# MODULI SPACES AND FORMAL OPERADS

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**Abstract**

*Let $\overline{\mathscr{M}}_{g,l}$ be the moduli space of stable algebraic curves of genus $g$ with $l$ marked points. With the operations that relate the different moduli spaces identifying marked points, the family $(\overline{\mathscr{M}}_{g,l})_{g,l}$ is a modular operad of projective smooth Deligne-Mumford stacks $\overline{\mathscr{M}}$. In this paper, we prove that the modular operad of singular chains $S_*(\overline{\mathscr{M}};\mathbb{Q})$ is formal, so it is weakly equivalent to the modular operad of its homology $H_*(\overline{\mathscr{M}};\mathbb{Q})$. As a consequence, the up-to-homotopy algebras of these two operads are the same. To obtain this result, we prove a formality theorem for operads analogous to the Deligne-Griffiths-Morgan-Sullivan formality theorem, the existence of minimal models of modular operads, and a characterization of formality for operads which shows that formality is independent of the ground field.*

**Contents**



## 1. Introduction

In recent years, moduli spaces of Riemann surfaces, such as the moduli spaces of stable algebraic curves of genus $g$ with $l$ marked points, $\overline{\mathscr{M}}_{g,l}$, have played an important role

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in the mathematical formulation of certain theories inspired by physics, such as the complete cohomological field theories.

In these developments, the operations that relate the different moduli spaces $\overline{\mathscr{M}}_{g,l}$ identifying marked points $\overline{\mathscr{M}}_{g,l} \times \overline{\mathscr{M}}_{h,m} \longrightarrow \overline{\mathscr{M}}_{g+h,l+m-2}$ and $\overline{\mathscr{M}}_{g,l} \longrightarrow \overline{\mathscr{M}}_{g+1,l-2}$ have been interpreted in terms of operads. With these operations, the spaces $\overline{\mathscr{M}}_{0,l}$, $l \geq 3$, form a cyclic operad of projective smooth varieties $\overline{\mathscr{M}}_0$ (see [GK1]), and the spaces $\overline{\mathscr{M}}_{g,l}$, $g, l \geq 0$, $2g - 2 + l > 0$, form a modular operad of projective smooth Deligne-Mumford stacks $\overline{\mathscr{M}}$ (see [GK2]). Therefore the homologies of these operads, $H_*(\overline{\mathscr{M}}_0; \mathbb{Q})$ and $H_*(\overline{\mathscr{M}}; \mathbb{Q})$, are cyclic and modular operads, respectively.

An important result in the algebraic theory of Gromov-Witten invariants is that if $X$ is a complex projective manifold and $\Lambda(X)$ is the Novikov ring of $X$, the cohomology $H^*(X; \Lambda(X))$ has a natural structure of an algebra over the modular operad $H_*(\overline{\mathscr{M}}; \mathbb{Q})$, and so it is a complete cohomological field theory ([Be]; see [M]).

But there is another modular operad associated with the geometric operad $\overline{\mathscr{M}}$: the modular operad $S_*(\overline{\mathscr{M}}; \mathbb{Q})$ of singular chains. Algebras over this operad have been studied in [GK2], [KSV], and [KVZ].

In this paper, we prove that the modular operad $S_*(\overline{\mathscr{M}}; \mathbb{Q})$ is formal, so it is weakly equivalent to the modular operad of its homology $H_*(\overline{\mathscr{M}}; \mathbb{Q})$. As a consequence, the *up-to-homotopy* algebras of these two operads are the same.

A paradigmatic example of an operad is the little 2-disc operad of Boardman and Vogt, $\mathscr{D}_2(l)$, of configurations of $l$ disjoint discs in the unit disc of $\mathbb{R}^2$. Our result can be seen as the analogue for $\overline{\mathscr{M}}$ of the Kontsevich-Tamarkin formality theorem for $S_*(\mathscr{D}_2; \mathbb{Q})$ (see [K] and [T]; moreover, [K] also explains the relation between this formality theorem, Deligne's conjecture in Hochschild cohomology, and Kontsevich's formality theorem in deformation quantization).

Our paper is organized as follows. In Section 2 we study symmetric monoidal functors between symmetric monoidal categories since they induce functors between the categories of their operads. After recalling some definitions and fixing some notation of operads and monoidal categories, we prove a symmetric de Rham theorem. We then introduce the notion of formal symmetric monoidal functor, and we see how this kind of functor produces formal operads.

In Section 3 as a consequence of Hodge theory, we prove that the singular chain functor on the category of compact Kähler manifolds $S_* : \mathbf{K\ddot{a}h} \longrightarrow \mathbf{C}_*(\mathbb{R})$ is a formal symmetric monoidal functor. It follows that if $X$ is an operad of compact Kähler manifolds, then the operad of chains $S_*(X; \mathbb{R})$ is formal. This is the analogue in the theory of operads to the Deligne-Griffiths-Morgan-Sullivan formality theorem in rational homotopy theory (see [DGMS, main theorem]).

The goal of Sections 4, 5, and 6 is to prove the descent of formality from $\mathbb{R}$ to $\mathbb{Q}$. In Section 4 we recall some results due to M. Markl [Ma] on minimal models of operads in the form that we use in order to generalize them to cyclic and modular operads.

In Section 5 we prove a characterization of formality of an operad in terms of the lifting of automorphisms of the homology of the operad to automorphisms of the operad itself, which is analogous in the operadic setting to the Sullivan criterion of formality for commutative differential graded (cdg) algebras (see [Su]).

The automorphism group of a minimal operad with homology of finite type is a proalgebraic group. This result, together with the previous characterization of formality, allows us to use the descent theory of algebraic groups to prove that formality is independent of the ground field (see Th. 6.2.1).

In Section 7 we show how the above results can be extended easily to cyclic operads. In particular, we obtain the formality of the cyclic operad $S_*(\overline{\mathscr{M}}_0; \mathbb{Q})$.

In Section 8 we go one step further and also prove the above results for modular operads. In particular, we introduce minimal models of modular operads and we prove their existence. Here our use of the modular dimension is inspired by Grothendieck's "jeu de Légo-Teichmüller" ([Gr, Sec. 2]), in which he builds the complete Teichmüller tower inductively on this dimension. Once the existence of minimal models is established, the proofs of the previous sections can be transferred to the modular context without difficulty. Finally, we conclude the formality of the modular operad $S_*(\overline{\mathscr{M}}; \mathbb{Q})$.

## 2. Formal operads

*2.1. Operads*

Let us recall some definitions and notation about operads (see [GiK], [KM], [MSS]).

*2.1.1*

Let $\Sigma$ be the *symmetric groupoid*, that is, the category whose objects are the sets $\{1, \dots, n\}$, $n \geq 1$, and whose only morphisms are those of the symmetric groups $\Sigma_n = \mathrm{Aut}\{1, \dots, n\}$.

*2.1.2*

Let $\mathscr{C}$ be a category. The category of contravariant functors from $\Sigma$ to $\mathscr{C}$ is called the category of $\Sigma$-*modules* and is denoted by $\mathbf{\Sigma Mod}_{\mathscr{C}}$, or just $\mathbf{\Sigma Mod}$ if $\mathscr{C}$ is understood. We identify its objects with sequences of objects in $\mathscr{C}$, $E = (E(l))_{l \geq 1}$ with a right $\Sigma_l$-action on each $E(l)$. If $e$ is an element of $E(l)$, $l$ is called the *arity* of $e$. If $E$ and $F$ are $\Sigma$-modules, a *morphism of* $\Sigma$-*modules* $f : E \longrightarrow F$ is a sequence of $\Sigma_l$-equivariant morphisms $f(l) : E(l) \longrightarrow F(l)$, $l \geq 1$.

*2.1.3*

Let $(\mathscr{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. A *unital* $\Sigma$*-operad* (an *operad* for short) in $\mathscr{C}$ is a $\Sigma$-module $P$ together with a family of structure morphisms: composition $\gamma_{l;m_1,\ldots,m_l} : P(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l) \longrightarrow P(m_1 + \cdots + m_l)$ and unit $\eta : \mathbf{1} \longrightarrow P(1)$, satisfying the axioms of equivariance, associativity, and unit. A *morphism of operads* is a morphism of $\Sigma$-modules compatible with structure morphisms. Let us denote by $\mathbf{Op}_{\mathscr{C}}$ or, simply, $\mathbf{Op}$ when $\mathscr{C}$ is understood, the category of operads in $\mathscr{C}$ and its morphisms.

*2.2. Symmetric monoidal categories and functors*

In the study of $\Sigma$-operads, the commutativity constraint plays an important role. In particular, the functors we are interested in are functors between symmetric monoidal categories which are compatible with the associativity, commutativity, and unit constraints.

*2.2.1*

The following are some of the symmetric monoidal categories we deal with in this paper. On the one hand, we have the geometric ones:

**Top**: the category of topological spaces.
**Dif**: the category of differentiable manifolds.
**Käh**: the category of compact Kähler manifolds.
$\mathbf{V}(\mathbb{C})$: the category of smooth projective $\mathbb{C}$-schemes.

On the other hand, we have the algebraic categories, which are variants of

$\mathbf{C}_*(\mathscr{A})$: the category of complexes with a differential of degree $-1$ of an abelian symmetric monoidal category $(\mathscr{A}, \otimes, \mathbf{1})$. The morphisms are called *chain maps*. If $\mathscr{A}$ is the category of $R$-modules for some ring $R$, we denote it by $\mathbf{C}_*(R)$. Operads in $\mathbf{C}_*(\mathscr{A})$ are also called *differential graded (dg) operads*.

In a symmetric monoidal category $(\mathscr{C}, \otimes, \mathbf{1})$, we usually denote the natural commutativity isomorphism by $\tau_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$. For example, in $\mathbf{C}_*(\mathscr{A})$, the natural commutativity isomorphism

$$\tau_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

includes the signs

$$\tau_{X,Y}(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x.$$

*2.2.2*

As usual, we move from a geometric category to an algebraic one through a functor. Let us recall (see [KS]) that a *monoidal functor*

$$(F, \kappa, \eta) : (\mathscr{C}, \otimes, \mathbf{1}) \longrightarrow (\mathscr{D}, \otimes, \mathbf{1}')$$

between monoidal categories is a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ together with a natural morphism of $\mathscr{D}$,

$$\kappa_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y),$$

for all objects $X, Y \in \mathscr{C}$, and a morphism of $\mathscr{D}$, $\eta : \mathbf{1}' \longrightarrow F\mathbf{1}$, compatible with the constraints of associativity and unit. We refer to $\kappa$ as the *Künneth morphism*.

If $\mathscr{C}$ and $\mathscr{D}$ are symmetric monoidal categories, a monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be *symmetric* if $\kappa$ is compatible with the commutativity constraint.

For example, the *homology functor* $H_* : \mathbf{C}_*(\mathscr{A}) \longrightarrow \mathbf{C}_*(\mathscr{A})$ is a symmetric monoidal functor, taking the usual Künneth morphism

$$H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y)$$

as $\kappa$.

Let $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ be two monoidal functors. A natural transformation $\phi : F \Rightarrow G$ is said to be *monoidal* if it is compatible with $\kappa$ and $\eta$.

*2.2.3*

Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a symmetric monoidal functor. It is easy to prove that applied componentwise, $F$ induces a functor between $\Sigma$-operads

$$\mathbf{Op}_F : \mathbf{Op}_{\mathscr{C}} \longrightarrow \mathbf{Op}_{\mathscr{D}},$$

which is also denoted by $F$.

In particular, for an operad $P \in \mathbf{Op}_{\mathbf{C}_*(\mathscr{A})}$, its homology is an operad $HP \in \mathbf{Op}_{\mathbf{C}_*(\mathscr{A})}$.

In the same way, if $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ are two symmetric monoidal functors, a monoidal natural transformation $\phi : F \Rightarrow G$ induces a natural transformation

$$\mathbf{Op}_\phi : \mathbf{Op}_F \Rightarrow \mathbf{Op}_G,$$

which is also denoted by $\phi$.

*2.3. Weak equivalences*

We use weak equivalences in several contexts.

*2.3.1*

Let $X$ and $Y$ be objects of $\mathbf{C}_*(\mathscr{A})$. A chain map $f : X \longrightarrow Y$ is said to be a *weak equivalence of complexes* if the induced morphism $f_* = Hf : HX \longrightarrow HY$ is an isomorphism.

*2.3.2*
Let $\mathscr{C}$ be a category, let $\mathscr{A}$ be an abelian category, and let $F, G : \mathscr{C} \rightrightarrows \mathbf{C}_*(\mathscr{A})$ be two functors. A natural transformation $\phi : F \Rightarrow G$ is said to be a *weak equivalence of functors* if the morphism $\phi(X) : F(X) \to G(X)$ is a weak equivalence for every object $X$ in $\mathscr{C}$.

*2.3.3*
A morphism $\rho : P \longrightarrow Q$ of operads in $\mathbf{C}_*(\mathscr{A})$ is said to be a *weak equivalence of operads* if $\rho(l) : P(l) \longrightarrow Q(l)$ is a weak equivalence of chain complexes for all $l$.

*2.3.4*
Let $\mathscr{C}$ be a category endowed with a distinguished class of morphisms called *weak equivalences*. We suppose that this is a saturated class of morphisms which contains all isomorphisms. Two objects $X$ and $Y$ of $\mathscr{C}$ are said to be *weakly equivalent* if there exists a sequence of weak equivalences of $\mathscr{C}$,

$$X \longleftarrow X_1 \longrightarrow \cdots \longleftarrow X_{n-1} \longrightarrow Y.$$

If $X$ and $Y$ are weakly equivalent, we say that $Y$ is a *model* of $X$.

*2.3.5*
The following proposition is an easy consequence of the definitions.

PROPOSITION 2.3.1
*If $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ are two weakly equivalent symmetric monoidal functors, the functors $\mathbf{Op}_F$ and $\mathbf{Op}_G$ are weakly equivalent. In particular, for every operad $P$ in $\mathscr{C}$, the operads $F(P)$ and $G(P)$ are weakly equivalent.*

*2.4. The symmetric de Rham theorem*
In this section, we formulate a de Rham theorem comparing the complex of singular chains and the de Rham complex of currents in terms of symmetric monoidal functors.

*2.4.1*
Denote by

$$S_* : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z})$$

the functor of *singular chains*. Recall that the *shuffle product* (see [EM])

$$\nabla_{X,Y} : S_*(X; \mathbb{Z}) \otimes S_*(Y; \mathbb{Z}) \longrightarrow S_*(X \times Y; \mathbb{Z})$$

is defined, for $c \in S_p(X;\mathbb{Z})$ and $d \in S_q(X;\mathbb{Z})$, by

$$c\nabla d = \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} s_{\nu_q} \cdots s_{\nu_1} c \times s_{\mu_p} \cdots s_{\mu_1} d,$$

where the sum is taken over all the $(p,q)$-shuffles $(\mu,\nu)$, $s_i$ denotes the degeneracy operators, and $(-1)^{\epsilon(\mu)}$ denotes the sign of the associated permutation. With this product and the obvious unit object, $S_*$ is a symmetric monoidal functor (see [EM, Th. 5.2]).

We denote as well by $S_*$ the symmetric monoidal functors of singular chains with coefficients in $\mathbb{Q}$ or $\mathbb{R}$.

Let $\mathscr{D}'_*(M)$ be the complex of de Rham's *currents* of a differentiable manifold $M$; that is, $\mathscr{D}'_*(M)$ is the topological dual of the complex $\mathscr{D}^*(M)$ of differential forms with compact support. Then the functor

$$\mathscr{D}'_* : \mathbf{Dif} \longrightarrow \mathbf{C}_*(\mathbb{R})$$

is a symmetric monoidal functor with the Künneth morphism

$$\kappa_{M,N} : \mathscr{D}'_*(M) \otimes \mathscr{D}'_*(N) \longrightarrow \mathscr{D}'_*(M \times N)$$

induced by the tensor product of currents. Thereby, if $S \in \mathscr{D}'_*(M)$ and $T \in \mathscr{D}'_*(N)$, then

$$\langle \kappa(S \otimes T),\ \pi_M^*(\omega) \wedge \pi_N^*(\nu) \rangle \ = \ \langle S,\ \omega \rangle \cdot \langle T,\ \nu \rangle$$

for all $\omega \in \mathscr{D}^*(M)$ and $\nu \in \mathscr{D}^*(N)$.

In order to compare the functor of currents with the functor of singular chains on differentiable manifolds, we consider the complex of chains $S_*^\infty(M;\mathbb{Z})$ generated by the $\mathscr{C}^\infty$-maps $\Delta^p \longrightarrow M$. Since the shuffle product of $\mathscr{C}^\infty$-singular chains is also a $\mathscr{C}^\infty$-singular chain, the functor of $\mathscr{C}^\infty$-*singular chains* $S_*^\infty : \mathbf{Dif} \longrightarrow \mathbf{C}_*(\mathbb{Z})$ is a symmetric monoidal functor, and the natural inclusion of $\mathscr{C}^\infty$-singular chains in singular ones defines a monoidal natural transformation $S_*^\infty \Rightarrow S_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{Z})$. From the approximation theorem, it follows that it is a weak equivalence of symmetric monoidal functors.

Let $M$ be a differentiable manifold, and let $c : \Delta^p \longrightarrow M$ be a $\mathscr{C}^\infty$-singular simplex. Integration along $c$, $\int_c \omega := \int_{\Delta^p} c^*(\omega)$, defines, by the Stokes theorem, a chain map

$$\int : S_*^\infty(M;\mathbb{R}) \longrightarrow \mathscr{D}'_*(M;\mathbb{R}), \qquad c \mapsto \int_c,$$

which is obviously natural in $M$ and is a weak equivalence by the de Rham theorem. Thus $\int : S_*^\infty \Rightarrow \mathscr{D}'_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{R})$ is a weak equivalence of functors.

Now we check that integration is compatible with the monoidal structure.

Denote by $\iota_p : \Delta^p \longrightarrow \Delta^p$ the identity map. If $c : \Delta^p \longrightarrow M$ and $d : \Delta^q \longrightarrow N$ are $\mathscr{C}^\infty$-singular simplexes, we have $c\nabla d = (c \times d)_*(\iota_p \nabla \iota_q)$. Thus, if $\omega \in \mathscr{D}^p(M)$ and $\eta \in \mathscr{D}^q(N)$, by naturality we have

$$\begin{aligned}\int_{c\nabla d} \pi_M^*(\omega) \wedge \pi_N^*(\eta) &= \int_{(c\times d)_*(\iota_p\nabla\iota_q)} \pi_M^*(\omega) \wedge \pi_N^*(\eta) \\ &= \int_{\iota_p\nabla\iota_q} (c \times d)^*\big(\pi_M^*(\omega) \wedge \pi_N^*(\eta)\big) .\end{aligned}$$

As noted in [EM], the simplexes of the shuffle product $\iota_p\nabla\iota_q$ form the standard triangulation of the simplicial polyhedron $\Delta^p \times \Delta^q$, and the corresponding signs are such that $\iota_p\nabla\iota_q$ is the fundamental chain of the product $\Delta^p \times \Delta^q$; that is, we have

$$\int_{\iota_p\nabla\iota_q} = \int_{\Delta^p\times\Delta^q} .$$

Therefore, by the Fubini theorem, we obtain

$$\begin{aligned}\int_{c\nabla d} \pi_M^*(\omega) \wedge \pi_N^*(\eta) &= \int_{\Delta^p\times\Delta^q} (c \times d)^*\big(\pi_M^*(\omega) \wedge \pi_N^*(\eta)\big) \\ &= \Big(\int_c \omega\Big)\Big(\int_d \eta\Big) \\ &= \Big(\kappa \circ \Big(\int_c \otimes \int_d\Big)\Big)\big(\pi_M^*(\omega) \wedge \pi_N^*(\eta)\big) .\end{aligned}$$

To sum up, we can state the following version of the de Rham theorem.

THEOREM 2.4.1
*The functors of singular chains and currents*

$$S_*, \mathscr{D}'_* : \mathbf{Dif} \rightrightarrows \mathbf{C}_*(\mathbb{R})$$

*are weakly equivalent symmetric monoidal functors.*

*Remark 2.4.1*
A similar result can be obtained with the functor of *cubic chains* or with the functor of *oriented cubic chains* used by Kontsevich [K].

*2.5. Formality*

*2.5.1*

The notion of formality has attracted interest since Sullivan's work on rational homotopy theory in [Su]. In the operadic setting, the notion of formality appears in [Ma] and [K].

*Definition 2.5.1*

An operad $P$ in $\mathbf{C}_*(\mathscr{A})$ is said to be *formal* if it is weakly equivalent to its homology $HP$.

More generally, we can give the following definition.

*Definition 2.5.2*

Let $\mathscr{C}$ be a category endowed with an idempotent endofunctor $H : \mathscr{C} \longrightarrow \mathscr{C}$, and take as weak equivalences the morphisms $f : X \longrightarrow Y$ such that $H(f)$ is an isomorphism. An object $X$ of $\mathscr{C}$ is said to be *formal* if $X$ and $HX$ are weakly equivalent.

*2.5.2*

In particular, a functor $F : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathscr{A})$ is *formal* if it is weakly equivalent to its homology $HF : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathscr{A})$. However, we use this notion in the context of symmetric monoidal functors. So in this case, the following is the definition of formality.

*Definition 2.5.3*

Let $\mathscr{C}$ be a symmetric monoidal category, and let $F : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathscr{A})$ be a symmetric monoidal functor. It is said that $F$ is a *formal symmetric monoidal functor* if $F$ and $HF$ are weakly equivalent in the category of symmetric monoidal functors.

The properties below follow immediately from the definitions.

PROPOSITION 2.5.4

*Let $U : \mathscr{B} \longrightarrow \mathbf{C}_*(\mathscr{A})$ be a formal symmetric monoidal functor. For every symmetric monoidal functor $G : \mathscr{C} \longrightarrow \mathscr{B}$, the composition $U \circ G : \mathscr{B} \longrightarrow \mathbf{C}_*(\mathscr{A})$ is a formal symmetric monoidal functor.*

PROPOSITION 2.5.5

*Let $F : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathscr{A})$ be a functor. If $F$ is a formal symmetric monoidal functor, then*

$$F : \mathbf{Op}_{\mathscr{C}} \longrightarrow \mathbf{Op}_{\mathbf{C}_*(\mathscr{A})}$$

*sends operads in $\mathscr{C}$ to formal operads in $\mathbf{C}_*(\mathscr{A})$.*

*2.5.3*
Let $R$ be a commutative ring, and let $R$-**cdga** be the category of commutative differential graded $R$-algebras or, simply, cdg $R$-algebras. It is a symmetric monoidal category. Then if $\mathscr{C}$ is a symmetric monoidal category and $F : \mathscr{C} \longrightarrow \mathbf{C}^*(R)$ is a symmetric monoidal functor, it is a well-known fact that $F$ induces a functor from the category of commutative monoids of $(\mathscr{C}, \otimes, \mathbf{1})$ to cdg $R$-algebras.

Besides, if $F$ is a formal symmetric monoidal functor, then $F$ sends commutative monoids to formal cdg $R$-algebras.

If $\mathscr{C}$ is a category with finite products and a final object $\mathbf{1}$, then $(\mathscr{C}, \times, \mathbf{1})$ is a symmetric monoidal category. In this case, every object $X$ of $\mathscr{C}$ is a comonoid object with diagonal $X \to X \times X$ and unit $X \to \mathbf{1}$. So we have the following proposition.

PROPOSITION 2.5.6
*Let $\mathscr{C}$ be a category with finite products and a final object $\mathbf{1}$. Every formal symmetric monoidal contravariant functor $F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{C}^*(R)$ sends objects in $\mathscr{C}$ to formal cdg $R$-algebras.*

## 3. Hodge theory implies formality

In [DGMS], Deligne, Griffiths, Morgan, and Sullivan prove the formality of the de Rham cdg algebra of a compact Kähler manifold. In this section, we see how their result can be mimicked for the singular chain complex of an operad of compact Kähler manifolds.

### *3.1. Formality of the de Rham functor*

In [DGMS], the first of the proofs of formality relies on the *Hodge decomposition* for the complex of forms and the Kähler identities. From them, the *$dd^c$-lemma* is proved and the existence of a diagram of complexes called the *$d^c$-diagram*,

$$\big(\mathscr{E}^*(M), d\big) \longleftarrow \big({}^c\mathscr{E}^*(M), d\big) \longrightarrow \big(H^*_{d^c}(M), d\big),$$

is deduced. Here $M$ is a compact Kähler manifold, $\mathscr{E}^*(M)$ is the real de Rham complex of $M$, ${}^c\mathscr{E}^*(M)$ is the subcomplex of $d^c$-closed forms, and $H^*_{d^c}(M)$ is the quotient complex ${}^c\mathscr{E}^*(M)/d^c(\mathscr{E}^*(M))$. In the $d^c$-diagram, both maps are weak equivalences of chain complexes, and the differential induced by $d$ on $H^*_{d^c}(M)$ is zero. Since ${}^c\mathscr{E}^*$ and $H^*_{d^c}$ are symmetric monoidal functors and the morphisms of the $d^c$-diagram are natural and monoidal, the functor $\mathscr{E}^*$ is formal. So the theorem of formality can also be stated with the above definitions as follows.

THEOREM 3.1.1
*The functor of differential forms* $\mathscr{E}^* : \mathbf{K\ddot{a}h}^{\mathrm{op}} \longrightarrow \mathbf{C}^*(\mathbb{R})$ *is a formal symmetric monoidal functor.*

This result, together with Proposition 2.5.6, implies the formality theorem for the de Rham cdg algebra in its usual formulation: the de Rham functor $\mathscr{E}^* : \mathbf{K\ddot{a}h}^{\mathrm{op}} \longrightarrow \mathbb{R}$-**cdga** sends objects in **Käh** to formal cdg $\mathbb{R}$-algebras.

*3.2. Formality of the current complex functor*
We claim that an analogous theorem of formality is obtained by replacing forms with currents.

THEOREM 3.2.1
*The functor of currents* $\mathscr{D}'_* : \mathbf{K\ddot{a}h} \longrightarrow \mathbf{C}_*(\mathbb{R})$ *is a formal symmetric monoidal functor.*

*Proof*
Let $M$ be a compact Kähler manifold. It is a classical result of Hodge theory (see [S]) that the Kähler identities between the operators $d$, $d^c$, $\Delta, \ldots$ of the de Rham complex of differential forms are also satisfied by the corresponding dual operators on the de Rham complex of currents. Hence we have the following $dd^c$-lemma.

LEMMA 3.2.2
*Let* $T$ *be a* $d^c$*-closed and* $d$*-exact current. Then there exists a current* $S$ *such that* $T = dd^c S$.

From this lemma, we can follow verbatim the first proof of the formality theorem of [DGMS], and we obtain a $d^c$-diagram for currents

$$\big(\mathscr{D}'_*(M), d\big) \longleftarrow \big({}^c\mathscr{D}'_*(M), d\big) \longrightarrow \big(H^{d^c}_*(M), d\big).$$

Here ${}^c\mathscr{D}'_*(M)$ denotes the subcomplex of $\mathscr{D}'_*(M)$ defined by the $d^c$-closed currents, and $H^{d^c}_*(M)$ is the quotient ${}^c\mathscr{D}'_*(M)/d^c(\mathscr{D}'_*(M))$. In this $d^c$-diagram, both maps are weak equivalences, and the differential induced by $d$ on the latter is zero. So we have $H^{d^c}_*(M) \cong H_*(\mathscr{D}'_*(M))$.

Now since $d^c$ satisfies the Leibniz rule, ${}^c\mathscr{D}'_*$ is a symmetric monoidal subfunctor of $\mathscr{D}'_*$.

Finally, since the morphisms of the above $d^c$-diagram are natural and compatible with the Künneth morphism, it follows that $\mathscr{D}'_*$ is a formal symmetric monoidal functor. □

As a consequence of the formality of the current complex functor and the symmetric de Rham theorem for currents (Th. 2.4.1), the formality of the singular chain functor for compact Kähler manifolds follows.

COROLLARY 3.2.3
*The functor of singular chains* $S_* : \mathbf{K\ddot{a}h} \longrightarrow \mathbf{C}_*(\mathbb{R})$ *is a formal symmetric monoidal functor.*

*3.3. Formality of Kählerian operads*
From Proposition 2.5.5 and Corollary 3.2.3, we obtain the operadic version of the formality DGMS theorem.

THEOREM 3.3.1
*If* $X$ *is an operad in* **Käh**, *then the operad of singular chains* $S_*(X;\mathbb{R})$ *is formal.*

This result, together with Proposition 2.5.5 and the descent theorem (Th. 6.2.1), implies the formality of the operad of singular chains with rational coefficients for every operad of compact Kähler manifolds (see Cor. 6.3.1).

*3.4. Formality of Deligne-Mumford operads*
The above results can be easily generalized to the category of *Deligne-Mumford (DM) projective and smooth stacks* over $\mathbb{C}$, which we denote by $\mathbf{DM}(\mathbb{C})$.

Indeed, every stack of this kind defines a compact Kähler $V$-manifold; and for such $V$-manifolds, we have the functors of singular chains, $\mathscr{C}^\infty$-singular chains and currents, and also Hodge theory (see [B]). This allows us to obtain an analogous result to Corollary 3.2.3.

THEOREM 3.4.1
*The functor of singular chains* $S_* : \mathbf{DM}(\mathbb{C}) \longrightarrow \mathbf{C}_*(\mathbb{R})$ *is a formal symmetric monoidal functor.*

From Proposition 2.5.5, we have the following theorem.

THEOREM 3.4.2
*If* $X$ *is an operad in* $\mathbf{DM}(\mathbb{C})$, *then the operad of singular chains* $S_*(X;\mathbb{R})$ *is formal.*

## 4. Minimal operads

In this section, $\mathbf{k}$ denotes a field of characteristic zero, and an operad is an operad in the category of dg vector spaces over $\mathbf{k}$, $\mathbf{C}_*(\mathbf{k})$. The category of these operads is denoted simply by **Op**. It is a complete and cocomplete category ([GJ]; see also [H]).

*4.1. Some preliminaries*

Let us start by recalling some basic results on minimal operads ([Ma]; see [MSS]).

*4.1.1*

A *minimal operad* is an operad of the form $(\Gamma(V), d_M)$, where $\Gamma : \mathbf{\Sigma Mod} \longrightarrow \mathbf{Op}$ is the free operad functor, $V$ is a $\Sigma$-module with zero differential, $V(1) = 0$, and the differential $d_M$ is *decomposable*.

The free operad functor $\Gamma : \mathbf{\Sigma Mod} \longrightarrow \mathbf{Op}$ is a right adjoint functor for the forgetful functor $U : \mathbf{Op} \longrightarrow \mathbf{\Sigma Mod}$.

A *minimal model* of an operad $P$ is a minimal operad $P_\infty$ together with a weak equivalence $P_\infty \longrightarrow P$.

Let $P = (P(l))_{l \geq 1}$ be an operad. M. Markl [Ma] has proved that if $HP(1) = \mathbf{k}$, $P$ has a minimal model $P_\infty$ with $P_\infty(1) = \mathbf{k}$ (see also [MSS, Part II, Th. 3.125]).

As observed in [MSS, Part II, Rem. 1.62], the category of operads $P$ with $P(1) = \mathbf{k}$ is equivalent to the category of *pseudo-operads* $Q$ with $Q(1) = \mathbf{0}$, the zero dg vector space (see [MSS, Part II, Def. 1.16]).

In the sequel, we work only with pseudo-operads with $HP(1) = 0$, and we simply call them operads. We denote by $\mathbf{Op}$ the category of these operads and by

$$\circ_i : P(l) \otimes P(m) \longrightarrow P(l+m-1), \quad 1 \leq i \leq l,$$

their composition operations.

*4.2. Truncated operads*

We now introduce the arity truncation and its right and left adjoints, which enable us to introduce in the operadic setting functors analogous to the skeleton and coskeleton functors in simplicial set theory.

Here we establish the results for the arity truncation in a form that can be easily translated to modular operads in Section 8.

*4.2.1*

Let $E = (E(l))_{l \geq 1}$ be a $\Sigma$-module, and let $n \geq 1$ be an integer. The grading of $E$ induces a decreasing filtration $(E\,(\geq l))_{l \geq 1}$ by the sub-$\Sigma$-modules

$$E\,(\geq l) := \big(E(i)\big)_{i \geq l}\,.$$

Let $P$ be an operad. We denote by $P \cdot P(n)$ the sub-$\Sigma$-module consisting of elements $\alpha \circ_i \beta$ with $\alpha \in P(l)$ and $\beta \in P(m)$ such that $l = n$ or $m = n$. It follows from the definitions that

$$P \cdot P(n) \subset P\,(\geq n).$$

If, moreover, $P(1) = 0$, then

$$P \cdot P(n) \subset P\,(\geq n+1).$$

The first property implies that $P\,(\geq n)$ is an ideal of $P$, so the quotient $P/P\,(\geq n+1)$ is an operad, which is zero in arities greater than $n$. This is a so-called $n$-truncated operad. However, we find it more natural to give the following definition of an $n$-truncated operad.

*Definition 4.2.1*
An *$n$-truncated operad* is a finite sequence of objects in $\mathbf{C}_*(\mathbf{k})$,

$$P = \bigl(P(1), \ldots, P(n)\bigr),$$

with a right $\Sigma_l$-action on each $P(l)$, together with a family of composition operations, satisfying those axioms of composition operations in $\mathbf{Op}$ that make sense for truncated operads. A *morphism of $n$-truncated operads* $f : P \longrightarrow Q$ is a finite sequence of morphisms of $\Sigma_l$-modules $f(l) : P(l) \longrightarrow Q(l),\ 1 \leq l \leq n$, which commutes with composition operations.

Let $\mathbf{Op}\,(\leq n)$ denote the category of *$n$-truncated operads* of $\mathbf{C}_*(\mathbf{k})$.

A *weak equivalence* of $n$-truncated operads is a morphism of $n$-truncated operads $\phi : P \longrightarrow Q$, which induces isomorphisms of graded $\mathbf{k}$-vector spaces, $H\phi(l) : HP(l) \longrightarrow HQ(l)$ for $l = 1, \ldots, n$.

Given an operad $P$, $t_n P := (P(1), \ldots, P(n))$ defines a *truncation functor*

$$t_n : \mathbf{Op} \longrightarrow \mathbf{Op}\,(\leq n).$$

*4.2.2*
For an $n$-truncated operad $P$, denote by $t_* P$ the $\Sigma$-module that is $\mathbf{0}$ in arities greater than $n$ and coincides with $P$ in arities less than or equal to $n$. Since $P \cdot P(n) \subset P\,(\geq n)$, $t_* P$, together with the structural morphisms of $P$ trivially extended, is an operad, and the proposition below follows easily from the definitions.

PROPOSITION 4.2.2
*Let $n \geq 1$ be an integer. Then we have the following.*
(1) *The functor $t_* : \mathbf{Op}\,(\leq n) \longrightarrow \mathbf{Op}$ is a right adjoint for $t_n$.*
(2) *There exists a canonical isomorphism $t_n \circ t_* \cong \mathrm{Id}_{\mathbf{Op}(\leq n)}$.*
(3) *The functor $t_*$ is fully faithful.*
(4) *The functor $t_*$ preserves limits.*

(5) *For $m \geq n$, there exists a natural morphism*

$$\psi_{m,n} : t_* t_m \longrightarrow t_* t_n$$

*such that $\psi_{l,n} = \psi_{m,n} \circ \psi_{l,m}$ for $l \geq m \geq n$. For an operad $P$, the family $(t_* t_n P)_n$, with the morphisms $\psi_{m,n}$, is an inverse system of operads. The family of unit morphisms of the adjunctions*

$$\psi_n : P \longrightarrow t_* t_n P$$

*induces an isomorphism $\psi : P \longrightarrow \varprojlim t_* t_n P$.*

(6) *Let $P$, $Q$ be operads. If $n \geq 2$, $t_{n-1} P = 0$, and $Q \cong t_* t_n Q$, then*

$$\mathrm{Hom}_{\mathbf{Op}}(P, Q) \cong \mathrm{Hom}_{\Sigma_n}\big(P(n), Q(n)\big).$$

*4.2.3*

On the other hand, the functor $t_n$ also has a left adjoint. For an $n$-truncated operad $P$, denote by $t_! P$ the operad obtained by freely adding to $P$ the operations generated in arities greater than $n$; that is,

$$t_! P = \Gamma(U t_* P)/J,$$

where $J$ is the ideal in $\Gamma(U t_* P)$ generated by the kernel of $t_n \Gamma(U t_* P) \longrightarrow P$.

PROPOSITION 4.2.3

*Let $n \geq 1$ be an integer. Then we have the following.*

(1) *The functor $t_! : \mathbf{Op}\,(\leq n) \longrightarrow \mathbf{Op}$ is a left adjoint for $t_n$.*

(2) *There exists a canonical isomorphism $t_n \circ t_! \cong \mathrm{Id}_{\mathbf{Op}(\leq n)}$.*

(3) *The functor $t_!$ is fully faithful.*

(4) *The functor $t_!$ preserves colimits.*

(5) *For $m \leq n$, there exists a natural morphism*

$$\phi_{m,n} : t_! t_m \longrightarrow t_! t_n$$

*such that $\phi_{l,n} = \phi_{m,n} \circ \phi_{l,m}$ for $l \leq m \leq n$. For an operad $P$, the family $(t_! t_n P)_n$ with the morphisms $\phi_{m,n}$ is a directed system of operads. The family of unit morphisms of the adjunctions*

$$\phi_n : t_! t_n P \longrightarrow P$$

*induces an isomorphism $\phi : \varinjlim t_! t_n P \longrightarrow P$.*

(6) *Let $P$, $Q$ be operads. If $t_{n-1}Q = 0$, $P \cong t_! t_n P$, and $P(1) = 0$, then*

$$\mathrm{Hom}_{\mathbf{Op}}(P, Q) \cong \mathrm{Hom}_{\Sigma_n}\big(P(n), Q(n)\big).$$

*Proof*
Part (1) follows from the definition of $t_!$, and the remaining parts follow from (1) and Proposition 4.2.2. □

We call the direct system of operads given by

$$0 \to t_! t_1 P \to \cdots \to t_! t_{n-1} P \to t_! t_n P \to \cdots$$

the *canonical tower* of $P$.

As an easy consequence of the existence of right and left adjoint functors for $t_n$, we obtain the following result.

COROLLARY 4.2.4
*The truncation functors $t_n$ preserve limits and colimits. In particular, they commute with homology, send weak equivalences to weak equivalences, and preserve formality.*

### *4.3. Principal extensions*

Next, we recall the definition of a principal extension of operads and show that the canonical tower of a minimal operad is a sequence of principal extensions. This allows us to extend these notions to the truncated setting.

*4.3.1*
To begin, we establish some notation on suspension and mapping cones of complexes in an additive category.

If $A$ is a chain complex and $n$ is an integer, we denote by $A[n]$ the complex defined by $A[n]_i = A_{i-n}$ with the differential given by $d_{A[n]} = (-1)^n d_A$.

For a chain map $\eta : B \longrightarrow A$, we denote by $C\eta$, or by $A \oplus_\eta B[1]$, the mapping cone of $\eta$, that is to say, the complex that is given in degree $i$ by $(C\eta)_i = A_i \oplus B_{i-1}$ with the differential $d(a, b) = (d_A a + \eta b, -d_B b)$. Therefore $C\eta$ comes with a canonical chain map $i_A : A \longrightarrow C\eta$ and a canonical homogeneous map of graded objects $j_B : B[1] \longrightarrow C\eta$.

For a chain complex $X$, a chain map $\phi : C\eta \longrightarrow X$ is determined by the chain map $\phi i_A : A \longrightarrow X$ together with the homogeneous map $\phi j_B : B[1] \longrightarrow X$. Conversely, if $f : A \longrightarrow X$ is a chain map and $g : B[1] \longrightarrow X$ is a homogeneous map such that $f\eta = d_X g + g d_B$, that is, if $g$ is a homotopy between $f\eta$ and zero, then there exists a unique chain map $\phi : C\eta \longrightarrow X$ such that $\phi i_A = f$ and $\phi j_B = g$. In other words,

$C\eta$ represents the functor $h_\eta : \mathbf{C}_*(\mathbf{k}) \longrightarrow \mathbf{Sets}$ defined, for $X \in \mathbf{C}_*(\mathbf{k})$, by

$$h_\eta(X) = \big\{(f, g);\ f \in \mathrm{Hom}_{\mathbf{C}_*(\mathbf{k})}(A, X),\ g \in \mathrm{Hom}_{\mathbf{k}}(B, X)_1,\ d_X g + g d_B = f\eta\big\},$$

where $\mathrm{Hom}_{\mathbf{k}}(B, X)_1$ denotes the set of homogeneous maps of degree 1 of graded $\mathbf{k}$-vectorial spaces.

*4.3.2*

Recall the construction of standard cofibrations in the category $\mathbf{Op}$ (see [H]). Let $P$ be an operad, let $V$ be a dg $\Sigma$-module, and let $\xi : V[-1] \longrightarrow P$ be a chain map of dg $\Sigma$-modules. The standard cofibration associated with these data is an operad that represents the functor $h_\xi : \mathbf{Op} \longrightarrow \mathbf{Sets}$ defined for $Q \in \mathbf{Op}$ by

$$h_\xi(Q) = \big\{(f, g);\ f \in \mathrm{Hom}_{\mathbf{Op}}(P, Q),\ g \in \mathrm{Hom}_{\mathbf{Gr\Sigma Mod}}(V, UQ)_0,\ d_Q g - g d_V = f\xi\big\},$$

where $\mathrm{Hom}_{\mathbf{Gr\Sigma Mod}}(V, UQ)_0$ denotes the set of homogeneous maps of degree zero of graded $\Sigma$-modules. When $V$ has zero differential, this construction is called a *principal extension* and denoted by $P *_\xi \Gamma(V)$ in [MSS]. For reasons that become clear at once, we denote it by $P \sqcup_\xi V$. From the definition, it follows that $P \sqcup_\xi V$ comes with a canonical morphism of operads $i_P : P \longrightarrow P \sqcup_\xi V$ and a canonical homogeneous map of degree zero of graded $\Sigma$-modules $j_V : V \longrightarrow P \sqcup_\xi V$.

Now one can express $P \sqcup_\xi V$ as a pushout. Let $C(V[-1])$ be the mapping cone of $\mathrm{id}_{V[-1]}$, let $S(V) = \Gamma(V[-1])$, let $T(V) = \Gamma(C(V[-1]))$, let $i_V : S(V) \longrightarrow T(V)$ be the morphism of operads induced by the canonical chain map $i : V[-1] \longrightarrow C(V[-1])$, and let $\widetilde{\xi} : S(V) \longrightarrow P$ be the morphism of operads induced by $\xi$. Then $P \sqcup_\xi V$ is isomorphic to the pushout of the following diagram of operads:

$$\begin{array}{ccc} S(V) & \xrightarrow{\widetilde{\xi}} & P \\ {\scriptstyle i_V}\big\downarrow & & \\ T(V) & & \end{array}$$

If $V$ is concentrated in arity $n$ and its differential is zero, the operad $P \sqcup_\xi V$ is called an *arity $n$ principal extension*.

Let us explicitly describe it in the case when $n \geq 2$, $P(1) = 0$, and $P \cong t_! t_n P$. First of all, since for a truncated operad $Q \in \mathbf{Op}(\leq n-1)$ there is a chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}\big(t_{n-1}(P \sqcup_\xi V), Q\big) &\cong \mathrm{Hom}(P \sqcup_\xi V, t_* Q) \\ &\cong \mathrm{Hom}(P, t_* Q) \\ &\cong \mathrm{Hom}(t_{n-1} P, Q), \end{aligned}$$

we have $t_{n-1}(P \sqcup_\xi V) \cong t_{n-1}P$. Next let $X$ be the $n$-truncated operad extending $t_n P$ defined by

$$X(i) = \begin{cases} P(i) & \text{if } i < n, \\ P(n) \oplus_\xi V & \text{if } i = n, \end{cases}$$

the composition operations involving $V$ being trivial because $P(1) = 0$. Then it is clear that $X$ represents the functor $h_\xi$ restricted to the category $\mathbf{Op}\,(\leq n)$, so $t_n(P \sqcup_\xi V) \cong X$. Finally, it is easy to check that $t_! X$ satisfies the universal property of $P \sqcup_\xi V$. Summing up, we have proven the following proposition.

PROPOSITION 4.3.1
*Let $n \geq 2$ be an integer. Let $P$ be an operad such that $P(1) = 0$ and $t_! t_n P \cong P$, let $V$ be a dg $\Sigma$-module concentrated in arity $n$ with zero differential, and let $\xi : V[-1] \longrightarrow P(n)$ be a chain map of $\Sigma_n$-modules. The principal extension $P \sqcup_\xi V$ satisfies the following.*

(1) *The $(n-1)$-truncated operad $t_{n-1}(P \sqcup_\xi V)$ is canonically isomorphic to $t_{n-1}P$.*

(2) *The chain complex $(P \sqcup_\xi V)(n)$ is canonically isomorphic to $C(\xi)$. In particular, there exists an exact sequence of complexes*

$$0 \longrightarrow P(n) \longrightarrow (P \sqcup_\xi V)(n) \longrightarrow V \longrightarrow 0.$$

(3) *The operad $P \sqcup_\xi V$ is canonically isomorphic to $t_! t_n(P \sqcup_\xi V)$.*

(4) *A morphism of operads $\phi : P \sqcup_\xi V \longrightarrow Q$ is determined by a morphism of $n$-truncated operads $f : t_n P \longrightarrow t_n Q$ and a homogeneous map of $\Sigma_n$-modules $g : V \longrightarrow Q(n)$ such that $f\xi = dg$.*

These results extend trivially to truncated operads.

*4.4. Minimal objects*

Now we can translate the definition of minimality of operads of dg modules in terms of the canonical tower.

PROPOSITION 4.4.1
*An operad $M$ is minimal if and only if $M(1) = 0$ and the canonical tower of $M$,*

$$0 = t_! t_1 M \to \cdots \to t_! t_{n-1} M \to t_! t_n M \to \cdots,$$

*is a sequence of principal extensions.*

*Proof*
Let $M = (\Gamma(V), d_M)$ be a minimal operad. Then (see [MSS, Part II, (3.89)])

$$t_! t_n M \cong \big(\Gamma(V(\leq n)), \partial_n\big),$$

and $t_! t_n M$ is an arity $n$ principal extension of $t_! t_{n-1} M$ defined by $\partial_n : V(n) \longrightarrow (t_! t_{n-1} M)(n)$.

Conversely, let us suppose that $M(1) = 0$ and that $t_! t_{n-1} M \longrightarrow t_! t_n M$ is an arity $n$ principal extension defined by a $\Sigma$-module $V(n)$ concentrated in arity $n$ and zero differential for each $n$. Then $M = \Gamma\big(\bigoplus_{n\geq 2} V(n)\big)$ and its differential is decomposable because $M(1) = 0$. So $M$ is a minimal operad. $\square$

*4.4.1*
We now give the definition of minimality for truncated operads.

For $m \leq n$, we have an obvious truncation functor

$$t_m : \mathbf{Op}(\leq n) \longrightarrow \mathbf{Op}(\leq m),$$

which has a right adjoint $t_*$ and a left adjoint $t_!$.

*Definition 4.4.2*
An $n$-truncated operad $M$ is said to be *minimal* if $M(1) = 0$ and the canonical tower

$$0 = t_! t_1 M \longrightarrow t_! t_2 M \longrightarrow \cdots \longrightarrow t_! t_{n-1} M \longrightarrow M$$

is a sequence of ($n$-truncated) principal extensions.

An operad $M$ is said to be *$n$-minimal* if the truncation $t_n M$ is minimal.

It follows from the definitions that an operad $M$ is $n$-minimal if and only if $t_! t_n M$ is minimal. It is clear that an operad $M$ is minimal if and only if it is $n$-minimal for every $n$ and that [MSS, Part II, Ths. 3.120, 3.123, 3.125] remain true in $\mathbf{Op}(\leq n)$ merely by replacing "minimal" with "$n$-minimal."

*4.4.2*
The category $\mathbf{Op}$ has a natural structure of closed model category (see [H]). For our present purposes, we do not need all the model structure but only a small piece: the notion of homotopy between morphisms of operads and the fact that minimal operads are cofibrant objects in $\mathbf{Op}$. This can be developed independently, as in [MSS, Part II, Sec. 3.10]. From these results, the next one follows easily.

PROPOSITION 4.4.3
*Let $M$ be a minimal operad, and let $P$ be a suboperad. If the inclusion $P \hookrightarrow M$ is a weak equivalence, then $P = M$.*

*Proof*
Let us call $i : P \hookrightarrow M$ the inclusion. By [MSS, Part II, Th. 3.123], we can lift up to homotopy the identity of $M$ in the diagram

$$\begin{tikzcd} & P \arrow[d, "i"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

So we obtain a morphism of operads $f : M \longrightarrow P$ such that $if$ is homotopic to id. Hence $if$ is a weak equivalence, and by [MSS, Part II, Prop. 3.120], it is an isomorphism. Therefore $i$ is an isomorphism, too. □

*4.5. Automorphisms of a formal minimal operad*
For an operad $P$, let $\mathrm{Aut}(P)$ denote the group of its automorphisms. The following lifting property from automorphisms of the homology of the operad to automorphisms of the operad itself is the first part of the characterization of formality that we establish in Theorem 5.2.3.

PROPOSITION 4.5.1
*Let $M$ be a minimal operad. If $M$ is formal, then the map $H : \mathrm{Aut}(M) \longrightarrow \mathrm{Aut}(HM)$ is surjective.*

*Proof*
Because $M$ is a formal operad, we have a sequence of weak equivalences

$$M \longleftarrow X_1 \longrightarrow X_2 \longleftarrow \cdots \longrightarrow X_{n-1} \longleftarrow X_n \longrightarrow HM.$$

By the lifting property of minimal operads (see [MSS, Part II, Th. 3.123]), there exists a weak equivalence

$$\rho : M \longrightarrow HM.$$

Let $\phi \in \mathrm{Aut}(HM)$. Again by the lifting property of minimal operads, given the diagram

$$\begin{tikzcd} & M \arrow[d, "\rho"] \\ M \arrow[r, "(H\rho)\phi(H\rho)^{-1}\rho"] & HM \end{tikzcd}$$

there exists a morphism $f : M \longrightarrow M$ such that $\rho f$ is homotopic to $(H\rho)\phi(H\rho)^{-1}\rho$. Since homotopic maps induce the same morphism in homology, it turns out that $f$ is a weak equivalence, and by ([MSS, Part II, Th. 3.120]), it is also an isomorphism because $M$ is minimal. Finally, from $(H\rho)(Hf) = (H\rho)\phi(H\rho)^{-1}(H\rho)$, $Hf = \phi$ follows. □

It is clear that Proposition 4.5.1 remains true in $\mathbf{Op}\,(\leq n)$ merely by replacing "minimal" and "formal" with "$n$-minimal" and "$n$-formal," respectively.

*4.6. Finiteness of the minimal model*

In this section, we show that we can transfer the finiteness conditions of the homology of an operad to the finiteness conditions of its minimal model.

*Definition 4.6.1*

A $\Sigma$-module $V$ is said to be of *finite type* if, for every $l$, $V(l)$ is a finite-dimensional $\mathbf{k}$-vector space. An operad $P$ is said to be *of finite type* if the underlying $\Sigma$-module, $UP$, is of finite type.

*Example 4.6.2*

If $V$ is a $\Sigma$-module of finite type such that $V(1) = 0$, then the free operad $\Gamma(V)$ is of finite type because

$$\Gamma(V)(n) \cong \bigoplus_{T \in \mathscr{T}ree(n)} V(T),$$

where $\mathscr{T}ree(n)$ is the finite set of isomorphism classes of $n$-labeled reduced trees and $V(T) = \bigotimes_{v \in \mathrm{Vert}(T)} V(\mathrm{In}(v))$ for every $n$-labeled tree $T$ (see [MSS, Part II, Rem. 1.84]). In particular, if $P$ is an $n$-truncated operad of finite type, then $t_! P \cong \Gamma(P)/J$ (see Prop. 4.2.3) is of finite type as well.

THEOREM 4.6.3

*Let $P$ be an operad. If the homology of $P$ is of finite type, then every minimal model $P_\infty$ of $P$ is of finite type.*

*Proof*

Let $M$ be a minimal operad such that $HM$ is of finite type. Since

$$M(n) = (t_! t_n M)(n),$$

it suffices to check that $t_! t_n M$ is of finite type. We proceed by induction. The first step of the induction is trivial because $M(1) = 0$. Then $t_! t_n M$ is an arity $n$ principal extension of the operad $t_! t_{n-1} M$ by the vector space

$$V(n) = HC\big((t_! t_{n-1} M)(n) \to M(n)\big).$$

Thus, by the induction hypothesis, $t_n\,(t_! t_n M)$ is finite-dimensional. Therefore, by the previous example, $t_! t_n M = t_!\,(t_n t_! t_n M)$ is also of finite type. □

## 5. A criterion of formality

In this section, we establish a criterion of formality for operads which is an adaptation of the Sullivan criterion of formality for cdg algebras (see [Su, Th. 12.7]) and is based on the existence of a lifting of a grading automorphism of the cohomology.

Throughout this section, we denote by $\mathbf{k}$ a field of characteristic zero, and an operad is an operad in $\mathbf{C}_*(\mathbf{k})$. We fix $\alpha \in \mathbf{k}^*$ to not be a root of unity.

### *5.1. Lifting of the grading automorphism*

Let $V$ be a $\mathbf{k}$-vector space of finite dimension, and let $f$ be an endomorphism of $V$. If $q(t) \in \mathbf{k}[t]$ is an irreducible polynomial, we denote by $\ker q(f)^\infty$ the primary component corresponding to the irreducible polynomial $q(t)$, that is, the union of the subspaces $\ker q(f)^s$, $s \geq 1$. The space $V$ decomposes as a direct sum of primary components $V = \bigoplus \ker q(f)^\infty$, where $q(t)$ runs through the set of all irreducible factors of the minimal polynomial of $f$.

Denote $V^n = \ker(f - \alpha^n)^\infty$; then we have a decomposition

$$V = C \oplus \bigoplus_n V^n,$$

where $C$ is the sum of the primary components corresponding to the polynomials different from $t - \alpha^n$ for all $n \in \mathbb{N}$. This decomposition is called the *$\alpha$-weight decomposition* of $V$, and $V^n$ is the *component* of $\alpha$-weight $n$. The $\alpha$-weight decomposition is obviously functorial on the category of pairs $(V, f)$.

Let $P$ be a complex of $\mathbf{k}$-vector spaces. The *grading automorphism* $\phi_\alpha^P$ (or, simply, $\phi_\alpha$) of $HP$ is defined by $\phi_\alpha^P = \alpha^i \cdot \mathrm{id}_{HP_i}$ on $HP_i$ for all $i \in \mathbb{Z}$. A chain endomorphism $f$ of $P$ is said to be a *lifting* of the grading automorphism if $H(f) = \phi_\alpha^P$.

### *5.2. Formality criterion*

Denote by $\mathbf{C}_*^\alpha(\mathbf{k})$ the category of couples $(P, f)$, where $P$ is a finite-type complex and $f$ is a lifting of $\phi_\alpha$.

THEOREM 5.2.1

*$\mathbf{C}_*^\alpha(\mathbf{k})$ is a symmetric monoidal category, and the forgetful functor*

$$\mathbf{C}_*^\alpha(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k}), \qquad (P, f) \mapsto P$$

*is a formal symmetric monoidal functor.*

*Proof*
By the Künneth theorem and elementary linear algebra, if $f$ is a lifting of $\phi_\alpha^P$ and $g$ is a lifting of $\phi_\alpha^Q$, then the tensor product $f \otimes g$ is a lifting of $\phi_\alpha^{P\otimes Q}$. Then it is easy to check that, with the product defined by

$$(P, f) \otimes (Q, g) := (P \otimes Q, f \otimes g),$$

$\mathbf{C}_*^\alpha(\mathbf{k})$ has a structure of symmetric monoidal category such that the assignment

$$U : \mathbf{C}_*^\alpha(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k}), \qquad (P, f) \mapsto P$$

is a symmetric monoidal functor, and such that the functor of homology $H : \mathbf{C}_*^\alpha(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k})$ is a symmetric monoidal functor as well.

In order to prove that $U$ is a formal symmetric monoidal functor, we use the $\alpha$-weight decomposition to define a symmetric monoidal functor $T : \mathbf{C}_*^\alpha(\mathbf{k}) \longrightarrow \mathbf{C}_*(\mathbf{k})$ and two monoidal weak equivalences of monoidal functors

$$U \longleftarrow T \longrightarrow H \circ U = H.$$

Let $(P, f)$ be an object $\mathbf{C}_*^\alpha(\mathbf{k})$. Since each $P_i$ is finite-dimensional, $P_i$ has an $\alpha$-weight decomposition $P_i = C_i \oplus \bigoplus_n P_i^n$. Then the components of $\alpha$-weight $n$,

$$P^n := \bigoplus_i P_i^n,$$

form a subcomplex of $P$, and the same is true for $C := \bigoplus C_i$. So $P$ has an $\alpha$-weight decomposition as a direct sum of complexes

$$P = C \oplus \bigoplus_n P^n.$$

Taking homology, we obtain

$$HP = HC \oplus \bigoplus_n HP^n.$$

Obviously, this decomposition is exactly the $\alpha$-weight decomposition of $HP$ with respect to $Hf$. Since $Hf = \phi_\alpha^P$, we have $HC = 0$ and $H(P^n) = H_n(P)$ for all $n \in \mathbb{Z}$. Hence the inclusion $\bigoplus_n P^n \to P$ is a weak equivalence.

Next, for every $n \in \mathbb{Z}$, the homology of the complex $P^n$ is concentrated in degree $n$. So there is a natural way to define a weak equivalence between the complex $P^n$ and its homology $H(P^n)$. Let $\tau_{\geq n} P^n$ be the canonical truncation in degree $n$ of $P^n$:

$$\tau_{\geq n} P^n := Z_n P^n \oplus \bigoplus_{i>n} P_i^n .$$

This is a subcomplex of $P^n$, and the inclusion $\tau_{\geq n} P^n \to P^n$ is a weak equivalence. Since $\tau_{\geq n} P^n$ is nontrivial only in degrees greater than or equal to $n$ and its homology is concentrated in degree $n$, the canonical projection $\tau_{\geq n} P^n \to H(P^n)$ is a chain map, which is a weak equivalence.

Define $T$ by

$$TP := \bigoplus_n \tau_{\geq n} P^n.$$

Since $T$ is an additive functor, we have $TP = \bigoplus_n T(P^n)$. Moreover, the canonical projection $TP \longrightarrow HP$ is a weak equivalence.

Now we prove that $T$ is a symmetric monoidal subfunctor of $U$. Let $P$ and $Q$ be objects in $\mathbf{C}_*^\alpha(\mathbf{k})$. Since $T$ is additive and $\sum_{i+j=n} P^i \otimes Q^j \subset (P \otimes Q)^n$, it suffices to show that $T(P^i) \otimes T(Q^j) \subset T(P^i \otimes Q^j)$. By the Leibnitz rule, we have an inclusion in degree $i + j$:

$$Z_i P^i \otimes Z_j Q^j \subset Z_{i+j}(P^i \otimes Q^j).$$

In the other degrees, the inclusion is trivially true. Since $T$ is stable by products, it is a symmetric monoidal subfunctor of $U$.

Finally, the projection to homology $TP \to HP$ is well defined and obviously compatible with the Künneth morphism, so the canonical projection $T \to H$ is a monoidal natural transformation. Therefore $U$ is a formal symmetric monoidal functor. □

COROLLARY 5.2.2
*Let $P$ be an operad with homology of finite type. If $P$ has a lifting of $\phi_\alpha$ (with respect to some nonroot of unity $\alpha \in \mathbf{k}^*$), then $P$ is a formal operad.*

*Proof*
If $P_\infty \to P$ is a minimal model of $P$, then $P_\infty$ is an operad of finite type by Theorem 4.6.3. From the lifting property in [MSS, Part II, Th. 3.123], there exists an induced endomorphism $f$ on $P_\infty$ such that $Hf = \phi_\alpha$. Thus $(P_\infty, f)$ is an operad of $\mathbf{C}_*^\alpha(\mathbf{k})$, and the corollary follows from Theorem 5.2.1 and Proposition 2.5.5. □

THEOREM 5.2.3
*Let $\mathbf{k}$ be a field of characteristic zero, and let $P$ be an operad with homology of finite type. The following statements are equivalent.*

(1) *$P$ is formal.*

(2) *There exists a model $P'$ of $P$ such that $H : \mathrm{Aut}(P') \longrightarrow \mathrm{Aut}(HP)$ is surjective.*

(3) *There exists a model $P'$ of $P$ and $f \in \mathrm{Aut}(P')$ such that $H(f) = \phi_\alpha$ for some $\alpha \in \mathbf{k}^*$ that is not a root of unity.*

*Proof*
We have the fact that (1) $\Rightarrow$ (2) is Proposition 4.5.1. Since $\Sigma$-actions and compositions $\circ_i$ are homogeneous maps of degree zero, every grading automorphism $\phi_\alpha$ is an endomorphism of the operad $HP$, so (2) $\Rightarrow$ (3). Finally, (3) $\Rightarrow$ (1) is Corollary 5.2.2. $\square$

## 6. Descent of formality

In this section, $\mathbf{k}$ denotes a field of characteristic zero, and *operad* means an operad in the category $\mathbf{C}_*(\mathbf{k})$ unless another category is mentioned. Using the characterization of formality of Theorem 5.2.3, we now prove that the formality of an operad can be checked on an extension of the ground field $\mathbf{k}$.

### *6.1. Automorphism group of a finite-type operad*

*6.1.1*
Let $P$ be an operad. Restricting the automorphism, we have an inverse system of groups $(\mathrm{Aut}(t_n P))_n$ and a morphism of groups $\mathrm{Aut}(P) \longrightarrow \varprojlim \mathrm{Aut}(t_n P)$. Because $P \cong \varprojlim t_* t_n P$, the following lemma is clear.

LEMMA 6.1.1
*The morphisms of restriction induce a canonical isomorphism of groups*

$$\mathrm{Aut}(P) \longrightarrow \varprojlim \mathrm{Aut}(t_n P).$$

*6.1.2*
In order to prove that the group of automorphisms of a finite-type operad is an algebraic group, we start by fixing some notation about group schemes. Let $\mathbf{k} \longrightarrow R$ be a commutative $\mathbf{k}$-algebra. If $P$ is an operad, its extension of scalars $P \otimes_{\mathbf{k}} R$ is an operad in $\mathbf{C}_*(R)$, and the correspondence

$$R \mapsto \mathbf{Aut}(P)(R) = \mathrm{Aut}_R(P \otimes_{\mathbf{k}} R),$$

where $\mathrm{Aut}_R$ means the set of automorphisms of operads in $\mathbf{C}_*(R)$, defines a functor

$$\mathbf{Aut}(P) : \mathbf{k} - \mathbf{alg} \longrightarrow \mathbf{Gr}$$

from the category $\mathbf{k} - \mathbf{alg}$ of commutative $\mathbf{k}$-algebras to the category $\mathbf{Gr}$ of groups. It is clear that

$$\mathbf{Aut}(P)(\mathbf{k}) = \mathrm{Aut}(P).$$

We denote by $\mathbb{G}_m$ the *multiplicative group scheme* defined over the ground field $\mathbf{k}$.

PROPOSITION 6.1.2
*Let $P$ be a truncated operad. If $P$ is of finite type, then we have the following.*
(1) $\mathrm{Aut}(P)$ *is an algebraic matrix group over* $\mathbf{k}$.
(2) $\mathbf{Aut}(P)$ *is an algebraic affine group scheme over* $\mathbf{k}$ *represented by the algebraic matrix group* $\mathrm{Aut}(P)$.
(3) *Homology defines a morphism* $\mathbf{H} : \mathbf{Aut}(P) \longrightarrow \mathbf{Aut}(HP)$ *of algebraic affine group schemes.*

*Proof*
Let $P$ be a finite-type, $n$-truncated operad. The sum $M = \sum_{l \le n} \dim P(l)$ is finite; hence $\mathrm{Aut}(P)$ is the closed subgroup of $\mathbf{GL}_M(\mathbf{k})$ defined by the polynomial equations that express compatibility with the $\Sigma$-action, the differential, and the bilinear compositions $\circ_i$. Thus $\mathrm{Aut}(P)$ is an algebraic matrix group. Moreover, $\mathbf{Aut}(P)$ is obviously the algebraic affine group scheme represented by the matrix group $\mathrm{Aut}(P)$.

Next, for every commutative $\mathbf{k}$-algebra $R$, the map

$$\mathbf{Aut}(P)(R) = \mathrm{Aut}_R(P \otimes_{\mathbf{k}} R) \longrightarrow \mathrm{Aut}_R(HP \otimes_{\mathbf{k}} R) = \mathbf{Aut}(HP)(R)$$

is a morphism of groups and is natural in $R$; thus (3) follows. □

THEOREM 6.1.3
*Let $P$ be a finite-type truncated operad. If $P$ is minimal, then*

$$\mathbf{N} = \ker\big(\mathbf{H} : \mathbf{Aut}(P) \longrightarrow \mathbf{Aut}(HP)\big)$$

*is a unipotent algebraic affine group scheme over* $\mathbf{k}$.

*Proof*
Since $\mathbf{Aut}(P)$ and $\mathbf{Aut}(HP)$ are algebraic by Proposition 6.1.2 and $\mathbf{k}$ has zero characteristic, the kernel $\mathbf{N}$ is represented by an algebraic matrix group defined over $\mathbf{k}$ (see [Bo]). So it suffices to verify that all elements in $\mathbf{N}(\mathbf{k})$ are unipotent.

Given $f \in \mathbf{N}(\mathbf{k})$, let $P^1 = \ker(f - \mathrm{id})^\infty$ be the primary component of $P$ corresponding to the eigenvalue 1 (see Sec. 5.1). Then $P^1$ is a suboperad of $P$, and the inclusion $P^1 \hookrightarrow P$ is a weak equivalence. Since $P$ is minimal, it follows from Proposition 4.4.3 that $P = P^1$; thus $f$ is unipotent. □

*6.2. A descent theorem*

After these preliminaries, let us prove the descent theorem of the formality for operads. In rational homotopy theory, this corresponds to the descent theorem of formality for cdg algebras of Sullivan ([Su]) and Halperin and Stasheff ([HS]; see also [Mo] and [R]).

THEOREM 6.2.1

*Let $\mathbf{k}$ be a field of characteristic zero, and let $\mathbf{k} \subset \mathbf{K}$ be a field extension. If $P$ is an operad in $\mathbf{C}_*(\mathbf{k})$ with homology of finite type, then the following statements are equivalent.*

(1) *$P$ is formal.*

(2) *$P \otimes \mathbf{K}$ is a formal operad in $\mathbf{C}_*(\mathbf{K})$.*

(3) *For every $n$, $t_n P$ is formal.*

*Proof*

Because the statements of the theorem only depend on the homotopy type of the operad, we can assume $P$ to be minimal and, by Theorem 4.6.3, of finite type. Moreover, minimality of $P$ is equivalent to the minimality of all its truncations, $t_n P$.

Let us consider the following additional statement.

(2′) For every $n$, $t_n P \otimes \mathbf{K}$ is formal.

We prove the sequence of implications

$$(1) \Rightarrow (2) \Rightarrow (2') \Rightarrow (3) \Rightarrow (1),$$

where (1) implies (2) because $\_ \otimes_{\mathbf{k}} \mathbf{K}$ is an exact functor.

If $P \otimes \mathbf{K}$ is formal, then so are all of its truncations $t_n(P \otimes \mathbf{K}) \cong t_n P \otimes \mathbf{K}$ because truncation functors are exact. So (2) implies (2′).

Let us see that (2′) implies (3). From the implication (1) $\Rightarrow$ (2), already proven, it is clear that we may assume $\mathbf{K}$ to be algebraically closed. So let $\mathbf{K}$ be an algebraically closed field, let $n$ be an integer, and let $P$ be a finite-type minimal operad such that $t_n P \otimes \mathbf{K}$ is formal. Since

$$\mathbf{Aut}(t_n P)(\mathbf{K}) \longrightarrow \mathbf{Aut}(H t_n P)(\mathbf{K})$$

is a surjective map, by Theorem 5.2.3 it follows that

$$\mathbf{Aut}(t_n P) \longrightarrow \mathbf{Aut}(H t_n P)$$

is a quotient map. Thus, by [W, Sec. 18.1], we have an exact sequence of groups

$$1 \longrightarrow \mathbf{N}(\mathbf{k}) \longrightarrow \mathbf{Aut}(t_n P)(\mathbf{k}) \longrightarrow \mathbf{Aut}(H t_n P)(\mathbf{k}) \longrightarrow H^1(\mathbf{K}/\mathbf{k}, \mathbf{N}) \longrightarrow \cdots .$$

Since $\mathbf{N}$ is unipotent by Theorem 6.1.3 and $\mathbf{k}$ has zero characteristic, it follows that $H^1(\mathbf{K}/\mathbf{k}, \mathbf{N})$ is trivial (see [W, Exam. 18.2.e]). So we have an exact sequence of groups

$$1 \to \mathbf{N}(\mathbf{k}) \longrightarrow \mathrm{Aut}(t_n P) \longrightarrow \mathrm{Aut}(H t_n P) \longrightarrow 1.$$

In particular, $\mathrm{Aut}(t_n P) \longrightarrow \mathrm{Aut}(H t_n P)$ is surjective. Hence, again by Theorem 5.2.3, $t_n P$ is a formal operad.

Finally, let us see that (3) implies (1). By Theorem 5.2.3, it suffices to prove that all the grading automorphisms have a lift. Let $\phi : \mathbb{G}_m \longrightarrow \mathbf{Aut}(H t_n P)$ be the grading representation that sends $\alpha \in \mathbb{G}_m$ to the grading automorphism $\phi_\alpha$ defined in Section 5.1. For every $n$, form the pullback of algebraic affine group schemes

$$\begin{tikzcd} \mathbf{F}_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[d, "\phi"] \\ \mathbf{Aut}(t_n P) \arrow[r, "\mathbf{H}"] & \mathbf{Aut}(H t_n P) \end{tikzcd}$$

That is to say, for every commutative $\mathbf{k}$-algebra $R$,

$$\mathbf{F}_n(R) = \left\{(f, \alpha) \in \mathbf{Aut}(t_n P)(R) \times \mathbb{G}_m(R)\,;\, Hf = \phi_\alpha\right\}.$$

By Lemma 6.1.1, we have a commutative diagram

$$\begin{tikzcd} \varprojlim \mathbf{F}_n(\mathbf{k}) \arrow[r] \arrow[d] & \mathbb{G}_m(\mathbf{k}) \arrow[d, "\phi"] \\ \mathrm{Aut}(P) \cong \varprojlim \mathrm{Aut}(t_n P) \arrow[r, "H"] & \mathrm{Aut}(HP) \cong \varprojlim \mathrm{Aut}(H t_n P) \end{tikzcd}$$

So to lift grading automorphisms, it suffices to verify that the map $\varprojlim \mathbf{F}_n(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is surjective. In order to prove this surjectivity, we first replace the inverse system $(\mathbf{F}_n(\mathbf{k}))_n$ with an inverse system $(\mathbf{F}'_n(\mathbf{k}))_n$ whose transition maps are surjective. Indeed, for all $p \geq n$, the restriction $\varrho_{p,n} : \mathbf{F}_p \longrightarrow \mathbf{F}_n$ is a morphism of algebraic affine group schemes that are represented by algebraic matrix groups, so by [W, Sec. 15.1], it factors as a quotient map and a closed embedding:

$$\mathbf{F}_p \longrightarrow \operatorname{im} \varrho_{p,n} \longrightarrow \mathbf{F}_n.$$

Denote $\mathbf{F}'_n := \bigcap_{p \geq n} \operatorname{im} \varrho_{p,n}$. Since $\{\operatorname{im} \varrho_{p,n}\}_{p \geq n}$ is a descending chain of closed subschemes of the Noetherian scheme $\mathbf{F}_n$, there exists an integer $N(n) \geq n$ such that

$$\mathbf{F}'_n = \operatorname{im} \varrho_{N(n),n}.$$

Thus the restrictions $\varrho_{n+1,n}$ induce quotient maps $\varrho_{n+1,n} : \mathbf{F}'_{n+1} \longrightarrow \mathbf{F}'_n$. So, applying again [W, Sec. 18.1], we have an exact sequence of groups

$$1 \longrightarrow \mathbf{N}'(\mathbf{k}) \longrightarrow \mathbf{F}'_{n+1}(\mathbf{k}) \longrightarrow \mathbf{F}'_n(\mathbf{k}) \longrightarrow H^1(\bar{\mathbf{k}}/\mathbf{k}, \mathbf{N}) \longrightarrow \cdots .$$

Here $\mathbf{N}'(\mathbf{k})$ is a closed subscheme of $\mathbf{N}(\mathbf{k})$ because for every $(f, \alpha) \in \mathbf{N}'(\mathbf{k})$, we have $\alpha = 1$, and so $Hf = 1$ in $Ht^*_{\leq n+1}P$, which means that $f \in \mathbf{N}(\mathbf{k})$. By Theorem 6.1.3, $\mathbf{N}'(\mathbf{k})$ is unipotent; thus, as in the previous implication, it follows that $\mathbf{F}'_{n+1}(\mathbf{k}) \longrightarrow \mathbf{F}'_n(\mathbf{k})$ is surjective for all $n \geq 2$.

Since in the inverse system $(\mathbf{F}'_n(\mathbf{k}))_n$ all the transition maps are surjective, the map

$$\varprojlim \mathbf{F}'_p(\mathbf{k}) \longrightarrow \mathbf{F}'_2(\mathbf{k})$$

is surjective as well. Moreover, $\mathbf{F}'_2(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is also surjective. Indeed, given $\alpha \in \mathbb{G}_m(\mathbf{k})$, and since $t_{N(2)}P$ is formal by hypothesis, by Theorem 5.2.3 we can lift the grading automorphism $\phi_\alpha \in \mathrm{Aut}(Ht_{N(2)}P)$ to an automorphism $f \in \mathrm{Aut}(t_{N(2)}P)$. So we have an element $(f, \alpha) \in \mathbf{F}_{N(2)}(\mathbf{k})$ whose image in $\mathbf{F}_2(\mathbf{k})$ is an element of $\mathbf{F}'_2(\mathbf{k})$ which projects onto $\alpha$.

We conclude that $\varprojlim \mathbf{F}'_p(\mathbf{k}) \longrightarrow \mathbb{G}_m(\mathbf{k})$ is surjective; hence $P$ is formal. □

*6.3. Applications*

As an immediate consequence of Theorem 6.2.1, Theorems 3.3.1 and 3.4.2 of formality over $\mathbb{R}$ imply, respectively, the following corollaries.

COROLLARY 6.3.1

*If $X$ is an operad in* **Käh**, *then the operad of singular chains $S_*(X; \mathbb{Q})$ is formal.*

COROLLARY 6.3.2

*If $X$ is an operad in* $\mathbf{DM}(\mathbb{C})$, *then the operad of singular chains $S_*(X; \mathbb{Q})$ is formal.*

Finally, we can apply Theorem 6.2.1 to the formality of the little $k$-discs operad. Let $\mathscr{D}_k$ denote the *little $k$-discs* operad of Boardman and Vogt. It is the topological operad with $\mathscr{D}_k(1) = \mathbf{pt}$, and for $l \geq 2$, $\mathscr{D}_k(l)$ is the space of configurations of $l$ disjoint discs inside the unit disc of $\mathbb{R}^k$.

M. Kontsevich [K] proved that the operad of singular chains with real coefficients $S_*(\mathscr{D}_k; \mathbb{R})$ is formal. Therefore, from Theorem 6.2.1, we obtain the following corollary.

COROLLARY 6.3.3

*The operad of singular chains with rational coefficients of the little $k$-discs operad $S_*(\mathscr{D}_k; \mathbb{Q})$ is formal.*

## 7. Cyclic operads

*7.1. Basic results*

Let us recall some definitions from [GK1] (see also [MSS]). For all $l \in \mathbb{N}$, the group $\Sigma_l^+ := \mathrm{Aut}\{0, 1, \ldots, l\}$ contains $\Sigma_l$ as a subgroup, and it is generated by $\Sigma_l$ and the cyclic permutation of order $l + 1$, $\tau_l : (0, 1, \ldots, l) \mapsto (1, 2, \ldots, l, 0)$.

Let $\mathscr{C}$ be a symmetric monoidal category. A *cyclic* $\Sigma$-module $E$ in $\mathscr{C}$ is a sequence $(E(l))_{l \geq 1}$ of objects of $\mathscr{C}$ together with an action of $\Sigma_l^+$ on each $E(l)$. Let $\mathbf{\Sigma^+ Mod}$ denote the category of cyclic $\Sigma$-modules. Forgetting the action of the cyclic permutation, we have a functor

$$U^- : \mathbf{\Sigma^+ Mod} \longrightarrow \mathbf{\Sigma Mod}.$$

A *cyclic operad* is a cyclic $\Sigma$-module $P$ whose underlying $\Sigma$-module $U^- P$ has the structure of an operad compatible with the action of the cyclic permutation (see [GK1, Def. 2.1]).

Let $\mathbf{Op}_{\mathscr{C}}^+$ denote the category of cyclic operads. We also have an obvious forgetful functor

$$U^- : \mathbf{Op}_{\mathscr{C}}^+ \longrightarrow \mathbf{Op}_{\mathscr{C}}.$$

If $\mathbf{k}$ is a field of characteristic zero, we denote by $\mathbf{Op}^+$ the category of cyclic dg operads $\mathbf{Op}^+_{C_*(\mathbf{k})}$. There are obvious extensions of the notions of free operad, homology, weak equivalence, minimality, and formality for cyclic dg operads. The localization of $\mathbf{Op}^+$ with respect to weak equivalences is denoted by $\mathrm{Ho}\,\mathbf{Op}^+$. The results in the previous sections can be easily transferred to the cyclic setting. We do not give the details here because we return to this issue in Section 8 for modular operads. We mention only that every cyclic dg operad $P$ with $HP(1) = 0$ has a minimal model $P_\infty$ such that $U^-(P_\infty)$ is a minimal model of $U^-(P)$ and that the formality of a cyclic dg operad can be checked on an extension of the ground field.

Let $\mathscr{A}$ be an abelian category. It is clear that a formal symmetric monoidal functor $F : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathscr{A})$ induces a functor of cyclic operads

$$F : \mathbf{Op}_{\mathscr{C}}^+ \longrightarrow \mathbf{Op}^+_{\mathbf{C}_*(\mathscr{A})}$$

which sends cyclic operads in $\mathscr{C}$ to formal cyclic operads in $\mathbf{C}_*(\mathscr{A})$. From Corollary 3.2.3 and the cyclic version of Theorem 6.2.1, it follows that $S_*(X; \mathbb{Q})$ is a formal cyclic operad for every cyclic operad $X$ in **Käh**.

*7.2. Formality of the cyclic operad $S_*(\overline{\mathscr{M}}_0; \mathbb{Q})$*

Let us apply the previous results to the configuration operad.

Let $\mathscr{M}_{0,l}$ be the moduli space of $l$ different labeled points on the complex projective line $\mathbb{P}^1$. For $l \geq 3$, let $\overline{\mathscr{M}}_{0,l}$ denote its *Grothendieck-Knudsen compactification*, that is, the moduli space of stable curves of genus zero with $l$ different labeled points.

For $l = 1$, put $\overline{\mathscr{M}}_0(1) = *$, a point; and for $l \geq 2$, let $\overline{\mathscr{M}}_0(l) = \overline{\mathscr{M}}_{0,l+1}$. The family of spaces $\overline{\mathscr{M}}_0 = (\overline{\mathscr{M}}_0(l))_{l \geq 1}$ is a cyclic operad in $\mathbf{V}(\mathbb{C})$ (see [GK1]; see also [KoM] and [MSS]). Applying the functor of singular chains componentwise, we obtain a dg cyclic operad $S_*(\overline{\mathscr{M}}_0; \mathbb{Q})$. So we have the following result.

COROLLARY 7.2.1
*The cyclic operad of singular chains $S_*(\overline{\mathscr{M}}_0; \mathbb{Q})$ is formal.*

*7.3. Homotopy algebras over a cyclic operad*
If $P$ is a dg operad, a homotopy $P$-algebra is generally understood to be an algebra over a minimal, or cofibrant, model of $P$. Nevertheless, we propose a slightly different definition that does not presume the existence of cofibrant or minimal models.

Let $P$ be a cyclic dg operad. Recall that a $P$-algebra is a finite-type chain complex $V$ with an inner product $B$ together with a morphism of cyclic operads $\rho : P \longrightarrow \mathscr{E}[V]$, where $\mathscr{E}[V]$ is the endomorphism cyclic operad associated with $(V, B)$ (see [GK1] and [GK2]). So notice that if $P_\infty \longrightarrow P$ is a minimal model, a $P_\infty$-algebra structure on $V$, $\rho : P_\infty \longrightarrow \mathscr{E}[V]$, induces a morphism $\widetilde{\rho} : P \longrightarrow \mathscr{E}[V]$ in $\operatorname{Ho} \mathbf{Op}^+$. Conversely, if $\widetilde{\rho} : P \longrightarrow \mathscr{E}[V]$ is a morphism in $\operatorname{Ho} \mathbf{Op}^+$, by the lifting property of minimal operads there exists a lifting of $\widetilde{\rho}$ to a morphism $\rho : P_\infty \longrightarrow \mathscr{E}[V]$ in $\mathbf{Op}^+$, which is unique up to homotopy. This leads us to give the following definition.

*Definition 7.3.1*
A *homotopy $P$-algebra* is a finite-type chain complex $V$ with an inner product $B$ together with a morphism $P \longrightarrow \mathscr{E}[V]$ in $\operatorname{Ho} \mathbf{Op}^+$.

Let $V$ and $W$ be homotopy $P$-algebras. A *morphism* of homotopy $P$-algebras $V \longrightarrow W$ is a chain map $f : V \longrightarrow W$ compatible with the inner products and such that the diagram

$$\begin{tikzcd} & \mathscr{E}[V] \arrow[dd, "f_*"] \\ P \arrow[ur] \arrow[dr] & \\ & \mathscr{E}[W] \end{tikzcd}$$

commutes in $\operatorname{Ho} \mathbf{Op}^+$.

The *homotopy invariance property* is an immediate consequence of Definition 7.3.1.

PROPOSITION 7.3.2
*Let $V$ and $W$ be finite-type chain complexes with inner product, and let $f : V \longrightarrow W$ be a chain homotopy equivalence compatible with the inner product. If $W$ is a homotopy $P$-algebra, then $V$ has a unique homotopy $P$-algebra structure such that $f$ becomes a morphism of homotopy $P$-algebras.*

A morphism $\phi : P \longrightarrow Q$ of cyclic operads induces an inverse image functor

$$\phi^* : \{\text{homotopy } Q\text{-algebras}\} \longrightarrow \{\text{homotopy } P\text{-algebras}\}.$$

The following proposition is also an obvious consequence of the definitions.

PROPOSITION 7.3.3
*If $\phi : P \longrightarrow Q$ is a weak equivalence of dg cyclic operads, then the functor $\phi^*$ is an equivalence of categories.*

Therefore we obtain the following corollary.

COROLLARY 7.3.4
*The categories of homotopy $S_*(\overline{\mathscr{M}}_0;\mathbb{Q})$-algebras and homotopy $H_*(\overline{\mathscr{M}}_0;\mathbb{Q})$-algebras are equivalent.*

## 8. Modular operads

*8.1. Preliminaries*
Let us recall some definitions and notation about modular operads (see [GK2] or [MSS] for details).

*8.1.1*
Let $\mathscr{C}$ be a symmetric monoidal category. A *modular $\Sigma$-module* of $\mathscr{C}$ is a bigraded object of $\mathscr{C}$, $E = \big(E((g,l))\big)_{g,l}$ with $g, l \geq 0,\ 2g-2+l > 0$ such that $E((g,l))$ has a right $\Sigma_l$-action. Let us denote the category of modular $\Sigma$-modules by $\mathbf{MMod}_{\mathscr{C}}$, or just by **MMod** if no confusion can arise.

*8.1.2*
A *modular operad* is a modular $\Sigma$-module $P$ together with *composition* morphisms

$$\circ_i : P((g,l)) \otimes P((h,m)) \longrightarrow P((g+h, l+m-2)), \quad 1 \leq i \leq l,$$

and *contraction* morphisms

$$\xi_{ij} : P((g,l)) \longrightarrow P((g+1,l-2)), \quad 1 \le i \ne j \le l,$$

which verify axioms of associativity, commutativity, and compatibility (see [GK2], [MSS]). Let us denote the category of modular operads by $\mathbf{MOp}_{\mathscr{C}}$, or just by $\mathbf{MOp}$ if no confusion can arise.

As for operads and cyclic operads, a symmetric monoidal functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ applied componentwise induces a functor

$$\mathbf{MOp}_F : \mathbf{MOp}_{\mathscr{C}} \longrightarrow \mathbf{MOp}_{\mathscr{D}},$$

and every monoidal natural transformation $\phi : F \Rightarrow G$ between symmetric monoidal functors induces a natural transformation $\mathbf{MOp}_\phi : \mathbf{MOp}_F \Rightarrow \mathbf{MOp}_G$.

*Example 8.1.1*

As Getzler and Kapranov proved in [GK2], the family $\overline{\mathscr{M}}((g,l)) := \overline{\mathscr{M}}_{g,l}$ of Deligne-Knudsen-Mumford moduli spaces of stable genus $g$ algebraic curves with $l$ marked points, with the maps that identify marked points, is a modular operad in the category of projective smooth DM stacks.

*8.2. dg modular operads*

From now on, $\mathbf{k}$ denotes a field of characteristic zero, and modular operads in $\mathbf{C}_*(\mathbf{k})$ are called simply *dg modular operads*.

An *ideal* of a dg modular operad $P$ is a modular $\Sigma$-submodule $I$ of $P$ such that $P \cdot I \subset I$, $I \cdot P \subset I$, and $I$ is closed under the contractions $\xi_{ij}$.

For any dg modular operad $P$ and any ideal $I$ of $P$, the quotient $P/I$ inherits a natural dg modular operad structure, and the projection $P \longrightarrow P/I$ is a morphism of dg modular operads.

If $P$ is a dg modular operad, its *homology* $HP$, defined by $(HP)((g,l)) = H\big(P((g,l))\big)$, is also a dg modular operad. A morphism $\rho : P \longrightarrow Q$ of dg modular operads is said to be a *weak equivalence* if $\rho((g,l)) : P((g,l)) \longrightarrow Q((g,l))$ is a weak equivalence for all $(g,l)$.

The localization of $\mathbf{MOp}_{\mathbf{C}_*(\mathbf{k})}$ with respect to weak equivalences is denoted by $\operatorname{Ho}\mathbf{MOp}_{\mathbf{C}_*(\mathbf{k})}$.

*Definition 8.2.1*

A dg modular operad $P$ is said to be *formal* if $P$ is weakly equivalent to its homology $HP$.

Clearly, for a formal symmetric monoidal functor $F : \mathscr{C} \longrightarrow \mathbf{C}_*(\mathbf{k})$, the induced functor

$$F : \mathbf{MOp}_{\mathscr{C}} \longrightarrow \mathbf{MOp}_{\mathbf{C}_*(\mathbf{k})}$$

transforms modular operads in $\mathscr{C}$ to formal modular operads in $\mathbf{C}_*(\mathbf{k})$.

*8.3. Modular dimension*

In order to study the homotopy properties of dg modular operads, we replace the arity truncation with the truncation with respect to the modular dimension.

Let $\mathscr{C}$ be a symmetric monoidal category.

Recall that the dimension as algebraic variety of the moduli space $\overline{\mathscr{M}}_{g,l}$ is $3g-3+l$. So the following definition is a natural one. The function $d : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$, given by $d(g,l) = 3g-3+l$, is called the *modular dimension* function.

*Definition 8.3.1*

Let $E$ be a modular $\Sigma$-module in $\mathscr{C}$. The modular dimension function induces a graduation $(E_n)_{n\geq 0}$ on $E$ by $E_n = \big(E((g,l))\big)_{d(g,l)=n}$ and a decreasing filtration $(E_{\geq n})_n$ of $E$ by

$$E_{\geq n} := \big(E((g,l))\big)_{d(g,l)\geq n}.$$

The following properties are easily checked.

PROPOSITION 8.3.2

*Let $P$ be a modular operad in $\mathscr{C}$. The modular dimension grading satisfies*

$$P \cdot P_n \subset P_{\geq n+1},$$

*where $P \cdot P_n$ is the set of meaningful products $\alpha \circ_i \beta$ with $\alpha \in P_m$, $\beta \in P_l$ and such that $l = n$ or $m = n$. On the other hand, the contraction maps satisfy*

$$\xi_{ij} : P_n \longrightarrow P_{\geq n+1}$$

*for all $i, j$.*

*8.4. Truncation of modular operads*

*Definition 8.4.1*

An *$n$-truncated modular operad* in a symmetric monoidal category $\mathscr{C}$ is a modular operad defined only up to modular dimension $n$, that is, a family of $\mathscr{C}$, $\{P((g,l));\ g,l \geq 0,\ 2g-2+l > 0,\ d(g,l) \leq n\}$, such that $P((g,l))$ has a right $\Sigma_l$-action with morphisms

$$\circ_i : P((g,l)) \otimes P((h,m)) \longrightarrow P((g+h, l+m-2)), \quad 1 \leq i \leq l,$$

and *contractions*

$$\xi_{ij} : P((g,l)) \longrightarrow P((g+1, l-2)), \quad 1 \leq i \neq j \leq l,$$

satisfying those axioms in **MOp** that make sense.

If $\mathbf{MOp}_{\leq n}$ denotes the category of *n-truncated dg modular operads*, we have a *modular dimension truncation* functor

$$t_n : \mathbf{MOp} \longrightarrow \mathbf{MOp}_{\leq n}$$

defined by $t_n(P) = \big(P((g,l))\big)_{d(g,l)\leq n}$.

Since the obvious forgetful functor

$$U : \mathbf{MOp} \longrightarrow \mathbf{MMod}$$

has a left adjoint, the *free modular operad functor*

$$\mathbb{M} : \mathbf{MMod} \longrightarrow \mathbf{MOp}$$

(see [GK2, (2.18)]), by Proposition 8.3.2 we can translate the truncation formalism developed in Section 4.2 to the setting of dg modular operads. So we have a sequence of adjunctions $t_! \dashv t_n \dashv t_*$, and Propositions 4.2.2 and 4.2.3 are still true, merely replacing "operad" with "modular operad" and "arity" with "modular dimension" shifted by $+2$. For instance, the arity truncation begins with $t_2$, whereas the modular dimension truncation begins with $t_0$.

If $P$ is a dg modular operad, the direct system of dg modular operads given by

$$0 \longrightarrow t_!t_0P \to \cdots \to t_!t_{n-1}P \to t_!t_nP \to \cdots$$

is called the *canonical tower* of $P$.

*8.5. Principal extensions*

Let us explicitly describe the construction of a principal extension in the context of modular operads. Let $P$ be a dg modular operad, let $V$ be a dg modular $\Sigma$-module concentrated in modular dimension $n \geq 0$ with zero differential, and let $\xi : V[-1] \longrightarrow P_n$ be a chain map. Then the principal extension of $P$ by $\xi$, $P \sqcup_\xi V$, is defined by a universal property, as in Section 4.3.1:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{MOp}}(P \sqcup_\xi V, Q) = \big\{(f, g);\ & f \in \mathrm{Hom}_{\mathbf{MOp}}(P, Q),\\ & g \in \mathrm{Hom}_{\mathbf{GrMMod}}(V, UQ)_0,\ d_Q g - g d_V = f\xi\big\}.\end{aligned}$$

In particular, we have

$$(P \sqcup_\xi V)_i = \begin{cases} P_i & \text{if } i < n, \\ P_n \oplus_\xi V & \text{if } i = n, \end{cases}$$

because in $t_n(P \sqcup_\xi V)$, by Proposition 8.3.2, all the structural morphisms involving $V$ are trivial.

Furthermore, the following property, analogous to Proposition 4.3.1, is satisfied.

PROPOSITION 8.5.1

*Let $n \geq 0$ be an integer. Let $P$ be a dg modular operad such that $t_! t_n P \cong P$, let $V$ be a dg modular $\Sigma$-module concentrated in modular dimension $n$ with zero differential, and let $\xi : V[-1] \longrightarrow P_n$ be a morphism of dg modular $\Sigma_n$-modules. The principal extension $P \sqcup_\xi V$ satisfies the following.*

(1) *The $(n-1)$-truncated modular operad $t_{n-1}(P \sqcup_\xi V)$ is canonically isomorphic to $t_{n-1}P$.*

(2) *The chain complex $(P \sqcup_\xi V)_n$ is canonically isomorphic to $C\xi$. In particular, there exists an exact sequence of complexes*

$$0 \longrightarrow P_n \longrightarrow (P \sqcup_\xi V)_n \longrightarrow V \longrightarrow 0.$$

(3) *The modular operad $P \sqcup_\xi V$ is canonically isomorphic to $t_! t_n(P \sqcup_\xi V)$.*

(4) *A morphism of dg modular operads $\phi : P \sqcup_\xi V \longrightarrow Q$ is determined by a morphism of $n$-truncated dg modular operads $f : t_n P \longrightarrow t_n Q$ and a homogeneous map $g : V \longrightarrow Q_n$ of modular $\Sigma$-modules such that $f\xi = dg$.*

*8.6. Minimal models*

*8.6.1. Minimal objects*

*Definition 8.6.1*

A dg modular operad $M$ is said to be *minimal* if the canonical tower

$$0 \longrightarrow t_! t_0 M \longrightarrow \cdots \longrightarrow t_! t_{n-1} M \longrightarrow t_! t_n M \longrightarrow \cdots$$

is a sequence of principal extensions.

A *minimal model* of a dg modular operad $P$ is a minimal dg modular operad $P_\infty$ together with a weak equivalence $P_\infty \longrightarrow P$ in $\mathbf{MOp}$.

From Proposition 8.5.1, by induction on the modular dimension, we have the following proposition.

PROPOSITION 8.6.2
*Let $M$, $N$ be minimal dg modular operads. If $\rho : M \longrightarrow N$ is a weak equivalence of dg modular operads, then $\rho$ is an isomorphism.*

*8.6.2. Existence of minimal models*
THEOREM 8.6.3
*Let* $\mathbf{k}$ *be a field of characteristic zero. Every modular operad $P$ in* $\mathbf{C}_*(\mathbf{k})$ *has a minimal model.*

*Proof*
We start in modular dimension zero. Let $M^0 = \mathbb{M}HP_0$, and let $s : HP_0 \longrightarrow ZP_0$ be a section of the canonical projection. Then $s$ induces a morphism of modular operads

$$\rho^0 : M^0 \longrightarrow P$$

which is a weak equivalence of modular operads up to modular dimension zero because $M^0_0 = HP_0$.

For $n \geq 1$, assume that we have already constructed a morphism of modular operads

$$\rho^{n-1} : M^{n-1} \longrightarrow P$$

such that
(1) $M^{n-1} \cong t_! t_{n-1} M^{n-1}$ is a minimal modular operad and
(2) $t_{n-1}(\rho^{n-1})$ is a weak equivalence.
To define the next step of the induction, we use the following statement, which contains the main homological part of the inductive construction of minimal models.

LEMMA 8.6.4
*Let*

$$\begin{tikzcd} & B \arrow[r, "\eta"] \arrow[d, "\lambda"'] & A \arrow[d, "\mu"] & \\ & (C\zeta)[-1] \arrow[r, "-p_Y"] & Y \arrow[r, "\zeta"] & X \end{tikzcd}$$

*be a commutative diagram of complexes of an additive category; then there exists a chain map* $\nu : C\eta \longrightarrow X$ *such that in the diagram*

$$\begin{tikzcd} B \arrow[r, "\eta"] \arrow[d, "\lambda"'] & A \arrow[r] \arrow[d, "\mu"] & C\eta \arrow[r] \arrow[d, "\nu"] & B[1] \arrow[d, "\lambda[1]"] \\ (C\zeta)[-1] \arrow[r, "-p_Y"] & Y \arrow[r, "\zeta"] & X \arrow[r] & C\zeta \end{tikzcd} \tag{8.6.3.1}$$

*the central square is commutative, and the right-hand-side square is homotopy commutative. Moreover, the rows of (8.6.3.1) are distinguished triangles, and the vertical maps define a morphism of triangles in the derived category.*

We have $C\eta = A \oplus_\eta B[1]$ and $C\zeta = X \oplus_\zeta Y[1]$. Let $(\lambda_X, \lambda_Y)$ be the components of $\lambda$; then one can check that $\nu(a, b) = \lambda_X(b) + \zeta\,\mu(a)$ with the homotopy $h(a, b) = (0, \mu(a))$ satisfies the conditions of the statement.

The upper row of diagram (8.6.3.1) is obviously a distinguished triangle. By axiom $(TR2)$ of a triangulated category, turning the distinguished triangle

$$Y \xrightarrow{\zeta} X \longrightarrow C\zeta \xrightarrow{p_Y} Y[1]$$

one step to the left, we obtain the fact that the lower row of diagram (8.6.3.1) is also a distinguished triangle.

Now we return to the proof of Theorem 8.6.3. Since $\mathbf{k}$ is a field of zero characteristic, the category of modular $\Sigma$-modules is semisimple, and $C\rho_n^{n-1}$ is a formal complex of modular $\Sigma$-modules. Therefore, if $V = HC\rho_n^{n-1}$, there exists a weak equivalence

$$s : V \longrightarrow C\rho_n^{n-1}.$$

In fact, $s$ can be obtained from a $\Sigma$-equivariant section of the canonical projection from cycles to homology.

Let $\xi$ be the composition

$$V[-1] \xrightarrow{s[-1]} (C\rho_n^{n-1})[-1] \xrightarrow{-p} M_n^{n-1},$$

where the second arrow is the opposite of the canonical projection. We have a commutative diagram of complexes

$$\begin{tikzcd} V[-1] \arrow[r, "\xi"] \arrow[d, "{s[-1]}"'] & M_n^{n-1} \arrow[d, "\mathrm{id}"] & \\ \left(C\rho_n^{n-1}\right)[-1] \arrow[r, "-p"] & M_n^{n-1} \arrow[r, "\rho_n^{n-1}"] & P_n \end{tikzcd}$$

By Lemma 8.6.4, there exists a chain map

$$\nu : C\xi \longrightarrow P_n$$

which completes the previous diagram in a diagram

$$\begin{tikzcd} V[-1] \arrow[r, "\xi"] \arrow[d, "{s[-1]}"'] & M_n^{n-1} \arrow[r] \arrow[d, "\mathrm{id}"'] & C\xi \arrow[r] \arrow[d, "\nu"'] & V \arrow[d, "s"] \\ \left(C\rho_n^{n-1}\right)[-1] \arrow[r, "-p"] & M_n^{n-1} \arrow[r, "\rho_n^{n-1}"] & P_n \arrow[r] & C\rho_n^{n-1} \end{tikzcd} \tag{8.6.3.2}$$

where the rows are distinguished triangles in the category of complexes, the central square is commutative, and the vertical maps define a morphism of triangles in the derived category.

The step $M^n$ is defined as the principal extension of $M^{n-1}$ by the attachment map $\xi : V[-1] \longrightarrow M_n^{n-1}$; that is to say,

$$M^n := M^{n-1} \sqcup_\xi V \,.$$

Let $\nu_V : V \longrightarrow P$ be the graded map

$$V \longrightarrow C\xi \xrightarrow{\nu} P_n,$$

where the first map is the canonical inclusion. Since $\rho^{n-1}\xi = d\nu_V$, the maps $\rho^{n-1}$ and $\nu_V$ define, according to the universal property of $M^n = M^{n-1} \sqcup_\xi V$, a morphism of modular operads

$$\rho^n : M^n \longrightarrow P$$

such that $t_{n-1}\rho^n = t_{n-1}\rho^{n-1}$ and $\rho_n^n = \nu$. By the inductive hypothesis, $\rho^n$ is a weak equivalence in modular dimensions less than $n$. Finally, in diagram (8.6.3.2), $s$ is a weak equivalence; hence $\nu$ is a weak equivalence as well. It follows that $t_n\rho^n$ is a weak equivalence, which finishes the induction. Therefore $\varinjlim M^n$ is a minimal model of $P$. □

*8.6.4. Finiteness of minimal models*

*Definition 8.6.5*

A modular $\Sigma$-module $V$ is said to be *of finite type* if, for every $(g, l)$, $V((g, l))$ is a finite-dimensional $\mathbf{k}$-vector space. A dg modular operad $P$ is said to be of *finite type* if $UP$ is of finite type.

Obviously, for every integer $n \geq 0$, there are only a finite number of pairs $(g, l)$ such that $g, l, 2g - 2 + l > 0$ and $d(g, l) = n$; thus a modular $\Sigma$-module $V$ is of finite type if and only if $V_n$ is finite-dimensional for every $n \geq 0$.

PROPOSITION 8.6.6

*If $V$ is a modular $\Sigma$-module of finite type, then $\mathbb{M}(V)$ is of finite type.*

*Proof*

Indeed, for every pair $(g, l)$, there is an isomorphism

$$\mathbb{M}(V)((g, l)) \cong \bigoplus_{\gamma \in \{\mathbf{\Gamma}((g,l))\}} V((\gamma))_{\mathrm{Aut}(\gamma)},$$

where $\{\mathbf{\Gamma}((g, l))\}$ denotes the set of equivalence classes of isomorphisms of stable $l$-labeled graphs of genus $g$, the subscript $\mathrm{Aut}(\gamma)$ denotes the space of coinvariants, and

$$V\big((\gamma)\big) = \bigotimes_{v \in \mathrm{Vert}(\gamma)} V\big((g(v), \mathrm{Leg}\,(v))\big).$$

By [GK2, Lem. 2.16], the set $\{\mathbf{\Gamma}((g, l))\}$ is finite for every pair $(g, l)$. Therefore the free modular operad $\mathbb{M}(V)$ is of finite type. □

As a consequence of Proposition 8.6.6, we obtain the finiteness result analogous to Theorem 4.6.3.

THEOREM 8.6.7

*Let $P$ be a dg modular operad. If $HP$ is of finite type, then every minimal model of $P$ is of finite type.*

*8.7. Lifting properties*

Analogously to [MSS, Part II, Def. 3.121], there exists a similarly defined path object and a notion of homotopy in the category of dg modular operads.

*8.7.1. Homotopy*

Let $\mathbf{I} := \mathbf{k}[t, \delta t]$ be the differential graded commutative $\mathbf{k}$-algebra generated by a generator $t$ in degree zero and its differential $\delta t$ in degree $-1$. For every dg modular operad $P$, the path object of $P$ is the dg modular operad $P \otimes \mathbf{I}$, obtained by extension of scalars.

The evaluations at zero and one define two morphisms of modular operads $\rho_0, \rho_1 : P \otimes \mathbf{I} \rightrightarrows P$ which are weak equivalences. An *elementary homotopy* between two morphisms of dg modular operads $f_0, f_1 : P \rightrightarrows Q$ is a morphism $H : P \longrightarrow Q \otimes \mathbf{I}$ of dg modular operads such that $\rho_i H = f_i$ for $i = 0, 1$. Elementary homotopy is a reflexive and symmetric relation, and the *homotopy relation* between morphisms is the equivalence relation generated by elementary homotopy. Homotopic morphisms induce the same morphism in $\mathrm{Ho}\,\mathbf{MOp}$.

*8.7.2. Lifting properties of minimal objects*

Obstruction theory ([MSS, Part II, Lem. 3.139]) and the homotopy properties of the minimal objects ([MSS, Part II, Ths. 3.120, 3.123]) are easily established in the context of modular operads. So we have the following lemma.

LEMMA 8.7.1

*Let $\rho : Q \longrightarrow R$ be a weak equivalence of dg modular operads, and let $\iota : P \longrightarrow P \sqcup_\xi V$ be a principal extension. For every homotopy commutative diagram in* $\mathbf{MOp}$

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\rho} \\ P \sqcup_\xi V & \xrightarrow{\psi} & R \end{array}$$

*there exists an extension $\overline{\phi} : P \sqcup_\xi V \longrightarrow Q$ of $\phi$ such that $\rho\overline{\phi}$ is homotopic to $\psi$. Moreover, $\overline{\phi}$ is unique up to homotopy.*

From this lemma, the lifting property of minimal modular operads follows by induction.

THEOREM 8.7.2

*Let $\rho : Q \longrightarrow R$ be a weak equivalence of dg modular operads, and let $M$ be a minimal modular operad. For every morphism $\psi : M \longrightarrow R$, there exists a morphism $\widetilde{\psi} : M \longrightarrow Q$ such that $\rho\widetilde{\psi}$ is homotopic to $\psi$. Moreover, $\widetilde{\psi}$ is unique up to homotopy.*

*8.7.3. Uniqueness of minimal models*
From Theorem 8.7.2 and Proposition 8.6.2, we obtain the following theorem.

THEOREM 8.7.3
*Two minimal models of a modular operad are isomorphic.*

The modular analogue of Proposition 4.4.3 follows in the same way.

PROPOSITION 8.7.4
*Let $M$ be a minimal dg modular operad, and let $P$ be a subobject of $M$. If the inclusion $P \hookrightarrow M$ is a weak equivalence, then $P = M$.*

*8.8. Formality*
From Theorems 8.7.2 and 8.6.2, the modular analogue of Proposition 4.5.1 follows easily.

PROPOSITION 8.8.1
*Let $M$ be a minimal dg modular operad. If $M$ is formal, then the map $H : \mathrm{Aut}(M) \longrightarrow \mathrm{Aut}(HM)$ is surjective.*

Now from Theorem 5.2.1 and Proposition 8.8.1, the formality criterion for modular operads follows with the same proof as that of Theorem 5.2.3.

THEOREM 8.8.2
*Let $\mathbf{k}$ be a field of characteristic zero, and let $P$ be a dg modular operad with homology of finite type. The following statements are equivalent.*
(1) *$P$ is formal.*
(2) *There exists a model $P'$ of $P$ such that $H : \mathrm{Aut}(P') \longrightarrow \mathrm{Aut}(HP)$ is surjective.*
(3) *There exists a model $P'$ of $P$, and $f \in \mathrm{Aut}(P')$ such that $Hf = \phi_\alpha$ for some $\alpha \in \mathbf{k}^*$ that is not a root of unity.*

Then using this result, the descent of formality for modular operads follows as in Theorem 6.2.1.

THEOREM 8.8.3
*Let $\mathbf{k}$ be a field of characteristic zero, and let $\mathbf{k} \subset \mathbf{K}$ be a field extension. If $P$ is a modular operad in $\mathbf{C}_*(\mathbf{k})$ with homology of finite type, then $P$ is formal if and only if $P \otimes \mathbf{K}$ is a formal modular operad in $\mathbf{C}_*(\mathbf{K})$.*

Finally, the result below follows from Section 8.2 and Theorems 3.4.1 and 8.8.3.

THEOREM 8.8.4
*Let $X$ be a modular operad in* $\mathbf{DM}(\mathbb{C})$*. Then* $S_*(X;\mathbb{Q})$ *is a formal modular operad.*

*8.9. Application to moduli spaces*
Applying these results to the modular operad of moduli spaces $\overline{\mathscr{M}}$ and taking into account Theorem 8.8.4, we have the following corollary.

COROLLARY 8.9.1
$S_*(\overline{\mathscr{M}};\mathbb{Q})$ *is a formal modular operad.*

Now for any dg modular operad, we can define its homotopy algebras as we did above for cyclic operads.

So if $P$ is a dg modular operad, a *homotopy $P$-algebra* is a finite-type chain complex $V$ with an inner product $B$ together with a morphism $P \longrightarrow \mathscr{E}[V]$ in Ho **MOp**, where $\mathscr{E}[V]$ is the endomorphism modular operad associated with $(V, B)$ (see [GK2]). By Theorems 8.6.3 and 8.7.2, if $P_\infty \longrightarrow P$ is a minimal model of $P$, this is equivalent to giving a homotopy class of morphisms $P_\infty \longrightarrow \mathscr{E}[V]$. A *morphism* of homotopy $P$-algebras is a chain map $f : V \longrightarrow W$ compatible with the inner products and such that the induced morphism $f_* : \mathscr{E}[V] \longrightarrow \mathscr{E}[W]$ commutes with the structural morphisms.

The results corresponding to Propositions 7.3.2 and 7.3.3 are also easily established in the context of modular operads. In particular, we have the following proposition.

PROPOSITION 8.9.2
*If $\phi : P \longrightarrow Q$ is a weak equivalence of dg modular operads, then the functor*

$$\phi^* : \{\textit{homotopy } Q\textit{-algebras}\} \longrightarrow \{\textit{homotopy } P\textit{-algebras}\}$$

*is an equivalence of categories.*

And finally, we obtain the following corollary.

COROLLARY 8.9.3
*Every* $H_*(\overline{\mathscr{M}};\mathbb{Q})$*-algebra structure lifts to a homotopy* $S_*(\overline{\mathscr{M}};\mathbb{Q})$*-algebra structure.*

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