A model for the transmission of contact forces in granular piles

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Abstract: Granular matter is fundamentally different from other, more conventional, phases of matter, such as solids and simple liquids. We propose a simple scalar model to study the distribution of weight and the formation of arches in a granular pile subject to gravity. This model simulates dense packaging conditions and introduces the usual heterogeneity observed in these systems. So far, this model was used to study grains in a silo. Here we focus our attention on granular piles in order to avoid the effects of walls and boundary conditions on the contact force distributions. In this pile geometry we can clearly identify a second order phase transition for a given value of the parameter that simulates the presence of slip events in the granular pile. This second order transition is akin to a percolation transition, since some of the critical exponents computed are similar to critical percolation exponents. Similar phenomena were observed in the model of a silo, but in this case wall effects appeared to have significant consequences for the behaviour of weight and slip fluctuations.

I. INTRODUCTION

Sand, gravel, powders, and pharmaceutical pills, are large aggregates of macroscopic particles or grains. Far from being simple materials with simple properties, they display an incredible range of complex behavior that defies their categorization as solid, liquid, or gas. The interaction of these grains is only contact between them, however, it's sufficient to produce unexpected effects. Phenomenology in these systems is very rich. Patterns, oscillations, segregation, avalanches, etc, at the macroscopic level, and long range correlations, lack of gaussianity in velocity distributions, violation of fluctuation-dissipation, etc at the microscopic level, are examples of some of the phenomenology in these systems. Contact between particles is very heterogeneous , thus we can't extrapolate results from the study of a pair of particles to the rest of them. Every particle has different form, and they aren't distributed equally in space. Any particle experiences interactions with a finite number of other particles, each of which defines a different direction. Therefore, the local environment of a particle is not isotropic. The existence of the preferred directions on the particle scale implies the possible emergence of force chains, chains of contacts along which the forces are transmitted. We call these chains with the name of arches. When we have a system where grains have contact with walls, these walls can absorb part of the weight and, if it is large enough, the absorbed weight can bring the failure of the system's walls. At Figure 1 , we can see the structure of these arches in experiment made by Iker Zuriguel and coworkers. These arches are completely different from one realization to another, leading to very different values of thr weight collected at the bottom of the granular pile. Furthermore, the geometry of these arches can be easily rearranged under small perturbations. Building upon this information, we can assert the importance of granular materials in applications as diverse as pharmaceutical industry, agriculture, and energy production. For this reason, the study of granular material is an excellent example of collaboration between various branches of science , from engineers to biologists , through physicists, chemists,... The system is far from equilibrium, and it has its own dynamic, but ordinary temperature has no effect on grain motion, because external forces such as gravity dominate the materials' behavior. So far, the silo is the most studied structure for its industrial utility, but here we focus on piles, such as the one illustrated in Figure 1 to avoid the weight loss caused by absorption at the walls. In this work, we will focus on the weight distribution and arch formations in a static granular pile. With this purpose in mind, we use a simple scalar model which accounts for disorder and arch formation, , as well as for the existence of slip events in the system. The model is an extension of the Scalar Arching Model (SAM in the following) proposed by Liu and Coppersmith ([1], [2]).

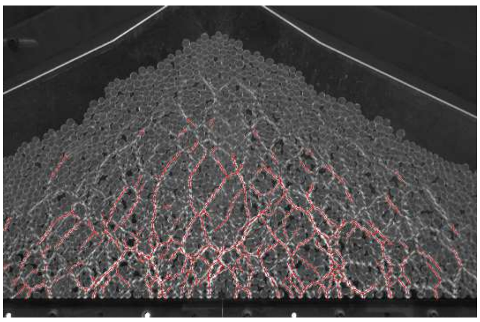

FIG. 1: Pile of round grains. Arches are showed using red lines. Image courtesy of Iker Zuriguel.

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II. THE MODEL

The SAM model describes weight transmission in a discrete 2D lattice which represents static configurations of high dense granular matter .The only force that appears in this model is gravity and the corresponding weight is heterogeneously distributed in space through grain contacts. We consider a stack of round particles (or grains) rows of monotonous increasing lenght in the downward (or gravity) direction. Each particle in the bulk has two particles over it and two particles under it. We can observe a characteristic local configuration in Figure 2. Each grain is labeled by two integer (i,j) giving its horizontal and vertical positions within the pile. In a real pile, the stacking of particles is heterogeneous, that is, contact between particles depends on a lot of parameters, like form, area, material,... Each particle has its own mass (m), and supports the weight of its two upstairs neighbors. Furthermore it shares its total load randomly between its two downstairs neighbors, as illustrated in Figure 2. Nevertheless, as stated before, not all

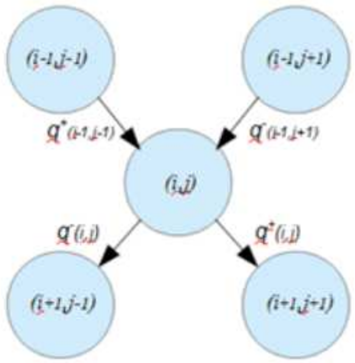

FIG. 2: Local structure of the SAM model and of the weight transmissions rules (arrows).

contacts are equal. When particle passes its load to its immediate neighbors at the row below, one has to include these differences (due to randomness in the local packing, size and shape of the grains, etc). For that reason, we introduce the random coefficients $q^{+}(i,j)$ and $q^{-}(i,j)$, independently thrown from a uniform distribution for each particle. These coefficients represents the portion of the weight that is passed to the left neighbor (q^{-}) and to right neighbor(q^{+}) particles below. Thus the coefficients are random variables between zero and one, subject to the mass conservation constraint $q^{+}(i,j) + q^{-}(i,j) = 1$. As a result, each particle supports a weight (W) (or force) due to its own mass and to the weight received from its upstairs neighbor particles (see eq.(1)).

$$W(i,j) = mg + q^{+}W(i-1,j-1) + q^{-}W(i-1,j+1) \quad (1)$$

Moreover, the SAM model also accounts for slip events, i.e. if a particle receives a bigger force from one side than from the other, this particle will be forced to pass its weight mostly in the same direction of the big force. This process can be justified because the bigger force bias the particle to have more contact with its downstairs neighbor along the same direction. Whenever the difference among the forces acting on one particle is bigger than a given slip condition, slip is produced. Slip implies passing all the weight to a single particle in the next row and the removal of the weight portion that should have been passed to the other neighbor particle. The slip condition is expressed in eqs. (2) and (3), i.e. when the difference between left and right contact forces is bigger than the total weight at a given particle multiplied by an arbitrary threshold, and slip event takes place. In the SAM model, heterogeneity is accounted for using a tunable threshold value R (related to the friction coefficient between grains). This parameter can be between 0 and 1, and for the rest af the analysis, it will be our control parameter. If it is bigger than 1, slip condition can't happen (the difference of portions can't be greater than the addition of portions and intrinsic mass). When the slip condition occurs, we make 0 and 1 the correspondingly q coefficients (eqs. (4) and (5)).

$$q^{+}W(i-1,j-1) - q^{-}W(i-1,j+1) > \; RW(i,j) \quad (2)$$
$$q^{-}W(i-1,j+1) - q^{+}W(i-1,j-1) > \; RW(i,j) \quad (3)$$

$$q^{+} = \; 0 \; q^{-} = \; 1 \quad (4)$$
$$q^{+} = \; 1 \; q^{-} = \; 1 \quad (5)$$

When the slip condition occurs continually, a chain is formed. This chain is also known as arch. These arches are important for weight distribution and avalanches. Originally this model was proposed for silo configurations, that is a stack of equivalent rows of granular material confined by rigid walls that, in generally, absorb part of the weight. In this article, we have focused our attention to the study of piles, where each row has one more particle than the previous one, and there is no need of walls.

III. SIMULATION RESULTS

A. Weight transmission and distribution

First, we describe the distribution of weight at the bottom of pile in function of R, the parameter which appears in slip condition (Figure 3). As we can see at Figure 3 , there are two main behaviors of the distribution of total weight at the bottom. Convex behavior, which is produced when the value of R is high, and accumulates the weight at the central part of the base. Concave behavior, which is produced when the value of R is low, and accumulates the weight at the sides of the base. Furthermore, there are a behaviour that can't be included in the previous two: when R is near 0.7 . In this case,

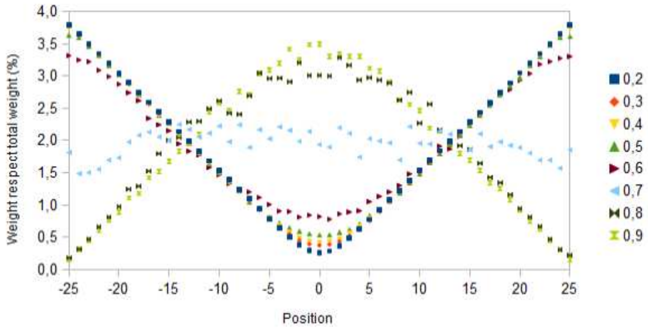

FIG. 3: Weight at each base particle normalized by total weight. Each data series represents a different R ,between 0.2-0.9. The pile has 51 rows. Every point is average of 1000 samples

we see the weight is equally distributed. This motivated us to make a closer study around this point (it's represented in Figure 4). Figure 3 and Figure 4 suggested to

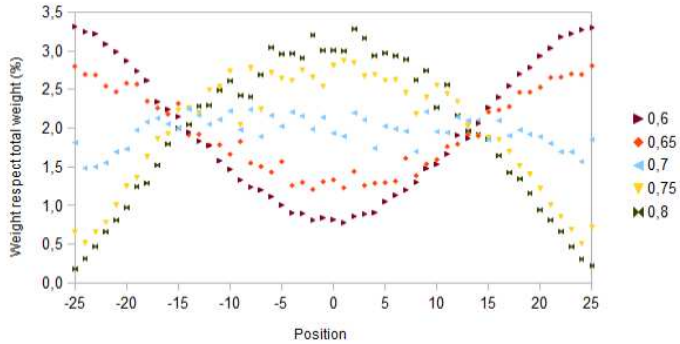

FIG. 4: Weight at each base particle normalized by total weight. Each data series represents a different R, between 0.6-0.8 The pile has 51 rows. It is an enlargement of Fig. 3, to show phase transition. Every point is average of 1000 samples.

us the possibility of continuous change of the total weight distribution at the bottom of the pile in function of R, between convex and concave behaviors. Furthermore, SAM model was proposed to study silos, and we wanted to explore the effect of walls on the distribution of weight at the bottom of this different structure. In pile, weight at the bottom is the addition of the weights of grains, but in silo we have rigid walls, which can absorb part of this weight. At Figure 5, where the total weight at the bottom is represented in function of R, we see that the walls absorb some of the weight, and that portion is a function of R. A property of this system are fluctuations , which become twice the average value. More accurate description of this system can be found in [5].

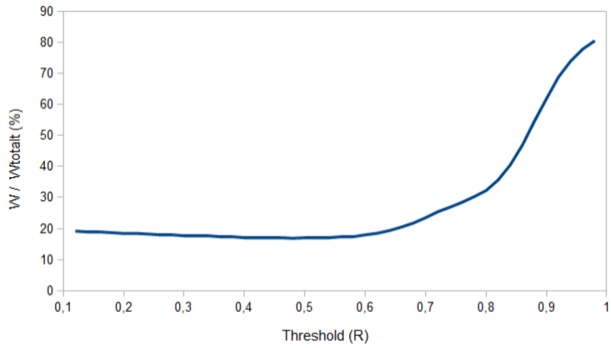

FIG. 5: Weight at the bottom in 51x101 silo in function of threshold, Rc, normalized by the weight of all grains. Each point is sn average of 1000 samples.

B. Critical point properties

Since the transition between the two behaviors appears to be smooth and continuous, one possible reason to observe this gradual changing is that we are in the neighborhood of a critical point or 2nd order transition. First, we had to find a good candidate for an order parameter that distinguishes the two phases. Inspired by the antiferromagnetic case, we thought of decomposing the lattice in two parts. We take only the left side of the pile , and separate the two lattices: the particles that achieve the slip condition and pass its weight to the left , and those who do to the right. Order parameter (m) is the average of the difference between the total number of particles in each lattice. It's calculated by using eq.(6). N_{sleft} and N_{sright} represents, respectively, slips to the left and to the right on the left side of the pile.

$$m = \ < N_{sleft} - N_{sright} > \qquad (6)$$

At Figure 6 , we represent order parameter (m) in function of R in a pile with 301 particles of base. We see the expected behavior of the order parameter, being different from zero below the critical point and zero above the critical point. In a second order phase transition, it is very important to study the fluctuations near the critical point, characterized by response function. In this case, in analogy to magnetism, we use χ as response function (susceptibility). In the neighborhood of the critical point, we expect the divergence of the response function. Actually, this divergence is infinite when we are in thermodynamic limit (infinite system), which is where the true phase transition occurs. When we study finite systems, we expect a finite divergence, which it grows with increasing system size. At Figure 7, we see the expected growth of maximum susceptibility as a function of system size. In view of these results, we now estimate the critical exponents of phase transition, which characterize the behavior at the critical point. To find these exponents, we use finite size scaling, that is a method to find

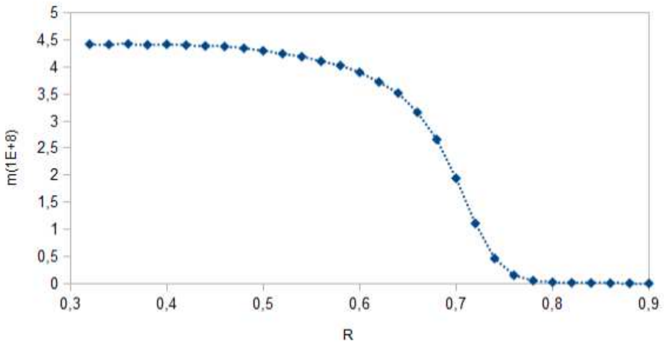

FIG. 6: Order parameter (m) in function of threshold (R) for pile with 301 particles on the base.

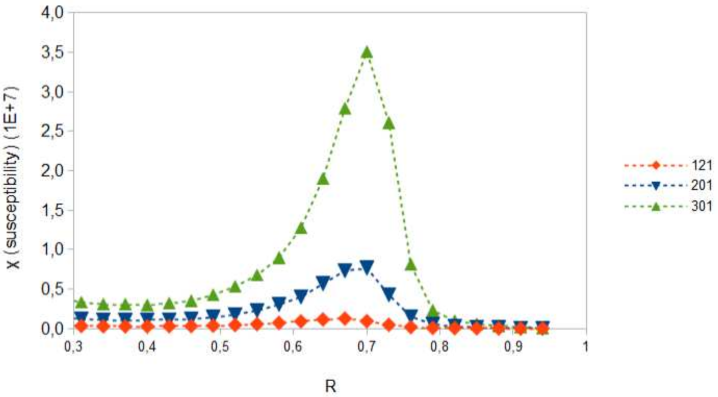

FIG. 7: (susceptibility) in function of R for different sizes. It shows the grown of maximum at critical point with the size of the system. It is according to theory, which in critical point the maximum is a discontinuity.

the values for the critical exponents and the transition threshold by observing how measured quantities vary for different lattice sizes. In analogy with usual 2nd order phase transitions, we use eqs. (7),(8) and (9) to find critical exponents.

$$m \propto \ (R - R_{crit})^{\beta} \tag{7}$$

$$L \propto \ (R(\infty) - R_{crit}(L))^{-}\nu \tag{8}$$

$$\chi \propto \mathrm{L}^{\frac{\gamma}{\nu}} \tag{9}$$

To improve the results, we use Maximum Likely Hood-Newmann method approximations to fit the exponents of power law distributions. Derivation of this method is explained with detail in [3]. The equation that we use to fit is eq. (10), where α is the critical exponent that we want to find, n is the number of different size data and x is order parameter or response function (it depends on the exponent we want to find) for every size data.

$$\alpha = \ 1 + n[\sum_{i=1}^{n} ln\frac{x_i}{x_{min}}]^{-1} \tag{10}$$

Exponents found are in Table 1. Critical point (when thermodynamic limit is valid) is located at $R = 0.7415$. To validate these exponents we do the collapse. Collapse is based on theory mentioned above (finite size scaling),and using different scale relations, we obtain equation (11). In the collapse, data of different sizes have to overlap. We see at Figure 8 that the collapse is better in the neighborhood of the critical point, and smallest system (51 particles of base) is the worse fit of the collapse. Bigger sizes collapse better, it was expectable, because big systems are closer to infinite length, the real phase transition.

Exponents	Our system	Percolation 2D
β	0.14 ± 0.01	0.14
ν	1.32 ± 0.08	1.33
γ	4.9 ± 0.3	2.39

TABLE I: Comparison between critical exponents of studied system and percolation 2d

$$\frac{\chi}{L^{\frac{\gamma}{\nu}}} \propto \ [R_c(L) - R(L)]L^{\frac{1}{\nu}} \tag{11}$$

We observe exponents similar to percolation exponents.

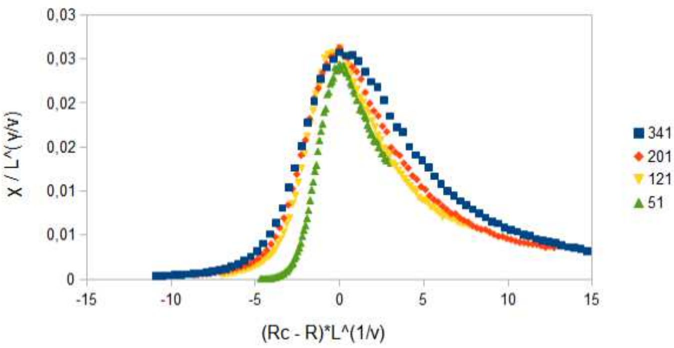

FIG. 8: Collapse of phase transition for different pile sizes. Exponents used are Table 1 exponents.

It shows us that there is some kind of relation among our system and the percolation propagation. Percolation is the measure of clusters and propagation in lattice structure. We observe we have two exponents very close to percolation (β , ν) and one (γ) that differs greatly. The next step is to applicate a force at the top of pile (besides gravity). We studied this new system, and we discovered the force applied conceals the phase transition. Effects are similar to applying magnetic field in magnetic phase transition. When the force applied is approximately 10 times the weight of the pile, we are be able to say there isnt phase transition.

IV. CONCLUSION

We have analyzed the distribution and transmission of weight on high density 2D lattice. Using SAM model, we discovered phase transition in pile geometry in funcion of R parameter, that represents the randomness that characterize these granular systems. We explain the phase transition like the competition among disorder and shape. When R is low, arches are easy to form, and shape leads them to go left on the left side and to go right on the right side. This accumulates weight at the ends. When R is high, disorder beats the design of the system and doesnt produce arches. The weight is distributed randomly, and there are more particles over the base at the center of pile, which provides more weight accumulation at this place. The transition is continuous (second order) and it has similarities with percolation 2d. Applying a force at the top conceals the phase transition. We are going to continue with the study of these systems. The distribution of forces at the low lines of piles of active particles will be analyzed because they guide the avalanches. These avalanches and distributions are function of the disorder and randomness, and concepts analyzed in this work is the basis from which we start.

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