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Potential Theory of Signed Riesz Kernels: Capacity and Hausdorff Measure

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1 Introduction

There has recently been substantial progress in the problem of understanding the nature of analytic capacity (see [4, 11, 21]). Recall that the analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup \left| f'(\infty) \right|, \tag{1.1}$$

where the supremum is taken over those analytic functions on $\mathbb{C} \setminus E$ such that $|f(z)| \leq 1$, for $z \notin E$. It is easily shown that sets of zero analytic capacity are the removable sets for bounded analytic functions.

In [4], one proves Vitushkin's conjecture, namely, the statement that among compact sets of finite length (one-dimensional Hausdorff measure), the sets of zero analytic capacity are precisely those that project into sets of zero length in almost all directions. Equivalently, by Besicovitch theory, these are the purely unrectifiable sets, that is, the sets that intersect each rectifiable curve in zero length. In [11], the Cantor sets of vanishing analytic capacity are characterized, and in [21], the semiadditivity of analytic capacity is proven.

When dealing with analytic capacity, one very often finds oneself working with the Cauchy kernel $1/z$ and not using analyticity at all. Indeed, analytic capacity itself can

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easily be expressed without making any reference to analyticity in the form

$$\gamma(E) = \sup |\langle T, 1\rangle|, \tag{1.2}$$

where the supremum is taken over all complex distributions T supported on E such that the Cauchy potential of T, $f = 1/z * T$, is a function in $L^\infty(\mathbb{C})$ satisfying $\|f\|_\infty \leq 1$. Then, it seems interesting to try to isolate properties of analytic capacity that depend only on the basic characteristics of the Cauchy kernel such as oddness or homogeneity. With this purpose in mind, we start in this paper the study of certain real variable versions of analytic capacity related to the Riesz kernels in $\mathbb{R}^n$. Their definition is as follows. Given $0 < \alpha < n$ and a compact subset E of $\mathbb{R}^n$, set

$$\gamma_\alpha(E) = \sup |\langle T, 1\rangle|, \tag{1.3}$$

where the supremum is taken over all real distributions T supported on E such that, for $1 \leq i \leq n$, the ith α-Riesz potential $T * x_i/|x|^{1+\alpha}$ of T is a function in $L^\infty(\mathbb{R}^n)$ and $\sup_{1\leq i\leq n} \|T * x_i/|x|^{1+\alpha}\|_\infty \leq 1$. When $n = 2$ and $\alpha = 1$, writing $1/z = x/|z|^2 - i(y/|z|^2)$ with $z = x + iy$, we obtain $\gamma_1(E) \leq \gamma(E)$, for all compact sets E. According to Tolsa's Theorem [21], one has

$$\gamma(E) \leq C\gamma_+(E) \tag{1.4}$$

for all compact sets E, where $\gamma_+(E)$ is defined by the supremum in (1.2) where one now requires T to be a positive measure supported on E (with Cauchy potential bounded almost everywhere by 1 on $\mathbb{C}$). Thus, on compact subsets of the plane, γ and γ_1 are comparable in the sense that, for some positive constant C, one has

$$C^{-1}\gamma_1(E) \leq \gamma(E) \leq C\gamma_1(E). \tag{1.5}$$

Therefore, our set function γ_α can be viewed as a real variable version of analytic capacity associated to the vector-valued kernel $x/|x|^{1+\alpha}$. Of course, one can think of other possibilities; for example, one can associate in a similar fashion a capacity γ_Ω to a scalar kernel of the form $K(x) = \Omega(x)/|x|^\alpha$, where Ω is a real-valued smooth function on $\mathbb{R}^n$, homogeneous of degree zero. We will not pursue this issue here.

In Section 3, we compare the capacity γ_α to Hausdorff content. We obtain quantitative statements that, in particular, imply that if E has zero α-dimensional Hausdorff measure, then it has also zero γ_α capacity. In the other direction, one gets that if E has Hausdorff dimension larger than α, then γ_α is positive. Then, the critical situation occurs in dimension α, in accordance with the classical case.

The main contribution of this paper is the discovery of an interesting special behaviour of γ_α for noninteger indexes α. When α is an integer and E is a compact subset of an α-dimensional smooth surface, then one can see that $\gamma_\alpha(E) > 0$ provided that $\mathcal{H}^\alpha(E) > 0$, with $\mathcal{H}^\alpha$ being α-dimensional Hausdorff measure (see [14], where it is shown that if E lies on a Lipschitz graph, then $\gamma_{n-1}(E)$ is comparable to the $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}(E)$). In particular, there are sets of finite α-dimensional Hausdorff measure $\mathcal{H}^\alpha(E)$ and positive $\gamma_\alpha(E)$. It turns out that this cannot happen when $0 < \alpha < 1$.

Theorem 1.1. Let $0 < \alpha < 1$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$. Then, $\gamma_\alpha(E) = 0$. □

Notice that the analogue of the above result in the limiting case $\alpha = 1$ is the difficult part of Vitushkin's conjecture: if E is a purely unrectifiable planar compact set of finite length, then $\gamma(E) = 0$. We do not know how to prove Theorem 1.1 for a noninteger $\alpha > 1$. Even for an integer $\alpha > 1$, we do not know if the natural analogue of Vitushkin's conjecture is true. However, we do have a result in the Ahlfors-David regular case. Recall that a closed subset E of $\mathbb{R}^n$ is said to be Ahlfors-David regular of dimension d if it has locally finite and positive d-dimensional Hausdorff measure in a uniform way:

$$C^{-1}r^d \le \mathcal{H}^d\big(E \cap B(x,r)\big) \le Cr^d, \quad \text{for } x \in E,\ r \le d(E), \tag{1.6}$$

where $B(x,r)$ is the open ball centered at x of radius r and $d(E)$ is the diameter of E. Notice that if E is a compact Ahlfors-David regular set of dimension d, then $\mathcal{H}^d(E) < \infty$.

Theorem 1.2. Let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of noninteger dimension α, $0 < \alpha < n$. Then, $\gamma_\alpha(E) = 0$. □

In proving Theorem 1.1, we use a deep recent result of Nazarov, Treil, and Volberg [18] on the L^2-boundedness of singular integrals with respect to very general measures (see Section 2 for a statement). As a technical tool, we also need a variant of the well-known symmetrization method relating Menger curvature (see Section 2 for a definition) and the Cauchy kernel (see [13, 15, 16]). Symmetrization of the kernel $x/|x|^{1+\alpha}$ leads to a nonnegative quantity, only for $0 < \alpha \le 1$. For $\alpha = 1$, this is Menger curvature and, for $0 < \alpha < 1$, a description can be found in Lemma 4.2. However, nonnegativity and homogeneity seem to be more relevant facts than having exact expressions for the symmetrized quantity. The lack of nonnegativity, for $\alpha > 1$, is the reason that explains the restriction on α in Theorem 1.1.

The proof of Theorem 1.2 follows the line of reasoning of a well-known result of Christ [3] stating that if an Ahlfors-David regular set E of dimension one in the plane has positive analytic capacity, then the Cauchy integral operator is bounded in $L^2(F, \mathcal{H}^1)$, where F is another Ahlfors-David regular set such that $\mathcal{H}^1(E \cap F) > 0$. The main difficulty for us lies in the fact that if α is noninteger, then, according to a result of Vihtilä [24], there are no Ahlfors-David regular sets E on which the α-dimensional Riesz operator is bounded in the space $L^2(E, \mathcal{H}^\alpha)$. This prevents us from directly adapting Christ's arguments.

Throughout the paper, the letter C will stand for an absolute constant that may change at different occurrences.

If $A(X)$ and $B(X)$ are two quantities depending on the same variable (or variables) X, we will say that $A(X) \approx B(X)$ if there exists $C \geq 1$ independent of X such that $C^{-1}A(X) \leq B(X) \leq CB(X)$, for every X.

In Section 2, one can find statements of some auxiliary results and the basic notation and terminology that will be used throughout the paper. As we have already mentioned above, in Section 3, we compare γ_α to Hausdorff content. Theorem 1.1 is proven in Section 4 and Theorem 1.2 in Section 5.

2 L^2-boundedness of singular integral operators

A function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$ is called a Calderón-Zygmund kernel if the following holds:

(1) $|K(x, y)| \leq C|x - y|^{-\alpha}$, for some $0 < \alpha < n$ (with α not necessarily integer) and some positive constant $C < \infty$,

(2) there exists $0 < \epsilon \leq 1$ such that, for some constant $0 < C < \infty$, if $|x - x_0| \leq |x - y|/2$,

$$\left|K(x, y) - K\left(x_0, y\right)\right| + \left|K(y, x) - K\left(y, x_0\right)\right| \leq C\frac{\left|x - x_0\right|^\epsilon}{|x - y|^{\alpha+\epsilon}}. \tag{2.1}$$

Let μ be a Radon measure on $\mathbb{R}^n$. Then, the Calderón-Zygmund operator T associated to the kernel K and the measure μ is formally defined as

$$Tf(x) = T(f\mu)(x) = \int K(x, y)f(y)d\mu(y). \tag{2.2}$$

This integral may not converge for many functions f because for $x = y$ the kernel K may have a singularity. For this reason, we introduce the truncated operators T_ϵ, $\epsilon > 0$:

$$T_\epsilon f(x) = T_\epsilon(f\mu)(x) = \int_{|x-y|>\epsilon} K(x, y)f(y)d\mu(y). \tag{2.3}$$

We say that the singular integral operator T is bounded in $L^2(\mu)$ if the operators T_ϵ are bounded in $L^2(\mu)$ uniformly in ϵ.

The maximal operator T^* is defined as

$$T^*f(x) = \sup_{\epsilon>0} \left|T_\epsilon f(x)\right|. \tag{2.4}$$

Let $0 < \alpha < n$ and consider the Calderón-Zygmund operator R_α associated to the antisymmetric vector-valued Riesz kernel $x/|x|^{1+\alpha}$.

For the proof of Theorem 1.1, a deep result of Nazarov, Treil, and Volberg will be needed. First, we introduce some more notation. We say that $B(x,r)$ is a non-Ahlfors disk with respect to some constant $M > 0$ if $\mu(B(x,r)) > Mr$. Let b be a bounded function. We say that a disk $B(x,r)$ is nonaccretive with respect to b if, for some fixed positive constant ϵ, we have $|\int_{B(x,r)} b\, d\mu| < \epsilon\mu(B(x,r))$.

Let ϕ be some nonnegative Lipschitz function with Lipschitz constant 1 and consider the antisymmetric Calderón-Zygmund operator K_ϕ associated to the suppressed kernel k_ϕ:

$$k_\phi(x,y) = \frac{\overline{x-y}}{|x-y|^2 + \phi(x)\phi(y)}. \tag{2.5}$$

The kernel k_ϕ has the very important property of being well suppressed (we are borrowing the terminology from [18]) at the points where $\phi > 0$, that is,

$$\left|k_\phi(x,y)\right| \le \frac{1}{\max\left\{\phi(x), \phi(y)\right\}}. \tag{2.6}$$

We will state now a $T(b)$ theorem of [18] for the Cauchy kernel.

Theorem 2.1. Let μ be a positive Radon measure on $\mathbb{C}$ with $\limsup_{r\to 0} \mu(B(x,r))/r < \infty$, for μ-almost all x, and b an $L^\infty(\mu)$ function with $|\int_{\mathbb{C}} b\, d\mu| = \gamma$. Let $M > 0$, $B > 0$, an open set $H \subset \mathbb{C}$ with $\mu(H^c) > 0$, and $\phi : \mathbb{C} \to [0,\infty)$ a Lipschitz function with constant 1 such that

(1) every non-Ahlfors disk and every nonaccretive disk are contained in H,
(2) $\phi(x) \ge \operatorname{dist}(x, H^c)$,
(3) $K_\theta^* b(x) \le B$, for μ-almost all x and for every Lipschitz function θ with constant 1 such that $\theta \ge \phi$.

Then, K_ϕ is bounded in $L^2(\mu)$. In particular, if $F = \{x : \phi(x) = 0\}$, the Cauchy transform is bounded in $L^2(\mu_{|F})$. □

One can use this result to give an alternative proof of Vitushkin's conjecture (see [18]).

To use their result for the α-Riesz transform R_α, $0 < \alpha < n$, we need an appropriate version of the suppressed kernels associated to the Riesz α-operator R_α. We have found that the following kernel does the job:

$$k_{\phi,\alpha}(x,y) = \frac{x-y}{|x-y|^{1+\alpha}} \left(\frac{|x-y|^2}{|x-y|^2 + \phi(x)\phi(y)} \right)^N, \tag{2.7}$$

where $N = \min\{m \in \mathbb{N} : \alpha \leq m\}$. That is, $N = \alpha$ if $\alpha \in \mathbb{N}$ and $N = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, where $[\alpha]$ denotes the integer part of α. Notice that $k_{\phi,1} = k_\phi$.

For the sake of completeness, we state the properties of the kernel $k_{\phi,\alpha}$ in a separate lemma.

Lemma 2.2. The kernel $k_{\phi,\alpha}(x,y)$ is an antisymmetric Calderón-Zygmund kernel and is also well suppressed at the points where $\phi > 0$, that is,

$$\left|k_{\phi,\alpha}(x,y)\right| \leq \frac{1}{\max\left\{\phi(x)^\alpha, \phi(y)^\alpha\right\}}. \tag{2.8}$$

□

Proof. It is easy to prove that this suppressed kernel satisfies $k_{\phi,\alpha}(x,y) = -k_{\phi,\alpha}(y,x)$ and $|k_{\phi,\alpha}(x,y)| \leq |x-y|^{-\alpha}$. We show now that $|k_{\phi,\alpha}(x,y)| \leq 1/\phi(x)^\alpha$, for all x, y. Observe first that $\phi(y) \geq \phi(x) - |x-y|$, which implies that

$$\begin{aligned} \left|k_{\phi,\alpha}(x,y)\right| &\leq \frac{1}{|x-y|^\alpha} \left(\frac{|x-y|^2}{|x-y|^2 + \phi(x)\big(\phi(x) - |x-y|\big)} \right)^N \\ &= \frac{1}{|x-y|^\alpha} \left(\frac{|x-y|^2}{|x-y|^2 + \phi(x)\big(\phi(x) - |x-y|\big)} \right)^{N-\alpha} \\ &\quad \times \left(\frac{|x-y|^2}{|x-y|^2 + \phi(x)\big(\phi(x) - |x-y|\big)} \right)^\alpha \\ &\leq \frac{1}{|x-y|^\alpha} \left(\frac{|x-y|^2}{|x-y|^2 + \phi(x)\big(\phi(x) - |x-y|\big)} \right)^\alpha \\ &= \frac{1}{|x-y|^\alpha} \left(\frac{|x-y|^2}{\phi(x)|x-y| + \big(\phi(x) - |x-y|\big)^2} \right)^\alpha \\ &\leq \frac{1}{|x-y|^\alpha} \left(\frac{|x-y|^2}{\phi(x)|x-y|} \right)^\alpha \\ &= \frac{1}{\phi(x)^\alpha}. \end{aligned} \tag{2.9}$$

Now, we only need to show that

$$\left|\nabla_x k_{\phi,\alpha}(x,y)\right| \leq \frac{4N+\alpha+3}{|x-y|^{1+\alpha}}. \tag{2.10}$$

Set $P_\phi(x,y) = |x-y|^2/(|x-y|^2+\phi(x)\phi(y))$ and write $\nabla_x k_{\phi,\alpha}(x,y) = A + B$, with

$$|A| = \left|P_\phi(x,y)\right|^N \left|\frac{|x-y|^{1+\alpha} - (1+\alpha)|x-y|^\alpha (x-y)}{|x-y|^{2(1+\alpha)}}\right| \leq \frac{\alpha+2}{|x-y|^{1+\alpha}} \tag{2.11}$$

and

$$\begin{aligned} |B| &= N\left|P_\phi(x,y)\right|^{N-1} \\ &\quad \times \frac{\left|2(x-y)\left(|x-y|^2+\phi(x)\phi(y)\right) - |x-y|^2\left(2(x-y)+\phi'(x)\phi(y)\right)\right|}{\left(|x-y|^2+\phi(x)\phi(y)\right)^2 |x-y|^\alpha} \\ &\leq N\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)\phi(y)}\right)^N \\ &\quad \times \frac{2\left(|x-y|^2+\phi(x)\phi(y)\right) + |x-y|\left(2|x-y|+\phi'(x)\phi(y)\right)}{|x-y|^{1+\alpha}\left(|x-y|^2+\phi(x)\phi(y)\right)} \\ &\leq N\left(\frac{|x-y|^2}{|x-y|^2+\phi(x)\phi(y)}\right)^N \frac{4|x-y|^2+2\phi(x)\phi(y)+\phi(y)|x-y|}{|x-y|^{1+\alpha}\left(|x-y|^2+\phi(x)\phi(y)\right)} \\ &\leq \frac{4N}{|x-y|^{1+\alpha}} + \frac{\phi(y)}{|x-y|^{1+\alpha}}\left|k_\phi(x,y)\right| \\ &\leq \frac{4N+1}{|x-y|^{1+\alpha}}, \end{aligned} \tag{2.12}$$

where one uses (2.6) in the last inequality. ■

Using this operators and adapting Theorem 2.1, one obtains the following result for the α-Riesz transform R_α.

Theorem 2.3. Let μ be a positive measure on $\mathbb{R}^n$ such that $\limsup_{r\to 0} \mu(B(x,r))/r^\alpha < +\infty$, for μ-almost all x, and b an $L^\infty(\mu)$ function such that $|\int b\, d\mu| = \gamma_\alpha$. Assume that $R^*_\alpha b(x) < +\infty$ for μ-almost all x. Then, there is a set F with $\mu(F) \geq \gamma_\alpha/4$ such that the α-Riesz potential R_α is bounded in $L^2(\mu_{|F})$. □

Remark 2.4. The set F in Theorem 2.3 corresponds to $\mathbb{C} \setminus H$. Namely, F is the set where there are no problems (every disk is Ahlfors and accretive and the maximal operator is uniformly bounded).

Remark 2.5 (Volberg, personal communication). Instead of using the Calderón-Zygmund operator related to the suppressed kernel defined in (2.7), one can also use the operator related to the following suppressed kernel:

$$k_{\phi,\alpha}(x,y) = \frac{k_\alpha(x,y)}{1+k_\alpha^2(x,y)\phi^\alpha(x)\phi^\alpha(y)}, \tag{2.13}$$

with $k_\alpha(x,y) = (x-y)/|x-y|^{1+\alpha}$.

For the proof of Theorem 1.2, we need to define some sets Q_β^k that will be the analogues of the Euclidean dyadic cubes. These "dyadic cubes" were introduced by Christ in [3].

Let $E \subset \mathbb{R}^n$ be an Ahlfors-David regular compact set with $\mathcal{H}^\alpha(E) < \infty$. Let $\mu = \mathcal{H}^\alpha_{|E}$ and let ρ be the Euclidean metric. Then, (E,ρ,μ) is a space of homogeneous type, that is, (E,ρ) is a metric space and μ is a doubling measure, that is, $\mu(B(x,2r)) \le C\mu(B(x,r))$ (see [3]).

Theorem 2.6 [3]**.** For a space of homogeneous type (E,ρ,μ) with μ as above, there exists a collection of Borel sets $\mathcal{Q}(E) = \{Q_\beta^k \subset E : k \in \mathbb{N},\ \beta \in \mathbb{N}\}$ and positive numbers $\delta \in (0,1)$, a_1, b_1, and η such that

(1) $\mu(E \setminus \bigcup_\beta Q_\beta^k) = 0$, for each k,
(2) if $l \ge k$, then either $Q_\gamma^l \subset Q_\beta^k$ or $Q_\gamma^l \cap Q_\beta^k = \varnothing$,
(3) for each (k,β) and each $l < k$, there is a unique γ such that $Q_\beta^k \subset Q_\gamma^l$,
(4) $d(Q_\beta^k) \le \delta^k$, where $d(Q_\beta^k)$ denotes the diameter of the cube Q_β^k,
(5) each Q_β^k contains some ball $B(Q_\beta^k) = E \cap B(z_\beta^k, a_1\delta^k)$,
(6) each cube Q_β^k has a "small boundary," that is, $\mu\{x \in Q_\beta^k : \rho(x, E\setminus Q_\beta^k) \le t\delta^k\} \le b_1 t^\eta \mu(Q_\beta^k)$, for every k, β and for every $t > 0$. □

We denote by $\mathcal{Q}^k(E) = \{Q_\beta^k \in \mathcal{Q}(E) : \beta \in \mathbb{N}\}$, $k \in \mathbb{N}$, the cubes of generation k in $\mathcal{Q}(E)$.

For the variant of the $T(b)$ theorem that we need (see [3, Theorem 20]), we require the definitions of a dyadic para-accretive function and a dyadic BMO function.

Definition 2.7. A function $b \in L^\infty(E)$ is said to be dyadic para-accretive if, for every $Q_\beta^k \in \mathcal{Q}(E)$, there exists $Q_\gamma^l \in \mathcal{Q}(E)$, $Q_\gamma^l \subset Q_\beta^k$, with $l \le k+N$ and

$$\left| \int_{Q_\gamma^l} b\, d\mu \right| \ge c\mu(Q_\gamma^l), \tag{2.14}$$

for some fixed constants $c > 0$ and $N \in \mathbb{N}$.

Definition 2.8. A locally μ integrable function f belongs to dyadic BMO(μ) if

$$\sup_{Q} \inf_{c \in \mathbb{C}} \frac{1}{\mu(Q)} \int_{Q} \big|f(z) - c\big| d\mu(z) < \infty, \tag{2.15}$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{Q}(E)$.

At the beginning of this section, we have defined Calderón-Zygmund operators and standard kernels in the Euclidean case. In the context of spaces of homogeneous type, one has a slightly different definition for them (see [2, pages 93–94]).

Theorem 2.9 [3]**.** Let E be a space of homogeneous type with underlying doubling measure μ, b a dyadic para-accretive function, and T a Calderón-Zygmund operator associated to an antisymmetric standard kernel. Suppose that $T(b)$ belongs to dyadic BMO(μ). Then, T is a bounded operator in $L^2(\mu)$. □

A recent new approach to a variety of $T(b)$ theorems can be found in [1].

For the proof of Theorem 1.2, the following result of Vihtilä will also be needed.

Theorem 2.10 [24]**.** Let μ be a nonzero Radon measure in $\mathbb{R}^n$ for which there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 r^{\alpha} \leq \mu\big(B(x,r)\big) \leq c_2 r^{\alpha}, \tag{2.16}$$

for all $x \in \operatorname{spt}(\mu)$ and $0 < r < d(\operatorname{spt}\mu)$. If R_α is a bounded operator in $L^2(\mu)$, then α is an integer. □

This theorem was proved by using an approach based on tangent measures.

3 Relation between γ_α and Hausdorff content

We need the following lemma.

Lemma 3.1. If a function $f(x)$ has compact support and has continuous derivatives up to order n, then it is representable, for $0 < \alpha < n$, in the form

$$f(x) = \sum_{i=1}^{n} \left(\varphi_i * \frac{x_i}{|x|^{1+\alpha}} \right)(x), \quad x \in \mathbb{R}^n, \tag{3.1}$$

where φ_i, $i = 1, \dots, n$, are defined by the formulas

$$\begin{aligned} \varphi_i &= c_{n,\alpha} \triangle^k \partial_i f * \frac{1}{|x|^{n-\alpha}}, \quad \text{for } n = 2k+1, \\ \varphi_i &= c_{n,\alpha} \triangle^k f * \frac{x_i}{|x|^{1+n-\alpha}}, \quad \text{for } n = 2k, \end{aligned} \tag{3.2}$$

in which $c_{n,\alpha}$ is a constant depending on n and α. □

Proof. Assume first that $n = 2k+1$. Taking Fourier transform of the right-hand side of (3.1), we get, for appropriate numbers $a_{n,\alpha}$ and $b_{n,\alpha}$,

$$\begin{aligned} \sum_{i=1}^{n} \widehat{\varphi_i}(\xi) a_{n,\alpha} \frac{\xi_i}{|\xi|^{1+n-\alpha}} &= \sum_{i=1}^{n} c_{n,\alpha} |\xi|^{2k} \xi_i \widehat{f}(\xi) \frac{b_{n,\alpha}}{|\xi|^{\alpha}} a_{n,\alpha} \frac{\xi_i}{|\xi|^{1+n-\alpha}} \\ &= c_{n,\alpha} a_{n,\alpha} b_{n,\alpha} \widehat{f}(\xi). \end{aligned} \tag{3.3}$$

Then, (3.1) follows by choosing $c_{n,\alpha}$ so that $c_{n,\alpha} a_{n,\alpha} b_{n,\alpha} = 1$.

A similar argument proves (3.1) in the case $n = 2k$. ■

We are now ready to describe the basic relationship between γ_α and Hausdorff content (the d-dimensional Hausdorff content will be denoted by M^d (see [9] for the definition and basic properties)).

Lemma 3.2. If $0 < \alpha < n$, then there exist constants C and C_ϵ such that

$$C_\epsilon M^{\alpha+\epsilon}(E)^{\alpha/(\alpha+\epsilon)} \le \gamma_\alpha(E) \le C M^\alpha(E), \tag{3.4}$$

for any compact set $E \subset \mathbb{R}^n$ and $\epsilon > 0$. □

Proof. We proof first the second inequality. Let $\{Q_j\}_j$ be a covering of E by dyadic cubes $Q_j \subset \mathbb{R}^n$ with disjoint interiors. By a well-known lemma (see [10, Lemma 3.1]), there exist functions $g_j \in C_0^\infty(2Q_j)$ satisfying $\sum_j g_j = 1$ in a neighborhood of $\cup_j Q_j$ and $|\partial^s g_j| \le C_s l(Q_j)^{-|s|}$, $|s| \ge 0$. Here, $s = (s_1, \dots, s_n)$, with $0 \le s_i \in \mathbb{Z}$, $|s| = s_1 + s_2 + \cdots + s_n$, and $\partial^s = (\partial/\partial x_i)^{s_1} \cdots (\partial/\partial x_n)^{s_n}$.

Let T be a distribution with compact support contained in E such that the ith α-Riesz potentials $T * x_i/|x|^{1+\alpha}$ of T are functions in $L^\infty(\mathbb{R}^n)$ with L^∞-norm not greater than 1, $1 \le i \le n$. Applying Lemma 3.1 to each g_j, we obtain functions φ_j^i satisfying (3.1) with f and φ_i replaced by g_j and φ_i^j, respectively. Thus,

$$\begin{aligned} |\langle T, 1\rangle| &= \left|\left\langle T, \sum_j g_j \right\rangle\right| \\ &\leq \sum_j |\langle T, g_j\rangle| \\ &= \sum_j \left|\left\langle T, \sum_{i=1}^n \varphi_i^j * \frac{x_i}{|x|^{1+\alpha}} \right\rangle\right| \\ &\leq \sum_j \sum_{i=1}^n \left|\left\langle T * \frac{x_i}{|x|^{1+\alpha}}, \varphi_i^j \right\rangle\right| \\ &\leq \sum_j \sum_{i=1}^n \int |\varphi_i^j(x)| dx. \end{aligned} \tag{3.5}$$

Take $n = 2k+1$ (for $n = 2k$, the argument is similar) and write $k_\alpha(x) = |x|^{-n+\alpha}$. Let Q_0 be the unit cube centered at 0. Integrating by parts to bring the $\Delta^k \partial_i$ derivatives from g_j to the kernel k_α, changing variables, and using $|\partial^s g_j| \leq C_s l(Q_j)^{-|s|}$, we get

$$\begin{aligned} |\langle T, 1\rangle| &\leq \sum_j \sum_{i=1}^n \int |\varphi_i^j(x)| dx \\ &= \sum_j \sum_{i=1}^n \int \left| \int_{2Q_j} \triangle^k \partial_i g_j(y) k_\alpha(x-y) dy \right| dx \\ &= \sum_j \sum_{i=1}^n \left\{ \int_{3Q_j} \left| \int_{2Q_j} \triangle^k \partial_i g_j(y) k_\alpha(x-y) dy \right| dx \right. \\ &\qquad \left. + \int_{\mathbb{R}^n \setminus 3Q_j} \left| \int_{2Q_j} g_j(y) \Delta^k \partial_i k_\alpha(x-y) dy \right| dx \right\} \\ &\leq n \sum_j l(Q_j)^\alpha \left\{ C_n \iint_{3Q_0 \times 2Q_0} k_\alpha(x-y) dy\, dx \right. \\ &\qquad \left. + C_0 \iint_{(\mathbb{R}^n \setminus 3Q_0) \times 2Q_0} \frac{1}{|x-y|^{2n-\alpha}} dy\, dx \right\} \\ &\leq C \sum_j l(Q_j)^\alpha. \end{aligned} \tag{3.6}$$

Thus, $\gamma_\alpha(E) \leq CM^\alpha(E)$.

For the reverse inequality, we use a standard argument that we reproduce for the reader's convenience. Suppose that $M^{\alpha+\epsilon}(E) > 0$, for some $\epsilon > 0$. By Frostman's Lemma (see [12, Theorem 8.8]), there exists a measure μ supported on E such that $\mu(E) \geq CM^{\alpha+\epsilon}(E) > 0$ and $\mu(B(x,r)) \leq r^{\alpha+\epsilon}$, $x \in \mathbb{R}^n$ and $r > 0$. Then, by a change of variables, we obtain

$$
\begin{aligned}
\left|\left(\mu * \frac{x_i}{|x|^{1+\alpha}}\right)(y)\right| &\leq \int \frac{d\mu(x)}{|x-y|^{\alpha}} \\
&= \int_0^{\infty} \mu\big(\{x : |x-y|^{-\alpha} \geq t\}\big)\,dt \\
&= \int_0^{\infty} \mu\big(B(y, t^{-1/\alpha})\big)\,dt \\
&= \alpha \int_0^{\infty} \frac{\mu\big(B(x,r)\big)}{r^{1+\alpha}}\,dr \\
&\leq \alpha\left(\int_0^{\mu(E)^{1/(\alpha+\epsilon)}} r^{\epsilon-1}\,dr + \int_{\mu(E)^{1/(\alpha+\epsilon)}}^{\infty} \frac{\mu(E)}{r^{1+\alpha}}\right) \\
&= \left(\frac{\alpha}{\epsilon} + 1\right)\mu(E)^{\epsilon/(\alpha+\epsilon)}.
\end{aligned}
\tag{3.7}
$$

Using this estimate, we get the desired inequality, namely,

$$
\begin{aligned}
\gamma_\alpha(E) &\geq \frac{\mu(E)}{\left\|\mu * \frac{x_i}{|x|^{1+\alpha}}\right\|_{\infty}} \\
&\geq \frac{\epsilon}{\alpha+\epsilon}\mu(E)^{1-\epsilon/(\alpha+\epsilon)} \\
&= C_\epsilon \mu(E)^{\alpha/(\alpha+\epsilon)} \\
&\geq C_\epsilon M^{\alpha+\epsilon}(E)^{\alpha/(\alpha+\epsilon)}.
\end{aligned}
\tag{3.8}
$$

■

Let $\dim(E)$ be the Hausdorff dimension of the set E. A qualitative version of Lemma 3.2 is the following corollary.

Corollary 3.3. Let $E \subset \mathbb{R}^n$ be compact.

(1) If $\dim(E) > \alpha$, then $\gamma_\alpha(E) > 0$.

(2) If $\dim(E) < \alpha$, then $\gamma_\alpha(E) = 0$. □

4 Proof of Theorem 1.1

4.1 Distributions that are measures

We start by a lemma that shows that certain distributions are actually measures.

Lemma 4.1. Let $0 < \alpha < n$, let $E \subset \mathbb{R}^n$ be compact with $\mathcal{H}^\alpha(E) < \infty$, and let T be a distribution with compact support contained in E such that $T * x_i/|x|^{1+\alpha}$ is bounded in $\mathbb{R}^n$, $1 \leq i \leq n$. Then, T is a measure which is absolutely continuous with respect to the restriction of $\mathcal{H}^\alpha$ to E and has a bounded density, that is,

$$T = h\mathcal{H}^\alpha, \quad \text{for some } h \in L^\infty\left(\mathcal{H}^\alpha\right) \text{ supported on } E. \tag{4.1}$$

□

Proof. We first show that T is a measure. For this, it is enough to prove that

$$\left|\langle T, f\rangle\right| \le C\mathcal{H}^\alpha(E)\|f\|_\infty, \quad f \in \mathcal{C}_0^\infty. \tag{4.2}$$

Given $\epsilon > 0$, we can cover the compact set E with open balls B_j of radius r_j, $j = 1, \dots, k$, such that $B_j \cap E \ne \varnothing$, $r_j < \epsilon$, and

$$\sum_{j=1}^{k} r_j^\alpha \le 2\mathcal{H}^\alpha(E) + \epsilon. \tag{4.3}$$

Let ψ be a function in $\mathcal{C}_0^\infty$ with $\operatorname{spt}\psi \subset B(0,1)$ and $\int \psi(x)dx = 1$. Define

$$\psi_\epsilon(x) = \frac{1}{\epsilon^n}\psi\left(\frac{x}{\epsilon}\right). \tag{4.4}$$

To prove (4.2), we can assume without loss of generality that $\operatorname{spt}(f) \subset \cup_j B_j$. This is so because if $\beta \in \mathcal{C}_0^\infty$, $\operatorname{spt}(\beta) \subset \cup_j B_j$, $0 \le \beta \le 1$, and $\beta(x) = 1$ in a neighborhood of E, then $\langle T, f\rangle = \langle T, f\beta\rangle$ and $\|\beta f\|_\infty \le \|f\|_\infty$.

Assume that $n = 2k+1$ (the argument for even dimensions is similar). Applying Lemma 3.1 to ψ_ϵ, using the boundedness of $T * x_i/|x|^{1+\alpha}$, for $1 \le i \le n$, and setting $k_\alpha(x) = |x|^{-n+\alpha}$, we have

$$\begin{aligned}
\left|\langle T, f * \psi_\epsilon\rangle\right| &\le C\sum_{i=1}^{n}\left|\left\langle T * \frac{x_i}{|x|^{1+\alpha}}, f * \triangle^k\partial_i\psi_\epsilon * k_\alpha\right\rangle\right| \\
&\le C\sum_{i=1}^{n}\int\left|\left(f * \triangle^k\partial_i\psi_\epsilon * k_\alpha\right)(x)\right|dx \\
&= C\sum_{i=1}^{n}\int\left|\int f(y)\left(\triangle^k\partial_i\psi_\epsilon * k_\alpha\right)(x-y)dy\right|dx \\
&\le C\|f\|_\infty\sum_j r_j^n\sum_{i=1}^{n}\int\left|\triangle^k\partial_i\psi_\epsilon * k_\alpha(z)\right|dz.
\end{aligned} \tag{4.5}$$

We will show that

$$\int\left|\triangle^k\partial_i\psi_\epsilon * k_\alpha(z)\right|dz \le C\epsilon^{-n+\alpha}, \tag{4.6}$$

where C is a constant depending on the L^1-norm of ψ and $\triangle^k\partial_i\psi$ but not on ϵ.

Then, using (4.3), we will have

$$\begin{aligned}\left|\langle T, f*\psi_\epsilon\rangle\right| &\le C\|f\|_\infty\,\epsilon^{-n+\alpha}\sum_j r_j^n\\ &\le C\|f\|_\infty\,\epsilon^{-n+\alpha}\sum_j \epsilon^{n-\alpha}r_j^\alpha\\ &= C\|f\|_\infty\sum_j r_j^\alpha\\ &\le C\big(\mathcal{H}^\alpha(E)+\epsilon\big)\|f\|_\infty,\end{aligned} \tag{4.7}$$

which proves (4.2) by letting $\epsilon \to 0$.

To prove (4.6), we use Fubini's Theorem and a change of variables:

$$\begin{aligned}&\int\left|(\triangle^k\partial_i\psi_\epsilon * k_\alpha)(z)\right|dz\\ &\quad= \int\left|\int \epsilon^{-2n}\triangle^k\partial_i\psi\left(\frac{z-x}{\epsilon}\right)k_\alpha(x)\,dx\right|dz\\ &\quad= \epsilon^{-n+\alpha}\int\left|(\triangle^k\partial_i\psi * k_\alpha)(z)\right|dz\\ &\quad\le \epsilon^{-n+\alpha}\int_{|z|\ge 2}\int_{|x|\le 1}\frac{|\psi(x)|}{|z-x|^{2n-\alpha}}\,dx\,dz + \epsilon^{n-\alpha}\int_{|z|\le 2}\int_{|x|\le 1}\frac{|\triangle^k\partial_i\psi(x)|}{|z-x|^{n-\alpha}}\,dx\,dz\\ &\quad= \epsilon^{-n+\alpha}\int_{|x|\le 1}|\psi(x)|\int_{|z|\ge 2}\frac{dz}{|z-x|^{2n-\alpha}}\,dx\\ &\qquad+ \epsilon^{n-\alpha}\int_{|x|\le 1}|\Delta^k\partial_i\psi(x)|\int_{|z|\le 2}\frac{dz}{|z-x|^{n-\alpha}}\,dx\\ &\quad\le C\epsilon^{-n+\alpha}\Big(\|\psi\|_1 + \left\|\triangle^k\partial_i\psi\right\|_1\Big)\\ &\quad= C\epsilon^{-n+\alpha}.\end{aligned} \tag{4.8}$$

Let B_0 be an open ball and let $\overline{B_0}$ denote its closure. Let $\mathcal{H}_E^\alpha$ stand for the restriction of $\mathcal{H}^\alpha$ to E. If we show that

$$\left|\mu(B_0)\right| \le C\mathcal{H}_E^\alpha(\overline{B_0}), \tag{4.9}$$

then, taking a sequence of open balls $B_0^i \downarrow \overline{B_0}$ and applying (4.9) to these balls, we will have

$$\left|\mu(\overline{B_0})\right| \le \lim_{i\to\infty}\left|\mu(B_0^i)\right| \le \lim_{i\to\infty} C\mathcal{H}_E^\alpha(\overline{B_0^i}) = C\mathcal{H}_E^\alpha(\overline{B_0}). \tag{4.10}$$

It is shown in [12, page 271] that, for $\alpha = 1$, (4.10) implies

$$|\mu(A)| \le C\mathcal{H}^\alpha(A), \quad \text{for sets } A \subset E \text{ with } \mathcal{H}^\alpha(A) < \infty. \tag{4.11}$$

The argument extends verbatim to any α and thus we can take (4.11) for granted, which gives (4.1) by Radon-Nikodym's Theorem.

It remains to prove (4.9). We know that, for every $\delta > 0$, there exists a compact set $K \subset E \setminus \overline{B_0}$ such that

$$\mathcal{H}^\alpha(K) > \mathcal{H}^\alpha\big(E \setminus \overline{B_0}\big) - \delta. \tag{4.12}$$

Let

$$\begin{aligned} J_1 &= \{j : B_j \cap \overline{B_0} \neq \varnothing\}, \\ J_2 &= \{j : B_j \cap K \neq \varnothing\}. \end{aligned} \tag{4.13}$$

Recall that the radii of the balls B_j satisfy $r_j < \epsilon$. For an appropriate $\epsilon > 0$, the following holds:

$$\sum_{j\in J_2} r_j^\alpha \ge 2\mathcal{H}^\alpha(K) - \delta, \tag{4.14}$$

$$\max_j r_j < \epsilon < \frac{\operatorname{dist}\big(K, \overline{B_0}\big)}{2}. \tag{4.15}$$

This last condition implies that, for $j_1 \in J_1$ and $j_2 \in J_2$, we have $\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing$. So, using inequalities (4.3), (4.14), and (4.12),

$$\begin{aligned} \sum_{j\in J_1} r_j^\alpha &\le \sum_j r_j^\alpha - \sum_{j\in J_2} r_j^\alpha \\ &\le 2\mathcal{H}^\alpha(E) + \epsilon - 2\mathcal{H}^\alpha(K) + \delta \\ &< 2\mathcal{H}^\alpha(E) + \epsilon - 2\mathcal{H}^\alpha\big(E \setminus \overline{B_0}\big) + \delta \\ &= 2\mathcal{H}^\alpha_E\big(\overline{B_0}\big) + \epsilon + \delta. \end{aligned} \tag{4.16}$$

If χ_{B_0} denotes the characteristic function of the ball B_0, then

$$\mu(B_0) = \langle \mu, \chi_{B_0}\rangle = \langle \mu, \chi_{B_0\cap E}\rangle = \lim_{\epsilon\to 0} \langle \mu, \chi_{B_0\cap E} * \psi_\epsilon\rangle. \tag{4.17}$$

Arguing as in (4.5), (4.6), and (4.7), we get

$$\big|\langle \mu, \chi_{B_0\cap E} * \psi_\epsilon\rangle\big| \le C\|\chi_{B_0\cap E}\|_\infty \sum_{j\in J_1} r_j^\alpha \le C\big(\mathcal{H}^\alpha_E\big(\overline{B_0}\big) + \epsilon + \delta\big), \tag{4.18}$$

and letting ϵ and δ tend to zero, we get (4.9). ∎

4.2 Symmetrization of the Riesz kernel

The symmetrization process of the Cauchy kernel introduced in [15] has been successfully applied in the last years to many problems of analytic capacity and L^2-boundedness of the Cauchy integral operator (see, e.g., [13, 16, 23]; the survey papers [5, 22] contain many other references). Given three distinct points z_1, z_2, and z_3 in the plane, one finds out, by an elementary computation, that

$$c(z_1, z_2, z_3)^2 = \sum_{\sigma} \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}, \tag{4.19}$$

where the sum is taken over the six permutations of the set $\{1, 2, 3\}$ and $c(z_1, z_2, z_3)$ is Menger curvature, that is, the inverse of the radius of the circle through z_1, z_2, and z_3. In particular, (4.19) shows that the sum on the right-hand side is a nonnegative quantity.

On the other hand, it has been proved in [7] that nothing similar occurs for the Riesz kernel $k_\alpha = x/|x|^{1+\alpha}$ with α integer and $1 < \alpha \le n$. In this section, we show that, for $0 < \alpha < 1$, we recover an explicit expression for the symmetrization of the Riesz kernel k_α and that the quantity one gets is also nonnegative. For $\alpha > 1$, the phenomenon of change of signs appears again.

For $0 < \alpha < n$, the quantity

$$\sum_{\sigma} \frac{x_{\sigma(2)} - x_{\sigma(1)}}{\left|x_{\sigma(2)} - x_{\sigma(1)}\right|^{1+\alpha}} \frac{x_{\sigma(3)} - x_{\sigma(1)}}{\left|x_{\sigma(3)} - x_{\sigma(1)}\right|^{1+\alpha}}, \tag{4.20}$$

where the sum is taken over the six permutations of the set $\{1, 2, 3\}$, is the obvious analogue of the right-hand side of (4.19) for the Riesz kernel k_α. Observe, however, that if σ is a transposition of two numbers in $\{1, 2, 3\}$, then the term one obtains is one of the three terms associated to the permutations $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$. Thus, (4.20) is exactly

$$2p_\alpha(x_1, x_2, x_3), \tag{4.21}$$

where $p_\alpha(x_1, x_2, x_3)$ is defined as the sum in (4.20) taken only on the three permutations $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$.

Lemma 4.2. Let $0 < \alpha < 1$, and let x_1, x_2, and x_3 be three distinct points in $\mathbb{R}^n$. Then,

$$\frac{2 - 2^\alpha}{L(x_1, x_2, x_3)^{2\alpha}} \le p_\alpha(x_1, x_2, x_3) \le \frac{2^{1+\alpha}}{L(x_1, x_2, x_3)^{2\alpha}}, \tag{4.22}$$

where $L(x_1, x_2, x_3)$ is the largest side of the triangle determined by x_1, x_2, and x_3. In particular, $p_\alpha(x_1, x_2, x_3)$ is a nonnegative quantity. □

Proof. If $n = 1$ and $x_1 < x_2 < x_3$, then

$$p_\alpha(x_1, x_2, x_3) = \frac{a^\alpha + b^\alpha - (a+b)^\alpha}{a^\alpha b^\alpha (a+b)^\alpha}, \tag{4.23}$$

where $a = x_2 - x_1$ and $b = x_3 - x_2$. An elementary estimate shows that (4.22) holds in this case, even with $2^{1+\alpha}$ replaced by 2^α in the numerator of the last term.

Note that if $x_1, x_2, x_3 \in \mathbb{R}^n$, one can write

$$p_\alpha(x_1, x_2, x_3) = \frac{\cos(\theta_{23})|x_2 - x_3|^\alpha + \cos(\theta_{13})|x_1 - x_3|^\alpha + \cos(\theta_{12})|x_1 - x_2|^\alpha}{|x_1 - x_2|^\alpha |x_1 - x_3|^\alpha |x_2 - x_3|^\alpha}, \tag{4.24}$$

where θ_{ij} is the angle opposite to the side $x_i x_j$ in the triangle determined by x_1, x_2, and x_3. Without loss of generality, we can assume that $\theta_{23}, \theta_{13} \in [0, \pi/2]$. Denote $l_{ij} = |x_i - x_j|$, for $i \neq j$, $i, j \in \{1, 2, 3\}$. We consider two different cases.

Case 1 $(0 \leq \theta_{12} \leq \pi/2)$. Without loss of generality, suppose that $l_{12} \geq l_{13} \geq l_{23}$. Then, we have

$$\begin{aligned} p_\alpha(x_1, x_2, x_3) &= \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{13})\frac{l_{13}^\alpha}{l_{23}^\alpha} + \cos(\theta_{12})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right) \\ &\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha}(\cos(\theta_{23}) + \cos(\theta_{13}) + \cos(\theta_{12})) \\ &\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha} \\ &\geq \frac{2 - 2^\alpha}{L(x_1, x_2, x_3)^{2\alpha}}. \end{aligned} \tag{4.25}$$

For the second inequality, one argues as follows:

$$\begin{aligned} &p_\alpha(x_1, x_2, x_3) \\ &= \frac{1}{l_{12}^{1+\alpha} l_{13}^{1+\alpha}}\left(\cos(\theta_{23}) l_{12} l_{13} + \cos(\theta_{13}) l_{12} l_{23}\frac{l_{13}^{1+\alpha}}{l_{23}^{1+\alpha}} + \cos(\theta_{12}) l_{13} l_{23}\frac{l_{12}^{1+\alpha}}{l_{23}^{1+\alpha}}\right) \\ &\leq \frac{1}{l_{12}^{1+\alpha} l_{13}^{1+\alpha}}\left(\cos(\theta_{23}) l_{12} l_{13} + \cos(\theta_{13}) l_{12} l_{23}\frac{l_{13}^{2}}{l_{23}^{2}} + \cos(\theta_{12}) l_{13} l_{23}\frac{l_{12}^{2}}{l_{23}^{2}}\right) \\ &= l_{12}^{1-\alpha} l_{13}^{1-\alpha} p_1(x_1, x_2, x_3) \\ &= l_{12}^{1-\alpha} l_{13}^{1-\alpha}\frac{1}{2R^2}, \end{aligned} \tag{4.26}$$

by (4.19), where R is the radius of the circle through x_1, x_2, and x_3. Since clearly $l_{ij} \leq 2R$, we conclude that

$$p_\alpha(x_1, x_2, x_3) \leq \frac{2}{l_{12}^\alpha l_{13}^\alpha} \leq \frac{2^{1+\alpha}}{L(x_1, x_2, x_3)^{2\alpha}}. \tag{4.27}$$

Case 2 ($\pi/2 \leq \theta_{12} \leq \pi$). We start by proving the first inequality in (4.22). Note that in this case, the largest side of the triangle is l_{12}. Assume without loss of generality that $l_{13} \geq l_{23}$ and denote $t = l_{13}/l_{23} \geq 1$. Write $\theta_{13} = \theta_{23} + a$, with $0 \leq a \leq \pi/2$. Then, by the triangle inequality, we have

$$\begin{aligned} p_\alpha(x_1, x_2, x_3) &= \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{23} + a)\frac{l_{13}^\alpha}{l_{23}^\alpha} + \cos(\theta_{12})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right) \\ &\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{23} + a)t^\alpha - \cos(2\theta_{23} + a)(1+t)^\alpha\right) \\ &\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(a, \theta_{23}, t), \end{aligned} \tag{4.28}$$

where

$$f(a, y, t) = \cos(y) + \cos(y + a)t^\alpha - \cos(2y + a)(1+t)^\alpha, \tag{4.29}$$

for $0 \leq 2y + a \leq \pi/2$, $a \geq 0$, and $y \geq 0$.

We claim that

$$f(a, y, t) \geq f(0, y, t) \geq f(0, 0, t), \tag{4.30}$$

for $0 \leq 2y + a \leq \pi/2$, $a \geq 0$, and $y \geq 0$. Notice that the inequality $f(a, y, t) \geq f(0, 0, t)$ in (4.30) means that the smallest value of p_α is attained when the three points x_1, x_2, and x_3 lie on a line.

If we assume that the claim is proved, then, going back to (4.28) and using that $t \geq 1$, we get

$$
\begin{aligned}
p_\alpha(x_1, x_2, x_3) &\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(a, \theta_{23}, t) \\
&\geq \frac{1}{l_{12}^\alpha l_{13}^\alpha} f(0, 0, t) \\
&= \frac{1}{l_{12}^\alpha l_{13}^\alpha} \left(1 + t^\alpha - (1+t)^\alpha\right) \\
&\geq \frac{2 - 2^\alpha}{l_{12}^\alpha l_{13}^\alpha} \\
&\geq \frac{2 - 2^\alpha}{L(x_1, x_2, x_3)^{2\alpha}}.
\end{aligned} \tag{4.31}
$$

To prove the first inequality in (4.30), we use that, for $0 \leq 2y + a \leq \pi/2$, $a \geq 0$, and $y \geq 0$, we have $\cos(y) - \cos(y + a) \leq \cos(2y) - \cos(2y + a)$. Thus, $\cos(y) - \cos(y + a) \leq (1 + 1/t)^\alpha(\cos(2y) - \cos(2y + a))$, which is $f(a, y, t) \geq f(0, y, t)$.

Finally, for each t, the function

$$
f(0, y, t) = \cos(y) + \cos(y)t^\alpha - \cos(2y)(1+t)^\alpha \tag{4.32}
$$

has a minimum at $y = 0$, and this proves the claim and thus the first inequality in (4.22).

We are now only left with the second inequality in (4.22) for $\theta_{12} \in [\pi/2, \pi]$. Recall that we can assume without loss of generality that $l_{23} \leq l_{13} \leq l_{12}$. We have

$$
\begin{aligned}
p_\alpha(x_1, x_2, x_3) &= \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \cos(\theta_{13})\frac{l_{13}^\alpha}{l_{23}^\alpha} - \cos(\theta_{23} + \theta_{13})\frac{l_{12}^\alpha}{l_{23}^\alpha}\right) \\
&\leq \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \left(\cos(\theta_{13}) - \cos(\theta_{23} + \theta_{13})\right)\frac{l_{13}^\alpha}{l_{23}^\alpha}\right).
\end{aligned} \tag{4.33}
$$

The function $g(x) = \cos x - \cos(x + y)$ is increasing for x, y, and $x + y$ in $[0, \pi/2]$. Thus, $g(x) \leq g(\pi/2) = \sin y$, for x, y, and $x + y$ in $[0, \pi/2]$. Moreover, using that $\sin(\theta_{23})/l_{23} = \sin(\theta_{13})/l_{13}$, we get

$$
\begin{aligned}
p_\alpha(x_1, x_2, x_3) &\leq \frac{1}{l_{12}^\alpha l_{13}^\alpha}\left(\cos(\theta_{23}) + \sin(\theta_{13})\frac{l_{23}^{1-\alpha}}{l_{13}^{1-\alpha}}\right) \\
&\leq \frac{2}{l_{12}^\alpha l_{13}^\alpha} \\
&\leq \frac{2^{1+\alpha}}{L(x_1, x_2, x_3)^{2\alpha}},
\end{aligned} \tag{4.34}
$$

which completes the proof of the lemma. ∎

4.3 The main step

Let $0 < \alpha < n$ and suppose that μ is a measure such that $\mu(B(x,r)) \le C_0 r^\alpha$, for some constant C_0 and for all balls $B(x,r) \subset \mathbb{R}^n$. We will now analyze what happens in a ball $B(x,r)$ satisfying the lower-density condition $\mu(B(x,r)) \ge \epsilon r^\alpha$ for a given number $\epsilon > 0$.

Lemma 4.3. There exist constants $a \ge 1$ and $b \ge 1$ depending only on C_0 and ϵ such that given any ball $B_0 = B(x,r)$ satisfying $\mu(B_0) \ge \epsilon r^\alpha$, there exist two balls $B_1 = B(x_1, r/a)$ and $B_2 = B(x_2, r/a)$, with $x_1, x_2 \in \operatorname{spt}\mu \cap B_0$, such that

(1) $|x_1 - x_2| \ge 6r/a$,

(2) $\mu(B_0 \cap B_i) \ge r^\alpha/b$, for $i = 1, 2$. □

Proof. Without loss of generality, we may assume that $B_0 = B(0,1)$. Let $a \ge 1$ and $b \ge 1$ be two constants to be chosen at the end of the construction and suppose that the lemma is not true. This means that given any pair of closed balls B_1 and B_2 of radius a^{-1} centered at $\operatorname{spt}\mu \cap B_0$, then either

$$|x_1 - x_2| < \frac{6}{a} \tag{4.35}$$

or one of the two balls, say B_i, satisfies

$$\mu(B_i \cap B_0) \le \frac{1}{b}. \tag{4.36}$$

Consider the covering of $\operatorname{spt}\mu \cap B_0$ by balls of radius a^{-1} centered at $\operatorname{spt}\mu \cap B_0$. Apply Besicovitch's covering lemma to this covering to obtain $N = N(n)$ families $\mathcal{B}_i$ of disjoint balls such that

$$\operatorname{spt}\mu \cap B_0 \subset \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{B}_i} B. \tag{4.37}$$

Notice that a simple estimate of the volume of the union of the balls in a given family reveals that each family contains no more than $(2a)^n$ balls. We have

$$\epsilon \le \mu(B_0) \le \mu\left(\bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{B}_i} B\right) \le \sum_{i=1}^{N} \sum_{B \in \mathcal{B}_i} \mu(B \cap B_0), \tag{4.38}$$

which means that there exists at least one family $\mathcal{B}_i$ such that

$$\sum_{B \in \mathcal{B}_i} \mu(B \cap B_0) \ge \frac{\epsilon}{N}. \tag{4.39}$$

Consider the set

$$\mathcal{M} = \left\{ B \in \mathcal{B}_i : \mu\big(B \cap B_0\big) > \frac{1}{b} \right\}. \tag{4.40}$$

Condition (4.35) implies that all balls in $\mathcal{M}$ are contained in a ball of radius $8/a$, and hence,

$$\sum_{B \in \mathcal{M}} \mu\big(B \cap B_0\big) \le C_0 \left(\frac{8}{a} \right)^{\alpha}, \tag{4.41}$$

using that $\mu(B(x,r)) \le C_0 r^{\alpha}$ holds for any ball $B(x,r)$ in $\mathbb{R}^n$.

The fact that each family $\mathcal{B}_i$ contains no more than $(2a)^n$ balls implies that

$$\sum_{\substack{B \in \mathcal{B}_i \\ B \notin \mathcal{M}}} \mu\big(B \cap B_0\big) \le \frac{(2a)^n}{b}, \tag{4.42}$$

and so we get

$$\epsilon \le N \sum_{B \in \mathcal{B}_i} \mu\big(B \cap B_0\big) \le N \left(\frac{(2a)^n}{b} + C_0 \left(\frac{8}{a} \right)^{\alpha} \right). \tag{4.43}$$

If a and b are appropriately chosen, this inequality gives a contradiction. ■

Let $0 \le \alpha < \infty$ and let μ be a positive Borel measure on $\mathbb{R}^n$. The upper and lower α-densities of μ at $x \in \mathbb{R}^n$ are defined by

$$\begin{aligned} \Theta^{*\alpha}(\mu, x) &= \limsup_{r \to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^{\alpha}}, \\ \Theta_*^{\alpha}(\mu, x) &= \liminf_{r \to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^{\alpha}}, \end{aligned} \tag{4.44}$$

respectively.

Theorem 4.4. Let $0 < \alpha < 1$ and let μ be a positive Borel measure with $0 < \Theta^{*\alpha}(\mu, x) < \infty$, for μ-almost all $x \in \mathbb{R}^n$. Then,

$$\iiint p_\alpha(x_1, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3) = +\infty. \tag{4.45}$$

□

Proof. Since $\Theta^{*\alpha}(\mu, x) < \infty$, for μ-almost all $x \in \mathbb{R}^n$, there exists a compact set $K_1 \subset \mathbb{R}$ with $\mu(K_1) > 0$ and a constant $c_1 > 0$ such that $\mu(K_1 \cap B(x,r)) \leq c_1 r^\alpha$, for every ball $B(x,r) \subset \mathbb{R}^n$. It is well known that $\Theta^{*\alpha}(\mu|K_1, x) = \Theta^{*\alpha}(\mu, x)$, for μ-almost all $x \in K_1$ (see [12, Theorems 6.2 and 6.9]), whence, replacing μ by $\mu|K_1$, we can assume that $\mu(B(x,r)) \leq c_1 r^\alpha$, for $x \in \mathbb{R}^n$.

From the fact that $\Theta^{*\alpha}(\mu, x) > 0$, for μ-almost all $x \in \mathbb{R}^n$, we deduce that there exists a compact set $K_2 \subset \mathbb{R}^n$, with $\mu(K_2) > 0$ and a constant $c_2 > 0$, such that, for each $x \in K_2$, there is a sequence $r_i(x) > 0$ with $\lim_{i\to\infty} r_i(x) = 0$ and $\mu(B(x, r_i(x))) \geq c_2 r_i(x)^\alpha$. Notice that truncating the sequences of radii appropriately, we can assume that $\sup_{x\in K_2} r_i(x) \to 0$, $i \to \infty$.

By the 5-covering Theorem (see [12, Theorem 2.1]), for each $i \in \mathbb{N}$, there are disjoint balls $B^i_j = B(a_j, r_i(a_j))$, $1 \leq j \leq m_i$, such that $K_2 \subset \bigcup_{j=1}^{m_i} 5B^i_j$. Then, we have

$$\mu(K_2) \leq \sum_{j=1}^{m_i} \mu\left(5B^i_j\right) \leq c_1 5^\alpha \sum_{j=1}^{m_i} r_i\left(a_j\right)^\alpha, \tag{4.46}$$

that is,

$$\sum_{j=1}^{m_i} r_i^\alpha\left(a_j\right) \geq \frac{\mu\left(K_2\right)}{5^\alpha c_1}. \tag{4.47}$$

Fix $i = 1$ and consider the disjoint balls B^1_j, for $1 \leq j \leq m_1$. For every B^1_j, we can use Lemma 4.3 twice to find three balls B_1, B_2, and B_3 centered at $\mathrm{spt}(\mu) \cap B^1_j$ enjoying the following properties: their mutual distances and their radii are comparable to $r(a_j)$ and the mass $\mu(B^1_j \cap B_l)$ is also comparable to $r(a_j)^\alpha$. The comparability constants in the above statements depend only on c_1, c_2, and n. Define a set of triples by

$$S_{j,1} = \left(B^1_j \cap B_1\right) \times \left(B^1_j \cap B_2\right) \times \left(B^1_j \cap B_3\right), \quad \text{for } 1 \leq j \leq m_1. \tag{4.48}$$

Applying Lemma 4.2, we obtain

$$\begin{aligned}
&\iiint_{(B^1_j)^3} p_\alpha\left(x_1, x_2, x_3\right) d\mu\left(x_1\right) d\mu\left(x_2\right) d\mu\left(x_3\right) \\
&\quad \geq \iiint_{S_{j,1}} p_\alpha\left(x_1, x_2, x_3\right) d\mu\left(x_1\right) d\mu\left(x_2\right) d\mu\left(x_3\right) \\
&\quad \geq C \iiint_{S_{j,1}} \frac{1}{\left|x_1 - x_3\right|^{2\alpha}} d\mu\left(x_1\right) d\mu\left(x_2\right) d\mu\left(x_3\right) \\
&\quad \geq C r_1\left(a_j\right)^\alpha.
\end{aligned} \tag{4.49}$$

Set

$$A_1 = \bigcup_{j=1}^{m_1} S_{j,1} \subset \bigcup_{j=1}^{m_1} \left(B_j^1 \times B_j^1 \times B_j^1\right),$$
$$d_j = \min\left\{\operatorname{dist}\left(B_j^1 \cap B^k,\ B_j^1 \cap B^l\right) : k, l \in \{1,2,3\},\ k \neq l\right\}, \tag{4.50}$$
$$t_1 = \min_{1 \le j \le m_1} d_j.$$

For $(x_1, x_2, x_3) \in A_1$, we then have $|x_i - x_j| > t_1$, for $i, j \in \{1,2,3\}$, $j \neq i$. Moreover, using (4.47) and (4.49),

$$\begin{aligned}&\iiint_{A_1} p_\alpha(x_1, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3)\\ &\quad = \sum_{j=1}^{m_1} \iiint_{S_{j,1}} p_\alpha(x_1, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3) \ge C \sum_{j=1}^{m_1} r_1(a_j)^\alpha \ge C.\end{aligned} \tag{4.51}$$

Let q be such that

$$\sup_{x \in K_2} r_q(x) \le \frac{t_1}{2} \tag{4.52}$$

and consider the balls of the qth generation, namely, B_j^q, for $1 \le j \le m_q$. Repeat the process described above, replacing B_j^1 by B_j^q. We then find balls B_1, B_2, and B_3 centered at points in $\operatorname{spt}\mu \cap B_j^q$, whose mutual distances and radii are comparable to $r_q(a_j)$ and such that $\mu(B_j^q \cap B_l)$ is also comparable to $r_q(a_j)^\alpha$, $l = 1, 2, 3$.

Set

$$S_{j,2} = \left(B_j^q \cap B_1\right) \times \left(B_j^q \cap B_2\right) \times \left(B_j^q \cap B_3\right),$$
$$A_2 = \bigcup_{j=1}^{m_q} S_{j,2} \subset \bigcup_{j=1}^{m_q} \left(B_j^q \times B_j^q \times B_j^q\right). \tag{4.53}$$

Hence, again by (4.52),

$$\iiint_{A_2} p_\alpha(x_1, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3) \ge C. \tag{4.54}$$

Notice that the sets of triples A_1 and A_2 are disjoint because of the definition of q. Define t_2 as we did before for t_1 so that, for $(x_1, x_2, x_3) \in A_2$, one has $|x_i - x_j| > t_2$, for $i, j \in \{1,2,3\}$, $i \neq j$. It becomes now clear that we can inductively construct disjoint sets of triples A_k, $k = 1, 2, \ldots$, such that

$$\iiint_{A_k} p_\alpha(x_1, x_2, x_3)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x_3) \ge C, \quad k = 1, 2, \ldots, \tag{4.55}$$

and therefore,

$$\begin{aligned}\iiint p_\alpha(x_1,x_2,x_3)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x_3)\\ \geq \sum_{k=1}^{\infty}\iiint_{A_k} p_\alpha(x_1,x_2,x_3)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x_3)\\ \geq \sum_{k=1}^{\infty} C = +\infty.\end{aligned} \tag{4.56}$$

■

4.4 End of proof of Theorem 1.1

Suppose that $\gamma_\alpha(E) > 0$, for $0 < \alpha < 1$. Applying Lemma 4.1, we find a measure of the form $\nu = b\mathcal{H}^\alpha$, with $b \in L^\infty(\mathcal{H}^\alpha, E)$ such that the α-Riesz potential $R_\alpha(\nu) = \nu * x/|x|^{1+\alpha}$ is in $L^\infty(\mathbb{R}^n)$ and $\int_E b\, d\mathcal{H}^\alpha = \gamma_\alpha(E)$. We can apply now Theorem 2.3 to get a set $F \subset E$ of positive $\mathcal{H}^\alpha$-measure such that the operator R_α is bounded in $L^2(\mathcal{H}^\alpha, F)$. On the other hand, since $\mathcal{H}^\alpha(F) < \infty$, we have $2^{-\alpha} \leq \Theta^{*\alpha}(\mathcal{H}^\alpha_{|F}, x) \leq 1$, for $\mathcal{H}^\alpha$-almost all $x \in \mathbb{R}^n$ (see [12, Theorem 6.2]). This means that we can apply Theorem 4.4 to obtain

$$\iiint p_\alpha(x_1,x_2,x_3)\,d\mathcal{H}^\alpha_{|F}(x_1)\,d\mathcal{H}^\alpha_{|F}(x_2)\,d\mathcal{H}^\alpha_{|F}(x_3) = +\infty. \tag{4.57}$$

This last fact contradicts the L^2-boundedness of R_α on $L^2(\mathcal{H}^\alpha, F)$ by a well-known argument that we now outline briefly (see [15, 16]).

Set $\mu = \mathcal{H}^\alpha_{|F}$. Then,

$$\int \left|R_{\alpha,\epsilon}(\mu)(x)\right|^2 d\mu(x) = \iiint_{T_\epsilon} R_\alpha(x-y)R_\alpha(x-z)\,d\mu(x)\,d\mu(y)\,d\mu(z), \tag{4.58}$$

where

$$T_\epsilon = \big\{(x,y,z) : |x-y| > \epsilon,\ |x-z| > \epsilon\big\}. \tag{4.59}$$

Interchanging the roles of x and y, and then of x and z, and estimating the error terms in a standard way, we obtain

$$\int \left|R_{\alpha,\epsilon}(\mu)(x)\right|^2 d\mu(x) = \frac{1}{3}\iiint_{S_\epsilon} p_\alpha(x,y,z)\,d\mu(x)\,d\mu(y)\,d\mu(z) + O\big(\mu(F)\big), \tag{4.60}$$

where

$$S_\epsilon = \big\{(x,y,z) : |x-y| > \epsilon,\ |x-z| > \epsilon,\ |y-z| > \epsilon\big\}. \tag{4.61}$$

Letting $\epsilon \to 0$, we get the promised contradiction.

Remark 4.5. Notice that if we knew that, for some $0 < \alpha < n$, the α-Riesz kernel never defines a bounded operator on a set of finite α-Hausdorff measure, then Theorem 1.1 would extend to this value of α. For any $0 < \alpha < 1$, this follows from the symmetrization method, as it is shown above. For $1 < \alpha < n$, to get such a result, we have to restrict ourselves to α-dimensional Ahlfors-David regular sets and noninteger α (see Theorem 2.10).

5 Proof of Theorem 1.2

As a tool to prove Theorem 1.2, consider the tangent measures that were introduced by Preiss in [20].

Let $T_{a,r}$ be the map that blows up $B(a, r)$ to $B(0, 1)$, that is,

$$T_{a,r}(x) = \frac{x - a}{r}. \tag{5.1}$$

The image of μ under $T_{a,r}$ is given by

$$T_{a,r\sharp}\mu(A) = \mu(rA + a), \quad A \subset \mathbb{R}^n. \tag{5.2}$$

Definition 5.1. Let μ be a Radon measure on $\mathbb{R}^n$. The measure σ is said to be a tangent measure of μ at a point $a \in \mathbb{R}^n$ if σ is a nonzero Radon measure on $\mathbb{R}^n$ and if there exist sequences $\{r_i\}$ and $\{c_i\}$ of positive numbers such that $r_i \to 0$ and $c_i T_{a,r_i\sharp}\mu \to \sigma$ weakly, as $i \to \infty$.

The set of all tangent measures to μ at a is denoted by $\mathrm{Tan}(\mu, a)$.

Remark 5.2. If $0 < \Theta^{\alpha}_{*}(\mu, a) \leq \Theta^{*\alpha}(\mu, a) < \infty$, then we may find a sequence $\{r_i\}$ such that

$$\sigma = d \lim_{i\to\infty} r_i^{-\alpha} T_{a,r_i\sharp}\mu, \tag{5.3}$$

for some positive number d (see [12, pages 187–188]).

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $0 < \alpha < n$ and let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of dimension α. Suppose that $\gamma_\alpha(E) > 0$. Then, there exists a distribution S with compact support contained in E, whose α-Riesz potential $S * x/|x|^{1+\alpha}$ is in $L^\infty(\mathbb{R}^n)$ and such that $\langle S, 1\rangle \neq 0$.

By Lemma 4.1, $S = h\mathcal{H}^\alpha$ with $h \in L^\infty(E, \mathcal{H}^\alpha)$. Thus, $\langle S, 1\rangle = \int_E h(x)d\mathcal{H}^\alpha(x) \neq 0$.

We will now construct an α-dimensional Ahlfors-David regular measure σ, that is, a measure such that, for some constant C,

$$C^{-1}r^{\alpha} \le \sigma(B(x,r)) \le Cr^{\alpha}, \quad x \in \operatorname{spt}(\sigma),\ 0 < r < d(\operatorname{spt}(\sigma)), \tag{5.4}$$

whose α-Riesz operator R_α is bounded in $L^2(\sigma)$. Then, applying Theorem 2.10, we will conclude that α must be an integer.

We first sketch briefly the main ideas involved in the construction of the measure σ. The first step will be to construct a set E' with $\mathcal{H}^\alpha(E \cap E') > 0$ and a doubling measure μ on E'. The pair (E', μ) is then endowed with a system of dyadic cubes $\mathcal{Q}(E')$ satisfying the properties of Theorem 2.6. We also define a bounded function b on E', which will be dyadic para-accretive with respect to the system of dyadic cubes $\mathcal{Q}(E')$ and such that the function $R_\alpha(b\mu)$ belongs to dyadic $\mathrm{BMO}(\mu)$. Therefore, the α-Riesz transform R_α associated to μ will be bounded on $L^2(E', \mu)$ by the $T(b)$ theorem on a space of homogeneous type (Theorem 2.9). The required Ahlfors-David regular measure σ will be a tangent measure of μ at some point of density of E inside E'. The fact that the α-Riesz transform R_α, associated to σ defines a bounded operator on $L^2(\sigma)$ will follow from the $L^2(\mu)$-boundedness of R_α associated to μ by taking weak limits.

Now, we turn to the construction of the set E' and the measures μ and σ. Let $\mathcal{Q}(E)$ be a system of dyadic cubes on E satisfying the properties (1) through (6) in Theorem 2.6. The first dyadic cube of E to examine is E itself. By hypothesis, there exists a function $h \in L^\infty(E)$ such that $\int_E h\, d\mathcal{H}^\alpha \neq 0$. Let $\epsilon_0 > 0$ be a sufficiently small constant to be fixed later such that $|\int_E h\, d\mathcal{H}^\alpha| > \epsilon_0 \mathcal{H}^\alpha(E)$. Then, for every positive integer k, there exists at least one cube Q^k_γ satisfying $|\int_{Q^k_\gamma} h\, d\mathcal{H}^\alpha| > \epsilon_0 \mathcal{H}^\alpha(Q^k_\gamma)$, since otherwise, for some k,

$$\left|\int_E h\, d\mathcal{H}^\alpha\right| = \left|\int_{\cup_\gamma Q^k_\gamma} h\, d\mathcal{H}^\alpha\right| \le \epsilon_0 \sum_\gamma \mathcal{H}^\alpha(Q^k_\gamma) = \epsilon_0 \mathcal{H}^\alpha(E), \tag{5.5}$$

which is a contradiction.

We now run a stopping-time procedure. Let $\epsilon > 0$ be another constant, much smaller than ϵ_0, to be chosen later. Take a dyadic cube $Q \in \mathcal{Q}^1(E)$ and check whether or not the condition

$$\left|\int_Q h\, d\mathcal{H}^\alpha\right| \le \epsilon \mathcal{H}^\alpha(Q) \tag{5.6}$$

is satisfied. If (5.6) holds for that cube Q, and Q has more than one child, we call it a stopping-time cube. If (5.6) holds but Q has only one child, then we look for the first descendent of Q with more than one child and we call it a stopping-time cube. Notice that (5.6) remains true for this descendent.

If (5.6) does not hold for Q, then we examine each child of Q and repeat the above procedure. After possibly infinitely many steps and possibly passing through all generations, we obtain a collection of pairwise disjoint stopping-time cubes $\{P_\gamma\}$ in E. Each P_γ has at least two children and satisfies the nonaccretivity condition (5.6) with Q replaced by P_γ.

Set $\|h\|_\infty = M$. Then,

$$\begin{aligned}
\mathcal{H}^\alpha\left(E \setminus \bigcup_\gamma P_\gamma\right) &= \int_{E\setminus\bigcup_\gamma P_\gamma} d\mathcal{H}^\alpha \\
&\geq \frac{1}{M}\int_{E\setminus\bigcup_\gamma P_\gamma} |h|\, d\mathcal{H}^\alpha \\
&\geq \frac{1}{M}\left|\int_{E\setminus\bigcup_\gamma P_\gamma} h\, d\mathcal{H}^\alpha\right| \\
&\geq \frac{1}{M}\left|\int_E h\, d\mathcal{H}^\alpha\right| - \frac{1}{M}\sum_\gamma \left|\int_{P_\gamma} h\, d\mathcal{H}^\alpha\right| \\
&> \frac{1}{M}\left(\epsilon_0\mathcal{H}^\alpha(E) - \epsilon\sum_\gamma \mathcal{H}^\alpha(P_\gamma)\right).
\end{aligned} \tag{5.7}$$

Therefore,

$$\sum_\gamma \mathcal{H}^\alpha(P_\gamma) \leq (1-\eta)\mathcal{H}^\alpha(E), \tag{5.8}$$

for $\eta = (\epsilon_0 - \epsilon)/(M-\epsilon)$.

We want to construct the set E' by excising from E the union of the stopping-time cubes P_γ and replacing each child R_β of P_γ by a certain ball B_β.

Property (5) of Theorem 2.6 gives us a constant $0 < a_1 < 1$, such that, for each $Q \in \mathcal{Q}^k(E)$, there exists z_Q with $B(Q) = B(z_Q, a_1\delta^k) \cap E \subset Q$, $\operatorname{dist}(B(Q), E-Q) \approx d(Q)$, and $\mathcal{H}^\alpha(B(Q)) \approx \mathcal{H}^\alpha(Q)$. Recall that $d(Q) \leq \delta^k$.

For each stopping-time cube $P_\gamma = \cup_\beta R_\beta$, set $F_\gamma = \cup_\beta B_\beta$, where, for each β, $B_\beta = B(z_{R_\beta}, c\delta^k)$, with k being the generation of R_β and c some small constant such that $B_\beta \subset B(z_{R_\beta}, a_1\delta^k/2)$.

In what follows, set $\delta^k = r_\beta$, where k is the generation of R_β. That is, for each γ, the sets F_γ replace the stopping-time cubes P_γ in the new set E'. In other words,

$$E' = \left(E \setminus \bigcup_\gamma P_\gamma\right) \cup \bigcup_\gamma F_\gamma. \tag{5.9}$$

We will define now a measure μ on this set E' as follows:

$$\mu = \begin{cases} \mathcal{H}^\alpha & \text{on } E \setminus \bigcup_\gamma P_\gamma, \\ \dfrac{\mathcal{H}^\alpha(R_\beta)}{\mathcal{L}^n(B_\beta)} \mathcal{L}^n_{|B_\beta} & \text{on } \bigcup_\gamma F_\gamma = \bigcup_\beta B_\beta, \end{cases} \tag{5.10}$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.

We will now check that there exist some positive constants M_0 and M_1 such that, for every $x \in E'$ and $r > 0$,

(1) the measure μ has α-growth, that is,

$$\mu\big(E' \cap B(x,r)\big) \leq M_0 r^\alpha, \tag{5.11}$$

(2) the measure μ is doubling, that is,

$$\mu\big(E' \cap B(x,2r)\big) \leq M_1 \mu\big(E' \cap B(x,r)\big). \tag{5.12}$$

To prove that μ has α-growth, first let $x \in E' \setminus \cup_\beta B_\beta$, $r > 0$, and let β be such that $B_\beta \cap B(x,r) \neq \varnothing$. Since $B_\beta \cap B(x,r) \neq \varnothing$, we have

$$\big|x - z_\beta\big| \leq d\big(B_\beta\big) + r \leq \frac{a_1 r_\beta}{2} + r. \tag{5.13}$$

On the other hand, since $x \in E' \setminus B_\beta$, then $x \notin R_\beta$ and property (5) in Theorem 2.6 gives us $|x - z_\beta| > a_1 r_\beta$. Thus, by the definition of r_β and property (4) in Theorem 2.6, we get

$$d\big(R_\beta\big) \leq \delta^k = r_\beta < \frac{2r}{a_1}, \tag{5.14}$$

which implies that $R_\beta \subset B(x, 5r/a_1)$. Since our initial set E is Ahlfors-David regular and $\mu(B_\beta) = \mathcal{H}^\alpha(R_\beta)$, we get, for some positive constant M_0,

$$\begin{aligned} \mu\big(E' \cap B(x,r)\big) &\leq \mu\big(E \cap B(x,r)\big) + \sum_{B_\beta \cap B(x,r) \neq \varnothing} \mu\big(B_\beta \cap B(x,r)\big) \\ &\leq \mathcal{H}^\alpha\big(E \cap B(x,r)\big) + \sum_{R_\beta \subset B(x,5r/a_1)} \mathcal{H}^\alpha\big(R_\beta\big) \\ &\leq M_0 r^\alpha. \end{aligned} \tag{5.15}$$

If, for some β, $x \in B_\beta$, then the above inequality follows in the same way because the diameter of B_β is less than the distance to its complement in E'. Thus, the measure μ satisfies (5.11).

To prove that μ is a doubling measure, take any $x \in E' \setminus \cup_\beta B_\beta$ and $r > 0$. Then, arguing as above, but with r replaced by $2r$, we obtain

$$\begin{aligned}\mu\big(E' \cap B(x,2r)\big) &\leq \mathcal{H}^\alpha\big(E \cap E' \cap B(x,2r)\big) + \sum_{B_\beta \cap B(x,2r) \neq \varnothing} \mu\big(B_\beta \cap B(x,2r)\big)\\ &\leq \mathcal{H}^\alpha\big(E \cap B(x,2r)\big) + \sum_{R_\beta \subset B(x,10r/a_1)} \mathcal{H}^\alpha(R_\beta)\\ &\leq \mathcal{H}^\alpha\big(E \cap B\big(x,10r/a_1\big)\big)\\ &\leq C\mathcal{H}^\alpha\big(E \cap B(x,r)\big)\end{aligned} \tag{5.16}$$

because our initial measure $\mathcal{H}^\alpha$ is doubling on E.

We claim that, for some positive constant M_1, the following holds:

$$\mathcal{H}^\alpha\big(E \cap B(x,r)\big) \leq C\mu\big(E' \cap B(x,r)\big), \tag{5.17}$$

which proves (5.12).

To prove (5.17), let $Q^* \in \mathcal{Q}(E)$ be the biggest cube such that $x \in Q^* \subset B(x,r)$ and let $Q = (Q^* \setminus \bigcup_\beta R_\beta) \cup (\bigcup_{R_\beta \subset Q^*} B_\beta)$. Then, $Q \subset E' \cap B(x,r)$, and due to the definition of μ, we have $\mathcal{H}^\alpha(Q^*) = \mu(Q)$ (see (5.34)). Hence, the doubling property for $\mathcal{H}^\alpha$ on E gives that

$$\mathcal{H}^\alpha\big(E \cap B(x,r)\big) \leq C\mathcal{H}^\alpha\big(Q^*\big) = C\mu(Q) \leq C\mu\big(E' \cap B(x,r)\big), \tag{5.18}$$

and proves claim (5.17).

If $x \in B_\beta$, for some β and $r \leq r_\beta/2$, then the doubling property for μ holds clearly. If $r > r_\beta/2$. Then, arguing as above, one gets the doubling property for μ on E'. Therefore, (5.12) holds.

For a system of dyadic cubes $\mathcal{Q}(E')$ on E' satisfying the properties of Theorem 2.6 with respect to the doubling measure μ, take all dyadic cubes $Q \in \mathcal{Q}(E)$ which are not contained in any stopping-time cube P_γ, together with each $F_\gamma = \cup_\beta B_\beta$ and with the dyadic cubes of $\mathcal{Q}(B_\beta)$ in each F_γ. Namely,

$$\mathcal{Q}(E') = \mathcal{Q}_1(E') \cup \mathcal{Q}_2(E'), \tag{5.19}$$

where $\mathcal{Q}_1(E') = \{(S \setminus \bigcup_{P_\gamma \subset S} P_\gamma) \cup (\bigcup_{P_\gamma \subset S} F_\gamma) : S \in \mathcal{Q}(E) \setminus \{P_\gamma\}\} \cup \{F_\gamma\}$ and $\mathcal{Q}_2(E')$ consists of the dyadic systems $\mathcal{Q}(B_\beta)$ associated to the balls B_β coming from all the F_γ. Hence, each F_γ is a dyadic cube in $\mathcal{Q}(E')$.

After defining the set E', the doubling measure μ, and the system of dyadic cubes $\mathcal{Q}(E')$, our next step will consist in modifying the function h on the union $\cup_\gamma F_\gamma$ in order to obtain a new function b, defined on E', bounded and dyadic para-accretive with respect to the system of dyadic cubes $\mathcal{Q}(E')$. In fact, we want b to satisfy

$$\int_{F_\gamma} b\, d\mu = \int_{P_\gamma} h\, d\mathcal{H}^\alpha, \quad \text{for each } \gamma. \tag{5.20}$$

Condition (5.20) does not seem to contribute to the accretivity of the new function b with respect to the measure μ because the cubes P_γ were chosen precisely because the mean of h on them became too small. But although our b has a small mean on F_γ, as h does on P_γ, we will have a satisfactory lower bound on the integral of b over each child B_β of F_γ. In this way, b becomes "more" accretive than h.

The function b is defined on E' by

$$b(x) = \begin{cases} h(x) & \text{if } x \in E \setminus \bigcup_\gamma P_\gamma, \\ \sum_\beta c_\beta \chi_{B_\beta}(x) & \text{on } \bigcup_\gamma F_\gamma = \bigcup_\beta B_\beta, \end{cases} \tag{5.21}$$

where the coefficients c_β are defined below to get the boundedness of the function b and (5.20).

Notice first that due to properties (5) and (6) of Theorem 2.6, $B_\beta \cap B_\eta = \emptyset$, for $\beta \neq \eta$, and $B_\beta \cap (E \setminus \cup_\gamma P_\gamma) = \emptyset$ so that the function b is well defined on E'.

To define the coefficients c_β, fix P_γ and let $N_\gamma = \sharp\{\beta : R_\beta \text{ is a child of } P_\gamma\}$. The number of children of the dyadic cubes is in between 2 and a fixed upper bound, that is, $2 \leq N_\gamma \leq c_1$, where c_1 is some constant independent of γ.

Order the children $\{R_\beta\}$ of P_γ starting with the cube R_β with the smallest $\mathcal{H}^\alpha$-measure and ending with the cube R_β with the biggest one. Write $\{R_\beta\} = \{R_\beta^j\}_{j=1}^{N_\gamma}$, where R_β^j stands for the jth child R_β in this ordering. We want to divide the children of P_γ into two nonempty collections I and II, each with the same number of elements (plus or minus one) in the following way:

$$\begin{aligned} I &= \left\{\beta : R_\beta = R_\beta^j, \text{ for } 1 \leq j \leq \left[\frac{N_\gamma}{2}\right]\right\}, \\ II &= \left\{\beta : R_\beta = R_\beta^j, \text{ for } \left[\frac{N_\gamma}{2}\right] + 1 \leq j \leq N_\gamma\right\}. \end{aligned} \tag{5.22}$$

Clearly,

$$\sum_{\beta \in II} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in I} \mathcal{H}^\alpha(R_\beta) \geq 0. \tag{5.23}$$

Let θ be $\int_{P_\gamma} h\, d\mathcal{H}^\alpha (|\int_{P_\gamma} h\, d\mathcal{H}^\alpha|)^{-1}$ if $\int_{P_\gamma} h\, d\mathcal{H}^\alpha \neq 0$ and let θ be 1 if $\int_{P_\gamma} h\, d\mathcal{H}^\alpha = 0$. Define the coefficients c_β as

$$c_\beta = \begin{cases} \theta & \text{if } \beta \in I, \\ -\theta \widetilde{c}_\beta & \text{if } \beta \in II, \end{cases} \tag{5.24}$$

where the $\widetilde{c}_\beta$ satisfy $\epsilon_0 \leq \widetilde{c}_\beta \leq 1$ and, moreover, a certain constraint specified below.

Notice that the above-defined function b is bounded:

$$\|b\|_\infty = \max\left\{\|h\|_\infty, |c_\beta|\right\} \leq \max\left\{\|h\|_\infty, 1\right\} \leq C. \tag{5.25}$$

Moreover, integrating b on F_γ with respect to the measure μ, we get

$$\int_{F_\gamma} b\, d\mu = \sum_\beta c_\beta \mathcal{H}^\alpha(R_\beta) = \sum_{\beta \in I} \theta \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in II} \theta \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta). \tag{5.26}$$

We claim that we can choose $\epsilon_0 > 0$ sufficiently small so that there exist numbers $\widetilde{c}_\beta$, $\epsilon_0 \leq \widetilde{c}_\beta \leq 1$, such that

$$\sum_{\beta \in I} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in II} \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta) = \left| \int_{P_\gamma} h\, d\mathcal{H}^\alpha \right|. \tag{5.27}$$

Once (5.27) is proved, we get the desired expression for the integral of b over F_γ, namely,

$$\int_{F_\gamma} b\, d\mu = \theta\left(\sum_{\beta \in I} \mathcal{H}^\alpha(R_\beta) - \sum_{\beta \in II} \widetilde{c}_\beta \mathcal{H}^\alpha(R_\beta) \right) = \theta \left| \int_{P_\gamma} h\, d\mathcal{H}^\alpha \right| = \int_{P_\gamma} h\, d\mathcal{H}^\alpha. \tag{5.28}$$

To show (5.27), let $N_2 = \sharp\{\beta : \beta \in II\}$ and define

$$\widetilde{c}_\eta = \frac{1}{N_2 \mathcal{H}^\alpha(R_\eta)} \left(\sum_{\beta \in I} \mathcal{H}^\alpha(R_\beta) - \left| \int_{P_\gamma} h\, d\mathcal{H}^\alpha \right| \right). \tag{5.29}$$

With this choice of the coefficients $\widetilde{c}_\eta$, equality (5.27) clearly holds. Thus, we only have to show that there exists $\epsilon_0 > 0$ such that $\epsilon_0 \leq \widetilde{c}_\eta \leq 1$, for all η.

The inequality $\widetilde{c}_\eta \leq 1$ is equivalent to

$$\frac{1}{N_2 \mathcal{H}^\alpha(R_\eta)} \sum_{\beta \in I} \mathcal{H}^\alpha(R_\beta) \leq 1 + \frac{1}{N_2 \mathcal{H}^\alpha(R_\eta)} \left| \int_{P_\gamma} h\, d\mathcal{H}^\alpha \right|. \tag{5.30}$$

Notice that, by the way the indexes were ordered, for all $\eta \in II$,

$$\sum_{\beta \in I} \mathcal{H}^{\alpha}(R_{\beta}) \leq N_2 \mathcal{H}^{\alpha}(R_{\eta}), \tag{5.31}$$

which implies $\widetilde{c}_{\eta} \leq 1$.

For the lower inequality (5.32), we have to choose ϵ_0 such that $\widetilde{c}_{\eta} \geq \epsilon_0$. Recall that, for P_{γ}, the stopping-time condition (5.6) holds with Q replaced by P_{γ}, and that the children of P_{γ} have comparable measures. Moreover, we know that there exists some (small) positive constant $0 < c < 1/2$ such that $\sum_{\beta \in I} \mathcal{H}^{\alpha}(R_{\beta}) \geq c\mathcal{H}^{\alpha}(P_{\gamma})$. Then, we have

$$\widetilde{c}_{\eta} \geq \frac{(c-\epsilon)\mathcal{H}^{\alpha}(P_{\gamma})}{N_2 \mathcal{H}^{\alpha}(R_{\eta})} \geq \frac{(c-\epsilon)\mathcal{H}^{\alpha}(P_{\gamma})}{N_{\gamma} \mathcal{H}^{\alpha}(P_{\gamma})} \geq \frac{c-\epsilon}{c_1}, \tag{5.32}$$

where $c_1 > 0$ is the upper bound for the number of children of a dyadic cube.

We have to choose ϵ_0 and ϵ such that $c - \epsilon \geq \epsilon_0 c_1$ holds. This can be achieved by requiring $\epsilon_0 c_1 \leq c/2$ and $\epsilon < \min(\epsilon_0, c/2)$. The identity (5.27) is now proved, and therefore, (5.20) holds.

In order to construct the function b, we have to carry out this procedure for each stopping-time cube P_{γ}.

The P_{γ} are the cubes where the accretivity condition for h fails. The function h_1 has the advantage that although $\int_{P_{\gamma}} h \, d\mathcal{H}^{\alpha} = \int_{F_{\gamma}} h_1 \, d\mathcal{H}^{\alpha}$, we have a satisfactory lower bound on the integral over each child B_{β} of F_{γ}. This is due to the definition of the coefficients c_{β}.

(1) If $\beta \in I$, then $|\int_{B_{\beta}} b \, d\mu| = |c_{\beta}|\mu(B_{\beta}) \geq \epsilon_0 \mu(B_{\beta})$.

(2) If $\beta \in II$, then $|\int_{B_{\beta}} b \, d\mu| \geq |\widetilde{c}_{\beta}|\mu(B_{\beta}) \geq \epsilon_0 \mu(B_{\beta})$ because $\epsilon_0 \leq \widetilde{c}_{\beta}$.

Thus, the function b satisfies the para-accretivity condition on the cubes F_{γ}.

For future reference, note that, for every cube $Q \in \mathcal{Q}(E')$, such that $Q \not\subseteq F_{\gamma}$ for all γ, there is a nonstopping time cube $Q^* \in \mathcal{Q}(E)$ uniquely associated to Q by the identity

$$Q = \left(Q^* \setminus \bigcup_{P_{\gamma} \subset Q^*} P_{\gamma}\right) \cup \left(\bigcup_{P_{\gamma} \subset Q^*} F_{\gamma}\right). \tag{5.33}$$

Moreover, one has

$$\mu(Q) = \mathcal{H}^{\alpha}(Q^*) - \sum_{P_{\gamma} \subset Q^*} \mathcal{H}^{\alpha}(P_{\gamma}) + \sum_{P_{\gamma} \subset Q^*} \mu(F_{\gamma}) = \mathcal{H}^{\alpha}(Q^*), \tag{5.34}$$

$$d(Q) \approx (Q^*). \tag{5.35}$$

We will check now that, by construction, the function b is dyadic para-accretive with respect to $\mathcal{Q}(E')$.

(1) If $Q \in \mathcal{Q}_2(E')$, then $Q = B_\beta$, for some β, or $Q \in \mathcal{Q}(B_\beta)$. In both cases, the para-accretivity of b follows as above due to the lower bound of $|c_\beta|$.

(2) If $Q \in \mathcal{Q}_1(E')$, the case $Q = F_\gamma$ has already been discussed, so we are only left with $Q \in \mathcal{Q}_1(E') \setminus \{F_\gamma\}$.

Let $Q^* \in \mathcal{Q}(E)$ be the cube defined in (5.33). Recall that Q^* is a nonstopping time cube. Then, due to (5.20) and (5.34), we can write

$$\begin{aligned}\left|\int_Q b\,d\mu\right| &= \left|\int_{Q^*\setminus \cup_{P_\gamma \subset Q^*} P_\gamma} h\,d\mathcal{H}^\alpha + \sum_{P_\gamma \subset Q^*} \int_{F_\gamma} b\,d\mu\right| \\ &= \left|\int_{Q^*\setminus \cup_{P_\gamma \subset Q^*} P_\gamma} h\,d\mathcal{H}^\alpha + \sum_{P_\gamma \subset Q^*} \int_{P_\gamma} h\,d\mathcal{H}^\alpha\right| \\ &= \left|\int_{Q^*} h\,d\mathcal{H}^\alpha\right| \geq \epsilon \mathcal{H}^\alpha(Q^*) = \epsilon\mu(Q).\end{aligned} \tag{5.36}$$

Hence, b is a dyadic para-accretive function with respect to $\mathcal{Q}(E')$.

We are still left with the fact that $R_\alpha(b\mu)$ belongs to dyadic $BMO(\mu)$. We postpone the proof of the BMO-boundedness and we continue with the argument.

At this point, we have constructed a set E' with a system of dyadic cubes $\mathcal{Q}(E')$, a function b dyadic para-accretive with respect to this system of dyadic cubes, and a measure μ which is doubling and has α-growth. Moreover, we are assuming that the function $R_\alpha(b\mu)$ belongs to dyadic $BMO(\mu)$. Therefore, by the $T(b)$ theorem (see Theorem 2.9), the Riesz α-operator R_α associated to the measure μ is bounded in $L^2(\mu)$.

Notice that since

$$\begin{aligned}&\int_E h\,d\mathcal{H}^\alpha = \int_{E\setminus \cup_\gamma P_\gamma} h\,d\mathcal{H}^\alpha + \int_{\cup_\gamma P_\gamma} h\,d\mathcal{H}^\alpha \neq 0, \\ &\int_{\cup_\gamma P_\gamma} h\,d\mathcal{H}^\alpha < \epsilon \sum_\gamma \mathcal{H}^\alpha(P_\gamma) < \epsilon \mathcal{H}^\alpha(E), \\ &\left|\int_E h\,d\mathcal{H}^\alpha\right| \geq \epsilon_0 \mathcal{H}^\alpha(E),\end{aligned} \tag{5.37}$$

we get

$$\left|\int_{E\setminus \cup_\gamma P_\gamma} h\,d\mathcal{H}^\alpha\right| \geq (\epsilon_0 - \epsilon)\mathcal{H}^\alpha(E) > 0 \tag{5.38}$$

from the choice of ϵ_0 and ϵ. This shows that $\mathcal{H}^\alpha(E \setminus \cup_\gamma P_\gamma) > 0$, and therefore, that $\mathcal{H}^\alpha(E \cap E') > 0$ because of the inclusion $E \setminus \cup_\gamma P_\gamma \subset E' \cap E$. In fact, from (5.8), we get the better lower bound $\mathcal{H}^\alpha(E \cap E') \geq \mathcal{H}^\alpha(E \setminus \bigcup_\gamma P_\gamma) \geq \eta \mathcal{H}^\alpha(E)$.

Set

$$E_{good} = E \setminus \bigcup_{\gamma} P_\gamma,$$
$$E_{bad} = E \setminus E_{good} = \bigcup_{\gamma} P_\gamma. \tag{5.39}$$

By density (see, e.g., [12, Corollary 2.14]), for $\mathcal{H}^\alpha_E$-almost all $x \in E_{good}$, the limit

$$\limsup_{r\to 0} \frac{\mathcal{H}^\alpha\big(E_{bad} \cap B(x,r)\big)}{r^\alpha} = 0. \tag{5.40}$$

Therefore, for such x, using the lower bound from the Ahlfors-David regularity of the set E, we obtain

$$\begin{aligned}\liminf_{r\to 0} &\frac{\mathcal{H}^\alpha\big(E_{good} \cap B(x,r)\big)}{r^\alpha} \\ &\geq \liminf_{r\to 0}\left(\frac{\mathcal{H}^\alpha\big(E \cap B(x,r)\big)}{r^\alpha} - \frac{\mathcal{H}^\alpha\big(E_{bad} \cap B(x,r)\big)}{r^\alpha}\right) \\ &\geq C^{-1} - \limsup_{r\to 0} \frac{\mathcal{H}^\alpha\big(E_{bad} \cap B(x,r)\big)}{r^\alpha} = C^{-1}.\end{aligned} \tag{5.41}$$

Moreover, the upper bound coming from the Ahlfors-David regularity of the set E implies that, for every $x \in E$, we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^\alpha\big(E_{good} \cap B(x,r)\big)}{r^\alpha} \leq \limsup_{r\to 0} \frac{\mathcal{H}^\alpha\big(E \cap B(x,r)\big)}{r^\alpha} \leq C. \tag{5.42}$$

Let $x_0 \in E_{good}$ be a point satisfying (5.41) and (5.42) and let $\sigma \in \mathrm{Tan}(\mathcal{H}^\alpha_{E_{good}}, x_0)$. Then, [12, Lemma 14.7] shows that there is a positive number C such that

$$C^{-1} r^\alpha \leq \sigma\big(B(x,r)\big) \leq C r^\alpha, \quad \text{for } x \in \operatorname{spt}\sigma,\ 0 < r < \infty, \tag{5.43}$$

which is the same as to say that σ is an Ahlfors-David regular measure.

Now, we only have to show that the α-Riesz operator associated to σ is bounded in $L^2(\sigma)$.

Notice first that due to Remark 5.2, there exists a sequence $r_i \to 0$ such that, for some positive number d,

$$\sigma = d \lim_{r_i\to 0} r_i^{-\alpha} T_{x_0,r_i\sharp}\mathcal{H}^\alpha_{E_{good}} = \lim_{r_i\to 0} \mathcal{H}^\alpha_{r_i}, \tag{5.44}$$

where the last identity is the definition of the measures $\mathcal{H}^\alpha_{r_i}$.

In what follows, we let T stand for the α-Riesz operator R_α.

Fix a radial function $\varphi \in \mathcal{C}^\infty$ such that $0 \le \varphi \le 1$, $\varphi = 0$ on $B(0, 1/2)$, and $\varphi = 1$ on $\mathbb{R}^n \setminus B(0,1)$. For $\epsilon > 0$, define the regularized operators $\tilde{T}_\epsilon$ as follows:

$$\tilde{T}_\epsilon(f\nu)(x) = \int \varphi\left(\frac{x-y}{\epsilon}\right)\frac{x-y}{|x-y|^{1+\alpha}} f(y)\,d\nu(y), \tag{5.45}$$

for complex Radon measures ν in $\mathbb{R}^n$. One can easily check that, for $\epsilon > 0$,

$$\left|\tilde{T}_\epsilon(f\nu)(x) - T_\epsilon(f\nu)(x)\right| \le CM(f\nu)(x), \tag{5.46}$$

where $M(f\nu)$ is the Hardy-Littlewood maximal function:

$$M(f\nu)(x) = \sup_{r>0} \frac{1}{\nu\big(B(x,r)\big)} \int_{B(x,r)} f(y)\,d\nu(y). \tag{5.47}$$

It is well known that M is bounded in L^2. Thus, the L^2-boundedness of the truncated operators T_ϵ is equivalent to that of $\tilde{T}_\epsilon$. If the measure we are considering is non-doubling, then the maximal function in (5.47) does not work, but instead of M, one can consider a modified maximal operator introduced in [17] to get the same equivalence.

Notice that the fact that the operator T with respect to $\mathcal{H}^\alpha_{E_{\text{good}}}$ is bounded in $L^2(\mathcal{H}^\alpha_{E_{\text{good}}})$ implies that, for each $r > 0$, the operator T with respect to $\mathcal{H}^\alpha_r$ is bounded in $L^2(\mathcal{H}^\alpha_r)$; namely, for f and g test functions, we have

$$\left|\left\langle \tilde{T}_\epsilon\left(f\mathcal{H}^\alpha_r\right), g\right\rangle\right| \le C\|f\|_{L^2(\mathcal{H}^\alpha_r)}\|g\|_{L^2(\mathcal{H}^\alpha_r)}. \tag{5.48}$$

Therefore,

$$\begin{aligned}\left|\left\langle \tilde{T}_\epsilon(f\sigma), g\sigma\right\rangle\right| &= \lim_{r_i\to 0}\left|\left\langle \tilde{T}_\epsilon\left(f\mathcal{H}^\alpha_{r_i}\right), g\mathcal{H}^\alpha_{r_i}\right\rangle\right| \\ &\le C \lim_{r_i\to 0}\|f\|_{L^2(\mathcal{H}^\alpha_{r_i})}\|g\|_{L^2(\mathcal{H}^\alpha_{r_i})} \\ &= C\|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)},\end{aligned} \tag{5.49}$$

which means that T is bounded in $L^2(\sigma)$.

We still have to show that $T(b\mu)$ is a BMO function. We claim that since the function $b \in L^\infty(E')$, it is enough to show the following L^1-inequality:

$$\left\|T\big(b\chi_Q\big)\right\|_{L^1(\mu_Q)} \le C\mu(Q), \tag{5.50}$$

for every $Q \in \mathcal{Q}(E')$, where μ_Q denotes the restriction of the measure μ to Q.

Suppose (5.50) holds for every $Q \in \mathcal{Q}(E')$, and let, for some positive constant A, $2Q = \{x \in E' : \operatorname{dist}(x, Q) \le Ad(Q)\}$. As a consequence of the "small boundary condition" for the dyadic cubes (see Theorem 2.6, property (6)), we have

$$\left\|T\left(b\chi_{2Q\setminus Q}\right)\right\|_{L^1(\mu_Q)} \le C\mu(Q) \tag{5.51}$$

(see the bound for the second integral in (5.64)). The standard estimates for the Calderón-Zygmund operators show that

$$\left\|T\left(b\chi_{(2Q)^c}\right)(x) - T\left(b\chi_{(2Q)^c}\right)(x_0)\right\|_{L^1(\mu_Q)} \le C\mu(Q), \tag{5.52}$$

where x_0 is a fixed point in Q. This implies that

$$\begin{aligned} &\int_Q \left|T(b)(x) - T\left(b\chi_{(2Q)^c}\right)(x_0)\right| d\mu(x) \\ &\quad \le \int_Q \left|T\left(b\chi_Q\right)(x)\right| d\mu(x) + \int_Q \left|T\left(b\chi_{2Q\setminus Q}\right)(x)\right| d\mathcal{H}^\alpha(x) \\ &\quad\quad + \int_Q \left|T\left(b\chi_{(2Q)^c}\right)(x) - T\left(b\chi_{(2Q)^c}\right)(x_0)\right| d\mu(x) \\ &\quad \le C\mu(Q), \end{aligned} \tag{5.53}$$

which proves the claim.

To prove (5.50), let $Q \in \mathcal{Q}(E')$ be some dyadic cube of E'. We distinguish now between two cases.

(1) For some β, let $Q = B_\beta$ or $Q \in \mathcal{Q}(B_\beta)$. Set $K(x) = x/|x|^{1+\alpha}$. Then, Fubini and a change of variables give us, for some constant c,

$$\begin{aligned} &\int_{B_\beta} \left| \int_{B_\beta} K(x-y) b(x) d\mu(x) \right| d\mathcal{L}^n(y) \\ &\quad \le \left|c_\beta\right| \frac{\mathcal{H}^\alpha\left(R_\beta\right)}{\mathcal{L}^n\left(B_\beta\right)} \int_{B_\beta} \int_{B_\beta} \frac{d\mathcal{L}^n(x)}{|x-y|^\alpha} d\mathcal{L}^n(y) \\ &\quad \le C \frac{\mathcal{H}^\alpha\left(R_\beta\right)}{\mathcal{L}^n\left(B_\beta\right)} \int_{B_\beta} \int_{B(x, c r_\beta)} \frac{d\mathcal{L}^n(y) d\mathcal{L}^n(x)}{|y-x|^\alpha} \\ &\quad = C \frac{\mathcal{H}^\alpha\left(R_\beta\right)}{\mathcal{L}^n\left(B_\beta\right)} \int_{B_\beta} \int_{B(0, c r_\beta)} \frac{d\mathcal{L}^n(z)}{|z|^\alpha} d\mathcal{L}^n(x) \\ &\quad = C\mathcal{H}^\alpha\left(R_\beta\right) \int_{B(0, c r_\beta)} \frac{d\mathcal{L}^n(z)}{|z|^\alpha} \\ &\quad \le C\mathcal{L}^n\left(B_\beta\right), \end{aligned} \tag{5.54}$$

where at the last step, we have used the fact that E is Ahlfors-David regular, and so $\mathcal{H}^\alpha(R_\beta) \approx d(R_\beta)^\alpha \approx r_\beta^\alpha$.

Since $\mu_{|B_\beta} = (\mathcal{H}^\alpha(R_\beta)/\mathcal{L}^n(B_\beta))\mathcal{L}^n_{|B_\beta}$, we get

$$\int_{B_\beta} |T(b\chi_{B_\beta})| d\mu \leq C\mathcal{H}^\alpha(R_\beta) = C\mu(B_\beta), \tag{5.55}$$

which is (5.50) in this case. If $Q \in \mathcal{Q}(B_\beta)$, for some β, (5.50) is obtained arguing in a similar way.

(2) Let $Q \in \mathcal{Q}_1(E')$. If $Q = F_\gamma$, for some γ, then one argues as in the previous case because, for each γ, the number of B_β involved in $\bigcup_\beta B_\beta = F_\gamma$ is bounded above by some constant independent of γ. Thus, let $Q \in \mathcal{Q}_1(E') \setminus \{F_\gamma\}_\gamma$ and let Q^* be the uniquely associated nonstopping dyadic cube in $\mathcal{Q}(E)$ defined before. Using (5.20), we can write

$$\begin{aligned} T(b\chi_Q) &= \int_{Q^*\setminus\cup_\beta R_\beta} h(x)K(x-y)d\mathcal{H}^\alpha(x) + \sum_{R_\beta\subset Q^*}\int_{B_\beta} b(x)K(x-y)d\mu(x) \\ &= T(h\chi_{Q^*}) + \sum_{R_\beta\subset Q^*}\int_{B_\beta} b(x)\big(K(x-y)-K(z_\beta-y)\big)d\mu(x) \\ &\quad + \sum_{R_\beta\subset Q^*}\int_{R_\beta} h(x)\big(K(z_\beta-y)-K(x-y)\big)d\mathcal{H}^\alpha(x). \end{aligned} \tag{5.56}$$

We claim that the following estimates hold for each β:

$$\int_{Q\setminus B_\beta}\int_{B_\beta} |K(x-y)-K(z_\beta-y)| d\mu(x)d\mu(y) \leq C\mu(B_\beta), \tag{5.57}$$

$$\int_{Q\setminus B_\beta}\int_{R_\beta} |K(z_\beta-y)-K(x-y)| d\mathcal{H}^\alpha(x)d\mu(y) \leq C\mathcal{H}^\alpha(R_\beta), \tag{5.58}$$

$$\int_{B_\beta}\left|\int_{R_\beta} h(x)K(x-y)d\mathcal{H}^\alpha(x)\right| d\mu(y) \leq C\mu(B_\beta). \tag{5.59}$$

Moreover,

$$\int_Q |T(h\chi_{Q^*})| d\mu \leq C\mu(Q). \tag{5.60}$$

If (5.60), (5.57), (5.58), and (5.59) hold, going back to (5.56) and using the boundedness of the functions h and b together with the fact that $\mathcal{H}^\alpha(R_\beta) = \mu(B_\beta)$ for each β, we can write

$$
\begin{aligned}
&\int_Q \left|T\left(b\chi_Q\right)\right| d\mu \\
&\quad \le C\mu(Q) + C\sum_{R_\beta \subset Q^*} \left\{ \int_{Q\setminus B_\beta}\int_{B_\beta} \left|K(x-y) - K\left(z_\beta - y\right)\right| d\mu(x)\,d\mu(y) \right. \\
&\qquad + \int_{Q\setminus B_\beta}\int_{R_\beta} \left|K\left(z_\beta - y\right) - K(x-y)\right| d\mathcal{H}^\alpha(x)\,d\mu(y) \\
&\qquad + \int_{B_\beta} \left| \int_{B_\beta} b(x)K(x-y)\,d\mu(x) \right| d\mu(y) \\
&\qquad + \int_{B_\beta} \left| \int_{R_\beta} h(x)K(x-y)\,d\mathcal{H}^\alpha(x) \right| d\mu(y) \\
&\qquad \left. + \int_{B_\beta} \frac{\mu\left(B_\beta\right)}{\left|z_\beta - y\right|^\alpha} d\mu(y) + \int_{B_\beta} \frac{\mathcal{H}^\alpha\left(R_\beta\right)}{\left|z_\beta - y\right|^\alpha} d\mu(y) \right\} \\
&\quad \le C\mu(Q) + C\sum_{R_\beta \subset Q^*} \mu\left(B_\beta\right) + C\sum_{R_\beta \subset Q^*} \mu\left(B_\beta\right) \int_{B_\beta} \frac{d\mu(y)}{\left|z_\beta - y\right|^\alpha}.
\end{aligned}
\tag{5.61}
$$

Since $\mathcal{H}^\alpha(R_\beta) \approx d(R_\beta)$, we get

$$
\int_{B_\beta} \frac{d\mu(y)}{\left|z_\beta - y\right|^\alpha} = C \frac{\mathcal{H}^\alpha\left(R_\beta\right)}{\mathcal{L}^n\left(B_\beta\right)} \frac{\mathcal{L}^n\left(B_\beta\right)}{d\left(R_\beta\right)^\alpha} \int_{B(0,1)} \frac{d\mathcal{L}^n(z)}{|z|^\alpha} \le C, \tag{5.62}
$$

which is (5.50) provided that inequalities (5.60), (5.57), (5.58), and (5.59) hold.

We deal first with (5.57). Notice that if $x \in B_\beta$ and $y \in Q\setminus B_\beta$, then $|x-y| \ge a_1 r_\beta/2$. Hence, the standard estimates for the Calderón-Zygmund operators and the α-growth of the measure μ give

$$
\begin{aligned}
&\int_{Q\setminus B_\beta}\int_{B_\beta} \left|K(x-y) - K\left(z_\beta - y\right)\right| d\mu(x)\,d\mu(y) \\
&\quad \le C\int_{B_\beta} \sum_{j=1}^{\infty} \int_{\{2^{j-1}a_1 r_\beta \le |y-x| \le 2^j a_1 r_\beta\}} \frac{\left|x - z_\beta\right|}{|x-y|^{1+\alpha}} d\mu(y)\,d\mu(x) \\
&\quad \le C\int_{B_\beta} \sum_{j=1}^{\infty} \frac{\mu\left(\left\{2^{j-1}a_1 r_\beta \le |y-x| \le 2^j a_1 r_\beta\right\}\right)}{\left(2^{j-1}a_1 r_\beta\right)^{1+\alpha}} r_\beta\,d\mu(x) \\
&\quad \le C\int_{B_\beta} \sum_{j=1}^{\infty} 2^{-j}\,d\mu(x) \le C\mu\left(B_\beta\right).
\end{aligned}
\tag{5.63}
$$

To show (5.58), notice that $(Q \setminus B_\beta) \cap R_\beta = \varnothing$. Therefore,

$$\begin{aligned}
&\int_{Q\setminus B_\beta}\int_{R_\beta}\left|\frac{z_\beta - y}{|z_\beta - y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(x)\,d\mu(y)\\
&\quad = \int_{(Q\setminus B_\beta)\setminus 2R_\beta}\int_{R_\beta}\left|\frac{z_\beta - y}{|z_\beta - y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(x)\,d\mu(y)\\
&\qquad + \int_{(Q\setminus B_\beta)\cap 2R_\beta\setminus R_\beta}\int_{R_\beta}\left|\frac{z_\beta - y}{|z_\beta - y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(x)\,d\mu(y).
\end{aligned} \tag{5.64}$$

The first integral in (5.64) may be estimated in the same way as (5.63). Thus, we get

$$\int_{(Q\setminus B_\beta)\setminus 2R_\beta}\int_{R_\beta}\left|\frac{z_\beta - y}{|z_\beta - y|^{1+\alpha}} - \frac{x-y}{|x-y|^{1+\alpha}}\right| d\mathcal{H}^\alpha(x)\,d\mu(y) \le C\mathcal{H}^\alpha(R_\beta). \tag{5.65}$$

To deal with the second integral in (5.64), let $j \in \mathbb{Z}$ and define the set

$$A_j = \left\{x \in R_\beta : 2^{j-1} r_\beta < \operatorname{dist}\left(x, 2R_\beta \setminus R_\beta\right) \le 2^j r_\beta\right\}. \tag{5.66}$$

Now, for $x \in A_j$, let $F_i(x) = \{y \in 2R_\beta \setminus R_\beta : 2^{i-1} r_\beta < |x-y| \le 2^i r_\beta\}$. Then, because of (5.11),

$$\begin{aligned}
\int_{2R_\beta\setminus R_\beta}\left|\frac{x-y}{|x-y|^{1+\alpha}}\right| d\mu(y) &= \sum_{i=j}^{1}\int_{F_i(x)}\left|\frac{x-y}{|x-y|^{1+\alpha}}\right| d\mu(y)\\
&\le \sum_{i=j}^{1}\int_{F_i(x)}\frac{1}{|x-y|^\alpha}\,d\mu(y)\\
&\le \sum_{i=j}^{1}\frac{C\left(2^i r_\beta\right)^\alpha}{\left(2^{i-1} r_\beta\right)^\alpha}\\
&\le C\sum_{i=j}^{1} 1 \le C\left(1+|j|\right).
\end{aligned} \tag{5.67}$$

Summing over j and using the "small boundary" condition stated in property (6) of Theorem 2.6 gives

$$\begin{aligned}\int_{R_\beta}\int_{2R_\beta\setminus R_\beta}\left|\frac{x-y}{|x-y|^{1+\alpha}}\right|d\mu(y)\,d\mathcal{H}^\alpha(x) &\le C\sum_{j=-\infty}^{0}(1+|j|)\int_{A_j}d\mathcal{H}^\alpha(x)\\ &= C\sum_{j=-\infty}^{0}(1+|j|)\mathcal{H}^\alpha(A_j)\\ &\le C\sum_{j=-\infty}^{0}(1+|j|)b_1 2^{\eta j}\mathcal{H}^\alpha(R_\beta)\\ &\le C\mathcal{H}^\alpha(R_\beta).\end{aligned} \tag{5.68}$$

Moreover, using (5.12) and (5.11), we obtain

$$\int_{R_\beta}\int_{2R_\beta\setminus R_\beta}\left|\frac{z_\beta-y}{\left|z_\beta-y\right|^{1+\alpha}}\right|d\mu(y)\,d\mathcal{H}^\alpha(x)\le\frac{\mathcal{H}^\alpha(R_\beta)\mu(2R_\beta\setminus R_\beta)}{(cr_\beta)^\alpha}\le C\mathcal{H}^\alpha(R_\beta). \tag{5.69}$$

Therefore, we have

$$\int_{2R_\beta\setminus R_\beta}\int_{R_\beta}\left|\frac{z_\beta-y}{\left|z_\beta-y\right|^{1+\alpha}}-\frac{x-y}{|x-y|^{1+\alpha}}\right|d\mathcal{H}^\alpha(x)\,d\mathcal{H}^\alpha(y)\le C\mathcal{H}^\alpha(R_\beta), \tag{5.70}$$

and so we are done with the estimate of the second integral in (5.64) and we get

$$\int_{Q\setminus B_\beta}\int_{R_\beta}\left|\frac{z_\beta-y}{\left|z_\beta-y\right|^{1+\alpha}}-\frac{x-y}{|x-y|^{1+\alpha}}\right|d\mathcal{H}^\alpha(x)\,d\mu(y)\le C\mathcal{H}^\alpha(R_\beta), \tag{5.71}$$

which is (5.58).

To show (5.59), let R_β^c be the complement of R_β. Then, using that $h\mathcal{H}^\alpha * K$ is a bounded function, we can write

$$\begin{aligned}&\int_{B_\beta}\left|\int_{R_\beta}h(x)K(x-y)\,d\mathcal{H}^\alpha(x)\right|d\mu(y)\\ &\quad\le C\mu(B_\beta)+\int_{B_\beta}\left|\int_{R_\beta^c}h(x)K(x-y)\,d\mathcal{H}^\alpha(x)\right|d\mu(y).\end{aligned} \tag{5.72}$$

Recall that $B_\beta=B(z_\beta,cr_\beta)$. Then, the boundedness of h, together with the upper bound in the Ahlfors-David regularity condition, implies that there exists a constant m

such that

$$\begin{aligned} &\frac{1}{\mathcal{L}^n(B_\beta)}\int_{B_\beta}\left|\int_{R_\beta} h(x)K(x-y)\,d\mathcal{H}^\alpha(x)\right| d\mathcal{L}^n(y) \\ &\quad\le \frac{C}{r_\beta^n}\int_{R_\beta}\int_{B(x,cmr_\beta)}\frac{1}{|x-y|^\alpha}\,d\mathcal{L}^n(y)\,d\mathcal{H}^\alpha(x)\le C. \end{aligned} \tag{5.73}$$

Therefore, there exists a point $y_\beta \in B_\beta$ such that

$$\left|\int_{R_\beta} h(x)K\big(x-y_\beta\big)\,d\mathcal{H}^\alpha(x)\right| \le C. \tag{5.74}$$

Consequently,

$$\left|\int_{R_\beta^c} h(x)K\big(x-y_\beta\big)\,d\mathcal{H}^\alpha(x)\right| \le C, \tag{5.75}$$

which gives

$$\begin{aligned} &\int_{B_\beta}\left|\int_{R_\beta^c} h(x)K(x-y)\,d\mathcal{H}^\alpha(x)\right| d\mu(y) \\ &\quad\le \int_{B_\beta}\left|\int_{R_\beta^c} h(x)K(x-y)\,d\mathcal{H}^\alpha(x)-\int_{R_\beta^c} h(x)K\big(x-y_\beta\big)\,d\mathcal{H}^\alpha(x)\right| d\mu(y) \\ &\quad\quad+\int_{B_\beta}\left|\int_{R_\beta^c} h(x)K\big(x-y_\beta\big)\,d\mathcal{H}^\alpha(x)\right| d\mu(y) \\ &\quad\le C\mu\big(B_\beta\big) \end{aligned} \tag{5.76}$$

by arguing similarly as in the proof of (5.63).

We are now left with the proof of (5.60). Notice that we can write

$$\begin{aligned} &\int_Q \big|T\big(h\chi_{Q^*}\big)\big|\,d\mu \\ &\quad=\int_{Q^*}\big|T\big(h\chi_{Q^*}\big)\big|\,d\mathcal{H}^\alpha+\sum_{R_\beta\subset Q^*}\left(\int_{B_\beta}\big|T\big(h\chi_{Q^*}\big)\big|\,d\mu-\int_{R_\beta}\big|T\big(h\chi_{Q^*}\big)\big|\,d\mathcal{H}^\alpha\right) \\ &\quad\le\int_{Q^*}\big|T\big(h\chi_{Q^*}\big)\big|\,d\mathcal{H}^\alpha+\sum_{R_\beta\subset Q^*}\int_{B_\beta}\big|T\big(h\chi_{Q^*}\big)\big|\,d\mu. \end{aligned} \tag{5.77}$$

To deal with the first integral in the last line of (5.77), set $g = h\chi_{E\setminus 2Q^*}$. Then, one has a BMO estimate for $T(g)$ restricted to Q^*; namely, there exists some constant c, depending on g and Q^*, such that

$$\left\|T(g) - c\right\|_{L^1(Q^*)} \leq C\mathcal{H}^\alpha\left(Q^*\right) = C\mu(Q) \tag{5.78}$$

(something similar was done before (5.52) to show that (5.50) suffices for the BMO estimate). Using the small boundary condition (see Theorem 2.6, property (6)), we also have

$$\left\|T\left(h\chi_{2Q^*\setminus Q^*}\right)\right\|_{L^1(Q^*)} \leq C\mathcal{H}^\alpha\left(Q^*\right) \tag{5.79}$$

(see the estimates for the second integral in (5.64)).

Thus, if we write

$$\begin{aligned}\int_{Q^*} T\left(h\chi_{Q^*}\right) d\mathcal{H}^\alpha = \int_{Q^*} T(h) d\mathcal{H}^\alpha - \int_{Q^*} T\left(h\chi_{2Q^*\setminus Q^*}\right) d\mathcal{H}^\alpha \\ - \int_{Q^*} \left(T(g) - c\right) d\mathcal{H}^\alpha - c\mathcal{H}^\alpha\left(Q^*\right)\end{aligned} \tag{5.80}$$

to show (5.60), it suffices to find an upper bound for $|c|$ independent of Q^* (recall that $T(h)$ is also bounded). For this purpose, consider the integral over Q^* of the product of $h\chi_{Q^*}$ with $T(h\chi_{Q^*})$. On the one hand, it is zero by antisymmetry. On the other hand, if we write $T(h\chi_{Q^*}) = T(h) - T(g) - T(h\chi_{2Q^*\setminus Q^*})$, it is equal to $\int_{Q^*} hT(h)d\mathcal{H}^\alpha - \int_{Q^*} h(T(g) - c)d\mathcal{H}^\alpha - c\int_{Q^*} h\, d\mathcal{H}^\alpha - \int_{Q^*} hT(h\chi_{2Q^*\setminus Q^*})d\mathcal{H}^\alpha$. Hence, due to (5.78), (5.79), and the boundedness of h and $T(h)$, we get

$$\begin{aligned}|c|\left|\int_{Q^*} h\, d\mathcal{H}^\alpha\right| &\leq \left|\int_{Q^*} hT(h)d\mathcal{H}^\alpha\right| + \left|\int_{Q^*} h\left(T(g) - c\right) d\mathcal{H}^\alpha\right| \\ &\quad + \left|\int_{Q^*} hT\left(h\chi_{2Q^*\setminus Q^*}\right) d\mathcal{H}^\alpha\right| \\ &\leq C\mathcal{H}^\alpha\left(Q^*\right).\end{aligned} \tag{5.81}$$

The upper bound on $|c|$ is obtained by using the fact that $Q^* \in \mathcal{Q}(E)$ is not a stopping-time cube, namely, that $\int_{Q^*} h\, d\mathcal{H}^\alpha > \epsilon\mathcal{H}^\alpha(Q^*)$. Therefore, using that $\mathcal{H}^\alpha(Q^*) = \mu(Q)$, we get

$$\int_{Q^*} \left|T\left(h\chi_{Q^*}\right)\right| d\mathcal{H}^\alpha \leq C\mathcal{H}^\alpha\left(Q^*\right) = C\mu(Q). \tag{5.82}$$

To estimate the second integral in (5.77), notice that, for each β, we can write

$$\int_{B_\beta} |T(h\chi_{Q^*})|\,d\mu \le \int_{B_\beta} |T(h\chi_{R_\beta})|\,d\mu + \int_{B_\beta} |T(h\chi_{Q^*\setminus R_\beta})|\,d\mu. \tag{5.83}$$

The first integral has been already estimated in (5.59). To deal with the second one, write Q^* as a finite union of cubes R_γ of the same generation as R_β, that is, such that $d(R_\gamma) \approx r_\beta$. Then, for each $\gamma \neq \beta$, if $x \in B_\beta$ and $y \in R_\gamma$, one has $|x-y| \ge Cr_\beta$. Hence,

$$\int_{B_\beta} |T(h\chi_{R_\gamma})|\,d\mu \le C\|h\|_\infty \frac{\mathcal{H}^\alpha(R_\gamma)}{r_\beta^\alpha}\mu(B_\beta) \approx \mu(B_\beta) \tag{5.84}$$

because of the facts that E is Ahlfors-David regular, and so $\mathcal{H}^\alpha(R_\gamma) \approx d(R_\gamma)^\alpha \approx r_\beta^\alpha$ and h is a bounded function. Plugging all these estimates in (5.77) and using that, for each β, $\mathcal{H}^\alpha(R_\beta) = \mu(B_\beta)$, we obtain

$$\begin{aligned}\int_Q |T(h\chi_{Q^*})|\,d\mu &\le C\mu(Q) + C\sum_\beta \mu(B_\beta)\\ &= C\mu(Q) + C\sum_\beta \mathcal{H}^\alpha(R_\beta)\\ &= C\mu(Q) + C\mathcal{H}^\alpha(Q^*)\\ &\le C\mu(Q),\end{aligned} \tag{5.85}$$

which finishes the proof of (5.60). Therefore, $T(b\mu)$ is a dyadic BMO function. ■

Remark 5.3. For a different proof of Theorem 1.2 without using tangent measures, see [19].

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References

[1] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, and C. Thiele, *Carleson measures, trees, extrapolation, and* $T(b)$ *theorems*, Publ. Mat. **46** (2002), no. 2, 257–325.

[2] M. Christ, *Lectures on Singular Integral Operators*, CBMS Regional Conference Series in Mathematics, vol. 77, American Mathematical Society, Rhode Island, 1990.

[3] ———, *A* $T(b)$ *theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60-61** (1990), no. 2, 601–628.

[4] G. David, *Unrectifiable* 1*-sets have vanishing analytic capacity*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 369–479.

[5] ———, *Analytic capacity, Calderón-Zygmund operators, and rectifiability*, Publ. Mat. **43** (1999), no. 1, 3–25.

[6] G. David and P. Mattila, *Removable sets for Lipschitz harmonic functions in the plane*, Rev. Mat. Iberoamericana **16** (2000), no. 1, 137–215.

[7] H. M. Farag, *The Riesz kernels do not give rise to higher-dimensional analogues of the Menger-Melnikov curvature*, Publ. Mat. **43** (1999), no. 1, 251–260.

[8] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, New York, 1969.

[9] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Math., vol. 297, Springer-Verlag, Berlin, 1972.

[10] R. Harvey and J. Polking, *Removable singularities of solutions of linear partial differential equations*, Acta Math. **125** (1970), 39–56.

[11] J. Mateu, X. Tolsa, and J. Verdera, *The planar Cantor sets of zero analytic capacity and the local* $T(b)$*-theorem*, J. Amer. Math. Soc. **16** (2003), no. 1, 19–28.

[12] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.

[13] P. Mattila, M. S. Melnikov, and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. (2) **144** (1996), no. 1, 127–136.

[14] P. Mattila and P. V. Paramonov, *On geometric properties of harmonic* Lip_1*-capacity*, Pacific J. Math. **171** (1995), no. 2, 469–491.

[15] M. S. Melnikov, *Analytic capacity: discrete approach and curvature of measure*, Sb. Math. **186** (1995), no. 6, 827–846.

[16] M. S. Melnikov and J. Verdera, *A geometric proof of the* L^2 *boundedness of the Cauchy integral on Lipschitz graphs*, Int. Math. Res. Not. **1995** (1995), no. 7, 325–331.

[17] F. Nazarov, S. Treil, and A. Volberg, *Weak-type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces*, Int. Math. Res. Not. **1998** (1998), no. 9, 463–487.

[18] ———, *How to prove Vitushkin's conjecture by pulling ourselves up by the hair*, preprint, 2000.

[19] L. Prat, *Capacitat analítica i nuclis de Riesz*, Ph.D. thesis, Universitat Autònoma de Barcelona, Barcelona, 2003.

[20] D. Preiss, *Geometry of measures in* $\mathbb{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math. (2) **125** (1987), no. 3, 537–643.

[21] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), no. 1, 105–149.

[22] J. Verdera, L^2 *boundedness of the Cauchy integral and Menger curvature*, Contemp. Math. **277** (2001), 139–158.

[23] ———, *A new elementary proof of* L^2 *estimates for the Cauchy integral on Lipschitz graphs*, A lecture given at the Conference on Geometric and Algebraic Aspects in Several Complex Variables, Cetraro, 1994.

[24] M. Vihtilä, *The boundedness of Riesz* s*-transforms of measures in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **124** (1996), no. 12, 3797–3804.

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