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Classical and modern results on interpolation of operators

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Introduction

The Project

The main purpose of this project is to study the classical theorems on interpolation of linear operators in order to analyse some modern results on interpolation of multilinear operators.

Following the approach of Bennett and Sharpley [3] to the classical interpolation theory of quasilinear operators, we gather all the results that will allow us to tackle the recent developments on multilinear interpolation theory, in particular, the result of Grafakos, Liu, Lu and Zhao [11].

Our goal is to fully understand the different real interpolation techniques presented by the previous authors, so we devote our time and efforts to give detailed, self-contained and complete proofs of the main interpolation results.

We focus on the study of real-variable methods and we start with one of the cornerstones of the classical interpolation theory: the Marcinkiewicz interpolation theorem.

We continue the study with the K-method of interpolation, which it may be regarded as a lifting of the Marcinkiewicz interpolation theorem from its classical context in spaces of measurable functions to an abstract Banach space setting.

Finally, we study multilinear interpolation theory, exposing the proof a version of Marcinkiewicz's interpolation theorem for bi-sublinear operators.

Personal Conclusions

There haven't been any remarkable issues concerning the mathematics of the project. Most of the basic tools that we have used were studied in the Master courses of Functional Analysis and Harmonic Analysis. This project has allowed to acquire a deeper comprehension of the results exposed in these courses, and also to understand more abstract ideas in the fields of Functional and Harmonic Analysis.

The most difficult part of the project has been to find time to work on it, since during its development I was attending the Master courses and giving lectures as a professor at the Faculty of Chemistry. Nevertheless, we have fulfilled our initial goals and obtained a document that can be regarded as an introduction to the theory of interpolation of operators.

My future plans are to continue studying multilinear operators, using this work as a bridge between the theory of interpolation of operators and the Rubio de Francia's extrapolation theory for multilinear operators. If I am awarded a Ph.D. grant during the following months, I will start the development of this new enterprise by the end of this year.

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I also place on record, my sense of gratitude to one and all, who directly or indirectly, have lent their hand in this venture.

An Overview of Interpolation Theory

Consider a $\mathbb{R}$-vector space X and a function $\rho : X \to \mathbb{R}$ such that $\forall f, g \in X$ and $\forall a \in \mathbb{R}$, we have that $\rho(af) = |a|\rho(f)$, $\rho(f) = 0 \Leftrightarrow f = 0$ and $\rho(f+g) \leq \rho(f) + \rho(g)$. Then, ρ is called a norm on X. A pair (X, ρ) is called a Banach space if X is complete with respect to ρ.

A first example of Banach spaces are the Lebesgue spaces L^p, for $1 \leq p \leq \infty$. Given (R, μ) a totally σ-finite measure space, $L^p(X, \mu)$ consists of all scalar-valued measurable functions for which

$$\|f\|_p = \left(\int_R |f|^p d\mu \right)^{\frac{1}{p}} < \infty,$$

or $\|f\|_\infty = \operatorname{ess\,sup}_R |f| < \infty$, for the case $p = \infty$, together with the norm $\|\cdot\|_p$.

Suppose that we have two Banach spaces $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ and an operator $T : X \to Y$. We say that T is bounded if there exists a constant $M > 0$ such that $\|Tx\|_Y \leq M\|x\|_X$, $\forall x \in X$. One of the main purposes of the theory of interpolation of operators is to determine whether a given operator between two Banach spaces is bounded or not.

The first interpolation theorem in the theory of operators for L^p spaces was obtained by Riesz in 1926 [21], and refined by Thorin in 1939 [24]. This theorem, known as the Riesz-Thorin convexity theorem, asserts that for $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $0 \leq \theta \leq 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and a linear operator T,

$$\left.\begin{array}{l} T : L^{p_0} \longrightarrow L^{q_0} \\ T : L^{p_1} \longrightarrow L^{q_1} \end{array}\right\} \Longrightarrow T : L^p \longrightarrow L^q.$$

Its proof involves techniques in complex analysis. This result is an important part of what it is known as the complex method of interpolation.

Consider the averaging operator A defined on $L^1(0,1)$ by

$$(Af)(t) = \frac{1}{t}\int_0^t f(s)ds, \; 0 < t < 1.$$

It follows from the first Hardy's inequality (see Lemma 1.1.11) that $A : L^p(0,1) \to L^p(0,1)$ is a bounded linear operator for $1 < p < \infty$. If we wish to establish this result by appealing to the Riesz-Thorin convexity theorem, we would first need to verify that A is bounded on $L^\infty(0,1)$ and on $L^1(0,1)$. The L^∞-boundedness follows from the definition of A. The problem is that A is not bounded on L^1, as may be seen by considering a decreasing function of the form $f(s) = s^{-1}(\log s)^{-2}$ near the origin and observing that Af fails to be integrable there. Thus, the Riesz-Thorin convexity theorem does not apply.

The desired interpolation can still be accomplished, but by a quite different technique introduced by J. Marcinkiewicz in 1939 [18]. The Marcinkiewicz interpolation theorem is best formulated in the larger context of a two-parameter family of spaces, the Lorentz $L^{p,q}$-spaces, for $0 < p, q \leq \infty$. Given (R,μ) a totally σ-finite measure space, $L^{p,q}(X,\mu)$ consists of all scalar-valued measurable functions for which

$$\|f\|_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

or $\|f\|_{p,\infty} = \sup_{0<t<\infty}(t^{1/p}f^*(t)) < \infty$, for the case $q = \infty$, where

$$f^*(t) = \inf\{\lambda \geq 0 : \mu\{x \in R : |f(x)| > \lambda\} \leq t\}, \; \forall t \geq 0.$$

The Marcinkiewicz interpolation theorem (see Theorem 1.2.36) asserts that for $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0 \neq q_1, r \leq \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$,

$1/q = (1-\theta)/q_0 + \theta/q_1$ and a quasilinear operator T,

$$\left.\begin{array}{c} T : L^{p_0,1} \longrightarrow L^{q_0,\infty} \\ T : L^{p_1,1} \longrightarrow L^{q_1,\infty} \end{array}\right\} \Longrightarrow T : L^{p,r} \longrightarrow L^{q,r},$$

and if $p_1 = \infty$,

$$\left.\begin{array}{c} T : L^{p_0,1} \longrightarrow L^{q_0,\infty} \\ T : L^{\infty} \longrightarrow L^{q_1} \end{array}\right\} \Longrightarrow T : L^{p,r} \longrightarrow L^{q,r}.$$

This result is an important part of what it is known as the real method of interpolation.

Returning to the example of the averaging operator A, we have that $A : L^1 \to L^{1,\infty}$ is bounded (since it is bounded from above by the Hardy-Littlewood maximal operator [3, Ch. 3]) and $A : L^\infty \to L^\infty$ is bounded, so by the Marcinkiewicz interpolation theorem, we conclude that $A : L^p \to L^p$ is bounded, for $1 < p \leq \infty$.

The theorems of Riesz-Thorin and Marcinkiewicz and other generalisations pertain to the Lebesgue spaces, the Lorentz spaces and other spaces closely related to them [3, Ch. 4]. The development of general interpolation theorems for families of abstract Banach spaces begun in 1958, and the works of Peetre played an essential role [20]. The proof of the Marcinkiewicz interpolation theorem is based on an idea of decomposition of a function in two pieces. This idea was generalised by Peetre, giving rise to the concept of the K-functional, which plays a central role in modern interpolation theory.

A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a compatible couple if there is some Hausdorff topological vector space $\mathscr{X}$ in which each of X_0 and X_1 is continuously embedded. For such a couple, the Peetre K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf_{f = f_0 + f_1} \{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1}, f_j \in X_j, j = 0, 1\}.$$

We can define new Banach spaces, denoted by $(X_0, X_1)_{\theta,q}$, for $0 < \theta < 1$, $1 \leq q < \infty$ or $0 \leq \theta \leq 1$, $q = \infty$, consisting of all f in $X_0 + X_1$ for which

$$\|f\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(f, t; X_0, X_1))^q \frac{dt}{t}\right)^{1/q} < \infty,$$

for $0 < \theta < 1, 1 \leq q$, or $\|f\|_{\theta,\infty} = \sup_{0<t<\infty} t^{-\theta} K(f, t; X_0, X_1) < \infty$, for $0 \leq \theta \leq 1, q = \infty$.

In this abstract setting, we have the following general interpolation result (see Theorem 2.4.7). For (X_0, X_1) and (Y_0, Y_1) compatible couples, and $0 < \theta_0 < \theta_1 <$

1, $0 \le \psi_0 \ne \psi_1 \le 1$, $0 < \theta < 1$, $(\theta', \psi') = (1-\theta)(\theta_0, \psi_0) + \theta(\theta_1, \psi_1)$, $1 \le q \le \infty$, $(X_0, X_1)_{\theta_j,1} \hookrightarrow \overline{X}_{\theta_j} \hookrightarrow (X_0, X_1)_{\theta_j,\infty}$, $(Y_0, Y_1)_{\psi_j,1} \hookrightarrow \overline{Y}_{\psi_j} \hookrightarrow (Y_0, Y_1)_{\psi_j,\infty}$ and T a linear operator, we have that

$$\left.\begin{array}{l} T : \overline{X}_{\theta_0} \longrightarrow \overline{Y}_{\psi_0} \\ T : \overline{X}_{\theta_1} \longrightarrow \overline{Y}_{\psi_1} \end{array}\right\} \Longrightarrow T : (X_0, X_1)_{\theta',q} \longrightarrow (Y_0, Y_1)_{\psi',q}.$$

The previous results involve operators in one variable, but we can think of operators of several variables which are linear in each of them. Consider, for example, the following operator, known as the bilinear Hilbert transform:

$$H(f_1, f_2)(x) = \lim_{\varepsilon \to 0^+} \int_{|t| \ge \varepsilon} f_1(x-t) f_2(x+t) \frac{dt}{t},$$

for $f_j \in L^{p_j}(\mathbb{R})$, $j = 1, 2$, with $1 < p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2$.

In 1999, Lacey and Thiele proved that $H : L^{p_1} \times L^{p_2} \to L^p$ is bounded, provided that $2/3 < p < \infty$ [15]. In particular, this resolves in the affirmative Calderón's conjecture that H is bounded from $L^2 \times L^2$ into L^1 [7]. However, the boundedness into L^p for $1/2 < p \le 2/3$ remains open as of this writing. This kind of problems motivated the development of a multilinear version of the theory of interpolation of operators.

In the literature we can find several multilinear interpolation theorems. In 1964, Lions and Peetre proved an interpolation theorem for bilinear operators defined over spaces $(X_0, X_1)_{\theta,q}$ [16].

In 1969, Strichartz proved a bilinear version of the Marcinkiewicz interpolation theorem for Lebesgue spaces $L^p(X, \mu)$ for arbitrary totally σ-finite measure spaces (X, μ) [23].

In 1978, Zafran generalized the work of Lions and Peetre and proved an interpolation theorem for multilinear operators defined over spaces $(X_0, X_1)_{\theta,q}$ [25].

In 2001, Grafakos and Kalton proved an extension of the classical Marcinkiewicz interpolation theorem to the multilinear setting and for Lorentz spaces $L^{p,q}$ over the measure space $(\mathbb{R}^+, m)$, where m denotes the Lebesgue measure [10].

In 2012, Grafakos, Liu, Lu and Zhao, proved a multilinear extension of the Marcinkiewicz interpolation theorem for Lorentz spaces $L^{p,q}$ over general measure spaces [11].

Structure of the Chapters

The chapters are organized as follows:

In Chapter 1 we recall the definition of Banach space and present the Lebesgue spaces L^p. For these spaces, we state the Hölder's inequality and use it to prove the Hardy's inequalities.

We continue with the definitions of the distribution function and the decreasing rearrangement, and the review of some of their properties. With this tools, we define the Lorentz spaces $L^{p,q}$ and prove some results concerning their structure.

We expose some notions concerning operators. We present the definitions of quasilinear, strong type, weak type and restricted weak type operators, and we study the Calderón operator and some of its properties.

In the last part of this chapter, we state and prove the Marcinkiewicz interpolation theorem for quasilinear operators on Lorentz spaces, the corresponding corollary for Lebesgue spaces and some degenerate cases.

In Chapter 2 we start working with abstract Banach spaces and operators defined on them. We present the Peetre K-functional and we prove several properties of it. This object allows us to define the general Banach spaces $(X_0, X_1)_{\theta,q}$. We also prove some of their structure properties. After that, we state and prove a basic interpolation theorem for Banach spaces $(X_0, X_1)_{\theta,q}$ and operators defined on them.

We devote the last part of this chapter to the theorem of Holmstedt and the reiteration theorem. Using them, we prove a general interpolation theorem for abstract Banach spaces and we compute examples of K-functionals for pairs of Lebesgue and Lorentz spaces.

In Chapter 3 we present a comparison between the main results on multilinear interpolation theory and we give a proof of the Marcinkiewicz interpolation theorem for bi-sublinear operators [11].

Chapter 1

A Classical Interpolation Theorem

1.1 Preliminaries

We devote this section to some basic definitions and results concerning Banach spaces. The details of the proofs can be found in the book of Bennett and Sharpley [3, Ch. 1].

Definition 1.1.1. Given a $\mathbb{R}$-vector space X, a function $\rho : X \to \mathbb{R}$ is called a *norm* on X if, for all $f, g \in X$, for all $a \in \mathbb{R}$, the following properties hold:

1. $\rho(af) = |a|\rho(f)$;
2. $\rho(f) = 0 \Leftrightarrow f = 0$;
3. $\rho(f + g) \leq \rho(f) + \rho(g)$.

Definition 1.1.2. Consider a $\mathbb{R}$-vector space X and $\rho : X \to \mathbb{R}$ a norm on X. The pair (X, ρ) is called a *Banach space* if X is complete with respect to ρ, that is, for every Cauchy sequence $\{f_n\}$ in X, there exists an element $f \in X$ such that $\lim_{n\to\infty} f_n = f$ or, equivalently, $\lim_{n\to\infty} \rho(f_n - f) = 0$.

Definition 1.1.3. Given a measure space (R, μ), we say that it is *totally σ-finite* if R is the countable union of sets of finite measure.

Remark 1.1.4. From now on, (R, μ) and (S, ν) will denote totally σ-finite measure spaces, if we do not specify otherwise.

Let $\mathscr{M}$ denote the collection of all scalar-valued μ-measurable functions on R and $\mathscr{M}_0$ the class of functions in $\mathscr{M}$ that are finite μ-a.e. As usual, any two functions coinciding μ-a.e. will be identified. The natural vector space operations are well defined on $\mathscr{M}_0$.

Definition 1.1.5. For all function $f \in \mathscr{M}_0(R,\mu)$ and for all $1 \leq p \leq \infty$, we define the quantity

$$\|f\|_p := \begin{cases} \left(\int_R |f|^p d\mu\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_R |f|, & p = \infty. \end{cases}$$

The *Lebesgue space* $L^p = L^p(R,\mu)$ consists of all $f \in \mathscr{M}_0(R,\mu)$ for which $\|f\|_p$ is finite.

Proposition 1.1.6. *Suppose* $1 \leq p \leq \infty$. *Then* $(L^p, \|\cdot\|_p)$ *is a Banach space.*

Remark 1.1.7. The triangular inequality for $\|\cdot\|_p$ is the classical Minkowski's inequality.

Another interesting inequality involving Lebesgue spaces is the so-called Hölder's inequality.

Lemma 1.1.8. *Suppose* $1 \leq p \leq \infty$ *and consider* p' *such that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then for all* $f \in L^p$ *and for all* $g \in L^{p'}$ *we have that* $\int_R |fg| d\mu \leq \|f\|_p \|g\|_{p'}$.

Remark 1.1.9. From now on, given $1 \leq p \leq \infty$, p' will denote the unique value in $[1,\infty]$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. This value is called the *conjugate exponent of* p.

Remark 1.1.10. This inequality is sharp in the sense that

$$\|g\|_{p'} = \sup\left\{\int_R |fg| d\mu : f \in L^p, \|f\|_p \leq 1\right\},$$

for all $g \in L^{p'}$ and for all p and p'.

We close this section with the so-called Hardy's inequalities, which can be proved using Hölder's inequality. We give here the complete proof, extending the one in [3, Ch. 3].

Lemma 1.1.11. *Let* ψ *be a nonnegative measurable function on* $(0,\infty)$ *and suppose* $\lambda < 1$ *and* $1 \leq q < \infty$. *Then*

$$\left(\int_0^\infty \left(t^{\lambda-1}\int_0^t \psi(s)ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \leq \frac{1}{1-\lambda}\left(\int_0^\infty (t^\lambda \psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and

$$\left(\int_0^\infty \left(t^{1-\lambda}\int_t^\infty \psi(s)\frac{ds}{s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \leq \frac{1}{1-\lambda}\left(\int_0^\infty (t^{1-\lambda} \psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Proof. Writing $\psi(s) = s^{-\lambda/q'} s^{\lambda/q'} \psi(s)$ and applying Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t \psi(s)ds &\leq \left(\frac{1}{t}\int_0^t s^{-\lambda}ds\right)^{1/q'} \left(\frac{1}{t}\int_0^t s^{\lambda q/q'}\psi(s)^q ds\right)^{1/q} \\ &= (1-\lambda)^{-1/q'} t^{-\lambda/q'-1/q} \left(\int_0^t s^{\lambda(q-1)}\psi(s)^q ds\right)^{1/q}.\end{aligned}$$

Hence, by an interchange in the order of integration,

$$\begin{aligned}\int_0^\infty \left(t^{\lambda-1}\int_0^t \psi(s)ds\right)^q \frac{dt}{t} &\leq (1-\lambda)^{1-q}\int_0^\infty t^{\lambda-2}\int_0^t s^{\lambda(q-1)}\psi(s)^q dsdt \\ &= (1-\lambda)^{1-q}\int_0^\infty s^{\lambda(q-1)}\psi(s)^q \int_s^\infty t^{\lambda-2}dtds.\end{aligned}$$

Performing the integration over t and taking q-th roots, we obtain the first inequality of the lemma. For the second inequality, writing $\frac{\psi(s)}{s} = s^{-1/q'} s^{(\lambda-1)/q'} s^{-1/q} s^{(1-\lambda)/q'} \psi(s)$ and applying Hölder's inequality, we obtain

$$\begin{aligned}\int_t^\infty \psi(s)\frac{ds}{s} &\leq \left(\int_t^\infty s^{\lambda-1}\frac{ds}{s}\right)^{1/q'} \left(\int_t^\infty s^{(1-\lambda)q/q'}\psi(s)^q\frac{ds}{s}\right)^{1/q} \\ &= (1-\lambda)^{-1/q'} t^{(\lambda-1)/q'} \left(\int_t^\infty s^{(1-\lambda)(q-1)}\psi(s)^q \frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Hence, by an interchange in the order of integration,

$$\begin{aligned}&\int_0^\infty \left(t^{1-\lambda}\int_t^\infty \psi(s)\frac{ds}{s}\right)^q \frac{dt}{t} \\ &\quad\leq (1-\lambda)^{1-q}\int_0^\infty t^{-\lambda}\int_t^\infty s^{(1-\lambda)(q-1)}\psi(s)^q\frac{ds}{s}dt \\ &\quad= (1-\lambda)^{1-q}\int_0^\infty s^{(1-\lambda)(q-1)}\psi(s)^q\int_0^s t^{-\lambda}dt\frac{ds}{s}.\end{aligned}$$

Performing the integration over t and taking q-th roots, we obtain the desired inequality. □

Remark 1.1.12. If $q = \infty$, we will consider the inequalities

$$\sup_{0<t<\infty} t^{\lambda-1}\int_0^t \psi(s)ds \leq \left(\frac{1}{1-\lambda}\right) \sup_{0<t<\infty} t^{\lambda}\psi(t)$$

and

$$\sup_{0<t<\infty} t^{1-\lambda}\int_t^\infty \psi(s)\frac{ds}{s} \leq \left(\frac{1}{1-\lambda}\right) \sup_{0<t<\infty} t^{1-\lambda}\psi(t).$$

1.2 The Marcinkiewicz Theorem

The Marcinkiewicz interpolation theorem is best formulated in the larger context of a two-parameter family of spaces $L^{p,q}$, the Lorentz spaces, which generalize the Lebesgue spaces L^p. Therefore, we first define the Lorentz spaces and expose some of their elementary properties [3, Ch. 2,4].

Definition 1.2.1. The *distribution function* μ_f of a function $f \in \mathscr{M}_0(R, \mu)$ is given by

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\}, \, \forall \lambda \geq 0.$$

Proposition 1.2.2. *Consider $f, g, f_n, (n = 1, 2, \ldots)$ in $\mathscr{M}_0(R, \mu)$ and let a be a nonzero scalar. The distribution function μ_f is nonnegative, decreasing and right-continuous on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \;\; \mu - a.e. \Rightarrow \mu_g \leq \mu_f;$$

$$\mu_{af}(\lambda) = \mu_f(\lambda/|a|), \;\; (\lambda \geq 0);$$
$$\mu_{f+g}(\lambda_1 + \lambda_2) \leq \mu_f(\lambda_1) + \mu_g(\lambda_2), \;\; (\lambda_1, \lambda_2 \geq 0);$$
$$|f_n| \uparrow |f| \;\; \mu - a.e. \Rightarrow \mu_{f_n} \uparrow \mu_f.$$

Definition 1.2.3. Suppose f belongs to $\mathscr{M}_0(R, \mu)$. The *decreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \, \forall t \geq 0.$$

Remark 1.2.4. We use here the convention that $\inf \emptyset = \infty$.

Proposition 1.2.5. *Consider $f, g, f_n, (n = 1, 2, \ldots)$ in $\mathscr{M}_0(R, \mu)$ and let a be any scalar. The decreasing rearrangement f^* is nonnegative, decreasing and right-continuous on $[0, \infty)$. Furthermore,*

$$|g| \leq |f| \;\; \mu - a.e. \Rightarrow g^* \leq f^*;$$

$$(af)^* = |a| f^*;$$
$$(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2), \;\; (t_1, t_2 \geq 0);$$
$$|f_n| \uparrow |f| \;\; \mu - a.e. \Rightarrow f_n^* \uparrow f^*,$$
$$f^*(\mu_f(\lambda)) \leq \lambda, \; (\mu_f(\lambda) < \infty),$$
$$\mu_f(f^*(t)) \leq t, \; (f^*(t) < \infty).$$

The next result gives alternative descriptions of the L^p-norm in terms of the distribution function and the decreasing rearrangement.

Proposition 1.2.6. *Let $f \in \mathscr{M}_0(R,\mu)$. If $0 < p < \infty$, then*

$$\int_R |f(x)|^p d\mu(x) = p\int_0^\infty \lambda^{p-1}\mu_f(\lambda)d\lambda = \int_0^\infty f^*(t)^p dt.$$

Furthermore, in the case $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in R} |f(x)| = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0).$$

While the decreasing rearrangement does not necessarily preserve products of functions, we have the following integral inequality due to Hardy and Littlewood.

Theorem 1.2.7. *If $f, g \in \mathscr{M}_0(R,\mu)$, then*

$$\int_R |f(x)g(x)|d\mu(x) \leq \int_0^\infty f^*(s)g^*(s)ds.$$

Corollary 1.2.8. *Let $f \in \mathscr{M}_0(R,\mu)$. If $E \subseteq R$ is a set of positive measure t, then*

$$\frac{1}{\mu(E)}\int_E |f(x)|d\mu(x) \leq \frac{1}{t}\int_0^t f^*(s)ds.$$

Proof. Take $g(x) = \chi_E(x)$, the characteristic function of E. Then, $g^*(s) = \chi_{[0,\mu(E))}$, and applying the previous theorem, we obtain

$$\begin{aligned}\int_E |f(x)|d\mu(x) &= \int_R |f(x)g(x)|d\mu(x)\\ &\leq \int_0^\infty f^*(s)g^*(s)ds = \int_0^{\mu(E)} f^*(s)ds.\end{aligned}$$

Dividing by $\mu(E)$ and writing $t = \mu(E)$, we get the desired result. □

Definition 1.2.9. Let $f \in \mathscr{M}_0(R,\mu)$. Then f^{**} will denote the *maximal function* of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \ (t > 0).$$

Some elementary properties of the *maximal operator* $f \to f^{**}$ are listed below.

Proposition 1.2.10. *Consider $f, g, f_n, (n = 1, 2, \ldots)$ in $\mathscr{M}_0(R,\mu)$ and let a be any scalar. Then f^{**} is nonnegative, decreasing and continuous on $(0,\infty)$. Furthermore,*

$$f^{**} \equiv 0 \Leftrightarrow f = 0 \ \mu - a.e.;$$

$$f^* \leq f^{**};$$

$$
\begin{gathered}
|g| \leq |f| \ \mu - a.e. \Rightarrow g^{**} \leq f^{**}; \\
(af)^{**} = |a| f^{**}; \\
(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \ (0 < t < \infty); \\
|f_n| \uparrow |f| \ \mu - a.e. \Rightarrow f_n^{**} \uparrow f^{**}.
\end{gathered}
$$

Now we can give the definition of the Lorentz spaces.

Definition 1.2.11. Suppose $0 < p, q \leq \infty$. The *Lorentz space* $L^{p,q} = L^{p,q}(R, \mu)$ consists of all $f \in \mathscr{M}_0(R, \mu)$ for which the quantity

$$
\|f\|_{p,q} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \dfrac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{0<t<\infty} (t^{1/p} f^*(t)), & q = \infty, \end{cases}
$$

is finite.

Remark 1.2.12. It is clear that the Lorentz space $L^{p,p}$ coincides with the Lebesgue space L^p, for $0 < p \leq \infty$, and $\|f\|_{p,p} = \|f\|_p$, for all $f \in L^p$. Note also that the space $L^{\infty,q}$, for finite q, contains only the zero-function, and the space $L^{\infty,\infty}$ coincides with the space L^∞.

The following result shows that, for any fixed p, the Lorentz spaces $L^{p,q}$ increase as the second exponent q increases.

Proposition 1.2.13. *Suppose* $0 < p \leq \infty$ *and* $0 < q \leq r \leq \infty$. *Then,* $\|f\|_{p,r} \leq c \|f\|_{p,q}$, *for all* $f \in \mathscr{M}_0(R, \mu)$, *where* c *is a constant depending only on* p, q *and* r. *In particular,* $L^{p,q} \hookrightarrow L^{p,r}$.

Proof. We may assume $p < \infty$ and $q < r$ since in the other cases there is nothing to prove. Using the fact that f^* is decreasing, we have

$$
\begin{aligned}
t^{1/p} f^*(t) &= \left(\frac{q}{p} \int_0^t (s^{1/p} f^*(t))^q \frac{ds}{s} \right)^{1/q} \leq \left(\frac{q}{p} \int_0^t (s^{1/p} f^*(s))^q \frac{ds}{s} \right)^{1/q} \\
&\leq \left(\frac{q}{p} \right)^{1/q} \|f\|_{p,q}.
\end{aligned}
$$

Hence, taking the supremum over all $t > 0$, we obtain

$$
\|f\|_{p,\infty} \leq \left(\frac{q}{p} \right)^{1/q} \|f\|_{p,q}.
$$

This establishes the result in the case $r = \infty$. In the remaining case where $r < \infty$, we have

$$\begin{aligned}
\|f\|_{p,r} &= \left(\int_0^\infty (t^{1/p} f^*(t))^{r-q+q} \frac{dt}{t} \right)^{1/r} \leq \|f\|_{p,\infty}^{1-q/r} \|f\|_{p,q}^{q/r} \\
&\leq \left(\frac{q}{p} \right)^{(r-q)/rq} \|f\|_{p,q} . \quad \square
\end{aligned}$$

Remark 1.2.14. Embedding relations among spaces $L^{p,q}$, with p varying, depend on the structure of the underlying measure space. On finite measure spaces, if $0 < p < r \leq \infty$ and $0 < q, s \leq \infty$, then $L^{r,s} \hookrightarrow L^{p,q}$.

Definition 1.2.15. Given a totally σ-finite measure space (X, μ), we denote by $\mathcal{S}(X)$ the linear space generated by the functions of the form

$$f(x) = \sum_{1 \leq j \leq N} \sum_{|k| \leq N} 2^k \chi_{E_{k,j}}(x),$$

for $N \in \mathbb{N}$ and $\{E_{k,j}\}_{k,j} \subseteq X$, subsets of finite measure.

We have the following result [5, 22].

Lemma 1.2.16. *$\mathcal{S}(X)$ is dense in $L^{p,q}(X, \mu)$, for every $0 < p, q < \infty$.*

Proof. Consider first a positive function $f \in L^{p,q}$. Call $E_k(f) = \{x : 2^k \leq f(x) < 2^{k+1}\}$ and $\tilde{f} = f - \sum_{k \in \mathbb{Z}} 2^k \chi_{E_k(f)}$. It holds that $0 \leq \tilde{f} < f/2$ almost everywhere. Define $f_0 = f$ and $f_{j+1} = \tilde{f}_j$, for $j \geq 0$. Call $E_{k,j} = E_k(f_j)$. We have that

$$\begin{aligned}
f_{n+1} &= f_n - \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,n}} = \tilde{f}_{n-1} - \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,n}} \\
&= f_{n-1} - \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,n-1}} - \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,n}} \\
&= \cdots = f - \sum_{1 \leq j \leq n} \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,j}} .
\end{aligned}$$

Hence, for every $n \geq 1$, it holds that

$$f = f_{n+1} - \sum_{1 \leq j \leq n} \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,j}} .$$

By induction on n, we get that $0 \leq f_{n+1} < \frac{f}{2^{n+1}}$ almost everywhere, for every $n \geq 0$. Taking the limit for $n \to \infty$, we obtain the equality

$$f(x) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} 2^k \chi_{E_{k,j}}(x), \text{ a.e. } x \in X.$$

Now, for $N \in \mathbb{N}$, put
$$f_N(x) = \sum_{1\le j\le N} \sum_{|k|\le N} 2^k \chi_{E_{k,j}}(x).$$
It holds that $f_N \in \mathcal{S}(X)$, for every N. By Proposition 1.2.5, $f_N^* \le f^*$, so $\|f_N\|_{p,q} \le \|f\|_{p,q} < \infty$, and $f_N^* \uparrow f^*$. Hence, by Lebesgue Dominated Convergence Theorem [2], we obtain that $f_N \to f$ in $L^{p,q}$. Finally, for an arbitrary $f \in L^{p,q}$, write $f = f^+ - f^-$, where $f^+ = f\chi_{\{f>0\}}$ and $f^- = |f|\chi_{\{f<0\}}$ are both positive. By the previous argument, taking $\{f_N^+\}_N$ and $\{f_N^-\}_N$ in $\mathcal{S}(X)$ such that $f_N^+ \to f^+$ and $f_N^- \to f^-$ in $L^{p,q}$, and observing that $f_N := f_N^+ - f_N^- \in \mathcal{S}(X)$ for every N, we obtain that $f_N \to f$ in $L^{p,q}$. □

We next wish to determine for which values of p and q the Lorentz space $L^{p,q}$ may be regarded as a Banach space. The functional $f \mapsto \|f\|_{p,q}$ is not always a norm. Assume $1 \le q \le p < \infty$. Lorentz [17] proved that if ϕ is a nonnegative function defined on $(0,\infty)$, not identically 0 and such that $\int_0^l \phi(t)dt < \infty$, $\forall l \in [0,\infty)$, then the functional

$$f \mapsto \|f\| := \left(\int_0^\infty \phi(t) f^*(t)^q dt\right)^{1/q}, \forall f \in \mathscr{M}_0(R,\mu),$$

is a norm if, and only if ϕ is decreasing. In particular, for the case of the functional $f \mapsto \|f\|_{p,q}$, $\phi(t) = t^{q/p-1}$, which is decreasing if, and only if $q \le p$.

Theorem 1.2.17. *Suppose $1 \le q \le p < \infty$ or $p = q = \infty$. Then, the functional $f \mapsto \|f\|_{p,q}$ is a norm.*

Although the restriction $q \le p$ in this result is necessary, it can be circumvented in the case $p > 1$ by replacing $\|\cdot\|_{p,q}$ with an equivalent functional which is a norm for all $q \ge 1$.

Definition 1.2.18. Suppose $1 < p \le \infty$ and $0 < q \le \infty$. If $f \in \mathscr{M}_0(R,\mu)$, let

$$\|f\|_{(p,q)} := \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} (t^{1/p} f^{**}(t)), & q = \infty. \end{cases}$$

Lemma 1.2.19. *If $1 < p \le \infty$ and $1 \le q \le \infty$, then*

$$\|f\|_{p,q} \le \|f\|_{(p,q)} \le p' \|f\|_{p,q}$$

for all $f \in \mathscr{M}_0(R,\mu)$.

Proof. The first inequality is an immediate consequence of the definitions of $\|\cdot\|_{p,q}$ and $\|\cdot\|_{(p,q)}$, and the fact that $f^* \leq f^{**}$. The second follows directly from the first Hardy's inequality. $\square$

We have the following result.

Theorem 1.2.20. *If* $1 < p < \infty$*,* $1 \leq q \leq \infty$ *or* $p = q = \infty$*, then* $(L^{p,q}, \|f\|_{(p,q)})$ *is a Banach space.*

Let us now expose some definitions related to operators.

Definition 1.2.21. Suppose $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ and $q_0 \neq q_1$. Let σ denote the *interpolation segment*

$$\sigma = \left[\left(\frac{1}{p_0}, \frac{1}{q_0}\right), \left(\frac{1}{p_1}, \frac{1}{q_1}\right)\right],$$

that is, the line segment in the unit square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ with endpoints $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. Let m denote the slope

$$m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}$$

of the line segment σ. For each measurable function f on $(0, \infty)$ and each $t > 0$, let

$$(S_\sigma f)(t) := t^{-1/q_0} \int_0^{t^m} s^{1/p_0} f(s) \frac{ds}{s} + t^{-1/q_1} \int_{t^m}^{\infty} s^{1/p_1} f(s) \frac{ds}{s}.$$

The operator $S_\sigma : f \mapsto S_\sigma f$ is the *Calderón operator* associated with the interpolation segment σ.

Here are some simple properties of the operator S_σ [3, Ch. 3].

Proposition 1.2.22. *If f is a nonnegative measurable function on the interval $(0, \infty)$, then $S_\sigma f$ is decreasing and for each $t > 0$, $(S_\sigma f)(t) = (S_\sigma f)^*(t) \leq S_\sigma(f^*)(t)$.*

Proposition 1.2.23. *Let f be a μ-measurable function on R. $S_\sigma(f^*)(t) < \infty$ for each $t > 0$ if, and only if, $S_\sigma(f^*)(1) < \infty$.*

Proof. Assume that $S_\sigma(f^*)(1) < \infty$. Then

$$0 \leq \int_0^1 s^{1/p_0} f^*(s) \frac{ds}{s}, \int_1^{\infty} s^{1/p_1} f^*(s) \frac{ds}{s} < \infty.$$

Fix $0 < u < 1$. Then

$$0 \leq \int_0^u s^{1/p_0} f^*(s) \frac{ds}{s} \leq \int_0^1 s^{1/p_0} f^*(s) \frac{ds}{s} < \infty.$$

Now,

$$\int_u^\infty s^{1/p_1} f^*(s) \frac{ds}{s} = \int_u^1 s^{1/p_1} f^*(s) \frac{ds}{s} + \int_1^\infty s^{1/p_1} f^*(s) \frac{ds}{s}$$

and

$$\int_u^1 s^{1/p_1} f^*(s) \frac{ds}{s} \leq u^{1/p_1 - 1/p_0} \int_u^1 s^{1/p_0} f^*(s) \frac{ds}{s} < \infty,$$

since $p_0 < p_1$, thus

$$0 \leq \int_u^\infty s^{1/p_1} f^*(s) \frac{ds}{s} < \infty.$$

Similarly, we have that

$$0 \leq \int_0^u s^{1/p_0} f^*(s) \frac{ds}{s}, \int_u^\infty s^{1/p_1} f^*(s) \frac{ds}{s} < \infty$$

in the case $u \geq 1$. Hence, $S_\sigma(f^*)(t) < \infty$, for each $t > 0$. $\square$

Definition 1.2.24. Let T be an operator whose domain is some linear subspace of $\mathscr{M}_0(R, \mu)$ and whose range is contained in the ν-measurable functions on S. Then T is said to be *quasilinear* if there is a constant $k \geq 1$ such that the relations

$$|T(f+g)| \leq k(|Tf| + |Tg|), \ |T(\lambda f)| = |\lambda||Tf|$$

hold ν-a.e. on S for all f and g in the domain of T and for all scalars λ. If these relations hold for $k = 1$, then T is said to be *sublinear*.

Definition 1.2.25. Suppose $1 \leq p_0 < p_1 \leq \infty$ and $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$. Let T be a quasilinear operator with respect to (R, μ) and (S, ν), and suppose Tf is defined for all μ-measurable functions f on R. Then T is said to be of *joint weak type* $(p_0, q_0; p_1, q_1)$ if there is a constant c such that $(Tf)^*(t) \leq c S_\sigma(f^*)(t), (0 < t < \infty)$, for all f for which $S_\sigma(f^*)(1) < \infty$.

Definition 1.2.26. Suppose $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let T be an operator defined on $L^{p,1}(R, \mu)$ and taking values in $\mathscr{M}_0(S, \nu)$. Then T is said to be of *restricted weak type* (p, q) if it is a bounded operator from $L^{p,1}(R, \mu)$ into $L^{q,\infty}(S, \nu)$, that is, if there is a constant M such that $\|Tf\|_{q,\infty} \leq M \|f\|_{p,1}$ for all $f \in L^{p,1}(R, \mu)$. The least constant M is called the *restricted weak type* (p, q) *norm* of T.

Remark 1.2.27. Suppose $1 \leq p, q < \infty$ and $1 \leq r, s \leq \infty$. Since $L^{p,1} \hookrightarrow L^{p,r}$ and $L^{q,s} \hookrightarrow L^{q,\infty}$, it follows that if T is a bounded operator from $L^{p,r}$ into $L^{q,s}$, then T is of restricted weak type (p, q).

Definition 1.2.28. Suppose $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Let T be an operator defined on $L^p(R,\mu)$ and taking values in $\mathscr{M}_0(S,\nu)$. Then T is said to be of *strong type* (p,q) if it is a bounded operator from $L^p(R,\mu)$ into $L^q(S,\nu)$, that is, if there is a constant M such that $\|Tf\|_q \leq M\|f\|_p$ for all $f \in L^p(R,\mu)$. The least constant M is called the *strong type* (p,q) *norm* of T.

We present the following result. We will expose its proof later, after showing some technical lemmas.

Theorem 1.2.29. *Suppose* $1 \leq p_0 < p_1 < \infty$ *and* $1 \leq q_0, q_1 \leq \infty$ *with* $q_0 \neq q_1$ *and let* T *be a quasilinear operator defined on* $(L^{p_0,1} + L^{p_1,1})(R,\mu)$ *and taking values in* $\mathscr{M}_0(S,\nu)$. *If* T *is of restricted weak types* (p_0,q_0) *and* (p_1,q_1), *then* T *is of joint weak type* $(p_0,q_0;p_1,q_1)$.

Remark 1.2.30. The converse of this result is also true [3, Ch. 4].

Lemma 1.2.31. *Consider a* μ*-measurable function* $f : R \longrightarrow \mathbb{R}$. *If the function* f *belongs to* $(L^{p_0,1} + L^{p_1,1})(R,\mu)$, *then* $S_\sigma(f^*)(t) < \infty$ *for all* $t > 0$.

Proof. Fix $t > 0$ and recall that

$$S_\sigma(f^*)(t) = t^{\frac{-1}{q_0}} \int_0^{t^m} s^{\frac{1}{p_0}} f^*(s)\frac{ds}{s} + t^{\frac{-1}{q_1}} \int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s)\frac{ds}{s}.$$

To prove the result, it suffices to show that both integrals are finite. Let us assume that $f \in L^{p_0,1}(R,\mu)$. We have $\int_0^{t^m} s^{\frac{1}{p_0}} f^*(s)\frac{ds}{s} \leq \|f\|_{p_0,1} < \infty$ and since $p_0 < p_1$, $\int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s)\frac{ds}{s} = \int_{t^m}^\infty s^{\frac{1}{p_1}-\frac{1}{p_0}} s^{\frac{1}{p_0}} f^*(s)\frac{ds}{s} \leq t^{\frac{m}{p_1}-\frac{m}{p_0}} \|f\|_{p_0,1} < \infty$. Thus $S_\sigma(f^*)(t) < \infty$. A similar argument establishes the result for $f \in L^{p_1,1}(R,\mu)$. Now if $f \in (L^{p_0,1} + L^{p_1,1})(R,\mu)$, there exist $f_0 \in L^{p_0,1}(R,\mu)$ and $f_1 \in L^{p_1,1}(R,\mu)$ such that $f = f_0 + f_1$. Applying the properties of decreasing rearrangements and performing a change of variables, we obtain

$$\begin{aligned} S_\sigma(f^*)(t) \leq{}& 2^{\frac{1}{p_0}} t^{\frac{-1}{q_0}} \int_0^{\frac{t^m}{2}} r^{\frac{1}{p_0}} f_0^*(r)\frac{dr}{r} + 2^{\frac{1}{p_0}} t^{\frac{-1}{q_0}} \int_0^{\frac{t^m}{2}} r^{\frac{1}{p_0}} f_1^*(r)\frac{dr}{r} \\ &+ 2^{\frac{1}{p_1}} t^{\frac{-1}{q_1}} \int_{\frac{t^m}{2}}^\infty r^{\frac{1}{p_1}} f_0^*(r)\frac{dr}{r} + 2^{\frac{1}{p_1}} t^{\frac{-1}{q_1}} \int_{\frac{t^m}{2}}^\infty r^{\frac{1}{p_1}} f_1^*(r)\frac{dr}{r}. \end{aligned}$$

Since $f_i \in L^{p_i,1}(R,\mu)$, it holds that $S_\sigma(f_i^*)(\frac{t}{2^{1/m}}) < \infty$, for $i = 0, 1$. Thus the four integrals in the expression above are finite and $S_\sigma(f^*)(t) < \infty$. □

Lemma 1.2.32. *Consider a* μ*-measurable function* $f : R \longrightarrow \mathbb{R}$ *and a real value* $k \geq 0$, *and define the functions* $g(x) := \min(|f(x)|, k)$ *and* $h(x) := \max(|f(x)| - k, 0)$, *for all* $x \in R$. *Then, their decreasing rearrangements satisfy* $g^*(s) = \min(f^*(s), k)$ *and* $h^*(s) = \max(f^*(s) - k, 0)$, *for all* $s > 0$.

Proof. Let us first consider the distribution function of the function g, $\mu_g(\lambda) = \mu\{x \in R : |g(x)| > \lambda\}$. Since g is a minimum, it holds that $\{x \in R : |g(x)| > \lambda\} = \{x \in R : |f(x)| > \lambda\} \cap \{x \in R : k > \lambda\}$. Fix $\lambda \geq 0$. If $\lambda < k$, then $\{x \in R : k > \lambda\} = R$ and $\mu_g(\lambda) = \mu_f(\lambda)$. Similarly, if $\lambda \geq k$, then $\{x \in R : k > \lambda\} = \emptyset$ and $\mu_g(\lambda) = 0$.

Fix $s \geq 0$ and consider the decreasing rearrangement of the function g, $g^*(s) = \inf\{\lambda \geq 0 : \mu_g(\lambda) \leq s\}$. It holds that $\{\lambda \geq 0 : \mu_g(\lambda) \leq s\} = \{\lambda \geq k : \mu_g(\lambda) \leq s\} \cup \{k > \lambda \geq 0 : \mu_g(\lambda) \leq s\} = \{\lambda \geq k : 0 \leq s\} \cup \{k > \lambda \geq 0 : \mu_f(\lambda) \leq s\} = [k, \infty) \cup \{k > \lambda \geq 0 : \mu_f(\lambda) \leq s\}$. Recall that for subsets A and B of $\mathbb{R}$ it holds that $\inf A \cup B = \min(\inf A, \inf B)$. Thus, $g^*(s) = \min(k, \inf\{k > \lambda \geq 0 : \mu_f(\lambda) \leq s\})$. Since $\min(k, \inf\{k > \lambda \geq 0 : \mu_f(\lambda) \leq s\}) = \min(k, \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq s\})$, it follows that $g^*(s) = \min(f^*(s), k)$.

Let us consider now the distribution function of h, $\mu_h(\lambda) = \mu\{x \in R : |h(x)| > \lambda\}$. Since h is a maximum, it holds that $\{x \in R : |h(x)| > \lambda\} = \{x \in R : |f(x)| - k > \lambda\} \cup \{x \in R : 0 > \lambda\}$. Thus, $\mu_h(\lambda) = \mu_f(k+\lambda)$ for all $\lambda \geq 0$. Fix $s \geq 0$ and consider the decreasing rearrangement of the function h, $h^*(s) = \inf\{\lambda \geq 0 : \mu_h(\lambda) \leq s\}$. Since $\inf\{\lambda \geq 0 : \mu_f(k+\lambda) \leq s\} = \max(\inf\{\lambda \geq 0 : \mu_f(\lambda) \leq s\} - k, 0)$, it follows that $h^*(s) = \max(f^*(s) - k, 0)$. $\square$

Lemma 1.2.33. *Consider a μ-measurable function $f : X \longrightarrow \mathbb{R}$ and a real value $t > 0$, and define the functions $g(x) := f(x)\chi_{\{|f| > f^*(t)\}}(x)$ and $h(x) := f(x)\chi_{\{|f| \leq f^*(t)\}}(x)$. Then, their decreasing rearrangements satisfy*

$$g^*(s) \leq \begin{cases} f^*(s), & 0 < s < t, \\ 0, & s \geq t, \end{cases}$$

and

$$h^*(s) \leq \begin{cases} f^*(t), & 0 < s < t, \\ f^*(s), & s \geq t, \end{cases}$$

for all $s > 0$.

Proof. Let us first consider the distribution function of the function g, $\mu_g(\lambda) = \mu\{x \in X : |g(x)| > \lambda\}$. It holds that $\{x \in X : |g(x)| > \lambda\} = \{x \in X : |f(x)| > \max\{\lambda, f^*(t)\}\}$, so

$$\begin{aligned} \mu_g(\lambda) &= \begin{cases} \mu\{x \in X : |f(x)| > \lambda\}, & \lambda \geq f^*(t), \\ \mu\{x \in X : |f(x)| > f^*(t)\}, & \lambda < f^*(t), \end{cases} \\ &= \begin{cases} \mu_f(\lambda), & \lambda \geq f^*(t), \\ \mu_f(f^*(t)), & \lambda < f^*(t), \end{cases} \\ &\leq \begin{cases} \mu_f(\lambda), & \lambda \geq f^*(t), \\ t, & \lambda < f^*(t), \end{cases} \end{aligned}$$

where we have used the fact that $\mu_f(f^*(t)) \leq t$.

Fix $s \geq 0$ and consider the decreasing rearrangement of the function g, $g^*(s) = \inf\{\lambda \geq 0 : \mu_g(\lambda) \leq s\}$. It holds that $\{\lambda \geq 0 : \mu_g(\lambda) \leq s\} = \{0 \leq \lambda < f^*(t) : \mu_g(\lambda) \leq s\} \cup \{\lambda \geq f^*(t) : \mu_g(\lambda) \leq s\} \supseteq \{0 \leq \lambda < f^*(t) : t \leq s\} \cup \{\lambda \geq f^*(t) : \mu_f(\lambda) \leq s\}$. Hence,

$$\begin{aligned} g^*(s) &\leq \inf\left(\{0 \leq \lambda < f^*(t) : t \leq s\} \cup \{\lambda \geq f^*(t) : \mu_f(\lambda) \leq s\}\right) \\ &= \begin{cases} \inf\{\lambda \geq f^*(t) : \mu_f(\lambda) \leq s\}, & 0 < s < t, \\ 0, & s \geq t, \end{cases} \\ &\leq \begin{cases} f^*(s), & 0 < s < t, \\ 0, & s \geq t, \end{cases} \end{aligned}$$

since for $s < t$, $f^*(t) \leq f^*(s)$ and $\mu_f(f^*(s)) \leq s$, so $f^*(s) \in \{\lambda \geq f^*(t) : \mu_f(\lambda) \leq s\}$.

Let us consider now the distribution function of h. It holds that

$$\begin{aligned} \mu_h(\lambda) &= \mu\{x \in X : |h(x)| > \lambda\} = \mu\{x \in X : \lambda < |f(x)| \leq f^*(t)\} \\ &\leq \begin{cases} 0, & \lambda \geq f^*(t), \\ \mu_f(\lambda), & \lambda < f^*(t). \end{cases} \end{aligned}$$

Fix $s \geq 0$ and consider the decreasing rearrangement of the function h, $h^*(s) = \inf\{\lambda \geq 0 : \mu_h(\lambda) \leq s\}$. It holds that $\{\lambda \geq 0 : \mu_h(\lambda) \leq s\} \supseteq \{0 \leq \lambda < f^*(t) : \mu_f(\lambda) \leq s\} \cup \{\lambda \geq f^*(t) : 0 \leq s\}$. Hence,

$$\begin{aligned} h^*(s) &\leq \inf\left(\{0 \leq \lambda < f^*(t) : \mu_f(\lambda) \leq s\} \cup \{\lambda \geq f^*(t) : 0 \leq s\}\right) \\ &= \min\{\inf\{0 \leq \lambda < f^*(t) : \mu_f(\lambda) \leq s\}, f^*(t)\} \\ &= \min\{\inf\{\lambda \geq 0 : \mu_f(\lambda) \leq s\}, f^*(t)\} = \min\{f^*(s), f^*(t)\} \\ &= \begin{cases} f^*(t), & 0 < s < t, \\ f^*(s), & s \geq t. \end{cases} \end{aligned}$$

This completes the proof. □

Now we can give the proof of Theorem 1.2.29.

Proof. (**Thm. 1.2.29**) Let $f \in (L^{p_0,1} + L^{p_1,1})(R, \mu)$ and fix $t > 0$. Consider

$$m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}$$

and define f_0 and f_1 on R by $f_1(x) = \min(|f(x)|, f^*(t^m)) \cdot \operatorname{sgn} f(x)$ and $f_0(x) = f(x) - f_1(x) = \max(|f(x)| - f^*(t^m), 0) \cdot \operatorname{sgn} f(x)$. Then, by the previous lemma,

$f_1^*(s) = \min(f^*(s), f^*(t^m))$ for all $s > 0$ and since f^* is decreasing,

$$\begin{aligned}\|f_1\|_{p_1,1} &= \int_0^\infty s^{\frac{1}{p_1}} \min(f^*(s), f^*(t^m))\frac{ds}{s} = \int_0^{t^m} s^{\frac{1}{p_1}} f^*(t^m)\frac{ds}{s} \\ &+ \int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s)\frac{ds}{s} = p_1 t^{\frac{m}{p_1}} f^*(t^m) + \int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s)\frac{ds}{s}.\end{aligned}$$

Similarly, $f_0^*(s) = \max(f^*(s) - f^*(t^m), 0)$ for all $s > 0$ and

$$\begin{aligned}\|f_0\|_{p_0,1} &= \int_0^\infty s^{\frac{1}{p_0}} \max(f^*(s) - f^*(t^m), 0)\frac{ds}{s} = \int_0^{t^m} s^{\frac{1}{p_0}} f^*(s)\frac{ds}{s} \\ &- \int_0^{t^m} s^{\frac{1}{p_0}} f^*(t^m)\frac{ds}{s} = \int_0^{t^m} s^{\frac{1}{p_0}} f^*(s)\frac{ds}{s} - p_0 t^{\frac{m}{p_0}} f^*(t^m).\end{aligned}$$

Since $f \in (L^{p_0,1} + L^{p_1,1})(R,\mu)$, it holds that $S_\sigma(f^*)(t^m) < \infty$, thus $\|f_i\|_{p_i,1} < \infty$ and $f_i \in L^{p_i,1}(R,\mu)$, for $i = 0, 1$.

Now suppose that T is quasilinear with constant K. Since $f = f_0 + f_1$, by the properties of decreasing rearrangements we have $(Tf)^*(t) \le (K(|Tf_0| + |Tf_1|))^*(t) \le K\left((Tf_0)^*\left(\frac{t}{2}\right) + (Tf_1)^*\left(\frac{t}{2}\right)\right)$. Furthermore, the restricted weak type hypotheses on T give $(Tf_i)^*\left(\frac{t}{2}\right) \le \left(\frac{t}{2}\right)^{\frac{-1}{q_i}} M_i \|f_i\|_{p_i,1}$, for $i = 0, 1$. Combining these estimates, we obtain

$$(Tf)^*(t) \le C\left(\frac{t^{\frac{-1}{q_0}}}{p_0}\|f_0\|_{p_0,1} + \frac{t^{\frac{-1}{q_1}}}{p_1}\|f_1\|_{p_1,1}\right),$$

with $C = K \cdot \max_i(p_i M_i 2^{\frac{1}{q_i}})$. Incorporating the expressions of the norms and observing that the terms in $f^*(t^m)$ cancel, we find that $(Tf)^*(t) \le C \cdot S_\sigma(f^*)(t)$. Since $f \in (L^{p_0,1} + L^{p_1,1})(R,\mu)$, it holds that $S_\sigma(f^*)(t) < \infty$ for all $t > 0$. Hence, T is of joint weak type $(p_0, q_0; p_1, q_1)$. □

Remark 1.2.34. A simple modification of the proof above shows that if T is of restricted weak type (p_0, q_0), $p_0 < \infty$, and strong type (∞, q_1), then T is of joint weak type $(p_0, q_0; \infty, q_1)$. The expression for $\|f_1\|_{p_1,1}$ is replaced by the property that $\|f_1\|_\infty = f^*(t^m)$ and then the strong type (∞, q_1) hypothesis is invoked in the form $T : L^\infty \to L^{q_1} \hookrightarrow L^{q_1,\infty}$, so $(Tf_1)^*\left(\frac{t}{2}\right) \le \left(\frac{t}{2}\right)^{\frac{-1}{q_1}} M_1 \|f_1\|_\infty$.

This proof and some previous results allow us to characterize the space $L^{p_0,1} + L^{p_1,1}$ in terms of the Calderón operator S_σ.

Corollary 1.2.35. *Consider a μ-measurable function $f : R \longrightarrow \mathbb{R}$. The following conditions are equivalent:*

1. *The function f belongs to $(L^{p_0,1} + L^{p_1,1})(R, \mu)$;*
2. $S_\sigma(f^*)(t) < \infty$ *for all* $t > 0$*;*
3. $S_\sigma(f^*)(1) < \infty$.

Now we can give the statement and the proof of the main theorem of this section, the Marcinkiewicz interpolation theorem.

Theorem 1.2.36. *Suppose $1 \leq p_0 < p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$. Let $0 < \theta < 1$ and define p and q by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T be a quasilinear operator defined on $(L^{p_0,1} + L^{p_1,1})(R, \mu)$ and taking values in $\mathscr{M}_0(S, \nu)$. Suppose T is of restricted weak types (p_0, q_0) and (p_1, q_1), with restricted weak type norms M_0 and M_1, respectively. If $1 \leq r \leq \infty$, then

$$T : L^{p,r} \longrightarrow L^{q,r}.$$

That is, there is a constant c, depending only on p_0, q_0, p_1, q_1 and r, such that, for all $f \in L^{p,r}$,

$$\|Tf\|_{q,r} \leq \frac{c}{\theta(1-\theta)} \max(M_0, M_1) \, \|f\|_{p,r} .$$

Proof. Because of Theorem 1.2.29, we know that T is of joint weak type $(p_0, q_0; p_1, q_1)$ with joint weak type norm $M \leq c \cdot \max(M_0, M_1)$, where c depends only on p_0, q_0, p_1, q_1. Assume, first, that $r < \infty$. We have

$$\|Tf\|_{q,r} \leq M \left(\int_0^\infty \left(t^{\frac{1}{q}} S_\sigma(f^*)(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} .$$

Applying Minkowski's inequality for L^r, we obtain

$$\begin{aligned} \|Tf\|_{q,r} \leq{}& M \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_0}} \int_0^{t^m} s^{\frac{1}{p_0}} f^*(s) \frac{ds}{s} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} \\ &+ M \left(\int_0^\infty \left(t^{\frac{1}{q} - \frac{1}{q_1}} \int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s) \frac{ds}{s} \right)^r \frac{dt}{t} \right)^{\frac{1}{r}} . \end{aligned}$$

Making the change of variables $u = t^m$ in each of the integrals and using the relations

$$\frac{1}{m} \left(\frac{1}{q} - \frac{1}{q_0} \right) = \frac{1}{p} - \frac{1}{p_0}, \quad \frac{1}{m} \left(\frac{1}{q} - \frac{1}{q_1} \right) = \frac{1}{p} - \frac{1}{p_1},$$

we have that

$$\|Tf\|_{q,r} \leq M|m|^{\frac{-1}{r}} \left(\int_0^\infty \left(u^{\frac{1}{p}-\frac{1}{p_0}} \int_0^u s^{\frac{1}{p_0}} f^*(s) \frac{ds}{s} \right)^r \frac{du}{u} \right)^{\frac{1}{r}}$$
$$+ M|m|^{\frac{-1}{r}} \left(\int_0^\infty \left(u^{\frac{1}{p}-\frac{1}{p_1}} \int_u^\infty s^{\frac{1}{p_1}} f^*(s) \frac{ds}{s} \right)^r \frac{du}{u} \right)^{\frac{1}{r}}.$$

Applying first and second Hardy's inequalities to the first and the second terms, respectively, the resulting estimate reduces to

$$\begin{aligned}\|Tf\|_{q,r} &\leq M|m|^{\frac{-1}{r}} c_1 \left(\int_0^\infty \left(u^{\frac{1}{p}} f^*(u) \right)^r \frac{du}{u} \right)^{\frac{1}{r}} \\ &+ M|m|^{\frac{-1}{r}} c_2 \left(\int_0^\infty \left(u^{\frac{1}{p}} f^*(u) \right)^r \frac{du}{u} \right)^{\frac{1}{r}} \\ &= M|m|^{\frac{-1}{r}} (c_1 + c_2) \|f\|_{p,r},\end{aligned}$$

with

$$\frac{1}{c_1} = \theta \left(\frac{1}{p_0} - \frac{1}{p_1} \right), \quad \frac{1}{c_2} = (1-\theta) \left(\frac{1}{p_0} - \frac{1}{p_1} \right).$$

Now assume that $r = \infty$. In this case, we have

$$\begin{aligned}\|Tf\|_{q,\infty} &= \sup_{0<t<\infty} t^{\frac{1}{q}} (Tf)^*(t) \leq M \sup_{0<t<\infty} t^{\frac{1}{q}} S_\sigma(f^*)(t) \\ &\leq M \sup_{0<t<\infty} \left(t^{\frac{1}{q}-\frac{1}{q_0}} \int_0^{t^m} s^{\frac{1}{p_0}} f^*(s) \frac{ds}{s} \right) \\ &+ M \sup_{0<t<\infty} \left(t^{\frac{1}{q}-\frac{1}{q_1}} \int_{t^m}^\infty s^{\frac{1}{p_1}} f^*(s) \frac{ds}{s} \right) \\ &\leq M \sup_{0<t<\infty} \left(u^{\frac{1}{p}-\frac{1}{p_0}} \int_0^u s^{\frac{1}{p_0}} f^*(s) \frac{ds}{s} \right) \\ &+ M \sup_{0<t<\infty} \left(u^{\frac{1}{p}-\frac{1}{p_1}} \int_u^\infty s^{\frac{1}{p_1}} f^*(s) \frac{ds}{s} \right) \\ &\leq M(c_1 + c_2) \|f\|_{p,\infty}.\end{aligned}$$

This completes the proof. $\square$

Corollary 1.2.37. *With parameters as above, suppose in addition that $p_0 \leq q_0$ and $p_1 \leq q_1$. If T is of restricted weak types (p_0, q_0) and (p_1, q_1), with norms M_0 and M_1, respectively, then*

$$T : L^p \longrightarrow L^q.$$

That is, there is a constant c, depending only on p_0, q_0, p_1, q_1, such that, for all $f \in L^p$,

$$\|Tf\|_q \leq \frac{c}{\theta(1-\theta)} \max(M_0, M_1) \|f\|_p .$$

Let us see what we can say when $p_0 = p_1$ and $q_0 \neq q_1$ or $p_0 < p_1$ and $q_0 = q_1$.

Lemma 1.2.38. *Suppose $1 \leq q_1 < q < q_0 \leq \infty$. If $1 \leq r \leq \infty$, then $L^{q_0,\infty} \cap L^{q_1,\infty} \hookrightarrow L^{q,r}$.*

Proof. Since $L^{q,1} \hookrightarrow L^{q,r}$, it suffices to show that $L^{q_0,\infty} \cap L^{q_1,\infty} \hookrightarrow L^{q,1}$. A measurable function f belongs to $L^{q_0,\infty} \cap L^{q_1,\infty}$ if, and only if $\sup_{0<t<\infty} t^{1/q_0} f^*(t) < \infty$ and $\sup_{0<t<\infty} t^{1/q_1} f^*(t) < \infty$. This condition holds if, and only if there exists a constant $c \geq 0$ such that

$$\begin{cases} f^*(t) \leq ct^{-1/q_0}, & (0 < t < 1), \\ f^*(t) \leq ct^{-1/q_1}, & (t \geq 1). \end{cases}$$

Now,

$$\begin{aligned} \|f\|_{q,1} &= \int_0^\infty t^{1/q-1} f^*(t) dt \leq c \int_0^1 t^{1/q-1-1/q_0} dt + c \int_1^\infty t^{1/q-1-1/q_1} dt \\ &= \frac{c}{\frac{1}{q} - \frac{1}{q_0}} + \frac{c}{\frac{1}{q_1} - \frac{1}{q}} < \infty, \end{aligned}$$

and hence, $f \in L^{q,1}$. $\square$

Remark 1.2.39. Under the hypotheses of Theorem 1.2.36 and in the case $p_0 = p_1$, this lemma implies that $T : L^{p_0,1} \to L^{q,r}$.

Lemma 1.2.40. *Suppose $1 \leq p_0 < p < p_1 < \infty$. If $1 \leq r \leq \infty$, then $L^{p,r} \hookrightarrow L^{p_0,1} + L^{p_1,1}$.*

Proof. Since $L^{p,r} \hookrightarrow L^{p,\infty}$, it suffices to show that $L^{p,\infty} \hookrightarrow L^{p_0,1} + L^{p_1,1}$. A measurable function f belongs to $L^{p,\infty}$ if, and only if $\sup_{0<t<\infty} t^{1/p} f^*(t) < \infty$ and this condition holds if, and only if there exists a constant $c \geq 0$ such that $f^*(t) \leq ct^{-1/p}$, for all $t > 0$. Now,

$$\begin{aligned} \int_0^1 t^{1/p_0} f^*(t) \frac{dt}{t} + \int_1^\infty t^{1/p_1} f^*(t) \frac{dt}{t} &\leq c \int_0^1 t^{1/p_0-1-1/p} dt + c \int_1^\infty t^{1/p_1-1-1/p} dt \\ &= \frac{c}{\frac{1}{p_0} - \frac{1}{p}} + \frac{c}{\frac{1}{p} - \frac{1}{p_1}} < \infty, \end{aligned}$$

and in virtue of Corollary 1.2.35, $f \in L^{p_0,1} + L^{p_1,1}$. $\square$

Remark 1.2.41. Under the hypotheses of Theorem 1.2.36 and in the case $q_0 = q_1$, this lemma implies that $T : L^{p,r} \to L^{q_0,\infty}$.

Remark 1.2.42. Theorem 1.2.36 holds also in the case $p_1 = \infty$ provided the restricted weak type (p_1, q_1) hypothesis is replaced by the strong type (p_1, q_1) hypothesis. In that case, the operator is of joint weak type $(p_0, q_0; \infty, q_1)$ and the proof of Theorem 1.2.36 carries over almost verbatim.

Remark 1.2.43. Under the hypotheses of Theorem 1.2.36 we obtain that $T : L^{p,r} \to L^{q,s}$ in the case $r \leq s$, but in general this result is not true if $r > s$. Consider the measure space $(\mathbb{R}^+, m)$, where m denotes the Lebesgue measure on $\mathbb{R}^+$. Take $p_0 = q_0, p_1 = q_1$ and consider the identity operator $T = Id : (L^{p_0,1} + L^{p_1,1})(\mathbb{R}^+, m) \to (L^{p_0,1} + L^{p_1,1})(\mathbb{R}^+, m)$, which is of restricted weak types (p_0, p_0) and (p_1, p_1). If $1 \leq s < r = \infty$, then $T : L^{p,\infty} \nrightarrow L^{p,s}$, since for the function $f(t) = t^{-1/p}$, $(t > 0)$, we have $f^* = f$, $\|f\|_{p,\infty} = 1$ and $\|f\|_{p,s} = \infty$.

Chapter 2

Real Interpolation Method

2.1 Preliminaries

The *K-method* of interpolation, or real method, may be regarded as a lifting of the Marcinkiewicz interpolation theorem from its classical context in spaces of measurable functions to an abstract Banach space setting. In order to study this method, we need some basic definitions and results concerning interpolation spaces. The details of the proofs can be found in the book of Bennett and Sharpley [3, Ch. 3].

Definition 2.1.1. A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a *compatible couple* if there is some Hausdorff topological vector space $\mathscr{X}$ in which each of X_0 and X_1 is continuously embedded.

Remark 2.1.2. The pair (L^1, L^∞) is a compatible couple because both L^1 and L^∞ are continuously embedded in the Hausdorff space $\mathscr{M}_0$ of measurable functions that are finite almost everywhere.

Definition 2.1.3. Let (X_0, X_1) be a compatible couple, with corresponding Hausdorff space $\mathscr{X}$. Let $X_0 + X_1$ denote the *sum* of X_0 and X_1, that is, the set of elements $x \in \mathscr{X}$ that are representable in the form $x = x_0 + x_1$, for some $x_0 \in X_0$ and $x_1 \in X_1$. For each x in $X_0 + X_1$, set

$$\|x\|_{X_0+X_1} = \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + \|x_1\|_{X_1}\},$$

where the infimum extends over all representations $x = x_0 + x_1$ of x with $x_0 \in X_0$ and $x_1 \in X_1$. For each element x in the intersection $X_0 \cap X_1$ of X_0 and X_1, set

$$\|x\|_{X_0 \cap X_1} = \max\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\}.$$

Theorem 2.1.4. *If (X_0, X_1) is a compatible couple, then $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces under the norms $\|\cdot\|_{X_0+X_1}$ and $\|\cdot\|_{X_0\cap X_1}$, respectively.*

Definition 2.1.5. If (X_0, X_1) is a compatible couple, then a Banach space X is said to be an *intermediate space* between X_0 and X_1 if X is continuously embedded between $X_0 \cap X_1$ and $X_0 + X_1$:

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

Observe that X_0 and X_1 are always intermediate spaces for the couple (X_0, X_1).

Now we turn our attention on operators defined on these spaces. We denote by $\mathscr{B}(X, Y)$ (or $\mathscr{B}(X)$, if $X = Y$) the space of bounded linear operators from a Banach space X into a Banach space Y. The space $\mathscr{B}(X, Y)$ is itself a Banach space under the operator norm

$$\|T\|_{\mathscr{B}(X,Y)} = \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}.$$

Definition 2.1.6. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples and let T be a linear operator defined on $X_0 + X_1$ and taking values in $Y_0 + Y_1$. Then T is said to be *admissible* with respect to the couples (X_0, X_1) and (Y_0, Y_1) if the restriction of T to X_i maps X_i into Y_i, for each $i = 0, 1$, and, furthermore, is a bounded operator from X_i into Y_i:

$$\|Tx\|_{Y_i} \leq \|T\|_{\mathscr{B}(X_i,Y_i)} \|x\|_{X_i}, \ \forall x \in X_i.$$

The class of admissible operators is denoted by

$$\mathscr{A} = \mathscr{A}(X_0, X_1; Y_0, Y_1).$$

The norm of an admissible operator T is given by

$$\|T\|_{\mathscr{A}} = \max_{i=0,1}\{\|T\|_{\mathscr{B}(X_i,Y_i)}\}.$$

Proposition 2.1.7. *Every admissible operator T is a bounded operator from $X_0 + X_1$ into $Y_0 + Y_1$, and*

$$\|T\|_{\mathscr{B}(X_0+X_1,Y_0+Y_1)} \leq \|T\|_{\mathscr{A}}.$$

Proof. Suppose T is admissible and consider $x \in X_0 + X_1$. Let $x = x_0 + x_1$ be any representation of x as a sum of elements $x_0 \in X_0$ and $x_1 \in X_1$. Then, by the previous definitions, we have

$$\begin{aligned}\|Tx\|_{Y_0+Y_1} &= \|Tx_0 + Tx_1\|_{Y_0+Y_1} \leq \|Tx_0\|_{Y_0} + \|Tx_1\|_{Y_1} \\ &\leq \|T\|_{\mathscr{B}(X_0,Y_0)} \|x_0\|_{X_0} + \|T\|_{\mathscr{B}(X_1,Y_1)} \|x_1\|_{X_1} \\ &\leq \|T\|_{\mathscr{A}} (\|x_0\|_{X_0} + \|x_1\|_{X_1}).\end{aligned}$$

Taking the infimum over all representations $x = x_0 + x_1$ of x, we obtain that

$$\|Tx\|_{Y_0+Y_1} \leq \|T\|_{\mathscr{A}} \|x\|_{X_0+X_1}.$$

This concludes the proof. □

Theorem 2.1.8. *The class $\mathscr{A}(X_0, X_1; Y_0, Y_1)$ of admissible operators is a Banach space when equipped with the norm $\|\cdot\|_{\mathscr{A}}$. Furthermore, $\mathscr{A}$ is continuously embedded in $\mathscr{B}(X_0 + X_1, Y_0 + Y_1)$.*

We finish this section with the formulation of the interpolation property.

Definition 2.1.9. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples. Let X and Y be intermediate spaces of the couples (X_0, X_1) and (Y_0, Y_1), respectively. The pair (X, Y) is said to have the *interpolation property* relative to (X_0, X_1) and (Y_0, Y_1) if every admissible operator maps X into Y.

2.2 The K-Functional

We begin this section with some definitions and elementary properties of the K-interpolation method.

Definition 2.2.1. Let (X_0, X_1) be a compatible couple of Banach spaces. The Peetre *K-functional* is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

Since

$$\min(1, t)\|f\|_{X_0+X_1} \leq K(f, t; X_0, X_1) \leq \max(1, t)\|f\|_{X_0+X_1},$$

the functionals $f \mapsto K(f, t; X_0, X_1)$, $t > 0$, define a family of mutually equivalent norms on $X_0 + X_1$.

Since every $f \in X_0$ has the trivial representation $f = f + 0$ as a member of $X_0 + X_1$, we have that

$$K(f, t; X_0, X_1) \leq \|f\|_{X_0}, \forall f \in X_0, t > 0.$$

Similarly, for $f \in X_1$, we have

$$K(f, t; X_0, X_1) \leq t\|f\|_{X_1}, \forall f \in X_1, t > 0.$$

Proposition 2.2.2. *For each $f \in X_0 + X_1$, the K-functional $K(f,t;\, X_0, X_1)$ is a nonnegative concave function of $t > 0$, and*

$$t^{-1}K(f,t;X_0,X_1) = K(f,t^{-1};X_1,X_0).$$

In particular, $K(f,t;X_0,X_1)$ is increasing on $(0,\infty)$ and $t^{-1}K(f,t;\, X_0,X_1)$ is decreasing.

Proof. All the properties follow immediately from the definition, except for concavity. To stablish this, consider $t_1, t_2 > 0$ and let t be the convex combination $t = \alpha_1 t_1 + \alpha_2 t_2$, where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. For each decomposition $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$, we have

$$\begin{aligned}
&\alpha_1 K(f,t_1;X_0,X_1) + \alpha_2 K(f,t_2;X_0,X_1) \leq \alpha_1(\|f_0\|_{X_0} + t_1\|f_1\|_{X_1}) \\
&+ \alpha_2(\|f_0\|_{X_0} + t_2\|f_1\|_{X_1}) = (\alpha_1+\alpha_2)\|f_0\|_{X_0} + (\alpha_1 t_1 + \alpha_2 t_2)\|f_1\|_{X_1} \\
&= \|f_0\|_{X_0} + t\|f_1\|_{X_1}.
\end{aligned}$$

Taking the infimum over all decompositions $f = f_0 + f_1$ of f, we obtain

$$\alpha_1 K(f,t_1;X_0,X_1) + \alpha_2 K(f,t_2;X_0,X_1) \leq K(f,t;X_0,X_1),$$

and the result follows. □

Proposition 2.2.3. *The K-functional $K(f,t;\, X_0,X_1)$ is subadditive, that is, given $f, g \in X_0 + X_1$ and $t > 0$, we have*

$$K(f+g,t;X_0,X_1) \leq K(f,t;X_0,X_1) + K(g,t;X_0,X_1).$$

Proof. It holds that

$$K(f+g,t;X_0,X_1) \leq \inf_{f=f_0+f_1} \inf_{g=g_0+g_1} \{\|f_0+g_0\|_{X_0} + t\|f_1+g_1\|_{X_1}\},$$

where the infimums are taken over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$, and all representations $g = g_0 + g_1$ of g with $g_0 \in X_0$ and $g_1 \in X_1$, respectively. Now, by the triangular inequalities for $\|\cdot\|_{X_0}$ and $\|\cdot\|_{X_1}$ and the properties of the infimum, we have that

$$\begin{aligned}
&\inf_{f=f_0+f_1} \inf_{g=g_0+g_1} \{\|f_0+g_0\|_{X_0} + t\|f_1+g_1\|_{X_1}\} \\
&\leq \inf_{f=f_0+f_1} \inf_{g=g_0+g_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1} + \|g_0\|_{X_0} + t\|g_1\|_{X_1}\} \\
&= \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\} + \inf_{g=g_0+g_1} \{\|g_0\|_{X_0} + t\|g_1\|_{X_1}\} \\
&= K(f,t;X_0,X_1) + K(g,t;X_0,X_1).
\end{aligned}$$

This completes the proof. □

Theorem 2.2.4. *Let (X_0, X_1) and (Y_0, Y_1) be compatible couples and suppose that $X_j \hookrightarrow Y_j$, $j = 0, 1$. Then, there exists a constant $c > 0$, depending only on the spaces X_0, X_1, Y_0, Y_1, such that*

$$K(f, t; Y_0, Y_1) \leq cK(f, t; X_0, X_1), \, \forall f \in X_0 + X_1, \, t > 0.$$

Proof. Since $X_0 + X_1 \hookrightarrow Y_0 + Y_1$, we have that

$$\begin{aligned} K(f, t; Y_0, Y_1) &= \inf_{f=\bar{f}_0+\bar{f}_1} \{\|\bar{f}_0\|_{Y_0} + t\,\|\bar{f}_1\|_{Y_1}, \bar{f}_j \in Y_j, j = 0, 1\} \\ &\leq \inf_{f=f_0+f_1} \{\|f_0\|_{Y_0} + t\,\|f_1\|_{Y_1}, f_j \in X_j, j = 0, 1\}. \end{aligned}$$

By hypothesis, there exist constants $c_0, c_1 > 0$ such that $\|\cdot\|_{Y_j} \leq c_j \|\cdot\|_{X_j}$, $j = 0, 1$. Taking $c = \max\{c_0, c_1\}$, we obtain

$$\begin{aligned} &\inf_{f=f_0+f_1} \{\|f_0\|_{Y_0} + t\,\|f_1\|_{Y_1}\} \leq \inf_{f=f_0+f_1} \{c_0\,\|f_0\|_{X_0} + c_1 t\,\|f_1\|_{X_1}\} \\ &\leq \max\{c_0, c_1\} \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1}\} = cK(f, t; X_0, X_1). \end{aligned}$$

This completes the proof. □

2.3 A Basic K-Interpolation Result

We have seen that the Marcinkiewicz interpolation theorem has a natural formulation in terms of the Lorentz $L^{p,q}$-spaces. Now, if $1 < p \leq \infty, 1 \leq q \leq \infty$, then $L^{p,q}$ is an intermediate space between L^1 and L^∞. Furthermore, and as we will see later, the $L^{p,q}$-norm can be defined entirely in terms of the K-functional for (L^1, L^∞). We impose now an analogous structure on any compatible couple (X_0, X_1) by defining a two-parameter family of intermediate spaces as follows:

Definition 2.3.1. Let (X_0, X_1) be a compatible couple and suppose $0 < \theta < 1$, $1 \leq q < \infty$ or $0 \leq \theta \leq 1$, $q = \infty$. The space $(X_0, X_1)_{\theta,q}$ consists of all f in $X_0 + X_1$ for which the functional

$$\|f\|_{\theta,q} := \begin{cases} \left(\displaystyle\int_0^\infty (t^{-\theta} K(f, t; X_0, X_1))^q \frac{dt}{t} \right)^{1/q}, & 0 < \theta < 1, 1 \leq q, \\ \displaystyle\sup_{0<t<\infty} t^{-\theta} K(f, t; X_0, X_1), & 0 \leq \theta \leq 1, q = \infty, \end{cases}$$

is finite.

Notation. We will use the notations $X_0 + \infty X_1 := (X_0, X_1)_{0,\infty}$ and $X_1 + \infty X_0 := (X_0, X_1)_{1,\infty}$.

Theorem 2.3.2. *Let (X_0, X_1) be a compatible couple of Banach spaces and suppose $0 < \theta < 1$, $1 \leq q < \infty$ or $0 \leq \theta \leq 1$, $q = \infty$. Then $(X_0, X_1)_{\theta,q}$ equipped with the norm $\|\cdot\|_{\theta,q}$ is a Banach space intermediate between X_0 and X_1.*

Given that the structure of the spaces (X_0, X_1) is modelled on that of the $L^{p,q}$-spaces, it is to be expected that they will satisfy similar embedding relations. We establish the following result.

Proposition 2.3.3. *If $0 < \theta < 1$ and $1 \leq q \leq r \leq \infty$, then*

$$(X_0, X_1)_{\theta,q} \hookrightarrow (X_0, X_1)_{\theta,r}.$$

Proof. We may assume $q < r$ since in the other case there is nothing to prove. Using the fact that $K(f, t; X_0, X_1)$ is increasing, we have

$$\begin{aligned} t^{-\theta} K(f, t; X_0, X_1) &= \left(\theta q \int_t^\infty (s^{-\theta} K(f, t; X_0, X_1))^q \frac{ds}{s} \right)^{1/q} \\ &\leq \left(\theta q \int_t^\infty (s^{-\theta} K(f, s; X_0, X_1))^q \frac{ds}{s} \right)^{1/q} \\ &\leq (\theta q)^{1/q} \|f\|_{\theta,q}. \end{aligned}$$

Hence, taking the supremum over all $t > 0$, we obtain

$$\|f\|_{\theta,\infty} \leq (\theta q)^{1/q} \|f\|_{\theta,q}.$$

This establishes the result in the case $r = \infty$. In the remaining case where $r < \infty$, we have

$$\begin{aligned} \|f\|_{\theta,r} &= \left(\int_0^\infty (t^{-\theta} K(f, t; X_0, X_1))^{r-q+q} \frac{dt}{t} \right)^{1/r} \leq \|f\|_{\theta,\infty}^{1-q/r} \|f\|_{\theta,q}^{q/r} \\ &\leq (\theta q)^{(r-q)/rq} \|f\|_{\theta,q}. \quad \square \end{aligned}$$

Theorem 2.3.4. *Let T be an admissible linear operator with respect to compatible couples (X_0, X_1) and (Y_0, Y_1). Then*

$$K(Tf, t; Y_0, Y_1) \leq M_0 K(f, tM_1/M_0; X_0, X_1), \ \forall f \in X_0 + X_1, t > 0.$$

Proof. The admissible operator T satisfies

$$\|Tf_i\|_{Y_i} \leq M_i \|f_i\|_{X_i}, \ \forall f_i \in X_i, i = 0, 1.$$

If $f \in X_0 + X_1$ and $f = f_0 + f_1$ is any decomposition of f with $f_0 \in X_0$ and $f_1 \in X_1$, then $Tf = Tf_0 + Tf_1$ and $Tf_i \in Y_i$, $i = 0, 1$. Hence,

$$\begin{aligned} K(Tf, t; Y_0, Y_1) &\leq \|Tf_0\|_{Y_0} + t\,\|Tf_1\|_{Y_1} \\ &\leq M_0 \left(\|f_0\|_{X_0} + t\frac{M_1}{M_0}\,\|f_1\|_{X_1} \right). \end{aligned}$$

Taking the infimum over all representations $f = f_0 + f_1$ of f, we obtain the desired result. $\square$

Applying this result to the (θ, q)-spaces, we obtain the following basic interpolation theorem.

Theorem 2.3.5. *Let (X_0, X_1) and (Y_0, Y_1) be compatible couples and let $0 < \theta < 1, 1 \leq q < \infty$ or $1 \leq \theta \leq 1, q = \infty$. Let T be an admissible linear operator with respect to (X_0, X_1) and (Y_0, Y_1), with*

$$\|Tf_i\|_{Y_i} \leq M_i\,\|f_i\|_{X_i}\,, \forall f_i \in X_i, i = 0, 1.$$

Then

$$T : (X_0, X_1)_{\theta,q} \longrightarrow (Y_0, Y_1)_{\theta,q},$$

with

$$\|Tf\|_{\theta,q} \leq M_0^{1-\theta} M_1^{\theta}\,\|f\|_{\theta,q}\,.$$

Proof. We start the discussion with the case $q = \infty$. In virtue of the previous theorem, and performing the change of variables $s = tM_1/M_0$ we have

$$\begin{aligned} \|Tf\|_{\theta,\infty} &= \sup_{t>0} t^{-\theta} K(Tf, t; Y_0, Y_1) \leq M_0 \sup_{t>0} t^{-\theta} K(f, tM_1/M_0; X_0, X_1) \\ &= M_0 \sup_{s>0} \left(s\frac{M_0}{M_1} \right)^{-\theta} K(f, s; X_0, X_1) = M_0^{1-\theta} M_1^{\theta}\,\|f\|_{\theta,\infty}\,. \end{aligned}$$

Similarly, for the case $q < \infty$ we have

$$\begin{aligned} \|Tf\|_{\theta,q} &= \left(\int_0^\infty (t^{-\theta} K(Tf, t; Y_0, Y_1))^q \frac{dt}{t} \right)^{1/q} \\ &\leq \left(M_0^q \int_0^\infty (t^{-\theta} K(f, tM_1/M_0; X_0, X_1))^q \frac{dt}{t} \right)^{1/q} \\ &= M_0^{1-\theta} M_1^{\theta} \left(\int_0^\infty (s^{-\theta} K(f, s; X_0, X_1))^q \frac{ds}{s} \right)^{1/q} \\ &= M_0^{1-\theta} M_1^{\theta}\,\|f\|_{\theta,q}\,. \end{aligned}$$

This completes the proof. $\square$

2.4 The General K-Interpolation Theorem

Let (X_0, X_1) be a compatible couple and consider two interpolation spaces

$$\overline{X}_{\theta_0} = (X_0, X_1)_{\theta_0,q_0}, \, \overline{X}_{\theta_1} = (X_0, X_1)_{\theta_1,q_1},$$

where $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0, q_1 \leq \infty$. Then $(\overline{X}_{\theta_0}, \overline{X}_{\theta_1})$ is itself a compatible couple. The following result, due to Holmstedt, relates the K-functionals of both couples.

Notation. We say that two positive quantities A and B are *equivalent* if there exist two constants $c_1, c_2 > 0$ independent of the essential parameters defining A and B, such that $c_1 A \leq B \leq c_2 A$. We use the notation $A \sim B$. If we only have $B \leq c_2 A$, we write $B \lesssim A$.

Theorem 2.4.1. *Let (X_0, X_1) be a compatible couple and suppose $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0, q_1 \leq \infty$. Let $\delta = \theta_1 - \theta_0$. Then,*

$$\begin{aligned} K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \sim & \left(\int_0^t (s^{-\theta_0} K(f, s; X_0, X_1))^{q_0} \frac{ds}{s} \right)^{1/q_0} \\ & + t^\delta \left(\int_t^\infty (s^{-\theta_1} K(f, s; X_0, X_1))^{q_1} \frac{ds}{s} \right)^{1/q_1}, \end{aligned}$$

for all $f \in \overline{X}_{\theta_0} + \overline{X}_{\theta_1}$ and all $t > 0$; if q_0 or q_1 is infinite, the corresponding integral in this expression is replaced by the supremum in the usual way.

Proof. For $j = 0, 1$, let

$$P_j g(t) := \left(\int_0^t (s^{-\theta_j} K(g, s; X_0, X_1))^{q_j} \frac{ds}{s} \right)^{1/q_j}$$

and

$$Q_j g(t) := \left(\int_t^\infty (s^{-\theta_j} K(g, s; X_0, X_1))^{q_j} \frac{ds}{s} \right)^{1/q_j},$$

with the usual modification if $q_j = \infty$. By the subadditivity of $K(g, s; \, X_0, X_1)$ and the Minkowski's inequality for L^{q_j}, it follows that P_j and Q_j are subadditive. Now, we can express the desired result in the form

$$K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \sim P_0 f(t) + t^\delta Q_1 f(t).$$

Suppose first that $f \in \overline{X}_{\theta_0} + \overline{X}_{\theta_1}$ and fix $t > 0$. Let $f = g + h$ be any representation of f with $g \in \overline{X}_{\theta_0}$ and $h \in \overline{X}_{\theta_1}$. Then, using the subadditivity of P_0 and Q_1, we obtain

$$P_0 f(t) + t^\delta Q_1 f(t) \leq \|g\|_{\overline{X}_{\theta_0}} + P_0 h(t) + t^\delta \left(Q_1 g(t) + \|h\|_{\overline{X}_{\theta_1}} \right).$$

Since $\overline{X}_{\theta_0} = (X_0, X_1)_{\theta_0,q_0} \hookrightarrow (X_0, X_1)_{\theta_0,\infty}$, there exists a constant $c_0 > 0$ depending only on θ_0 and q_0 such that

$$\sup_{s>0} s^{-\theta_0} K(g, s; X_0, X_1) \leq c_0 \|g\|_{\overline{X}_{\theta_0}} .$$

Hence, for all $s > 0$ we have that

$$K(g, s; X_0, X_1) \leq c_0 s^{-\theta_0} \|g\|_{\overline{X}_{\theta_0}} .$$

With this estimate and performing the integration, we find that

$$\begin{aligned} Q_1 g(t) &\leq c_0 \|g\|_{\overline{X}_{\theta_0}} \left(\int_t^\infty s^{(\theta_0-\theta_1)q_1-1} ds \right)^{1/q_1} \\ &= c_0 \left(\frac{1}{(\theta_1 - \theta_0)q_1} \right)^{1/q_1} t^{-\delta} \|g\|_{\overline{X}_{\theta_0}} . \end{aligned}$$

Similarly, there exists a constant $c_1 > 0$ depending only on θ_1 and q_1 such that

$$P_0 h(t) \leq c_1 \left(\frac{1}{(\theta_1 - \theta_0)q_0} \right)^{1/q_0} t^{\delta} \|h\|_{\overline{X}_{\theta_1}} .$$

Combining all these estimates, we obtain that there exists a constant $c > 0$ depending only on $\theta_0, \theta_1, q_0, q_1$ such that

$$P_0 f(t) + t^\delta Q_1 f(t) \leq c \left(\|g\|_{\overline{X}_{\theta_0}} + t^\delta \|h\|_{\overline{X}_{\theta_1}} \right) .$$

Hence, passing to the infimum over all such representations $f = g + h$ of f, we conclude that

$$P_0 f(t) + t^\delta Q_1 f(t) \leq cK(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}).$$

Conversely, suppose that $f \in X_0 + X_1$ and that $P_0 f(t)$ and $Q_1 f(t)$ are finite. We shall show that $f \in \overline{X}_{\theta_0} + \overline{X}_{\theta_1}$ and

$$K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \leq c \left(P_0 f(t) + t^\delta Q_1 f(t) \right) ,$$

with $c > 0$ a constant depending only on $\theta_0, \theta_1, q_0, q_1$. Let $f = g + h$ be a representation of f with $g \in X_0$, $h \in X_1$, and

$$\|g\|_{X_0} + t \|h\|_{X_1} \leq 2K(f, t; X_0, X_1).$$

Then, for all $s > 0$ we obtain that

$$K(g, s; X_0, X_1) \leq \|g\|_{X_0} \leq 2K(f, t; X_0, X_1)$$

and

$$K(h,s;X_0,X_1) \le s\,\|h\|_{X_1} \le \frac{2s}{t}K(f,t;X_0,X_1).$$

With these estimates and performing the integration, we find that

$$\begin{aligned} Q_0g(t) &= \left(\int_t^\infty (s^{-\theta_0}K(g,s;X_0,X_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} \\ &\le 2K(f,t;X_0,X_1)\left(\int_t^\infty s^{-\theta_0 q_0-1}ds\right)^{1/q_0} \\ &= 2\left(\frac{1}{\theta_0 q_0}\right)^{1/q_0} t^{-\theta_0}K(f,t;X_0,X_1). \end{aligned}$$

Using the fact that $t^{-1}K(f,t;X_0,X_1)$ is decreasing on $(0,\infty)$, we get that

$$\begin{aligned} P_0f(t) &= \left(\int_0^t (s^{1-\theta_0-1}K(f,s;X_0,X_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} \\ &\ge \left(\int_0^t (s^{1-\theta_0}t^{-1}K(f,t;X_0,X_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} \\ &= t^{-1}K(f,t;X_0,X_1)\left(\int_0^t s^{(1-\theta_0)q_0-1}ds\right)^{1/q_0} \\ &= \left(\frac{1}{(1-\theta_0)q_0}\right)^{1/q_0} t^{-\theta_0}K(f,t;X_0,X_1). \end{aligned}$$

Hence,

$$Q_0g(t) \le 2\left(\frac{1-\theta_0}{\theta_0}\right)^{1/q_0} P_0f(t).$$

Similarly, we obtain that

$$\begin{aligned} P_0h(t) &= \left(\int_0^t (s^{-\theta_0}K(h,s;X_0,X_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} \\ &\le 2t^{-1}K(f,t;X_0,X_1)\left(\int_0^t s^{(1-\theta_0)q_0-1}ds\right)^{1/q_0} \\ &\le 2\left(\frac{1}{(1-\theta_0)q_0}\right)^{1/q_0} t^{-\theta_0}K(f,t;X_0,X_1) \le 2P_0f(t). \end{aligned}$$

Since $g=f-h$, by the subadditivity of P_0 we obtain that $P_0g \le P_0f + P_0h$. The previous estimates show that there exists a constant $c_0>0$ depending only on θ_0 and q_0 such that

$$\|g\|_{\overline{X}_{\theta_0}} = (P_0+Q_0)g(t) \le c_0P_0f(t) < \infty,$$

which proves, in particular, that $g \in \overline{X}_{\theta_0}$.

Using the fact that $K(f, t;\, X_0, X_1)$ is increasing on $(0, \infty)$, we get that

$$\begin{aligned}
Q_1 f(t) &= \left(\int_t^\infty (s^{-\theta_1} K(f, s; X_0, X_1))^{q_1} \frac{ds}{s} \right)^{1/q_1} \\
&\geq K(f, t; X_0, X_1) \left(\int_t^\infty s^{-\theta_1 q_1 - 1} ds \right)^{1/q_1} \\
&= \left(\frac{1}{\theta_1 q_1} \right)^{1/q_1} t^{-\theta_1} K(f, t; X_0, X_1).
\end{aligned}$$

Following the previous arguments, we obtain that

$$\begin{aligned}
P_1 h(t) &= \left(\int_0^t (s^{-\theta_1} K(h, s; X_0, X_1))^{q_1} \frac{ds}{s} \right)^{1/q_1} \\
&\leq 2t^{-1} K(f, t; X_0, X_1) \left(\int_0^t s^{(1-\theta_1)q_1 - 1} ds \right)^{1/q_1} \\
&\leq 2 \left(\frac{1}{(1-\theta_1) q_1} \right)^{1/q_1} t^{-\theta_1} K(f, t; X_0, X_1) \\
&\leq 2 \left(\frac{\theta_1}{1-\theta_1} \right)^{1/q_1} Q_1 f(t)
\end{aligned}$$

and

$$\begin{aligned}
Q_1 g(t) &= \left(\int_t^\infty (s^{-\theta_1} K(g, s; X_0, X_1))^{q_1} \frac{ds}{s} \right)^{1/q_1} \\
&\leq 2K(f, t; X_0, X_1) \left(\int_t^\infty s^{-\theta_1 q_1 - 1} ds \right)^{1/q_1} \\
&= 2 \left(\frac{1}{\theta_1 q_1} \right)^{1/q_1} t^{-\theta_1} K(f, t; X_0, X_1) \leq 2 Q_1 f(t).
\end{aligned}$$

Since $h = f - g$, by the subadditivity of Q_1 we have that $Q_1 h \leq Q_1 f + Q_1 g$. The previous estimates show that there exists a constant $c_1 > 0$ depending only on θ_1 and q_1 such that

$$\|h\|_{\overline{X}_{\theta_1}} = (P_1 + Q_1) h(t) \leq c_1 Q_1 f(t) < \infty,$$

and so $h \in \overline{X}_{\theta_1}$. Combining all the previous results we get that there exists a constant $c > 0$ depending only on $\theta_0, \theta_1, q_0, q_1$ such that

$$K(f, t^\delta; \overline{X}_0, \overline{X}_1) \leq \|g\|_{\overline{X}_{\theta_0}} + t^\delta \|h\|_{\overline{X}_{\theta_1}} \leq c \left(P_0 f(t) + t^\delta Q_1 f(t) \right),$$

which is the desired estimate. □

Remark 2.4.2. This theorem also holds for the case $0 < q_0, q_1 < 1$ and for a more general type of spaces rather than Banach spaces, the so-called *quasi-normed* spaces [13].

Definition 2.4.3. Suppose $0 \leq \theta \leq 1$. An intermediate space X of a compatible couple (X_0, X_1) is said to be of *class* θ if $0 < \theta < 1$ and $(X_0, X_1)_{\theta,1} \hookrightarrow X \hookrightarrow (X_0, X_1)_{\theta,\infty}$, or $\theta = 0$ and $X_0 \hookrightarrow X \hookrightarrow X_0 + \infty X_1$, or $\theta = 1$ and $X_1 \hookrightarrow X \hookrightarrow X_1 + \infty X_0$.

The following result corresponds to the extreme cases $\theta_0 = 0$ and $\theta_1 = 1$ of the previous theorem [13, 3, Ch. 5].

Theorem 2.4.4. *Let (X_0, X_1) be a compatible couple and let $\overline{X}_0$ and $\overline{X}_1$ be intermediate spaces of (X_0, X_1) of class 0 and 1, respectively. Suppose $0 < \theta < 1$, $1 \leq q \leq \infty$ and let $X_{\theta,q} = (X_0, X_1)_{\theta,q}$. Then*

$$K(f, t^\theta; \overline{X}_0, X_{\theta,q}) \sim t^\theta \left(\int_t^\infty (s^{-\theta} K(f, s; X_0, X_1))^q \frac{ds}{s} \right)^{1/q},$$

for all $f \in \overline{X}_0 + X_{\theta,q}$ and all $t > 0$, and

$$K(f, t^{1-\theta}; X_{\theta,q}, \overline{X}_1) \sim \left(\int_0^t (s^{-\theta} K(f, s; X_0, X_1))^q \frac{ds}{s} \right)^{1/q},$$

for all $f \in X_{\theta,q} + \overline{X}_1$ and all $t > 0$; with the usual modifications if $q = \infty$.

Now we stablish the reiteration theorem, which shows that the interpolation spaces $(\overline{X}_{\theta_0}, \overline{X}_{\theta_1})_{\theta,q}$ can be obtained as interpolation spaces from the original couple (X_0, X_1).

Theorem 2.4.5. *Let (X_0, X_1) be a compatible couple and suppose $0 < \theta_0 < \theta_1 < 1$. Let $\overline{X}_{\theta_j}$ be an intermediate space of (X_0, X_1) of class θ_j, $j = 0, 1$. If $0 < \theta < 1$ and $1 \leq q \leq \infty$, then*

$$(\overline{X}_{\theta_0}, \overline{X}_{\theta_1})_{\theta,q} = (X_0, X_1)_{\theta',q},$$

with equivalent norms, where $\theta' = (1 - \theta)\theta_0 + \theta\theta_1$.

Proof. Let $\delta = \theta_1 - \theta_0$. By hypothesis,

$$(X_0, X_1)_{\theta_j,1} \hookrightarrow \overline{X}_{\theta_j} \hookrightarrow (X_0, X_1)_{\theta_j,\infty},\ j = 0, 1.$$

By Theorem 2.4.1 with $q_0 = q_1 = \infty$, we have that

$$t^{-\theta_0} K(f, t; X_0, X_1) \lesssim K(f, t^\delta; (X_0, X_1)_{\theta_0,\infty}, (X_0, X_1)_{\theta_1,\infty}),$$

and since $\overline{X}_{\theta_j} \hookrightarrow (X_0, X_1)_{\theta_j,\infty}$, $j = 0, 1$, we obtain that

$$K(f, t^\delta; (X_0, X_1)_{\theta_0,\infty}, (X_0, X_1)_{\theta_1,\infty}) \lesssim K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}).$$

If q is finite, this, with the change of variables $s = t^\delta$ and the definition of θ' gives

$$\begin{aligned}
\|f\|^q_{(X_0,X_1)_{\theta',q}} &= \int_0^\infty (t^{\theta_0-\theta'-\theta_0} K(f, t; X_0, X_1))^q \frac{dt}{t} \\
&\lesssim \int_0^\infty (t^{\theta_0-\theta'} K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}))^q \frac{dt}{t} \\
&\sim \int_0^\infty (s^{(\theta_0-\theta')/(\theta_1-\theta_0)} K(f, s; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}))^q \frac{ds}{s} \\
&= \int_0^\infty (s^{-\theta} K(f, s; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}))^q \frac{ds}{s} = \|f\|^q_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,q}} .
\end{aligned}$$

Similarly, for the case $q = \infty$ we have

$$\begin{aligned}
\|f\|_{(X_0,X_1)_{\theta',\infty}} &= \sup_{t>0} t^{\theta_0-\theta'-\theta_0} K(f, t; X_0, X_1) \lesssim \sup_{t>0} t^{\theta_0-\theta'} K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \\
&= \sup_{s>0} s^{-\theta} K(f, s; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) = \|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,\infty}} .
\end{aligned}$$

These estimates imply that $(\overline{X}_{\theta_0}, \overline{X}_{\theta_1})_{\theta,q} \hookrightarrow (X_0, X_1)_{\theta',q}$. To complete the proof, we need to establish the embedding in the opposite direction.

Using that $(X_0, X_1)_{\theta_j,1} \hookrightarrow \overline{X}_{\theta_j}$, $j = 0, 1$, and applying Theorem 2.4.1 with $q_0 = q_1 = 1$, we obtain

$$\begin{aligned}
&K(f, t^\delta; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \lesssim K(f, t^\delta; (X_0, X_1)_{\theta_0,1}, (X_0, X_1)_{\theta_1,1}) \\
&\lesssim \int_0^t s^{-\theta_0} K(f, s; X_0, X_1) \frac{ds}{s} + t^\delta \int_t^\infty s^{-\theta_1} K(f, s; X_0, X_1) \frac{ds}{s}.
\end{aligned}$$

If q is finite, this, with the change of variables $u = t^\delta$ and the Minkowski's inequality for L^q, gives

$$\begin{aligned}
&\|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,q}} = \left(\int_0^\infty (u^{-\theta} K(f, u; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}))^q \frac{du}{u} \right)^{1/q} \\
&\lesssim \left(\int_0^\infty \left(t^{-\delta\theta} \int_0^t s^{-\theta_0} K(f, s; X_0, X_1) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&+ \left(\int_0^\infty \left(t^{\delta(1-\theta)} \int_t^\infty s^{-\theta_1} K(f, s; X_0, X_1) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} .
\end{aligned}$$

For the first summand, applying the first of Hardy's inequalities with $\lambda = 1-\delta\theta < 1$ and exponent q, and taking into account the definition of θ', we obtain

$$\left(\int_0^\infty \left(t^{-\delta\theta}\int_0^t s^{-\theta_0}K(f,s;X_0,X_1)\frac{ds}{s}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_0^\infty (t^{-\delta\theta-\theta_0}K(f,t;X_0,X_1))^q\frac{dt}{t}\right)^{1/q} = \|f\|_{(X_0,X_1)_{\theta',q}}.$$

For the second summand, applying the second of Hardy's inequalities with $\lambda = 1-\delta(1-\theta) < 1$ and exponent q, and taking into account the definition of θ', we obtain

$$\left(\int_0^\infty \left(t^{\delta(1-\theta)}\int_t^\infty s^{-\theta_1}K(f,s;X_0,X_1)\frac{ds}{s}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_0^\infty (t^{\delta(1-\theta)-\theta_1}K(f,t;X_0,X_1))^q\frac{dt}{t}\right)^{1/q} = \|f\|_{(X_0,X_1)_{\theta',q}}.$$

This establishes the desired result for finite q. Similarly, for the case $q=\infty$, we have

$$\begin{aligned}
\|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,\infty}} &= \sup_{u>0} u^{-\theta}K(f,u;\overline{X}_{\theta_0},\overline{X}_{\theta_1}) \\
&\lesssim \sup_{t>0} t^{-\delta\theta}\int_0^t s^{-\theta_0}K(f,s;X_0,X_1)\frac{ds}{s} \\
&+ \sup_{t>0} t^{\delta(1-\theta)}\int_t^\infty s^{-\theta_1}K(f,s;X_0,X_1)\frac{ds}{s} \\
&\lesssim \sup_{t>0} t^{-\delta\theta-\theta_0}K(f,t;X_0,X_1) + \sup_{t>0} t^{\delta(1-\theta)-\theta_1}K(f,t;X_0,X_1) \\
&\sim \|f\|_{(X_0,X_1)_{\theta',\infty}}.
\end{aligned}$$

This completes the proof. □

Remark 2.4.6. This theorem also holds for the cases $0=\theta_0<\theta_1<1$, $0<\theta_0<\theta_1=1$ and $\theta_0=0, \theta_1=1$, but the proof is slightly different [3, Ch. 5].

Via the reiteration theorem, the basic K-interpolation theorem of the previous section can be generalized as follows:

Theorem 2.4.7. *Let (X_0,X_1) and (Y_0,Y_1) be compatible couples and suppose $0 \leq \theta_0 < \theta_1 \leq 1$, $0 \leq \psi_0, \psi_1 \leq 1$ with $\psi_0 \neq \psi_1$. Let $\overline{X}_{\theta_j}$ and $\overline{Y}_{\psi_j}$ be intermediate spaces of (X_0,X_1) and (Y_0,Y_1) of class θ_j and ψ_j, respectively, for $j=0,1$. Let T be a linear operator satisfying*

$$\|Tf\|_{\overline{Y}_{\psi_j}} \leq M_j \|f\|_{\overline{X}_{\theta_j}},\ j=0,1.$$

If $0 < \theta < 1$ *and* $1 \leq q \leq \infty$, *then*

$$\|Tf\|_{(Y_0,Y_1)_{\psi',q}} \leq cM_0^{1-\theta}M_1^{\theta}\|f\|_{(X_0,X_1)_{\theta',q}},$$

where $(\theta', \psi') = (1-\theta)(\theta_0, \psi_0) + \theta(\theta_1, \psi_1)$.

Proof. The hypotheses imposed over T, together with Theorem 2.3.4 give

$$K(Tf, t; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1}) \leq M_0 K\left(f, t\frac{M_1}{M_0}; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right).$$

In the case $q = \infty$, and performing the change of variables $s = tM_1/M_0$, we have

$$\begin{aligned}
\|Tf\|_{(\overline{Y}_{\psi_0},\overline{Y}_{\psi_1})_{\theta,\infty}} &= \sup_{t>0} t^{-\theta} K(Tf, t; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1}) \\
&\leq M_0 \sup_{t>0} t^{-\theta} K\left(f, t\frac{M_1}{M_0}; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right) \\
&= M_0 \sup_{s>0} \left(s\frac{M_0}{M_1}\right)^{-\theta} K(f, s; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) \\
&= M_0^{1-\theta}M_1^{\theta}\|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,\infty}}.
\end{aligned}$$

Similarly, for the case $q < \infty$ we have

$$\begin{aligned}
\|Tf\|_{(\overline{Y}_{\psi_0},\overline{Y}_{\psi_1})_{\theta,q}} &= \left(\int_0^\infty (t^{-\theta}K(Tf, t; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1}))^q \frac{dt}{t}\right)^{1/q} \\
&\leq \left(M_0^q \int_0^\infty \left(t^{-\theta} K\left(f, t\frac{M_1}{M_0}; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right)\right)^q \frac{dt}{t}\right)^{1/q} \\
&= M_0^{1-\theta}M_1^{\theta}\left(\int_0^\infty (s^{-\theta}K(f, s; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}))^q \frac{ds}{s}\right)^{1/q} \\
&= M_0^{1-\theta}M_1^{\theta}\|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,q}}.
\end{aligned}$$

Applying the reiteration theorem and taking into account the definition of (θ', ψ'), we obtain that

$$\begin{aligned}
&\|Tf\|_{(Y_0,Y_1)_{\psi',q}} \lesssim \|Tf\|_{(\overline{Y}_{\psi_0},\overline{Y}_{\psi_1})_{\theta,q}} \\
&\leq M_0^{1-\theta}M_1^{\theta}\|f\|_{(\overline{X}_{\theta_0},\overline{X}_{\theta_1})_{\theta,q}} \lesssim M_0^{1-\theta}M_1^{\theta}\|f\|_{(X_0,X_1)_{\theta',q}}.
\end{aligned}$$

This completes the proof. $\square$

Remark 2.4.8. The Calderón operator S_σ for the interpolation segment

$$\sigma = [(1-\theta_0, 1-\psi_0), (1-\theta_1, 1-\psi_1)]$$

is lurking in the background here. Indeed, since $\overline{Y}_{\psi_j} \hookrightarrow (Y_0, Y_1)_{\psi_j,\infty}$ for $j = 0, 1$, the K-functional $K(Tf, t; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1})$ can be estimated from below applying Theorem 2.4.1 with $q_0 = q_1 = \infty$, obtaining

$$\begin{aligned} t^{-\psi_0} K(Tf, t; Y_0, Y_1) &\lesssim K(Tf, t^{\psi_1 - \psi_0}; (Y_0, Y_1)_{\psi_0,\infty}, (Y_0, Y_1)_{\psi_1,\infty}) \\ &\lesssim K(Tf, t^{\psi_1 - \psi_0}; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1}). \end{aligned}$$

Similarly, if $0 < \theta_j < 1$, the embedding $(X_0, X_1)_{\theta_j,1} \hookrightarrow \overline{X}_{\theta_j}$, $j = 0, 1$, and Theorem 2.4.1 with $q_0 = q_1 = 1$, give raise to the estimates

$$\begin{aligned} K(f, t; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}) &\lesssim K(f, t; (X_0, X_1)_{\theta_0,1}, (X_0, X_1)_{\theta_1,1}) \\ &\lesssim \int_0^{t^{1/(\theta_1-\theta_0)}} s^{-\theta_0} K(f, s; X_0, X_1) \frac{ds}{s} \\ &\quad + t \int_{t^{1/(\theta_1-\theta_0)}}^{\infty} s^{-\theta_1} K(f, s; X_0, X_1) \frac{ds}{s}. \end{aligned}$$

Recall that

$$K(Tf, t^{\psi_1-\psi_0}; \overline{Y}_{\psi_0}, \overline{Y}_{\psi_1}) \leq M_0 K\left(f, t^{\psi_1-\psi_0} \frac{M_1}{M_0}; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right).$$

Combining all these estimates, we obtain that

$$\begin{aligned} &\frac{K(Tf, t; Y_0, Y_1)}{t} \\ &\lesssim M_0 t^{\psi_0 - 1} \int_0^{\left(\frac{M_1}{M_0}\right)^{\frac{1}{\theta_1-\theta_0}} t^{\frac{\psi_1-\psi_0}{\theta_1-\theta_0}}} s^{1-\theta_0} \frac{K(f, s; X_0, X_1)}{s} \frac{ds}{s} \\ &+ M_1 t^{\psi_1 - 1} \int_{\left(\frac{M_1}{M_0}\right)^{\frac{1}{\theta_1-\theta_0}} t^{\frac{\psi_1-\psi_0}{\theta_1-\theta_0}}}^{\infty} s^{1-\theta_1} \frac{K(f, s; X_0, X_1)}{s} \frac{ds}{s} \end{aligned}$$

The right-hand side is essentially the S_σ-operator applied to the function $K(f, s; X_0, X_1)/s$. Defining $M = (M_1/M_0)^{1/(\theta_1-\theta_0)}$ and $m = (\psi_1 - \psi_0)/(\theta_1 - \theta_0)$, multiplying each side by $t^{1-\psi'}$, taking the $L^q(dt/t)$-norm and applying the Minkowski's inequality for L^q, performing the chancge of variables $u = Mt^m$, applying the first of Hardy's inequalities with $\lambda = 1 + (\psi_0 - \psi')/m < 1$ and exponent q, and the second of Hardy's inequalities with $\lambda = 1 - (\psi_1 - \psi')/m < 1$ and exponent q, and taking into account the definition of (θ', ψ') we get that

$$\begin{aligned}
&\|Tf\|_{(Y_0,Y_1)_{\psi',q}} \\
&\lesssim M_0 \left(\int_0^\infty \left(t^{\psi_0 - \psi'} \int_0^{Mt^m} s^{-\theta_0} K(f,s;X_0,X_1) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&+ M_1 \left(\int_0^\infty \left(t^{\psi_1 - \psi'} \int_{Mt^m}^\infty s^{-\theta_1} K(f,s;X_0,X_1) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&\lesssim M_0^{1-\theta} M_1^\theta \left(\int_0^\infty \left(u^{(\psi_0 - \psi')/m} \int_0^u s^{-\theta_0} K(f,s;X_0,X_1) \frac{ds}{s} \right)^q \frac{du}{u} \right)^{1/q} \\
&+ M_0^{1-\theta} M_1^\theta \left(\int_0^\infty \left(u^{(\psi_1 - \psi')/m} \int_u^\infty s^{-\theta_1} K(f,s;X_0,X_1) \frac{ds}{s} \right)^q \frac{du}{u} \right)^{1/q} \\
&\lesssim M_0^{1-\theta} M_1^\theta \left(\int_0^\infty (u^{-\theta'} K(f,u;X_0,X_1))^q \frac{du}{u} \right)^{1/q} \\
&= M_0^{1-\theta} M_1^\theta \|f\|_{(X_0,X_1)_{\theta',q}},
\end{aligned}$$

where the constants implicit in the symbol $\lesssim$ do not depend on M_0, M_1. This is exactly the procedure used previously to establish the Marcinkiewicz interpolation theorem. It illustrates once again how closely the abstract K-method is modelled on the classical Marcinkiewicz theory.

2.5 Some Examples of K-Functionals

Since the essence of the K-interpolation structure resides in the K-functional itself, we devote this section to determine in concrete terms the K-functionals for a variety of specific pairs of Banach spaces. We start computing the K-functional for the compatible couple (L^1, L^∞).

Theorem 2.5.1. *Let (R,μ) be a totally σ-finite measure space. Then, for each $f \in (L^1 + L^\infty)(R,\mu)$,*

$$K(f,t;L^1,L^\infty) = \int_0^t f^*(s)ds = tf^{**}(t),\ t > 0.$$

Proof. The second equality follows from the definition of f^{**}. Fix $f \in L^1 + L^\infty$ and $t > 0$. We show first that

$$\int_0^t f^*(s)ds \leq K(f,t;L^1,L^\infty).$$

Let $f = g + h$ be any representation of f with $g \in L^1$ and $h \in L^\infty$. The subadditivity of f^{**} gives

$$\int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds + \int_0^t h^*(s)ds$$

and hence, by Proposition 1.2.6 and the fact that f^* is decreasing,

$$\int_0^t f^*(s)ds \leq \int_0^\infty g^*(s)ds + th^*(0) = \|g\|_1 + t\,\|h\|_\infty .$$

Taking the infimum over all possible representations $f = g + h$, we obtain the desired estimate.

For the reverse inequality

$$K(f, t; L^1, L^\infty) \leq \int_0^t f^*(s)ds,$$

it will suffice to construct functions $g \in L^1$ and $h \in L^\infty$ such that $f = g + h$ and

$$\|g\|_1 + t\,\|h\|_\infty \leq \int_0^t f^*(s)ds.$$

Clearly, the right-hand side may be assumed to be finite, otherwise there is nothing to prove. Then Corollary 1.2.8 guarantees the integrability of f over any subset of R of measure at most t. Thus, if we let $E = \{x : |f(x)| > f^*(t)\}$ and set $t_0 = \mu(E)$, Proposition 1.2.5 gives $t_0 \leq t$ and so f is integrable over E. In particular, for the functions

$$g(x) = \max\{|f(x)| - f^*(t), 0\} \cdot \mathrm{sgn}\ f(x)$$

and

$$h(x) = \min\{|f(x)|, f^*(t)\} \cdot \mathrm{sgn}\ f(x),$$

we have

$$\begin{aligned}\|g\|_1 &= \int_0^\infty \max\{|f(x)| - f^*(t), 0\}d\mu(x) = \int_E |f(x)|d\mu(x) - \mu(E)f^*(t) \\ &\leq \int_0^{t_0} f^*(s)ds - t_0 f^*(t) \leq \int_0^t f^*(s)ds < \infty\end{aligned}$$

and

$$\|h\|_\infty \leq f^*(t) < \infty,$$

so $g \in L^1$ and $h \in L^\infty$. Moreover,

$$\|g\|_1 + t\,\|h\|_\infty \leq \int_0^{t_0} f^*(s)ds + (t - t_0)f^*(t).$$

But by Proposition 1.2.5, $f^*(s)$ is constant and equal to $f^*(t)$ whenever $t_0 \leq s \leq t$, so the last estimate in fact coincides with the desired result. Since $f = g + h$, the proof is complete. □

As a consequence of this theorem and the definition of the $L^{p,q}$-norm in terms of f^{**}, we have the following result.

Theorem 2.5.2. *If $0 < \theta < 1$ and $1 \leq q \leq \infty$, then*

$$(L^1, L^\infty)_{\theta,q} = L^{p,q},$$

where $1/p = 1 - \theta$.

Combining these results with Theorem 2.4.1 and its corollaries, we obtain descriptions of the K-functionals for the Lebesgue and Lorentz spaces.

Theorem 2.5.3. *Suppose $1 < p < r < \infty$ and $1 \leq q, s \leq \infty$. Let $\delta = 1/p - 1/r$. Then,*

$$K(f, t; L^{p,q}, L^{r,s}) \sim \left(\int_0^{t^{1/\delta}} (u^{1/p} f^{**}(u))^q \frac{du}{u} \right)^{1/q} + t \left(\int_{t^{1/\delta}}^\infty (u^{1/r} f^{**}(u))^s \frac{du}{u} \right)^{1/s},$$

for all $f \in L^{p,q} + L^{r,s}$ and all $t > 0$; if q or s is infinite, the corresponding integral in this expression is replaced by the supremum in the usual way.

Corollary 2.5.4. *Suppose $1 < p < r < \infty$. Let $\delta = 1/p - 1/r$. Then,*

$$K(f, t; L^p, L^r) \sim \left(\int_0^{t^{1/\delta}} (f^{**}(u))^p du \right)^{1/p} + t \left(\int_{t^{1/\delta}}^\infty (f^{**}(u))^r du \right)^{1/r},$$

for all $f \in L^p + L^r$ and all $t > 0$.

Theorem 2.5.5. *Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Then*

$$K(f, t; L^1, L^{p,q}) \sim t \left(\int_{t^{p/(p-1)}}^\infty (s^{1/p} f^{**}(s))^q \frac{ds}{s} \right)^{1/q},$$

for all $f \in L^1 + L^{p,q}$ and all $t > 0$, and

$$K(f, t; L^{p,q}, L^\infty) \sim \left(\int_0^{t^p} (s^{1/p} f^{**}(s))^q \frac{ds}{s} \right)^{1/q},$$

for all $f \in L^{p,q} + L^\infty$ and all $t > 0$; with the usual modifications if $q = \infty$.

Corollary 2.5.6. *Suppose* $1 < p < \infty$. *Then*

$$K(f,t;L^1,L^p) \sim t\left(\int_{t^{p/(p-1)}}^{\infty} (f^{**}(s))^p ds\right)^{1/p},$$

for all $f \in L^1 + L^p$ *and all* $t > 0$, *and*

$$K(f,t;L^p,L^\infty) \sim \left(\int_0^{t^p} (f^{**}(s))^p ds\right)^{1/p},$$

for all $f \in L^p + L^\infty$ *and all* $t > 0$.

Remark 2.5.7. These results also hold for parameters $0 < p < r \leq \infty$ and $0 < q, s \leq \infty$, as shown by Holmstedt [13]. Moreover, the function f^{**} can be replaced by f^* in the expressions above and the corresponding descriptions of the K-functionals are also true [1, 13, 14, 20].

Remark 2.5.8. The previous theorems provide descriptions of the K-functionals up to certain multiplicative constants implicit in the symbol $\sim$. Several authors have computed exact formulas for some pairs of Lorentz spaces. Nilsson and Peetre gave a formula for the couple (L^1, L^p) with $1 < p < \infty$ [19], and Ericsson proved formulas for the couples $(L^{p,1}, L^\infty)$ and $(L^{p,\infty}, L^\infty)$ with $0 < p < \infty$, and $(L^{p/q,1}, L^{p,q})$ with $1 < q \leq p < \infty$ [8].

Chapter 3

Multilinear Interpolation Theory

3.1 Brief History of Multilinear Interpolation

Multilinear interpolation is a powerful tool that yields intermediate estimates from a finite set of initial estimates for operators of several variables. In particular, the real multilinear interpolation method yields strong type bounds for multilinear or multi-sublinear operators as a consequence of initial restricted weak type estimates. We start this section describing what is a multilinear operator.

Definition 3.1.1. Let $n \geq 1$ be an integer. For $1 \leq j \leq n$, let (X_j, μ_j) and (Y, ν) be totally σ-finite measure spaces. Let $\mathscr{S}(X_j)$ be the space of simple functions on X_j. Let T be a map defined on $\mathscr{S}(X_1) \times \cdots \times \mathscr{S}(X_n)$ that takes values in the measure space (Y, ν). Then T is called *multilinear* if for all f_j, g_j in $\mathscr{S}(X_j)$ and all scalars λ we have

$$T(f_1, \ldots, \lambda f_j, \ldots, f_n) = \lambda T(f_1, \ldots, f_j, \ldots, f_n),$$

$$T(f_1, \ldots, f_j + g_j, \ldots, f_n) = T(f_1, \ldots, f_j, \ldots, f_n) + T(f_1, \ldots, g_j, \ldots, f_n).$$

The operator T is called *multi-quasilinear* if there exists a constant $K \geq 1$ such that

$$|T(f_1, \ldots, \lambda f_j, \ldots, f_n)| = |\lambda||T(f_1, \ldots, f_j, \ldots, f_n)|,$$

and

$$|T(f_1, \ldots, f_j + g_j, \ldots, f_n)| \leq K(|T(f_1, \ldots, f_j, \ldots, f_n)| + |T(f_1, \ldots, g_j, \ldots, f_n)|).$$

In the case where $K = 1$, T is called *multi-sublinear*. If $n = 2$, we talk about *bilinear, bi-quasilinear* and *bi-sublinear* operators, respectively.

If we try to generalize the classical Marcinkiewicz interpolation theorem to the multilinear setting, we will run into trouble because of the following. Suppose that $(A_0, A_1), (B_0, B_1)$ and (C_0, C_1) are compatible couples of Banach spaces, and that T is a bilinear operator defined on $(A_0 + A_1) \times (B_0 + B_1)$, and mapping this product continuously into $C_0 + C_1$ and such that T maps $A_0 \times B_0$ continuously into C_0 and $A_1 \times B_1$ into C_1. Take $a \in A_0 + A_1$, $b \in B_0 + B_1$, and write $a = a_0 + a_1$, $b = b_0 + b_1$, with $a_i \in A_i$, $b_i \in B_i$, for $i = 0, 1$. At this point we would like to invoke the bilinearity of T but if we do so, the terms $T(a_0, b_1)$ and $T(a_1, b_0)$ will appear, and we don't know how to control them with the given hypotheses. A way to bypass this problem is to impose embedding relations on the spaces involved. This is precisely what Lions and Peetre did, and in 1964 they proved the following interpolation theorem for bilinear operators [16].

Theorem 3.1.2. *Let (A_0, A_1), (B_0, B_1) and (C_0, C_1) be interpolation pairs, with $A_i \subseteq B_i$, $i = 0, 1$. Let T be a bilinear operator bounded from $A_1 \times B_1$ into C_1 with norm ω_1, and such that the restriction $T : A_0 \times B_0 \longrightarrow C_0$ is bounded, with norm ω_0. Let $1 \leq p, q \leq \infty$ such that $1/r := 1/p + 1/q - 1 \geq 0$, and $0 < \theta < 1$. Then, $T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (C_0, C_1)_{\theta,r}$ is bounded, with norm at most $\omega_0^{1-\theta} \omega_1^{\theta}$.*

The proof of this result is based on the J-method, which is equivalent to the K-method [6, Ch. 3].

In 1969, Strichartz proved the following result, which is a bilinear version of the Marcinkiewicz interpolation theorem [23]. This result can be regarded as a specialization of the previous result of Lions and Peetre to the case of the Lebesgue spaces $L^p(X, \mu)$ for arbitrary totally σ-finite measure spaces (X, μ). Observe that in this case, we start from three weak type estimates for the operator involved, instead of the two hypotheses imposed in the previous result.

Theorem 3.1.3. *Let $T(f_1, \dots, f_m)$ be a bilinear transformation from*

$$\begin{aligned}\{L^{p_{1,1}}(X_1, \mu_1) \times L^{p_{1,2}}(X_2, \mu_2)\} + \{L^{p_{2,1}}(X_1, \mu_1) \times L^{p_{2,2}}(X_2, \mu_2)\} \\ + \{L^{p_{3,1}}(X_1, \mu_1) \times L^{p_{3,2}}(X_2, \mu_2)\}\end{aligned}$$

to measurable function on (Y, ν). Suppose T satisfies the weak type estimates

$$\nu\{z : |T(f_1, f_2)(z)| \geq \alpha\} \leq \left(\frac{M_i \|f_1\|_{p_{k,1}} \|f_2\|_{p_{k,2}}}{\alpha} \right)^{q_k},$$

with $p_{k,1}, p_{k,2}, q_k \geq 1$, for $k = 1, 2, 3$, and the q_k's pairwise different. Suppose the points $(1/p_{k,1}, 1/p_{k,2})$ in $\mathbb{R}^2$ span a nondegenerate simplex, and let $(1/p_1, 1/p_2)$ be a point in the interior of the simplex. In barycentric coordinates

$$\frac{1}{p_1} = \sum_k \frac{\eta_k}{p_{k,1}}, \ \frac{1}{p_2} = \sum_k \frac{\eta_k}{p_{k,2}},$$

where $\sum_k \eta_k = 1$ *and* $0 < \eta_k < 1$. *Let* $1/q = \sum_k \eta_k/q_k$. *Suppose that for just one of the* p_j*'s, say* p_i*, we have* $p_i \leq q$. *Then* T *satisfies the strong type estimate*

$$\|T(f_1, f_2)\|_q \leq M \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

The proof involves arguments based on the Riesz-Thorin interpolation theorem [3, Ch. 4] and the ordinary Marcinkiewicz interpolation theorem.

In 1978, Zafran generalized the work of Lions and Peetre [16] and proved the following interpolation theorems for multilinear operators [25]. There are two important facts in this result. The first one is that there is no need to assume any kind of embedding hypothesis on the spaces involved. The second one is that for all $n \geq 1$ and T a n-linear operator, we always start with two weak type hypotheses imposed on T.

Theorem 3.1.4. *Let* (B_j^0, B_j^1), (C^0, C^1) *be interpolation pairs,* $1 \leq j \leq n$. *Let* T *be a multilinear operator from* $\bigoplus_{j=1}^n B_j^0 \cap B_j^1$ *into* $C^0 \cap C^1$ *such that*

$$\|T(x_1, \ldots, x_n)\|_{C^k} \leq M_k \prod_{j=1}^n \|x_j\|_{B_j^k},$$

for $k = 0, 1$ *and for all* $(x_1, \ldots, x_n) \in \bigoplus_{j=1}^n B_j^0 \cap B_j^1$. *Let* $0 < s < 1$, $1 \leq p_j \leq \infty$, *and suppose* $1/q = \sum_{j=1}^n 1/p_j - n + 1 \geq 0$. *Then,*

$$\|T(x_1, \ldots, x_n)\|_{(C^0,C^1)_{s,q}} \leq M_0^{1-s} M_1^s \prod_{j=1}^n \|x_j\|_{(B_j^0,B_j^1)_{s,p_j}},$$

for all $(x_1, \ldots, x_n) \in \bigoplus_{j=1}^n B_j^0 \cap B_j^1$. *In particular, if* $p_j < \infty$, $1 \leq j \leq n$, *then* T *has a unique extension as a bounded multilinear operator from* $\bigoplus_{j=1}^n (B_j^0, B_j^1)_{s,p_j}$ *into* $(C^0, C^1)_{s,q}$ *of norm at most* $M_0^{1-s} M_1^s$.

As a corollary of this theorem, we obtain the corresponding result for the Lebesgue spaces. In contraposition to the result of Strichartz, in this case it is enough to impose only two weak type hypotheses on the operator involved, at the price of the condition $0 \leq 1/q \leq \sum_{j=1}^n 1/p_j - n + 1$. If this condition is removed, the interpolation result fails, in general. We will present an example of this fact in the next section.

Corollary 3.1.5. *Let* (X_j, μ_j) *and* (Y, ν) *denote totally* σ*-finite measure spaces,* $1 \leq j \leq n$, *and denote by* $\mathscr{S}_j$ *the integrable simple functions on* X_j, *and by* $\mathscr{M}$ *the measurable functions on* Y. *Let* $1 \leq p_{1,j} \neq p_{2,j} \leq \infty$, $1 \leq q_1 \neq q_2 \leq \infty$, *and* $0 < s < 1$. *Define* $1/p_j = (1-s)/p_{1,j} + s/p_{2,j}$ *and* $1/q = (1-s)/q_1 + s/q_2$

and suppose $0 \le 1/q \le \sum_{j=1}^n 1/p_j - n + 1$. *Let* T *be a multilinear operator from* $\bigoplus_{j=1}^n \mathscr{S}_j$ *into* $\mathscr{M}$ *such that*

$$\|T(f_1, \dots, f_n)\|_{L^{q_k,\infty}(\nu)} \le M_k \prod_{j=1}^n \|f_j\|_{L^{p_{k,j},1}(\mu_j)},$$

for all $(f_1, \dots, f_n) \in \bigoplus_{j=1}^n \mathscr{S}_j$. *Then,*

$$\|T(f_1, \dots, f_n)\|_{L^q(\nu)} \le c M_0^{1-s} M_1^s \prod_{j=1}^n \|f_j\|_{L^{p_j}(\mu_j)},$$

for all $(f_1, \dots, f_n) \in \bigoplus_{j=1}^n \mathscr{S}_j$, *where* c *is a constant depending only on the* $p_{k,j}, q_k, n$ *and* s. *In particular,* T *has a unique extension to* $\bigoplus_{j=1}^n L^{p_j}(\mu_j)$ *satisfying the previous estimate.*

The proof of this result is based on the J-method [6, Ch. 3].

In 2001, Grafakos and Kalton proved the following extension of the classical Marcinkiewicz interpolation theorem to the multilinear setting [10]. This result holds for Lorentz spaces $L^{p,q}$ over the measure space $(\mathbb{R}^+, m)$, where m denotes the Lebesgue measure. In this case, for all $n \ge 1$ and T a n-linear operator, we start with $n+1$ weak type hypotheses imposed on T.

Theorem 3.1.6. *Let* $0 < p_{k,j} \le \infty$ *for* $1 \le k \le n+1$ *and* $1 \le j \le n$, *and also let* $0 < q_k \le \infty$ *for* $1 \le k \le n+1$. *Suppose that a locally continuous* n*-linear map* $T : \mathcal{E}^n \to L_0(0, \infty)$ *satisfies*

$$\|T(\chi_{E_1}, \dots, \chi_{E_n})\|_{q_k,\infty} \le M \prod_{j=1}^n m(E_j)^{1/p_{k,j}},$$

for all sets E_j *of finite measure and all* $1 \le k \le n+1$. *Assume that the system*

$$\begin{pmatrix} 1/p_{1,1} & 1/p_{1,2} & \cdots & 1/p_{1,n} & 1 \\ 1/p_{2,1} & 1/p_{2,2} & \cdots & 1/p_{2,n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_{n,1} & 1/p_{n,2} & \cdots & 1/p_{n,n} & 1 \\ 1/p_{n+1,1} & 1/p_{n+1,2} & \cdots & 1/p_{n+1,n} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \\ -\tau \end{pmatrix} = \begin{pmatrix} 1/q_1 \\ 1/q_2 \\ \vdots \\ 1/q_n \\ 1/q_{n+1} \end{pmatrix}$$

has a unique solution $(\sigma_1, \dots, \sigma_n, -\tau) \in \mathbb{R}^{n+1}$ *with not all* $\sigma_j = 0$. *Suppose that* $(1/p_1, \dots, 1/p_n, 1/q)$ *lies in the open convex hull of the points* $(1/p_{k,1}, \dots, 1/p_{k,n}, 1/q_k)$ *in* $\mathbb{R}^{n+1}$ *and let* $0 < s_j, s \le \infty$ *satisfy*

$$\frac{1}{s} = \sum_{1 \le j \le n:\, \sigma_j \ne 0} \frac{1}{s_j}.$$

Then T extends to a bounded n-linear map

$$T : \prod_{j=1}^{n} L^{p_j,s_j}(0,\infty) \longrightarrow L^{q,s}(0,\infty),$$

with norm a multiple of M.

The proof of this result is based on the work of Boyd [4].

In 2012, Grafakos, Liu, Lu and Zhao, proved a multilinear extension of the Marcinkiewicz real method interpolation theorem [11]. Their result is similar to the theorem of Grafakos and Kalton but with the difference that it works for general measure spaces rather than $\mathbb{R}^+$. Just as in the previous theorem, for a n-linear operator T, we impose $n+1$ weak type hypotheses on T.

In order to state this theorem, we introduce some notation.

Definition 3.1.7. Let n be a positive integer. For $1 \le k \le n+1$ and $1 \le j \le n$, we are given $0 < p_{k,j} < \infty$ and $0 < q_k < \infty$. We define determinants γ_j as follows:

$$\gamma_0 = \det \begin{pmatrix} 1/p_{1,1} & 1/p_{1,2} & \cdots & 1/p_{1,n} & 1 \\ 1/p_{2,1} & 1/p_{2,2} & \cdots & 1/p_{2,n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_{n,1} & 1/p_{n,2} & \cdots & 1/p_{n,n} & 1 \\ 1/p_{n+1,1} & 1/p_{n+1,2} & \cdots & 1/p_{n+1,n} & 1 \end{pmatrix},$$

and for each j, we define

$$\gamma_j = \det \begin{pmatrix} 1/p_{1,1} & 1/p_{1,2} & \cdots & -1/q_1 & \cdots & 1/p_{1,n} & 1 \\ 1/p_{2,1} & 1/p_{2,2} & \cdots & -1/q_2 & \cdots & 1/p_{2,n} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/p_{n,1} & 1/p_{n,2} & \cdots & -1/q_n & \cdots & 1/p_{n,n} & 1 \\ 1/p_{n+1,1} & 1/p_{n+1,2} & \cdots & -1/q_{n+1} & \cdots & 1/p_{n+1,n} & 1 \end{pmatrix},$$

where the j-th column of the determinant defining γ_j is obtained by replacing the j-th column of the determinant defining γ_0 by the vector $Q := (-1/q_1, \dots, -1/q_{n+1})$.

Remark 3.1.8. These determinants have a geometric meaning. For $1 \le k \le n+1$, let $P_k := (1/p_{k,1}, \dots, 1/p_{k,n})$ be points in $\mathbb{R}^n$. Let

$$\mathbb{H} := \left\{ \sum_{k=1}^{n+1} \eta_k P_k : \forall k, \eta_k \in (0,1), \sum_{k=1}^{n+1} \eta_k = 1 \right\}$$

be the open convex hull of the points P_k, $1 \le k \le n+1$. Then, $\mathbb{H}$ is an open subset of $\mathbb{R}^n$ whose n-dimensional volume is $n!|\gamma_0|$. Hence, the condition $\gamma_0 \neq 0$

is equivalent to the fact that $\mathbb{H}$ is a nontrivial open simplex in $\mathbb{R}^n$. The geometric meaning for the remaining γ_j's is similar. That is, the condition $\gamma_j \neq 0$ is equivalent to the fact that the open convex hull of $P_1, \ldots, P_{j-1}, Q, P_{j+1}, \ldots, P_{n+1}$ is a nontrivial open simplex in $\mathbb{R}^n$.

We now state the theorem of Grafakos, Liu, Lu and Zhao.

Theorem 3.1.9. *Let T be a n-sublinear operator defined on $\mathscr{S}(X_1)\times\cdots\times\mathscr{S}(X_n)$ and taking values in the set of measurable functions of (Y,ν). For $1 \leq k \leq n+1$ and $1 \leq j \leq n$, we are given $1 \leq p_{k,j}$ and $1 < q_k$. Suppose that $\gamma_0 \neq 0$. Assume that T satisfies*

$$\|T(\chi_{E_1}, \ldots, \chi_{E_n})\|_{q_k,\infty} \leq B_k \prod_{j=1}^{n} \mu_j(E_j)^{1/p_{k,j}},$$

for all $1 \leq k \leq n+1$ and for all subsets E_j of X_j with $\mu_j(E_j) < \infty$. Let

$$P = \left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right) = \sum_{k=1}^{n+1} \eta_k P_k,$$

for some $\eta_k \in (0,1)$ such that $\sum_{k=1}^{n+1} \eta_k = 1$, and define

$$\frac{1}{q} = \sum_{k=1}^{n+1} \frac{\eta_k}{q_k}.$$

For each j, let $1 \leq s_j$, and let

$$\frac{1}{s} = \sum_{1\leq j\leq n;\, \gamma_j\neq 0} \frac{1}{s_j}.$$

Under these assumptions, there is a positive finite constant c such that

$$\|T(f_1, \ldots, f_n)\|_{q,s} \leq c \left(\prod_{k=1}^{n+1} B_k^{\eta_k}\right) \prod_{j=1}^{n} \|f_j\|_{p_j,s_j},$$

for all $f_j \in L^{p_j,s_j}(X_j)$.

For simplicity, we have removed the description of the constant c. Observe that the linear system in the theorem of Grafakos and Kalton will have a unique solution if, and only if $\gamma_0 \neq 0$, and in this case, $\sigma_j = -\gamma_j/\gamma_0$, so the choice of the parameter s is the same as in the previous theorem.

3.2 A Modern Bi-Sublinear Interpolation Theorem

We will devote this section to the proof of Theorem 3.1.9 for the case of bi-sublinear operators. For convenience, we restate the result.

Theorem 3.2.1. *Let T be a bi-sublinear operator defined on $\mathscr{S}(X_1)\times\mathscr{S}(X_2)$ and taking values in the set of measurable functions of (Y,ν). For $1\leq k\leq 3$ and $j=1,2$, we are given $1\leq p_{k,j}$ and $1<q_k$. Suppose that $\gamma_0\neq 0$. Assume that T satisfies*

$$\|T(\chi_{E_1},\chi_{E_2})\|_{q_k,\infty}\leq B_k\mu_1(E_1)^{1/p_{k,1}}\mu_2(E_2)^{1/p_{k,2}},$$

for all $1\leq k\leq 3$ and for all subsets E_j of X_j with $\mu_j(E_j)<\infty$. Let

$$P=\left(\frac{1}{p_1},\frac{1}{p_2}\right)=\sum_{k=1}^{3}\eta_k P_k,$$

for some $\eta_k\in(0,1)$ such that $\sum_{k=1}^{3}\eta_k=1$, and define

$$\frac{1}{q}=\sum_{k=1}^{3}\frac{\eta_k}{q_k}.$$

For each $j=1,2$, let $1\leq s_j$, and let

$$\frac{1}{s}=\sum_{1\leq j\leq 2;\,\gamma_j\neq 0}\frac{1}{s_j}.$$

Under these assumptions, there is a positive finite constant c such that

$$\|T(f_1,f_2)\|_{q,s}\leq c\,\|f_1\|_{p_1,s_1}\,\|f_2\|_{p_2,s_2},$$

for all $f_j\in L^{p_j,s_j}(X_j)$.

As a preliminary to this theorem, we state the following lemmas.

Lemma 3.2.2. *Let $p\geq 1$, $q>1$ and T be a sublinear operator defined on the characteristic functions χ_E, with $E\subseteq X$ and $\mu(E)<\infty$. Assume that for some constant $M>0$ and for all such measurable subsets E, we have*

$$\|T(\chi_E)\|_{q,\infty}\leq M\mu(E)^{1/p}.$$

Then, $T: L^{p,1}\to L^{q,\infty}$ is bounded.

Proof. Let $f \in \mathcal{S}(X)$ and positive. Then, f is of the form

$$f(x) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^k \chi_{E_{k,j}}(x),$$

where the sums are, in fact, finite almost everywhere (see Definition 1.2.15).

Recall that for $q > 1$, $\|\cdot\|_{(q,\infty)}$ is a norm and $\|\cdot\|_{q,\infty} \leq \|\cdot\|_{(q,\infty)} \lesssim \|\cdot\|_{q,\infty}$. Hence, by the sublinearity of T, we have

$$\begin{aligned}\|T(f)\|_{q,\infty} \leq \|T(f)\|_{(q,\infty)} &\leq \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^k \left\|T(\chi_{E_{k,j}})\right\|_{(q,\infty)} \\ &\lesssim \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^k \left\|T(\chi_{E_{k,j}})\right\|_{q,\infty} \lesssim M \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^k \mu(E_{k,j})^{1/p}.\end{aligned}$$

Since $E_{k,j} = \{x : 2^k \leq f_j(x) < 2^{k+1}\}$ and $f_j < \frac{f}{2^j}$, we obtain that $E_{k,j} \subseteq \{x : f(x) > 2^{k+j}\}$, so $\mu(E_{k,j}) \leq \mu_f(2^{k+j})$. Therefore,

$$\begin{aligned}\|T(f)\|_{q,\infty} &\lesssim M \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} 2^k \mu_f(2^{k+j})^{1/p} = M \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{k\in\mathbb{Z}} 2^{k+j} \mu_f(2^{k+j})^{1/p} \\ &\lesssim M \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{k\in\mathbb{Z}} 2^{k+1} \mu_f(2^{k+1})^{1/p} \lesssim M \sum_{k\in\mathbb{Z}} 2^{k+1} \mu_f(2^{k+1})^{1/p} \\ &\lesssim M \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \mu_f(s)^{1/p} ds \lesssim M \int_0^{\infty} \mu_f(s)^{1/p} ds \lesssim M \|f\|_{p,1},\end{aligned}$$

where we have used that μ_f is decreasing, so

$$\int_{2^k}^{2^{k+1}} \mu_f(s)^{1/p} ds \geq 2^k \mu_f(2^{k+1})^{1/p},$$

and that $L^{p,1} \hookrightarrow L^p$.

For an arbitrary $f \in \mathcal{S}(X)$, the same result holds by decomposing f in its positive and negative parts. Finally, T extends uniquely to the whole space $L^{p,1}$ by density, and the theorem follows. $\square$

Lemma 3.2.3. *Under the assumptions of Theorem 3.2.1,*

$$\|T(\chi_{E_1}, \chi_{E_2})\|_{q,\infty} \leq B_1^{\eta_1} B_2^{\eta_2} B_3^{\eta_3} \mu_1(E_1)^{1/p_1} \mu_2(E_2)^{1/p_2},$$

for all subsets E_j of X_j with $\mu_j(E_j) < \infty$.

Proof. By hypothesis,

$$\|T(\chi_{E_1},\chi_{E_2})\|_{q_k,\infty} \leq B_k \mu_1(E_1)^{1/p_{k,1}} \mu_2(E_2)^{1/p_{k,2}},$$

for $1 \leq k \leq 3$, hence

$$\begin{aligned}
\prod_{k=1}^{3} \|T(\chi_{E_1},\chi_{E_2})\|_{q_k,\infty}^{\eta_k} &\leq \prod_{k=1}^{3} B_k^{\eta_k} \mu_1(E_1)^{\eta_k/p_{k,1}} \mu_2(E_2)^{\eta_k/p_{k,2}} \\
&= \left(\prod_{k=1}^{3} B_k^{\eta_k}\right) \prod_{k=1}^{3} \mu_1(E_1)^{\eta_k/p_{k,1}} \mu_2(E_2)^{\eta_k/p_{k,2}} \\
&= \left(\prod_{k=1}^{3} B_k^{\eta_k}\right) \mu_1(E_1)^{1/p_1} \mu_2(E_2)^{1/p_2}.
\end{aligned}$$

Now, for any measurable function f, and since $\sum_{k=1}^{3} \eta_k = 1$, we have

$$\begin{aligned}
\|f\|_{q,\infty} &= \sup_{t>0} t^{1/q} f^*(t) = \sup_{t>0} t^{\sum_{k=1}^{3} \eta_k/q_k} f^*(t) \\
&= \sup_{t>0} \left(t^{1/q_1} f^*(t)\right)^{\eta_1} \left(t^{1/q_2} f^*(t)\right)^{\eta_2} \left(t^{1/q_3} f^*(t)\right)^{\eta_3} \\
&\leq \|f\|_{q_1,\infty}^{\eta_1} \|f\|_{q_2,\infty}^{\eta_2} \|f\|_{q_3,\infty}^{\eta_3}.
\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}
\|T(\chi_{E_1},\chi_{E_2})\|_{q,\infty} &\leq \prod_{k=1}^{3} \|T(\chi_{E_1},\chi_{E_2})\|_{q_k,\infty}^{\eta_k} \\
&\leq \left(\prod_{k=1}^{3} B_k^{\eta_k}\right) \mu_1(E_1)^{1/p_1} \mu_2(E_2)^{1/p_2}. \quad \square
\end{aligned}$$

Lemma 3.2.4. *Let T be as in Theorem 3.2.1. Let $1 \leq p_1, p_2$ and $1 < q$. Suppose that for some constant $M > 0$, we have*

$$\|T(\chi_{E_1},\chi_{E_2})\|_{q,\infty} \leq M \mu_1(E_1)^{1/p_1} \mu_2(E_2)^{1/p_2},$$

for all subsets E_j of X_j with $\mu_j(E_j) < \infty$. Then, $T : L^{p_1,1} \times L^{p_2,1} \to L^{q,\infty}$ is bounded.

Proof. Fix $F \subseteq X_2$ with $\mu_2(F) < \infty$ and consider the operator $T_F := T(\cdot, \chi_F)$. Since T is bi-sublinear, T_F is sublinear. Moreover,

$$\|T_F(\chi_E)\|_{q,\infty} \leq M \mu_1(E)^{1/p_1} \mu_2(F)^{1/p_2},$$

for all subsets E of X_1 with $\mu_1(E_1) < \infty$. Applying Lemma 3.2.2, we obtain that $T_F : L^{p_1,1} \to L^{q,\infty}$ is bounded. In particular,

$$\|T(f, \chi_F)\|_{q,\infty} \lesssim M\mu_2(F)^{1/p_2} \|f\|_{p_1,1},$$

for all $f \in L^{p_1,1}$. Now, fix $f \in L^{p_1,1}$ and consider the operator $T_f := T(f, \cdot)$. Since T is bi-sublinear, T_f is sublinear, and by the previous estimate and Lemma 3.2.2, we obtain that $T_f : L^{p_2,1} \to L^{q,\infty}$ is bounded. In particular,

$$\|T(f, g)\|_{q,\infty} \lesssim M \|f\|_{p_1,1} \|g\|_{p_2,1},$$

for all $g \in L^{p_2,1}$. Hence, $T : L^{p_1,1} \times L^{p_2,1} \to L^{q,\infty}$ is bounded. $\square$

Combining the previous lemmas, we obtain the following result.

Corollary 3.2.5. *Under the assumptions of Theorem 3.2.1, we have that*

$$T : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{q,\infty}$$

is bounded.

In the sequel we will make use of the set

$$S_2 := \{(\sigma_{\ell,1}, \sigma_{\ell,2}) : \ell = 1, \dots, 4\}$$

of all possible pairs of the form $(\pm 1, \pm 1)$. Under the assumptions of Theorem 3.2.1, since all $p_j < \infty$ and $P = (1/p_1, 1/p_2)$ lies in the open convex hull $\mathbb{H}$, we can choose $\varepsilon > 0$ small enough such that the points $R_\ell := P + \varepsilon(\sigma_{\ell,1}, \sigma_{\ell,2})$ belong to $\mathbb{H}$, for $\ell = 1, \dots, 4$. Hence, for each ℓ we can write

$$\left(\frac{1}{r_{\ell,1}}, \frac{1}{r_{\ell,2}}\right) := R_\ell = \sum_{k=1}^{3} \theta_{\ell,k} P_k,$$

for some $\theta_{\ell,k} \in (0,1)$ such that $\sum_{k=1}^{3} \theta_{\ell,k} = 1$. Observe that for each $\ell = 1, \dots, 4$, and $j = 1, 2$, $r_{\ell,j} < \infty$. Also, we have

$$\frac{1}{r_{\ell,j}} - \frac{1}{p_j} = \varepsilon\sigma_{\ell,j}.$$

For each $\ell = 1, \dots, 4$, we also define

$$\frac{1}{r_\ell} := \sum_{k=1}^{3} \frac{\theta_{\ell,k}}{q_k}.$$

It holds that

$$\frac{1}{q} - \frac{1}{r_\ell} = \frac{\gamma_1}{\gamma_0}\left(\frac{1}{r_{\ell,1}} - \frac{1}{p_1}\right) + \frac{\gamma_2}{\gamma_0}\left(\frac{1}{r_{\ell,2}} - \frac{1}{p_2}\right).$$

We will also need the following two lemmas.

Lemma 3.2.6. *Consider the set* $\Lambda := \{1 \leq j \leq 2 : \gamma_j \neq 0\}$. *For all* $j = 1, 2$, *fix functions* $f_j \in \mathcal{S}(X_j)$ *and for any* $t > 0$, *write* $f_j = f_{j,1,t} + f_{j,-1,t}$, *with* $f_{j,1,t} = f_j \chi_{\{|f_j| > f_j^*(t^{-\gamma_j/\gamma_0})\}}$ *and* $f_{j,-1,t} = f_j \chi_{\{|f_j| \leq f^*(t^{-\gamma_j/\gamma_0})\}}$. *For* $j \in \Lambda$ *and* $\ell = 1, \ldots, 4$, *if* $p_j > r_{\ell,j}$, *we have that*

$$\left\| t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left\| f_{j,1,t} \right\|_{r_{\ell,j},1} \right\|_{L^{s_j}\left(\frac{dt}{t}\right)} \lesssim \left\| f_j \right\|_{p_j,s_j},$$

and if $p_j < r_{\ell,j}$, *we have*

$$\left\| t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left\| f_{j,-1,t} \right\|_{r_{\ell,j},1} \right\|_{L^{s_j}\left(\frac{dt}{t}\right)} \lesssim \left\| f_j \right\|_{p_j,s_j}.$$

Proof. Assume, first, that $p_j > r_{\ell,j}$. For the case $s_j < \infty$, by Lemma 1.2.33, the change of variables $u = t^{-\gamma_j/\gamma_0}$ and the first of Hardy's inequalities with $\lambda = 1 + (1/p_j - 1/r_{\ell,j}) < 1$ and exponent $s_j \geq 1$, we have that

$$\begin{aligned}
&\left\| t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left\| f_{j,1,t} \right\|_{r_{\ell,j},1} \right\|_{L^{s_j}\left(\frac{dt}{t}\right)} \\
&= \left(\int_0^\infty t^{s_j \frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left(\int_0^\infty v^{1/r_{\ell,j}} f_{j,1,t}^*(v) \frac{dv}{v} \right)^{s_j} \frac{dt}{t} \right)^{1/s_j} \\
&\lesssim \left(\int_0^\infty u^{-s_j\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left(\int_0^u v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \right)^{s_j} \frac{du}{u} \right)^{1/s_j} \\
&\lesssim \left(\int_0^\infty (u^{1/p_j} f_j^*(u))^{s_j} \frac{du}{u} \right)^{1/s_j} = \left\| f_j \right\|_{p_j,s_j}.
\end{aligned}$$

Similarly, for the case $s_j = \infty$ we have

$$\begin{aligned}
&\sup_{t>0} t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \left\| f_{j,1,t} \right\|_{r_{\ell,j},1} \\
&\lesssim \sup_{u>0} u^{-\left(\frac{1}{r_{\ell,j}} - \frac{1}{p_j}\right)} \int_0^u v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \\
&\lesssim \sup_{u>0} u^{1/p_j} f_j^*(u) = \left\| f_j \right\|_{p_j,\infty}.
\end{aligned}$$

Now, assume that $p_j < r_{\ell,j}$. For the case $s_j < \infty$, applying Minkowski's inequality for L^{s_j}, using Lemma 1.2.33, performing the change of variables $u = t^{-\gamma_j/\gamma_0}$, evaluating directly the first summand and applying the second of Hardy's inequalities with $\lambda = 1 + (1/r_{\ell,j} - 1/p_j) < 1$ and exponent $s_j \geq 1$ to the second summand, we have that

$$\begin{aligned}
&\left\| t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \|f_{j,-1,t}\|_{r_{\ell,j},1} \right\|_{L^{s_j}\left(\frac{dt}{t}\right)} \\
&\lesssim \left(\int_0^\infty u^{-s_j\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \left(f_j^*(u) \int_0^u v^{1/r_{\ell,j}} \frac{dv}{v} \right)^{s_j} \frac{du}{u} \right)^{1/s_j} \\
&+ \left(\int_0^\infty u^{-s_j\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \left(\int_u^\infty v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \right)^{s_j} \frac{du}{u} \right)^{1/s_j} \lesssim \|f_j\|_{p_j,s_j}.
\end{aligned}$$

Similarly, for the case $s_j = \infty$ we have

$$\begin{aligned}
&\sup_{t>0} t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \|f_{j,-1,t}\|_{r_{\ell,j},1} \lesssim \sup_{u>0} u^{-\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} f_j^*(u) \int_0^u v^{1/r_{\ell,j}} \frac{dv}{v} \\
&+ \sup_{u>0} u^{-\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \int_u^\infty v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \lesssim \sup_{u>0} u^{1/p_j} f_j^*(u) = \|f_j\|_{p_j,\infty}.
\end{aligned}$$

This concludes the proof. □

Lemma 3.2.7. *Consider the set $\Lambda' := \{1 \leq j \leq 2 : \gamma_j = 0\}$. For $j \in \Lambda'$ and $\ell = 1, \dots, 4$, if $p_j > r_{\ell,j}$, we have that*

$$\|f_{j,1,1}\|_{r_{\ell,j},1} \lesssim \|f_j\|_{p_j,\infty},$$

and if $p_j < r_{\ell,j}$, we have

$$\|f_{j,-1,1}\|_{r_{\ell,j},1} \lesssim \|f_j\|_{p_j,\infty}.$$

Proof. When $j \in \Lambda'$, we have $\gamma_j = 0$ and $f_{j,1,1} = f_j \chi_{\{|f_j| > f_j^*(1)\}}$ and $f_{j,-1,1} = f_j \chi_{\{|f_j| \leq f^*(1)\}}$. If $p_j > r_{\ell,j}$, applying Lemma 1.2.33 we obtain

$$\begin{aligned}
\|f_{j,1,1}\|_{r_{\ell,j},1} &\leq \int_0^1 v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \\
&\leq \left(\int_0^1 v^{1/r_{\ell,j}-1/p_j} \frac{dv}{v} \right) \|f_j\|_{p_j,\infty} \lesssim \|f_j\|_{p_j,\infty}.
\end{aligned}$$

Now, assume that $p_j < r_{\ell,j}$. By Lemma 1.2.33 we get that

$$\begin{aligned}
\|f_{j,-1,1}\|_{r_{\ell,j},1} &\leq \int_0^1 v^{1/r_{\ell,j}} f_j^*(1) \frac{dv}{v} + \int_1^\infty v^{1/r_{\ell,j}} f_j^*(v) \frac{dv}{v} \\
&\leq \left(\int_0^1 v^{1/r_{\ell,j}} \frac{dv}{v} + \int_1^\infty v^{1/r_{\ell,j}-1/p_j} \frac{dv}{v} \right) \|f_j\|_{p_j,\infty} \lesssim \|f_j\|_{p_j,\infty}.
\end{aligned}$$

This concludes the proof. □

Now we can give the proof of Theorem 3.2.1.

Proof. (**Thm. 3.2.1**) For all $j = 1, 2$, fix functions $f_j \in \mathcal{S}(X_j)$ and for any $t > 0$ write $f_j = f_{j,1,t/4} + f_{j,-1,t/4}$. Proposition 1.2.5, together with Minkowski's inequality for L^s with $s \geq 1$, the bi-sublinearity of the operator T and the quasilinearity of the L^s-quasi-norm with $0 < s < 1$ [9], imply that

$$\begin{aligned}
\|T(f_1, f_2)\|_{q,s} &= \left\| t^{1/q} T(f_1, f_2)^*(t) \right\|_{L^s\left(\frac{dt}{t}\right)} \\
&\leq \left\| t^{1/q} \left(\sum_{i_1, i_2 \in \{1,-1\}} |T(f_{1,i_1,t/4}, f_{2,i_2,t/4})| \right)^* (t) \right\|_{L^s\left(\frac{dt}{t}\right)} \\
&\leq \left\| t^{1/q} \sum_{i_1, i_2 \in \{1,-1\}} (|T(f_{1,i_1,t/4}, f_{2,i_2,t/4})|)^*(t/4) \right\|_{L^s\left(\frac{dt}{t}\right)} \\
&\lesssim \sum_{i_1, i_2 \in \{1,-1\}} \left\| t^{1/q} (|T(f_{1,i_1,t}, f_{2,i_2,t})|)^*(t) \right\|_{L^s\left(\frac{dt}{t}\right)} \\
&= \sum_{\ell=1}^{4} \left\| t^{1/q} (|T(f_{1,\sigma_{\ell,1},t}, f_{2,\sigma_{\ell,2},t})|)^*(t) \right\|_{L^s\left(\frac{dt}{t}\right)},
\end{aligned}$$

because each pair (i_1, i_2) with $i_j \in \{1, -1\}$ corresponds to a unique ℓ such that $(i_1, i_2) = \sigma_\ell \in S_2$.

It follows from Corollary 3.2.5 that for $\ell = 1, \ldots, 4$, we have

$$\|T(f_1, f_2)\|_{r_\ell,\infty} \lesssim \|f_1\|_{r_{\ell,1},1} \|f_2\|_{r_{\ell,2},1},$$

for all functions $f_j \in \mathcal{S}(X_j)$, and since

$$\frac{1}{q} - \frac{1}{r_\ell} = \frac{\gamma_1}{\gamma_0} \left(\frac{1}{r_{\ell,1}} - \frac{1}{p_1} \right) + \frac{\gamma_2}{\gamma_0} \left(\frac{1}{r_{\ell,2}} - \frac{1}{p_2} \right),$$

we obtain that for all $\ell = 1, \dots, 4$, and $t > 0$,

$$\begin{aligned}
&t^{1/q}(|T(f_{1,\sigma_{\ell,1},t}, f_{2,\sigma_{\ell,2},t})|)^*(t)\\
&\leq t^{\frac{1}{q}-\frac{1}{r_\ell}} \sup_{s>0} s^{1/r_\ell}(|T(f_{1,\sigma_{\ell,1},t}, f_{2,\sigma_{\ell,2},t})|)^*(s)\\
&= t^{\frac{1}{q}-\frac{1}{r_\ell}} \left\| T(f_{1,\sigma_{\ell,1},t}, f_{2,\sigma_{\ell,2},t}) \right\|_{r_\ell,\infty}\\
&\lesssim t^{\frac{1}{q}-\frac{1}{r_\ell}} \left\| f_{1,\sigma_{\ell,1},t} \right\|_{r_{\ell,1},1} \left\| f_{2,\sigma_{\ell,2},t} \right\|_{r_{\ell,2},1}\\
&= \left(t^{\frac{\gamma_1}{\gamma_0}\left(\frac{1}{r_{\ell,1}}-\frac{1}{p_1}\right)} \left\| f_{1,\sigma_{\ell,1},t} \right\|_{r_{\ell,1},1} \right) \left(t^{\frac{\gamma_2}{\gamma_0}\left(\frac{1}{r_{\ell,2}}-\frac{1}{p_2}\right)} \left\| f_{2,\sigma_{\ell,2},t} \right\|_{r_{\ell,2},1} \right)\\
&= \left(\prod_{j\in\Lambda} t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \left\| f_{j,\sigma_{\ell,j},t} \right\|_{r_{\ell,j},1} \right) \left(\prod_{j\in\Lambda'} \left\| f_{j,\sigma_{\ell,j},1} \right\|_{r_{\ell,j},1} \right),
\end{aligned}$$

where we made use of the observation that for $j \in \Lambda'$ we have $\gamma_j = 0$ and hence, for all $t > 0$,

$$\left\| f_{j,\sigma_{\ell,j},t} \right\|_{r_{\ell,j},1} = \left\| f_{j,\sigma_{\ell,j},1} \right\|_{r_{\ell,j},1}.$$

In virtue of Hölder's inequality with exponents $1 = \sum_{j\in\Lambda} \frac{s}{s_j}$, the fact that $L^{p_j,s_j} \hookrightarrow L^{p_j,\infty}$ and applying Lemma 3.2.6 when $j \in \Lambda$ or Lemma 3.2.7 when $j \in \Lambda'$, we get that

$$\begin{aligned}
&\left\| t^{1/q}(|T(f_{1,\sigma_{\ell,1},t}, f_{2,\sigma_{\ell,2},t})|)^*(t) \right\|_{L^s\left(\frac{dt}{t}\right)}\\
&\lesssim \left(\prod_{j\in\Lambda} \left\| t^{\frac{\gamma_j}{\gamma_0}\left(\frac{1}{r_{\ell,j}}-\frac{1}{p_j}\right)} \left\| f_{j,\sigma_{\ell,j},t} \right\|_{r_{\ell,j},1} \right\|_{L^{s_j}\left(\frac{dt}{t}\right)} \right) \left(\prod_{j\in\Lambda'} \left\| f_{j,\sigma_{\ell,j},1} \right\|_{r_{\ell,j},1} \right)\\
&\lesssim \left(\prod_{j\in\Lambda} \|f_j\|_{p_j,s_j} \right) \left(\prod_{j\in\Lambda'} \|f_j\|_{p_j,\infty} \right) \lesssim \|f_1\|_{p_1,s_1} \|f_2\|_{p_2,s_2}
\end{aligned}$$

and hence,

$$\|T(f_1, f_2)\|_{q,s} \lesssim \|f_1\|_{p_1,s_1} \|f_2\|_{p_2,s_2},$$

for all $f_j \in \mathcal{S}(X_j)$. Finally, T extends uniquely to $L^{p_1,s_1} \times L^{p_2,s_2}$ by its bi-sublinearity and the density of the space $\mathcal{S}(X_j)$ in $L^{p_j,s_j}(X_j)$, and the theorem follows. □

Remark 3.2.8. The general version of Theorem 3.2.1 is stated for multi-quasi-linear operators, with parameters $0 < p_{k,j}, q_k, s_j \leq \infty$ and the constant c is given explicitly. If we don't take into account the estimates of the constants, the proof of the general theorem follows almost verbatim [11].

Corollary 3.2.9. *Suppose that in Theorem 3.2.1 we have all $\gamma_j \neq 0$ and*

$$\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}.$$

Then, there exists a positive constant c such that T satisfies the strong bound

$$\|T(f_1, f_2)\|_q \leq c \|f_1\|_{p_1} \|f_2\|_{p_2},$$

for all $f_j \in L^{p_j}(X_j)$.

Proof. For $j = 1, 2$, we take $s_j = p_j$ and define s by $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2}$. By hypothesis, $q \geq s$, so $L^{q,s} \hookrightarrow L^q$ and we have

$$\|T(f_1, f_2)\|_q \lesssim \|T(f_1, f_2)\|_{q,s}.$$

The required boundedness holds by Theorem 3.2.1. □

Remark 3.2.10. Let $X_1 = X_2 = Y = (0, \infty)$ and $\mu_1 = \mu_2 = \nu = m$, the Lebesgue measure. Let

$$T(f, g)(z) = \int_0^\infty f(xz)g(x)dx.$$

Then, by Hölder's inequality,

$$|T(f, g)(z)| \leq z^{-1/p} \|f\|_p \|g\|_{p'},$$

so $\|T(f, g)\|_{p,\infty} \leq \|f\|_p \|g\|_{p'}$, for all $1 \leq p \leq \infty$. But if $p < \infty$ we never have the strong type estimate, for if we choose g to be a positive function and consider the linear operator

$$S(f)(z) = \int_0^\infty f(xz)g(x)dx = \int_0^\infty f(x)g\left(\frac{x}{z}\right)\frac{dx}{z},$$

it is an integral operator with positive kernel homogeneous of degree -1 and hence it will be bounded in L^p if, and only if $\int_0^\infty g(x)x^{-1/p}dx < \infty$ [12]. Now, for the function

$$g(x) = \frac{1}{x^{1/p'}(|\log x| + 1)},$$

we have that $g \in L^{p'}$ and $\int_0^\infty g(x)x^{-1/p}dx = \infty$. This example shows that if in Corollary 3.1.5 we remove the hypothesis $0 \leq 1/q \leq \sum_{j=1}^n 1/p_j - n + 1$, or if in Theorem 3.2.1 we remove the hypothesis $\gamma_0 \neq 0$, then the interpolation results are no longer true, in general.

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