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## A SINGULAR STOCHASTIC INTEGRAL EQUATION

DAVID NUALART AND MARTA SANZ

**ABSTRACT.** This note is devoted to the discussion of the stochastic differential equation  $X dX + Y dY = 0$ ,  $X$  and  $Y$  being continuous local martingales. A method to construct solutions of this equation is given.

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space and  $\{\mathcal{F}_t, t \geq 0\}$  a filtration on it satisfying the usual properties. That means,  $\mathcal{F}_t$  is right continuous, and  $\mathcal{F}_0$  contains the null sets of  $\mathcal{F}$ .

Let  $X = \{X_t, t \geq 0\}$  be a continuous local martingale with respect to the filtration  $\{\mathcal{F}_t, t \geq 0\}$ . The continuous local martingale  $M_t = \int_0^t X_s dX_s$  has an associated increasing process given by  $\langle M \rangle_t = \int_0^t X_s^2 d\langle X \rangle_s$ . Denoting by  $\langle M \rangle$  and  $\langle X \rangle$  the measures on  $\mathbf{R}$  induced by the sample paths  $\langle M \rangle_t$  and  $\langle X \rangle_t$ , respectively, we obviously have  $\langle M \rangle \ll \langle X \rangle$ , and  $d\langle M \rangle/d\langle X \rangle = X^2$ . Reciprocally,  $\langle X \rangle \ll \langle M \rangle$ , and  $d\langle X \rangle/d\langle M \rangle = 1/X^2$ . In fact, it is known that  $\langle X \rangle$  does not charge the set  $\{X = 0\}$ .

By Itô's formula  $X_t^2 - X_0^2 = 2M_t + \langle X \rangle_t$ . So, applying the preceding result we have

$$X_t^2 - X_0^2 = 2M_t + \int_0^t \frac{1}{X_s^2} d\langle M \rangle_s.$$

Therefore, a continuous local martingale  $Y$  is a solution of the stochastic differential equation

$$(1) \quad X dX + Y dY = 0,$$

if and only if

$$(2) \quad Y_t^2 - Y_0^2 = -2M_t + \int_0^t \frac{1}{Y_s^2} d\langle M \rangle_s.$$

Equation (1) arises in a natural way in the theory of two-parameter martingales with path independent variation adapted to the  $\sigma$ -fields generated by two independent couples of two-dimensional brownian motions (see [4]).

First we prove a lemma that will be used to construct  $Y_t^2$ .

**LEMMA 1.** *Let  $b(t)$  be a continuous real function defined on  $\mathbf{R}_+$  such that  $b(0) \geq 0$  and  $\mu$  a continuous measure on  $\mathbf{R}_+$ . Assume that  $b(t)$  takes constant values on every interval  $[a, c]$  such that  $\mu([a, c]) = 0$ . Then the integral equation*

$$(3) \quad r(t) = b(t) + \int_0^t \frac{1}{r(s)} d\mu_s$$

*has a unique, nonnegative, continuous solution.*

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PROOF. The case  $d\mu_s = ds$  was proved by McKean in [3]. In the general case define  $G(s) = \inf\{t: F(t) > s\}$ ,  $F$  being the distribution function of  $\mu$ .  $G$  is right continuous and  $F \circ G = \text{Id}$ .

$b \circ G$  is continuous. Indeed, fix a point  $s > 0$ , and suppose that  $G(s) = c$ ,  $G(s^-) = a$ . Then  $F(t) = s$  for all  $t$  in  $[a, c]$  and, by hypothesis,  $b$  is constant on  $[a, c]$ , which proves the continuity of  $b \circ G$  on  $s$ .

Using McKean's result, we know that equation

$$r'(t) = b(G(t)) + \int_0^t \frac{1}{r'(s)} ds$$

has a unique, nonnegative, continuous solution.

Then, if we define  $r = r' \circ F$ ,  $r$  is a solution of (3). In fact,

$$r'(F(t)) = b(G(F(t))) + \int_0^{F(t)} \frac{1}{r'(s)} ds,$$

but

$$\int_0^{F(t)} \frac{1}{r'(s)} ds = \int_0^{F(t)} \frac{1}{r'(F(G(s)))} ds = \int_0^t \frac{1}{r'(F(s))} d\mu_s,$$

and  $b \circ G \circ F = b$  as easily follows from the assumption made on  $b$ .

Now we can state the main result.

**THEOREM 1.** *Let  $X$  be a continuous local martingale. Assume that there exists a sequence  $\phi_n$  of independent random variables, with  $\phi_n \in \{-1, 1\}$  and  $E(\phi_n) = 0$ , such that they are independent of  $X$  and  $\mathcal{F}_0$ -measurable. Then there exists a continuous local martingale  $Y$  such that  $\int_0^t (X_s dX_s + Y_s dY_s) = 0$  for all  $t \geq 0$ .*

PROOF. First we choose an integrable random variable  $Y_0$  which will be the value of  $Y$  at the origin.

Let us consider the equation

$$(4) \quad r(t) = Y_0^2 - 2M(t) + \int_0^t \frac{1}{r(s)} d\langle M \rangle_s,$$

where  $M_t = \int_0^t X_s dX_s$ .

It is well known (see [2]) that there exists a null set  $N \subset \Omega$  such that for all  $\omega \notin N$ ,  $\langle M \rangle([a, c]) = 0$  implies that  $M$  takes a constant value on the interval  $[a, c]$ .

So, for  $\omega \notin N$  fixed, we can apply Lemma 1 to  $b(t) = Y_0^2 - 2M(t)$  and  $\mu = \langle M \rangle$ , and state the existence of a unique, nonnegative, continuous solution of (4).

Now  $r(t)$  is a local submartingale because it is the sum of a local martingale and the increasing process  $\int_0^t (1/r(s)) d\langle M \rangle_s$ . We want to show that  $\sqrt{r(t)}$  also is a local submartingale. To do this, apply Itô's formula to  $f(r(t)) = \sqrt{r(t) + \lambda}$ , where  $\lambda$  is any real positive number,

$$\begin{aligned} \sqrt{r(t) + \lambda} &= \sqrt{r(0) + \lambda} - \int_0^t (r(s) + \lambda)^{-1/2} dM(s) \\ &\quad + \int_0^t \frac{1}{2}(r(s) + \lambda)^{-1/2} \left( \frac{1}{r(s)} - \frac{1}{r(s) + \lambda} \right) d\langle M \rangle_s. \end{aligned}$$

Let  $\{T_n, n \in N\}$  be an increasing sequence of stopping times such that  $T_n \uparrow \infty$  and  $M^{T_n}$  is a martingale bounded by  $n$ . Set  $R_n = \inf\{t, r(t) \geq n\}$  and  $S_n = T_n \wedge$

$R_n$ . Then  $\sqrt{r(t \wedge S_n) + \lambda}$  is a positive submartingale, and letting  $\lambda$  tend to zero we obtain, by monotone convergence, the submartingale property of  $\sqrt{r(t \wedge S_n)}$ . Therefore  $\sqrt{r(t)}$  is a local submartingale because  $S_n \uparrow \infty$ .

The increasing process given by the Doob decomposition of  $\sqrt{r(t)}$  is  $A_t = \sqrt{r(t)} - \sqrt{r(0)} + \int_0^t (r(s))^{-1/2} dM(s)$ . In fact, for any  $\lambda > 0$ ,  $\sqrt{r(t \wedge S_n) + \lambda} - \sqrt{r(0) + \lambda} + \int_0^{t \wedge S_n} (r(s) + \lambda)^{-1/2} dM(s)$  is increasing.

We next show that  $\int_0^t 1_{\{\sqrt{r(s)} > 0\}} dA_s = 0$ . Indeed, we have

$$r(t) = (A_t + \sqrt{r(0)})^2 - 2(A_t + \sqrt{r(0)}) \int_0^t (r(s))^{-1/2} dM(s) + \left( \int_0^t (r(s))^{-1/2} dM(s) \right)^2.$$

Computing the bounded variation part of each term we obtain

$$r(0) + \int_0^t \frac{1}{r(s)} d\langle M \rangle_s = (A_t + \sqrt{r(0)})^2 - 2 \int_0^t \left( \int_0^s (r(u))^{-1/2} dM(u) \right) dA_s + \int_0^t \frac{1}{r(s)} d\langle M \rangle_s.$$

Therefore,

$$0 = (A_t + \sqrt{r(0)})^2 - r(0) - 2 \int_0^t \left( \int_0^s (r(u))^{-1/2} dM(u) \right) dA_s = 2 \int_0^t \sqrt{r(s)} dA_s,$$

which implies the assertion.

Henceforth, for any  $n \geq 1$ ,  $\{\rho_n(t) = \sqrt{r(S_n + t) \wedge S_{n+1}}, \mathcal{F}_{S_n+t}, t \geq 0\}$  is a submartingale which satisfies  $\int_0^t 1_{\{\rho_n(s) > 0\}} dA_s^n = 0$ , where

$$A_t^n = A((S_n + t) \wedge S_{n+1})$$

is the increasing process associated to  $\rho_n(t)$ . Using Barlow's procedure (see [1]) it is possible to find a martingale  $M_n(t)$  such that  $|M_n(t)| = \rho_n(t)$  and  $M_n(0) = M_{n-1}(S_n)$  (we take  $S_0 = 0$  and  $M_0 = Y_0$ ). Then  $Y = \sum_n M_n 1_{\llbracket T_n, T_{n+1} \rrbracket}$  is a nonnegative local submartingale whose absolute value is  $\sqrt{r(t)}$ . Note that according to Barlow's method, the sign of  $Y$  is defined in terms of the sequence  $\phi_n$ . So  $Y_t^2 = r(t)$  satisfies equation (2) and this finishes the proof of the theorem.  $\square$

REMARKS. 1. Instead of assuming that the  $\sigma$ -field  $\mathcal{F}_0$  is rich enough to contain the sequence  $\phi_n$ , we can adjoin a new probability space and show the existence of the local martingale  $Y$  in the product space.

2. There is no uniqueness of the solution because  $-Y$  also is a solution. Moreover the random variable  $Y_0$  is arbitrary.

3. Let  $T = \inf\{t, Y_t = 0\}$ . If the initial value  $Y_0$  is given, there is uniqueness of the solution in  $[0, T]$ . Furthermore, to construct the solution in this interval we do not need the sequence  $\phi_n$ .

4. Suppose that  $Y_0 = X_0$ . In this case, the processes  $X^2$  and  $Y^2$  may have the same law. A sufficient condition for it to hold is the equality between the law of  $\{(M_t, \langle M \rangle_t), t \geq 0\}$  and that of  $\{(-M_t, \langle M \rangle_t), t \geq 0\}$ . For example, this condition holds if  $M_t$  is a Wiener process.

5. As a consequence of Theorem 1 we obtain the existence of nonnull two-dimensional martingales with increasing norm.

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