothesis $f \in Z_{\mathcal{R}(X,L^\infty)}(I^n)$, and hence $g \in \mathcal{R}(X, L^\infty)$. So, using (3.43) and (3.44), we get

$$\|g\|_{Z(I^n)} \lesssim \|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}.$$

But, by Lemma 2.1.3, g and f are equimeasurable functions, and hence we obtain

$$\|f\|_{Z(I^n)} \lesssim \|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}.$$

From this, we get that any $f \in Z_{\mathcal{R}(X,L^\infty)}(I^n)$, with $\lambda_f(0) \leq \omega_n r^n$, belongs to $Z(I^n)$. The general case can be proved as in Theorem 3.2.3. Thus, the proof is complete. □

We shall give now a corollary of Theorem 3.3.8. In particular, we shall see that the Fournier's embedding (3.35) cannot be improved within the class of r.i. spaces. This should be understood as follows: if we replace the range space in

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n),$$

by a smaller r.i. space, say $Y(I^n)$, then the resulting embedding

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow Y(I^n).$$

can no longer be true.

Corollary 3.3.9. *Let $1 < p_1 < \infty$ and $1 \leq q_1 \leq \infty$ or $p_1 = q_1 = 1$. Then, the Lorentz space $L^{n'p_1,q_1}(I^n)$ is the smallest r.i. space satisfying*

$$\mathcal{R}(L^{p_1,q_1}, L^\infty) \hookrightarrow L^{n'p_1,q_1}(I^n).$$

Proof. It follows from Theorems 3.3.8, using $L^{p_1,q_1}(I^{n-1})$ instead of $X(I^{n-1})$. □

3.4 Embeddings between mixed norm spaces and r.i. spaces

In this section, our analysis will focus on embeddings of the form

$$Z(I^n) \hookrightarrow \mathcal{R}(X, L^1).$$

In particular, for a fixed r.i. space $Z(I^n)$ we shall find the smallest mixed norm of the form $\mathcal{R}(X, L^1)$ into which $Z(I^n)$ is continuously embedded.

Theorem 3.4.1. *Let $Z(I^n)$ be an r.i. space. Then,*

$$Z(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1) \iff Z(I^n) = L^\infty(I^n).$$

Proof. In view of Lemma 3.2.2, it suffices to prove the necessary part of this result. In fact, we shall see that

$$Z(I^n) \neq L^\infty(I^n) \Longrightarrow Z(I^n) \not\hookrightarrow \mathcal{R}(L^\infty, L^1).$$

To this end, we suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. We fix $f \in Z(I^n)$, such that $f \notin L^\infty(I^n)$. Next, we define

$$g(x) = \begin{cases} f^*(\omega_{n-1}|x_n|^{n-1}), & (\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times I, \\ 0, & \text{otherwise.} \end{cases}$$

Using now Lemma 2.1.3, we deduce that

$$\psi_n^*(g, L^1)(t) = \begin{cases} f^*(t), & 0 \leq t < \min(\lambda_f(0), \omega_{n-1} r^{n-1}), \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, our assumption on f implies that $g \notin \mathcal{R}_n(L^\infty, L^1)$, and so, by definition, $g \notin \mathcal{R}(L^\infty, L^1)$. On the other hand, we have

$$g^*(t) = \begin{cases} f^*(t), & 0 \leq t < \min(\lambda_f(0), \omega_{n-1} r^{n-1}), \\ 0, & \text{otherwise,} \end{cases}$$

and hence, we get

$$\|g\|_{Z(I^n)} \leq \|f\|_{Z(I^n)}.$$

By hypothesis, $f \in Z(I^n)$, and so, g also belongs to $Z(I^n)$. Hence, the proof is complete. □

Now, for a fixed r.i. space $Z(I^n)$ we would like to describe the smallest mixed norm space of the form $\mathcal{R}(X, L^1)$ for which the embedding

$$Z(I^n) \hookrightarrow \mathcal{R}(X, L^1)$$

holds. To this end, let us introduce a new space, denoted by $X_{Z,L^1}(I^{n-1})$, consisting of those measurable functions f on $\mathcal{M}(I^{n-1})$ such that

$$\|f\|_{X_{Z,L^1}(I^{n-1})} = \|f^*\|_{\overline{Z}(0,1)} < \infty. \tag{3.45}$$

Observe that $X_{Z,L^1}(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{X_{Z,L^1}(I^{n-1})}$.

Theorem 3.4.2. *Let $Z(I^n)$ be an r.i. space and let $X_{Z,L^1}(I^{n-1})$ be the r.i. defined in (3.45). Then, the embedding*

$$Z(I^n) \hookrightarrow \mathcal{R}(X_{Z,L^1}, L^1) \tag{3.46}$$

holds. Moreover, $\mathcal{R}(X_{Z,L^1}, L^1)$ is the smallest space of the form $\mathcal{R}(X, L^1)$ that verifies (3.46).

Proof. We fix any $f \in Z(I^n)$. Then, by Theorem 2.1.17 together with the endpoint embeddings

$$L^\infty(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1) \ \text{ and } \ L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^1) = L^1(I^n),$$

we have that

$$K(f, t; L^1, \mathcal{R}(L^\infty, L^1)) \lesssim K(f, t; L^1, L^\infty), \quad 0 < t < 1.$$

Therefore, using Theorem 3.1.11 and Theorem 2.1.18, we get

$$\int_0^t \psi_k^*(f, L^1)(s)ds \lesssim \int_0^t f^*(s)ds, \quad 0 < t < 1, \quad k \in \{1, \dots, n\},$$

and hence Theorem 2.1.11 implies that

$$\left\|\psi_k(f, L^1)\right\|_{X_{Z,L^1}(I^{n-1})} = \left\|\psi_k^*(f, L^1)\right\|_{\overline{Z}(0,1)} \lesssim \|f^*\|_{\overline{Z}(0,1)},$$

from which (3.46) follows.

Now, let us see that $\mathcal{R}(X_{Z,L^1}, L^1)$ is the smallest mixed norm space in (3.46), i.e., let us see that if a mixed norm space $\mathcal{R}(Y, L^1)$ verifies

$$Z(I^n) \hookrightarrow \mathcal{R}(Y, L^1), \tag{3.47}$$

then

$$\mathcal{R}(X_{Z,L^1}, L^1) \hookrightarrow \mathcal{R}(Y, L^\infty). \tag{3.48}$$

As before, assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $0 < r < \min(a, b)$. We fix $f \in X_{Z,L^1}(I^{n-1})$, with $\lambda_f(0) \leq \omega_{n-1} r^{n-1}$, and we define

$$h(x) = \begin{cases} f^*(\omega_{n-1}|\widehat{x_n}|^{n-1}), & \text{if } (\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times I, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2.1.3 and (3.45), we get

$$\|h\|_{Z(I^n)} = \|f^*\|_{\overline{Z}(0,1)} = \|f\|_{X_{Z,L^1}(I^{n-1})}, \tag{3.49}$$

and so our assumption on f ensures that h belongs to $X_{Z,L^1}(I^{n-1})$. As a consequence, combining (3.47) with (3.49), we have that

$$\|h\|_{\mathcal{R}_n(X,L^1)} \lesssim \|f\|_{X_{Z,L^1}(I^{n-1})}. \tag{3.50}$$

On the other hand, it holds that

$$\psi_n(f, L^1)(\widehat{x_n}) = \begin{cases} f^*(\omega_{n-1}|\widehat{x_n}|^{n-1}), & \text{if } \widehat{x_n} \in B_{n-1}(0,r), \\ 0, & \text{otherwise,} \end{cases}$$

and so, using again Lemma 2.1.3 with (3.50), we get

$$\|f^*\|_{\overline{X}(0,1)} \lesssim \|f^*\|_{\overline{Z}(0,1)}.$$

From this, we obtain that any $f \in X_{Z,L^1}(I^{n-1})$, with $\lambda_f(0) \le \omega_{n-1} r^{n-1}$, belongs to $Y(I^n)$. The general case can be proved as in the proof of Theorem 3.2.3. Thus, using Lemma 3.2.2, we get (3.48). □

Finally, let us study the optimal range for concrete examples of r.i. spaces.

Corollary 3.4.3. *Let $1 < p < \infty$ and $1 \le q \le \infty$ or $p = q = \infty$. Then,*

$$L^{p,q}(I^n) \hookrightarrow \mathcal{R}(L^{p,q}, L^1).$$

Moreover, $\mathcal{R}(L^{p,q}, L^1)$ is the smallest mixed norm space for which this embedding holds.

Proof. It is a consequence of Theorem 3.4.2, with $Z(I^n)$ replaced by $L^{p,q}(I^n)$. □

The following theorem establishes a similar result, but for Lorentz-Zygmund spaces. Before that, we recall that the Lorentz Zygmund spaces $L^{p,q;\alpha}$ are r.i. spaces (up to equivalent norms) if and only if one of the following conditions is satisfied (see [7, 52, 53] for more details):

$$\begin{cases} p = q = 1; \\ 1 < p < \infty, 1 \le q \le \infty, \alpha \in \mathbb{R}; \\ p = \infty, 1 \le q < \infty, \alpha + 1/q < \infty; \\ p = q = \infty, \alpha \le 0. \end{cases} \tag{3.51}$$

Corollary 3.4.4. *Let $1 \le p, q \le \infty$ and $\alpha \in \mathbb{R}$ as in (3.51). Then, the mixed norm space $\mathcal{R}(L^{p,q;\alpha}, L^1)$ is the smallest range of the form $\mathcal{R}(X, L^1)$ that verifies*

$$L^{p,q;\alpha}(I^n) \hookrightarrow \mathcal{R}(L^{p,q;\alpha}, L^1).$$

Proof. It follows from Theorem 3.4.2. □

Chapter 4

The n-dimensional Hardy operator

Important intrinsic properties of the Hardy operator:

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds, \quad f \in \mathcal{M}_+(0,1), \ t > 0,$$

such as boundedness and compactness, on various function spaces, have been intensively studied over almost a century. In particular, various authors, including Hardy, Littlewood and Pólya [34], Muckenhoupt [49], Bradley [13], Maz'ya [45] and Sinnamon [57] determined for which non-negative measurable functions w and v on $(0,\infty)$ and for which parameters $p; q \in (0,\infty]$, it holds that

$$\left(\int_0^\infty \left(\int_0^t f(s)ds\right)^q w(t)\right)^{1/q} \lesssim \left(\int_0^\infty (f(t))^p v(t)dt\right)^{1/p}.$$

Since the above result was first proved, many other proofs and different extensions to the higher dimensional case have appeared in the literature. In particular, a complete characterization of weights for which the n-dimensional Hardy operator

$$P_n f(x) = \int_0^1 f(sx)ds, \quad f \in \mathcal{M}_+(\mathbb{R}^n), \quad x \in \mathbb{R}^n$$

is bounded on weighted Lebesgue spaces was given by Sinnamon [56].

The principal goal of this chapter is to characterize Hardy inequalities in the context of mixed norm spaces.

The sections are organized as follows: we first study some basic properties of Hardy-type operators that will be useful to get several estimates in the next chapters. Here, among other things, we describe the smallest r.i. space into which the operator

$$H_{\alpha,\beta}f(t) = \int_{t^\beta}^1 s^{\alpha-1}f(s)ds, \quad t \in (0,1) \text{ and } \alpha, \beta > 0$$

is bounded from a given r.i. space $X(0,1)$.

In the second section, our analysis focuses on the n-dimensional Hardy operator. To be more precise, we describe for what pairs of parameters p and q the operator (4.5) is bounded from $\mathcal{R}(L^p, L^\infty)$ to $L^q(\mathbb{R}^n)$ (see Theorem 4.2.2). Furthermore, we study the boundedness of

$$P_n : \mathcal{R}(L^{p_1,q_1}, L^\infty) \to L^{p_2,q_2}(\mathbb{R}^n),$$

for more general exponents $p_1, p_2, q_1, q_2 > 0$ (see Theorem 4.2.3).

4.1 1-dimensional case

The starting point of our theory are the so-called Hardy's inequalities [8].

Theorem 4.1.1. *Let $1 \leq p < \infty$, $\alpha > 0$, and let f be any non-negative measurable on $(0, \infty)$ function Then,*

$$\left(\int_0^\infty t^{-\alpha-1}\left[\int_0^t f(s)ds\right]^p dt\right)^{1/p} \leq \frac{p}{\alpha}\left(\int_0^\infty t^{p-\alpha-1} f(t)^p dt\right)^{1/p},$$

and

$$\left(\int_0^\infty t^{\alpha-1}\left[\int_t^\infty f(s)ds\right]^p dt\right)^{1/p} \leq \frac{p}{\alpha}\left(\int_0^\infty t^{p+\alpha-1} f(t)^p dt\right)^{1/p}.$$

This result has been generalized in many directions. In particular, various authors, including Hardy, Littlewood and Pólya [34], Muckenhoupt [49], Bradley [13], Maz'ya [45] and Sinnamon [57], focused on the characterizations of weights, for which such operator is bounded on weighted Lebesgue spaces.

Theorem 4.1.2. *Let $1 \leq p \leq q \leq \infty$. Then,*

$$\left(\int_0^\infty \left[w(x)\int_0^x f(t)dt\right]^q dx\right)^{1/q} \lesssim \left(\int_0^\infty [f(x)v(x)]^p dx\right)^{1/p},$$

if and only if

$$\sup_{r>0}\left(\int_r^\infty [w(x)]^q dx\right)^{1/q}\left(\int_0^r [v(x)]^{-p'} dx\right)^{1/p'} < \infty.$$

Now, we fix $\alpha, \beta > 0$. Our aim is to study some properties of the Hardy type operators:

$$H_{\alpha,\beta} f(t) = \int_{t^\beta}^1 s^{\alpha-1} f(s)ds, \quad t \in (0,1) \tag{4.1}$$

and

$$H'_{\alpha,\beta} f(t) = t^{\alpha-1}\int_0^{t^{1/\beta}} f(s)ds, \quad t \in (0,1), \tag{4.2}$$

which will be used in the forthcoming discussions. Let us start with an auxiliary lemma. The proof, based on a classical interpolation result due to Calderón (see [8, Theorem III.2.12]), follows the scheme of [19, Lemma 4.1].

Lemma 4.1.3. *Let $X(0,1)$ be an r.i. space.*

(i) If $\alpha, \beta > 0$, with $\alpha + 1/\beta \geq 1$, then

$$H_{\alpha,\beta} : X(0,1) \to X(0,1),$$

where $H_{\alpha,\beta}$ is the operator as in (4.1).

(ii) If $\gamma, \beta, \alpha > 0$, with $\alpha \leq \gamma/\beta$, then

$$\left\| t^{\gamma} \int_{t^{\beta}}^{1} s^{-\alpha-1} f(s) ds \right\|_{X(0,1)} \lesssim \|f\|_{X(0,1)}, \quad f \in X(0,1).$$

Proof. Our assumption on α and β implies that the operator $H_{\alpha,\beta}$ is bounded in $L^1(0,1)$. In fact, if $f \in L^1(0,1)$, then, using Fubini's theorem, we get

$$\|H_{\alpha,\beta} f\|_{L^1(0,1)} \leq \int_0^1 \left(\int_{t^{\beta}}^{1} s^{\alpha-1} |f(s)| ds \right) dt = \int_0^1 s^{\alpha+1/\beta-1} |f(s)| ds.$$

As a consequence, using that $\alpha + 1/\beta - 1 \geq 0$, we obtain that

$$\|H_{\alpha,\beta} f\|_{L^1(0,1)} \leq \|f\|_{L^1(0,1)}.$$

On the other hand, it is easy to see that $H_{\alpha,\beta}$ is also bounded in $L^{\infty}(0,1)$. Hence, using an interpolation theorem of Calderón (see [8, Theorem III.2.12]), we get

$$\|H_{\alpha,\beta} f\|_{X(0,1)} \lesssim \|f\|_{X(0,1)}, \quad f \in X(0,1),$$

i.e., $H_{\alpha,\beta} : X(0,1) \to X(0,1)$ and so statement (i) is proved. Finally, following the same arguments as before, we get (ii). Thus, the proof is complete. □

Now, we fix an r.i. space $X(0,1)$. We shall find the largest r.i. space $Y_X(0,1)$ from which the operator (4.2) is bounded into $X'(0,1)$. That is,

$$H'_{\alpha,\beta} : Y_X(0,1) \to X'(0,1).$$

For this, let us introduce a new space

$$Y_X(0,1) = \left\{ f \in \mathcal{M}(0,1) : \|f\|_{Y_X(0,1)} = \left\| t^{\alpha+1/\beta-1} f^{**}(t^{1/\beta}) \right\|_{X'(0,1)} < \infty \right\}. \tag{4.3}$$

The following result will be important for our purposes.

Lemma 4.1.4. *Let $\alpha, \beta \in (0, \infty)$. Let $X(0,1)$ and $Y(0,1)$ be r.i. spaces. Then, the following statements are equivalent:*

(i) $\left\| \int_{t^{\beta}}^{1} s^{\alpha-1} g(s) ds \right\|_{Y(0,1)} \lesssim \|g\|_{X(0,1)}$;

(ii) $\left\| s^{\alpha-1} \int_0^{s^{1/\beta}} g(t) dt \right\|_{X'(0,1)} \lesssim \|g\|_{Y'(0,1)}$;

(iii) $\left\| s^{\alpha-1} \int_0^{s^{1/\beta}} g^*(t) dt \right\|_{X'(0,1)} \lesssim \|g\|_{Y'(0,1)}$.

Proof. Using a duality argument, we get $(i) \Leftrightarrow (ii)$. In fact, let us suppose that (i) holds. Then, by Fubini's theorem, we get

$$\begin{aligned}\left\| s^{\alpha-1} \int_0^{s^{1/\beta}} g(t)dt \right\|_{X'(0,1)} &\leq \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^1 s^{\alpha-1}|f(s)| \left(\int_0^{s^{1/\beta}} |g(t)|dt \right) ds \\ &= \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^1 |g(t)| \left(\int_{t^\beta}^1 s^{\alpha-1}|f(s)|ds \right) dt.\end{aligned}$$

As consequence, by Hölder's inequality and (ii), we get

$$\begin{aligned}\left\| s^{\alpha-1} \int_0^{s^{1/\beta}} g(t)dt \right\|_{X'(0,1)} &\leq \sup_{\|f\|_{X(0,1)}\leq 1} \|g\|_{Y'(0,1)} \left\| \int_{t^\beta}^1 s^{\alpha-1}|f(s)|ds \right\|_{Y(0,1)} \\ &\lesssim \sup_{\|f\|_{X(0,1)}\leq 1} \|g\|_{Y'(0,1)} \|f\|_{X(0,1)} \leq \|g\|_{Y'(0,1)},\end{aligned}$$

as we wanted to prove. Now, if (ii) holds, then, using again Fubini's theorem, we obtain

$$\begin{aligned}\left\| \int_{t^\beta}^1 s^{\alpha-1}g(s)ds \right\|_{Y(0,1)} &\leq \sup_{\|f\|_{Y'(0,1)}\leq 1} \int_0^1 |f(t)| \left(\int_{t^\beta}^1 s^{\alpha-1}|g(s)|ds \right) dt \\ &= \sup_{\|f\|_{Y'(0,1)}\leq 1} \int_0^1 |f(s)|s^{\alpha-1} \left(\int_0^{s^{1/\beta}} |g(t)|dt \right) ds.\end{aligned}$$

Therefore, Hölder's inequality and (i) imply that

$$\begin{aligned}\left\| \int_{t^\beta}^1 s^{\alpha-1}g(s)ds \right\|_{Y(0,1)} &\leq \sup_{\|f\|_{Y'(0,1)}\leq 1} \|g\|_{X(0,1)} \left\| s^{\alpha-1} \int_0^{s^{1/\beta}} |f(t)|dt \right\|_{X'(0,1)} \\ &\lesssim \sup_{\|f\|_{Y'(0,1)}\leq 1} \|g\|_{X(0,1)} \|f\|_{Y'(0,1)} \leq \|g\|_{X(0,1)},\end{aligned}$$

from which (i) follows.

On the other hand, to prove $(ii) \Rightarrow (iii)$ we just have to apply the hypothesis to decreasing functions. Finally, the implication $(iii) \Rightarrow (ii)$ follows from Hardy-Littlewood inequality (see Theorem 2.1.5), with $g(t) = \chi_{(0,s^{1/\beta})}(t)$. □

Corollary 4.1.5. *Let $X(0,1)$ be an r.i. space. Then, the space $Y_X(0,1)$ defined in (4.3) is the largest r.i. space satisfying*

$$H'_{\alpha,\beta} : Y_X(0,1) \to X'(0,1), \tag{4.4}$$

where $H'_{\alpha,\beta}$ is the Hardy type operator given in (4.2).

Proof. Let us see that the space $Y_X(0,1)$ is a r.i. space. To this end, let us check properties (A1)-(A6) for

$$\|f\|_{Y_X(0,1)} = \left\| t^{\alpha+1/\beta-1} f^{**}(t^{1/\beta}) \right\|_{X'(0,1)}.$$

(A1) Clearly, if $f = 0$ then $\|f\|_{Y_X(0,1)} = 0$ and, for every $\gamma \in \mathbb{R}_+$, we have $\|\gamma f\|_{Y_X(0,1)} = \gamma \|f\|_{Y_X(0,1)}$. Now, if $\|f\|_{Y_X(0,1)} = 0$, then we have that

$$0 = \|f\|_{Y_X(0,1)} \gtrsim f^{**}(1) \int_0^1 t^{\alpha+1/\beta-1} dt \approx \int_0^1 f^*(t) dt = \int_0^1 |f(x)| dx.$$

This implies that $f = 0$. On the other hand, using subadditivity property of f^{**} (see Proposition 2.1.4) and the triangle inequality for $\|\cdot\|_{X'(0,1)}$, we get

$$\|f + g\|_{Y_X(0,1)} \leq \|f\|_{Y_X(0,1)} + \|g\|_{Y_X(0,1)}.$$

(A2) If $0 \leq f \leq g$ a.e., then $f^{**} \leq g^{**}$ (see Proposition 2.1.4). As a consequence, using the fact that $\|\cdot\|_{X(0,1)}$ is an r.i. norm, we obtain

$$\|f\|_{Y_X(0,1)} = \left\| t^{\alpha+1/\beta-1} f^{**}(t^{1/\beta}) \right\|_{X'(0,1)} \leq \left\| t^{\alpha+1/\beta-1} g^{**}(t^{1/\beta}) \right\|_{X'(0,1)} = \|g\|_{Y_X(0,1)}.$$

(A3) If $0 \leq f_j \uparrow f$ a.e., then $f_j^{**} \uparrow f^{**}$ (see Proposition 2.1.4). Hence, by the monotone converge theorem, we have that

$$t^{\alpha+1/\beta-1} f_j^{**}(t^{1/\beta}) \uparrow t^{\alpha+1/\beta-1} f^{**}(t^{1/\beta}),$$

from which the Fatou property for $\|\cdot\|_{Y_X(0,1)}$ follows.

(A4) It requires only the corresponding axioms for $\|\cdot\|_{X(0,1)}$.

(A5) We have to argue as in the proof of property (A1).

(A6) It is easy to see that $\|f\|_{Y_X(0,1)} = \|f^*\|_{Y_X(0,1)}$.

So far, we have proved that $Y_X(0,1)$ is an r.i. space. Moreover, by Hardy-Littlewood inequality (see Theorem 2.1.5), with $g(t) = \chi_{(0,s^{1/\beta})}(t)$, we have

$$\left\| s^{\alpha-1} \int_0^{s^{1/\beta}} f(t) dt \right\|_{X'(0,1)} \leq \left\| s^{\alpha-1} \int_0^{s^{1/\beta}} f^*(t) dt \right\|_{X'(0,1)} = \|f\|_{Y_X(0,1)},$$

and so, (4.4) holds. Thus, it only remains to show the optimality of $Y_X(0,1)$. For this, we consider any r.i. space $Y(0,1)$ such that $H'_{\alpha,\beta} : Y(0,1) \to X'(0,1)$ is bounded. Then, using Lemma 4.1.4 together with (4.3), we get

$$\|f\|_{Y_X(0,1)} = \left\| s^{\alpha-1} \int_0^{s^{1/\beta}} f^*(t) dt \right\|_{X'(0,1)} \lesssim \|f\|_{Y(0,1)}, \quad f \in Y(0,1),$$

i.e., $Y(0,1) \hookrightarrow Y_X(0,1)$, and so the proof is complete. □

Now, let us describe the smallest r.i. space into which the operator (4.1) is bounded from a given r.i. space $X(0,1)$.

Corollary 4.1.6. *Let $X(0,1)$ be an r.i. space and let $Y_X(0,1)$ be the r.i. space defined in (4.3). Then, $Y'_X(0,1)$ is the smallest r.i. space that verifies*

$$H_{\alpha,\beta} : X(0,1) \to Y'_X(0,1),$$

where $H_{\alpha,\beta}$ is the Hardy type operator given in (4.1).

Proof. It is a direct consequence of Lemma 4.1.4 and Corollary 4.1.5. In fact, using Corollary 4.1.5, we have that

$$H'_{\alpha,\beta} : Y_X(0,1) \to X'(0,1),$$

and so, Lemma 4.1.4 implies that

$$H_{\alpha,\beta} : X(0,1) \to Y'_X(0,1).$$

Now, suppose that $Y(0,1)$ is another r.i. space such that

$$H_{\alpha,\beta} : X(0,1) \to Y(0,1)$$

is bounded. Then, using again Lemma 4.1.4, we obtain that

$$H'_{\alpha,\beta} : Y'(0,1) \to X'(0,1)$$

is also bounded. Therefore, by Corollary 4.1.5, we deduce that

$$Y'(0,1) \hookrightarrow Y_X(0,1),$$

and hence, by [8, Proposition I.2.10], we conclude that

$$Y'_X(0,1) \hookrightarrow Y(0,1),$$

as we wanted to prove. □

Finally, let us recall a result concerning the supremum operator which will be useful in the study of the Sobolev-type inequality. For further information on this topic see [36, 20].

Theorem 4.1.7. *Let $\alpha \in (0,1)$ and $\beta > 0$, with $\beta(1-\alpha) < 1$. Let $X(0,1)$ be an r.i. space. Then,*

$$\left\| t^{\alpha-1} \int_0^{t^{1/\beta}} s^{\beta(1-\alpha)-1} \Big(\sup_{s<y<1} y^{1-\beta(1-\alpha)} f^*(y) \Big) ds \right\|_{X(0,1)} \lesssim \left\| t^{\alpha-1} \int_0^{t^{1/\beta}} f^*(s) ds \right\|_{X(0,1)}.$$

4.2 n-dimensional case

Our aim in this section is to study the n-dimensional Hardy operator, $n \geq 2$:

$$P_n f(x) = \int_0^1 f(sx)ds, \quad f \in \mathcal{M}(\mathbb{R}^n), \ x \in \mathbb{R}^n. \tag{4.5}$$

In particular, we shall describe for what pairs of parameters p and q the operator (4.5) is bounded from $\mathcal{R}(L^p, L^\infty)$ to $L^q(\mathbb{R}^n)$. Let us start with a preliminary lemma:

Lemma 4.2.1. *For any $1 \leq p_1, p_2, p_3 \leq \infty$, if the operator $P_n f$ satisfies*

$$P_n : \mathcal{R}(L^{p_1}, L^{p_2}) \to L^{p_3}(\mathbb{R}^n), \tag{4.6}$$

then $1/p_3 = 1/(n'p_1) + 1/(np_2)$.

Proof. Given any $\alpha > 0$ and $f \in \mathcal{R}(L^{p_1}, L^{p_2})$, we define

$$f_\alpha(x) = f(\alpha x), \quad x \in \mathbb{R}^n.$$

We observe that

$$\|f_\alpha\|_{\mathcal{R}(L^{p_1},L^{p_2})} = \alpha^{-(n-1)/p_1-1/p_2}\|f\|_{\mathcal{R}(L^{p_1},L^{p_2})},$$

and

$$\|P_n f_\alpha\|_{L^{p_3}(\mathbb{R}^n)} = \alpha^{-n/p_3}\|P_n f\|_{L^{p_3}(\mathbb{R}^n)},$$

and so, if (4.6) is true, then we have

$$\|P_n f\|_{L^{p_3}(\mathbb{R}^n)} \lesssim \alpha^{n/p_3-(n-1)/p_1-1/p_2}\|f\|_{\mathcal{R}(L^{p_1},L^{p_2})}, \quad \alpha > 0.$$

Now, if one had $n/p_3 > (n-1)/p_1 + 1/p_2$, then by letting α tend to 0 one would deduce the contradiction $\|f\|_{\mathcal{R}(L^{p_1},L^{p_2})} = 0$, for all $f \in \mathcal{R}(L^{p_1}, L^{p_2})$. Similarly, if $n/p_3 < (n-1)/p_1 + 1/p_2$, we get the same contradiction by letting α tend to ∞. Thus, (4.6) is possible only if

$$\frac{1}{p_3} = \frac{1}{n'p_1} + \frac{1}{np_2},$$

as we wanted to show. ☐

Theorem 4.2.2. *Let $n \geq 2$ and $n-1 < p \leq \infty$. Then,*

$$P_n : \mathcal{R}(L^p, L^\infty) \to L^{n'p}(\mathbb{R}^n).$$

Proof. We fix any $f \in \mathcal{R}(L^p, L^\infty)$. Then, using Corollary 3.1.5, we get

$$\begin{aligned}\|P_n f\|_{L^{n'p}(\mathbb{R}^n)} &= \left(\int_0^\infty [(P_n f)^*(s)]^{n'p} ds\right)^{1/(n'p)} \\ &\leq \sum_{k=1}^n \left(\int_0^\infty [\psi_k^*(P_n f, L^\infty)(s^{1/n'})]^{n'p} ds\right)^{1/(n'p)},\end{aligned}$$

and hence, by a change of variables, we obtain that

$$\|P_n f\|_{L^{n'p}(\mathbb{R}^n)} \lesssim \sum_{k=1}^n \left(\int_0^\infty s^{1/(n-1)}[\psi_k^*(P_n f, L^\infty)(s)]^{n'p} ds\right)^{1/(n'p)}. \tag{4.7}$$

Now, we observe that

$$\left|P_n f(x)\right| = \int_0^1 |f(sx)| ds \leq \int_0^1 \psi_k(f, L^\infty)(s\widehat{x_k}) ds, \quad k \in \{1, \dots, n\},$$

and so, it holds that

$$\psi_k(P_n f, L^\infty)(\widehat{x_k}) \leq P_{n-1}\psi_k(f, L^\infty)(\widehat{x_k}), \quad k \in \{1, \dots, n\}. \tag{4.8}$$

Now, we fix $t > 0$ and $k \in \{1, \dots, n\}$. Then, by [61, Lemma III.3.17] and Fubini's theorem, we have that

$$\int_0^t P_{n-1}^* \psi_k(f, L^\infty)(s) ds = \sup_{|E| \le t} \int_0^1 \int_E \psi_k(f, L^\infty)(\lambda y) dy d\lambda,$$

and so, by Hardy-Littlewood inequality (see Theorem 2.1.5), we obtain that

$$\int_0^t P_{n-1}^* \psi_k(f, L^\infty)(s) ds \le \int_0^1 \Big(\int_0^t \psi_k^*(f, L^\infty)(\lambda^{n-1} s) ds \Big) d\lambda. \tag{4.9}$$

Using now a change of variables, we get

$$\begin{aligned} \int_0^1 \Big(\int_0^t \psi_k^*(f, L^\infty)(\lambda^{n-1} s) ds \Big) d\lambda &\approx \int_0^t s^{-1/(n-1)} \int_0^s v^{1/(n-1)-1} \psi_k^*(f, L^\infty)(v) dv \, ds \\ &\lesssim t^{-1/(n-1)+1} \int_0^t v^{1/(n-1)-1} \psi_k^*(f, L^\infty)(v) dv, \end{aligned}$$

and hence, taking into account (4.9), we obtain that

$$\int_0^t P_{n-1}^* \psi_k(f, L^\infty)(s) ds \lesssim t^{-1/(n-1)+1} \int_0^t v^{1/(n-1)-1} \psi_k^*(f, L^\infty)(v) dv. \tag{4.10}$$

As a consequence, combining (4.7) and (4.10), we have that

$$\|P_n f\|_{L^{n'p}(\mathbb{R}^n)} \lesssim \sum_{k=1}^n \left(\int_0^\infty t^{(-n'p+1)/(n-1)} \left[\int_0^t v^{1/(n-1)-1} \psi_k^*(f, L^\infty)(v) dv \right]^{n'p} dt \right)^{1/(n'p)}.$$

Thus, by Theorem 4.1.2 with $w(t) = t^{(-n'p+1)/(n'p(n-1))}$ and $v(t) = t^{1/(n-1)'}$, we get

$$\|P_n f\|_{L^{n'p}(\mathbb{R}^n)} \lesssim \sum_{k=1}^n \left(\int_0^\infty [\psi_k^*(f, L^\infty)(t)]^p dt \right)^{1/p},$$

from which the result follows. □

Now, let us study sufficient conditions for the weak-type boundedness of the dimensional operator (4.5).

Theorem 4.2.3. *Let $n \ge 2$ and let $n - 1 \le p \le \infty$.*

(i) If $p > n - 1$, then

$$P_n : \mathcal{R}(L^{p,\infty}, L^\infty) \to L^{pn',\infty}(\mathbb{R}^n).$$

(ii) If $p = n - 1$, then

$$P_n : \mathcal{R}(L^{n-1,1}, L^\infty) \to L^{n,\infty}(\mathbb{R}^n).$$

Proof. Let us see that (i) holds. For this, we fix any $f \in \mathcal{R}(L^{p,\infty}, L^\infty)$. Then, using Corollary 3.1.4, we get

$$\|P_n f\|_{L^{pn',\infty}(\mathbb{R}^n)} = \sup_{t>0} t\,(\lambda_{P_n f}(t))^{1/(pn')} \leq \sup_{t>0} t \left(\prod_{k=1}^{n} \lambda_{\psi_k(P_n f, L^\infty)}(t) \right)^{1/pn},$$

and so, the geometric-arithmetic mean inequality implies that

$$\begin{aligned} \|P_n f\|_{L^{pn',\infty}(\mathbb{R}^n)} &\lesssim \sum_{k=1}^{n} \sup_{t>0} t\,(\lambda_{\psi_k(P_n f, L^\infty)}(t))^{1/p} \\ &= \sum_{k=1}^{n} \big\|\psi_k(P_n f, L^\infty)\big\|_{L^{p,\infty}(\mathbb{R}^{n-1})}. \end{aligned} \tag{4.11}$$

Now, using (4.8), we have that

$$\psi_k(P_n f, L^\infty) \leq P_{n-1}\psi_k(f, L^\infty), \quad k \in \{1, \ldots, n\},$$

and hence, by (4.11), we obtain that

$$\|P_n f\|_{L^{pn',\infty}(\mathbb{R}^n)} \lesssim \sum_{k=1}^{n} \big\|P_{n-1}\psi_k(f, L^\infty)\big\|_{L^{p,\infty}(\mathbb{R}^{n-1})}. \tag{4.12}$$

Thus, using (4.10) and Hölder's inequality, we get

$$\begin{aligned} \|P_n f\|_{L^{pn',\infty}(\mathbb{R}^n)} &\lesssim \sum_{k=1}^{n} \sup_{t>0} t^{1/p} P^{**}_{n-1}\psi_k(f, L^\infty)(t) \\ &\lesssim \sum_{k=1}^{n} \sup_{t>0} t^{1/p-1/(n-1)} \int_0^t v^{1/(n-1)-1}\psi_k^*(f, L^\infty)(v)\,dv \\ &\leq \sum_{k=1}^{n} \big\|\psi_k^*(f, L^\infty)\big\|_{L^{p,\infty}(\mathbb{R}^{n-1})}, \end{aligned}$$

from which (i) follows. Finally, applying the same arguments as before, we get (ii). □

Theorem 4.2.4. *Let $n \geq 2$. Let $Y(\mathbb{R}^n)$ be an r.i. space and let $X(\mathbb{R}^{n-1})$ be an r.i. space such that*

$$X(\mathbb{R}^{n-1}) \hookrightarrow L^{(n-1),1}(\mathbb{R}^{n-1}) + L^\infty(\mathbb{R}^{n-1}).$$

Then, the following statements are equivalent:

(i) $P_n f : \mathcal{R}(X, L^\infty) \to Y(\mathbb{R}^n)$;

(ii) $$\left\| t^{-1/n} \int_0^{t^{1/n'}} s^{1/(n-1)} f^*(s)\,ds \right\|_{\overline{Y}(0,\infty)} \lesssim \|f^*\|_{\overline{X}(0,\infty)}, \quad f \in X(\mathbb{R}^{n-1}).$$

Proof. $(i) \Rightarrow (ii)$ Given any $f \in X(\mathbb{R}^{n-1})$, we define

$$g(x) = f^*(\omega_n^{1/n'}|x|^{n-1}).$$

Then, we have that

$$\psi_k(g, L^\infty)(\widehat{x_k}) = f^*(\omega_n^{1/n'}|\widehat{x_k}|^{n-1}), \quad \text{for any } k \in \{1, \dots, n\},$$

and so, by the boundedness of the dilation operator in r.i. spaces and Lemma 2.1.3, we get

$$\|g\|_{\mathcal{R}(X,L^\infty)} \approx \|f^*(\omega_n^{1/n'}\omega_{n-1}^{-1}t)\|_{\overline{X}(0,\infty)} \lesssim \|f\|_{X(\mathbb{R}^{n-1})}. \tag{4.13}$$

Therefore, using (i) together with (4.13), we obtain that

$$\|P_n g\|_{Y(\mathbb{R}^n)} \lesssim \|f\|_{X(\mathbb{R}^{n-1})}. \tag{4.14}$$

Next, we observe that

$$P_n g(x) = \int_0^1 f^*(\omega_n^{1/n'}s^{n-1}|x|^{n-1})ds \approx |x|^{-1}\int_0^{\omega_n^{1/n'}|x|^{n-1}} s^{1/(n-1)-1} f^*(s)ds,$$

and so, again by Lemma 2.1.3, we have

$$\|P_n f\|_{Y(\mathbb{R}^n)} \approx \left\| t^{-1/n}\int_0^{t^{1/n'}} s^{1/(n-1)} f^*(s)ds \right\|_{\overline{Y}(0,\infty)}. \tag{4.15}$$

Hence, combining (4.14) and (4.15), we get (ii).

$(ii) \Rightarrow (i)$ We fix any $f \in \mathcal{R}(X, L^\infty)$. Then, using Theorem 4.2.3 together with

$$P_n : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

we conclude that

$$K(Pf, t^{1/n}; L^{n,\infty}, L^\infty) \lesssim K(f, Ct^{1/n}; \mathcal{R}(L^{(n-1),1}, L^\infty), L^\infty), \quad t > 0. \tag{4.16}$$

Now, by Theorem 3.1.7, we have

$$K(Ct^{1/n}, f; \mathcal{R}(L^{(n-1),1}, L^\infty), L^\infty) = \sum_{k=1}^n \int_0^{t^{1/n'}} s^{-1/(n-1)}\psi_k^*(f, L^\infty)(s)ds. \tag{4.17}$$

and, on the other hand, by Holmstedt's formulas (see Theorem 2.1.18), we get

$$K(P_n f, t^{1/n}; L^{n,\infty}, L^\infty) = \sup_{0<s<t} s^{1/n}P_n^* f(s). \tag{4.18}$$

Hence, taking into account (4.16), (4.17) and (4.18), we obtain that

$$P_n^* f(t) \lesssim \sum_{k=1}^n t^{-1/n}\int_0^{t^{1/n'}} s^{-1/(n-1)}\psi_k^*(f, L^\infty)(s)ds.$$

From this, we deduce that

$$\|P_n^* f\|_{\overline{Y}(0,\infty)} \lesssim \left\| t^{-1/n} \int_0^{t^{1/n'}} s^{-1/(n-1)} \psi_k^*(f, L^\infty)(s) ds \right\|_{\overline{Y}(0,\infty)},$$

and so, using (ii), we get

$$\|P_n^* f\|_{\overline{Y}(0,\infty)} \lesssim \|f^*\|_{\overline{X}(0,\infty)},$$

as we wanted to see. □

Consequently, we can completely solve the boundedness of the n-dimensional Hardy operator (4.5) when $n = 2$.

Corollary 4.2.5. *Let $X(\mathbb{R})$ and $Y(\mathbb{R}^n)$ be r.i. spaces. Then, the following statements are equivalent:*

(i) $P_2 f : \mathcal{R}(X, L^\infty) \to Y(\mathbb{R}^2)$;

(ii) $$\left\| t^{-1/2} \int_0^{t^{1/2}} f^*(s) ds \right\|_{\overline{Y}(0,\infty)} \lesssim \|f^*\|_{\overline{X}(0,\infty)}, \quad f \in X(\mathbb{R}).$$

Proof. It is an immediate consequence of the continuous inclusion

$$X(\mathbb{R}^2) \hookrightarrow L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2),$$

(for more detail see [8, Theorem II.6.6]) and of Theorem 4.2.4. □

Chapter 5

Sobolev embedding in $\mathcal{R}(X, L^\infty)$

Throughout this chapter, we completely characterize the Sobolev-type estimate given by Gagliardo [29] and Nirenberg [50] in the setting of r.i. spaces. In particular we concentrate on seeking the optimal domains and the optimal ranges for these embeddings between r.i. spaces and mixed norm spaces. As a consequence, we prove that well-known inequalities for the standard Sobolev space W^1L^p by Poornima [55] and Peetre [54] ($1 \le p < n$), and by Hansson [33], Brezis and Wainger[15] and Maz'ya [45] ($p = n$) can be further strengthened by considering mixed norms on the target spaces.

This chapter is organized as follows: In the first section, we establish necessary and sufficient conditions on the r.i. spaces $X(I^n)$ and $Z(I^n)$ for which the higher-order Sobolev embedding of the form

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty) \tag{5.1}$$

holds (see Theorem 5.1.5). Then, this relation will be a handy tool for characterizing the optimal domain and the optimal range for (5.1) between r.i. spaces and mixed norm spaces.

After this discussion, our analysis focuses on giving explicit constructions of such optimal spaces. In particular, in the second section, we provide a description of the smallest space of the form $\mathcal{R}(X, L^\infty)$ in (5.1), once the r.i. space $Z(I^n)$ is given (see Theorem 5.2.1). Moreover, in the third section, for a fixed mixed norm space $\mathcal{R}(X, L^\infty)$, we describe the largest r.i. space $Z(I^n)$ for which (5.1) holds (see Theorem 5.3.2).

All these results are then employed to establish classical Sobolev embeddings in the context of mixed norm spaces. Thus, for instance, we recover the classical estimate proved by Gagliardo [29] and Nirenberg [50]

$$W^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty), \tag{5.2}$$

and, as a new contribution, we show that $\mathcal{R}(L^1, L^\infty)$ is the smallest mixed norm space of the form $\mathcal{R}(X, L^\infty)$ satisfying (5.2).

As we have pointed out in the Introduction, the optimal range problem for the Sobolev embedding was studied in [36] within the class of r.i. spaces. To be more specific, for a fixed r.i. domain space $Z(I^n)$ they determined the smallest r.i. range space $X^{\mathrm{op}}(I^n)$ satisfying

$$W^m Z(I^n) \hookrightarrow X^{\mathrm{op}}(I^n).$$

Motivated by this problem, in the fourth section we compare the optimal r.i. range space with the optimal mixed norm space. In particular, Theorem 5.4.3 proves that the following chain of embeddings holds:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n),$$

with $\mathcal{R}(X, L^\infty)$ the mixed norm space constructed in Theorem 5.2.1. Consequently, it turns out that it is still possible to further improve the classical Sobolev embeddings by means of mixed norm spaces.

Most of the results of this chapter are included in [24].

5.1 Necessary and sufficient conditions

Let $n, m \in \mathbb{N}$, with $n \geq 2$ and $m \in \mathbb{N}$. Now, our main purpose is to find necessary and sufficient conditions on $X(I^{n-1})$ and $Z(I^n)$ under which we have

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty). \tag{5.3}$$

For this, we shall establish the equivalence between (5.3) and the boundedness of a suitable Hardy type operator, via an argument used by Kerman and Pick [36] (see also [27, 19]) to characterize higher-order Sobolev embeddings in r.i. spaces. Then, this relation will be a key tool in determining the largest r.i. space and the smallest mixed norm space for (5.3).

Let us start by analyzing the case $m \geq n$.

Lemma 5.1.1. *Let $m, n \in \mathbb{N}$, with $m \geq n$. Then, for any r.i. spaces $Z(I^n)$ and $X(I^{n-1})$, it holds that*

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Proof. Observe that if $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \partial^n f(y_1, \ldots, y_n) dy_1 \ldots dy_n,$$

and hence, it holds that

$$|f(x)| \leq \int_{\mathbb{R}^n} |\partial^n f(y)| dy.$$

From this, we deduce that

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{\mathcal{R}(L^\infty, L^\infty)} \leq \|f\|_{W^n L^1(\mathbb{R}^n)}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

and so, using that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^n L^1(\mathbb{R}^n)$ (for more details see [1, 45]), we obtain

$$W^n L^1(\mathbb{R}^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(\mathbb{R}^n). \tag{5.4}$$

Next, using (5.4) together with the fact that

$$W^m L^1(\mathbb{R}^n) \hookrightarrow W^n L^1(I^n), \quad m \geq n,$$

we conclude that

$$W^m L^1(\mathbb{R}^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(\mathbb{R}^n), \quad m \geq n. \tag{5.5}$$

Now, we shall check that (5.5) is valid for the Sobolev space $W^m L^1(I^n)$. In order to do that, we recall that since the boundary of I^n satisfies nice smoothness properties, by [60, Theorem VI.5], there exists a bounded linear operator

$$E : \ W^m L^1(I^n) \to W^m L^1(\mathbb{R}^n),$$

such that, for any $f \in W^m L^1(I^n)$, the restriction of Ef to I^n is f. Therefore, using this fact together with (5.5), we get

$$\|f\|_{\mathcal{R}(L^\infty,L^\infty)} \leq \|Ef\|_{\mathcal{R}(L^\infty,L^\infty)} \lesssim \|Ef\|_{W^m L^1(\mathbb{R}^n)} \lesssim \|f\|_{W^m L^1(I^n)}.$$

That is,

$$W^m L^1(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n), \quad m \geq n,$$

and hence, taking into account (2.1) together with Theorem 3.2.8, we get

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty), \quad m \geq n,$$

for every r.i. spaces $Z(I^n)$ and $X(I^{n-1})$ and the result follows. □

As we have seen in Lemma 5.1.1, the Sobolev embeddings (5.3) are uninteresting in the case when $m \geq n$, since they hold for any r.i. spaces $X(I^{n-1})$ and $Z(I^n)$. Consequently, from now on, we suppose $1 \leq m < n$.

Now, let us consider an extension of the estimate (5.2) due to Gagliardo [29] and Nirenberg [50] for higher order derivatives.

Theorem 5.1.2. *Let $m \in \mathbb{N}$, with $1 \leq m \leq n-1$. Then,*

$$W^m L^1(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty). \tag{5.6}$$

Proof. It suffices to prove the result for $1 < m \leq n-1$ since the case $m = 1$ follows from the classical estimate (5.2) due to Gagliardo [29] and Nirenberg [50]. To this end, we shall see that the following chain of embeddings hold:

$$W^m L^1(I^n) \hookrightarrow W^1 L^{n/(n-m+1),1}(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty).$$

In fact, since the classical estimate due to Poornima [55] claims that

$$W^{m-1}L^1(I^n) \hookrightarrow L^{n/(n-m+1),1}(I^n),$$

we have that

$$W^m L^1(I^n) \hookrightarrow W^1 L^{n/(n-m+1),1}(I^n). \tag{5.7}$$

On the other hand, we fix any $f \in W^1 L^{n/(n-m+1),1}(I^n)$. Combining the classical embedding on Lorentz spaces (see Theorem 5.1.3 below)

$$W^1 L^{n,1}(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n),$$

with Gagliardo-Nirenberg embedding

$$W^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty)$$

and then applying Theorem 2.2.1 we get

$$K(f,t;\mathcal{R}(L^1,L^\infty),L^\infty) \lesssim K(f,Ct;W^1L^1,W^1L^{n,1}), \quad 0<t<1. \tag{5.8}$$

But, by Theorem 3.1.7,

$$K(f,t;\mathcal{R}(L^1,L^\infty),L^\infty) \approx \sum_{k=1}^{n} \int_0^t \psi_k^*(f,L^\infty)(s)ds. \tag{5.9}$$

Moreover, using now Theorem 2.1.17, we get

$$\begin{aligned} K(f,Ct;W^1L^1,W^1L^{n,1}) &\approx \int_0^{(Ct)^{n'}} |D^1 f|^*(s)ds + Ct \int_{(Ct)^{n'}}^1 s^{-1/n'} |D^1 f|^*(s)ds \\ &\approx \int_0^{(Ct)^{n'}} s^{-1/n} \bigg(\int_s^1 y^{-1/n'} |D^1 f|^*(y)dy \bigg) ds. \end{aligned}$$

So, by a change of variables, we obtain

$$K(f,Ct;W^1L^1,W^1L^{n,1}) \approx \int_0^t \bigg(\int_{Cs^{n'}}^1 y^{-1/n'} |D^1 f|^*(y)dy \bigg) ds. \tag{5.10}$$

Therefore, taking into account (5.8), (5.9), and (5.10), we obtain

$$\int_0^t \psi_k^*(f,L^\infty)(s)ds \lesssim \int_0^t \bigg(\int_{Cs^{n'}}^1 y^{-1/n'} |D^1 f|^*(y)dy \bigg) ds, \quad k \in \{1,\ldots,n\}.$$

Hence, using Hardy-Littlewood-Pólya Principle (see Theorem 2.1.11) together with Fubini's theorem, we get

$$\begin{aligned} \|f\|_{\mathcal{R}(L^{(n-1)/(n-m),1},L^\infty)} &\lesssim \int_0^1 t^{(n-m)/(n-1)-1} \bigg(\int_{Ct^{n'}}^1 y^{-1/n'} |D^1 f|^*(y)dy \bigg) dt \\ &\approx \int_0^1 y^{(n-m+1)/(n-1)} |D^1 f|^*(y)dy. \end{aligned}$$

From this we conclude that

$$W^1 L^{n/(n-m+1),1}(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty),$$

and so, taking into account (5.7), we get

$$W^m L^1(I^n) \hookrightarrow W^1 L^{n/(n-m+1),1}(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty),$$

as we wanted to prove. □

The next lemma will be needed to prove the connection between Sobolev embedding and boundedness of a suitable Hardy type operator. Its proof involves two main ingredients: Theorem 5.1.2 and the following classical result [62, 51, 21, 36]:

Theorem 5.1.3. *Let $n, m \in \mathbb{N}$, with $1 \leq m \leq n-1$. Then, the Lorentz space $L^{n/m,1}(I^n)$ is the largest r.i. space satisfying*

$$W^m L^{n/m,1}(I^n) \hookrightarrow L^\infty(I^n).$$

Lemma 5.1.4. *Let $k \in \{1, \dots, n\}$ and let $f \in W^m L^1(I^n)$. Then, for any $t > 0$,*

$$\int_0^t s^{-(m-1)/(n-1)} \psi_k^*(f, L^\infty)(s)ds \lesssim \int_0^t s^{-(m-1)/(n-1)} \left(\int_{Cs^{n'}}^1 v^{m/n-1} |D^m f|^*(v)dv \right) ds.$$

Proof. We fix any $f \in W^m L^1(I^n)$. Then, combining Theorem 5.1.2 and Theorem 5.1.3 we conclude that, for any $t > 0$,

$$\begin{aligned} &K(f, t^{(n-m)/(n-1)}; \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty), L^\infty) \\ &\lesssim K(f, Ct^{(n-m)/(n-1)}; W^m L^1, W^m L^{n/m,1}). \end{aligned} \tag{5.11}$$

Using now Theorem 3.1.7, we get

$$K(f, t^{(n-m)/(n-1)}; \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty), L^\infty) = \int_0^t s^{-(m-1)/(n-1)} \psi_k^*(f, L^\infty)(s)ds. \tag{5.12}$$

On the other hand, by Theorem 2.2.1, we have that

$$K(f, Ct^{(n-m)/(n-1)}; W^m L^1, W^m L^{n/m,1}) \approx \int_0^{Ct^{n'}} s^{-m/n} \left(\int_s^1 v^{m/n-1} |D^m f|^*(v)dv \right) ds. \tag{5.13}$$

Hence, taking into account (5.11), (5.12) and (5.13), we deduce that

$$\int_0^t s^{-(m-1)/(n-1)} \psi_k^*(f, L^\infty)(s)ds \lesssim \int_0^{Ct^{n'}} s^{-m/n} \left(\int_s^1 v^{m/n-1} |D^m f|^*(v)dv \right) ds.$$

As a consequence, by a change of variables, we obtain that

$$\int_0^t s^{-(m-1)/(n-1)} \psi_k^*(f, L^\infty)(s)ds \lesssim \int_0^t s^{-(m-1)/(n-1)} \left(\int_{Cs^{n'}}^1 v^{m/n-1} |D^m f|^*(v)dv \right) ds,$$

as we wanted to show. □

Theorem 5.1.5. *Let $n \geq 2$ and $1 \leq m \leq n-1$. Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces. Then, the following statements are equivalent:*

(i) $W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty)$;

(ii) $\left\| \int_{t^{n'}}^1 s^{m/n-1} f(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$

Proof. *(i)* $\Rightarrow$ *(ii)* We suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$ and $r \in \mathbb{R}$, with $0 < r < \min(a, b)$. Given any non-negative $f \in \overline{Z}(0,1)$, with $\lambda_f(0) \leq \omega_{n-1}^{n'} r^n$, we define

$$u(x) = \begin{cases} \int_{\omega_{n-1}^{n'}|x|^n}^1 \int_{s_1}^1 \int_{s_2}^1 \cdots \int_{s_{m-1}}^1 s_m^{-m+m/n} f(s_m) ds_m \dots ds_1, & \text{if } x \in B_n(0,r), \\ 0, & \text{otherwise.} \end{cases}$$

We observe that if we apply Fubini's theorem $m-1$ times, then we get

$$u(x) \approx \int_{\omega_{n-1}^{n'}|x|^n}^1 s_m^{-m+m/n} f(s_m)(s_m - \omega_{n-1}^{n'}|x|^n)^{(m-1)} ds_m, \quad x \in B_n(0,r),$$

and $u(x) = 0$, otherwise. Therefore, we conclude that

$$u(x) \lesssim \begin{cases} \int_{\omega_{n-1}^{n'}|x|^n}^1 s_m^{-1+m/n} f(s_m) ds_m, & x \in B_n(0,r), \\ 0, & \text{otherwise,} \end{cases}$$

and so using Lemma 2.1.3, the boundedness of the dilation operator in r.i. spaces and Lemma 4.1.3, we get

$$\|u\|_{Z(I^n)} \lesssim \left\| \int_s^1 s_m^{-1+m/n} f(s_m) ds_m \right\|_{\overline{Z}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \tag{5.14}$$

Now, we fix any $\alpha \in (\mathbb{N} \cup \{0\})^{n-1}$. Then, following the same arguments as in [36, Theorem A], we conclude that, for a.e. $x \in B_n(0,r)$,

$$|\partial^{|\alpha|} u(x)| \lesssim \sum_{j=1}^{|\alpha|} |x|^{jn-|\alpha|} \int_{\omega_{n-1}^{n'}|x|^n}^1 f(y) y^{-j+m/n-1} dy, \quad 1 \leq |\alpha| \leq m-1, \tag{5.15}$$

and

$$|\partial^m u(x)| \lesssim f(\omega_{n-1}^{n'}|x|^n) + \sum_{j=1}^{m-1} |x|^{jn-m} \int_{\omega_{n-1}^{n'}|x|^n}^1 f(y) y^{-j+m/n-1} dy. \tag{5.16}$$

Moreover,

$$\partial^{|\alpha|} u(x) = 0, \quad \text{a.e. } x \notin B_n(0,r),$$

for any $\alpha \in (\mathbb{N} \cup \{0\})^{n-1}$, with $1 \le |\alpha| \le m$. Let us suppose that $1 \le |\alpha| \le m-1$. Then, using Lemma 2.1.3 together with (5.15), we get

$$\begin{aligned}
\|\partial^{|\alpha|} u\|_{Z(I^n)} &\lesssim \sum_{j=1}^{|\alpha|} \left\| \left(|\cdot|^{jn-|\alpha|} \int_{\omega_{n-1}^{n'}|\cdot|^n}^1 f(y) y^{-j+m/n-1} dy \right)^*(s) \right\|_{\overline{Z}(0,1)} \\
&= \sum_{j=1}^{|\alpha|} \left\| \left(\omega_n^{-jn+|\alpha|} t^{j-|\alpha|/n} \int_{\omega_n^{-1}\omega_{n-1}^{n'} t}^1 f(y) y^{-j+m/n-1} dy \right)^*(s) \right\|_{\overline{Z}(0,1)} \\
&\approx \sum_{j=1}^{|\alpha|} \left\| t^{j-|\alpha|/n} \int_{\omega_n^{-1}\omega_{n-1}^{n'} t}^1 f(y) y^{-j+m/n-1} dy \right\|_{\overline{Z}(0,1)}.
\end{aligned}$$

As a consequence, the boundedness of the dilation operator and Lemma 4.1.3 imply

$$\|\partial^{|\alpha|} u\|_{Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad 1 \le |\alpha| \le m-1. \tag{5.17}$$

Now, if $|\alpha| = m$ then, by (5.16) and Lemma 2.1.3, we have

$$\|\partial^m u\|_{Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)} + \sum_{j=1}^{m-1} \left\| t^{j-m/n} \int_{\omega_n^{-1}\omega_{n-1}^{n'} t}^1 f(y) y^{-j+m/n-1} dy \right\|_{\overline{Z}(0,1)}.$$

Therefore, applying the same arguments as before, we deduce that

$$\|\partial^{|\alpha|} u\|_{Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad |\alpha| = m. \tag{5.18}$$

Thus, combining (5.14), (5.17), and (5.18), we conclude that

$$\|u\|_{W^m Z(I^n)} = \sum_{0 \le |\alpha| \le m} \|\partial^{|\alpha|} u\|_{Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)} \tag{5.19}$$

Therefore, using (i) together with (5.19), we obtain that

$$\|u\|_{\mathcal{R}(X, L^\infty)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{5.20}$$

Now, we fix any $k \in \{1, \dots, n\}$. Then, using Lemma 2.1.3, we have that

$$\psi_k^*(u, L^\infty)(t) = \int_{t^{n'}}^1 \int_{s_1}^1 \int_{s_2}^1 \cdots \int_{s_{m-1}}^1 s_m^{-m+m/n} f(s_m) ds_m \dots ds_1,$$

for any $0 \le t < \omega_{n-1} r^{n-1}$ and $\psi_k^*(u, L^\infty)(t) = 0$ otherwise and so, Fubini's theorem $m-1$ times implies

$$\psi_k^*(u, L^\infty)(t) = \begin{cases} \displaystyle\int_{t^{n'}}^1 s_m^{-1+m/n} f(s_m)(1 - t^{n'}/s_m)^{(m-1)} ds_m, & 0 \le t < \omega_{n-1} r^{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we observe that if $0 \le t < \omega_{n-1} r^{n-1}/2$, then

$$\begin{aligned}
\int_{2^{n'} t^{n'}}^1 s_m^{m/n-1} f(s_m) ds_m &\le \int_{2^{n'} t^{n'}}^1 s_m^{m/n-1} \big(1 - t^{n'}/s_m\big) f(s_m) ds_m \\
&\le \int_{t^{n'}}^1 s_m^{m/n-1} \big(1 - t^{n'}/s_m\big) f(s_m) ds_m \\
&= \psi_k(u, L^\infty)(t),
\end{aligned}$$

and, so, by the boundedness of the dilation operator, we get

$$\left\|\int_{t^{n'}}^{1} s_m^{m/n-1} f(s_m)ds_m\right\|_{\overline{X}(0,1)} \lesssim \left\|\chi_{(0,\omega_{n-1}r^{n-1}/2)}\int_{2^{n'}t^{n'}}^{1} s_m^{m/n-1} f(s_m)ds_m\right\|_{\overline{X}(0,1)}$$
$$\leq \|\psi_k(u,L^\infty)\|_{\overline{X}(0,1)}.$$

As a consequence, using (5.20), we deduce that

$$\left\|\int_{t^{n'}}^{1} s_m^{m/n-1} f(s_m)ds_m\right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{5.21}$$

This proves (ii), for any non-negative $f\in\overline{Z}(0,1)$, with $\lambda_f(0)\leq \omega_{n-1}^{n'}r^n$. Now, let us consider any $f\in\overline{Z}(0,1)$. We define

$$f_1(x)=\max\left[\left(|f(x)|-f^*(\omega_{n-1}^{n'}r^n\right),0\right]\operatorname{sgn} f(x),$$

and

$$f_2(x)=\min\left[|f(x)|,f^*(\omega_{n-1}^{n'}r^n)\right]\operatorname{sgn} f(x).$$

Since $\lambda_{f_1}(0)\leq\omega_{n-1}^{n'}r^n$ and $f_1\leq f$ a.e., using inequality (5.21), with f replaced by f_1, we get

$$\left\|\int_{t^{n'}}^{1} s^{m/n-1} f_1(s)ds\right\|_{\overline{X}(0,1)} \lesssim \|f\|_{Z(0,1)}. \tag{5.22}$$

On the other hand, Hölder's inequality implies that

$$\left\|\int_{t^{n'}}^{1} s^{m/n-1} f_2(s)ds\right\|_{\overline{X}(0,1)} \lesssim f^{**}(\omega_{n-1}^{n'}r^n)\lesssim \|f\|_{Z(0,1)}. \tag{5.23}$$

As a consequence, using (5.22) and (5.23), we get

$$\begin{aligned}\left\|\int_{t^{n'}}^{1} s^{-1+m/n} f(s)ds\right\|_{\overline{X}(0,1)} &= \left\|\int_{t^{n'}}^{1} s^{m/n-1}(f_1(s)+f_2(s))ds\right\|_{\overline{X}(0,1)}\\ &\leq \left\|\int_{t^{n'}}^{1} s^{m/n-1} f_1(s)ds\right\|_{\overline{X}(0,1)}\\ &\quad+\left\|\int_{t^{n'}}^{1} s^{m/n-1} f_2(s)ds\right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)},\end{aligned}$$

as we wanted to show.

In order to prove $(ii)\Rightarrow(i)$, let us introduce a new space

$$Y(I^{n-1})=\left\{f\in\mathcal{M}(I^{n-1}):\left\|t^{(m-1)/n}f^{**}(t^{1/n'})\right\|_{\overline{Z}'(0,1)}<\infty\right\}.$$

Using the same ideas as we did in the proof of Corollary 4.1.5, it follows that $Y(I^{n-1})$ is an r.i. space. Moreover, by Corollary 4.1.6, its associate space $Y'(I^{n-1})$ is the smallest r.i. space satisfying

$$\left\|\int_{t^{n'}}^{1} s^{m/n-1} f(s)ds\right\|_{\overline{Y}'(0,1)} \lesssim \|f\|_{Z(0,1)},\quad f\in\overline{Z}(0,1). \tag{5.24}$$

Next, we shall see that the following chain of embeddings holds

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(Y', L^\infty) \hookrightarrow \mathcal{R}(X, L^\infty).$$

In fact, by hypothesis, the inequality (5.24) also holds when $\overline{Y}'(0,1)$ is replaced by $\overline{X}(0,1)$. As a consequence, Corollary 4.1.6 and Theorem 3.2.8 implies that

$$\mathcal{R}(Y', L^\infty) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Thus, it only remains to see that

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(Y', L^\infty). \tag{5.25}$$

To this end, we fix any $k \in \{1, \dots, n\}$ and $f \in W^m Z(I^n)$. Then, combining Lemma 5.1.4 and Hardy-Littlewood-Pólya Principle (see Theorem 2.1.11), we get

$$\int_0^1 t^{-(m-1)/(n-1)} h(t) \psi_k^*(f, L^\infty)(t)dt \lesssim \int_0^1 t^{-(m-1)/(n-1)} h(t) \times \left(\int_{Ct^{n'}}^1 s^{m/n-1} |D^m f|^*(s) ds \right) dt,$$

for any non-negative decreasing function $h \in \mathcal{M}(0,1)$ Thus, taking

$$h(t) = \sup_{t<s<1} s^{(m-1)/(n-1)} g^*(s), \qquad g \in \mathcal{M}(I^{n-1}),$$

and defining

$$Sg(t) = t^{-(m-1)/(n-1)} h(t),$$

we obtain that

$$\int_0^1 Sg(t) \psi_k^*(f, L^\infty)(t)dt \lesssim \int_0^1 Sg(t) \left(\int_{Ct^{n'}}^1 s^{m/n-1} |D^m f|^*(s) ds \right) dt.$$

From this inequality and the fact that $g^*(t) \le Sg(t)$, we conclude that

$$\begin{aligned}
\|f\|_{\mathcal{R}_k(Y', L^\infty)} &= \sup_{\|g\|_{Y(I^{n-1})} \le 1} \int_0^1 g^*(t) \psi_k^*(f, L^\infty)(t)dt \\
&\le \sup_{\|g\|_{Y(I^{n-1})} \le 1} \int_0^1 Sg(t) \psi_k^*(f, L^\infty)(t)dt \\
&\lesssim \sup_{\|g\|_{Y(I^{n-1})} \le 1} \int_0^1 t^{m/n-1} Sg(t) \left(\int_{Ct^{n'}}^1 s^{m/n-1} |D^m f|^*(s) ds \right) dt.
\end{aligned}$$

As a consequence, by Hölder's inequality, we obtain that

$$\|f\|_{\mathcal{R}_k(Y', L^\infty)} \le \sup_{\|g\|_{Y(I^{n-1})} \le 1} \left\| t^{-(m-1)/(n-1)} \sup_{t<s<1} s^{(m-1)/(n-1)} g^*(s) \right\|_{\overline{Y}(0,1)} \times \left\| \int_{Ct^{n'}}^1 s^{m/n-1} |D^m f|^*(s) ds \right\|_{\overline{Y}'(0,1)}.$$

But, by Theorem 4.1.7, we have

$$\left\|t^{-(m-1)/(n-1)} \sup_{t<s<1} s^{(m-1)/(n-1)} g^*(s)\right\|_{\overline{Y}(0,1)} \lesssim \|g^*\|_{\overline{Y}(0,1)},$$

and hence, using (5.24) we get

$$\|f\|_{\mathcal{R}_k(Y',L^\infty)} \lesssim \||D^m f|^*\|_{\overline{Z}(0,1)}, \quad k \in \{1,\dots,n\},$$

from which (5.25) follows. Thus, the proof is complete. □

5.2 Characterization of the optimal range

Now, we fix an r.i. space $Z(I^n)$. We shall provide a description of the smallest space of the form $\mathcal{R}(X, L^\infty)$ satisfying

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

It is important to note that Theorem 5.1.5 relates this problem with that of finding the smallest r.i. space into which the Hardy operator:

$$H_{m/n,n'} f(t) = \int_{t^{n'}}^1 s^{m/n} f(s) ds, \quad t \in (0,1) \tag{5.26}$$

is bounded from $\overline{Z}(0,1)$. Hence, in view of Corollary 4.1.6, it is natural to consider the space:

$$Y(I^{n-1}) = \left\{ f \in \mathcal{M}(I^{n-1}) : \|f\|_{Y(I^{n-1})} = \left\|t^{(m-1)/n} f^{**}(t^{1/n'})\right\|_{\overline{Z}'(0,1)} < \infty \right\}. \tag{5.27}$$

Using the same ideas as we did in the proof Corollary 4.1.5, one can deduce that $Y(I^{n-1})$ is an r.i. space. Moreover, by Corollary 4.1.6, we have that its associate space $Y'(I^{n-1})$ is the smallest r.i. space satisfying

$$H_{m/n,n'} : \overline{Z}(0,1) \to \overline{Y}'(0,1).$$

In order to clarify the notation used later, note that if we denote by

$$X_{W^m Z, L^\infty}(I^{n-1}) := Y'(I^{n-1}), \tag{5.28}$$

then, Theorem 2.1.9 implies that

$$Y(I^{n-1}) = (Y')'(I^{n-1}) = X'_{W^m Z, L^\infty}(I^{n-1}).$$

Theorem 5.2.1. *Let $n, m \in \mathbb{N}$, with $1 \le m \le n-1$. Let $Z(I^n)$ be an r.i. space and let $X_{W^m Z, L^\infty}(I^{n-1})$ be the r.i. space defined in (5.28). Then, the Sobolev embedding*

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X_{W^m Z, L^\infty}, L^\infty), \tag{5.29}$$

holds. Moreover, $\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)$ is the smallest space of the form $\mathcal{R}(X, L^\infty)$ that verifies (5.29).

Proof. The embedding (5.29) follows directly from Theorem 5.1.5 together with Corollary 4.1.6. In fact, by Corollary 4.1.6, we have that

$$\left\| \int_{t^{n'}}^{1} s^{m/n-1} f(s) ds \right\|_{\overline{X}_{W^m Z, L^\infty}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1),$$

and so, using Theorem 5.1.5, we get (5.29). Thus, it only remains to see that $\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)$ is the smallest space of the form $\mathcal{R}(X, L^\infty)$ satisfying (5.29). For this, we shall show that if a mixed norm space $\mathcal{R}(X, L^\infty)$ verifies

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty), \tag{5.30}$$

then

$$\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty) \hookrightarrow \mathcal{R}(X, L^\infty). \tag{5.31}$$

We fix any $g \in X'(I^{n-1})$. Then, combining (5.30) with Lemma 4.1.4, we get

$$\left\| t^{m/n-1} \int_0^{t^{1/n'}} g^*(s) ds \right\|_{\overline{Z}'(0,1)} \lesssim \|g\|_{X'(I^{n-1})}.$$

Therefore, using now (5.27), we obtain

$$X'(I^{n-1}) \hookrightarrow X'_{W^m Z, L^\infty}(I^{n-1}),$$

and so, [8, Proposition I.2.10] implies that

$$X_{W^m Z, L^\infty}(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Therefore, using Theorem 3.2.8, we have that the embedding (5.31) holds, as we wanted to show. □

Now, we shall present some applications of Theorem 5.2.1. In particular, we shall see that (5.6) cannot be improved within the class of spaces of the form $\mathcal{R}(X, L^\infty)$. This should be understood as follows: if we replace the range space in

$$W^m L^1(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-m),1}, L^\infty),$$

by a smaller mixed norm space, say $\mathcal{R}(X, L^\infty)$, then the resulting embedding

$$W^m L^1(I^n) \hookrightarrow \mathcal{R}(X, L^\infty)$$

cannot longer be true.

Corollary 5.2.2. *Let Let $n, m \in \mathbb{N}$, with $1 \leq m \leq n-1$ and $1 \leq p < n/m$. Then, $\mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty)$ is the smallest mixed norm space of the form $\mathcal{R}(X, L^\infty)$ satisfying*

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty).$$

Proof. We prove this result only when $1 < p < n/m$ ($p = 1$ is easier). Theorem 5.2.1, with $Z(I^n)$ replaced by $L^p(I^n)$, gives

$$\begin{aligned}\|f\|_{X'_{W^mL^p,L^\infty}(I^{n-1})} &= \left\| t^{m/n-1} \int_0^{t^{1/n'}} f^*(s)ds \right\|_{L^{p'}(0,1)} \\ &\approx \left(\int_0^1 t^{-(n-mp)/((n-1)(p-1))-1} \left(\int_0^t f^*(s)ds \right)^{p'} dt \right)^{1/p'}.\end{aligned}$$

Since $1 < p < n/m$, we may apply Theorem 4.1.1 to get

$$\|f\|_{X'_{W^mL^p,L^\infty}(I^{n-1})} \lesssim \|f\|_{L^{p(n-1)/((n-1)p-n+mp),p'}(I^{n-1})}. \tag{5.32}$$

On the other hand, we have

$$\|f\|_{X'_{W^1L^p,L^\infty}(I^{n-1})} \gtrsim \|f\|_{L^{p(n-1)/((n-1)p-n+mp),p'}(I^{n-1})}. \tag{5.33}$$

As a consequence, combining (5.32) and (5.33), we get

$$X'_{W^1L^p,L^\infty}(I^{n-1}) = L^{p(n-1)/(p(n-1)-n+mp),p'}(I^{n-1}),$$

and so

$$X_{W^1L^p,L^\infty}(I^{n-1}) = L^{p(n-1)/(n-mp),p}(I^{n-1}),$$

as we wanted to prove. □

Now, we shall apply Theorem 5.2.1 to find the optimal mixed norm space into which the Sobolev space $W^mL^{n/m}(I^n)$ is continuously embedded.

Corollary 5.2.3. *Let $n, m \in \mathbb{N}$, with $1 \le m \le n-1$. Then, the mixed norm space $\mathcal{R}(L^{\infty,n/m;-1}, L^\infty)$ is the smallest space of the form $\mathcal{R}(X, L^\infty)$ satisfying*

$$W^mL^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty). \tag{5.34}$$

The proof will be an immediate consequence of Theorem 5.2.1 and the following result given in [27].

Theorem 5.2.4. *Let $1 < p < \infty$ and let v be a weight on $(0,1)$ satisfying the following properties:*

(i) $\int_0^1 v(t)dt < \infty$;

(ii) $\int_0^1 t^{-p}v(t)^p dt = \infty$;

(iii) $\displaystyle\int_0^r v(t)^p dt \lesssim r^p\Big(1 + \int_r^1 t^{-p}v(t)^p dt\Big)$, $0 < r < 1$.

Then, the r.i. norm defined as

$$\|f\|_{X(0,1)} = \|v(t)f^{**}(t)\|_{L^p(0,1)}, \quad f \in \mathcal{M}(0,1)$$

has associate norm

$$\|g\|_{X'(0,1)} = \|w(t)g^{*}(t)\|_{L^{p'}(0,1)}, \quad g \in \mathcal{M}(0,1),$$

where

$$w(t)^{p'} = \frac{d}{dt}\left[\left(1 + \int_t^1 s^{-p} v(s)^p ds\right)^{1-p'}\right], \quad 0 < t < 1.$$

Proof of Corollary 5.2.3. By Theorem 5.2.1, with $Z(I^n) = L^{n/m}(I^n)$, we get

$$\begin{aligned}\|f\|_{X'_{W^mL^{n/m},L^\infty}(I^{n-1})} &= \left\| t^{m/n-1} \int_0^{t^{1/n'}} f^*(s)ds \right\|_{L^{n/(n-m)}(0,1)} \\ &\approx \left\| t^{m/n} f^{**}(t) \right\|_{L^{n/(n-m)}(0,1)}.\end{aligned}$$

Consequently, Theorem 5.2.4 implies that

$$\|f\|_{X_{W^mL^{n/m},L^\infty}(I^{n-1})} = \left\| t^{-1} \log(e/t) f^*(t) \right\|_{L^{n/m}(0,1)} = \|f\|_{L^{\infty,n/m;-1}(I^{n-1})},$$

from which it follows that

$$\mathcal{R}(X_{W^mL^n,L^\infty} L^\infty) = \mathcal{R}(L^{\infty,n/m;-1}, L^\infty),$$

as we wanted to prove. □

Corollary 5.2.5. *Let $n, m \in \mathbb{N}$, with $1 \le m \le n-1$, and $p > n/m$. Then, the mixed norm space $\mathcal{R}(L^{\infty,\infty}, L^\infty) = L^\infty(I^n)$ is the smallest space of the form $\mathcal{R}(X, L^\infty)$ satisfying*

$$W^m L^p(I^n) \hookrightarrow \mathcal{R}(L^{\infty,\infty}, L^\infty) = L^\infty(I^n). \tag{5.35}$$

Proof. By the Sobolev embedding theorem, we have that the estimate (5.35) holds. On the other hand, by (2.1), we have that it is the best possible as far as range spaces are concerned. Observe that it can be also proved by means of Theorem 5.2.1. □

5.3 Characterization of the optimal domain

We now focus on the problem of determining the largest r.i. domain space satisfying (5.3) for a fixed range space $\mathcal{R}(X, L^\infty)$. Observe that the equivalences proved in Theorem 5.2.1 suggest that in order to solve this problem, we should find the largest r.i. space $Z(I^n)$ such that

$$H_{m/n,n'} : \overline{Z}(0,1) \to \overline{X}(0,1)$$

is bounded, where $H_{m/n,n'}$ is the Hardy type operator defined in (5.26). Hence, it is natural to consider a new space $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ defined by

$$Z_{\mathcal{R}(X,L^\infty)}(I^n) = \left\{ f \in \mathcal{M}(I^n) : \|f\|_{Z_{\mathcal{R}(X,L^\infty)}} = \left\| \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)} < \infty \right\}. \tag{5.36}$$

The next lemma will be needed later. Its proof follows the same arguments used in [27, Theorem 4.4], with small modifications.

Lemma 5.3.1. *Let $X(I^{n-1})$ be an r.i space, with $\overline{\alpha}_X < (n-m)/(n-1)$. Then,*

$$\left\| \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)} \approx \left\| \int_{t^{n'}}^1 f^{*}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)}, \quad f \in \mathcal{M}(0,1).$$

Proof. Let $f \in \mathcal{M}(I^n)$. Since $f^* \leq f^{**}$, we will be done if we can prove

$$\left\| \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)} \lesssim \left\| \int_{t^{n'}}^1 f^{*}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)}.$$

We fix $g \in X'(I^{n-1})$, with $\|g\|_{X'(I^{n-1})} \leq 1$. Then, by a change of variables and Fubini's theorem, we obtain

$$\begin{aligned} I &= \int_0^1 g^*(t) \left(\int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \right) dt \\ &= \int_0^1 \left(\int_0^1 g^*(t) \left(\int_{t^{n'}}^1 f^*(sv) s^{m/n-1} ds \right) dt \right) dv \\ &\approx \int_0^1 v^{-m/n} \left(\int_0^1 g^*(t) \left(\int_{t^{n'}v}^v f^*(z) z^{m/n-1} dz \right) dt \right) dv. \end{aligned}$$

Using now and Hölder's inequality, we get

$$I \lesssim \int_0^1 v^{-m/n} \|g\|_{X'(I^{n-1})} \left\| \int_{t^{n'}v}^1 f^*(z) z^{m/n-1} dz \right\|_{\overline{X}(0,1)} dv.$$

But $\|g\|_{X'(I^{n-1})} \leq 1$, and hence

$$I \lesssim \int_0^1 v^{-m/n} \left\| \int_{t^{n'}v}^1 f^*(z) z^{m/n-1} dz \right\|_{\overline{X}(0,1)} dv.$$

Therefore, the boundedness of the dilation operator in r.i. spaces gives

$$\begin{aligned} I &\lesssim \left\| \int_{t^{n'}}^1 f^*(z) z^{m/n-1} dz \right\|_{\overline{X}(0,1)} \left(\int_0^1 v^{-m/n} h_X(v^{-1/n'}) dv \right) \\ &\approx \left\| \int_{t^{n'}}^1 f^*(z) z^{m/n-1} dz \right\|_{\overline{X}(0,1)} \left(\int_0^\infty v^{-(n-m)/(n-1)-1} h_X(v) dv \right). \end{aligned}$$

But, by hypothesis, $\overline{\alpha}_X < (n-m)/(n-1)$, and hence [8, Lemma III, 5.9] implies that

$$\int_0^1 g^*(t)\Big(\int_{t^{n'}}^1 f^{**}(s)s^{m/n-1}ds\Big)dt \lesssim \Big\|\int_{t^{n'}}^1 f^*(z)z^{m/n-1}dz\Big\|_{\overline{X}(0,1)}.$$

Therefore, we conclude that

$$\begin{aligned}\Big\|\int_{t^{n'}}^1 f^{**}(z)z^{m/n-1}dz\Big\|_{\overline{X}(0,1)} &= \sup_{\|g\|_{X'(I^{n-1})}\le 1}\int_0^1 g^*(t)\Big(\int_{t^{n'}}^1 f^{**}(s)s^{m/n-1}ds\Big)dt\\ &\lesssim \Big\|\int_{t^{n'}}^1 f^*(z)z^{m/n-1}dz\Big\|_{\overline{X}(0,1)},\end{aligned}$$

as we wanted to show. □

Theorem 5.3.2. *Let $X(I^{n-1})$ be an r.i space, with $\overline{\alpha}_X < (n-m)/(n-1)$. Then, the space $Z_{\mathcal{R}(X,L^\infty)}(I^n)$ given in (5.36) is the largest r.i. domain space satisfying*

$$W^m Z_{\mathcal{R}(X,L^\infty)}(I^n) \hookrightarrow \mathcal{R}(X, L^\infty). \tag{5.37}$$

Proof. Let us check properties (A1)-(A6) for

$$\|f\|_{Z_{\mathcal{R}(X,L^\infty)}} = \Big\|\int_{t^{n'}}^1 f^{**}(s)s^{m/n-1}ds\Big\|_{\overline{X}(0,1)}.$$

(A1) Clearly, if $f = 0$ then $\|f\|_{Z_{\mathcal{R}(X,L^\infty)}} = 0$ and, for every $\alpha \in \mathbb{R}_+$, we have

$$\|\alpha f\|_{Z_{\mathcal{R}(X,L^\infty)}} = \alpha\|f\|_{Z_{\mathcal{R}(X,L^\infty)}}.$$

Now, if $\|f\|_{Z_{\mathcal{R}(X,L^\infty)}} = 0$, then we have that

$$\begin{aligned} 0 = \|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} &\ge f^{**}(1)\Big\|\int_{t^{n'}}^1 s^{m/n-1}ds\Big\|_{\overline{X}(0,1)}\\ &\gtrsim f^{**}(1)\int_0^{1/2}\int_{t^{n'}}^1 s^{m/n-1}ds\,dt \gtrsim f^{**}(1)\int_{2^{-n'}}^1 s^{m/n-1}ds\\ &\approx \int_0^1 f^*(t)dt = \int_{I^n}|f(x)|dx.\end{aligned}$$

This implies that $f = 0$. On the other hand, using Proposition 2.1.4 together with the triangle inequality for $\|\cdot\|_{\overline{X}(0,1)}$, it follows that

$$\|f+g\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} \le \|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} + \|g\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}.$$

(A2) If $0 \le f \le g$ a.e., then $f^{**} \le g^{**}$ (see Proposition 2.1.4). As a consequence, using the fact that $\|\cdot\|_{\overline{X}(0,1)}$ is an r.i. norm, we obtain

$$\begin{aligned}\|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} &= \Big\|\int_{t^{n'}}^1 f^{**}(s)s^{m/n-1}ds\Big\|_{\overline{X}(0,1)}\\ &\le \Big\|\int_{t^{n'}}^1 g^{**}(s)s^{m/n-1}ds\Big\|_{\overline{X}(0,1)} = \|g\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}.\end{aligned}$$

(A3) If $0 \le f_j \uparrow f$ a.e., then $f_j^{**} \uparrow f^{**}$ (see Proposition 2.1.4). Hence, by the monotone converge theorem, we have that

$$\int_{t^{n'}}^1 f_j^{**}(s) s^{m/n-1} ds \uparrow \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds.$$

from which the Fatou property for $\|\cdot\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}$ follows.

(A4) It requires only the corresponding axioms for $\|\cdot\|_{\overline{X}(0,1)}$.

(A5) We have to argue as in the proof of property (A1).

(A6) It is easy to see that $\|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} = \|f^*\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)}$.

Now, let us see that (5.37) holds. We fix any $f \in \overline{Z}_{\mathcal{R}(X,L^\infty)}(0,1)$. We observe that if $0 < t < 2^{-(n-1)/n}$ then, we have

$$\begin{aligned}\int_{2t^{n'}}^1 |f(s)| s^{m/n-1} ds &\approx \int_{2t^{n'}}^1 |f(s)| \Big(\int_{s/2}^s v^{m/n-2} dv \Big) ds \\ &\le \int_{2t^{n'}}^1 |f(s)| \Big(\int_{s/2}^1 v^{m/n-2} dv \Big) ds,\end{aligned}$$

and so, using Fubini's theorem, we get

$$\begin{aligned}\int_{2t^{n'}}^1 |f(s)| s^{m/n-1} ds &\lesssim \int_{t^{n'}}^{1/2} v^{m/n-2} \Big(\int_{2t^{n'}}^{2v} |f(s)| ds \Big) dv \\ &\quad + \int_{1/2}^1 v^{m/n-2} \Big(\int_{2t^{n'}}^1 |f(s)| ds \Big) dv \\ &\lesssim \int_{t^{n'}}^{1/2} v^{m/n-2} \Big(\int_0^{2v} |f(s)| ds \Big) dv + \|f\|_{L^1(0,1)}.\end{aligned}$$

Therefore, using the Hardy-Littlewood inequality (see Theorem 2.1.5) together with a change of variables, we obtain

$$\begin{aligned}\int_{2t^{n'}}^1 |f(s)| s^{m/n-1} ds &\lesssim \int_{2t^{n'}}^1 v^{m/n-1} f^{**}(v) dv + \|f\|_{L^1(0,1)} \\ &\le \int_{t^{n'}}^1 v^{m/n-1} f^{**}(v) dv + \|f\|_{L^1(0,1)},\end{aligned}$$

for any $t \in (0, 2^{-(n-1)/n})$. Hence, using the boundedness of the dilation operator in r.i. spaces, we get

$$\begin{aligned}\Big\| \int_{t^{n'}}^1 f(s) s^{m/n-1} ds \Big\|_{\overline{X}(0,1)} &\lesssim \Big\| \chi_{(0,2^{-(n-1)/n})}(t) \int_{2t^{n'}}^1 f(s) s^{m/n-1} ds \Big\|_{\overline{X}(0,1)} \\ &\lesssim \Big\| \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \Big\|_{\overline{X}(0,1)} + \|f\|_{L^1(0,1)}.\end{aligned}$$

But, by hypothesis, we have that

$$\|f\|_{\overline{Z}_{\mathcal{R}(X,L^\infty)}(0,1)} = \left\| \int_{t^{n'}}^1 f^{**}(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)},$$

and so, we conclude that

$$\left\| \int_{t^{n'}}^1 f(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}_{\mathcal{R}(X,L^\infty)}(0,1)} + \|f\|_{L^1(0,1)}, \quad f \in \overline{Z}_{\mathcal{R}(X,L^\infty)}(0,1).$$

Therefore, using Theorem 5.1.5 together with (2.1), we deduce that (5.37) holds.

Finally, for the optimality, we consider any $Z(I^n)$ such that

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Then, using Theorem 5.1.5 together with Lemma 5.3.1, we obtain that

$$\|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} \approx \left\| \int_{t^{n'}}^1 f^*(s) s^{m/n-1} ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{Z(I^n)}.$$

So, we conclude that

$$Z(I^n) \hookrightarrow Z_{\mathcal{R}(X,L^\infty)}(I^n),$$

from which the result follows. □

Now, we shall present some applications of Theorem 5.3.2.

Corollary 5.3.3. *Let $1 < p < n/m$. Then, the Lebesgue space $L^p(I^n)$ is the largest r.i. space satisfying*

$$W^1 L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty).$$

Proof. In view of Theorem 5.3.2, we have

$$\begin{aligned} \|f\|_{Z_{\mathcal{R}(L^{p(n-1)/(n-mp),p},L^\infty)}(I^n)} &\approx \left\| \int_{t^{n'}}^1 f^*(s) s^{m/n-1} ds \right\|_{L^{p(n-1)/(n-mp),p}(0,1)} \\ &\approx \left(\int_0^1 t^{(n-mp)/n-1} \left(\int_t^1 f^*(s) s^{m/n-1} ds \right)^p dt \right)^{1/p}. \end{aligned}$$

Using now Theorem 4.1.1 we obtain

$$\|f\|_{Z_{\mathcal{R}(L^{p(n-1)/(n-mp),p},L^\infty)}(I^n)} \lesssim \|f\|_{L^p(I^n)}. \tag{5.38}$$

On the other hand, again Theorem 4.1.1 give us that

$$\|f\|^p_{Z_{\mathcal{R}(L^{p(n-1)/(n-p),p},L^\infty)}(I^n)} \gtrsim \quad \int_0^1 t^{-p(n-m)/n} \left(\int_0^t \int_v^1 f^*(s) s^{m/n-1} ds\, dv \right)^p dt \Bigg).$$

Next, we observe that if $0 < t < 1$, then

$$\int_0^t \int_v^1 f^*(s)s^{m/n-1}ds\,dv = t\int_t^1 f^*(s)s^{m/n-1}ds + \int_0^t f^*(v)v^{m/n}dv \gtrsim f^*(t)t^{m/n+1},$$

and so, we have that

$$\|f\|_{Z_{\mathcal{R}(L^{p(n-1)/(n-mp),p},L^\infty)}(I^n)} \gtrsim \|f\|_{L^p(I^n)}. \tag{5.39}$$

Therefore, combining (5.38) and (5.39), we obtain

$$L^p(I^n) = Z_{\mathcal{R}(L^{p(n-1)/(n-mp),p},L^\infty)}(I^n),$$

as we wanted to show. □

Finally, we shall see that a nontrivial improvement of the domain in (5.34) is possible among r.i. spaces. Before that, it will be convenient to give a technical lemma. Even though it was proved in [7, 52, 53], we shall present a simple proof by using the theory of weights.

Lemma 5.3.4. *Let $n \in \mathbb{N}$, with $n \geq 1$, and let $1 < p < \infty$. Then $\overline{\alpha}_{L^{\infty,p;-1}} = 0$.*

Proof. It follows immediately from [18, Theorem 3.1] that

$$h_{L^{\infty,p;-1}}(s) = \big[\log(se)\big]^{1/p'}, \quad s > 1.$$

Consequently, using (2.4), we get

$$\overline{\alpha}_{L^{\infty,p;-1}} = \lim_{s\to\infty} \frac{\log\big(h_{L^{\infty,p;-1}}(s)\big)}{\log(s)} = \frac{1}{p'}\lim_{s\to\infty}\frac{\log\big(1+\log(s)\big)}{\log(s)} = 0,$$

as we wanted to show. □

Corollary 5.3.5. *The r.i. space $Z_{\mathcal{R}(L^{\infty,n/m;-1},L^\infty)}(I^n)$, with norm given by*

$$\|f\|_{Z_{\mathcal{R}(L^{\infty,n/m;-1},L^\infty)}(I^n)} \approx \left\| \int_t^1 s^{m/n-1} f^*(s)ds \right\|_{L^{\infty,n/m;-1}(I^n)},$$

is the largest r.i. domain space that verifies

$$W^m Z_{\mathcal{R}(L^{\infty,n/m;-1},L^\infty)}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty).$$

Proof. According to Lemma 5.3.4, we may apply Theorem 5.3.2 to obtain

$$\|f\|_{Z_{\mathcal{R}(L^{\infty,n/m;-1},L^\infty)}(I^n)} \approx \left\| \int_{t^{n'}}^1 s^{m/n-1} f^*(s)ds \right\|_{L^{\infty,n;-1}(I^n)}.$$

Then, the result follows using a change of variables and [18, Theorem 3.1]. □

5.4 Comparison with the optimal r.i. range

As we have mentioned before, Kerman and Pick [36] studied the optimal range problem for Sobolev embedding within the class of r.i. spaces. Namely, for a fixed r.i. domain space $Z(I^n)$, they determined the smallest r.i. space $X^{\mathrm{op}}(I^n)$, satisfying

$$W^m Z(I^n) \hookrightarrow X^{\mathrm{op}}(I^n). \tag{5.40}$$

In our setting, we recall that in Theorem 5.2.1 we have studied an analogous problem in the context of mixed norm spaces. More precisely, we have found the smallest space of the form $\mathcal{R}(X, L^\infty)$, namely $\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)$, that verifies

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X_{W^m Z, L^\infty}, L^\infty). \tag{5.41}$$

Now, our goal is to compare the optimal r.i. range space with the optimal mixed norm space. We will show in Theorem 5.4.3 that the following chain of embeddings holds:

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X_{W^m Z, L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n).$$

To this end, we first need to recall the two following results [36].

Theorem 5.4.1. *Let $Y(I^n)$ and $Z(I^n)$ be r.i. spaces. Then, the Sobolev embedding*

$$W^m Z(I^n) \hookrightarrow Y(I^n)$$

holds if and only if

$$\left\| \int_t^1 t^{m/n-1} f(t)dt \right\|_{\overline{Y}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$$

Remark 5.4.2. We would also like to emphasize that Theorem 5.4.1 was used in [36] to prove Sobolev estimates as well as to give the following characterization of the optimal range space when the domain space is given:

$$(X^{\mathrm{op}})'(I^n) = \Big\{ f \in \mathcal{M}(I^n) : \|f\|_{(X^{\mathrm{op}})'(I^n)} = \big\| t^{m/n} f^{**}(t) \big\|_{\overline{Z}'(0,1)} < \infty \Big\}. \tag{5.42}$$

Thus, for instance, they recovered the classical estimates by Poornima [55] and Peetre [54]

$$W^m L^p(I^n) \hookrightarrow L^{np/(n-mp),p}(I^n), \tag{5.43}$$

and the so-called limiting or critical case of Sobolev embedding due to Hansson [33], Brezis and Wainger [15] and Maz'ya [45]

$$W^m L^{n/m}(I^n) \hookrightarrow L^{\infty, n/m; -1}(I^n). \tag{5.44}$$

Furthermore, as a new contribution, the authors showed that the range spaces $L^{np/(n-mp),p}(I^n)$ and $L^{\infty,n/m;-1}(I^n)$ in (5.43) and (5.44) respectively, are the best possible among r.i. spaces. We now see that we can further improve these results.

Theorem 5.4.3. *Let $Z(I^n)$ be an r.i. space, let $X^{\mathrm{op}}(I^n)$ be the optimal r.i. space in (5.40) and let $\mathcal{R}(X_{W^mZ,L^\infty}, L^\infty)$ be the smallest space of the form $\mathcal{R}(X, L^\infty)$ that verifies (5.41). Then,*

$$W^mZ(I^n) \hookrightarrow \mathcal{R}(X_{W^mZ,L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n).$$

Moreover, $X^{\mathrm{op}}(I^n)$ is the smallest r.i. space that verifies

$$\mathcal{R}(X_{W^mZ,L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n).$$

Proof. We fix $\mathcal{R}(X_{W^mZ,L^\infty}, L^\infty)$. Using now Theorem 3.3.8, we construct the smallest r.i. space, denoted by $Y_{\mathcal{R}(X_{W^mZ,L^\infty},L^\infty)}(I^n)$, that verifies

$$\mathcal{R}(X_{W^mZ,L^\infty}, L^\infty) \hookrightarrow Y_{\mathcal{R}(X_{W^mZ,L^\infty},L^\infty)}(I^n). \tag{5.45}$$

Then, by (5.41), it follows that

$$W^mZ(I^n) \hookrightarrow Y_{\mathcal{R}(X_{W^mZ,L^\infty},L^\infty)}(I^n),$$

and hence, our assumption on $X^{\mathrm{op}}(I^n)$ implies that

$$X^{\mathrm{op}}(I^n) \hookrightarrow Y_{\mathcal{R}(X_{W^mZ,L^\infty},L^\infty)}(I^n). \tag{5.46}$$

On the other hand, if $f \in X^{\mathrm{op}}(I^n)$, then, using a change of variables, we get

$$\begin{aligned}
\|f\|_{X^{\mathrm{op}}(I^n)} &= \sup_{\|g\|_{(X^{\mathrm{op}})'(I^n)}\le 1} \int_0^1 f^*(t)g^*(t)dt \\
&\le \sup_{\|g\|_{(X^{\mathrm{op}})'(I^n)}\le 1} \int_0^1 f^*(t)\, t^{-m/(n-1)} \sup_{t<s<1} s^{m/n} g^*(s)dt \\
&\approx \sup_{\|g\|_{(X^{\mathrm{op}})'(I^n)}\le 1} \int_0^1 f^*(t^{n'})\, t^{-(m-1)/(n-1)} \sup_{t^{n'}<s<1} s^{m/n} g^*(s)dt
\end{aligned}$$

and hence, by Hölder's inequality, we obtain that

$$\begin{aligned}
\|f\|_{X^{\mathrm{op}}(I^n)} \lesssim \sup_{\|g\|_{(X^{\mathrm{op}})'(I^n)}\le 1} & \|f^*(t^{n'})\|_{\overline{X}_{W^1Z,L^\infty}(0,1)} \\
& \times \left\| t^{\frac{-(m-1)}{(n-1)}} \sup_{t^{n'}<s<1} s^{m/n} g^*(s) \right\|_{\overline{X}'_{W^1Z,L^\infty}(0,1)}.
\end{aligned} \tag{5.47}$$

But, combining Theorem 5.2.1, Theorem 4.1.7 and (5.42), we have that

$$\begin{aligned}
\left\| t^{\frac{-(m-1)}{(n-1)}} \sup_{t^{n'}<s<1} s^{m/n} g^*(s) \right\|_{\overline{X}'_{W^1Z,L^\infty}(0,1)} &\approx \left\| t^{m/n-1} \int_0^t y^{-m/n} \sup_{y<s<1} s^{m/n} g^*(s)dy \right\|_{\overline{Z}'(0,1)} \\
&= \|t^{-m/n} \sup_{t<s<1} s^{m/n} g^*(s)\|_{\overline{X^{\mathrm{op}}}'(0,1)} \\
&\lesssim \|g\|_{(X^{\mathrm{op}})'(I^n)}.
\end{aligned}$$

Hence, by (5.47) and Theorem 3.3.8, we deduce that

$$\|f\|_{X^{\mathrm{op}}(I^n)} \lesssim \|f^*(t^{n'})\|_{\overline{X}_{W^m Z, L^\infty}(0,1)} = \|f\|_{Y_{\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)}(I^n)},$$

from which it follows that

$$Y_{\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)}(I^n) \hookrightarrow X^{\mathrm{op}}(I^n). \tag{5.48}$$

As a consequence, combining (5.46) and (5.48) yields

$$X^{\mathrm{op}}(I^n) = Y_{\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)}(I^n),$$

and so, (5.41) and (5.45) imply that

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X_{W^m Z, L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n) = Y_{\mathcal{R}(X_{W^m Z, L^\infty}, L^\infty)}(I^n),$$

as we wanted to show. □

As a consequence of Theorem 5.4.3, we shall see that the classical estimates for the standard Sobolev space $W^m L^p$ by Poornima [55] and Peetre [54] $(1 \le p < n/m)$, and by Hansson [33] and Brezis and Wainger [15] and Maz'ya [45] $(p = n/m)$ can be improved considering mixed norms on the target spaces.

Corollary 5.4.4. *Let $n, m \in \mathbb{N}$, with $1 \le m \le n-1$ and let $1 \le p < n/m$. Then,*

$$W^1 L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty) \underset{\neq}{\hookrightarrow} L^{pn/(n-mp),p}(I^n).$$

Proof. If $Z(I^n) = L^p(I^n)$, with $1 \le p < n$, then, by Remark 5.4.2 and Corollary 5.2.2, we have

$$X^{\mathrm{op}}(I^n) = L^{pn/(n-mp),p}(I^n) \quad \text{and} \quad \mathcal{R}(X_{W^m Z, L^\infty}, L^\infty) = \mathcal{R}(L^{p(n-1)/(n-mp),p}, L^\infty).$$

Therefore, using Theorem 5.4.3 and Theorem 3.3.3, the result follows immediately. □

Corollary 5.4.5. *Let $n, m \in \mathbb{N}$, with $1 \le m \le n-1$. Then,*

$$W^1 L^{n/m}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n/m;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n/m;-1}(I^n).$$

Proof. Use Corollary 5.2.3 instead of Corollary 5.2.2 and argue as in the proof of Corollary 5.4.4. □

Chapter 6

Hansson-Brézis-Wainger embedding

As we have pointed out in Chapter 5, the classical estimate (5.44) with $m = 1$ due to Hansson [33], Brezis, and Wainger [15] and Maz'ya [45] is the best possible as far as r.i. range spaces are concerned. However, Bastero, Milman, and Ruiz [5], and Malý and Pick [43] proved that it could be further strengthened by considering non-linear r.i. spaces on the target spaces. To this end, they introduced a new non-linear function class $L(\infty, n)(I^n)$ and they showed that

$$W_0^1 L^n(I^n) \hookrightarrow L(\infty, n)(I^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n). \tag{6.1}$$

In our setting, we recall that in Chapter 5 we derived new estimates, with improved target spaces of the form $\mathcal{R}(X, L^\infty)$, for standard Sobolev spaces. Thus, for instance, in Corollary 5.2.3, we concluded that the mixed norm space $\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ satisfying

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty). \tag{6.2}$$

Motivated by all these approaches, a question which thus naturally arises in this regard is to find an improvement of (6.2) within the class of non-linear spaces of the form $\mathcal{R}(X, L^\infty)$.

This chapter is organized as follows: In the first section, we present a brief review on the so-called limiting or critical case of the classical Sobolev embedding theorem.

In the second section, we introduce the function class of mixed norm spaces $\mathcal{R}(L(\infty, n), L^\infty)$ and we establish some of its basic properties (see Lemma 6.2.6). Thus, for instance, we show that it is not a linear set and, we also deduce that

$$\mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Consequently, following the same approach as in [5, 43], we get (Theorem 6.2.7)

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Moreover, taking into account that the classical Sobolev embeddings can be further improved by means of mixed norm spaces, we compare the non-linear space

$L(\infty, n)$ with the optimal mixed norm space corresponding to the domain space $L^n(I^n)$. In fact, we conclude that these spaces are not comparable (see Theorem 6.3.2 and Theorem 6.3.3).

Most of the results of this chapter are included in [22].

6.1 Review on the critical case of the classical Sobolev embedding

The classical Sobolev embedding theorem claims that if $1 \le p < n$, then

$$W_0^1 L^p(I^n) \hookrightarrow L^{p^*}(I^n), \quad p^* = pn/(n-p),$$

where $W_0^1 L^p(I^n)$ denotes the closure of $C_c^\infty(I^n)$ in $W^1 L^p(I^n)$ (see Definition **??**). Although $p^* \to \infty$ as $p \to n_-$, the space $W_0^1 L^n(I^n)$ contains unbounded functions (see [60]). Thus, in the limiting case, the only information which can be formulated in the Lebesgue spaces setting is

$$W_0^1 L^n(I^n) \hookrightarrow L^q(I^n), \quad 1 \le q < \infty. \tag{6.3}$$

Consequently, it is necessary to go outside the Lebesgue scale to find the optimal conditions satisfied by functions in $W_0^1 L^n(I^n)$.

An early result in this direction was obtained by Trudinger [63] who found a refinement of (6.3) expressed in terms of Orlicz spaces of exponential type (see (2.7)):

$$W_0^1 L^n(I^n) \hookrightarrow L_A(I^n), \qquad \text{where } A(t) = e^{t^{n'}}. \tag{6.4}$$

After the contribution of [63], an improved version of (6.4) was obtained by Hansson [33], and independently by Brézis and Wainger [15]

$$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n). \tag{6.5}$$

Moreover, Hansson [33], and later on, using a general method, Kerman and Pick [36], showed that $L^{\infty,n;-1}(I^n)$ is the optimal range space for (6.5) within the class of r.i. spaces.

On the other hand, Bastero, Milman, and Ruiz [5], and Malý and Pick [43], proved that if the requirement that the target space should be a linear space is abandoned, then a further improvement of (6.5) is still possible. To this end, they introduced a new function class defined by

$$L(\infty, n)(I^n) = \left\{ f \in \mathcal{M}(I^n) : \|f\|_{L(\infty,n)(I^n)} = \left[\int_0^1 \left(f^{**}(t) - f^*(t)\right)^n \frac{dt}{t} \right]^{1/n} < \infty \right\},$$

and they established some of its basic properties and relations with known function spaces. Thus, for instance, they showed that it is not a linear set and

$$L(\infty, n)(I^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n).$$

As a consequence, using a weak version of the Sobolev-Gagliardo-Nirenberg embedding together with a truncation argument due to Mazýa, they concluded that the chain of embeddings (6.1) holds.

6.2 Non-linear mixed norm spaces

Taking into account the results given in [5, 43], we are now interested in finding an improvement of (6.2) within the class of non-linear spaces of the form $\mathcal{R}(X, L^\infty)$. For this, we shall see that similar arguments to those used in [5, 43] for the sharp version (6.1), can be carried with mixed norm spaces as well. To be more precise, we shall establish a weak version of the classical estimate due to Gagliardo [29] and Nirenberg [50] and then we shall use a truncation argument due to Mazýa.

To this end, we first need to recall the following results [1]:

Lemma 6.2.1. *Let $y \in \mathbb{R}^{n-1}$ and let $G \subset \mathbb{R}^{n-1}$ be a set, with positive finite measure. Then,*

$$\int_G |x-y|^{(-n+1)/n'} dx \lesssim |G|^{1/n}.$$

Lemma 6.2.2. *Let $u \in W_0^1 L^1(I^n)$, let f be a function satisfying a Lipschitz condition on $\mathbb{R}$ and $g(x) = f(|u(x)|)$. Then, for any $i \in \{1, \dots, n\}$,*

$$\partial_{x_i} g(x) = \operatorname{sgn}(u(x)) f'(|u(x)|) \partial_{x_i} f(x)\,, \text{ a.e. } x \in I^n.$$

Proposition 6.2.3. *Let $k \in \{1, \dots, n\}$ and let $f \in W_0^1 L^n(I^n)$. Then,*

$$\sup_{s>0} s \Big| \Big\{ y \in I^{n-1} : \psi_k(f, L^\infty)(y) > s \Big\} \Big|^{1/n'} \lesssim \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy.$$

Proof. We fix any $f \in C_c^\infty(I^n)$. Then, extended by zero outside I^n, one can consider $f \in C_c^\infty(\mathbb{R}^n)$. Using [60, Chapter V, pag. 125], we have

$$|f(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} |\nabla f(y)| |x-y|^{-n+1} dy, \quad x \in \mathbb{R}^n.$$

Now, we set $\alpha_k = |\widehat{x_k} - \widehat{y_k}|$, for any $\widehat{x_k}, \widehat{y_k} \in \mathbb{R}^{n-1}$ fixed. Then, using Young's inequality, we get

$$\begin{aligned}
\Big\| \int_{\mathbb{R}} \frac{|\nabla f(\widehat{y_k}, y_k)|}{(|\cdot - y_k| + \alpha_k)^{n-1}} dy_k \Big\|_{L^\infty(\mathbb{R})} &\leq \psi_k(|\nabla f|, L^n)(\widehat{y_k}) \Big(\int_{\mathbb{R}} (|y| + \alpha_k)^{-n} dy \Big)^{1/n'} \\
&\approx \psi_k(|\nabla f|, L^n)(\widehat{y_k}) \alpha_k^{-(n-1)/n'} \\
&= \psi_k(|\nabla f|, L^n)(\widehat{y_k}) |\widehat{x_k} - \widehat{y_k}|^{-(n-1)/n'}.
\end{aligned}$$

So, for any $\widehat{x_k} \in \mathbb{R}^{n-1}$, we obtain

$$\begin{aligned}
\psi_k(f, L^\infty)(\widehat{x_k}) &\lesssim \int_{\mathbb{R}^{n-1}} \Big\| \int_{\mathbb{R}} \frac{|\nabla f(\widehat{y_k}, y_k)|}{(|\cdot - y_k| + \alpha_k)^{n-1}} dy_k \Big\|_{L^\infty(\mathbb{R})} d\widehat{x_k} \\
&\lesssim \int_{\mathbb{R}^{n-1}} \psi_k(|\nabla f|, L^n)(\widehat{x_k}) |\widehat{x_k} - \widehat{y_k}|^{-(n+1)/n'} d\widehat{y_k}. \qquad (6.6)
\end{aligned}$$

Now, we fix any $s > 0$ and we denote

$$G = \Big\{ \widehat{x_k} \in \mathbb{R}^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) > s \Big\}.$$

Then, combining (6.6) with Chebyshev's inequality and Fubini's Theorem, we obtain

$$s|G| \lesssim \int_{\mathbb{R}^{n-1}} \psi_k(|\nabla f|, L^n)(\widehat{y}_k) \int_G |\widehat{x}_k - \widehat{y}_k|^{-(n+1)/n'} d\widehat{x}_k d\widehat{y}_k.$$

But, by Lemma 6.2.1, we have

$$\int_G |\widehat{x}_k - \widehat{y}_k|^{-(n+1)/n'} d\widehat{x}_k \lesssim |G|^{1/n},$$

and hence, we get

$$s|G|^{1/n'} \lesssim \int_{\mathbb{R}^{n-1}} \psi_k(|\nabla f|, L^n)(\widehat{y}_k) d\widehat{y}_k.$$

As a consequence, for any $f \in C_c^\infty(I^n)$,

$$\|f\|_{\mathcal{R}_k(L^{n',\infty}, L^\infty)} \lesssim \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy. \tag{6.7}$$

Now, we shall extend the validity of (6.7) to all functions in $W_0^1 L^n(I^n)$. In fact, given $f \in W_0^1 L^n(I^n)$, we select $f_j \in C_c^\infty(I^n)$ such that $f_j(x) \to f(x)$ a.e. and $f_j \to f$ in $W^1 L^n(I^n)$. Using now Fatou's lemma and (6.7), we get

$$\begin{aligned}
\|f\|_{\mathcal{R}_k(L^{n',\infty}, L^\infty)} &\leq \liminf_j \|f_j\|_{\mathcal{R}_k(L^{n',\infty}, L^\infty)} \lesssim \liminf_j \int_{I^{n-1}} \psi_k(|\nabla f_j|, L^n)(y) dy \\
&\leq \liminf_j \int_{I^{n-1}} \psi_k(|\nabla (f_j - f)|, L^n)(y) dy + \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy.
\end{aligned}$$

So, Hölder's inequality implies that

$$\begin{aligned}
\|f\|_{\mathcal{R}_k(L^{n',\infty}, L^\infty)} &\lesssim \liminf_j \|\nabla (f_j - f)\|_{L^n(I^n)} + \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy \\
&\leq \liminf_j \|f - f_j\|_{W^1 L^n(I^n)} + \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy \\
&= \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy.
\end{aligned}$$

Thus, the proof is complete. □

As a consequence of Proposition 6.2.3, we have the following result.

Corollary 6.2.4. *Let $k \in \{1, \dots, n\}$ and let $f \in W_0^1 L^n(I^n)$. Then,*

$$s \Big| \Big\{ y \in I^{n-1} : \psi_k(f, L^\infty)(y) \geq s \Big\} \Big|^{1/n'} \lesssim \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y) dy, \quad s > 0.$$

Proof. Let $s > 0$. Given any $0 < \varepsilon < s$, we define

$$A_j = \Big\{ x \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x}_k) > s - \varepsilon/j \Big\}, \quad j \in \mathbb{N}.$$

The sets A_j form a decreasing sequence, as j increases and $|A_1| < \infty$. Consequently, using Proposition 6.2.3, we obtain

$$\begin{aligned} s\Big|\Big\{y \in I^{n-1} : \psi_k(f, L^\infty)(y) \geq s\Big\}\Big|^{1/n'} &= \lim_j (s - \varepsilon/j)|\cap_j A_j|^{1/n'} \\ &= \lim_j (s - \varepsilon/j)|A_j|^{1/n'} \\ &\leq \|f\|_{\mathcal{R}_k(L^{n',\infty}, L^\infty)} \\ &\lesssim \int_{I^{n-1}} \psi_k(|\nabla f|, L^n)(y)dy. \end{aligned}$$

This completes the proof. □

Theorem 6.2.5. *Let $k \in \{1, \dots, n\}$ and let $0 < t_1 < t_2 < \infty$. Then, for any $f \in W_0^1 L^n(I^n)$,*

$$(t_2 - t_1)|D_{t_2}|^{1/n'} \lesssim \lambda^{1/n'}_{\psi_k(f, L^\infty)}(t_1)\bigg(\int_{\{x \in \mathbb{R}^n : t_1 < |f(x)| \leq t_2\}} |\nabla f(x)|^n dx\bigg)^{1/n},$$

where

$$D_{t_2} = \Big\{\widehat{x_k} \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) \geq t_2\Big\}.$$

Proof. Given any $f \in W_0^1 L^n(I^n)$, we define

$$F(x) = \begin{cases} t_2 - t_1, & \text{if } x \in A_{t_2}, \\ |f(x)| - t_1, & \text{if } x \in B_{t_1,t_2}, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$A_{t_2} = \Big\{x \in I^n : |f(x)| > t_2\Big\} \quad \text{and} \quad B_{t_1,t_2} = \Big\{x \in I^n : t_1 < |f(x)| \leq t_2\Big\}.$$

Using now Lemma 6.2.2, we have

$$|\nabla F(x)| = |\nabla f(x)|\chi_{B_{t_1,t_2}}(x), \quad \text{for a.e. } x \in I^n,$$

and so, our assumption on f implies that $F \in W_0^1 L^n(I^n)$. As a consequence, we may apply Corollary 6.2.4, with f replaced by F, and get

$$\begin{aligned} t\Big|\Big\{\widehat{x_k} \in I^{n-1} : \psi_k(F, L^\infty)(\widehat{x_k}) \geq t\Big\}\Big|^{1/n'} &\lesssim \int_{I^{n-1}} \psi_k(|\nabla F|, L^n)(\widehat{x_k})d\widehat{x_k} \\ &= \int_{\Pi_k^* B_{t_1,t_2}} \psi_k(|\nabla f|, L^n)(\widehat{x_k})d\widehat{x_k}. \end{aligned} \tag{6.8}$$

Let us estimate the left-hand side of (6.8). First of all, let us compute $\psi_k(F, L^\infty)$. We observe that if $\widehat{x_k} \in I^{n-1} \setminus (\Pi_k^* A_{t_2} \cup \Pi_k^* B_{t_1,t_2})$, then

$$\lambda_{F_{\widehat{x_k}}}(s) = 0, \quad \text{if } \widehat{x_k} \in I^{n-1} \setminus (\Pi_k^* A_{t_2} \cup \Pi_k^* B_{t_1,t_2}).$$

Now, if $\widehat{x_k} \in \Pi_k^* A_{t_2} \setminus \Pi_k^* B_{t_1,t_2}$, then

$$\lambda_{F_{\widehat{x_k}}}(s) = |\{\widehat{x_k} \in A_{t_2}(\widehat{x_k}) : F_{\widehat{x_k}}(x_k) > s\}| = \begin{cases} \lambda_{f_{\widehat{x_k}}}(s), & 0 \le s < t_2 - t_1, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence,

$$\psi_k(F, L^\infty)(\widehat{x_k}) = t_2 - t_1, \quad \widehat{x_k} \in \Pi_k^* A_{t_2} \setminus \Pi_k^* B_{t_1,t_2}. \tag{6.9}$$

Let us suppose that $\widehat{x_k} \in \Pi_k^* B_{t_1 t_2} \setminus \Pi_k^* A_{t_2}$. Then,

$$\lambda_{F_{\widehat{x_k}}}(s) = |\{\widehat{x_k} \in B_{t_1,t_2}(\widehat{x_k}) : F_{\widehat{x_k}}(x_k) > s\}| = \begin{cases} \lambda_{f_{\widehat{x_k}}}(s + t_1), & 0 \le s < t_2 - t_1, \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\psi_k(F, L^\infty)(\widehat{x_k}) = \psi_k(f, L^\infty)(\widehat{x_k}) - t_1, \quad \widehat{x_k} \in \Pi_k^* B_{t_1 t_2} \setminus \Pi_k^* A_{t_2}. \tag{6.10}$$

Finally, if $\widehat{x_k} \in \Pi^* A_{t_2} \cap \Pi^* B_{t_1,t_2}$, then

$$\begin{aligned} \lambda_{F_{\widehat{x_k}}}(s) &= |\{\widehat{x_k} \in A_{t_2}(\widehat{x_k}) : F_{\widehat{x_k}}(x_k) > s\}| + |\{\widehat{x_k} \in B_{t_1 t_2}(\widehat{x_k}) : F_{\widehat{x_k}}(x_k) > s\}| \\ &= \begin{cases} \lambda_{f_{\widehat{x_k}}}(s + t_1), & 0 \le s < t_2 - t_1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence,

$$\psi_k(F, L^\infty)(\widehat{x_k}) = t_2 - t_1, \quad \widehat{x_k} \in \Pi^* A_{t_2} \cap \Pi^* B_{t_1,t_2}. \tag{6.11}$$

Thus, combining (6.9), (6.10) and (6.11), we get

$$\psi_k(f, L^\infty)(\widehat{x_k}) = \begin{cases} t_2 - t_1, & \widehat{x_k} \in \Pi_k^* A_{t_2}, \\ \psi_k(f, L^\infty)(\widehat{x_k}) - t_1, & \widehat{x_k} \in \Pi_k^* B_{t_1,t_2} \setminus \Pi_k^* A_{t_2}, \\ 0, & \text{otherwise.} \end{cases}$$

But, using Lemma 3.1.3, we have

$$\Pi_k^* A_{t_2} = \{\widehat{x_k} \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) > t_2\},$$

and

$$\Pi_k^* B_{t_1,t_2} = \{\widehat{x_k} \in I^{n-1} : t_1 < \psi_k(f, L^\infty)(\widehat{x_k}) \le t_2\},$$

and so

$$\psi_k(f, L^\infty)(\widehat{x_k}) = \begin{cases} t_2 - t_1, & \widehat{x_k} \in \widetilde{A}_{t_2}, \\ \psi_k(f, L^\infty)(\widehat{x_k}) - t_1, & \widehat{x_k} \in \widetilde{B}_{t_1,t_2}, \\ 0, & \text{otherwise,} \end{cases}$$

with

$$\widetilde{A}_{t_2} = \Big\{ \widehat{x_k} \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) > t_2 \Big\},$$

and

$$\widetilde{B}_{t_1,t_2} = \Big\{ \widehat{x_k} \in I^{n-1} : t_1 < \psi_k(f, L^\infty)(\widehat{x_k}) \leq t_2 \Big\}.$$

Therefore, applying (6.8) to $t = t_2 - t_1$, we arrive at

$$(t_2 - t_1)|D_{t_2}|^{1/n'} \lesssim \int_{\Pi_k^* B_{t_1,t_2}} \psi_k(|\nabla f|, L^n)(y)dy,$$

where

$$D_{t_2} = \Big\{ \widehat{x_k} \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) \geq t_2 \Big\}.$$

Using now Hölder's inequality, we have

$$\begin{aligned}\int_{\Pi_k^* B_{t_1,t_2}} \psi_k(|\nabla f|, L^n)(y)dy &= \int_{\Pi_k^* B_{t_1,t_2}} \left(\int_{B_{t_1,t_2}(\widehat{x_k})} |\nabla f(\widehat{x_k}, x_k)|^n dx_k \right)^{1/n} d\widehat{x_k} \\ &\leq |\Pi_k^* B_{t_1,t_2}|^{1/n'} \left(\int_{B_{t_1,t_2}} |\nabla f(x)|^n dx \right)^{1/n}.\end{aligned}$$

But,

$$\Pi_k^* B_{t_1,t_2} \subseteq \Big\{ \widehat{x_k} \in I^{n-1} : \psi_k(f, L^\infty)(\widehat{x_k}) \geq t_1 \Big\},$$

and so

$$\int_{\Pi_k^* B_{t_1,t_2}} \psi_k(|\nabla f|, L^n)(y)dy \leq \lambda^{1/n'}_{\psi_k(f,L^\infty)}(t_1) \left(\int_{B_{t_1,t_2}} |\nabla f(x)|^n dx \right)^{1/n}. \tag{6.12}$$

As a consequence, using (6.8) and (6.12), we obtain

$$(t_2 - t_1)|D_{t_2}|^{1/n'} \lesssim \lambda^{1/n'}_{\psi_k(f,L^\infty)}(t_1) \left(\int_{\{x \in \mathbb{R}^n : t_1 < |f(x)| \leq t_2\}} |\nabla f(x)|^n dx \right)^{1/n}.$$

This completes the proof. □

We are now in a position to give an non-trivial improvement of

$$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n),$$

in the context of non-linear mixed norm spaces. Before that, let us mention some basic properties of the function space $\mathcal{R}(L(\infty,n), L^\infty)$.

Lemma 6.2.6. *Let $n \in \mathbb{N}$, with $n \geq 2$. Then,* *(i) For any $f \in \mathcal{R}(L(\infty,n),L^\infty)$, it holds that*

$$\|f\|_{\mathcal{R}(L(\infty,n),L^\infty)} \approx \sum_{k=1}^{n} \bigg(\int_0^1 s^{-1} \Big[\psi_k^*(f,L^\infty)(s/2) - \psi_k^*(f,L^\infty)(s) \Big]^n ds \bigg)^{1/n};$$

(ii) $L^\infty(I^n) \underset{\neq}{\hookrightarrow} \mathcal{R}(L(\infty,n),L^\infty)$;

(iii) $\mathcal{R}(L(\infty,n),L^\infty)$ *is not a linear set;*

(iv) $\mathcal{R}(L(\infty,n),L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1},L^\infty)$.

Proof. To prove (i), (ii) and (iv), we have to argue as in [43, 5]. We only need to use $\psi_k(f,L^\infty)$ instead of f^*. Thus, it only remains to see (iii). To this end, we define

$$f(x) = \begin{cases} \big[\log(e\, r^{n-1}|x|^{-n+1})\big]^\alpha, & \text{if } x \in B_n(0,r), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \sum_{i=1}^{\infty} i\, \chi_{A_i}(x), \quad \text{where } A_i = \Big\{ x \in I^n : i-1 < f(x) \le i \Big\}.$$

We set

$$h(x) = g(x) + 1.$$

Let us see that $h \notin \mathcal{R}(L(\infty,n),L^\infty)$. In fact, given any $k \in \{1,\dots,n\}$, we fix $\widehat{x_k} \in I^{n-1}$. We set

$$\alpha_k = \min \Big\{ \ell \in \mathbb{Z} : \ell \ge \psi(f,L^\infty)(\widehat{x_k}) \Big\}.$$

Then, it holds that

$$\begin{aligned} \lambda_{h_{\widehat{x_k}}}(\alpha_k) &= \Big| \bigcup_{i=\alpha_k+1}^{\infty} A_i(\widehat{x_k}) \Big| = \Big| \Big\{ y \in I : f_{\widehat{x_k}}(y) > \alpha_k \Big\} \Big| \\ &\le \Big| \Big\{ y \in I : f_{\widehat{x_k}}(y) > \psi(f,L^\infty)(\widehat{x_k}) \Big\} \Big| = 0. \end{aligned}$$

Thus, $\lambda_{h_{\widehat{x_k}}}(s) = 0$, for any $s \ge \alpha_k$. On the other hand, we have

$$\alpha_k - 1 < \psi(f,L^\infty)(\widehat{x_k}).$$

Thus,

$$\lambda_{h_{\widehat{x_k}}}(\alpha_k - 1) = \Big| \bigcup_{i=\alpha_k}^{\infty} A_i(\widehat{x_k}) \Big| = \Big| \Big\{ y \in I : f_{\widehat{x_k}}(y) > \alpha_k - 1 \Big\} \Big| > 0.$$

As a consequence, $\lambda_{h_{\widehat{x_k}}}(s) > 0$, for any $s \geq \alpha_k - 1$. Hence,

$$\psi_k(h, L^\infty)(\widehat{x_k}) = \alpha_k = \min\Big\{\ell \in \mathbb{Z} : \ell \geq \psi(f, L^\infty)(\widehat{x_k})\Big\} = \sum_{i=i}^{\infty} i\, \chi_{A_i}(\widehat{x_k}),$$

where

$$A_i = \Big\{\widehat{x_k} \in I^{n-1} : i - 1 < \psi_k(f, L^\infty)(\widehat{x_k}) \leq i\Big\},$$

and

$$\psi_k(f, L^\infty)(\widehat{x_k}) = \begin{cases} [\log(e\, r^{n-1}|\widehat{x_k}|^{-n+1})]^\alpha, & \text{if } \widehat{x_k} \in B_{n-1}(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

That is, we have that $\psi_k(h, L^\infty)$ is an integer-valued unbounded function. Therefore, Theorem 6.3.1 implies that

$$h \notin \mathcal{R}_k(L(\infty, n), L^\infty),$$

and so

$$h \notin \mathcal{R}(L(\infty, n), L^\infty). \tag{6.13}$$

On the other hand, if $v(x) = h(x) - f(x)$, then $v \leq 1$ and so, using property (ii), we have $v \in \mathcal{R}(L(\infty, n), L^\infty)$. Moreover, $f \in \mathcal{R}(L(\infty, n), L^\infty)$. Hence, taking into account (6.13), we conclude that $\mathcal{R}(L(\infty, n), L^\infty)$ is not a linear set, as we wanted to show. □

Theorem 6.2.7. $W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty)$.

Proof. Let $k \in \{1, \ldots, n\}$. Given any $f \in W_0^1 L^n(I^n)$, we define

$$t_j = 2^{1-j}, \quad \alpha_j = \psi_k^*(f, L^\infty)(t_j), \quad j \in \mathbb{N}.$$

Using now Lemma 6.2.6, we get

$$\|f\|^n_{\mathcal{R}_k(L(\infty,n),L^\infty)} \approx \int_0^1 \Big[\psi_k^*(f, L^\infty)(s/2) - \psi_k^*(f, L^\infty)(s)\Big]^n \frac{ds}{s},$$

and so, discretizing the integral we can write

$$\begin{aligned} \|f\|^n_{\mathcal{R}_k(L(\infty,n),L^\infty)} &\approx \sum_{j=1}^{\infty} \Big[\psi_k^*(f, L^\infty)(t_j/2) - \psi_k^*(f, L^\infty)(t_j)\Big]^n \\ &= \sum_{j=1}^{\infty} (\alpha_{j+1} - \alpha_j)^n. \end{aligned} \tag{6.14}$$

Now, we fix any $j \in \{1, \ldots, n\}$. Then, applying Theorem 6.2.5 to $t_1 = \alpha_j$ and $t_2 = \alpha_{j+1}$, we arrive at

$$t_{j+1}^{1/n'}(\alpha_{j+1} - \alpha_j) \lesssim t_j^{1/n'} \bigg(\int_{\{x \in \mathbb{R}^n : \alpha_j < |f(x)| \leq \alpha_{j+1}\}} |\nabla f(x)|^n dx\bigg)^{1/n}.$$

But, $t_j = 2t_{j+1}$, and so we get

$$(\alpha_{j+1} - \alpha_j)^n \lesssim \int_{\{x\in\mathbb{R}^n:\alpha_j<|f(x)|\le|\alpha_{j+1}\}} |\nabla f(x)|^n dx.$$

As a consequence, using (6.14), we obtain

$$\|f\|^n_{\mathcal{R}_k(L(\infty,n),L^\infty)} \lesssim \sum_{j=1}^{\infty} \int_{\{x\in I^n:\alpha_j<|f(x)|\le\alpha_{j+1}\}} |\nabla f(x)|^n dx \le \int_{I^n} |\nabla f(x)|^n dx.$$

Therefore, we conclude that

$$\|f\|_{\mathcal{R}(L(\infty,n),L^\infty)} \lesssim \|f\|_{W^1_0 L^n(I^n)},$$

as we wanted to prove. □

6.3 Relation with the optimal mixed norm space

As we have pointed out in Chapter 5, it is still possible to further improve the classical Sobolev embeddings (6.5) by means of mixed norm spaces. Precisely, we proved that the following chain of embeddings holds (for more details see Corollary 5.4.5):

$$W^1_0 L^n(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n). \tag{6.15}$$

Now, taking into account (6.1), we focus on the relation between the mixed norm space $\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ and $L(\infty, n)$. In fact, we shall show in Theorem 6.3.2 and Theorem 6.3.3 that these spaces are not comparable. The following result will be fundamental for our purposes [5, 43]:

Theorem 6.3.1. *Let $n \in \mathbb{N}$, with $n \ge 2$. Then,*

(i) $L^\infty(I^n) \underset{\neq}{\hookrightarrow} L(\infty, n)(I^n)$.

(ii) Each integer-valued function belongs to $L(\infty, n)(I^n)$ if and only if it is bounded.

(iii) $L(\infty, n)$ *is not a linear set.*

(iv) $L(\infty, n)(I^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n)$.

(v) For any $f \in L(\infty, n)(I^n)$, it holds that

$$\|f\|_{L(\infty,n)(I^n)} \approx \left(\int_0^1 \left[f^*(t/2) - f^*(t)\right]^n \frac{dt}{t} \right)^{1/n}.$$

Theorem 6.3.2. $\mathcal{R}(L^{\infty,n;-1}, L^\infty) \not\hookrightarrow L(\infty, n)$.

Proof. We suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. We fix $0 < \alpha < 1/n'$ and we define

$$g(x) = \begin{cases} \left\lfloor \left[\log \left(r^{n-1} e / |x|^{(n-1)} \right) \right]^\alpha \right\rfloor, & \text{if } x \in B_n(0, r), \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Then, g is an unbounded integer valued function and so, Theorem 6.3.1 tells us that $g \notin L(\infty, n)(I^n)$. On the other hand, for any $k \in \{1, \ldots, n\}$, it holds that

$$\psi_k(g, L^\infty)(\widehat{x_k}) \leq \left[\log(r^{n-1} e / |x|^{(n-1)}) \right]^\alpha, \quad \widehat{x_k} \in B_{n-1}(0, r),$$

and $\psi_k(g, L^\infty)(\widehat{x_k}) = 0$ otherwise. Consequently, we have that

$$\|g\|^n_{\mathcal{R}_k(L^{\infty,n;-1}, L^\infty)} \leq \int_0^{\omega_{n-1} r^{n-1}} t^{-1} [\log(r^{n-1} \omega_{n-1} e / t)]^{(\alpha-1)n} dt.$$

Hence, our assumption on α implies that $g \in \mathcal{R}_k(L^{\infty,n;-1}, L^\infty)$, $k \in \{1, \ldots, n\}$. Therefore, we have that $g \in \mathcal{R}(L^{\infty,n;-1}, L^\infty)$, from which the result follows. □

Theorem 6.3.3. *Let $X(I^{n-1})$ be an r.i. space. Then,*

$$L(\infty, n)(I^n) \not\hookrightarrow \mathcal{R}(X, L^\infty).$$

Proof. As before, we suppose that $I = (-a, b)$, $a, b \in \mathbb{R}^+$, and $0 < r < \min(a, b)$. Given any $0 < \alpha < 1/n'$, we define

$$f(x) = \begin{cases} \left[\log(2^{-1} r \, |x_n|^{-1}) \right]^\alpha, & \text{if } (\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times (-r/2, r/2), \\ 0, & \text{otherwise.} \end{cases}$$

Next, we observe that $f \notin \mathcal{R}_n(X, L^\infty)$, and so, by definition, $f \notin \mathcal{R}(X, L^\infty)$. On the other hand, defining

$$g(t) = \begin{cases} [\log(t^{-1})]^\alpha, & t \in (0, 1) \\ 0, & \text{otherwise,} \end{cases}$$

we have that, for any $s \geq 0$,

$$\begin{aligned} \lambda_f(s) &= \left| \left\{ (\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times (-r/2, r/2) : g(2 r^{-1} |x_n|) > s \right\} \right| \\ &= \left| B_{n-1}(0, r) \times \left\{ x_n \in (-r/2, r/2) : g(2 r^{-1} |x_n|) > s \right\} \right| \\ &= \omega_{n-1} r^n \lambda_g(s), \end{aligned}$$

and so, we obtain

$$f^*(t) = \begin{cases} \left[\log(\omega_{n-1} r^n t^{-1}) \right]^\alpha, & 0 < t < \omega_{n-1} r^n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, it holds that

$$f^*(t/2) - f^*(t) \approx \int_{t/2}^{t} y^{-1}\big[\log(\omega_{n-1}r^n y^{-1})\big]^{\alpha-1} dy \leq \log(2)\big[\log(\omega_{n-1}r^n t^{-1})\big]^{\alpha-1},$$

and hence, using Theorem 6.3.1, we get

$$\begin{aligned}\|f\|^n_{L(\infty,n)(I^n)} &\approx \int_0^{\omega_{n-1}r^n} t^{-1}\big[f^*(t/2) - f^*(t)\big]^n dt + \int_{\omega_{n-1}r^n}^{1} t^{-1}[f^*(t/2)]^n dt \\ &\lesssim \int_0^{\omega_{n-1}r^n} t^{-1}\big[\log(\omega_{n-1}r^n t^{-1})\big]^{(\alpha-1)n} dt + [\log(2)]^{n\alpha}\big[\log(\omega_n^{-1}r^{-n})\big].\end{aligned}$$

By hypothesis, we have that $0 < \alpha < 1/n'$, and finally we deduce that f belongs to $L(\infty, n)(I^n)$. Thus, the proof is complete. □

Remark 6.3.4. Note that if $f \in L(\infty, n)(I^n)$ then, by Theorem (6.3.3), we have that $f \notin \mathcal{R}(L^{\infty,n;-1}, L^\infty)$. Hence, using now Lemma 6.2.6, we conclude that f does not belong to $\mathcal{R}(L(\infty, n), L^\infty)$. As a consequence, we get

$$L(\infty, n)(I^n) \not\hookrightarrow \mathcal{R}(L(\infty, n), L^\infty).$$

Chapter 7

Sobolev embedding in $\mathcal{R}(X, L^1)$

In Chapter 5, we established an extension of the Sobolev-type estimates due to Gagliardo [29] and Nirenberg [50] to the case where Lebesgue spaces are replaced by any r.i. spaces. In particular, we concentrated on seeking the optimal domains and the optimal ranges for these embeddings between r.i. spaces and mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$.

In this chapter, we consider an analogous problem for mixed norm spaces of the form $\mathcal{R}(X, L^1)$. Namely, our aim is to study the following Sobolev-type estimate:

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1). \tag{7.1}$$

In particular, we are interested in the following problems:

(i) We would like to find the smallest space of the form $\mathcal{R}(X, L^1)$ in (7.1), for a given r.i. $Z(I^n)$.

(ii) On the other hand, given a fixed range space $\mathcal{R}(X, L^1)$, we would like to provide a characterization of the largest r.i. domain space $Z(I^n)$ satisfying (7.1).

This chapter is organized as follows: In the first section, we obtain necessary and sufficient conditions for (7.1) to hold (see Theorem 7.1.6). For this, we apply a method, which relies on studying embeddings between Sobolev space built upon mixed norm spaces. Specifically, we prove that the Sobolev embeddings of the form

$$W^1 \mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1) \tag{7.2}$$

hold if and only if the following Sobolev type estimates are fulfilled (see Theorem 7.1.3):

$$W^1 Z(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Such equivalence is then crucial in the forthcoming sections where an exhaustive study of the optimal domain-range problem for (7.1) and (7.2) is given.

To be more precise, in the second section we answer the question of the best possible target of the form $\mathcal{R}(X, L^1)$ for (7.1) when the domain space is fixed (see Theorem 7.2.1). Furthermore, given $\mathcal{R}(Z, L^1)$, we characterize the optimal, i.e. the smallest, range space $\mathcal{R}(X, L^1)$ for which (7.2) holds (see Theorem 7.2.2).

In the third section, we solve the converse problem. Namely, given a mixed norm space $\mathcal{R}(X, L^1)$, find its optimal domain r.i. partner $Z(I^n)$ such that (7.1) holds and $Z(I^n)$ is the largest r.i. possible (see Theorem 7.3.2).

As we have pointed out in Chapter 3, the mixed norm spaces of the forms $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$ are function spaces, having non-trivial intersection (for example, $L^\infty(I^n)$ is contained in both). Motivated by this fact, we now consider $Z(I^n) = L^p(I^n)$ and we compare its optimal ranges of the form $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$. Consequently, we conclude that there are examples, for instance the standard Sobolev spaces W^1L^p, showing that, in fact, such mixed norm spaces are not comparable.

As we have pointed out in the Introduction, the authors in [36] determined the smallest r.i. range space $X^{\mathrm{op}}(I^n)$ satisfying

$$W^1L^p(I^n) \hookrightarrow X^{\mathrm{op}}(I^n), \quad 1 \le p \le \infty.$$

Motivated by this problem, in the fourth section we compare the latter space with the optimal mixed norm space of the form $\mathcal{R}(X, L^1)$ corresponding to the domain space $L^p(I^n)$. In particular, we deduce that such spaces in general are not comparable.

Most of the results of this chapter are included in [25].

7.1 Necessary and sufficient conditions

Let $n, m \in \mathbb{N}$, with $n \ge 2$. Now, our analysis focuses on finding necessary and sufficient conditions on $X(I^{n-1})$ and $Z(I^n)$ under which we have the embedding (7.1). To this end, we shall apply a method, which relies on a characterization of embeddings of the form

$$W^m\mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1), \quad m \in \mathbb{N}. \tag{7.3}$$

Let us start with two technical results.

Lemma 7.1.1. *Let $n, m \in \mathbb{N}$, with $n \ge 2$ and $m \ge n-1$. Then, for any pair of r.i. spaces $X(I^{n-1})$ and $Z(I^n)$,*

$$W^mZ(I^n) \hookrightarrow \mathcal{R}(X, L^1).$$

In particular, it holds that

$$W^{n-1}L^1(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Proof. Let $k \in \{1, \dots, n\}$. Observe that if $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{k-1}} \int_{-\infty}^{x_{k+1}} \cdots \int_{-\infty}^{x_n} \partial^{n-1} f(\widehat{y}_k, x_k)\, d\widehat{y}_k.$$

Consequently, we get

$$\psi_k(f, L^1)(\widehat{x}_k) = \int_{\mathbb{R}} |f(\widehat{x}_k, x_k)|\, dx_k \le \int_{\mathbb{R}^n} |\partial^{n-1} f(x)|\, dx, \quad \widehat{x}_k \in \mathbb{R}^{n-1},$$

and hence, we have that

$$\|f\|_{\mathcal{R}_k(L^\infty, L^1)} = \|\psi_k(f, L^1)\|_{L^\infty(\mathbb{R}^{n-1})} \leq \|f\|_{W^{n-1}L^1(\mathbb{R}^n)}, \quad f \in C_c^\infty(\mathbb{R}^n).$$

Thus, applying the same technique as in the proof of Lemma 5.1.1, we deduce that

$$W^{n-1}L^1(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Therefore, using (2.1) and taking into account the fact that

$$W^m L^1(I^n) \hookrightarrow W^{n-1}L^1(I^n), \quad m \geq n-1,$$

we conclude that

$$W^m Z(I^n) \hookrightarrow \mathcal{R}(X, L^1), \quad m \geq n-1,$$

for every r.i. spaces $Z(I^n)$ and $X(I^{n-1})$ and the result follows. □

Lemma 7.1.2. *Let $n, m \in \mathbb{N}$, with $n \geq 2$ and $m \geq n-1$. Then, for any pair of r.i. spaces $X(I^{n-1})$ and $Z(I^{n-1})$,*

$$W^m \mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1).$$

Proof. By Lemma 7.1.1, we have

$$W^m L^1(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1), \quad m \geq n-1,$$

and so, using (2.1) together with Lemma 3.2.2, we get

$$W^m \mathcal{R}(Z, L^1) \hookrightarrow W^m L^1(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1) \hookrightarrow \mathcal{R}(X, L^1), \quad m \geq n,$$

from which the result follows. □

As we have seen in Lemma 7.1.1 and Lemma 7.1.2, the Sobolev embeddings of the form (7.1) and (7.3), respectively, are uninteresting in the case when $n \geq 2$ and $m \geq n-1$, since they hold for any couple of r.i. spaces. Consequently, from now on, we shall suppose that $n > 2$ and $1 \leq m < n-1$.

Theorem 7.1.3. *Let $n \geq 3$. Let $X(I^{n-1})$ and $Z(I^{n-1})$ be r.i. spaces. Then, the following statements are equivalent:*

(i) $W^1 Z(I^{n-1}) \hookrightarrow X(I^{n-1})$;

(ii) $W^1 \mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1)$;

(iii) $$\left\| \int_t^1 s^{1/(n-1)-1} f(s)ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$$

Furthermore, if $1 \leq m \leq n-2$ then

$$W^m \mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1) \Longrightarrow W^m Z(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Proof. $(i) \Rightarrow (ii)$ Let $f \in W^1\mathcal{R}(Z, L^1)$ and $k \in \{1, \dots, n\}$. We fix any $i \in \{1, \dots, n\}$, with $i \neq k$.Then, for any $\theta \in C^\infty(I^{n-1})$, we have

$$\begin{aligned}\int_{I^{n-1}} \psi_k(f, L^1)(\widehat{x_k})\partial_{x_i}\theta(\widehat{x_k})d\widehat{x_k} &= \int_I \left(\int_{I^{n-1}} |f(\widehat{x_k}, x_k)|\partial_{x_i}\theta(\widehat{x_k})d\widehat{x_k}\right)dx_k \\ &= -\int_I \left(\int_{I^{n-1}} (\operatorname{sgn}[f])\partial_{x_i} f(\widehat{x_k}, x_k)\theta(\widehat{x_k})d\widehat{x_k}\right)dx_k \\ &= -\int_{I^{n-1}} \left(\int_I (\operatorname{sgn}[f])\partial_{x_i} f(\widehat{x_k}, x_k)dx_k\right)\theta(\widehat{x_k})d\widehat{x_k}.\end{aligned}$$

That is, it holds that

$$\partial_{x_i}\psi_k(f, L^1)(\widehat{x_k}) = \int_I (\operatorname{sgn}[f])\partial_{x_i} f(\widehat{x_k}, x_k)dx_k \quad \text{a.e. } \widehat{x_k} \in I^{n-1},$$

and so,

$$|\partial_{x_i}\psi_k(f, L^1)(\widehat{x_k})| \leq \psi_k(\partial_{x_i} f, L^1)(\widehat{x_k}) \quad \text{a.e. } \widehat{x_k} \in I^{n-1},$$

for any $i \in \{1, \dots, n\}$, with $i \neq k$. Thus, since $f \in W^1\mathcal{R}(Z, L^1)$, we have that $\psi_k(f, L^1) \in W^1Z(I^{n-1})$. Hence, by (i), we get

$$\|\psi_k(f, L^1)\|_{X(I^{n-1})} \lesssim \|\psi_k(f, L^1)\|_{W^1Z(I^{n-1})}, \quad k \in \{1, \dots, n\},$$

from which (ii) follows.

Now, let us see that $(ii) \Rightarrow (i)$. In particular, we shall suppose that $1 \leq m \leq n-2$ and we shall prove that if the Sobolev embedding

$$W^m\mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1)$$

holds then

$$W^m Z(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

As before, we assume that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Given any non-negative function $f \in \overline{Z}(0, 1)$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$, we define

$$g(y) = \begin{cases} \displaystyle\int_{\omega_{n-1}|y|^{n-1}}^1 \int_{s_1}^1 \int_{s_2}^1 \cdots \int_{s_{m-1}}^1 s_m^{-m+m/(n-1)} f(s_m)ds_m \dots ds_1, & y \in B_{n-1}(0, r), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u(x) = g(\widehat{x_n})\theta(x_n),$$

where $\theta \in C_c^\infty(I)$ such that $0 \leq \theta \leq 1$. Now, applying Fubini's theorem $m-1$ times, we get

$$u(x) \approx \theta(x_n) \int_{\omega_{n-1}|\widehat{x_n}|^{n-1}}^1 s_m^{-m+m/(n-1)} f(s_m)(s_m - \omega_{n-1}|\widehat{x_n}|^{n-1})^{(m-1)}ds_m,$$

for any $(\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times I$, and so it holds that

$$u(x) \lesssim \begin{cases} \theta(x_n) \displaystyle\int_{\omega_{n-1}|\widehat{x_n}|^{n-1}}^{1} s_m^{-1+m/(n-1)} f(s_m) ds_m, & (\widehat{x_n}, x_n) \in B_{n-1}(0, r) \times I, \\ 0, & \text{otherwise,} \end{cases}$$

Therefore, Lemma 4.1.3 implies that

$$\|u\|_{\mathcal{R}_n(Z,L^1)} \lesssim \|\theta\|_{L^1(I^n)} \|f\|_{\overline{Z}(0,1)}. \tag{7.4}$$

Now, we fix any $k \in \{1, \dots, n-1\}$. Then, by Fubini's theorem, we obtain that

$$\psi_k(u, L^1)(\widehat{x_k}) \lesssim \theta(x_n) \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{n-1}}^{1} s_m^{-1+(m+1)/(n-1)} f(s_m) ds_m$$

for any $(\widehat{x_{n,k}}, x_n) \in B_{n-2}(0, r) \times I$ and $\psi_k(f, L^1)(\widehat{x_k}) = 0$ otherwise. Hence, the boundedness of the dilation operator in r.i. spaces gives

$$\|u\|_{\mathcal{R}_k(Z,L^1)} \lesssim \left\| \int_{t^{(n-1)'}}^{1} s_m^{-1+(m+1)/(n-1)} f(s_m) ds_m \right\|_{\overline{Z}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \tag{7.5}$$

for any $k \in \{1, \dots, n-1\}$. Therefore, combining (7.4) and (7.5), we have that

$$\|u\|_{\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)} \tag{7.6}$$

Next, we fix any $\alpha \in (\mathbb{N} \cup \{0\})^n$, with $1 \le |\alpha| \le m$. Then, we have

$$\partial^{|\alpha|} u(x) = \theta^{(\alpha_n)}(x_n) \partial^{|\widehat{\alpha_n}|} g(\widehat{x_n}),$$

and therefore, following the same arguments as in [36, Theorem A], we deduce that:

(a) if $|\widehat{\alpha_n}| = 0$ then

$$|\partial^{|\alpha|} u(x)| \lesssim |\theta^{(\alpha_n)}(x_n)| g(\widehat{x_n}); \tag{7.7}$$

(b) if $1 \le |\widehat{\alpha_n}| \le m-1$ then

$$|\partial^{|\alpha|} u(x)| \lesssim |\theta^{(\alpha_n)}(x_n)| \sum_{\ell=1}^{|\widehat{\alpha_n}|} |\widehat{x_n}|^{\ell(n-1)-|\widehat{\alpha_n}|} \int_{\omega_{n-1}|\widehat{x_n}|^{n-1}}^{1} s^{-\ell+m/(n-1)-1} f(s) ds; \tag{7.8}$$

(c) if $|\widehat{\alpha_n}| = m$ then

$$\begin{aligned} |\partial^{|\alpha|} u(x)| \lesssim |\theta(x_n)| \sum_{\ell=1}^{m-1} |\widehat{x_n}|^{\ell(n-1)-m} \int_{\omega_{n-1}|\widehat{x_n}|^{n-1}}^{1} s^{-\ell+m/(n-1)-1} f(s) ds \\ + |\theta(x_n)| f(\omega_{n-1}|\widehat{x_n}|^{n-1}). \end{aligned} \tag{7.9}$$

Now, using the same ideas as before, but for the function $\theta^{(\alpha_n)}g$, we get

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad 1 \le \alpha_m \le m, \quad |\widehat{\alpha_m}| = 0. \tag{7.10}$$

Let us suppose that $1 \le |\widehat{\alpha_n}| \le m-1$. Then, Lemma 2.1.3 implies that

$$\begin{aligned}\|\partial^{|\alpha|}u\|_{\mathcal{R}_n(Z,L^1)} &\approx \sum_{\ell=1}^{|\widehat{\alpha_n}|} \left\| \left(|\cdot|^{\ell(n-1)-|\widehat{\alpha_n}|} \int_{\omega_{n-1}|\cdot|^{n-1}}^1 s^{-\ell+m/(n-1)-1} f(s)ds \right)^*(y) \right\|_{\overline{Z}(0,1)} \\ &= \sum_{\ell=1}^{|\widehat{\alpha_n}|} \left\| \left(\omega_{n-1}^{-\ell(n-1)-|\widehat{\alpha_n}|} y^{\ell-|\widehat{\alpha_n}|/(n-1)} \int_y^1 s^{-\ell+m/(n-1)-1} f(s)ds \right)^*(y) \right\|_{\overline{Z}(0,1)} \\ &\approx \sum_{\ell=1}^{|\widehat{\alpha_n}|} \left\| y^{\ell-|\widehat{\alpha_n}|/(n-1)} \int_y^1 s^{-\ell+m/(n-1)-1} f(s)ds \right\|_{\overline{Z}(0,1)}\end{aligned}$$

Therefore, Lemma 4.1.3 gives

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_n(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad 1 \le |\widehat{\alpha_n}| \le m-1. \tag{7.11}$$

Now, we fix any $k \in \{1, \dots, n-1\}$. Then, using Fubini's theorem, we have

$$\begin{aligned}\psi_k(\partial^\alpha u, L^1) &\lesssim |\theta(x_n)| \\ &\quad \times \sum_{\ell=1}^{|\widehat{\alpha_n}|} \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{(n-1)}}^1 t^{\ell+(-|\widehat{\alpha_n}|+1)/(n-1)-1} \left(\int_t^1 s^{-\ell+m/(n-1)-1} f(s)ds \right) dt \\ &\lesssim |\theta(x_n)| \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{(n-1)}}^1 t^{(-|\widehat{\alpha_n}|+m+1)/(n-1)-1} f(t)dt.\end{aligned}$$

Therefore, using again Lemma 2.1.3, we get

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_k(Z,L^1)} \lesssim \left\| \int_{t^{(n-1)'}}^1 s^{(-|\widehat{\alpha_n}|+m+1)/(n-1)-1} f(s) \right\|_{\overline{Z}(0,1)},$$

and so, Lemma 4.1.3 implies

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_k(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad k \in \{1, \dots, n-1\}. \tag{7.12}$$

Thus, combing (7.11) and (7.12), we deduce that

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad 1 \le |\widehat{\alpha_n}| \le m-1. \tag{7.13}$$

Finally, let us assume that $|\widehat{a_n}| = m$. Then, as before, we have

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_n(Z,L^1)} \lesssim \sum_{\ell=1}^{m-1} \left\| t^{\ell-m/(n-1)} \int_t^1 s^{-\ell+m/(n-1)-1} f(s)ds \right\|_{\overline{Z}(0,1)} + \|f\|_{\overline{Z}(0,1)}.$$

and so Lemma 4.1.3 gives

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_n(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad |\widehat{a_n}| = m. \tag{7.14}$$

Let $k \in \{1,\dots,n-1\}$ be fixed. Then, by (7.9) and Fubini's theorem, we have

$$\begin{aligned}
&\psi_k(\partial^{|\alpha|}u,L^1)(\widehat{x_k})\\
&\lesssim |\theta(x_n)| \times \sum_{\ell=1}^{m-1} \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{(n-1)}}^1 t^{\ell+(-m+1)/(n-1)-1}\Big(\int_t^1 s^{-\ell+m/(n-1)-1} f(s)ds\Big)dt\\
&\quad + |\theta(\widehat{x_n})| \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{(n-1)}}^1 t^{1/(n-1)-1} f(t)dt\\
&\lesssim |\theta(\widehat{x_n})| \int_{\omega_{n-1}|\widehat{x_{n,k}}|^{(n-1)}}^1 t^{1/(n-1)-1} f(t)dt
\end{aligned}$$

for any $(\widehat{x_{n,k}}, x_n) \in B_{n-2}(0,r) \times I$ and $\psi_k(\partial^{|\alpha|}u, L^1)(\widehat{x_k}) = 0$ otherwise. As a consequence, Lemma 2.1.3 and Lemma 4.1.3 give

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}_k(Z,L^1)} \lesssim \Big\| \int_{t^{(n-1)'}}^1 t^{1/(n-1)-1} f(t)dt \Big\|_{\overline{Z}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \tag{7.15}$$

for any $k \in \{1,\dots,n-1\}$ and $\alpha \in (\mathbb{N}\cup\{0\})^n$, with $|\widehat{\alpha_n}| = m$. Therefore, by (7.14) and (7.15), we conclude that

$$\|\partial^{|\alpha|}u\|_{\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad |\widehat{\alpha_n}| = m. \tag{7.16}$$

Therefore, using (7.6), (7.10), (7.13), and (7.16), we deduce that

$$\begin{aligned}
\|u\|_{W^m\mathcal{R}(Z,L^1)} &= \|u\|_{\mathcal{R}(Z,L^1)} + \sum_{\substack{1\le\alpha_m\le m\\ |\widehat{\alpha}_n|=0}} \|\partial^\alpha u\|_{\mathcal{R}(Z,L^1)} + \sum_{\substack{1\le|\widehat{\alpha}_n|\le m-1\\ 0\le\alpha_m\le m-|\widehat{\alpha_m}|}} \|\partial^\alpha u\|_{\mathcal{R}(Z,L^1)}\\
&\quad + \sum_{\substack{\alpha_m=0\\ |\widehat{\alpha}_n|=m}} \|\partial^\alpha u\|_{\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}.
\end{aligned} \tag{7.17}$$

By hypothesis, the Sobolev embedding

$$W^m\mathcal{R}(Z,L^1) \hookrightarrow \mathcal{R}(X,L^1)$$

holds and hence, taking into account (7.17), we get

$$\|u\|_{\mathcal{R}(X,L^1)} \le\lesssim \|u\|_{W^m\mathcal{R}(Z,L^1)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{7.18}$$

But, using now Fubini's theorem $m-1$ times, we obtain

$$\psi_n^*(u,L^1)(t) = \begin{cases} \displaystyle\int_t^1 s_m^{-1+m/(n-1)} f(s_m)(1-t/s_m)^{(m-1)}ds_m, & 0 \le t < \omega_{n-1}r^{n-1},\\ 0, & \text{otherwise.}\end{cases}$$

and so, applying the same arguments as in the proof of Theorem 5.1.5, we get

$$\Big\| \int_t^1 s_m^{m/(n-1)-1} f(s_m)ds_m \Big\|_{\overline{X}(0,1)} \lesssim \|u\|_{\mathcal{R}_n(X,L^1)} \le \|u\|_{\mathcal{R}(X,L^1)}. \tag{7.19}$$

Therefore, combining (7.18) and (7.19), we deduce that

$$\left\| \int_t^1 s_m^{m/(n-1)-1} f(s_m)ds_m \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)},$$

for any non-negative $f \in \overline{Z}(0,1)$, with $\lambda_f(0) \leq \omega_{n-1}r^{n-1}$. Then, the techniques used in the proof of Theorem 5.1.5 allow us to extend the validity of the previous inequality to all functions in $\overline{Z}(0,1)$. That is,

$$\left\| \int_t^1 s^{m/(n-1)-1} f(s)ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$$

Therefore, by Theorem 5.4.1 (see also [36, Theorem A]), we deduce that the Sobolev embedding

$$W^m Z(I^{n-1}) \hookrightarrow X(I^{n-1})$$

holds, as we wanted to prove.

Finally, the equivalence between (i) and (iii) is a direct consequence of Theorem 5.4.1 (see also [36]). Thus, the proof is complete. □

Before going on, we shall reformulate Theorem 7.1.3 in terms of mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$. But first let us consider a concrete example.

Proposition 7.1.4. *Let $n, m \in \mathbb{N}$, with $n \geq 3$, and $1 \leq m \leq n-2$. Then,*

$$W^m\mathcal{R}(L^1, L^\infty) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-1-m),1}, L^\infty).$$

Proof. For any $f \in L^{n',1}(0,1)$, we have that

$$\begin{aligned}\left\| \int_{t^{n'}}^1 s^{m/n-1} f(s)ds \right\|_{L^{(n-1)/(n-m-1),1}(0,1)} &\leq \int_0^1 t^{(n-m-1)/(n-1)-1}\Big(\int_{t^{n'}}^1 s^{m/n-1}|f(s)|ds\Big)dt \\ &\approx \int_0^1 s^{1/n'-1}|f(s)|ds \leq \|f\|_{L^{n',1}(0,1)}.\end{aligned}$$

As a consequence, Theorem 5.1.5 implies that

$$W^m L^{n',1}(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-1-m),1}, L^\infty). \tag{7.20}$$

Using now the classical estimate due to Fournier [28]

$$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(I^n),$$

we deduce that

$$W^m\mathcal{R}(L^1, L^\infty) \hookrightarrow W^m L^{n',1}(I^n). \tag{7.21}$$

Hence, combining (7.20) and (7.21), we conclude that

$$W^m\mathcal{R}(L^1, L^\infty) \hookrightarrow W^m L^{n',1}(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)/(n-1-m),1}, L^\infty),$$

from which the result follows. □

Now, we shall give an analogous version of Theorem 7.1.3, although with different exponents.

Proposition 7.1.5. *Let $n,m\in\mathbb{N}$, with $n\geq 3$ and $1\leq m\leq n-2$. Then,*

$$W^m\mathcal{R}(Z,L^\infty)\hookrightarrow\mathcal{R}(X,L^\infty)\Longrightarrow W^{m+1}Z(I^{n-1})\hookrightarrow X(I^{n-1}).$$

Proof. We assume that $I=(-a,b)$, with $a,b\in\mathbb{R}_+$ and $0<r<\min(a,b)$. Given any non-negative function $f\in\overline{Z}(0,1)$, with $\lambda_f(0)\leq\omega_{n-1}r^{n-1}$, we define

$$u(x)=\begin{cases}\displaystyle\int_{\omega_{n-1}|x|^{n-1}}^1\int_{s_1}^1\int_{s_2}^1\cdots\int_{s_{m-1}}^1 s_m^{-m+m/(n-1)}F(s_m)ds_m\dots ds_1, & \text{if } x\in B_n(0,r),\\ 0, & \text{otherwise,}\end{cases}$$

where

$$F(s_m)=\begin{cases}\displaystyle\int_{s_m}^1 s_m^{-1+1/(n-1)}f(s)ds, & \text{if } s_m\in(0,1),\\ 0, & \text{otherwise.}\end{cases}\tag{7.22}$$

Then, techniques used in the proof of Theorem 5.1.5 allow us to get

$$u(x)\lesssim\begin{cases}\displaystyle\int_{\omega_{n-1}|x|^{n-1}}^1 s^{-1+(m+1)/(n-1)}f(s)ds, & x\in B_n(0,r),\\ 0, & \text{otherwise,}\end{cases}$$

and hence using Lemma 2.1.3 and Lemma 4.1.3, we get

$$\|u\|_{\mathcal{R}(Z,L^\infty)}\lesssim\left\|\int_s^1 s^{-1+(m+1)/(n-1)}f(s)ds\right\|_{\overline{Z}(0,1)}\lesssim\|f\|_{\overline{Z}(0,1)}.\tag{7.23}$$

Now, we fix $\alpha\in(\mathbb{N}\cup\{0\})^n$, with $1\leq|\alpha|\leq m$. Then, in view of Theorem 5.1.5, we have that, for a.e. $x\in B_n(0,r)$:

(a) if $1\leq|\alpha|\leq m-1$ then,

$$|\partial^{|\alpha|}u(x)|\lesssim\sum_{j=1}^{|\alpha|}|x|^{j(n-1)-|\alpha|}\int_{\omega_{n-1}|x|^{n-1}}^1 F(y)y^{-j+m/(n-1)-1}dy,\tag{7.24}$$

(b) if $|\alpha|=m$ then,

$$|\partial^m u(x)|\lesssim F(\omega_{n-1}|x|^{n-1})+\sum_{j=1}^{m-1}|x|^{j(n-1)-m}\int_{\omega_{n-1}|x|^{n-1}}^1 F(y)y^{-j+m/(n-1)-1}dy.\tag{7.25}$$

Moreover, for any $\alpha \in (\mathbb{N} \cup \{0\})^{n-1}$, with $1 \le |\alpha| \le m$, it holds that

$$\partial^{|\alpha|} u(x) = 0, \quad \text{a.e. } x \notin B_n(0,r).$$

Let us suppose that $1 \le |\alpha| \le m-1$. Then, using (7.24) together with (7.22), we have that, for any $k \in \{1, \dots, n\}$,

$$|\partial^{|\alpha|} u(x)| \lesssim |x|^{m-|\alpha|} \int_{\omega_{n-1}|x|^{n-1}}^{1} f(s) s^{1/(n-1)-1} ds \lesssim \int_{\omega_{n-1}|x|^{n-1}}^{1} f(s) s^{1/(n-1)-1} ds,$$

and so, it holds that

$$\psi_k(\partial^{|\alpha|} u, L^\infty)(\widehat{x_k}) \lesssim \int_{\omega_{n-1}|\widehat{x_k}|^{n-1}}^{1} f(s) s^{1/(n-1)-1} ds, \quad k \in \{1, \dots, n\}.$$

As a consequence, Lemma 4.1.3 implies that

$$\|\partial^{|\alpha|} u\|_{\mathcal{R}(Z,L^\infty)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad 1 \le |\alpha| \le m-1. \tag{7.26}$$

Now, if $|\alpha| = m$ then, by (7.25) and (7.22), we have

$$|\partial^{|\alpha|} u(x)| \lesssim \int_{\omega_{n-1}|x|^{n-1}}^{1} f(s) s^{1/(n-1)-1} ds.$$

Therefore, applying the same arguments as before, we deduce that

$$\|\partial^m u\|_{\mathcal{R}(Z,L^\infty)} \lesssim \|f\|_{\overline{Z}(0,1)} \tag{7.27}$$

Thus, combining (7.23), (7.26), and (7.27), we conclude that

$$\|u\|_{W^m Z(I^n)} = \sum_{0 \le |\alpha| \le m} \|\partial^{|\alpha|} u\|_{Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{7.28}$$

But, by hypothesis, the Sobolev embedding

$$W^m \mathcal{R}(Z, L^\infty) \hookrightarrow \mathcal{R}(X, L^\infty)$$

holds and so, taking into account (7.28), we obtain that

$$\|u\|_{\mathcal{R}(X,L^\infty)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{7.29}$$

But, following the ideas of Theorem 5.1.5, we have that

$$\left\| \int_t^1 s^{(m+1)/(n-1)-1} f(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|u\|_{\mathcal{R}(X,L^\infty)},$$

and so, by (7.29), we conclude that

$$\left\| \int_t^1 s^{(m+1)/(n-1)-1} f(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)},$$

for any non-negative $f \in \overline{Z}(0,1)$, with $\lambda_f(0) \le \omega_{n-1} r^{n-1}$. Then, techniques used in the proof of Theorem 5.1.5 allow us to extend the validity of the previous inequality to all functions in $\overline{Z}(0,1)$. That is,

$$\left\| \int_t^1 s^{(m+1)/(n-1)-1} f(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$$

Therefore, by Theorem 5.4.1 (see also [36, Theorem A]), we complete the proof. □

Now, we shall apply Theorem 7.1.3 to give a complete characterization of the following first-order Sobolev type embeddings:

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1). \tag{7.30}$$

In particular, our approach will consist in showing that any Sobolev-type estimate of the form (7.30) can be reduced to a one-dimensional inequality for a Hardy type operator. Then, such reduction principle will be crucial to describe the optimal range and optimal domain for (7.30) between mixed norm spaces and r.i. spaces, respectively.

Theorem 7.1.6. *Let $n \in \mathbb{N}$, with $n \geq 3$. Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces. Then, the following statements are equivalent:*

(i) $W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1)$;

(ii) $\left\| \int_t^1 s^{1/(n-1)-1} f(s)ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1).$

Proof. $(i) \Rightarrow (ii)$ As before, we may suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$. Let $r \in \mathbb{R}$, $0 < r < \min(a, b)$. Given any non-negative function $f \in \overline{Z}(0,1)$, with $\lambda_f(0) \leq \omega_{n-1} r^{n-1}$, we define

$$u(\widehat{x_n}, x_n) = g(\widehat{x_n})\, \theta(x_n), \quad (\widehat{x_n}, x_n) \in I^n,$$

where $\theta \in C_c^\infty(I)$, with $0 \leq \theta \leq 1$, and

$$g(\widehat{x_n}) = \begin{cases} \displaystyle\int_{\omega_{n-1}|\widehat{x_n}|^{n-1}}^1 s^{1/(n-1)-1} f(s)ds, & \text{if } \widehat{x_n} \in B_{n-1}(0,r), \\ 0, & \text{otherwise.} \end{cases}$$

Then, using similar arguments to those in Theorem 5.1.5, one can see that

$$\|u\|_{\mathcal{R}_n(X,L^1)} \lesssim \|u\|_{W^1 Z(I^n)} \lesssim \|f\|_{\overline{Z}(0,1)}. \tag{7.31}$$

But,

$$\psi_n(u, L^1)(\widehat{x_n}) \approx \begin{cases} \displaystyle\int_t^1 s^{1/(n-1)-1} f(s)ds, & \widehat{x_n} \in I^{n-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and hence using (7.31), we get

$$\left\| \int_t^1 s^{1/(n-1)-1} f(s)ds \right\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}.$$

This proves (ii), for any $f \in \overline{Z}(0,1)$, with $\lambda_f(0) \leq r^{n-1}\omega_{n-1}$. The general case can be treated as at the end of the proof of Theorem 5.1.5.

In order to prove $(ii) \Rightarrow (i)$, let us consider the optimal domain space given in Theorem 3.4.2, i.e., the smallest mixed norm space $\mathcal{R}(X_{Z,L^1}, L^1)$ that verifies

$$Z(I^n) \hookrightarrow \mathcal{R}(X_{Z,L^1}, L^1). \tag{7.32}$$

Now, in view of (3.45), we observe that if $h \in \overline{X}_{Z,L^1}(0,1)$, then $E_1 h \in \overline{Z}(0,1)$, where E_1 is the dilation operator defined in (2.3). Therefore, using (ii) with f replaced by $E_1 h$, we get

$$\left\| \int_t^1 s^{1/(n-1)-1} E_1 h(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|h^*\|_{\overline{Z}(0,1)}.$$

So, using the definition of $\|\cdot\|_{X_{Z,L^1}(I^{n-1})}$, we obtain

$$\left\| \int_t^1 s^{1/(n-1)-1} h(s) ds \right\|_{\overline{X}(0,1)} \lesssim \|h\|_{\overline{X}_{Z,L^1}(0,1)}, \quad h \in \overline{X}_{Z,L^1}(0,1).$$

So, Theorem 7.1.3 and (7.32) imply that the following chain of embeddings holds:

$$W^1 Z(I^n) \hookrightarrow W^1 \mathcal{R}(X_{Z,L^1}, L^1) \hookrightarrow \mathcal{R}(X, L^1),$$

and the proof is complete. □

As a consequence we obtain the following result:

Corollary 7.1.7. *Let $X(I^{n-1})$ and $Z(I^n)$ be r.i. spaces and let $\mathcal{R}(X_{Z,L^1}, L^1)$ be as in Theorem 3.4.2. Then, the following statements are equivalent:*

(i) $W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1)$;

(ii) $W^1 \mathcal{R}(X_{Z,L^1}, L^1) \hookrightarrow \mathcal{R}(X, L^1)$;

(iii) $W^1 X_{Z,L^1}(I^{n-1}) \hookrightarrow X(I^{n-1})$.

Proof. A similar argument as at the end of the proof of Theorem 7.1.6 shows that $(i) \Rightarrow (ii)$. On the other hand, the implication $(ii) \Rightarrow (i)$ is an immediate consequence of Theorem 3.4.2. Finally, $(ii) \Leftrightarrow (iii)$ is given in Theorem 7.1.3. □

7.2 Characterization of the optimal range

We shall provide a characterization of the smallest space of the form $\mathcal{R}(X, L^1)$ in (7.1) when the domain space $Z(I^n)$ is given. To this end, as in Section 5.2, we introduce the space

$$Y(I^{n-1}) = \left\{ f \in \mathcal{M}(I^{n-1}) : \|f\|_{Y(I^{n-1})} = \left\| t^{1/(n-1)} f^{**}(t) \right\|_{\overline{Z}'(0,1)} < \infty \right\}. \tag{7.33}$$

It is easy to see that $Y(I^{n-1})$ is an r.i. space equipped with the norm $\|\cdot\|_{Y(I^{n-1})}$. Moreover, according to Corollary 4.1.6, its associate space $Y'(I^{n-1})$ is the smallest r.i. space such that

$$H_{1/(n-1),1} : \overline{Z}(0,1) \to \overline{Y}'(0,1) \tag{7.34}$$

is bounded, where for which $H_{1/(n-1),1}$ is the Hardy operator defined by

$$H_{1/(n-1),1}f(t) = \int_t^1 s^{1/(n-1)-1} f(s)ds, \quad t \in (0,1).$$

In order to clarify the notation used later, as in Section 5.2, we shall denote

$$X_{W^1Z,L^1}(I^{n-1}) := Y'(I^{n-1}). \tag{7.35}$$

Theorem 7.2.1. *Let $Z(I^n)$ be an r.i. space and let $X_{W^1Z,L^1}(I^{n-1})$ be the r.i. space defined in (7.35). Then, the Sobolev embedding*

$$W^1Z(I^n) \hookrightarrow \mathcal{R}(X_{W^1Z,L^1}, L^1) \tag{7.36}$$

holds. Moreover, $\mathcal{R}(X_{W^1Z,L^1}, L^1)$ is the smallest space of the form $\mathcal{R}(X, L^1)$ that verifies (7.36).

Proof. In view of (7.34) and (7.35), we have that

$$\left\| \int_t^1 s^{1/(n-1)-1} f(s)ds \right\|_{\overline{X}_{W^1Z,L^1}(0,1)} \lesssim \|f\|_{\overline{Z}(0,1)}, \quad f \in \overline{Z}(0,1),$$

and so, taking into account Theorem 7.1.6, we conclude that the embedding (7.36) holds. Now, we suppose that $\mathcal{R}(X, L^1)$ is a mixed norm space such that

$$W^1Z(I^n) \hookrightarrow \mathcal{R}(X, L^1).$$

Then, again using Theorem 7.1.6 together with Lemma 4.1.4, we have that

$$\left\| t^{1/(n-1)-1} \int_0^t g^*(s)ds \right\|_{\overline{Z}'(0,1)} \lesssim \|g^*\|_{X'(I^{n-1})}, \quad g \in X'(I^{n-1}).$$

Consequently, using now (7.33), we obtain that

$$\|g\|_{X'_{W^1Z,L^1}(I^{n-1})} \lesssim \|g\|_{X'(I^{n-1})}.$$

Hence, [8, Proposition I.2.10] implies that

$$X_{W^1Z,L^1}(I^{n-1}) \hookrightarrow X(I^{n-1}),$$

and therefore, Lemma 3.2.2 gives

$$\mathcal{R}(X_{W^1Z,L^1}, L^1) \hookrightarrow \mathcal{R}(X, L^1),$$

from which the result follows. □

Now, we shall see that the optimal range given in Theorem 7.2.1 can be also characterized with the help of the $(n-1)$-dimensional target space constructed in [36]. For this purpose, for a given mixed norm space $\mathcal{R}(Z, L^1)$, we shall find the smallest space of the form $\mathcal{R}(X, L^1)$ into which $W^1\mathcal{R}(Z, L^1)$ is continuously embedded.

Theorem 7.2.2. *Let $Z(I^{n-1})$ be an r.i. space and let $X^{\mathrm{op}}(I^{n-1})$ be the smallest r.i. space satisfying*

$$W^1 Z(I^{n-1}) \hookrightarrow X^{\mathrm{op}}(I^{n-1}).$$

Then,

$$W^1\mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X^{\mathrm{op}}, L^1). \tag{7.37}$$

Moreover, $\mathcal{R}(X^{\mathrm{op}}, L^1)$ is the smallest range of the form $\mathcal{R}(X, L^1)$ for which the embedding (7.37) holds.

Proof. The embedding (7.37) follows directly from Theorem 7.1.3. For the optimality, we consider any mixed norm space $\mathcal{R}(X, L^1)$ such that

$$W^1\mathcal{R}(Z, L^1) \hookrightarrow \mathcal{R}(X, L^1).$$

Then, Theorem 7.1.3 gives

$$W^1 Z(I^{n-1}) \hookrightarrow X(I^{n-1}),$$

and so, using the fact that $X^{\mathrm{op}}(I^{n-1})$ is the optimal range corresponding to $Z(I^{n-1})$ together with Lemma 3.2.2, we get

$$\mathcal{R}(X^{\mathrm{op}}, L^1) \hookrightarrow \mathcal{R}(X, L^1),$$

as we wanted to show. □

Before going on, for the sake of completeness, let us see now some applications of Theorem 7.2.2. In particular, we shall derive new embeddings, with improved (optimal) target spaces of the form $\mathcal{R}(X, L^1)$, for Sobolev spaces $W^1\mathcal{R}(L^p, L^1)$.

Corollary 7.2.3. *Let $n \in \mathbb{N}$, with $n \geq 3$ and $1 \leq p \leq \infty$. Then,*

(i) if $1 \leq p < n-1$, then $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ is the smallest mixed norm space that verifies

$$W^1\mathcal{R}(L^p, L^1) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1);$$

(ii) if $p = n-1$, then $\mathcal{R}(L^{\infty,n-1;-1}, L^1)$ is the optimal range for

$$W^1\mathcal{R}(L^{n-1}, L^1) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1);$$

(iii) if $p > n-1$, then $\mathcal{R}(L^\infty, L^1)$ is the smallest mixed norm space satisfying

$$W^1\mathcal{R}(L^{n-1}, L^1) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Proof. If $1 \leq p < n-1$, then, by Remark 5.4.2, we have that

$$X^{\mathrm{op}}(I^{n-1}) = L^{p(n-1)/(n-1-p),p}(I^{n-1}),$$

and so, using Theorem 7.2.2, we get (i). With a completely similar proof we obtain (ii). Finally, when $p > n-1$, we have to argue as before, using Theorem 5.1.3 instead of Remark 5.4.2. □

Proposition 7.2.4. *Let $Z(I^n)$ be an r.i. space, let $\mathcal{R}(X_{Z,L^1}, L^1)$ be as in Theorem 3.4.2, and let $X^{\rm op}_{Z,L^1}(I^{n-1})$ be the optimal r.i. range for*

$$W^1 X_{Z,L^1}(I^{n-1}) \hookrightarrow X^{\rm op}_{Z,L^1}(I^{n-1}).$$

Then,

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X^{\rm op}_{Z,L^1}, L^1), \tag{7.38}$$

and $\mathcal{R}(X^{\rm op}_{Z,L^1}, L^1)$ is the smallest space of the form $\mathcal{R}(X, L^1)$ that satisfies (7.38).

Proof. The embedding (7.38) is a direct consequence of Corollary 7.1.7. Now, we suppose that $\mathcal{R}(X, L^1)$ is another mixed norm such that

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1), \tag{7.39}$$

and let us see that

$$\mathcal{R}(X^{\rm op}_{Z,L^1}, L^1) \hookrightarrow \mathcal{R}(X, L^1).$$

Since (7.39) holds, Corollary 7.1.7 implies that

$$W^1 X_{Z,L^1}(I^{n-1}) \hookrightarrow X(I^{n-1}),$$

and hence our assumption on $X^{\rm op}_{Z,L^1}(I^n)$ gives

$$X^{\rm op}_{Z,L^1}(I^{n-1}) \hookrightarrow X(I^{n-1}).$$

Therefore, using Lemma 3.2.2, we conclude that

$$\mathcal{R}(X^{\rm op}_{Z,L^1}, L^1) \hookrightarrow \mathcal{R}(X, L^1),$$

and the proof is complete. □

Now, we shall give examples of Theorem 7.2.1 and Proposition 7.2.4. Thus, for instance, we shall recover the estimate due to Algervik and Kolyada [2]:

$$W^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^{(n-1)',1}, L^1), \tag{7.40}$$

and we prove its optimality that had not been known before. This means that if an embedding of type (7.40) holds with $\mathcal{R}(L^{(n-1)',1}, L^1)$ replaced by some other mixed norm space, then the latter must contain $\mathcal{R}(L^{(n-1)',1}, L^1)$.

Corollary 7.2.5. *Let $n \in \mathbb{N}$, with $n \geq 3$ and $1 \leq p \leq \infty$. We have that:*

(i) if $1 \leq p < n-1$, then, $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ is the smallest space of the form $\mathcal{R}(X, L^1)$ that verifies

$$W^1 L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1);$$

(ii) *if* $p = n-1$, *then* $\mathcal{R}(L^{\infty,n-1;-1}, L^1)$ *is the optimal range of the form* $\mathcal{R}(X, L^1)$ *such that*

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1);$$

(iii) *if* $p > n-1$, *then,* $\mathcal{R}(L^\infty, L^1)$ *is the smallest mixed norm space of the form* $\mathcal{R}(X, L^1)$ *satisfying*

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Proof. If $1 \le p < n-1$, then Remark 5.4.2 claims that $L^{p(n-1)/(n-1-p),p}(I^{n-1})$ is the smallest r.i. space satisfying

$$W^1L^p(I^{n-1}) \hookrightarrow L^{p(n-1)/(n-1-p),p}(I^{n-1}),$$

and so, using Corollary 3.4.3 and Proposition 7.2.4 with $Z(I^n)$ replaced by $L^p(I^n)$, we get (i). With a completely analogous proof, we obtain (ii). Similarly, we prove (iii). We only need to use Theorem 5.1.3 instead of Remark 5.4.2. □

7.3 Characterization of the optimal domain

We now focus on the construction of the largest r.i. domain space for a fixed range space $\mathcal{R}(X, L^1)$. Using similar arguments to those in Section 5.2 for the case $\mathcal{R}(X, L^\infty)$, we now introduce the space $Z_{\mathcal{R}(X,L^1)}(I^n)$ defined by

$$Z_{\mathcal{R}(X,L^1)}(I^n) = \left\{ f \in \mathcal{M}(I^n) : \|f\|_{Z_{\mathcal{R}(X,L^1)}} = \left\| \int_t^1 f^{**}(s) s^{1/(n-1)-1} ds \right\|_{\overline{X}(0,1)} < \infty \right\}. \tag{7.41}$$

With a similar proof to that of Theorem 5.3.2, it is easy to see that $Z_{\mathcal{R}(X,L^1)}(I^n)$ is an r.i. space. The next lemma will be needed later. Its proof follows the scheme of Lemma 5.3.1 (see also [27]), so we do not include it here.

Lemma 7.3.1. *Let* $X(I^{n-1})$ *be an r.i space, with* $\overline{\alpha}_X < (n-2)/(n-1)$. *Then,*

$$\left\| \int_t^1 f^{**}(s) s^{1/(n-1)-1} ds \right\|_{\overline{X}(0,1)} \approx \left\| \int_t^1 f^{*}(s) s^{1/(n-1)-1} ds \right\|_{\overline{X}(0,1)}, \quad f \in \mathcal{M}(I^n).$$

Theorem 7.3.2. *Let* $X(I^{n-1})$ *be an r.i space,* $\overline{\alpha}_X < (n-2)/(n-1)$, *and let* $Z_{\mathcal{R}(X,L^1)}(I^n)$ *be the r.i. space defined in* (7.41). *Then, the Sobolev embedding*

$$W^1 Z_{\mathcal{R}(X,L^1)}(I^n) \hookrightarrow \mathcal{R}(X, L^1) \tag{7.42}$$

holds. Moreover, $Z_{\mathcal{R}(X,L^1)}(I^n)$ *is the largest domain space satisfying* (7.42).

Proof. We proceed as we did in Theorem 5.3.2. We fix any $f \in \overline{Z}_{\mathcal{R}(X,L^1)}(0,1)$ and we observe that if $0 < t < 1/2$ then, we have

$$\begin{aligned}\int_{2t}^{1} |f(s)| s^{1/(n-1)-1} ds &\approx \int_{2t}^{1} |f(s)| \Big(\int_{s/2}^{s} v^{1/(n-1)-2} dv \Big) ds \\ &\le \int_{2t}^{1} |f(s)| \Big(\int_{s/2}^{1} v^{1/(n-1)-2} dv \Big) ds,\end{aligned}$$

and so, using Fubini's theorem, we get

$$\begin{aligned}\int_{2t}^{1} |f(s)| s^{1/(n-1)-1} ds &\lesssim \int_{t}^{1/2} v^{1/(n-1)-2} \Big(\int_{2t}^{2v} |f(s)| ds \Big) dv \\ &\quad + \int_{1/2}^{1} v^{1/(n-1)-2} \Big(\int_{2t}^{1} |f(s)| ds \Big) dv \\ &\lesssim \int_{t}^{1/2} v^{1/(n-1)-2} \Big(\int_{0}^{2v} |f(s)| ds \Big) dv + \|f\|_{L^1(0,1)}.\end{aligned}$$

Therefore, using the Hardy-Littlewood inequality (see Theorem 2.1.5) together with a change of variables, we obtain

$$\begin{aligned}\int_{2t}^{1} |f(s)| s^{1/(n-1)-1} ds &\lesssim \int_{2t}^{1} v^{1/(n-1)-1} f^{**}(v) dv + \|f\|_{L^1(0,1)} \\ &\le \int_{t}^{1} v^{1/(n-1)-1} f^{**}(v) dv + \|f\|_{L^1(0,1)},\end{aligned}$$

for any $t \in (0, 1/2)$. Hence, using the boundedness of the dilation operator in r.i. spaces, we get

$$\begin{aligned}\Big\| \int_{t}^{1} f(s) s^{1/(n-1)-1} ds \Big\|_{\overline{X}(0,1)} &\lesssim \Big\| \chi_{(0,1/2)}(t) \int_{2t}^{1} f(s) s^{1/(n-1)-1} ds \Big\|_{\overline{X}(0,1)} \\ &\lesssim \Big\| \int_{t}^{1} f^{**}(s) s^{1/(n-1)-1} ds \Big\|_{\overline{X}(0,1)} + \|f\|_{L^1(0,1)}.\end{aligned}$$

But, by hypothesis, we have that

$$\|f\|_{\overline{Z}_{\mathcal{R}(X,L^1)}(0,1)} = \Big\| \int_{t}^{1} f^{**}(s) s^{1/(n-1)-1} ds \Big\|_{\overline{X}(0,1)},$$

and so, we conclude that

$$\Big\| \int_{t}^{1} f(s) s^{1/(n-1)-1} ds \Big\|_{\overline{X}(0,1)} \lesssim \|f\|_{\overline{Z}_{\mathcal{R}(X,L^1)}(0,1)} + \|f\|_{L^1(0,1)}, \quad f \in \overline{Z}_{\mathcal{R}(X,L^1)}(0,1).$$

Therefore, using Theorem 7.1.6 together with (2.1), we deduce that (7.42) holds.

Thus, it only remains to show that $Z_{\mathcal{R}(X,L^1)}(I^n)$ is the largest r.i. space satisfying (7.42). In fact, if an r.i. space $Z(I^n)$ verifies

$$W^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^1),$$

then, by Theorem 7.1.6, we get

$$\left\| \int_t^1 f^*(s) s^{1/(n-1)-1} ds \right\|_{\overline{X}(0,1)} \lesssim \|f^*\|_{\overline{Z}(0,1)}, \quad f \in Z(I^n).$$

As a consequence, using now Lemma 7.3.1, we obtain

$$\|f\|_{Z_{\mathcal{R}(X,L^1)}(I^n)} \approx \left\| \int_t^1 f^*(s) s^{1/(n-1)-1} ds \right\|_{\overline{X}(0,1)} \lesssim \|f^*\|_{\overline{Z}(0,1)}, \quad f \in Z(I^n),$$

from which the result follows. □

Now, we present some consequences of the previous theorem, In particular, we shall describe the optimal domain spaces corresponding to the mixed norm spaces given in Corollary 7.2.5.

Corollary 7.3.3. *Let $n \in \mathbb{N}$, with $n \geq 3$. We that that:*

(i) If $1 < p < n-1$, then the Lebesgue space $L^p(I^n)$ is the largest r.i. space satisfying

$$W^1 L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1).$$

(ii) The r.i. space $Z_{\mathcal{R}(L^{\infty,n-1;-1}, L^1)}(I^n)$, with norm given by

$$\|f\|_{Z_{\mathcal{R}(L^{\infty,n-1;-1}, L^1)}(I^n)} \approx \left\| \int_t^1 s^{1/(n-1)-1} f^*(s) ds \right\|_{L^{\infty,n-1;-1}(I^n)}$$

is the largest r.i. domain space that verifies

$$W^1 Z_{\mathcal{R}(L^{\infty,n-1;-1}, L^1)}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1).$$

(iii) The Lorentz space $L^{n-1,1}(I^n)$ is the largest r.i. space such that

$$W^1 L^{n-1,1}(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Proof. Applying the same technique as in the proof of Corollary 5.3.3 and using Theorem 7.3.2 instead of Theorem 5.3.2, we get (i). The proof of (ii) is similar to that of Corollary 5.3.5. The only difference is that we have to apply Theorem 7.3.2 instead of Theorem 5.3.2. Finally, (iii) is an immediate consequence of Theorem 7.3.2. □

7.4 Comparison with the optimal range $\mathcal{R}(X, L^\infty)$

As we have pointed out in Chapter 3, the mixed norm spaces of the forms $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$ are function spaces, having non-trivial intersection (for example, $L^\infty(I^n)$ is contained in both). Motivated by this fact, we consider $Z(I^n) = L^p(I^n)$ and we compare its optimal ranges of the form $\mathcal{R}(X, L^\infty)$ and $\mathcal{R}(X, L^1)$.

An earlier result in this direction was obtained by Algervik and Kolyada [2] who proved that the embedding (7.40) can be strengthened on allowing spaces of a different nature, such as mixed norm spaces of the form $\mathcal{R}(X, L^\infty)$. Namely, it was shown that:

$$W^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{(n-1)',1}, L^1). \tag{7.43}$$

It seems natural to expect that this chain of embeddings should hold for any Sobolev type estimate of first order. However, we shall see that there are examples, including inequalities for standard Sobolev spaces W^1L^p showing that such mixed norm spaces are not comparable. First we need the following technical lemma.

Lemma 7.4.1. *Let $n \in \mathbb{N}$, $n \geq 3$. Let $Z(I^n)$ be an r.i. space, let $\mathcal{R}(X_{W^1Z,L^\infty}, L^\infty)$ be the optimal mixed norm of the form $\mathcal{R}(X, L^\infty)$*

$$W^1Z(I^n) \hookrightarrow \mathcal{R}(X_{W^1Z,L^\infty}, L^\infty),$$

and let $\mathcal{R}(X_{W^1Z,L^1}, L^1)$ be the smallest space of the form $\mathcal{R}(X, L^1)$ that verifies

$$W^1Z(I^n) \hookrightarrow \mathcal{R}(X_{W^1Z,L^1}, L^1).$$

Then,

$$\mathcal{R}(X_{W^1Z,L^1}, L^1) \not\hookrightarrow \mathcal{R}(X_{W^1Z,L^\infty}, L^\infty).$$

Proof. It is a direct consequence of Theorem 3.2.3. □

Example 7.4.2. Now, let us study some concrete examples.

(i) If $1 < p < (n-1)$, then Theorem 3.2.14 ensures that the mixed norm space $\mathcal{R}(L^{p(n-1)(n-1)'/(n-p),p}, L^1)$ is the smallest space of the form $\mathcal{R}(X, L^1)$ satisfying

$$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty) \hookrightarrow \mathcal{R}(L^{p(n-1)(n-1)'/(n-p),p}, L^1). \tag{7.44}$$

On the other hand, by Corollary 5.2.2, we have that

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty),$$

and so, using (7.44), we deduce that

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)(n-1)'/(n-p),p}, L^1). \tag{7.45}$$

But, by Corollary 7.2.5, the mixed norm space $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ is the optimal range of the form $\mathcal{R}(X, L^1)$, and hence, using (7.45), we conclude that

$$\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1) \hookrightarrow \mathcal{R}(L^{p(n-1)(n-1)'/(n-p),p}, L^1). \tag{7.46}$$

Therefore, combining (7.44) with (7.46), we have that

$$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty) \not\hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1).$$

From this fact together with Lemma 7.4.1, we conclude that the optimal mixed norm spaces $\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ and $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ are not comparable.

(ii) If $n \in \mathbb{N}$, with $n \geq 3$ and $p = n - 1$, then Corollary 5.2.2 claims that $\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ for

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty).$$

On the other hand, by Corollary 7.2.5, we have that $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ is the optimal range for

$$W^1L^{n-1}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n-1;-1}, L^1).$$

Therefore, applying the same arguments as before, we obtain that such optimal spaces are not comparable.

(iii) If $n - 1 \leq p \leq n$, then following the same procedure, we can see that they are not comparable neither in this case.

(iv) If $p > n$, then $\mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ in

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty),$$

and $\mathcal{R}(L^\infty, L^1)$ is the optimal range for

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Consequently, using Lemma 3.2.2 and Theorem 3.2.3, we conclude that the following chain of embeddings holds:

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^\infty, L^1).$$

7.5 Comparison with the optimal r.i. range

Now, we focus on the relation between the optimal mixed norm space of the form $\mathcal{R}(X, L^1)$ and the optimal r.i. range given in [36]. In fact, we shall show that these spaces are not comparable in general.

(i) If $n \in \mathbb{N}$, with $n \geq 3$, and $1 \leq p < (n - 1)$, then thanks to Corollary 3.4.3, we have that $\mathcal{R}(L^{np/(n-p),p}, L^1)$ is the smallest mixed norm of the form $\mathcal{R}(X, L^1)$ such that

$$L^{np/(n-p),p}(I^n) \hookrightarrow \mathcal{R}(L^{np/(n-p),p}, L^1). \tag{7.47}$$

As a consequence, using that $L^{np/(n-p),p}(I^n)$ is the optimal r.i. range space for

$$W^1L^p(I^n) \hookrightarrow L^{np/(n-p),p}(I^n),$$

we get

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{np/(n-p),p}, L^1). \tag{7.48}$$

On the other hand, by Corollary 7.2.5, we have that $\mathcal{R}(L^{(n-1)/(n-1-p),p}, L^1)$ is the smallest mixed norm of the form $\mathcal{R}(X, L^1)$ satisfying

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1),$$

and so, using (7.48), we deduce that

$$\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1) \hookrightarrow \mathcal{R}(L^{pn/(n-p),p}, L^1). \tag{7.49}$$

Therefore, combining (7.47) and (7.49), we get

$$L^{n/(n-p),p}(I^n) \not\hookrightarrow \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1), \quad 1 \leq p < n-1. \tag{7.50}$$

Now, we fix $1 < p < n-1$, and we suppose that $I = (-a, b)$, with $a, b \in \mathbb{R}_+$, and $0 < r < \min(a, b)$. Taking

$$g(x) = \begin{cases} |x|^{-\alpha}, & x \in B_n(0, r), \\ 0, & \text{otherwise}, \end{cases}$$

with $(n-p)/p < \alpha < ((n-1)-p)/p + 1$, it is easy to see that

$$g \notin L^{np/(n-p),p}(I^n) \quad \text{and} \quad g \in \mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1).$$

So, we have that

$$\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1) \not\hookrightarrow L^{np/(n-p),p}(I^n), \quad 1 < p < (n-1). \tag{7.51}$$

Therefore, using (7.50) with (7.51), we have that $\mathcal{R}(L^{p(n-1)/(n-1-p),p}, L^1)$ and $L^{np/(n-p),p}(I^n)$ are not comparable, for $1 < p < n-1$.

(ii) If $n-1 \leq p < n$, then applying the same arguments as before, we can see that neither in this case the corresponding optimal range spaces are comparable.

(iii) Finally, if $p > n$, then $\mathcal{R}(L^\infty, L^\infty) = L^\infty(I^n)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ in

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty),$$

and $\mathcal{R}(L^\infty, L^1)$ is the optimal range for

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^1).$$

Consequently, using Lemma 3.2.2 and Theorem 3.2.3, we conclude that the following chain of embeddings holds:

$$W^1L^p(I^n) \hookrightarrow \mathcal{R}(L^\infty, L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^\infty, L^1).$$

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$D^m u$, vector of partial derivatives, 1
$E(\widehat{x_k})$, x_k-section of E, 18
$I \subset \mathbb{R}$, interval, 1
$L^{p,q,\alpha}$, Lorentz-Zygmund spaces, 3
$L^{p,q}$, Lorentz spaces, 3
L^p, Lebesgue spaces, 3
L_A, Orlicz spaces, 3
X' associate space, 11
Λ_{φ_X}, classical Lorentz spaces, 13
$\Pi_k^* E$, essential projection of E, 18
$\hookrightarrow$, continuous embedding, 1
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$\mathcal{R}_k f$, rearrangement with respect to x_k, 37
$\mathcal{M}$, set of measurable functions, 9
$\overline{X}$, representation space, 11
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