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Insights into the nucleolus of the assignment game

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Abstract: We show that the family of assignment matrices which give rise to the same nucleolus form a compact join-semilattice with one maximal element, which is always a valuation (see p.43, Topkis (1998)). We give an explicit form of this valuation matrix. The above family is in general not a convex set, but path-connected, and we construct minimal elements of this family. We also analyze the conditions to ensure that a given vector is the nucleolus of some assignment game.

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1. Introduction

Combinatorial optimization games, also known as OR-games (Curiel, 1997; Borm et al., 2001) analyze cooperative situations where the worth of a coalition of players is the optimal result of a well-known operations research problem. Among others we mention linear production games (Owen, 1975), minimum cost spanning tree games (Granot and Huberman, 1981), inventory games (Hartman and Dror, 1996; Hartman et al., 2000; Meca et al. 2004), minimum coloring games (Deng et al. 2000; Bietenhader and Okamoto, 2006), supply chain management games (Nagarajan and Sošić, 2008).

Matching in graphs are combinatorial optimization problems. Because of its importance they have been studied in depth (Lovász and Plummer, 1986; Korte and Vygen, 2000). In a pioneering paper, Shapley and Shubik (1972) analyze the bipartite graph case as a cooperative problem. It is called the assignment game. The assignment game (Shapley and Shubik, 1972) is the cooperative viewpoint of a two-sided market. There are two sides of the market, i.e. two disjoint sets of agents, buyers and sellers, who can trade. The profits are collected in the edges of the graph as the weights, or can be represented in a matrix, the assignment matrix. The problem is that the maximal weight matching or the gain of the market is to be shared fairly among the agents. The allocation of the optimal profit should be such that no coalition has incentives to depart from the grand coalition and act on its own. In doing so, a first game-theoretical analysis of cooperation focuses on the core of the game. Shapley and Shubik show that the core of any assignment game is always non-empty. It coincides with the set of solutions of the linear program, dual to the classical optimal assignment problem. Assignment games have been widely studied in the literature (Quint, 1991; Granot and Granot, 1992; Martínez-de-Albéniz et al., 2011a, 2011b).

Among other solutions, the nucleolus (Schmeidler, 1969) is a "fair" solution in the general context of cooperative games. It is a unique core-selection that lexicographically minimizes the excesses[1] arranged in a nondecreasing way. The standard procedure for computing the nucleolus proceeds by solving a finite (but large) number of related linear programs. As a solution concept, the nucleolus has been analyzed and computed in many OR-games, for instance Okamoto (2008), Solymosi et al. (2005), Kern and Paulusma (2003), Faigle et al. (1998), Granot et al. (1996), Deng and Papadimitriou (1994), or Granot and Huberman (1981). An interesting survey on the nucleolus and its computational complexity is given in Greco et al. (2015).

For matching games, the general non-bipartite case, the complexity of the computation of the nucleolus is still an open problem. Some special cases have been studied, for instance, balanced matching games (Biró et al., 2011) or cardinality matching games, with unitary weights (Kern and Paulusma, 2003). In all these cases it is proved the nucleolus can be computed in polynomial time, what can be viewed as a generalization of the first result with an algorithm for the computation of the nucleolus of the assignment game, the bilateral case (Solymosi and Raghavan, 1994). Recently Martínez-de-Albéniz et al. (2013) provides a new procedure to compute the nucleolus of the assignment game. From a geometric point of view, Llerena and Núñez (2011) have characterized the nucleolus of a square assignment game, essential for our purposes. Llerena et al. (2015) gives an axiomatic approach of the nucleolus of the assignment game.

In this paper we focus on the structure of matrices, that is the weight system on bipartite graphs, that give rise to the same nucleolus.

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[1] Given a coalition $S \subseteq N$, and an allocation $x \in \mathbb{R}^N$ the excess of a coalition is defined as $e(S,x) := v(S) - \sum_{i \in S} x_i$. Note they can be considered as complaints.

To illustrate the problem, consider the assignment matrix

$$A = \begin{pmatrix} 8 & 6 \\ 4 & 4 \end{pmatrix}.$$

The worth to share is $v^* = 12$, and its nucleolus is $(5,2,3,2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, but matrix

$$B = \begin{pmatrix} 8 & 4 \\ 0 & 4 \end{pmatrix}$$

has also the same nucleolus. Now we draw the core of the associated assignment games and their nucleolus. We depict the projection on the buyers' (first) coordinates of the core of both games in Figure 1. The core of the first one $C(w_A)$ is in dark shading and the second one $C(w_B)$ in light shading. From Llerena and Núñez (2011) the nucleolus of matrix A is the unique core point such that the distances over some segments to the core's walls are equal: $\overline{A'N} = \overline{NB'}, \overline{C'N} = \overline{ND'}$ and $\overline{EN} = \overline{NF}$. For matrix B we have $\overline{AN} = \overline{NB}, \overline{CN} = \overline{ND}$ and $\overline{EN} = \overline{NF}$.

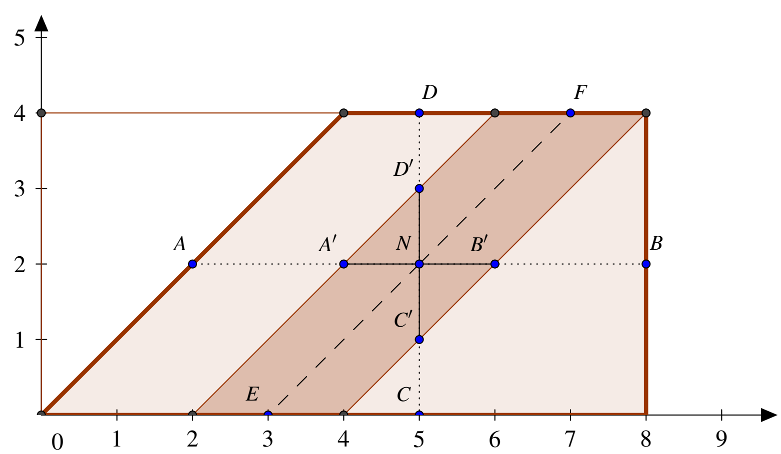

Figure 1: Two cores with the same nucleolus, $(5,2;3,2)$.

From the above geometric illustration we may expect a large class of assignment matrices sharing a given vector as their nucleolus.

The main contributions of the paper are the following:

- The family of matrices with the same nucleolus forms a join-semilattice, i.e. closed by entry-wise maximum. The family has a unique maximum element which is always a valuation matrix and we give its explicit form (Section 3).
- We show that the above family is a path-connected set, and give the precise path. We construct some minimal elements of the family (Section 4).
- We give conditions to characterize the non-emptiness of the family, i.e. conditions on a vector to be the nucleolus of some assignment game (Section 3).

2. Preliminaries and notation

An assignment market (M, M', A) is defined to be two disjoint finite sets: M the set of buyers and M' the set of sellers, and a nonnegative matrix $A = (a_{ij})_{i\in M, j\in M'}$ which represents the profit obtained by each mixed-pair $(i, j) \in M \times M'$. To distinguish the j-th seller from the j-th buyer we will write the former as j' when needed. The assignment market is called square whenever $|M| = |M'|$. Usually we denote by $m = |M|$ and $m' = |M'|$. M^+_m denotes the set of nonnegative square matrices with m rows and columns, and $\mathrm{M}^+_{m\times m'}$ the set of nonnegative matrices with m rows and m' columns.

Recall that $\mathrm{M}^+_{m\times m'}$ forms a lattice with the usual ordering $\leq$ between matrices. Given an ordered subset of matrices $(\mathscr{F}, \leq)$, $\mathscr{F} \subseteq \mathrm{M}^+_{m\times m'}$, we say matrix $C \in \mathscr{F}$ is a minimal element of $(\mathscr{F}, \leq)$ if there is no matrix $D \in \mathscr{F}$, with $D \neq C$ and

$D \leq C$. A matrix $A \in \mathrm{M}^{+}_{m\times m'}$ is a *valuation matrix*[2] if for any $i_1, i_2 \in \{1,\ldots,m\}$ and $j_1, j_2 \in \{1,\ldots,m'\}$ we have $a_{i_1 j_1} + a_{i_2 j_2} = a_{i_1 j_2} + a_{i_2 j_1}$.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M'_0 \subseteq M'$ with $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. If for some buyer $i \in M$ there is no seller $j \in M'$ satisfying $(i, j) \in \mu$ we say buyer i is unmatched by μ and similarly for sellers. The set of all matchings from M to M' is represented by $\mathscr{M}(M,M')$. A matching $\mu \in \mathscr{M}(M,M')$ is *optimal* for (M,M',A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for any $\mu' \in \mathscr{M}(M,M')$. We denote by $\mathscr{M}^*_A(M,M')$ the set of all optimal matchings.

Shapley and Shubik (1972) associate any assignment market with a game in coalitional form $(M \cup M', w_A)$ called the *assignment game* in which the worth of a coalition formed by $S \subseteq M$ and $T \subseteq M'$ is $w_A(S \cup T) = \max\limits_{\mu\in\mathscr{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$, and any coalition formed only by buyers or sellers has a worth of zero.

The main goal is to allocate the total worth among the agents, and one of the prominent solutions for cooperative games is the core. Shapley and Shubik (1972) prove that the core of the assignment game is always nonempty. Given an optimal matching $\mu \in \mathscr{M}^*_A(M,M')$, the *core* of the assignment game, $C(w_A)$, can be easily described as the set of non-negative payoff vectors $(x,y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ satisfying

$$x_i + y_j = a_{ij} \text{ for all } (i, j) \in \mu, \tag{1}$$

$$x_i + y_j \geq a_{ij} \text{ for all } (i, j) \in M \times M', \tag{2}$$

and all agents unmatched by μ get a null payoff.

Now we define the nucleolus (Schmeidler, 1969) of an assignment game, taking into account that its core is always nonempty. The excess of a coalition $\emptyset \neq R \subseteq M \cup M'$ with respect to an allocation in the core, $(x,y) \in C(w_A)$, is defined

[2] Following Topkis (1998), a function is a valuation if it is submodular and supermodular.

as $e(R,(x,y)) := w_A(R) - \sum_{i\in R\cap M} x_i - \sum_{j\in R\cap M'} y_j$. By the bilateral nature of the market, it is known that the only coalitions that matter are the individual and mixed-pair ones. Given an allocation $(x,y) \in C(w_A)$, define the excess vector $\theta(x,y) = (\theta_k)_{k=1,\ldots,r}$ as the vector of individual and mixed-pair coalitions excesses arranged in a non-increasing order, i.e. $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_r$. Then the *nucleolus* of the game $(M \cup M', w_A)$ is the unique core allocation $\nu(w_A) \in C(w_A)$ which minimizes $\theta(x,y)$ with respect to the lexicographic order[3] over the whole set of core allocations. For ease of notation we will use, for $A \in \mathrm{M}^+_{m\times m'}$, $\nu(A)$ instead of $\nu(w_A)$ if no confusion arises.

Solymosi and Raghavan (1994) use the excess vector to describe an algorithm to compute the nucleolus and Martínez-de-Albéniz et al. (2013) give a new procedure to compute it for any assignment game. Here, we use the characterization of the nucleolus of a square assignment game of Llerena and Núñez (2011), see also Llerena et al. (2015), to study some properties of the nucleolus. To this end we define the maximum transfer from a coalition to another coalition. Given any square assignment game $(M \cup M', w_A)$, and two arbitrary coalitions of the same cardinality $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$, with $|S| = |T|$ we define

$$\delta^A_{S,T}(x,y) := \min_{i\in S, j\in M'\setminus T} \left\{x_i, x_i + y_j - a_{ij}\right\}, \tag{3}$$

$$\delta^A_{T,S}(x,y) := \min_{j\in T, i\in M\setminus S} \left\{y_j, x_i + y_j - a_{ij}\right\}, \tag{4}$$

for any core allocation $(x,y) \in C(w_A)$.

The interpretation of expression (3) is the following: the largest same amount that can be transferred from players in S to players in T with respect to the core

[3]The *lexicographic order* $\geq_{lex}$ on $\mathbb{R}^d$ is defined in the following way: $x \geq_{lex} y$, where $x,y \in \mathbb{R}^d$, if $x = y$ or if there exists $1 \leq t \leq d$ such that $x_k = y_k$ for all $1 \leq k < t$ and $x_t > y_t$.

allocation (x,y) while remaining in the core, that is,

$$\delta^A_{S,T}(x,y) = \max\left\{\varepsilon \geq 0 \mid \left(x - \varepsilon 1^S, y + \varepsilon 1^T\right) \in C(w_A)\right\}, \tag{5}$$

where 1^S and 1^T represent the characteristic vectors[4] associated with coalition $S \subseteq M$ and $T \subseteq M'$, respectively.

Llerena and Núñez (2011) prove that the nucleolus of a square assignment game is characterized as the unique core allocation $(x,y) \in C(w_A)$ where

$$\delta^A_{S,T}(x,y) = \delta^A_{T,S}(x,y) \tag{6}$$

for any $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$. In certain cases, the number of equalities can be reduced. Indeed, note that if $T \neq \mu(S)$ for some $\mu \in \mathscr{M}^*_A(M,M')$, then it holds $\delta^A_{S,T}(x,y) = \delta^A_{T,S}(x,y) = 0$. Therefore, for this characterization we only have to check (6) for the cases $T = \mu(S)$ for some optimal matching $\mu \in \mathscr{M}^*_A(M,M')$ and any $\emptyset \neq S \subseteq M$, i.e.

$$\delta^A_{S,\mu(S)}(x,y) = \delta^A_{\mu(S),S}(x,y), \text{for any } \emptyset \neq S \subseteq M. \tag{7}$$

3. Assignment games with the same nucleolus

Different assignment games may have the same nucleolus. As a simple illustrative example, matrices $A, B \in \mathrm{M}^+_2$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{8}$$

satisfy $\nu(A) = \nu(B) = (1,1,1,1)$.

Notice also that matrices with the same nucleolus must have the same worth for the grand coalition even if they do not have any optimal matching in common.

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[4] Given $S \subseteq \{1,\ldots,n\}$, $1^S \in \mathbb{R}^n$ is such that $1^S_i = 1$, if $i \in S$, and zero otherwise.

We focus now on an interesting property of the family of assignment matrices that share the same nucleolus: they form a join-semilattice. That is, given two matrices with the same nucleolus, their maximum, defined entry-wise, has also the same nucleolus[5].

Theorem 3.1. *Let $A, A' \in \mathrm{M}^{+}_{m\times m'}$ be two matrices sharing the same nucleolus, $\nu(A) = \nu(A') \in \mathbb{R}^{M}_{+} \times \mathbb{R}^{M'}_{+}$. Then, $A \vee A'$ has the same nucleolus, i.e. $\nu(A \vee A') = \nu(A) = \nu(A')$.*

Proof. If $m \neq m'$, we add zero rows or columns to make the matrices square. It is known that these rows or columns correspond to dummy players which obtain zero payoff at any core allocation, and also in the nucleolus. Therefore we can assume from now on that matrices are square. We have $A, A' \leq A \vee A'$, and also $C(w_A) \cap C(w_{A'}) \neq \emptyset$, since both games share the nucleolus. We claim

$$C(w_A) \cap C(w_{A'}) = C(w_{A\vee A'}).$$

To see it, take any $(x,y) \in C(w_A) \cap C(w_{A'})$. It is clear $x_i + y_j \geq \max\{a_{ij}, a'_{ij}\}$ for all $(i,j) \in M \times M'$. Then for any optimal matching μ of $A \vee A'$ we have

$$w_{A\vee A'}(M \cup M') = \sum_{(i,j)\in\mu} \max\{a_{ij}, a'_{ij}\} \leq$$
$$\sum_{(i,j)\in\mu} [x_i + y_j] = w_A(M \cup M') = w_{A'}(M \cup M').$$

As a consequence $w_{A\vee A'}(M \cup M') = w_A(M \cup M') = w_{A'}(M \cup M')$. Now it is easy to see $(x,y) \in C(w_{A\vee A'})$. The other inclusion is straightforward.

[5] This property also holds with respect to the core (Martínez-de-Albéniz et al., 2011a), but curiously it is worth noting this property does not hold for the kernel, a set-solution defined by Davis and Maschler (1965). This can be easily seen by using the above matrices. Note that the nucleolus always belongs to the kernel, and Driessen (1998) proves that for assignment games, the kernel is included in the core.

Now to see $\nu(A) = \nu(A') = (x,y)$ is the nucleolus of $w_{A \vee A'}$, just note that, for all $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$,

$$\delta_{S,T}^{A \vee A'}(x,y) = \min\left\{ \delta_{S,T}^{A}(x,y), \delta_{S,T}^{A'}(x,y) \right\}, \text{ and}$$
$$\delta_{T,S}^{A \vee A'}(x,y) = \min\left\{ \delta_{T,S}^{A}(x,y), \delta_{T,S}^{A'}(x,y) \right\}.$$

As a consequence, since (x,y) is the nucleolus of w_A and $w_{A'}$, we obtain the equality $\delta_{S,T}^{A \vee A'}(x,y) = \delta_{T,S}^{A \vee A'}(x,y)$. $\square$

The previous result shows that the set of matrices with the same nucleolus is a join-semilattice. Now we introduce the family of matrices with a given nucleolus $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$,

$$\mathscr{F}_\nu(x,y) := \left\{ A \in M_{m \times m'}^+ \mid \nu(A) = (x,y) \right\}.$$

In this section we prove that the above family of assignment matrices forms a compact join-semilattice with a unique maximal element which is always a valuation (Topkis, 1998).

Firstly notice that not any vector is a candidate to be a nucleolus. For instance, the vector $(3,2,1,4) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$ can never be the nucleolus of any 2×2 assignment game. For any candidate $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ with $|M| = |M'|$, to be the nucleolus of an assignment game with matrix $A \in M_m^+$, by (6) it must satisfy

$$\delta_{M,M'}^{A}(x,y) = \min_{i \in M}\{x_i\} = \min_{j \in M'}\{y_j\} = \delta_{M',M}^{A}(x,y). \tag{9}$$

In our case $\min\{x_1, x_2\} = 2 \neq 1 = \min\{y_1, y_2\}$.

Moreover, let us see that condition (9) turns out to be a simple characterization of the non-emptiness of the family $\mathscr{F}_\nu(x,y)$ if we deal with the square assignment case $|M| = |M'|$. To see it, just define the square matrix $V = (v_{ij})_{1 \leq i,j \leq m}$ defined by $v_{ij} := x_i + y_j$, for all $(i,j) \in M \times M'$ being $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ with $|M| = |M'|$ and

$\min_{i \in M} \{x_i\} = \min_{j \in M'} \{y_j\}$. Indeed, any matching is optimal in V and the vector $(x, y) \in C(w_V)$. Therefore $\delta^V_{S,T}(x, y) = \delta^V_{T,S}(x, y) = 0$ for all $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$, and $S \neq M$. Moreover $\delta^V_{M,M'}(x, y) = \delta^V_{M',M}(x, y)$ by assumption. Hence we have $\nu(V) = (x, y)$. Summarizing we have the following result.

Theorem 3.2 (Condition for the nucleolus in the square case). *Let $(x, y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ be a vector, with $|M| = |M'|$. The following statements are equivalent:*

1. $\mathscr{F}_\nu(x, y) \neq \emptyset$,
2. $\min_{i \in M} \{x_i\} = \min_{j \in M'} \{y_j\}$.

To analyze the non-square case we need an important result relating the nucleolus of a non-square assignment game with the nucleolus of a suitable square assignment game, which will be used later. Its proof is in the Appendix. This is a result of independent interest to deal with non-square assignment games, since the usual approach is to add null rows or columns in order to make the matrix square. Firstly we need some definitions.

Let $A \in \mathrm{M}^+_{m \times m'}$ be a non-square assignment matrix, with $m = |M| < |M'| = m'$, and let $\mu \in \mathscr{M}^*_A(M, M')$ be an optimal matching. Define the vector $a^\mu = \left(a^\mu_i\right)_{i \in M} \in \mathbb{R}^M_+$ by

$$a^\mu_i := \max_{j \in M' \setminus \mu(M)} \left\{a_{ij}\right\} \text{ for each buyer } i \in M, \tag{10}$$

and define the square matrix $A^\mu \in \mathrm{M}^+_m$ by

$$a^\mu_{ij} := \max\left\{0, a_{ij} - a^\mu_i\right\}, \text{ for } (i, j) \in M \times \mu(M). \tag{11}$$

Lemma 3.1. *Let $A \in \mathrm{M}^+_{m \times m'}$ be a non-square assignment matrix, with $|M| < |M'|$, and let $\mu \in \mathscr{M}^*_A(M, M')$ be an optimal matching. Let $a^\mu \in \mathbb{R}^M_+$ and $A^\mu \in \mathrm{M}^+_m$ be as in (10) and (11), and let $(x, y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ and $(x', y') \in \mathbb{R}^M_+ \times \mathbb{R}^{\mu(M)}_+$ be related by*

$$x'_i = x_i - a^\mu_i, \text{ for } i \in M,$$

$$y'_j = y_j, \text{ for } j \in \mu(M), \text{ and } y_j = 0 \text{ for } j \in M' \setminus \mu(M).$$

Then $\quad (x,y) \in C(w_A)$ *if and only if* $(x',y') \in C(w_{A^\mu})$.

Moreover, $\quad \nu(A) = (x,y)$ *if and only if* $\nu(A^\mu) = (x',y')$.

Since it is well known that the nucleolus of a non-square assignment game gives zero payoff to all non-optimally assigned players, then a candidate vector must assign zero to some players. The next result is the precise necessary and sufficient condition. Its proof is in the Appendix

Theorem 3.3 (**Condition for the nucleolus in the non-square case**). *Let* $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ *be a vector, with* $|M| < |M'|$, *and let* $Z_0 = \left\{ j \in M' \mid y_j = 0 \right\}$. *The following statements are equivalent:*

1. $\mathscr{F}_\nu(x,y) \neq \emptyset$,
2. (a) *there exists* $Z_0' \subseteq Z_0$ *with* $|Z_0'| = |M'| - |M|$, *and*
   (b) $\min_{i \in M} \{x_i\} \geq \min_{j \in M' \setminus Z_0'} \{y_j\}$.

Notice that from Theorem 3.2 and 3.3, the vector $(3,2,1,4) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$ can never be the nucleolus of any 2×2 assignment game, but the vector $(3,2,1,4,0) \in \mathbb{R}_+^2 \times \mathbb{R}_+^3$ is the nucleolus of some assignment game, see (12).

Now we turn to the structure of the matrices that share the same nucleolus, and describe its maximum element.

Theorem 3.4. *The family of matrices with a given nucleolus* $\mathscr{F}_\nu(x,y)$ *forms a compact set, where* $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}, |M| \leq |M'|$. *Moreover, if it is nonempty, it has a unique maximum element, which is described by the valuation matrix* $V \in \mathscr{F}_\nu(x,y)$ *given by*

$$v_{ij} = \begin{cases} x_i + y_j & \text{if} \quad i \in M, \text{ and } j \in M' \setminus Z_0', \\ x_i - \min\limits_{j \in M' \setminus Z_0'} \{y_j\} & \text{if} \quad i \in M, \text{ and } j \in Z_0', \end{cases} \tag{12}$$

where Z_0' *is any subset of* $Z_0 = \left\{ j \in M' \mid y_j = 0 \right\}$ *with cardinality* $|Z_0'| = |M'| - |M|$.

Proof. Let $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ be a vector with $\mathscr{F}_\nu(x,y) \neq \emptyset$. For each $A \in \mathscr{F}_\nu(x,y)$ consider the square matrix $\widetilde{A}$ which has $(m'-m)$ zero rows, i.e. dummy buyers, to make matrix A square. If $m'=m$, $\widetilde{A}=A$. Let us denote by $\widetilde{M}$ the new set of buyers associated to the assignment market $\widetilde{A}$. This new assignment matrix $\widetilde{A}$ has an extended nucleolus with zero payoffs to the added agents, namely $\nu(\widetilde{A})=(x,0;y)$.

Now, let us see $\mathscr{F}_\nu(x,y)$ is a compact set, i.e. a bounded and closed subset of $M^+_{m\times m'}$. It is bounded since $0 \le a_{ij} \le x_i + y_j$ for all $(i,j) \in M \times M'$ and $A \in \mathscr{F}_\nu(x,y)$. It is closed because the functions $\delta^{\widetilde{A}}_{S,T}(x,0;y)$ and $\delta^{\widetilde{A}}_{T,S}(x,0;y)$ are continuous in $\widetilde{A}$ for all $\emptyset \neq S \subseteq \widetilde{M}$, $\emptyset \neq T \subseteq M'$ and $|S|=|T|$, and they must satisfy (6).

Next we show that the given matrix V is the maximum element in $\mathscr{F}_\nu(x,y)$, i.e. $A \le V$ for all $A \in \mathscr{F}_\nu(x,y)$. It should first be noted that V is well-defined because if there exists different Z'_0 they give the same matrix V. Furthermore, from the proof of Theorems 3.2 and 3.3, we have already shown that the nucleolus of matrix V is the vector (x,y) as well as V is a valuation matrix.

Let $A \in \mathscr{F}_\nu(x,y)$ be a matrix, $\mu \in \mathscr{M}^*_A(M,M')$ be an optimal matching, and $Z'_0 = M' \setminus \mu(M)$. Then $Z'_0 \subseteq Z_0, |Z'_0| = |M'|-|M|$ and let V be the matrix defined in the statement. We have to prove $v_{ij} \ge a_{ij}$ for all $(i,j) \in M \times M'$. Clearly, since $(x,y) \in C(w_A)$, for $i \in M$ and $j \in M' \setminus Z'_0$, we have $v_{ij} = x_i + y_j \ge a_{ij}$. Now recall (10) and Lemma 3.1 and since μ is an optimal matching such that $\mu(M) = M' \setminus Z'_0$, we have $(x',y') = \nu(A^\mu)$. Denote $\varepsilon = \delta^{A^\mu}_{M,M'\setminus Z'_0}(x',y') = \delta^{A^\mu}_{M'\setminus Z'_0,M}(x',y') = \min_{j\in M'\setminus Z'_0}\{y_j\}$. Since $(x',y') = \nu(A^\mu)$ and taking into account (5), we have $\left(x' - \varepsilon 1^M, y' + \varepsilon 1^{M'\setminus Z'_0}\right) \in C(w_{A^\mu})$. Then for all $i \in M$ we have $x'_i - \varepsilon \ge 0$, that is, $x_i - \varepsilon \ge a^\mu_i$. From here, for all $i \in M$ and $j \in Z'_0$, we have $v_{ij} = x_i - \min_{j\in M'\setminus Z'_0}\{y_j\} \ge a^\mu_i \ge a_{ij}$. This finishes the proof. □

4. Properties of the join-semilattice

It is interesting to point out that the family $\mathscr{F}_V(x,y)$ is not in general a convex set. Just take the matrices in (8) and their midpoint.

Now we prove an interesting property. There is a continuous piecewise linear path (maybe not unique) between any matrix in $\mathscr{F}_V(x,y)$ and its maximum element V. From here it is clear that the family $\mathscr{F}_V(x,y)$ is a path-connected set.

Theorem 4.1. *Let $\mathscr{F}_V(x,y)$ be a nonempty family of matrices with a given nucleolus, where $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}, |M| \le |M'|$, and $V \in \mathscr{F}_V(x,y)$ be its maximum given in (12). Then for any $A \in \mathscr{F}_V(x,y)$ there exists an increasing piecewise linear path*[6] *from A to V inside $\mathscr{F}_V(x,y)$.*

As a consequence, $\mathscr{F}_V(x,y)$ is a path-connected set.

Proof. First we analyze the square case. We can assume $|M| \ge 2$. Let it be $A \in \mathscr{F}_V(x,y)$. Let us define the following set, formed by the distances that appear in the characterization of the nucleolus, see (6), except for the grand coalition,

$$\Delta(A) = \left\{ \delta^A_{S,T}(x,y) \mid S \subseteq M, T \subseteq M', |S| = |T|, S \neq \emptyset, M, \text{ and } T \neq \emptyset, M' \right\}.$$

The elements of $\Delta(A) = \{\delta_0, \delta_1, \ldots, \delta_r\}$ can be ordered increasingly: $0 = \delta_0 < \delta_1 < \ldots < \delta_r$.

Note that for all $(i,j) \in M \times M'$ satisfying $x_i + y_j - a_{ij} \notin \Delta(A)$ we can raise the worth of a_{ij} to a^0_{ij} in a way that $x_i + y_j - a^0_{ij}$ equals to the closest element in $\Delta(A)$, and set $a^0_{ij} = a_{ij}$ otherwise. The nucleolus of this new matrix A^0 is also (x,y). We may choose different increasing linear paths from A to A^0.

[6] A path in $\mathscr{X} \subseteq M^+_{m\times m'}$ from A to B, $A, B \in \mathscr{X}$, is a continuous function f from the unit interval $I = [0,1]$ to $\mathscr{X}$, i.e. $f : [0,1] \to \mathscr{X}$, with $f(0) = A$ and $f(1) = B$. Moreover a subset $\mathscr{X} \subseteq M^+_{m\times m'}$ is path-connected if for any two elements $A, B \in \mathscr{X}$ there exists a path from A to B entirely contained in $\mathscr{X}$.

Now we have a matrix $A^0 \in \mathscr{F}_\nu(x,y)$ such that $x_i + y_j - a^0_{ij} \in \Delta(A^0)$ for all $(i,j) \in M \times M'$. Note that $\Delta(A^0) = \Delta(A)$ and if $\delta^A_{S,T}(x,y) = \delta_r$, we have, for all $i \in S$ and $j \notin T$, $x_i + y_j - a^0_{ij} = \delta_r$.

If $r = 0$, $A^0 = V$ and we are done. Otherwise, for all $(i,j) \in M \times M'$ such that $x_i + y_j - a^0_{ij} = \delta_r$ raise linearly a^0_{ij} to a^1_{ij} defined by the equality $x_i + y_j - a^1_{ij} = \delta_{r-1}$. We obtain a new matrix $A^1 \in \mathscr{F}_\nu(x,y)$, defined for all $i \in M$ and $j \in M'$ by

$$a^1_{ij} = \begin{cases} x_i + y_j - \delta_{r-1} & \text{if} \quad x_i + y_j - a^0_{ij} = \delta_r, \\ a^0_{ij} & \text{otherwise.} \end{cases}$$

We have $\Delta(A^1) \subsetneq \Delta(A^0)$.

Now, in a finite number of steps, proceed sequentially raising all entries until for all $(i,j) \in M \times M'$ we have $x_i + y_j - a^r_{ij} = 0$. This is matrix V for the square case.

For the non-square case, $|M| < |M'|$, let $A \in \mathscr{F}_\nu(x,y)$, and let $\mu \in \mathscr{M}^*_A(M,M')$ be an optimal matching. Notice first that matrix A can be modified without changing its nucleolus in the following way: for all $(i,j) \in M \times \mu(M)$ if $a_{ij} < a^\mu_i$ then raise these entries to a^μ_i, see (10); for all $(i,j) \in M \times (M' \setminus \mu(M))$ raise entries a_{ij} to a^μ_i, and we do not modify the rest of entries. This new matrix $\overline{A} \in \mathscr{F}_\nu(x,y)$ and gives rise to the same square matrix $A^\mu \in M^+_m$, i.e. $\overline{A}^\mu = A^\mu$, see (11). The relationship between $\overline{A}$ and $\overline{A}^\mu$ is

$$\overline{a}^\mu_{ij} = \overline{a}_{ij} - a^\mu_i \;\text{ for all }\; (i,j) \in M \times \mu(M). \tag{13}$$

From Lemma 3.1 applied to matrix $\overline{A}$ we know $\nu\left(\overline{A}\right) = \nu(A) = (x,y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ if and only if $\nu\left(\overline{A}^\mu\right) = (x',y') \in \mathbb{R}^M_+ \times \mathbb{R}^{\mu(M)}_+$, with $x'_i = x_i - a^\mu_i$ for $i \in M$, and $y'_j = y_j$ for $j \in \mu(M)$.

We can apply the previous procedure for square matrices to obtain an increasing piecewise linear path from $\overline{A}^\mu$ to its maximum matrix in $\mathscr{F}_\nu(x',y')$. This path, applied to matrix $\overline{A}_{|_{M\times\mu(M)}}$, see (13), induces a path from $\overline{A}_{|_{M\times\mu(M)}}$ to $V_{|_{M\times\mu(M)}}$, where V is the maximum of $\mathscr{F}_v(x,y)$.

Moreover, for $(i,j)\in M\times(M'\setminus\mu(M))$ recall from (12) that $v_{ij}=x_i-\min_{j\in\mu(M)}\left\{y_j\right\}$. From Theorem 3.2 we know that $\min_{i\in M}\{x'_i\} = \min_{j\in\mu(M)}\{y'_j\} = \min_{j\in\mu(M)}\left\{y_j\right\}$, and then for some $i^*\in M$ we have $x'_{i^*}=x_{i^*}-a^{\mu}_{i^*}=\min_{j\in\mu(M)}\left\{y_j\right\}$. That is, for $i^*\in M$ we have $v_{i^*j}=a^{\mu}_{i^*}$ for all $j\in M'\setminus\mu(M)$. For any $i\neq i^*$ such that $x'_i>\min_{i\in M}\{x'_i\}$ or equivalently $x'_i=x_i-a^{\mu}_i>\min_{i\in M}\{x'_i\}=\min_{j\in\mu(M)}\left\{y_j\right\}$, that is $a^{\mu}_i<x_i-\min_{j\in\mu(M)}\left\{y_j\right\}$, we can raise at the same time entries $\overline{a}_{ij}=a^{\mu}_i$ to $v_{ij}=x_i-\min_{j\in\mu(M)}\left\{y_j\right\}$ without changing the nucleolus, as the reader can check applying Lemma 3.1. This ends the proof. □

Finally, with respect to the minimal elements of the semilattice $(\mathscr{F}_v(x,y),\leq)$ our next result reveals the existence of many of them. Basically we obtain a minimal matrix each time we fix an appropriate optimal matching. Notice that any minimal matrix in the family has at least one optimal matching in common with matrix V, the maximum element of the family. Therefore, it is natural to ask for a minimal matrix whenever an optimal matching for matrix V has been fixed.

Curiously enough, not any optimal matching can be used. For instance, take the nucleolus $(x,y)=(0,3,2,0)\in\mathbb{R}^2_+\times\mathbb{R}^2_+$. Note that $\mathscr{F}_v(x,y)\neq\emptyset$. and $\min\{x_1,x_2\}=0=\min\{y_1,y_2\}$. The valuation matrix

$$V=\begin{pmatrix}2&0\\5&3\end{pmatrix}$$

has two optimal matchings. The first one, $\mu_1=\{(1,1),(2,2)\}$ cannot be preserved if we look for minimality, but the second one $\mu_2=\{(1,2),(2,1)\}$ can. Indeed,

$$C=\begin{pmatrix}0&0\\5&1\end{pmatrix}$$

is the desired minimal matrix. Differences between both matchings are subtle and they will be specified in the next definition.

To go on, let $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ with $|M| \leq |M'|$ such that $\mathscr{F}_\nu(x,y) \neq \emptyset$, and let $V \in \mathscr{F}_\nu(x,y)$ be its maximum given in (12). We say that an optimal matching $\mu \in \mathscr{M}_V^*(M,M')$ is a *minimal-matrix compatible matching (m2-compatible)* if $\min\limits_{j\in\mu(M)}\{y_j\} = 0$ then there exists a buyer $i^* \in M$ such that $x_{i^*} = \min\limits_{i\in M}\{x_i\}$ and his optimal partner receives $y_{\mu(i^*)} = \min\limits_{j\in\mu(M)}\{y_j\} = 0$. The set of all m2-compatible matchings is denoted by $\mathscr{M}_m(V)$.

Notice that in the square case, if $\min\limits_{i\in M}\{x_i\} = \min\limits_{j\in M'}\{y_j\} > 0$, all matchings are m2-compatible. As a consequence, $m!$ minimal matrices may appear. A similar result holds for the non-square case, as the next theorem implies.

Theorem 4.2 (Computation of a minimal matrix). *Let $\mathscr{F}_\nu(x,y)$ be a nonempty family of matrices with a given nucleolus, where $(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ and $|M| \leq |M'|$. Let $V \in \mathscr{F}_\nu(x,y)$ be the maximum of the family.*

For any minimal-matrix compatible matching $\mu \in \mathscr{M}_m(V)$ there exists matrix $C \in \mathscr{F}_\nu(x,y)$ with $\mu \in \mathscr{M}_C^*(M,M')$ and C is minimal in $(\mathscr{F}_\nu(x,y), \leq)$. Moreover, if $|M| \geq 3$ then $C \neq V$ whenever (x,y) is not the null vector.

Proof. For ease of the proof and w.l.o.g. we adopt the following normalization conditions, maybe by reordering agents:

i. We assume $x_1 \leq x_2 \leq \cdots \leq x_m$.
ii. The original matching μ is in the main diagonal $\mu = \{(1,1), \ldots, (m,m)\}$.

Moreover, once i. and ii. have been achieved we also ask for an additional condition:

iii. If there exists $i^* \in M$ such that $x_{i^*} = \min\{x_1, \ldots, x_m\}$ and $y_{i^*} = \min\{y_1, \ldots, y_m\}$, then $x_1 = \min\{x_1, \ldots, x_m\}$ and $y_1 = \min\{y_1, \ldots, y_m\}$.

Condition iii. has the following interpretation. If two optimally matched partners receive minimum payoff among the matched agents of their side, it happens also in the first place.

Now we analyze the square case, $|M| = |M'|$. We define matrix $C \in M_m^+$ by

$$
\begin{aligned}
c_{ii} &= x_i + y_i && \text{for } i = 1, \ldots, m, \\
c_{i1} &= x_i + y_1 - y_i && \text{for } i = 2, \ldots, m, \quad \text{and } x_i > y_i, \\
c_{1j} &= x_1 + y_j - x_j && \text{for } j = 2, \ldots, m, \quad \text{and } x_j < y_j, \\
c_{ij} &= 0 && \text{otherwise.}
\end{aligned}
$$

Clearly the main diagonal is an optimal matching for C. Moreover the vector $(x, y) \in C(w_C)$.

Now we prove $\nu(C) = (x, y)$. For simplicity we write S' instead of $\mu(S)$ for $S \subseteq M$. Therefore, we prove $\delta^C_{S,S'}(x,y) = \delta^C_{S',S}(x,y)$ for all $\emptyset \neq S \subseteq M$, see (7).

Notice first that for $i \neq j$ we have

$$
x_i + y_j - c_{ij} = \begin{cases} y_i & \text{if } \ j = 1 \text{ and } x_i > y_i, \\ x_j & \text{if } \ i = 1 \text{ and } x_j < y_j, \\ x_i + y_j & \text{otherwise.} \end{cases}
$$

Let $\emptyset \neq S \subseteq M$ be an arbitrary coalition. We distinguish two cases.

Case (a): Buyer $i = 1$ does not belong to the coalition S, i.e. $1 \notin S$. Then,

$$
\begin{aligned}
\delta^C_{S,S'}(x,y) &= \min\left\{ \min_{i \in S}\{x_i\}, \min_{i \in S\, j \notin S'}\{x_i + y_j - c_{ij}\} \right\} \\
&= \min\left\{ \min_{i \in S}\{x_i\}, \min_{i \in S,\, x_i > y_i}\{y_i\} \right\} = \min\left\{ \min_{i \in S}\{x_i\}, \min_{j \in S'}\{y_j\} \right\}.
\end{aligned}
$$

Similarly,

$$
\delta^C_{S',S}(x,y) = \min\left\{ \min_{j \in S'}\{y_j\}, \min_{i \in S}\{x_i\} \right\}.
$$

Case (b): Buyer $i = 1$ belongs to the coalition S, i.e. $1 \in S$. Then,

$$
\begin{aligned}
\delta^C_{S,S'}(x,y) &= \min\left\{ \min_{i \in S}\{x_i\}, \min_{i \in S\ j \notin S'}\{x_i + y_j - c_{ij}\} \right\} \\
&= \min\left\{ \min_{i \in S}\{x_i\}, \min_{j \notin S', x_j < y_j}\{x_j\} \right\} = x_1.
\end{aligned}
$$

Moreover, we have

$$\begin{aligned}\delta^C_{S',S}(x,y) &= \min\left\{\min_{j\in S'}\{y_j\},\ \min_{i\notin S\, j\in S'}\{x_i+y_j-c_{ij}\}\right\}\\ &= \min\left\{\min_{j\in S'}\{y_j\},\ \min_{i\notin S,\, y_i<x_i}\{y_i\}\right\} = \min_{j\in M'}\{y_j\} = x_1.\end{aligned}$$

To see the last but one equality, let $j^*\in M'$ be such that $y_{j^*}=\min_{j\in M'}\{y_j\}$. Clearly $y_{j^*}=\min_{j\in M'}\{y_j\}=\min\limits_{i\in M}\{x_i\}\le x_{j^*}$. Then the only case we have to analyze is $y_{j^*}=x_{j^*}=x_1$. In this case, the normalization conditions imply $y_1=y_{j^*}=x_{j^*}$, and the equality holds, since $1'\in S'$.

As a consequence, $\nu(C)=(x,y)$.

It remains to prove matrix C is minimal in $(\mathscr{F}_\nu(x,y),\le)$. Suppose on the contrary that there exists matrix $D\in M^+_m, D\le C, D\ne C$ and $\nu(D)=(x,y)$.

First we claim that for any optimal matching $\mu'\in\mathscr{M}^*_C(M,M')$, if $x_i>0$ and $y_i>0$ then $(i,i)\in\mu'$. Indeed and since $\nu(C)=(x,y)\in C(w_C)$, we know $x_k+y_{\mu'(k)}=c_{k\mu'(k)}$ for any buyer $k\in M$ and μ' an optimal matching of C. Now if $\mu'(i)\ne i'$ and since $x_i>0$ we know that either $i=1$ or $\mu'(i)=1'$. If $i=1$, by the normalization conditions, we know $x_k>0$ for all $k\in M$ and since $c_{1\mu'(1)}\ne 0$ we obtain $c_{1\mu'(1)}=x_1+y_{\mu'(1)}-x_{\mu'(1)}$ and then $x_{\mu'(1)}=0$, a contradiction. If $\mu'(i)=1'$ and since $c_{i1}\ne 0$ we obtain $c_{i1}=x_i+y_1-y_i$ and then $y_i=0$, a contradiction. Therefore the claim holds.

Note now that all optimal matchings in D are optimal also in C, since $D\le C$ and they have the same nucleolus. We distinguish two possibilities and recall $\mu=\{(1,1),\ldots,(m,m)\}\in\mathscr{M}^*_C(M,M')$.

(a) $\mu\in\mathscr{M}^*_D(M,M')$, i.e. $d_{ii}=c_{ii}$ for $i=1,\ldots,m$. Since $D\ne C$ there must exist $0\le d_{i1}<c_{i1}$ or $0\le d_{1j}<c_{1j}$ with $i\ge 2$ or $j\ge 2$. We analyze the first case, and the other is left to the reader. From $0\le d_{i1}<c_{i1}$, we achieve $c_{i1}\ne 0, i\ge 2$ and then $x_i>y_i$. Moreover by the definition of C, we have

$c_{1i}=0$, which implies $d_{1i}=0$. Now

$$\delta^D_{\{i\},\{i'\}}(x,y)=\min\left\{x_i,\min_{k\neq i}\{x_i+y_k-d_{ik}\}\right\}=\min\{x_i,x_i+y_1-d_{i1}\},\text{ and}$$
$$\delta^D_{\{i'\},\{i\}}(x,y)=\min\left\{y_i,\min_{k\neq i}\{x_k+y_i-d_{ki}\}\right\}=y_i,$$

but then we get $x_i+y_1-d_{i1}=y_i$, which implies $c_{i1}=d_{i1}$, a contradiction.

(b) $\mu\notin\mathscr{M}^*_D(M,M')$. Therefore there exists another matching μ' which is optimal, $\mu'\in\mathscr{M}^*_D(M,M')$, and then $\mu'\in\mathscr{M}^*_C(M,M')$. There exists $i\in M$, $\mu'(i)\neq\mu(i)=i'$ with $0\leq d_{ii}<c_{ii}$, that is $d_{ii}=c_{ii}-\varepsilon$ with $\varepsilon>0$, and by the claim, either $x_i>0$ and $y_i=0$ or $x_i=0$ and $y_i>0$. Notice that since μ is an m2-compatible matching and the normalization conditions we have $i\geq 2$ and $x_1=y_1=0$. We analyze the first case $x_i>0$ and $y_i=0$ and the second case is similar. Now we claim $\mu'(i)=1'$. If $\mu'(i)\neq 1'$ and being $\mu'(i)\neq i'$ and $i\geq 2$ we have $c_{i\mu'(i)}=0$, and then $0=c_{i\mu'(i)}<x_i+y_{\mu'(i)}$, contradicting the optimality of μ', since $\nu(C)=(x,y)$.

Now we prove $\delta^D_{\{i\},\{\mu'(i)\}}(x,y)\neq\delta^D_{\{\mu'(i)\},\{i\}}(x,y)$, for $\mu'\in\mathscr{M}^*_D(M,M')$. Recall that $\mu'(i)=1', d_{11}=c_{11}=x_1+y_1=0$ and $d_{ik}=0$ for $k\neq 1,i$. Hence

$$\delta^D_{\{i\},\{1'\}}(x,y)=\min\left\{x_i,\min_{k\neq 1}\{x_i+y_k-d_{ik}\}\right\}=\min\{x_i,\varepsilon\},\text{ and}$$
$$\delta^D_{\{1'\},\{i\}}(x,y)=\min\left\{y_1,\min_{k\neq i}\{x_k+y_1-d_{k1}\}\right\}=y_1=0,$$

and they are different, a contradiction with $\nu(D)=(x,y)$.

Therefore matrix C is minimal.

Now we analyze the non-square case, $|M|<|M'|$. From the normalization conditions $x_1\leq x_2\leq\cdots\leq x_m$, and $\mu=\{(1,1),\ldots,(m,m)\}$.

We define matrix $C \in \mathrm{M}^+_{m\times m'}$ by

$$\begin{aligned}
c_{ii} &= x_i + y_i && \text{for } i = 1,\ldots,m,\\
c_{i1} &= x_i + y_1 - y_i && \text{for } i = 2,\ldots,m, \quad \text{and } x_i > y_i,\\
c_{1j} &= x_1 + y_j - x_j && \text{for } j = 2,\ldots,m, \quad \text{and } x_j < y_j,\\
c_{1\,m+1} &= x_1 - \min_{j\in\mu(M)}\{y_j\} &&\\
c_{ij} &= 0 && \text{otherwise.}
\end{aligned}$$

The proof that this is the desired matrix in the non-square case is similar to the square case and can be found in the Appendix.

It is clear from the definition of matrix C that if $(x,y) \neq (0,0) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ and $|M| \geq 3$ then $C \neq V$. $\square$

As a direct consequence of Theorems 4.1 and 4.2 we obtain an interesting result on the cardinality of the family $\mathscr{F}_V(x,y)$.

Corollary 4.1. *For any vector $(x,y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ either*

(a) $\mathscr{F}_V(x,y) = \emptyset$,

(b) $\mathscr{F}_V(x,y)$ *is a singleton, or*

(c) $\mathscr{F}_V(x,y)$ *has a continuum of elements.*

We have characterized when the family $\mathscr{F}_V(x,y)$ is non-empty, and from Theorem 4.2 we know that for matrices with at least three agents in each side, the family has an infinite number of elements, since a minimal matrix does not coincide with the valuation matrix of the family and by the path-connectedness we can construct an infinite number of matrices between them. If we want to look when the family is a singleton, and apart the trivial cases of a sector having only one agent or the original vector being the null vector, we must seek a 2×2 case. It is easy to see that for the square case 2×2 the only non-trivial case is given by the vector $(0,0,0,k)\in\mathbb{R}^2_+\times\mathbb{R}^2_+$ for $k>0$ and its permutations. The proof is left to the reader, but in this case for $k>0$

$$\mathscr{F}_v(0,0,0,k)=\left\{\left(\begin{array}{cc}0 & k\\ 0 & k\end{array}\right)\right\}.$$

5. Appendix

Proof of Lemma 3.1

Proof. Let $A\in \mathrm{M}^+_{m\times m'}$ and let $\mu\in\mathscr{M}^*_A(M,M')$. Without loss of generality, we can assume that $\mu=\{(1,1),(2,2),\dots,(m,m)\}$ is an optimal matching of matrix A.

We claim that μ is an optimal matching of A^μ, defined by (10) and (11). To see it, consider any $(x,y)\in C(w_A)$. Clearly $x_i\geq a^\mu_i$ for all $i\in M$, and then $x_i-a^\mu_i\geq 0$, and $(x_i-a^\mu_i)+y_j\geq 0$, for all $(i,j)\in M\times\mu(M)$. Moreover, for all $(i,j)\in M\times\mu(M)$, we have $(x_i-a^\mu_i)+y_j\geq a_{ij}-a^\mu_i$, and therefore $(x_i-a^\mu_i)+y_j\geq a^\mu_{ij}$. Since $\mu=\{(1,1),(2,2),\dots,(m,m)\}$ is an optimal matching for A, then $a_{ii}\geq a^\mu_i$ for all $i\in M$, and we obtain $(x_i-a^\mu_i)+y_i=a_{ii}-a^\mu_i=a^\mu_{ii}$, for all $i\in M$. From this it is easy to see our claim: for any $\mu'\in\mathscr{M}(M,\mu(M))$,

$$\sum_{(i,j)\in\mu}a^\mu_{ij}=\sum_{i=1}^m a^\mu_{ii}=\sum_{i=1}^m(x_i-a^\mu_i)+y_i=\sum_{(i,j)\in\mu'}(x_i-a^\mu_i)+y_j\geq\sum_{(i,j)\in\mu'}a^\mu_{ij}.$$

Define the following square matrix $A^0\in M^+_{m'}$ obtained from the original matrix A by adding $m'-m$ zero rows, that is $m'-m$ dummy players, and let $M^0=M\cup\{m+1,\dots,m'\}$ be the new set of buyers and $A^0=\left(a^0_{ij}\right)_{1\leq i,j\leq m'}$ where

$$a^0_{ij}=\begin{cases}a_{ij} & \text{if}\quad (i,j)\in M\times M',\\ 0 & \text{if}\quad (i,j)\in(M^0\backslash M)\times M'.\end{cases}$$

We know that the matching $\mu^0 = \mu \cup \{(m+1, m+1), \dots, (m', m')\}$ is optimal for matrix A^0, i.e. $\mu^0 \in \mathscr{M}^*_{A^0}\left(M^0, M'\right)$.

For each $(x, y) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ denote now by $\left(x^0, y^0\right) \in \mathbb{R}^{M^0}_+ \times \mathbb{R}^{M'}_+$ the vector defined by $x^0_k = x_k$ if $k \in M$ and $x^0_k = 0$ if $k \in M^0 \backslash M$ and $y^0_k = y_k$ if $k \in M'$.

It is well-known that $\nu(A) = (x, y)$ if and only if $\nu\left(A^0\right) = \left(x^0, y^0\right)$.

We claim that $(x, y) \in C(w_A)$ if and only if $\left(x^0, y^0\right) \in C\left(w_{A^0}\right)$, and also that $(x, y) \in C(w_A)$ if and only if $(x', y') \in C\left(w_{A^\mu}\right)$, where the relationship between their coordinates is $x'_i = x_i - a^\mu_i$ for $i \in M$, and $y'_j = y_j$ for $j \in \mu(M)$. Notice that $y_j = 0$ for $j \in M' \setminus \mu(M)$. This claim is immediate from the previous comments.

Take any $(x, y) \in C(w_A)$ and let $\emptyset \neq S \subseteq M$ be an arbitrary coalition. For ease of notation we denote $S' = \mu^0(S) = \mu(S)$. We obtain

$$\delta^{A^0}_{S,S'}\left(x^0, y^0\right) = \min_{i \in S \;\; j \in M' \backslash S'} \left\{x_i, x_i + y_j - a_{ij}\right\} =$$

since for $j \in M' \setminus \mu(M)$ we have $y_j = 0$, and (10)

$$= \min_{i \in S \;\; j \in \mu(M) \backslash S'} \left\{x_i, x_i + y_j - a_{ij}, x_i - a^\mu_i\right\} =$$

since $x_i \geq x_i - a^\mu_i$ for all $i \in S$,

$$= \min_{i \in S \;\; j \in \mu(M) \backslash S'} \left\{x_i - a^\mu_i, [x_i - a^\mu_i] + y_j - a_{ij} + a^\mu_i\right\} =$$

whenever $a_{ij} \leq a^\mu_i$ we have $[x_i - a^\mu_i] + y_j - a_{ij} + a^\mu_i \geq [x_i - a^\mu_i]$,

$$\begin{aligned} &= \min_{i \in S \;\; j \in \mu(M) \backslash S'} \left\{x_i - a^\mu_i, [x_i - a^\mu_i] + y_j - a^\mu_{ij}\right\} = \\ &= \delta^{A^\mu}_{S,S'}\left(x', y'\right). \end{aligned}$$

Similarly we obtain

$$\delta^{A^0}_{S',S}\left(x^0, y^0\right) = \delta^{A^\mu}_{S',S}\left(x', y'\right).$$

Moreover, for any $(x^0, y^0) \in C(w_{A^0})$ and any $\emptyset \neq S \subseteq M^0$ such that $S \cap (M^0 \backslash M) \neq \emptyset$, we have

$$\delta^{A^0}_{S,S'}\left(x^0, y^0\right) = 0,$$
$$\delta^{A^0}_{S',S}\left(x^0, y^0\right) = 0.$$

Now, by using (7), it is immediate to prove that $\nu(A) = (x, y)$, if and only if $\nu(A^\mu) = (x', y')$. $\square$

Proof of Theorem 3.3

Proof. To prove the 'if' part, let $A \in \mathrm{M}^+_{m \times m'}$ be a matrix and let $(x, y) = \nu(A)$ be its nucleolus.

Let $\mu \in \mathscr{M}^*_A(M, M')$ be an optimal matching. Clearly, non-assigned sellers by μ get zero payoffs in the nucleolus. Therefore, let Z'_0 be the set of non-assigned sellers by μ, i.e. $Z'_0 = M' \setminus \mu(M)$.

Now apply Lemma 3.1 and $\nu(A^\mu) = (x', y')$, with $x'_i = x_i - a^\mu_i$ for $i \in M$, and $y'_j = y_j$ for $j \in \mu(M)$ where vector $a^\mu = \left(a^\mu_i\right)_{i \in M}$ and matrix A^μ are defined as in (10) and (11). Then, applying Theorem 3.2,

$$\min_{i \in M} \{x_i\} \geq \min_{i \in M} \left\{x_i - a^\mu_i\right\} = \min_{j \in M' \backslash Z'_0} \left\{y_j\right\}.$$

This is condition 2.

To prove the converse implication we define matrix $V \in \mathrm{M}^+_{m \times m'}$ by

$$v_{ij} := \begin{cases} x_i + y_j & \text{if} \quad i \in M, \text{ and } j \in M' \setminus Z'_0, \\ x_i - \min\limits_{j \in M' \backslash Z'_0} \left\{y_j\right\} & \text{if} \quad i \in M, \text{ and } j \in Z'_0. \end{cases} \tag{14}$$

Note that any matching between M and $M' \setminus Z'_0$ is optimal for V, i.e. $\mathscr{M}(M, M' \setminus Z'_0) \subseteq \mathscr{M}^*_V(M, M')$. This matrix $V \in \mathrm{M}^+_{m \times m'}$ is, in fact, a valuation matrix. The proof is left to the reader.

We must prove now that vector (x,y) is the nucleolus of this matrix V. By Lemma 3.1, $(x,y)=\nu(V)$ if and only if $\nu(V^{\mu})=(x',y')$, with $x'_i=x_i-v^{\mu}_i$ for $i\in M$, and $y'_j=y_j$ for $j\in\mu(M)$, for some $\mu\in\mathscr{M}(M,M'\setminus Z'_0)$. Indeed, all of them are optimal.

By (10), $v^{\mu}_i=x_i-\min_{j\in M'\setminus Z'_0}\{y_j\}$ for all $i\in M$ and then $x'_i=x_i-v^{\mu}_i=\min_{j\in M'\setminus Z'_0}\{y_j\}$ for all $i\in M$. Therefore matrix V^{μ} satisfies, for all $(i,j)\in M\times(M'\setminus Z'_0)$,

$$v^{\mu}_{ij}=\max\left\{0,y_j+\min_{j\in M'\setminus Z'_0}\{y_j\}\right\}=y_j+\min_{j\in M'\setminus Z'_0}\{y_j\}=x'_i+y'_j.$$

Since $\min_{i\in M}\{x'_i\}=\min_{i\in M}\left\{\min_{j\in M'\setminus Z'_0}\{y_j\}\right\}=\min_{j\in M'\setminus Z'_0}\left\{y'_j\right\}$ and V^{μ} is a square valuation matrix, we obtain $\nu(V^{\mu})=(x',y')$. $\square$

Proof of the non-square case in Theorem 4.2

Proof. Clearly the main diagonal is an optimal matching for C. Moreover $\nu(C)=(x,y)$. Indeed, apply Lemma 3.1 and notice that matrix C^{μ} is just the minimal matrix already stated in the square case for the nucleolus $(x',y')\in\mathbb{R}^M_+\times\mathbb{R}^{\mu(M)}_+$ defined by $x'_1=\min_{j\in\mu(M)}\{y_j\},x'_i=x_i$ for $i=2,\dots,m$ and $y'_j=y_j$ for $j\in\mu(M)$. As a side effect C^{μ} is minimal in $\mathscr{F}_{\nu}(x',y')$.

To see that C is minimal, assume $D\in\mathrm{M}^+_{m\times m'},D\leq C,D\neq C$ and $(x,y)=\nu(w_D)$. Recall that all optimal matchings for D are also optimal for C. We distinguish two cases.

(1) $x_1=\min_{j\in\mu(M)}\{y_j\}$. Notice that $d_{1\,m+1}=c_{1\,m+1}=0$. We are essentially in the square case and we are done, $D=C$.

(2) $x_1>\min_{j\in\mu(M)}\{y_j\}$. In this case we know $x_i>0$ for all $i\in M$. Then, there are two possibilities:

(2a) An optimal matching for D, $\mu'\in\mathscr{M}^*_D(M,M')$ satisfies $\mu'(M)=\mu(M)$.

By Lemma 3.1 the vector $(x',y')\in\mathbb{R}^M_+\times\mathbb{R}^{\mu(M)}_+$ defined by $x'_1=x_1-d_{1\,m+1},x'_i=x_i$ for $i=2,\dots,m$ and $y'_j=y_j$ for all $j\in\mu(M)$ is the nucleolus of matrix $D^{\mu'}$. This implies $\min_{i\in M}\{x'_i\} = \min_{j\in\mu'(M)}\{y'_j\}$ or $x_1 - d_{1\,m+1} = \min_{j\in\mu'(M)}\{y'_j\}$. Similarly $x_1 - c_{1\,m+1} = \min_{j\in\mu(M)}\{y'_j\}$, but then $d_{1\,m+1} = c_{1\,m+1}$. From here it is easy to see that $D^{\mu'} \le C^{\mu}$, both matrices share the same nucleolus (x',y') and C^{μ} is minimal in $\mathscr{F}_v(x',y')$. Therefore $D^{\mu'} = C^{\mu}$.

Now we prove $D = C$. For $i = 2,\dots,m$ and $j \in \mu(M)$ it is obvious that $d_{ij} = c_{ij}$, since $d^{\mu'}_{ij} = d_{ij}$ and $c^{\mu}_{ij} = c_{ij}$. To see $d_{11} = c_{11}$, notice that $d_{1\mu'(1)} = c_{1\mu'(1)} = x_1 + y_{\mu'(1)} > 0$ and this is only possible if $\mu'(1) = 1'$ by the definition of matrix C. Now we analyze $i = 1, j \in \{2,\dots,m\}$ and $c_{1j} \neq 0$. In this case we have $c_{1j} > c_{1\,m+1}$, and then $c^{\mu}_{1j} > 0$ and since $D^{\mu'} = C^{\mu}$ we have $d^{\mu'}_{1j} = c^{\mu}_{1j} > 0$. From here we deduce $d_{1j} - d_{1\,m+1} = c_{1j} - c_{1\,m+1}$ and $d_{1j} = c_{1j}$. The rest of entries are clearly equal.

(2b) All optimal matchings for D satisfy $\mu'(M) \neq \mu(M)$.

In this case there must exist an optimal matching $\mu' \in \mathscr{M}^*_D(M,M')$ such that $\mu'(1) = m+1$ and $\mu'(M\setminus\{1\}) \subset \mu(M)$. This matching μ' is also optimal for matrix C. Then $d_{1\,m+1} = c_{1\,m+1} = x_1 + y_{m+1} = x_1$, and $\min_{j\in\mu(M)}\{y_j\} = 0$. Now, since matching μ is m2-compatible and Condition iii., we get $y_1 = 0$.

As a consequence $c_{11} = c_{1\,m+1}$. We distinguish two cases.

(2b-1) There exists a buyer $i \in \{2,\dots,m\}$ such that $\mu'(i) = 1'$.

Since μ' is optimal for C all buyers different from 1 and i are matched equally by μ' and μ and $c_{11} + c_{ii} = c_{i1} + c_{1\,m+1}$. Therefore $y_i = 0$ and by definition $c_{1i} = 0$.

Now we apply Lemma 3.1 to matrix D. Firstly notice that $d^{\mu'}_k = 0$ for all $k \in M, k \neq i$, and $d^{\mu'}_i = d_{ii}$. Moreover, $\min_{k\in M}\left\{x_k - d^{\mu'}_k\right\} = \min_{j\in\mu'(M)}\left\{y_j\right\} = 0$, which implies $d_{ii} = x_i$. The nucleolus of $D^{\mu'}$ is $(x',y') \in \mathbb{R}^M_+ \times \mathbb{R}^{\mu'(M)}_+$ with $x'_k = x_k$ for $k \neq i$ and $x'_i = 0$ and $y'_k = y_k$ for $k \in \mu'(M)$. Notice that for all $k \in \mu'(M), k \neq (m+1)'$, we know $d^{\mu'}_{1k} = \max\{0, d_{1k} - d^{\mu'}_1\} = d_{1k}$.

Then we get a contradiction:

$$\begin{aligned}
\delta^{D^{\mu'}}_{\{1\},\{(m+1)'\}}(x',y') &= \min\left\{x_1, \min_{1\le k\le m, k\ne i}\{x_1+y_k-d^{\mu'}_{1k}\}\right\} \\
&= \min\left\{x_1, \min_{1\le k\le m, k\ne i}\{x_1+y_k-d_{1k}\}\right\} = x_1, \text{ and} \\
\delta^{D^{\mu'}}_{\{(m+1)'\},\{1\}}(x',y') &= \min\left\{y_{m+1}, \min_{1\le k\le m, k\ne 1}\{x_k+y_{m+1}-d^{\mu'}_{k\,m+1}\}\right\} = 0,
\end{aligned}$$

where we have used the fact that $x_1+y_k-d_{1k}\ge x_1+y_k-c_{1k}$ and if $x_k\ge y_k, c_{1k}=0$ and if $x_k<y_k, c_{1k}=x_1+y_k-x_k$. Also that $y_{m+1}=0$.

The last case to analyze is the following one.

(2b-2) Seller $j=1$ is unmatched by μ'.

Since μ' is optimal for C all buyers different from 1 are matched equally by μ' and μ. That is, matching $\mu'=\{(1,m+1),(2,2),\ldots,(m,m)\}$. In this case $d_{11}<c_{11}=x_1$ since we are assuming matching μ is not optimal for D. Now we apply Lemma 3.1 to matrix D. Firstly notice that $d^{\mu'}_k=d_{k1}$ for all $k\in M$, and the nucleolus of $D^{\mu'}$ is $(x',y')\in\mathbb{R}^M_+\times\mathbb{R}^{\mu'(M)}_+$ with $x'_k=x_k-d_{k1}$ for $k\in M$ and $y'_k=y_k$ for $k\in\mu'(M)$. Recall that $x'_1=x_1-d_{11}>0$.

This implies $\min_{i\in M}\{x'_i\}=\min_{j\in\mu'(M)}\{y'_j\}=0$. Then there exists a buyer $i\in\{2,\ldots,m\}$ such that $x'_i=x_i-d_{i1}=0$. Thus, $d_{i1}=x_i>0$ and $c_{i1}\ge d_{i1}>0$. From the definition of matrix C, we obtain $y_i=0$ and $d_{ii}=x_i$. Therefore $d_{i1}=d_{ii}$ and matching

$$\mu''=\{(1,m+1),(2,2),\ldots,(i-1,i-1),(i,1),(i+1,i+1),\ldots,(m,m)\}$$

is optimal for D. We are in case (2b-1). □

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