PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 1, Pages 23–29
S 0002-9939(05)08273-0
Article electronically published on August 15, 2005

ON VECTORIAL POLYNOMIALS AND COVERINGS IN CHARACTERISTIC 3

TERESA CRESPO AND ZBIGNIEW HAJTO

(Communicated by Jonathan I. Hall)

ABSTRACT. For K a field containing the finite field $\mathbb{F}_9$ we give explicitly the whole family of Galois extensions of K with Galois group $2S_4 * Q_8$ or $2S_4 * D_8$ and determine the discriminant of such an extension.

1. INTRODUCTION

The motivation of this work is the problem of resolution of singularities in positive characteristic, more precisely the ideas presented by S.S. Abhyankar in [4]. Following Abhyankar, loc. cit. Section 18, let $N^d_{k,t}$ denote a neighborhood of a simple point on a d-dimensional algebraic variety over an algebraically closed field k of characteristic p from which we have deleted a divisor having a t-fold normal crossing at the simple point and let $\pi^L_A(N^d_{k,t})$ be the set of all Galois groups of finite unramified local Galois coverings of $N^d_{k,t}$. In his landmark paper [1], Abhyankar, while working on local uniformization of algebraic varieties in a positive characteristic, proved the inclusion $\pi^L_A(N^d_{k,t}) \subset P_t(p)$, where $P_t(p)$ denotes the set of finite groups G such that the quotient $G/p(G)$ of G by the subgroup $p(G)$ generated by its p-Sylow subgroups is abelian, generated by t generators. Later, using so-called projective and vectorial polynomials, he proved (see [2, 4]) that $\pi^L_A(N^d_{k,t})$ contains $\mathrm{PGL}(m,q)$ and $\mathrm{GL}(m,q)$, for every integer $m > 1$ and every power $q > 1$ of p. Recently D. Harbater et al. [7] proved that for a group G to belong to $\pi^L_A(N^d_{k,t})$ it is necessary that $p(G)$ admit an abelian supplement in G of rank $\leq t$. In [4], Abhyankar exhibited some examples due to G. Stroth of groups contained in $P_t(p)$ but not satisfying the abelian supplement condition. In characteristic 3, and for $t = 3$, the Stroth groups are the groups $2S_4 * H$, where $2S_4$ denotes a double cover of the symmetric group S_4, H is either the quaternion group Q_8 or the dihedral group D_8 of order 8 and $*$ denotes central product. In this paper, for K a field containing the finite field $\mathbb{F}_9$ of nine elements, we give explicitly the whole family of Galois extensions of K with Galois group $2S_4 * H$, and determine the discriminant of such an extension. We note that in [5], the first author provided an explicit construction of $2S_4 * Q_8$-extensions of fields containing $\mathbb{F}_9$ using her previous results on Galois embedding problems based on Serre's trace formula, [9]. Here we use a different method of construction combining Abhyankar's embedding criterion [3]

Received by the editors August 4, 2004.
2000 *Mathematics Subject Classification.* Primary 12F12.
This work was partially supported by grant BFM2003-01898, Spanish Ministry of Education.

and Serre's trace formula, and reach a more explicit and simple formula both for $2S_4 * Q_8$- and $2S_4 * D_8$-extensions as well as an explicit formula for the discriminant of such extensions. The explicit determination of the discriminant of these extensions is a step towards local uniformization for three-dimensional varieties in positive characteristic.

2. Preliminaries

Let us first recall the definitions and fix the notation. We denote by $2S_n$ one of the two double covers of the symmetric group S_n reducing to the nontrivial double cover $2A_n$ of the alternating group A_n, and by H either the quaternion group Q_8 or the dihedral group D_8, double covers of the Klein group V_4. Let K be a field of characteristic different from 2 and let $\widetilde{L}|K$ be a Galois extension with Galois group the group $2S_4 * H$. Then if L is the field fixed by the center of $2S_4 * H$, we have $\mathrm{Gal}(L|K) \simeq S_4 \times V_4$, and for L_1, L_2 the fixed subfields of L by V_4 and S_4, respectively, we have $\mathrm{Gal}(L_1|K) \simeq S_4$ and $\mathrm{Gal}(L_2|K) \simeq V_4$. Therefore we obtain the whole family of Galois extensions with Galois group $2S_4 * H$ of a field K by constructing the whole family of $2S_4 * H$-extensions containing a given arbitrary S_4-extension of the field K. Let us now be given a polynomial $f(X) \in K[X]$ of degree 4 with Galois group S_4 and splitting field L_1 over K. We want to determine when L_1 is embeddable in a Galois extension of K with Galois group $2S_4 * H$. This fact is equivalent to the existence of a Galois extension $L_2|K$ with Galois group V_4, disjoint from L_1, and such that, if L is the compositum of L_1 and L_2, the Galois embedding problem

$$2S_4 * H \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K) \tag{1}$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension $\widetilde{L}$ of the field L, which is a Galois extension of K with Galois group $2S_4 * H$ and such that the restriction epimorphism between the Galois groups $\mathrm{Gal}(\widetilde{L}|K) \twoheadrightarrow \mathrm{Gal}(L|K)$ agrees with the given epimorphism $2S_4 * H \twoheadrightarrow S_4 \times V_4$. If $\widetilde{L} = L(\sqrt{\gamma})$ is a solution, then the general solution is $L(\sqrt{r\gamma}), r \in K^*$. Given a Galois extension $L_1|K$ with Galois group S_4, in order to obtain all $2S_4 * H$-extensions of K containing L_1, we have to determine all V_4-extensions L_2 of K, disjoint from L_1, and such that the embedding problem (1) is solvable.

Let us consider the double covers $2S_4 \to S_4$ and $H \to V_4$ and let $\varepsilon_1 \in H^2(S_4, \pm 1)$, $\varepsilon_2 \in H^2(V_4, \pm 1)$ denote the corresponding cohomology elements. Let $\pi_1 : S_4 \times V_4 \to S_4$ and $\pi_2 : S_4 \times V_4 \to V_4$ be the two projections and let π_1^*, π_2^* be the induced morphisms between the 2-cohomology groups. Then the element $\varepsilon = \pi_1^*(\varepsilon_1) \cdot \pi_2^*(\varepsilon_2) \in H^2(S_4 \times V_4, \{\pm 1\})$ corresponds to the double cover $2S_4 * H$ of $S_4 \times V_4$. This implies that the element in $H^2(G_K, \{\pm 1\})$ giving the obstruction to the solvability of the embedding problem (1) is equal to the product of the elements giving the obstructions to the solvability of the embedding problems $2S_4 \to S_4 \simeq \mathrm{Gal}(L_1|K)$ and $H \to V_4 \simeq \mathrm{Gal}(L_2|K)$.

Let us now specify notation by writing 2^+S_n or 2^-S_n depending on whether transpositions in S_n lift in the double cover to involutions or to elements of order 4. Let $E = K[X]/(f(X))$, for $f(X)$ the polynomial of degree 4 realizing L_1, let Q_E denote the trace form of the extension $E|K$, i.e. $Q_E(x) = \mathrm{Tr}_{E|K}(x^2)$, and let d be the discriminant of the polynomial $f(X)$. Let $L_2 = K(\sqrt{a}, \sqrt{b})$. We denote by w the Hasse-Witt invariant of a quadratic form and by $(\cdot, \cdot)$ a Hilbert symbol.

By [9] the obstruction to the solvability of the embedding problem $2^{\pm}S_4 \to S_4 \simeq \mathrm{Gal}(L_1|K)$ is equal to $w(Q_E).(\pm 2, d) \in H^2(G_K, \{\pm 1\})$. By [10], the obstruction to the solvability of $Q_8 \to V_4 \simeq \mathrm{Gal}(L_2|K)$ is equal to $(a,b).(-1,ab) \in H^2(G_K,\{\pm 1\})$ and by e.g. [6], the obstruction to the solvability of $D_8 \to V_4 \simeq \mathrm{Gal}(L_2|K)$ is equal to $(a,b) \in H^2(G_K,\{\pm 1\})$ (here we assume that the order 4 elements of D_8 are mapped on the nontrivial element in $\mathrm{Gal}(L_2|K)$ fixing $\sqrt{ab}$).

From now on, we assume that K is a field of characteristic 3. We write $f(X) = X^4 + s_2X^2 - s_3X + s_4$. By computation of the trace form Q_E, we obtain

$$w(Q_E) = (ds_2, (s_2^2 - s_4)s_2) \cdot (-1, s_2^2 - s_4).$$

If we further assume that K contains $\mathbb{F}_9$, i.e. that $-1 \in K^2$, the solvability of the embedding problem (1) is equivalent to

$$(ds_2, (s_2^2 - s_4)s_2) = (a,b), \tag{2}$$

that is, the equality of two Hilbert symbols.

We now recall the isomorphisms $S_4 \simeq \mathrm{PGL}(2,3)$ and $2^+S_4 \simeq \mathrm{GL}(2,3)$ and state Abhyankar's Embedding Criterion [3] (1.1), and Polynomial Theorem [3] (2.1), (3.7), in our particular case.

Proposition 1. *Let K be a field of characteristic 3, and let $M|K$ be a Galois extension with Galois group $\mathrm{PGL}(2,3)$. The embedding problem*

$$\mathrm{GL}(2,3) \to \mathrm{PGL}(2,3) \simeq \mathrm{Gal}(M|K) \tag{3}$$

is solvable $\Leftrightarrow$ $M|K$ is the splitting field of a projective polynomial $Y^4 + c_3Y + c_4 \in K[Y]$. Moreover, if $|K| \geq 9$, the splitting field of the vectorial polynomial $Y(Y^8 + c_3Y^2 + c_4)$ is a solution to the embedding problem (3).

3. Main results

Under the hypothesis $char K = 3$, the two equivalent conditions to the solvability of the Galois embedding problem $2^+S_4 \to S_4 \simeq \mathrm{Gal}(M|K)$ obtained by applying Serre's trace formula and Abhyankar's Embedding Criterion can directly be seen to be equivalent. Indeed, let $M|K$ be a Galois extension with the Galois group S_4 given as the splitting field of a polynomial $f(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$, let d be the discriminant of $f(X)$, let x be a root of $f(X)$ in M and let $E = K(x)$. Then M is the splitting field of a polynomial of the form $Y^4 + c_3Y + c_4 \in K[Y]$ if and only if there exists elements $a_0, a_1, a_2, a_3 \in K$ such that the irreducible polynomial over K of the element $y = a_0 + a_1x + a_2x^2 + a_3x^3$ has such a form. By computation, this is equivalent to the conditions $a_0 = -a_2s_2$ and $Q(a_1,a_2,a_3) := s_2a_1^2 + (s_2^2 - s_4)a_2^2 + s_2^3a_3^2 + (s_2^2 + s_4)a_1a_3 + 2s_2s_3a_2a_3 = 0$. Now the quadratic trace form Q_E, for $E = K(x)$, is equivalent to $1 + Q$, for Q the quadratic form in a_1, a_2, a_3 in the second condition. If we assume $w(Q_E) = (2,d)$, then we have $Q_E \sim \langle 1,1,2,2d\rangle$ (see [9] 3.2) which implies $Q \sim \langle 1,2,2d\rangle$ and this last quadratic form represents 0 over any field K of characteristic 3. Reciprocally, assume that Q represents 0 over K. Diagonalizing Q, we obtain $\langle s_2, m, s_2md\rangle$, with $m = s_2^2 - s_4$, and so we have $s_2b_1^2 + mb_2^2 + s_2mdb_3^2 = 0$, for some $b_1, b_2, b_3 \in K$, which implies $(-ds_2, -ms_2) = 1$, and so $(ds_2, ms_2) = (-1, md)\cdot(-1,-1)$. Hence we get $w(Q_E) = (ds_2, ms_2)\cdot(-1,m) = (-1,d) = (2,d)$.

Theorem 1. *Let K be a field of characteristic 3 containing $\mathbb{F}_9$, and let $f(X) = X^4+s_2X^2-s_3X+s_4 \in K[X]$, with Galois group S_4 and L_1 the splitting field of $f(X)$*

over K. Let $d = s_4^3 + s_2^2 s_4^2 + s_2^4 s_4 - s_2^3 s_3^2$ be the discriminant of the polynomial $f(X)$. The family of elements a, b in K such that $(a, b) = (ds_2, ms_2)$, where $m = s_2^2 - s_4$, can be given in terms of an arbitrary invertible matrix $P = (p_{ij})_{1\leq i,j,\leq 3} \in \mathrm{GL}(3, K)$ as $a = dA$, $b = s_2 mF$, where

$$\begin{aligned} A &= s_2 p_{11}^2 + m p_{21}^2 + dms_2 p_{31}^2, \\ F &= dm P_{13}^2 + ds_2 P_{23}^2 + P_{33}^2, \quad \text{with } P_{ij} = \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}. \end{aligned}$$

Let $L_2 = K(\sqrt{a}, \sqrt{b})$ and assume that $L_2|K$ has Galois group V_4 and $L_1 \cap L_2 = K$ (i.e. that the elements a, b, ab, da, db, dab are not squares in K). Let $L = L_1 \cdot L_2$. For x a root of the polynomial $f(X)$, take $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, with

$$\begin{aligned} a_0 &= -s_2 a_2, \\ a_1 &= dm\sqrt{-1}(ns_2 p_{11} P_{23} - p_{21} P_{33} + mnp_{21} P_{13} - ds_2 p_{31} P_{23}) + m\sqrt{a}(dP_{13} + nP_{33}), \\ a_2 &= ds_2\sqrt{-1}(p_{11} P_{33} - s_2^2 s_3 p_{11} P_{23} - ms_2 s_3 p_{21} P_{13} - dmp_{31} P_{13}) - s_2\sqrt{a}(s_2 s_3 P_{33} + dP_{23}), \\ a_3 &= dms_2\sqrt{-1}(s_2 p_{11} P_{23} + mp_{21} P_{13}) + ms_2\sqrt{a} P_{33}, \end{aligned}$$

where $n = s_2^2 + s_4$. Then $L(\sqrt{ry}), r \in K^*$, is the general solution to the embedding problem

$$2^+ S_4 * D_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K).$$

Proof. By [8], 3.2, the equality of Hilbert symbols (2) is equivalent to the K-equivalence of quadratic forms

$$\langle ds_2, ms_2, dm\rangle \sim \langle a, b, ab\rangle. \tag{4}$$

The family of quadratic forms K-equivalent to $R := \langle ds_2, ms_2, dm\rangle$ is given by $P^T RP$, for P running over $\mathrm{GL}(3, K)$. By diagonalizing $P^t RP$, we obtain $\langle dA, s_2 mF, dAs_2 mF\rangle$, with A and F as in the statement. Let $a = dA$, $b = s_2 mF$. Now, we have $(a, b) = 1 \in H^2(G_{K(\sqrt{a})}, \{\pm 1\})$ and, as $a \notin K^2$ and $L_1 \cap K(\sqrt{a}) = K$, the extension $L_1(\sqrt{a})|K(\sqrt{a})$ has Galois group S_4, and the Galois embedding problem $2^+ S_4 \to S_4 \simeq \mathrm{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$ is solvable. By the argument preceding Theorem 1, there exist $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$, and for the element $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ we have that $\mathrm{Irr}(y, K(\sqrt{a})$ is a projective polynomial. Also, by Abhyankar's Polynomial Theorem (see Proposition 1), the splitting field of the vectorial polynomial $Y.\mathrm{Irr}(\sqrt{y}, K(\sqrt{a}))$, that is, the field $L_1(\sqrt{a})(\sqrt{y})$, is a solution to the Galois embedding problem $2^+ S_4 \to S_4 \simeq \mathrm{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$. Now our aim is to compute explicitly such elements a_i. Diagonalizing Q, we obtain $\langle s_2, m, s_2 md\rangle$ and from (4) we get that $\langle s_2, m, s_2 md\rangle \sim \langle A, s_2 mAF, s_2 mFd\rangle$ and the basis change matrix can be written down explicitly in terms of the matrix P. Now the vector $(0, d\sqrt{-1}, \sqrt{a}) \in K(\sqrt{a})^3$ anihilates the quadratic form $\langle A, s_2 mAF, s_2 mFd\rangle$, and from it we obtain the values for $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$.

Now we want to see that $L(\sqrt{y})|K$ is a Galois extension with Galois group $2^+ S_4 * D_8$. By the assumption $L_1 \cap L_2 = K$, we have $\mathrm{Gal}(L(\sqrt{y})|L_2) \simeq 2^+ S_4$. We now consider the behaviour of y under the action of $\mathrm{Gal}(L_2|K)$. Let r, s, t be the nontrivial elements of $\mathrm{Gal}(L_2|K)$ fixing respectively $\sqrt{ab}, \sqrt{b}, \sqrt{a}$. By computation we obtain $y^s y = d^2 h^2 b$, where $h = ms_2 p_{31} x^3 + (p_{21} - s_2^2 s_3 p_{31})x^2 + (mnp_{31} + p_{11})x + s_2^3 s_3 p_{31} - s_2 p_{21}$. Now $y \in K(\sqrt{a})(x)$, so $y^t = y$ and $y^r = y^s$, so $L(\sqrt{y})$ is Galois over

K. Now we have $(dh\sqrt{b})^s = dh\sqrt{b}$ and $(dh\sqrt{b})^r = -dh\sqrt{b}$, so $\mathrm{Gal}(L(\sqrt{y})|L_1) \simeq D_8$, with $L(\sqrt{y})|L_1(\sqrt{ab})$ cyclic; hence $\mathrm{Gal}(L(\sqrt{y})|K) \simeq 2^+S_4 * D_8$. □

Remark 1. For the element y given by Theorem 1, we have $\mathrm{Irr}(y, K(\sqrt{a})) = Y^4 + c_3Y + c_4$, where

$$\begin{aligned} c_3 &= s_3a_1^3 + ma_1^2a_2 + s_2s_3a_1^2a_3 + s_2s_3a_1a_2^2 + ms_2a_1a_2a_3 + ms_3a_1a_3^2 + s_3^2a_2^3 \\ &\quad -ns_3a_2^2a_3 + (m^2 + s_2s_3^2)a_2a_3^2 + s_3^3a_3^3, \\ c_4 &= \frac{d}{s_2^2}(a_1a_3 - a_2^2 - s_2a_3^2)^2. \end{aligned}$$

Theorem 2. *Let the fields K and L and the elements d, a, b and y be as in Theorem 1, let $\mu = d + (d+1)\sqrt{d}$ and let $\rho = ab + (ab+1)\sqrt{ab}$. Then*

1. *$L(\sqrt{r\mu y}), r \in K^*$, is the general solution to the embedding problem*
$$2^-S_4 * D_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K).$$
2. *$L(\sqrt{r\rho y}), r \in K^*$, is the general solution to the embedding problem*
$$2^+S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K).$$
3. *$L(\sqrt{r\mu\rho y}), r \in K^*$, is the general solution to the embedding problem*
$$2^-S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K).$$

Proof. For $\sigma \in S_4 \setminus A_4$, we have $\mu^\sigma\mu = -(d-1)^2d$ and so, $L(\sqrt{\mu y})|K$ Galois. Now $(d-1)\sqrt{-d}$ changes sign under the action of σ, so $\mathrm{Gal}(L(\sqrt{\mu y})|L_2) \simeq 2^-S_4$, hence $\mathrm{Gal}(L(\sqrt{\mu y})|K) \simeq 2^-S_4 * D_8$.

For $r, s, t \in \mathrm{Gal}(L_2|K)$ fixing $\sqrt{ab}, \sqrt{b}, \sqrt{a}$, resp., we have $\rho^s\rho = \rho^t\rho = -(ab-1)^2ab$ and $\rho^r\rho = \rho^2$, so $L(\sqrt{\rho y})|K$ Galois. Now $(ab-1)\sqrt{-ab}$ changes sign under the action of s and under the action of t, so $\mathrm{Gal}(L(\sqrt{\rho y})|L_1) \simeq Q_8$, hence $\mathrm{Gal}(L(\sqrt{\rho y})|K) \simeq 2^+S_4 * Q_8$.

Combining both arguments, we obtain the third statement in the theorem. □

Proposition 2. *Let the fields K and L and the elements s_2, s_3, s_4, d, a, b, m, p_{ij} and y be as in Theorem 1; μ, ρ as in Theorem 2. We have*

$$\begin{aligned} \mathrm{disc}(L(\sqrt{y})|K) &= d^{104}a^{96}b^{100}D^2, \\ \mathrm{disc}(L(\sqrt{\mu y})|K) &= d^{152}a^{96}b^{100}D^2(d-1)^{48}, \\ \mathrm{disc}(L(\sqrt{\rho y})|K) &= d^{104}a^{144}b^{148}D^2(ab-1)^{48}, \\ \mathrm{disc}(L(\sqrt{\mu\rho y})|K) &= d^{152}a^{144}b^{148}D^2(d-1)^{48}(ab-1)^{48}, \end{aligned}$$

where

$$\begin{aligned} D &= s_4p_{11}^4 - s_2s_3p_{11}^3p_{21} + ms_2p_{11}^2p_{21}^2 - ms_3p_{11}p_{21}^3 + (m^2 - s_2s_3^2)p_{21}^4 \\ &\quad + dp_{31}(-p_{11}^3 + ms_2^2p_{11}^2p_{31} - s_3p_{21}^3 + m^2s_2p_{31}p_{21}^2) + d^2p_{31}^3(s_2s_3p_{21} + mp_{11}) + d^3p_{31}^4. \end{aligned}$$

Proof. We have $\mathrm{disc}(L(\sqrt{y})|K) = \mathrm{disc}(L|K)^2 \cdot N_{L|K}(y)$ and $\mathrm{disc}(L|K) = (dab)^{48}$. Now $N_{L|K}(y) = (N_{L_1(\sqrt{a})|K}(y))^2 = (N_{K(\sqrt{a})|K}(c_4))^2$, for c_4 the degree 0 coefficient in the irreducible polynomial of y over $K(\sqrt{a})$. By computation we obtain $c_4 = \frac{d}{s_2^2}(a_1a_3 - a_2^2 - s_2a_3^2)^2$ and, by substituting the values of a_1, a_2, a_3 and computing the norm, $N_{K(\sqrt{a})|K}(c_4) = d^8b^4D^2$ for D as in the statement.

To obtain the other three discriminants, it is now enough to compute $N_{L|K}(\mu) = N_{K(\sqrt{d})|K}(\mu)^{48} = (d-1)^{48}d^{48}$ and $N_{L|K}(\rho) = N_{K(\sqrt{ab})|K}(\rho)^{48} = (ab-1)^{48}(ab)^{48}$. □

4. Examples

Let $K = k((Z_1, Z_2, Z_3))$ be the quotient field of the formal power series ring in 3 variables over a field k containing $\mathbb{F}_9$. We consider the polynomial

$$f(X) = X^4 + Z_1 X^2 + Z_2 X + Z_3 \in K[X],$$

i.e. we are taking $s_2 = Z_1, s_3 = -Z_2, s_4 = Z_3$. We can check that the polynomial f has Galois group S_4 over K and let L_1 be the splitting field of f over K. We consider the extension $L_2|K$ generated by the elements $\sqrt{ds_2}, \sqrt{ms_2}, \sqrt{dm}$, which corresponds to taking the matrix P in Theorem 1 to be one of the matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can check that the elements $ds_2, ms_2, dm, s_2, dms_2, m$ are not squares in K, and so $L_2|K$ has Galois group V_4 and is disjoint with $L_1|K$. Let $L = L_1 \cdot L_2$. We denote by y_i the element y given by Theorem 1 for each of the matrices P_i, $i = 1, 2, 3$. Then we have

$$\mathrm{Gal}(L(\sqrt{y_i})|K) \simeq 2^+ S_4 * D_8, \qquad i = 1, 2, 3,$$

with $L(\sqrt{y_1})|L_1(\sqrt{dm})$, $L(\sqrt{y_2})|L_1(\sqrt{ms_2})$, $L(\sqrt{y_3})|L_1(\sqrt{ds_2})$ cyclic. The factors appearing in $\mathrm{disc}(L(\sqrt{y_1})|K)$ are d, s_2, m and s_4, the factors appearing in $\mathrm{disc}(L(\sqrt{y_2})|K)$ are d, s_2, m and $m^2 - s_2 s_3^2$, and the factors appearing in $\mathrm{disc}(L(\sqrt{y_3})|K)$ are d, s_2, m. In particular the discriminantial locus remains unchanged when going from L to $L(\sqrt{y_3})$.

We observe that the elements $d - 1$ and $ab - 1$, for each choice of a, b among ds_2, ms_2, dm, are invertible elements in the ring $k[[Z_1, Z_2, Z_3]]$, and so the discriminant locus will not change if we realize any other of the groups $2S_4 * H$ over the same field K by means of Theorem 2.

References

[1] S.S. Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. **77** (1955), 572-592. MR0071851 (17:193c)

[2] S.S. Abhyankar, *Local fundamental groups of algebraic varieties*, Proc. Amer. Math. Soc. **125** (1997), 1635-1641. MR1403110 (97h:14032)

[3] S.S. Abhyankar, *Galois embeddings for linear groups*, Trans. Amer. Math. Soc. **352** (2000), 3881-3912. MR1650057 (2000m:12003)

[4] S.S. Abhyankar, *Resolution of singularities and modular Galois theory*, Bull. Amer. Math. Soc. **38** (2001), 131-169. MR1816069 (2002a:14013)

[5] T. Crespo, *$2S_4 * Q_8$-extensions in characteristic 3*, Proc. Amer. Math. Soc. **132** (2004), 691-695. MR2019944 (2004j:12003)

[6] A. Fröhlich, *Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants*, J. Reine Angew. Math. **360** (1985), 84-123. MR0799658 (87h:11028)

[7] D. Harbater, M. van der Put and R. Guralnick, *Valued fields and covers in characteristic p*, Fields Institute Communications **32** (2002), 175-204. MR1928369 (2003i:12008)

[8] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin-Cummings Publ. Co., Reading, MA, 1973. MR0396410 (53:277)

[9] J.-P. Serre, *L'invariant de Witt de la forme* $\mathrm{Tr}(x^2)$, Comment. Math. Helv. **59** (1984), 651-676. MR0780081 (86k:11067)

[10] E. Witt, *Konstruktion von galoischen Körpern der Charakteristik p zu vorgegebener Gruppe der Ordnung* p^f, J. Reine Angew. Math. **174** (1936), 237-245.

Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

E-mail address: teresa.crespo@ub.edu

Zakład Matematyki, Akademia Rolnicza, al. Mickiewicza 24/28, 30-059 Kraków, Poland

E-mail address: rmhajto@cyf-kr.edu.pl

Current address: Instytut Matematyki, Politechnika Krakowska, ul. Warszawska 24, 31-155 Kraków, Poland