The physics of Chandrasekhar and Super-Chandrasekhar White Dwarfs

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Abstract: We consider white dwarfs to be composed by a degenerate electron gas at zero temperature and study the underlying physics. The electron pressure can only balance the gravitational force up to a maximum mass of the white dwarf, the so-called Chandrasekhar mass. By writing our own code, we have found the value of this limiting mass to be 1.417 solar masses, in accordance with most observations. However, recent discoveries of overluminous type Ia supernovae explosions suggest white dwarf masses above the Chandrasekhar limit. Motivated by recent literature, we show in this work that the presence of an intense magnetic field in the star leads to a higher value of the maximum mass, which could explain these new observations.

I. INTRODUCTION

White dwarfs (WDs) are the remnants of ordinary stars which are unable to continue with the nuclear fusion processes in their cores. After all the hydrogen is burnt up to helium in the core of a star, the latter shrinks due to the gravitational force not being balanced by the nuclear reactions in the core. This shrinking leads to an increase in temperature until it is high enough for helium nuclear fusion to occur, returning the star to a state of hydrostatic equilibrium where the inward gravitational force is balanced by the outward force due to the thermal pressure gradient. Similar processes can take place for heavier elements as long as the star mass is above a certain value. For masses lower than about $8M_0$, where M_0 stands for solar mass, the gravitational force is too weak to reach the minimum density and temperature needed for carbon burning, as, before this happens, the degeneracy pressure of electrons becomes important enough to counteract the core shrinking. At this point, the outer layers of the star are blown away due to strong stellar winds, and what remains is a hot compact core made of carbon and oxygen, or what we know as a WD.

Due to the high densities that can be reached in WDs, the pressure in this kind of objects can be assumed to come entirely from the degenerate electrons. Two fermions can not have the same quantum state, and thus in a WD electrons have to move to higher energy states by increasing their momenta, giving origin to what is called a degeneracy pressure. Due to its specific dependence with density, electron degeneracy pressure can only balance the gravitational force up to a maximum WD mass, known as the Chandrasekhar mass [3].

A WD in a close binary system can accrete matter from its companion star, increasing its mass in the process. This mass income results in an increase in the star temperature and, if it is high enough, a point is reached where temperature allows for carbon fusion in the core. The weakly temperature dependent electron pressure is unable to cool the star by expanding it just as happens with ordinary stars. This leads to a runaway fusion reaction that ends up blowing up the whole star in a type Ia supernova explosion. Due to the existence of a maximum WD mass, type Ia supernovae have a very well defined luminosity peak and can be used as standard candles [9]. These explosions were of prime importance in the discovery of the accelerated expansion of the universe, awarded with the Nobel Prize in 2011.

However, recent observations [1, 10] of peculiar type Ia supernovae with exceptionally high luminosities suggest the existence of Super-Chandrasekhar WDs, i.e. WDs with a higher mass than the Chandrasekhar limit. One proposed explanation to these observations is the presence of a strong magnetic field in the star, which would increase the maximum possible mass [7, 9].

In this work, the physics concerning WDs is studied. Section II provides the structural equations of a star in hydrostatic equilibrium. The equation of state for both non- and highly- magnetized WDs are discussed in section III. Section IV focuses on the obtained numerical results and we conclude with a summary and outlook in section V.

II. STRUCTURE EQUATIONS

II.1. Newtonian gravity

There are two main forces acting on a star. One of them is the inward directed gravitation and the other one results from a pressure gradient. In hydrostatic equilibrium these two forces are equal in magnitude, giving the first structure equation of the star [4]. For Newtonian gravity we have:

$$\frac{dP}{dr}=-\frac{G\,\varepsilon(r)\,m(r)}{c^2 r^2} \quad , \tag{1}$$

where P is the pressure, G the Newton's gravitational constant, c the speed of light and r the distance from the centre of the star. $m(r)$ is the mass contained within a radius r and $\varepsilon(r)=\rho(r)c^2$ is the energy density, with $\rho(r)$ being the mass density. The second structure equation comes from mass conservation:

$$\frac{dm}{dr}=\frac{4\pi r^2 \varepsilon(r)}{c^2} \quad . \tag{2}$$

II.2. General relativity corrections

When the star is very compact, general relativity effects must be taken into account. In this case, eq. (1) is replaced by the Tolman-Oppenheimer-Volkoff (TOV) equation [2]:

$$\frac{dP}{dr}=-\frac{G\,\varepsilon(r)\,m(r)}{c^2 r^2}\left[1+\frac{P(r)}{\varepsilon(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1-\frac{2G\,m(r)}{c^2 r}\right]^{-1} . \tag{3}$$

This equation adds three correction factors to Newtonian gravity. As the three factors are greater than 1, they strengthen the effect of gravity on the star. The importance of these corrections will be evaluated numerically in section IV.

III. EQUATIONS OF STATE

III.1. Non-magnetized WDs

In order to be able to solve the previous equations we need to find a relation between the pressure and the energy density, i.e. an equation of state (EoS) of the matter. Matter in WDs can be considered to be composed by atomic nuclei and an ideal Fermi gas of degenerate electrons. The energy distribution of the electrons is given by the Fermi-Dirac statistic, where we can set the temperature to zero [4], even when WDs core temperatures can be as high as 10^7 K [5], due to the huge densities present in the star.

Ignoring the electrostatic interactions, the number density of electrons has the following expression [4]:

$$n_e=\int_0^{p_F}\frac{2}{(2\pi\hbar)^3}d^3p=\frac{p_F^3}{3\pi^2\hbar^3} \quad , \tag{4}$$

where p_F is the Fermi momentum.

We can now write the expression for the total energy density:

$$\varepsilon=n_e m_N \frac{A}{Z}c^2+\varepsilon_{elec}(k_F) \quad , \tag{5}$$

where m_N is the nucleon mass and A and Z are the mass and atomic numbers, respectively. The first term in this equation corresponds to the contribution to the energy density of the rest mass of the nucleons assuming electrical neutrality in the star. The second term is the energy density of the electrons themselves. We do not include here the kinetic energy of the nucleons as it is very low compared to their rest mass. As a consequence, nucleons do not give a significant contribution to the pressure of the system and we can consider it to come entirely from the fast moving electrons. On the other hand, as we will see in the next section, electrons do not give an important contribution to the mass density of the star unless extremely high central densities are attained, where their Fermi momentum becomes very large.

With the electron energy

$$E(p)=\sqrt{p^2c^2+m_e^2c^4} \quad , \tag{6}$$

we can write ε_{elec} as follows [4]:

$$\begin{aligned}\varepsilon_{elec}(p_F)&=\frac{8\pi}{(2\pi\hbar)^3}\int_0^{p_F}E(p)\,p^2\,dp\\ &=\frac{\varepsilon_0}{8}[(2x^3+x)\sqrt{1+x^2}-\ln(x+\sqrt{1+x^2})]\end{aligned} \quad , \tag{7}$$

where we have defined ε_0 and x as:

$$\varepsilon_0=\frac{m_e^4c^5}{\pi^2\hbar^3} \quad , \tag{8}$$

$$x=\frac{p_F}{m_e c} \quad . \tag{9}$$

The pressure of an electron gas with an isotropic distribution of momenta is given by [4]:

$$\begin{aligned}P(p_F)&=\frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3}\int_0^{p_F}\frac{p^2c^2}{E(p)}p^2\,dp\\ &=\frac{\varepsilon_0}{24}[(2x^3-3x)\sqrt{1+x^2}-3\ln(x+\sqrt{1+x^2})]\end{aligned} \quad . \tag{10}$$

Eqs. (7) and (10) relate the energy density and pressure with the Fermi momentum, and thus, they provide the equation of state.

III.2. Magnetized WDs

If a magnetic field is considered, the electron motion in the plane perpendicular to it becomes determined by the field and quantized into Landau orbitals (see [6] or [7] for more details). For sufficiently high magnetic fields, the cyclotron energy, $\hbar\omega_c=\hbar(eB/m_ec)$, is comparable to the electron rest mass, m_ec^2, and the electrons become relativistic. This defines a critical magnetic field:

$$B_c = \frac{m_e^2 c^3}{\hbar e} = 4.414 \times 10^{13} G \quad . \qquad (11)$$

The Fermi energy of the electrons now gets an extra contribution due to the magnetic field, such that for a Landau level *ν* it has the following expression [6]:

$$E_F^2 = p_F(\nu)^2 c^2 + m_e^2 c^4 (1 + 2\nu B_D) \quad , \qquad (12)$$

where a dimensionless field $B_D = B/B_c$ is introduced.

The upper limit for *ν* is obtained by introducing the condition $p_F(\nu)^2 \geq 0$ in eq. (12). Defining the dimensionless maximum Fermi energy of a system (in our case the value of the Fermi energy at the centre of the star) $\varepsilon_{Fmax} = E_{Fmax}/m_e c^2$, we get:

$$\nu_m = \frac{\varepsilon_{Fmax}^2 - 1}{2B_D} \quad , \qquad (13)$$

where the nearest lowest integer must be taken. Note that $\varepsilon_{Fmax} \geq 1$, since the Fermi energy of the electrons is larger than their rest mass energy $m_e c^2$. Thus, for a given E_{Fmax} value, the stronger the magnetic field, the lower the number of Landau levels that can be occupied.

The main effect of the magnetic field is to modify the available density states for the electrons n_e, and consequently the EoS. The new expression for n_e is [7]:

$$n_e = \frac{B_D \varepsilon_0}{2 m_e c^2} \sum_{\nu=0}^{\nu_m} g_\nu x(\nu) \quad , \qquad (14)$$

where ε_0 and x have been defined in eqs. (8) and (9), and g_ν is the degeneracy of that level, such that $g_\nu=1$ for $\nu=0$ and $g_\nu=2$ for $\nu>0$.

The additional factor in eq. (12) due to the field modifies the electron energy density at zero temperature, and subsequently the pressure exerted by the electrons. In the presence of a magnetic field, eqs. (7) and (10) are replaced by these new expressions [6]:

$$\varepsilon_{elec}(p_F) = \frac{B_D \varepsilon_0}{4} \sum_{\nu=0}^{\nu_m} g_\nu (1 + 2\nu B_D) \eta_+ \left(\frac{x(\nu)}{\sqrt{1 + 2\nu B_D}} \right) \quad , \qquad (15)$$

$$P(p_F) = \frac{B_D \varepsilon_0}{4} \sum_{\nu=0}^{\nu_m} g_\nu (1 + 2\nu B_D) \eta_- \left(\frac{x(\nu)}{\sqrt{1 + 2\nu B_D}} \right) \quad , \qquad (16)$$

where $\eta_\pm(z) = z\sqrt{1+z^2} \pm \ln(z + \sqrt{1+z^2})$.

IV. NUMERICAL RESULTS

We have written numerical codes to compute several physical relations for WDs using the structure equations (1)-(3) and the EoS for non-magnetized and magnetized WDs.

IV.1. Non-magnetized WDs

In this work, a fourth-order Runge-Kutta method has been implemented to integrate numerically the structure equations. Starting with the initial conditions for the mass and the Fermi momentum at the centre of the star, $m(r=0)= 0$ and $x(r=0)= x_0$, we can compute iteratively the value of these two magnitudes at a radius r+dr, where dr is the integration step that we choose. We can find the dependence of x with the radius as $dx/dr= dx/dP*dP/dr$, where the first derivative can be computed analytically from eq. (10). As pressure decreases with radius, we will reach a point where P, and thus x, will become zero or negative. This point determines the radius R of the star and its mass $M= m(r=R)$.

The central Fermi momentum x_0 also determines the central density of the star, according to eqs. (4), (5) and (7). Table 1 illustrates the variation of the central density, central pressure, mass and radius of the WD as a function of x_0. We have set A/Z=2 in (5), as corresponds to a WD made of carbon and oxygen, and taking into account general relativity corrections (TOV eq. (3)).

As seen in Fig. 1, for the TOV equation the mass-density relation shows a maximum at x_0=23 where the mass is $M_{Ch}=1.417M_0$. This is what we know as the Chandrasekhar limit. If we do not take into account GR corrections, the curve behaves slightly different, and this difference becomes very clear at high central densities. For the "Newtonian gravity" case, the Chandrasekhar mass is $M_{Ch}=1.438M_0$ and corresponds to a central Fermi momentum x_0=34. We will be working with the TOV equation from now on.

Note that the solutions in Fig. 1 for central densities beyond the Chandrasekhar limit correspond to unstable cases. For a discussion about stability we refer the reader to [4]. We highlight, however, that this relation shows a strict maximum because the contribution to the density coming from the kinetic energy of the electrons has been taken into account.This contribution becomes more important for high densities and makes gravitational effects stronger (i.e. the mass decreases for a fixed density with respect to the case where only the nucleon mass has been taken into account). If we neglect $\varepsilon_{elec}(p_F)$ in (5), a maximum mass is never achieved, although similar results may be obtained by discarding non-realistic solutions.

x_0	ρ_0 (g/cm^3)	P_0 (erg/cm^3)	M/M_0	R(km)
0.1	$1.95\cdot10^{3}$	$9.57\cdot10^{17}$	0.022	31716
1.0	$1.95\cdot10^{6}$	$7.38\cdot10^{22}$	0.510	9701
5.0	$2.44\cdot10^{8}$	$7.23\cdot10^{25}$	1.268	3510
10	$1.95\cdot10^{9}$	$1.19\cdot10^{27}$	1.384	2069
23	$2.38\cdot10^{10}$	$3.35\cdot10^{28}$	1.417	1021
100	$1.99\cdot10^{12}$	$1.20\cdot10^{31}$	1.345	258

Table 1. Calculated values for different central Fermi momenta

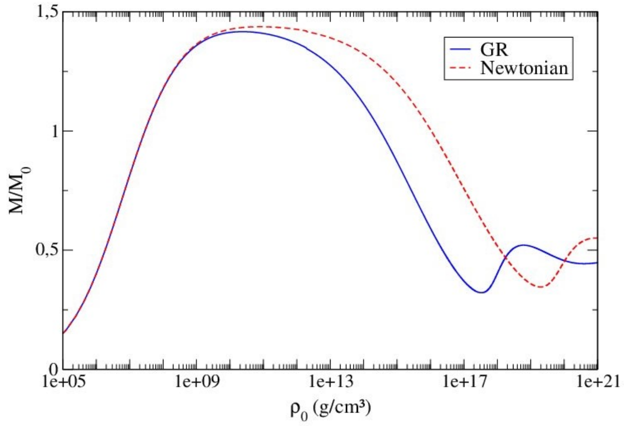

Fig. 1. Computed masses for C-O WDs as a function of central density.

We have studied the mass-radius relation of WDs in Fig. 2, where each point in the curve represents the solution for a specific x_0. The effect that the star composition may have on its mass is also studied in Fig. 2, where we have considered a C-O WD (with A/Z=2) and a star composed by iron (with A/Z=56/26=2.15), which is the last step that can be reached in a core fusion chain. We see that the Chandrasekhar mass for the Fe case is lower than for the C-O WD, due to the gravitational contribution of the extra neutrons. Specifically, for an iron WD, M_{Ch}=1.223M_0.

It is also interesting to see the density profile inside of the star. Fig. 3 shows how density decreases with the radial coordinate for different central values. Even with small variations in the central Fermi momentum, the difference in density becomes much more important. This can be explained by the cubic dependence of density with p_F seen in eqs. (4) and (5).

IV.1. Magnetized WDs

We have also written a code to solve the case where a strong magnetic field is present in the WD. For that purpose, we need to use the EoS provided by eqs. (15) and (16). In this work, we want to prove the possibility for a highly magnetized WD to exceed the Chandrasekhar mass limit, as

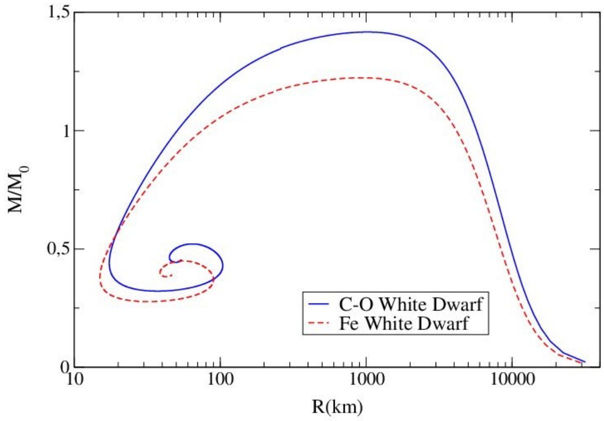

Fig. 2. Mass-radius relation of WDs for different compositions

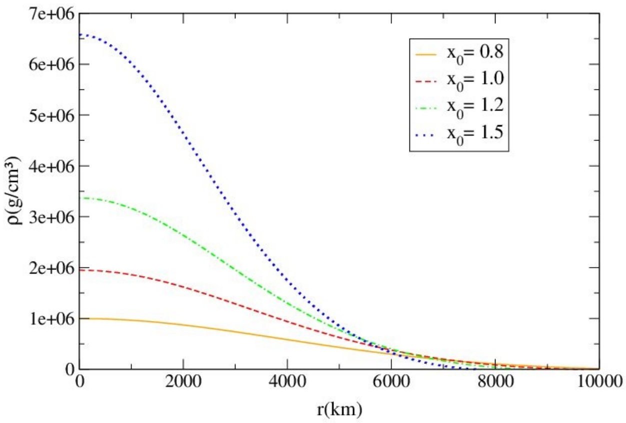

Fig. 3. Density profile

suggested in the literature [7, 9]. For this reason, we have studied the case where only the ground Landau level is occupied, and thus ν_m in (13) is set to ν_m<1 (implying ν=0). This condition ensures that, for a given E_{Fmax} at the centre of the star, we will have a high enough magnetic field such that from the centre to the surface of the WD it only allows the ground Landau level to be filled. This field is considered constant throughout the whole star.

Taking all the above considerations into account, we can compute the central Fermi momentum x_0 from eq. (12) and proceed as we did in the non-magnetized WD case, but with the use of eqs. (15) and (16), to obtain the mass and radius of the star. Fig. 4 shows the radius-mass relation for WD with different central Fermi energies. Note that in this figure B_D is a lower limit to the field strengths that ensure that only the ground Landau level is populated, as B_D has been obtained by setting ν_m=1 following the discussion by Lai and Shapiro [6]. This guarantees that, for any value of the field higher than B_D, the right hand side of eq. (13) is between 0 and 1, and thus only the ground Landau level can be occupied.

It is clear from Fig. 4 that, in a wide range of ε_{Fmax} values, the mass of the star greatly exceeds the Chandrasekhar limit. The largest achieved mass in the plot corresponds to a maximum dimensionless central energy ε_{Fmax}= 20 and has a value of 2.487M$_0$ (the magnetic field is $8.81\cdot10^{15}$ G). This mass is compatible with the calculated masses for the peculiar type Ia supernovae explosions, which lie in the range 2.1-2.8 M$_0$, depending on the chosen model to estimate the nickel mass [9, 10].

As we mentioned, the mass observed in the plot for ε_{Fmax}= 20 is not necessarily the limiting mass for B_D= 199.5, but the mass that a WD would have when only the level ν=0 is totally filled for this field. Increasing the Fermi energy from that point (and thus exceeding ε_{Fmax} associated with ν=0) would start filling higher Landau levels for B_D= 199.5 and, if enough of these levels are occupied, one would expect a strict maximum in the M-R relation for this B_D (a similar comment applies to other field values), as happens in Fig. 2.

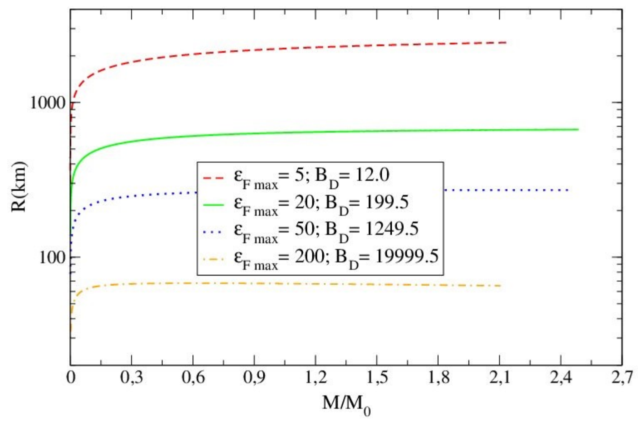

Fig. 4. Radius-mass relation for different central Fermi energies. We recall that B_D=B/B_c, with B_c= $4.414\cdot10^{13}$ G.

The WD radius evolution observed in Fig. 4 shows a fast increase for low masses and then remains practically constant. However, for higher energies (ε_{Fmax}>100), a bending towards lower radii can be observed, suggesting that a maximum mass may be reached eventually.

V. SUMMARY AND CONCLUSIONS

Considering WDs to be electron-pressure dominated unequivocally leads to a maximum mass for this kind of stars. Up to now, most observations of type Ia supernovae are in very good accordance with the Chandrasekhar mass obtained in section IV.1, with a value of 1.417M$_0$.

However, recent discoveries of overluminous peculiar type Ia supernovae suggesting higher WD masses have led us to consider the effect that a magnetic field would have on the limiting mass of the WDs. Without actually having searched for a new maximum mass, we have shown that the presence of an intense field can increase the mass of the WD well above the Chandrasekhar limit, with values compatible with the ones that are inferred from the peculiar supernovae.

Motivated by Das and Mukhopadhyay work [8], we leave for future study calculations accounting for Landau levels above the ground level, as well as the inclusion of a radial profile for the magnetic field in the star and the free field contributions to the pressure and energy density [7]. With all these considerations, we would expect to find a new limiting mass for highly magnetized WDs.

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[1] R.A. Scalzo et al., *Astrophys. J.* **713** 1073 (2010)

[2] S.L. Shapiro & S.A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects*. New York: Wiley (1983)

[3] S. Chandrasekhar, *Astrophys. J.* **74** 81 (1931)

[4] I. Sagert et al., *Eur. J. Phys.* **27** 577 (2006)

[5] M.L. Kutner, *Astronomy: A physical perspective*. Cambridge University Press. p.189 (2003)

[6] D. Lai & S.L. Shapiro, *Astrophys. J.* **383** 745 (1991)

[7] U. Das & B. Mukhopadhyay, *Phys. Rev. D* **86** 042001 (2012)

[8] U. Das & B. Mukhopadhyay, *J. Cosmol. Astropart. P.* **06** 050 (2014)

[9] U. Das and B. Mukhopadhyay, *Phys. Rev. Let.* **110** (2013)

[10] D.A. Howell et al., *Nature* **443** 308 (2006)