The Kuramoto model for Synchronization

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Abstract: The aim of this project is to reformulate Kuramoto's equations for N-coupled oscillators in a cycle limited boundary, both for the noise-free case ($D = 0$) and for the noisy one ($D \neq 0$) . We will see that solving integro-differential equations is needed to finally get the approximate value of K_c, which sets the phase transition. To prove this phenomena I will also program a simulation and compare both results.

I. INTRODUCTION

Synchronization is a phenomenon hugely found in nature and in our daily life. It involves large populations of units displaying a high level of "coherence" in their temporal activity. In every case, these feats of synchronicity befall spontaneously, almost as if nature had a yearning for order.

Blinking at unison fireflies is probably the example which motivated the study of synchronization [5]. A huge amount of fireflies from the Southeast Asia were spotted blinking at unison. For 300 years travelers from this region had been returning with tales about this behavior. However, there was a general disbelief. How could thousands of fireflies blink at synchronicity so precisely and on such a wide scale? No one could explain this event until the end of the 1960. Nowadays is known that each firefly delays or advances its internal clock, or natural frequency, to finally synchronize with their collective. A kind of interaction is needed so a firefly could modify its inner clock. In this particular instance, the interaction is the sight of their companions flashes.

Nevertheless, fireflies are not the only example of synchronization. A social example could be fads. Fads are contagious ideas competing for survival, with the winners proliferating through a cultural version of natural selection. We all have experienced fads, and that is a phenomenon which will always exist. When something is new, clothes, technologies, experiences etc, people talk about it and spread their desire. Finally, this experience will be a trend and will be synchronized, since a huge amount of people is doing, wearing or using it.

But, How do they do it? Are these two last examples related? These are the main questions that I will attempt to answer in my project. A.T.Winfree spent part of his life studying and modeling different kinds of phase and frequency synchronization [3]. He was able to formulate a mathematical framework where this collective behavior could be studied analytically. However, in 1975 Yoshiki Kuramoto brought simplicity to this problem. He solely considered the specific case of all-to-all interaction with oscillators synchronized at the same phase and frequency, and proposed the simplest model until that moment. What makes the Kuramoto model so interesting is the theoretical work that can be done with it. Despite being a non-linear model, it is fully solvable. He was not only able to prove that there will be a phase transition to synchronization, but also to find an equation that gives the critical coupling strength necessary to achieve it. We found this model to be a certainly good description of some synchronizing systems.

In the upcoming sections we will travel over the Kuramoto model to finally demonstrate it with a simulation.

II. THE KURAMOTO MODEL

In a further simplification, Kuramoto proposed this problem to be set with a weakly coupled, nearly identical and limit-cycle oscillators. These assumptions helped him to do an easier calculus.

At first, Kuramoto suggested the following approximated equation [1].

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \tag{1}$$

where N is the number of particles, K is the coupling strength, w_i are the natural frequencies and Γ_{ij} are the interaction functions between each oscillator. This is the simplest proposal of all-to-all purely sinusoidal coupling, the so-called *Kuramoto model* where $\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} sin(\theta_j - \theta i)$.

The dynamics of the system will be governed by the dominating factor of equation (1). In absence of interactions, the population runs incoherently driven by natural frequencies ω_i. On the other hand, the second term (coupling function) makes the population converge to the same phase leading to synchronization [9]. For a large value of the coupling constant K, a phase transition from incoherence to synchronization emerges spontaneously.

Therefore, the dynamics will be given by the next equation.

$$\dot{\theta}_i = w_i + \frac{K}{N} \sum_{j=1}^{N} sin(\theta_j - \theta_i) \tag{2}$$

To make the physics more understandable, Kuramoto introduced the order parameter, a value which shows us

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how synchronized the system is. Considering that the oscillators are running around the unit circle in the complex plane, we can express the order parameter as

$$re^{i(\Psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)} \quad (3)$$

Where r is the mean module of all positions and its phase corresponds to the mean phase of all oscillators. I represented the dynamics of the system for a better comprehension on Figure 1, where r is represented as a blue arrow.

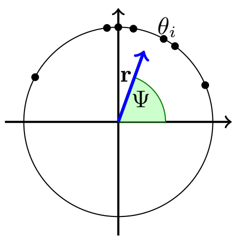

Figure 1: *Representation of the order parameter r and the average phase Ψ*

Now, we will rewrite the equation (2) in terms of the order parameter r and the mean phase Ψ. Therefore, it will give us a better visualization of the system dynamics. Since we can express the exponential function of equation (3) into sinus and cosinus functions using the Euler formulation, we will take the imaginary part corresponding to the sinus and will replace it into the equation (2). Finally we obtain the following expression.

$$\dot{\theta}_i = \omega_i - Krsin(\Psi-\theta_i) \quad i = 1, ..., N. \quad (4)$$

In the last equation we can perceive a clear proportionality of coupling and coherence. As the population becomes more coherent, r grows and so does the coupling factor Kr. Simultaneously θ_i slowly approximates to the mean phase Ψ. Otherwise, if the population becomes incoherent $r \to 0$ and $\dot{\theta}_i = \omega_i$, which means that each oscillator will move with its own natural frequency and will not be synchronicity.

This phase transition from incoherent to coherent state is set at a critical value K_c. Thus, we expect a $r > 0$ when $K \geq K_c$, where the population acts like a giant oscillator, and $r \approx 0$ when $K \leq K_c$ and oscillators are scattered around the circle. On Figure II we can observe the evolution of $r(t)$ for both cases synchronization and incoherence [2].

The speed evolution of $r(t)$ also depends on the density function $g(\omega)$ and the natural frequencies of each oscillator. Remember that a density function is defined as a function which gives us the number of oscillators with a natural frequency between ω and $\omega + \delta\omega$. This oscillators with its natural frequency near the center of the

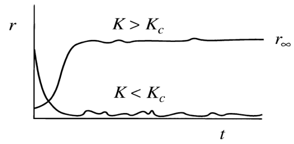

Figure 2: *Evolution of the order parameter $r(t)$ in front of the initial coupling parameter K*

frequency distribution would be the first ones to lock. Hence, the natural frequencies belonging to the tails of the density would be harder to recruit, so we will need a higher value of the coupling parameter to add them to the synchronized population.

III. RESOLUTION OF THE KURAMOTO MODEL

My objective in this section is to go through the different steps which lead Kuramoto find the exact expressions for the critical coupling strength and the order parameter.

From now on, we will consider a large number of oscillators $N \to \infty$, so the evolution of the system will be described by densities and we will apply what it is called the continuum limit. The density function $\rho(\theta, t, w)$ refers to the fraction of oscillators with frequency ω placed between θ and $\theta + \delta\theta$ at a time t. We will solve this problem for two different cases, the deterministic one on which the density evolution is governed by the continuity equation $\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\theta}(\rho v)$, and the non-deterministic where the density evolution turns to the Fokker-Planck equation $\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial\theta^2} - \frac{\partial}{\partial\theta}(\rho v)$ [7]. Notice that the Fokker-Planck equation change into the continuity equation for $D = 0$.

A. The deterministic case for D=0

As I introduced in the last section, when the number of oscillators increase we will not talk about discrete oscillators anymore, we will rather operate with densities.

Hence, we will have to rewrite the order parameter defined at the continuum limit such as

$$re^{i\Psi} = \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\rho(\theta, t, \omega)g(\omega)d\omega\theta \quad (5)$$

where $\rho(\theta, t, \omega)$ is the density distribution, and $g(\omega)$ the frequency distribution function. To solve the preview equation, it is necessary to determine the density distribution function.

First we will consider the deterministic case with the noise strength in the Fokker-Planck equation $D = 0$, consequently we will recover the continuity equation. Its stationary states are the steady solutions that Kuramoto proposed, where the density distribution do not evolves. As we expect, this function satisfies the normalization condition $\int_0^{2\pi} \rho(\theta, t, w)d\theta = 1$. We notice that $\partial\rho/\partial t = 0$ implies $\rho v = C(\omega)$. When $C(\omega) \neq 0$ the density will be inversely proportional to the speed, and applying the normalization condition over $[-\pi, \pi]$ we find $C(\omega) = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}$, and $\rho(\omega, \theta) = \frac{C}{|\omega - Krsin\theta|}$ [2]. This result agrees with experience. At the most populated places the flow is slower, and in free places the flow does not get interrupted and goes faster. For instance, this is analogue to the transit flow.

Therefore, after replacing the density distribution function into the equation (5) and considering the drifting and locked oscillators separately[2], the resulting equation is

$$r = Kr\int_{-\pi/2}^{\pi/2} cos^2\theta g(Krsin\theta)d\theta \tag{6}$$

After solving this last equation for $r = 0$ to find the critical coupling strength, we easily obtain this general expression

$$K_c = \frac{2}{\pi g(0)} \tag{7}$$

where $g(0)$ is a general distribution function evaluated at zero. For a specific distribution, in this case the Lorentzian, we will get $K_c = 2\gamma$ and $r = \sqrt{1 - \frac{K_c}{K}}$ [2].

We can discern that the expression for the order parameter is consistent with our statements. For $K \geq K_c$ it grows reaching values $r > 0$. Otherwise, with $K \leq K_c$ the system has not reached the threshold, so the state will be in the incoherent state where for definition $r = 0$.

B. Continuum limit for D≠0

In this section I will aboard the problem for $D \neq 0$. Instead of the continuity equation now we will require the Fokker-Planck equation. To solve this problem we will study the stability of the incoherent state [4]. First we rewrite $\rho(\theta, t\omega) = \frac{1}{2\pi} + \epsilon\eta(\theta, t, \omega)$, where $\epsilon \ll 1$, $\eta(\theta, t, \omega)$ is a perturbation function and the first term corresponds to the density at the incoherent state. This density is easily found just replacing $r = 0$ into the density function expression from the deterministic case. The perturbation function can be rewritten as a Fourier series, on this wise $\eta(\theta, t, \omega) = c(t, \omega)e^{i\theta} + c.c + \eta^{\perp}(\theta, t, \omega)$. From this last expression $c(\theta, t, \omega)$, the fundamental mode, is the only term which contributes to the system [2].

After substituting this last expression into the Fokker-Planck equation, we obtain the following integro-differential equation for the fundamental mode.

$$\frac{\partial c}{\partial t} = -(D + i\omega)c + \frac{K}{2}\int_{-\infty}^{\infty} c(t, \omega)g(\omega)d\omega \tag{8}$$

This last equation has both a discrete and continuous spectrum. Therefore, on the following sections I will solve each of them separately.

1. Discrete Spectrum

To solve the discrete spectrum we will propose solutions such as $c(t, \omega) = b(\omega)e^{\lambda t}$, where λ is the eigenvalue. As we can see in this last expression, this eigenvalue governs the linear stability of the system. To make this mode stable, λ must be negative, since the exponential will decay and when $t \to \infty \implies c(\omega, t) \to 0$. Hence, the unstable case comes when λ is positive. After replacing the proposed solution we get the following equation.

$$\lambda b = -(D + i\omega)b + \frac{K}{2}\int_{-\infty}^{\infty} b(\omega')g(\omega')d\omega' \tag{9}$$

Paying attention to the equation (9), we can prove that has at most one solution for λ, and if it exists must be necessary real [8]. The second term in the RHS of the equality do not depends on ω, so its is a constant. Thus, isolating $b(\omega)$ and replacing it again into the constant expression we obtain [2]

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda + D}{(\lambda + D)^2 + \omega^2}g(\omega)d\omega \tag{10}$$

It is important to notice that λ must always satisfy $\lambda > -D$. Otherwise the value of the coupling strength K would be negative, and as known, K must be positive by definition. Taking into account this last issue, for $D = 0$ we notice that the system will be unstable, since λ must be positive, and if $\lambda > 0$ the fundamental mode diverges. Thus, the critical point, in which the stability is bifurcated, is set at $\lambda = 0$. After substituting it into the expression (10) we get

$$K_c = 2\left[\int_{-\infty}^{\infty} \frac{D}{D^2 + \omega^2}g(\omega)\right]^{-1} \tag{11}$$

To prove that for the free-noise case the incoherent solution goes unstable with $K > K_c = 2/[\pi g(0)]$, as conjectured by Kuramoto, we set D=0 and let $\lambda \to 0^+$. If we first solve the integral by the residues method, just considering the positive pol ($i\lambda$), we finally obtain $1 = (K/2)\pi g(0)$ since $\lambda > 0$ for $K > 2/[\pi g(0)]$.

2. Continuous Spectrum

To find the continuous spectrum we apply the operator L to the equation (8) as follows.

$$Lb = -(D + i\omega)b + \frac{K}{2}\int_{-\infty}^{\infty} b(\omega)g(\omega)d\omega \tag{12}$$

Notice that the continuous spectrum of L is defined as the set of complex numbers λ such that the operator $L - \lambda I$ is not surjective. Recall that for surjective we mean inversible, so $Det \mid L - \lambda I \mid = 0$ [6]. Thus, the spectrum of L consists of all non regular values or neutral modes. This modes can be interpreted by imagining the perturbation of one part of the oscillators population, for instance these with $\omega = \omega_0$ and leaving the rest in their perfectly incoherent state. The corresponding amplitude would be $c(0, \omega) = 0$ for all $\omega \neq \omega_0$, and we can choose $c(0, \omega_0) = 1$. The point is that the integral from the equation (8) vanishes for this perturbation and it reduces to $\frac{\partial c}{\partial t} = i\omega_0 c$. Hence, $c(0, \omega)$ is an eigenfunction with pure imaginary eigenvalue $i\omega_0$ [1].

Now, adding $-\lambda b$ at each side of the equality we get

$$-(\lambda + D + i\omega)b + \frac{K}{2}\int_{-\infty}^{\infty} b(\omega)g(\omega)d\omega = f(\omega) \quad (13)$$

where $f(\omega)$ is an arbitrary function that satisfies $(L - \lambda I)b = f(\omega)$. If $\lambda + D + i\omega = 0$ for ω in the support of $g(\omega)$, then the equation is not solvable in general. Hence, the continuous spectrum contains the set $\{-D - i\omega : \omega \in Support(g(\omega))\}$. This last set is all of the continuous spectrum just supposing that λ is not in the support of $g(\omega)$, then the equation is solvable. Substituting the integral by A, isolating $b(\omega)$ and replacing again in the A equation we get

$$A\left(1 - \frac{K}{2}\int_{-\infty}^{\infty} \frac{g(\omega)}{\lambda + D + i\omega}d\omega\right) = \quad (14)$$

$$-\frac{-K}{2}\int_{-\infty}^{\infty} \frac{f(\omega)g(\omega)}{\lambda + D + i\omega}d\omega \quad (15)$$

By assumption, λ is not in the discrete spectrum, and $A \neq 0$ (we do not consider the trivial solution). Thus, equation (14) can be solved for A. Hence, the set considered before is the continuous spectrum. We notice that if $D = 0$ the spectrum lies in the imaginary axis, and if $D < 0$ it is set in the left-plane. This continuous and imaginary spectrum are related with other kind of motion waves that would need more accurate study. This is not the aim of my project, therefore I will not focus deeper on that issue.

In the Figure III B 2 is represented the discrete and continuous spectrum for a frequency density $g(\omega)$ with support $[-\gamma, \gamma]$. Notice that the eigenvalue is born at $\lambda = -D$, which follows the definition $\lambda > -D$.

For the specific case of the Lorentzian distribution, the exact solution for the eigenvalue is $\lambda = \frac{K}{2} - D - \gamma$, whose consistency can be easily checked. We will consider the free-noise case with $D = 0$, so the eigenvalue must be absorbed by the continuous spectrum at $\lambda = 0$. Since the eigenvalue must satisfy $\lambda > -D$, in this particular case $\lambda > 0$. Finally, replacing $K_c = 2\gamma$ in this last expression we recover $\lambda = 0$ [2].

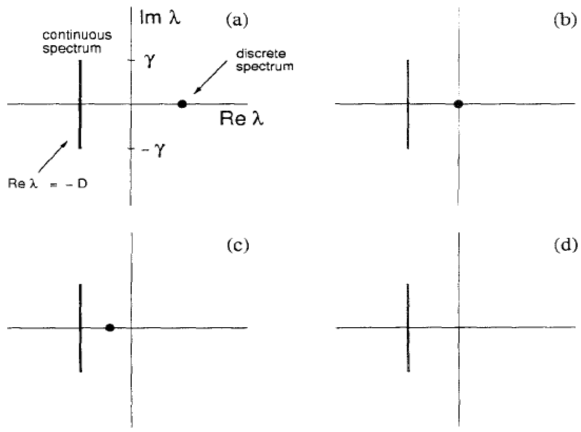

Figure 3: *Continuous and discrete spectrum for the linear operator from equation (12) for the noisy case $D > 0$. (a) $K > K_c$, the fundamental mode is unstable since $\lambda > 0$. The continuous spectrum lies in the left-plane. (b) $K = K_c$ we are in the critical point, so $\lambda = 0$ (c) $K^* < K < K_c$. Here the fundamental mode is stable since $\lambda < 0$ (d) $K^* = K$. K^* is the value at which $\lambda = -D$, and where the discrete value is absorbed by the continuous spectrum.*

IV. NUMERICAL RESULTS

To demonstrate the Kuramoto theory for the deterministic case with $D = 0$, I implemented a Python program which simulates oscillators interaction in a unit circle. In this specific example I used the forth order Runge-Kutta method with an uniform density between $[\pi + 0.5, \pi - 0.5]$. In Figures IV and IV I represented the order parameter evolution for $K > K_c$ and $K < K_c$ with a population of $N = 1000$.

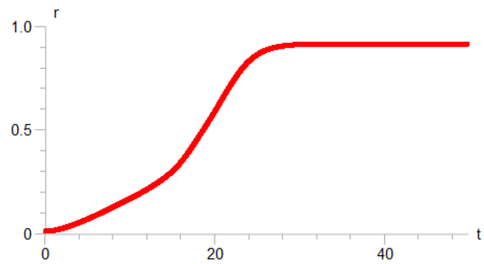

Figure 4: *Evolution of the order parameter $r(t)$ for $K = 0.8$ for a population of $N = 1000$ oscillators and a uniform distribution*

This results show the validity of the theory. We can observe in Figure IV that above the critical point, set over $K_c \approx 0.6$, the order parameter increases until its saturation value r_∞. The critical value is determined thanks to the equation (7), where in this specific case $g(0) = 1$.

On the other hand, in Figure IV we can observe that with $K < K_c$ the order parameter keeps constant at $r \approx 0$

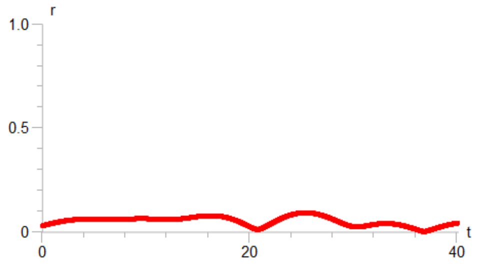

Figure 5: *Evolution of the order parameter* $r(t)$ *for* $K = 0.4$ *for a population of* $N = 1000$ *oscillators and a uniform distribution*

describing fluctuations of size $O(\frac{1}{\sqrt{N}})$.

For its simplicity, the initial condition is set at the incoherent state $\rho(t, \omega)$. If we impose a larger restriction over the initial positions, the starting r would be higher and we would see a quick decay until $r \approx 0$.

We also have solved the stochastic version of the model, with $D \neq 0$, by considering $\dot{\theta}_i = \omega_i - Krsin(\Psi - \theta_i) + \eta_i(t)$. This last expression is the Kuramoto model with an additional white noise function which satisfies $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t)\eta_j(t') \rangle = 2D\delta_{ij}\delta(t - t')$. To solve this differential equation we have implemented a first order Euler method. The results are the same except for the critical coupling K_c, which is higher due to the additional impediment caused by the white noise function. I have not added these results due to the lack of space.

V. CONCLUSIONS

Observing spontaneous synchronization in nature is not only beautiful but enigmatic. In 1960s, thanks to the Kuramoto model scientists were able to finally uncover this mystery. A mathematical description was proposed, and it was not only beautiful by its good description of this phenomena, but for its simplicity. With a few simplifications Kuramoto was able to predict the exact condition at which synchronization would emerge.

In this project I attempted to solve the Kuramoto model analytically to finally find the critical coupling constant, which sets the phase transition from the incoherent state to synchronization. To archive it we had to solve a non-linear differential equation, which in this specific case it is completely solvable. That is one of the reasons that makes this model so fascinating.

I solved the problem for both situations, the deterministic case governed by the continuity equation and the non-deterministic with $D \neq 0$. To determine the solution of the non-deterministic case, ruled by the Fokker-Planck equation, I studied the stability of the incoherent state solving an eigenvalue problem.

To finally test the veracity of the theory, I programmed a simulation and compared the numerical results. As we expected, I found concordance with the model's predictions. The order parameter tends to a constant value when $K > K_c$, which means synchronization, and decays to zero at $K < K_c$ for the incoherent state. After testing, I am able to verify that the critical value also agrees with the theoretical one ruled by equation (7) for an specific density function.

Overall, the Kuramoto model provides a large mathematical realm to explore. Therefore, synchronization for different varieties of interactions would be a stimulating innovation for a further research.

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