First-passage times for non-Markovian processes: Correlated impacts on a free process

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We develop a method to obtain first-passage-time statistics for non-Markovian processes driven by dichotomous fluctuations. The fluctuations themselves need not be Markovian. We calculate analytic first-passage-time distributions and mean first-passage times for exponential, rectangular, and long-tail temporal distributions of the fluctuations.

I. INTRODUCTION

The statistical analysis of the occurrence of extreme events is a venerable subject in the area of stochastic processes.[1–8] For example, in many applications in the physical sciences and engineering it is necessary to estimate the time at which a process first crosses a preassigned value, i.e., the first-passage time. Examples include "false alarm" problems in which an alarm is triggered when a fluctuating current or voltage exceeds a given value;[2,9–11] chemical rate processes that require a reactant to cross a potential barrier;[12–14] mechanical structures that collapse if they vibrate at an amplitude exceeding a stability threshold;[2,15] the fatigue failure of ductile materials due to cracks,[2,16] and measures of statistical fluctuations in lasers.[17]

The importance of analytic procedures for calculating mean first-passage times (and other first-passage-time moments) has long been recognized. Such techniques are particularly important in this case because the numerical analysis of "extreme" events (e.g., via simulations or Monte Carlo methods) is usually prohibitively expensive and time consuming:[17,18] extreme events are, by definition, rare and require a large number of long runs to provide significant statistics. It is this same rarity of occurrence that helps one in the theoretical analysis of such processes because many tools of asymptotic analysis can be brought to bear on the problem.[2,3,6–8,19–22]

The existing theory of first-passage times (and other extrema statistics) is well understood in only a limited number of situations. The theory is fully developed for independent random processes.[1,2] For dependent processes, on the other hand, most theories deal only with diffusive single-variable Markov processes, i.e., those that are described by a Fokker-Planck or master equation in which the phase space is specified by only one variable.[2–22] Recently a great deal of progress has been made in the theory of first-passage times for multivariable diffusive Markov processes using the methods of singular perturbation theory.[8,23–26] Some progress has also been made for nondiffusive (e.g., Lévy) Markov processes, but here the results are so far approximate and restricted to one-dimensional processes with no restoring force.[2,5,27]

Recently there has been a great deal of interest in the extension of the theory of first-passage times to non-Markov processes, i.e., to processes whose dynamics are influenced by memory effects.[28–32] The physical motivation for this interest is clear: in many systems the time scales of variation of the fluctuations induced by the environment are not well separated from those of the system of interest. Such time-scale effects have been a recurrent theme in areas as diverse as chemical rate processes in fluids,[33–35] the stochastic description of the dynamics of large scale hydrodynamic structures,[36,37] the description of energy transport in condensed media,[38] and the statistical analysis of the light emitted by a dye laser near threshold, to name a few.[17,39,40] In spite of this interest, the theoretical progress in the area has been limited. In some theories, an effective Markovian equation to approximate the non-Markovian system is used, and the usual methods are then brought to bear.[30,32] One expects that these methods are perhaps appropriate when the memory effects are small, i.e., in some perturbative sense. These methods are never exact except in the Markov limit. In other theories a formalism has been presented whose application depends on the construction of operators for which a readily implemented prescription is not available.[28] These theories have in fact only been implemented for systems driven by dichotomous Markovian fluctuations.[31]

In this paper we consider the simplest dynamical system driven by external dichotomous (not necessarily Markovian) fluctuations with a finite correlation time.[41] In this system we are able to obtain analytic expressions for the mean first-passage time to cross a given amplitude for arbitrary correlation times of the fluctuations. To our knowledge, ours are the first such exact analytic results available for non-Markovian fluctuations. Our method is different from those used before and relies on the explicit construction of trajectories. We are able to keep track of each trajectory and in particular we follow those that cross the given amplitude at each time interval. We stress that the problem we shall consider does not describe a thermodynamically closed system, i.e., the correlated fluctuations are external ones.

In Sec. II we outline the formal model. In Sec. III we obtain the equations satisfied by the first-passage-time probability density. In Sec. IV the formalism is applied to various specific examples and explicit first-passage distributions and mean first-passage times are obtained. The results are discussed in Sec. V.

II. THE MODEL

Consider a one-dimensional random process $X(t)$ whose dynamical evolution is specified by the differential equation

$$\dot{X}=f(X)+g(X)F(t) . \tag{2.1}$$

The random variable $F(t)$ is a dichotomous (not necessarily Markovian) process, alternately taking on the value a and $-b$ with $a,b>0$. The times that the variable $F(t)$ retains the values a and $-b$ are respectively governed by the distributions $\psi_a(t)$ and $\psi_b(t)$.[42] If $F(t)$ is a dichotomous Markov process, then these distributions are exponential,

$$\psi_a(t)=\lambda_a e^{-\lambda_a t}, \quad \psi_b(t)=\lambda_b e^{-\lambda_b t} , \tag{2.2}$$

where λ_a^{-1} and λ_b^{-1} are the average residence times in the states $F(t)=a$ and $-b$. Thus λ_a^{-1} and λ_b^{-1} are average times between switches, and $a\lambda_b=b\lambda_a$. Our first-passage-time theory is the first *not* to be restricted to these forms.

In this paper we shall restrict ourselves to processes for which $f(X)=0$ and $g(X)=1$, i.e., to unconstrained "Einstein" processes, such as a free process subjected to random impacts:

$$\dot{X}=F(t) . \tag{2.3}$$

The generalization to bound processes will be presented elsewhere.[43] The random process $X(t)$ can take on all real values $-\infty \le X(t) \le \infty$, and we wish to calculate the distribution of times for $X(t)$ to first cross the levels $\pm z$. In particular, we are interested in the mean value of this distribution, i.e., in the mean first-passage time to $|X(t)|=z$. Let us begin the process at $X(t=0)=x_0$, with $-z \le x_0 \le z$. Our procedure is based on the fact that the process evolves from this initial state in a series of steps that can be used to construct an actual trajectory by direct integration for any particular realization of $F(t)$. Suppose, for example, that $F(0)=a$. Then we have the following trajectory:

$$X(t)=\begin{cases} x_0+at, \quad 0\le t\le t_1 , \\ x_0+at_1-b(t-t_1), \quad t_1\le t\le t_1+t_2 , \\ x_0+at_1-bt_2+a(t-t_2-t_1), \quad t_1+t_2\le t\le t_1+t_2+t_3 , \\ x_0+at_1-bt_2+at_3-b(t-t_3-t_2-t_1), \quad \sum_{j=1}^{3} t_j \le t \le \sum_{j=1}^{4} +_j , \\ x_0+at_1-bt_2+at_3-bt_4+a\left[t-\sum_{j=1}^{4} t_j\right], \quad \sum_{j=1}^{4} t_j\le t\le \sum_{j=1}^{5} t_j , \\ \dots, \quad \dots \\ x_0-bt+(a+b)\sum_{l=1}^{k/2} t_{2l-1}, \quad \sum_{j=1}^{k-1} t_j \le t \le \sum_{j=1}^{k} t_j, \text{ with } k \text{ even} , \\ x_0+at-(a+b)\sum_{l=1}^{k/2} t_{2l} , \quad \sum_{j=1}^{k} t_j\le t\le \sum_{j=1}^{k+1} t_j, \text{ with } k \text{ even } (k\ge 2) , \\ \dots, \quad \dots . \end{cases} \tag{2.4}$$

The time intervals t_n are governed by the distributions $\psi_a(t)$ and $\psi_b(t)$. One such trajectory is shown in Fig. 1, where the levels $\pm z$ are also indicated.

Our goal is to calculate the conditional first-passage-time probability density $p(t;x_0)$ defined as follows:

$p(t;x_0)dt=$ probability that the processes $X(\tau)$ crosses z or $-z$ in the time range $t\le\tau\le t+dt$ without ever having crossed either of these levels during the time span $0\le\tau<t$. (2.5)

To calculate $p(t;x_0)$ it is useful to denote each time range

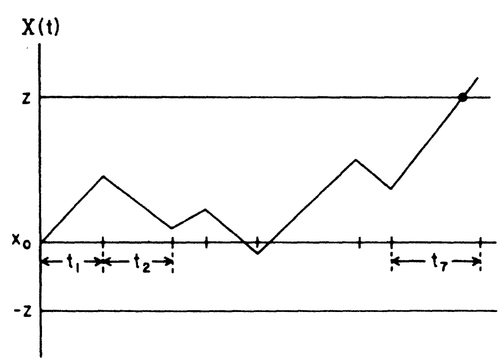

FIG. 1. A typical trajectory of $X(t)$ with $F(0)=a$. The first crossing of z or $-z$ occurs during the seventh interval.

t_n between switches as an "interval" and to define the auxiliary probability,

$p_n(t;x_0)dt=$ probability that the first crossing of z or $-z$ occurs during the nth interval and in the time range $(t,t+dt)$. (2.6)

Clearly

$$p(t;x_0)=\sum_{n=1}^{\infty} p_n(t;x_0) . \tag{2.7}$$

The probability densities $p_n(t;x_0)$ can be constructed explicitly from the trajectories (2.4). To illustrate this construction, let us consider a realization that begins with $F(0)=a$, as detailed in Eq. (2.4). We wish to insure that no crossing of $\pm z$ occurred in the first $(n-1)$ intervals and that a crossing does occur during the nth interval. During the first interval

$$X(t)=x_0+at \tag{2.8}$$

and the level z is not crossed if the switch to $F(t)=-b$ occurs sufficiently early, i.e., if

$$X(t_1)=x_0+at_1<z \tag{2.9}$$

or, equivalently, if

$$t_1<(z-x_0)/a\equiv\tau_1 . \tag{2.10}$$

The probability that the inequality (2.10) holds is

$$\text{Prob}(t_1<\tau_1)=\int_0^{\tau_1}\Phi_a(t_1)dt_1 , \tag{2.11}$$

where $\Phi_a(t)$ is the probability density for the first interval. For example, in a "modified renewal process" [42] a frequent choice is

$$\Phi_a(t)=\lambda_a\int_{-\infty}^{0}\psi_a(t-\tau)d\tau . \tag{2.12}$$

In general, $\Phi_a(t)$ depends on the preparation of the system. To insure that the second interval does not lead to a crossing of level $-z$ we must require that

$$X(t_2)=x_0+at_1-bt_2>-z , \tag{2.13}$$

i.e., that

$$t_2<(z+x_0+at_1)/b\equiv\tau_2 . \tag{2.14}$$

The probability that this inequality is satisfied is

$$\text{Prob}(t_2<\tau_2)=\int_0^{\tau_2}\psi_b(t_2)dt_2 . \tag{2.15}$$

Similar conditions can be written for the probability that each successive interval up to and including the $(n-1)$st does not lead to a crossing. To proceed with our explicit illustration we must choose the parity of n: if it is even then a crossing can only occur at $-z$ during the nth interval, while odd n can only lead to a crossing at z. We select the former and note that the level $-z$ *will* be crossed during the $2m$th interval if

$$X(t_{2m})=x_0+at_1-bt_2+\cdots+at_{2m-1}-bt_{2m}<-z . \tag{2.16}$$

The probability that this inequality is satisfied is

$$\text{Prob}(t_{2m}>\tau_{2m})=\int_{\tau_{2m}}^{\infty}\psi_b(t_{2m})dt_{2m} , \tag{2.17}$$

where

$$\tau_{2m}=(z+x_0+at_1-bt_2+\cdots+at_{2m-1})/b . \tag{2.18}$$

Finally, we must specify *when* during the $2m$th interval the crossing actually occurs. For the crossing to occur at time t it is necessary that

$$X(t)=x_0+at_1-bt_2+\cdots+at_{2m-1}-b\Delta_{2m}=-z , \tag{2.19}$$

where

$$\Delta_{n+1}=t-(t_1+t_2+t_3+\cdots+t_n) . \tag{2.20}$$

The probability density for this crossing event is the delta function $\delta(\tau_{2m}-\Delta_{2m})$. Collecting the results (2.11), (2.15), and (2.17) along with this delta function immediately gives the following integral form for the density $p_{2m}(t;x_0)$:

$$p_{2m}^{(a)}(t;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-1}}dt_{2m-1}\,\psi_a(t_{2m-1})\int_{\tau_{2m}}^{\infty}dt_{2m}\psi_b(t_{2m})\delta(\tau_{2m}-\Delta_{2m}) \tag{2.21}$$

for $m\geq 1$, where

$$\tau_{2m-1}\equiv(z-x_0-at_1+bt_2+\cdots+bt_{2m-2})/a , \tag{2.22}$$

and we have explicitly indicated the initial value $F(0)=a$. For odd n similar reasoning leads to

$$p_{2m-1}^{(a)}(t;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-2}}dt_{2m-2}\,\psi_b(t_{2m-2})\times\int_{\tau_{2m-1}}^{\infty}dt_{2m-1}\,\psi_a(t_{2m-1})\delta(\tau_{2m-1}-\Delta_{2m-1}) \tag{2.23}$$

for $m\geq 2$, and

$$p_1^{(a)}(t;x_0)=\int_{\tau_1}^{\infty}dt_1\,\Phi_a(t_1)\delta(t-\tau_1) . \tag{2.24}$$

Similar expressions can clearly be obtained for $F(0)=-b$, with

$$\bar{\tau}_{2m}=(z-x_0+bt_1-at_2+\cdots+bt_{2m-1})/a \tag{2.25}$$

and

$$\bar{\tau}_{2m-1}=(z+x_0-bt_1+at_2-\cdots+bt_{2m-2})/b\ , \tag{2.26}$$

replacing τ_{2m} and τ_{2m-1}, respectively.

III. EVOLUTION OF THE FIRST-PASSAGE-TIME PROBABILITY DENSITY

The next step in our procedure is to Laplace transform Eqs. (2.21)–(2.24) according to the definition

$$\tilde{p}(s;x_0)=\int_0^\infty dt\, e^{-st}p(t;x_0) \tag{3.1}$$

and to establish an integral recursion relation to connect the nth and $(n+2)$nd densities. The recursion relations for even n and for odd n must thus be constructed separately. The appropriate Laplace transforms are

$$\tilde{p}_1^{(a)}(s;x_0)=e^{-s\tau_1}\int_{\tau_1}^\infty dt_1\,\Phi_a(t_1)\ , \tag{3.2}$$

$$\tilde{p}_{2m-1}^{(a)}(s;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-2}}dt_{2m-2}\,\psi_b(t_{2m-2})\int_{\tau_{2m-1}}^\infty dt_{2m-1}\,\psi_a(t_{2m-1})\times e^{-s(t_1+t_2+\cdots+t_{2m-2}+\tau_{2m-1})}\ , \tag{3.3}$$

$$\tilde{p}_{2m}^{(a)}(s;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-1}}dt_{2m-1}\,\psi_a(t_{2m-1})\int_{\tau_{2m}}^\infty dt_{2m}\,\psi_b(t_{2m})\times e^{-s(t_1+t_2+\cdots+t_{2m-1}+\tau_{2m})}\ . \tag{3.4}$$

It is clear that these repeated nested integrals can be encapsulated in a recursion relation. To achieve this we define the auxiliary functions

$$\tilde{I}_{2m-2}^{(a)}(s;x_0+at_1)=\int_0^{\tau_2}dt_2\,\psi_b(t_2)\int_0^{\tau_3}dt_3\,\psi_a(t_3)\cdots\int_0^{\tau_{2m-2}}dt_{2m-2}\,\psi_b(t_{2m-2})\times\int_{\tau_{2m-1}}^\infty dt_{2m-1}\,\psi_a(t_{2m-1})e^{-s(t_2+\cdots+t_{2m-2}+\tau_{2m-1})} \tag{3.5a}$$

and

$$\tilde{I}_{2m-1}^{(a)}(s;x_0+at_1)=\int_0^{\tau_2}dt_2\,\psi_b(t_2)\int_0^{\tau_3}dt_3\,\psi_a(t_3)\cdots\int_0^{\tau_{2m-1}}dt_{2m-1}\,\psi_a(t_{2m-1})\times\int_{\tau_{2m}}^\infty dt_{2m}\,\psi_b(t_{2m})e^{-s(t_2+\cdots+t_{2m-1}+\tau_{2m})} \tag{3.5b}$$

in terms of which

$$\tilde{p}_n^{(a)}(s;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)e^{-st_1}\tilde{I}_{n-1}^{(a)}(s;x_0+at_1),\quad n\ge 2\ . \tag{3.6}$$

We note that in the special case of an "ordinary renewal process"[42] wherein $\Phi_a(t)=\psi_a(t)$, the auxiliary functions are

$$\tilde{I}_n^{(a)}(s;x_0+at_1)=\tilde{p}_n^{(-b)}(s;x_0+at_1)\ . \tag{3.7}$$

The functions $\tilde{I}_n^{(a)}$ clearly satisfy the recursion relation

$$\tilde{I}_{n+2}^{(a)}(s;x_1)=\int_0^{\tau_2}dt_2\,\psi_b(t_2)\int_0^{\tau_3}dt_3\,\psi_a(t_3)e^{-s(t_2+t_3)}\tilde{I}_n^{(a)}(s;x_1-bt_2+at_3),\quad n\ge 1\ . \tag{3.8}$$

If we sum this recursion relation from $n=1$ to ∞ we obtain the integral equation

$$\tilde{I}^{(a)}(s;x_1)=\tilde{I}_1^{(a)}(s;x_1)+\tilde{I}_2^{(a)}(s;x_1)+\int_0^{\tau_2}dt_2\,\psi_b(t_2)\int_0^{\tau_3}dt_3\,\psi_a(t_3)e^{-s(t_2+t_3)}\tilde{I}^{(a)}(s;x_1-bt_2+at_3) \tag{3.9}$$

where

$$\tilde{I}^{(a)}(s;x_1)\equiv\sum_{n=1}^\infty \tilde{I}_n^{(a)}(s;x_1)\ . \tag{3.10}$$

In terms of this function the Laplace transform of the first-passage-time probability density for $F(0)=a$ is

$$\tilde{p}^{(a)}(s;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)e^{-st_1}\tilde{I}^{(a)}(s;x_0+at_1)+\tilde{p}_1^{(a)}(s;x_0)\ . \tag{3.11}$$

An analogous argument for $F(0)=-b$ leads to

$$\tilde{p}^{(-b)}(s;x_0)=\int_0^{\bar{\tau}_1}dt_1\,\Phi_b(t_1)e^{-st_1}\tilde{I}^{(-b)}(s;x_0-bt_1)+\tilde{p}_1^{(-b)}(s;x_0) \tag{3.12}$$

where $\tilde{I}^{(-b)}(s;x_1)$ satisfies the integral equation

$$\tilde{I}^{(-b)}(s;x_1)=\tilde{I}_1^{(-b)}(s;x_1)+\tilde{I}_2^{(-b)}(s;x_1)+\int_0^{\tilde{\tau}_2}dt_2\,\psi_a(t_2)\int_0^{\tilde{\tau}_3}dt_3\,\psi_b(t_3)e^{-s(t_2+t_3)}\tilde{I}^{(-b)}(s;x_1+at_2-bt_3)\,. \tag{3.13}$$

Finally, if we define the probabilities $w_0(a\mid x_0)$ and $w_0(-b\mid x_0)$ that $F(0)=a$ and $-b$, respectively, given that $X(0)=x_0$, we find the Laplace transform of the first-passage-time probability density for arbitrary $F(0)$ to be

$$\tilde{p}(s;x_0)=\tilde{p}^{(a)}(s;x_0)w_0(a\mid x_0)+\tilde{p}^{(-b)}(s;x_0)w_0(-b\mid x_0)\,. \tag{3.14}$$

Thus the entire problem has been reduced to the solution of the integral equation (3.9), which is the central result of our method. Such equations can in general not be solved for arbitrary forms of $\psi_a(t)$ and $\psi_b(t)$, but they lend themselves to approximation schemes appropriate to specific forms of these functions. There are, however, situations when the integral equations *can* be solved exactly for certain forms of $\psi_a(t)$ and $\psi_b(t)$, and we here give three such examples.

It is worthwhile to note a further simplification in these results when the system is prepared in such a way that $\Phi_a(t)=\psi_a(t)$ and $\Phi_b(t)=\psi_b(t)$. In this case it is no longer necessary to define the auxiliary functions $\tilde{I}$ because of the relation (3.7). The probability densities $\tilde{p}^{(a)}$ and $\tilde{p}^{(-b)}$ themselves now satisfy the integral equations

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)+\int_0^{\tau_1}dt_1\int_0^{\tau_2}dt_2\,\psi_a(t_1)\psi_b(t_2)e^{-s(t_1+t_2)}\tilde{p}^{(a)}(s;x_0+at_1-bt_2)\,, \tag{3.15}$$

$$\tilde{p}^{(-b)}(s;x_0)=\tilde{p}_1^{(-b)}(s;x_0)+\tilde{p}_2^{(-b)}(s;x_0)+\int_0^{\tilde{\tau}_1}dt_1\int_0^{\tilde{\tau}_2}dt_2\,\psi_b(t_1)\psi_a(t_2)e^{-s(t_1+t_2)}\tilde{p}^{(-b)}(s;x_0-bt_1+at_2)\,. \tag{3.16}$$

IV. APPLICATIONS

In this section we evaluate the first-passage time properties of the process (2.3) for various forms of $\psi_a(t)$ and $\psi_b(t)$ for which the integral equations (3.9) and (3.13) [or (3.15) and (3.16)] can be converted to an equivalent differential equation. The differential equation together with boundary conditions (one of which turns out to be rather different from those traditionally imposed) can be solved analytically. For simplicity we take $\Phi(t)=\psi(t)$ in our examples. Other forms of $\Phi(t)$ can be easily incorporated.

A. Dichotomous Markov process $F(t)$

A dichotomous Markov process $F(t)$ is characterized by the exponential switching time distributions (2.2) (see Fig. 2). The condition that the fluctuations be zero-centered is insured by the choice of parameters $a\lambda_b=b\lambda_a$.

To find the differential equation satisfied by $\tilde{p}^{(a)}(s;x_0)$ we differentiate (3.15) twice with respect to x_0 to obtain

$$\left[\frac{d^2}{dx_0^2}+s\left[\frac{1}{b}-\frac{1}{a}\right]\frac{d}{dx_0}-\frac{s(s+\lambda_a+\lambda_b)}{ab}\right]\tilde{p}^{(a)}(s;x_0)=0\,. \tag{4.1}$$

The details of the derivation of (4.1) are given in the Appendix. The boundary conditions, also obtained in the Appendix, are

$$\tilde{p}^{(a)}(s;z)=1\,, \tag{4.2}$$

$$\left.\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)\right|_{x_0=-z}-\frac{s+\lambda_a}{a}\tilde{p}^{(a)}(s;-z)=-\frac{\lambda_a}{a}\,. \tag{4.3}$$

Condition (4.2) insures that a process initiated at the upper boundary with positive slope is immediately absorbed with certainty. Condition (4.3) is not of the usual form for a Fokker-Planck process in which initiation at the lower boundary would also guarantee immediate absorption $[\tilde{p}^{(a)}(s;-z)=1]$. Its physical interpretation is not straightforward.

Equation (4.1) is homogeneous, second order, and has constant coefficients. Its solution is straightforwardly obtained and is[41]

$$\tilde{p}^{(a)}(s;x_0)=[(\alpha+r)e^{2rz}-(\alpha-r)e^{-2rz}]^{-1}\left[e^{-\beta(z-x_0)}[(\alpha+r)e^{r(z+x_0)}-(\alpha-r)e^{-r(z+z_0)}]+\frac{\lambda_a}{a}e^{\beta(z+x_0)}[e^{r(z-x_0)}-e^{-r(z-x_0)}]\right], \tag{4.4}$$

where

$$\alpha=\frac{a\lambda_b+b\lambda_a+(a+b)s}{2ab}\,,\quad \beta=\frac{1}{2}\left[\frac{s+\lambda_a}{a}-\frac{s+\lambda_b}{b}\right], \tag{4.5}$$

and

$$r=[4\beta^2+4s(s+\lambda_a+\lambda_b)/ab]^{1/2}\,. \tag{4.6}$$

The mean first-passage time to z or $-z$ is given by

$$T_1^{(a)}(z,x_0)=\int_0^\infty dt\,tp^{(a)}(t;x_0) \tag{4.7}$$

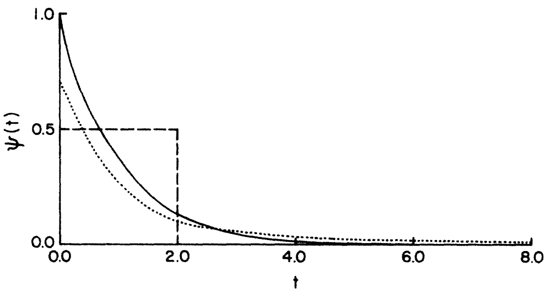

FIG. 2. Distribution $\psi(t)$ of time intervals between switches of $F(t)$ from one value to another, with $\lambda=1$. Solid curve: exponential $\psi(t)$ corresponding to a dichotomous Markov process $F(t)$. Dashed curve: a rectangular $\psi(t)$. Dotted curve: $\psi(t)$ with a long-time tail (power law) with $z=0.5$, $a=b=1$ and $\alpha=1.5$ in Eq. (4.29).

and satisfies the simple differential equation

$$\frac{d^2}{dx_0^2}T_1^{(a)}(z,x_0)=-\frac{\lambda_a+\lambda_b}{ab} \tag{4.8}$$

as can be obtained directly from (4.1) using the relation

$$T_1^{(a)}(z,x_0)=-\frac{d}{ds}\tilde{p}^{(a)}(s;x_0)\bigg|_{s=0} . \tag{4.9}$$

The boundary conditions constraining the solution of (4.8) are obtained from (4.2) and (4.3) using (4.9):

$$T_1^{(a)}(z,z)=0 , \tag{4.10}$$

$$\frac{d}{dx_0}T_1^{(a)}(z,x_0)\bigg|_{x_0=-z}=\frac{\lambda_a T_1^{(a)}(z,-z)-1}{a} . \tag{4.11}$$

One can either solve (4.8) subject to (4.10) and (4.11) or in the present case since we have the analytic form for $\tilde{p}^{(a)}(s;x_0)$ in Eq. (4) we can apply (4.9) directly to it to obtain[41]

$$T_1^{(a)}(z,x_0)=\frac{z^2-x_0^2}{2D}+\frac{(z-x_0)}{a}\frac{\left[\frac{a}{\lambda_a}+z\frac{(a+b)}{b}\right]}{\left[\frac{a}{\lambda_a}+2z\right]} , \tag{4.12}$$

where

$$D=\frac{ab}{a+b}\frac{a}{\lambda_a} . \tag{4.13}$$

This result was obtained by Hänggi and Talkner[31] using an entirely different procedure restricted to a dichotomous Markov process $F(t)$. For the special case $a=b$ and $\lambda_a=\lambda_b\equiv\lambda$, the mean first-passage time (4.12) reduces to the simpler form

$$T_1^{(a)}(z,x_0)=\frac{z^2-x_0^2}{2D}+\frac{z-x_0}{a} \tag{4.14}$$

where now $D=a^2/2\lambda$.

The corresponding results for the initial value $F(0)=-b$ are obtained from (4.4) and (4.12) with the interchanges $a\leftrightarrow b$, $\lambda_a\leftrightarrow\lambda_b$, and $-x_0\leftrightarrow x_0$. Note that the proportion in which $T_1^{(a)}(z,x_0)$ and $T_1^{(-b)}(z,x_0)$ contribute to the total mean first-passage time is determined by the initial weights $w_0(a\,|\,x_0)$ and $w_0(-b\,|\,x_0)$ appearing in Eq. (3.14).

The first term in (4.12) or in (4.14) is the mean first-passage time in the limit of Gaussian white noise, in which $b,a\to\infty$, $\lambda_b,\lambda_a\to\infty$, and $D=$const, i.e., the well-known diffusion case.[44] The remaining contribution in these equations gives the deviation from the diffusive limit and leads to a discontinuous change in the mean first-passage time at $x_0=-z$. This finite value of $T_1^{(a)}(z,-z)$ is the reason why the first-passage-time problem in the presence of colored noise cannot be described in terms of the standard boundary conditions but rather involves the mixed boundary condition (4.11).[31] Equation (4.14) is shown in Figs. 3(a) and 4(a), as is the Gaussian white-

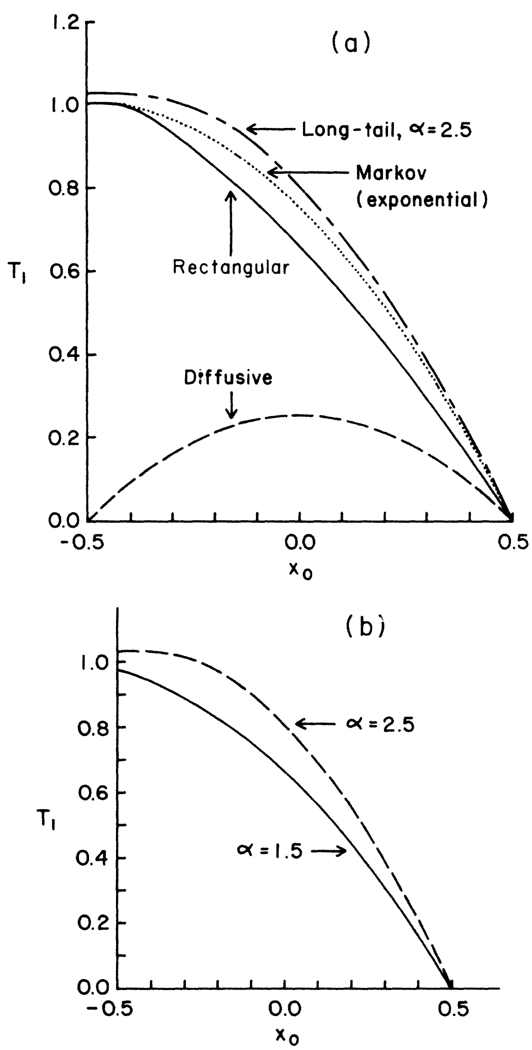

FIG. 3. (a) Mean first-passage time T_1 to ± 0.5 vs initial position x_0 for a process that begins with positive slope. We choose parameter values $\lambda=a=b=1$. Dotted curve: dichotomous Markov fluctuations. Solid curve: rectangular fluctuations. Dot-dashed curve: long-tail process with $\alpha=2.5$. Dashed curve: diffusive process with $2D=1$. (b) Mean first-passage time T_1 to ± 0.5 vs initial positive x_0 for a long-tail process that begins with a positive value, with $\lambda=a=b=1$. Solid curve: $\alpha=1.5$. Dashed curve: $\alpha=2.5$ [also shown as dot-dashed curve in Fig. 3(a)].

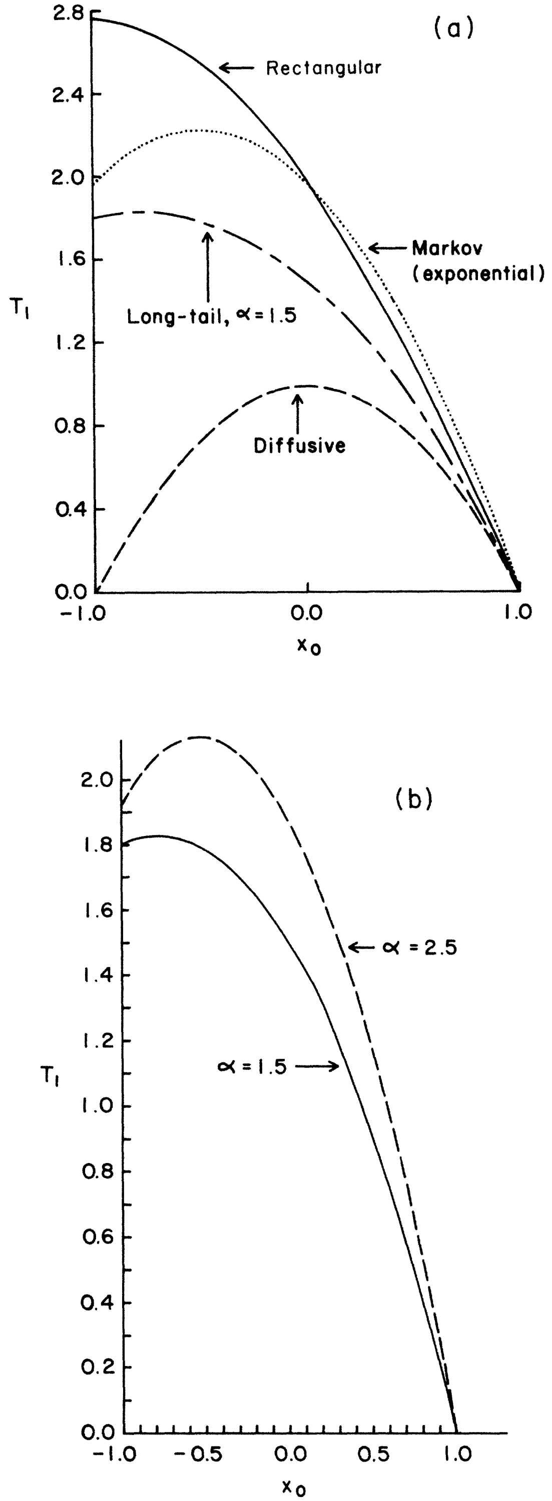

FIG. 4. (a) Mean first-passage time T_1 to ± 0.99 vs initial position x_0 for a process that begins with a positive slope. The parameter values are $\lambda = a = b = 1$. Dotted curve: dichotomous Markov fluctuations. Solid curve: rectangular fluctuations. Dot-dashed curve: long-tail process with $\alpha = 1.5$. Dashed curve: diffusive process with $2D = 1$. (b) Mean first-passage time T_1 to ± 0.99 vs initial position x_0 for a long-tail process that begins with a positive value, with $\lambda = a = b = 1$. Solid curve: $\alpha = 1.5$ [also shown as dot-dashed curve in Fig. 4(a)]. Dashed curve: $\alpha = 2.5$.

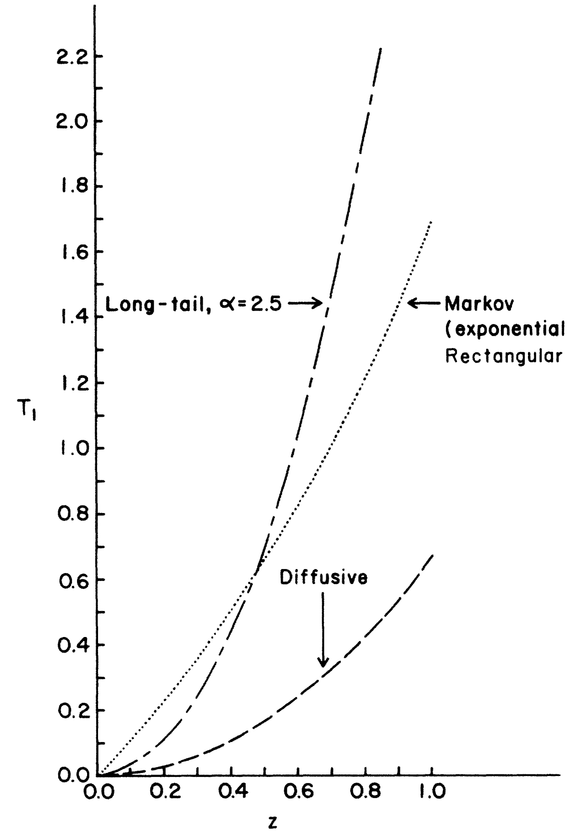

FIG. 5. Mean first-passage time averaged over initial position, vs barrier height z. Parameter values are $\lambda = a = b = 1$. Dotted curve indicates both the dichotomous Markov fluctuations and the rectangular process. Dot-dashed curve: long-tail process with $\alpha = 2.5$. Dashed curve: diffusive process with $2D = 1$.

noise result for comparison. A further discussion of these results is given in Sec. V.

An equally interesting quantity is the mean first-passage time to z or $-z$ averaged over a distribution of initial states. For a uniform initial distribution in the interval $(-z, z)$ we find

$$T_1^{(a)}(z) \equiv \frac{1}{2z} \int_{-z}^{z} dx_0 \, T_1^{(a)}(z, x_0) = \frac{z^2}{3D} + \frac{z}{a} \; . \tag{4.15}$$

Equation (4.14) is shown in Fig. 5, as is the Gaussian white-noise result for comparison.

B. A rectangular process $F(t)$

One of the simplest temporally localized non-Markov dichotomous processes is a rectangular, i.e., one in which the distributions $\psi_a(t)$ and $\psi_b(t)$ have the form shown in Fig. 2. For $\mu = a, b$ we take

$$\psi_\mu(t) = \begin{cases} \dfrac{\lambda_\mu}{2}, & 0 \le t \le \dfrac{2}{\lambda_\mu} \\ 0, & \text{otherwise} \; . \end{cases} \tag{4.16}$$

Once again the mean times between switches are λ_a^{-1} and λ_b^{-1}. We shall restrict ourselves to the parameter domain $a/\lambda_a = b/\lambda_b > z$ so as to allow the process a finite probability of reaching the boundary z or $-z$ even in the first time interval. This restriction is chosen for convenience but can easily be relaxed. We shall further restrict the analysis to the symmetric case $a=b$, so that $\lambda_a = \lambda_b \equiv \lambda$. This restriction can also be easily relaxed. As Fig. 2 shows, the relative probability of very frequent switches *and* of very infrequent switches of $F(t)$ from one value to another is greater for a dichotomous Markov process than for a rectangular process. On the other hand, the rectangular process allows for more intervals of an intermediate ($\Delta t \sim \lambda^{-1}$) length. Since one does not *a priori* know which interval lengths contribute most to the first-passage-time behavior, it is difficult to predict the effects of this redistribution of relative weights.

In the Appendix we show that with the rectangular process $F(t)$ the following differential equation is satisfied by the transform $\tilde{p}^{(a)}(s;x_0)$:

$$\left[\frac{d^2}{dx_0^2} + \frac{1}{a^2}\left[\frac{\lambda^2}{4} - s^2\right]\right]\tilde{p}^{(a)}(s;x_0) = \frac{\lambda s}{a^2}e^{-(z-x_0)s/a} + \frac{\lambda^2}{4a^2}e^{(z+x_0)s/a} . \quad (4.17)$$

The boundary conditions for Eq. (4.17) are

$$\tilde{p}^{(a)}(s;z) = 1 \quad (4.18)$$

and

$$\left.\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)\right|_{x_0=-z} - \frac{s}{a}\tilde{p}^{(a)}(s;-z) = \frac{\lambda}{2a}(e^{-2zs/a} - 1) . \quad (4.19)$$

As before, condition (4.18) insures certain absorption of a process that begins at the upper boundary with positive slope. The second boundary condition, (4.19), differs from its counterpart (4.3) in the previous example and is thus seen to be process specific. Its physical interpretation is not as straightforward as that for a Fokker-Planck process.

Equation (4.17) is an inhomogeneous second-order equation with constant coefficients. Again, its solution is straightforward[41]

$$\tilde{p}^{(a)}(s;x_0) = \left[\left[\frac{s}{a}+\beta\right]e^{2\beta z} - \left[\frac{s}{a}-\beta\right]e^{-2\beta z}\right]^{-1} \times \left[1 - 4\frac{s}{\lambda} - e^{-2sz/a}\right]\left[\frac{\lambda}{2a}(e^{\beta(z-x_0)} - e^{-\beta(z-x_0)}) + \left[\frac{s}{a}+\beta\right]e^{\beta(z+x_0)} - \left[\frac{s}{a}-\beta\right]e^{-\beta(z-x_0)}\right] + e^{-x(z+x_0)/a} + 4\frac{s}{\lambda}e^{-s(z-x_0)/a} \quad (4.20)$$

where

$$\beta = \frac{i}{2a}(\lambda^2 - 4s^2)^{1/2} . \quad (4.21)$$

The mean first-passage time in general satisfies the simple differential equation

$$\left[\frac{d^2}{dx_0^2} + \frac{\lambda_a\lambda_b}{4ab}\right]T_1^{(a)}(z,x_0) = -\frac{\lambda_a}{2a}\left[\frac{a+b}{ab}\right] + \frac{\lambda_a\lambda_b}{4ab^2}(z+x_0) \quad (4.22)$$

with

$$T_1^{(a)}(z,z) = 0 \quad (4.23)$$

and

$$\left.\frac{d}{dx_0}T_1^{(a)}(z,x_0)\right|_{x_0=-z} = \frac{\lambda_a z}{a^2} - \frac{1}{a} \quad (4.24)$$

and where we have not imposed the parameter restrictions $a=b$ and $\lambda_a = \lambda_b = \lambda$. The solution of (4.22) with (4.23) and (4.24) is[41]

$$T_1^{(a)}(z,x_0) = \frac{1}{\cos(2\alpha z)}\left\{\left[\frac{2}{\lambda_b}\left[1+\frac{b}{a}\right] - \frac{2z}{b}\right]\cos[\alpha(z+x_0)] + \frac{1}{\alpha}\left[\left[\frac{a+b}{ab}\right] - \frac{\lambda_a z}{a^2}\right]\sin[\alpha(z-x_0)]\right\} - \frac{2}{\lambda_b}\left[1+\frac{b}{a}\right] + \frac{1}{b}(z+x_0) \quad (4.25)$$

where

$$\alpha \equiv \frac{1}{2}\left[\frac{\lambda_a\lambda_b}{ab}\right]^{1/2} . \tag{4.26}$$

For the special case $a=b$ and $\lambda_a=\lambda_b=\lambda$ (4.24) reduces to the simpler form

$$T_1^{(a)}(z,x_0)=\frac{2}{a\lambda}\frac{(2a-\lambda z)}{\cos(\lambda z/a)}\left[\sin\left[\frac{\lambda(z-x_0)}{2a}\right]+\cos\left[\frac{\lambda(z+x_0)}{2a}\right]\right]-\frac{1}{a}\left[\frac{4a}{\lambda}-z-x_0\right] . \tag{4.27}$$

Again, $T_1^{(-b)}(z,x_0)$ can be obtained from (4.25) with the interchanges $a\leftrightarrow b$, $\lambda_a\leftrightarrow\lambda_b$, and $-x_0\leftrightarrow x_0$. Equation (4.27) is shown in Figs. 3(a) and 4(a) and is discussed further in Sec. V.

We can average $T_1^{(a)}(z,x_0)$ over the initial state of the process. For a uniform initial distribution one obtains

$$T_1^{(a)}(z)=\frac{2}{\lambda^2 z}\frac{(2a-\lambda z)}{\cos(\lambda z/a)}[1-\cos(\lambda z/a)+\sin(\lambda z/a)] -\frac{4}{\lambda}+\frac{z}{a} . \tag{4.28}$$

This result is nearly identical to that depicted in Fig. 5 for a dichotomous Markov process with $\lambda=a=1$ and $0\le z<1$.

C. A process $F(t)$ with a long-time tail

It is interesting to investigate the effects of long-time memories of the fluctuations on the first-passage-time properties of the system. A particular choice of distribution that is analytically tractable within our theory is one that decays exponentially for short times and algebraically for long times:

$$\psi_\mu(t)=\begin{cases} A_\mu e^{-\lambda_\mu t}, & 0\le t\le \dfrac{2z}{\mu} \\ \dfrac{B_\mu}{t^{\alpha_\mu}}, & \dfrac{2z}{\mu}<t<\infty \end{cases} \tag{4.29}$$

where $\mu=a,b$ and $\alpha_\mu>1$. This distribution is shown in Fig. 2. The requirement that each $\psi_\mu(t)$ be normalized and be continuous at $2z/\mu$ imposes the following relations among the parameters:

$$\frac{A_\mu}{\lambda_\mu}(1-e^{-2\lambda_\mu z/\mu})+\frac{B_\mu}{\alpha_\mu-1}\left[\frac{2z}{\mu}\right]^{1-\alpha_\mu}=1 , \tag{4.30}$$

$$A_\mu e^{-2\lambda_\mu z/\mu}=B_\mu(2z/\mu)^{-\alpha_\mu} . \tag{4.31}$$

We shall restrict further discussion to the symmetric case $a=b$, $\lambda_a=\lambda_b=\lambda$. It should be noted that in this example λ^{-1} is no longer the mean time between switches. The latter time, denoted as τ_s, is given by

$$\tau_s=\int_0^\infty dt\, t\psi(t) \tag{4.32}$$

and is only finite if $\alpha_a=\alpha_b\equiv\alpha>2$, whence

$$\tau_s=\frac{A}{\lambda^2}+\frac{2z}{a}\left[1-\frac{A}{\lambda}\right]\left[\frac{\alpha-1}{\alpha-2}\right] +\frac{A}{a\lambda}\left[-\frac{a}{\lambda}+\frac{2z}{\alpha-2}\right]e^{-2\lambda z/a} . \tag{4.33}$$

The normalization condition (4.30) and continuity constraint impose relations among the parameters that depend on the value of z. One finds, for instance, that

$$A=\frac{\lambda}{1-\left[1-\dfrac{2z\lambda}{a(\alpha-1)}\right]e^{-2\lambda z/a}} . \tag{4.34}$$

These conditions can be relaxed (without affecting the analytic manipulation) by changing the location of the exponential-algebraic boundary in (4.29) from $2z/\mu$ to $2y/\mu$ where $y\ge z$. This modification would lead to a quantitative but not qualitative change in the results. With the choice made in (4.29), one finds that (4.30) and (4.31) impose the following behavior on $\psi(t)$: when α decreases for a fixed z, A decreases so that the weight is shifted from very short to very long switching intervals. When α is fixed, on the other hand, a decrease in z causes an increase in A so that short intervals are emphasized.

In the Appendix we show that the differential equation now satisfied by the transform $\tilde{p}^{(a)}(s;x_0)$ is

$$\left[\frac{d^2}{dx_0^2}+\frac{1}{a^2}[A^2-(\lambda+s)^2]\right]\tilde{p}^{(a)}(s;x_0) =\left[\frac{A-\lambda}{a^2}\right][(\lambda+2s)^{-s(z-x_0)/a}+Ae^{-s(z+x_0)/a}] \tag{4.35}$$

where we have used the parameter relations (4.30) and (4.31) to eliminate B and α. The boundary conditions for (4.35) are

$$\tilde{p}^{(a)}(s;z)=1 \tag{4.36}$$

and

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)\bigg|_{x_0=-z}-\frac{(s+\lambda)}{a}\tilde{p}^{(a)}(s;-z) =\left[\frac{A-\lambda}{a}\right]e^{-2sz/a}-\frac{A}{a} . \tag{4.37}$$

Equation (4.35) is once again an inhomogeneous second-order equation with constant coefficients and can be solved without major difficulty. The mean first-passage time satisfies the equation

$$\left[\frac{d^2}{dx_0^2}+\frac{(A^2-\lambda^2)}{a^2}\right]T_1^{(a)}(z,x_0)=-\frac{2\lambda}{a^2}-\frac{(A-\lambda)}{a^3}[2a+(\lambda-A)x_0-(\lambda+A)z] \tag{4.38}$$

with

$$T_1^{(a)}(z,z)=0 \tag{4.39}$$

and

$$\left.\frac{d}{dx_0}T_1^{(a)}(z,x_0)\right|_{x_0=-z}-\frac{\lambda}{a}T_1^{(a)}(z,-z)=\frac{2z}{a^2}(A-\lambda)-\frac{1}{a}\ . \tag{4.40}$$

The solution of (4.38)–(4.40) has two forms depending on whether $\lambda>A$ or $\lambda<A$. When $\lambda>A$

$$T_1^{(a)}(z,x_0)=\frac{1}{\Delta}\frac{2A}{\lambda+A}\left[\frac{1}{\lambda-A}+\frac{z}{a}\right]\left\{\left[\left[\rho+\frac{\lambda}{a}\right]e^{\rho z}-\frac{A}{a}e^{-\rho z}\right]e^{\rho x_0}+\left[\frac{A}{a}e^{\rho z}+\left[\rho-\frac{\lambda}{a}\right]e^{-\rho z_0}\right]e^{-\rho x_0}\right\}+\frac{2A}{\lambda^2-A^2}+\frac{z}{a}-\frac{\lambda-A}{a(\lambda+A)}x_0\ , \tag{4.41}$$

where

$$\rho\equiv(\lambda^2-A^2)^{1/2}/a \tag{4.42}$$

and

$$\Delta\equiv-\left[\rho+\frac{\lambda}{a}\right]e^{2\rho z}-\left[\rho-\frac{\lambda}{a}\right]e^{-2\rho z}\ . \tag{4.43}$$

For $\lambda<A$,

$$T_1^{(a)}(z,x_0)=\frac{1}{\Delta'}\frac{2A}{A+\lambda}\left[\frac{z}{a}-\frac{1}{A-\lambda}\right]\left[\left[\frac{(A-\lambda)}{a}\cos(\rho' z)+\rho'\sin(\rho' z)\right]\sin(\rho' x_0)-\left[\rho'\cos(\rho' z)+\frac{(A+\lambda)}{a}\sin(\rho' z)\right]\cos(\rho' x_0)\right]-\frac{2A}{A^2-\lambda^2}+\frac{z}{a}+\frac{A-\lambda}{a(A+\lambda)}x_0\ , \tag{4.44}$$

where

$$\rho'\equiv(A^2-\lambda^2)^{1/2}/a \tag{4.45}$$

and

$$\Delta'\equiv\rho'\cos(2\rho' z)+\frac{\lambda}{a}\sin(2\rho' z)\ . \tag{4.46}$$

Equations (4.41) and (4.44) are shown in Figs. 3 and 4 and discussed further in Section V.

The mean first-passage time averaged over a uniform initial distribution is again dependent on the relative size of λ and A. For $A<\lambda$

$$T_a^{(a)}(z)=\frac{1}{\Delta\rho}\frac{2A}{\lambda+a}\left[\frac{1}{\lambda-A}+\frac{z}{A}\right]\left[\left[\frac{\lambda+A}{a}+\rho\right]e^{2\rho z}+\left[\frac{\lambda+A}{a}-\rho\right]e^{-2\rho z}-2\left[\frac{\lambda+A}{a}\right]\right]+2z\left[\frac{2A}{\lambda^2-A^2}+\frac{z}{a}\right] \tag{4.47}$$

and for $A>\lambda$

$$T_a^{(a)}(z)=\frac{-1}{\Delta'}\frac{2A}{A+\lambda}\left[\frac{z}{A}-\frac{1}{A-\lambda}\right]\left[\sin(2\rho' z)+\frac{2}{\rho'}\left[\frac{A+\lambda}{a}\right]\sin^2(\rho' z)\right]+2z\left[\frac{z}{a}-\frac{2A}{A^2-\lambda^2}\right]\ . \tag{4.48}$$

These results are shown in Fig. 5 and discussed in Sec. V.

V. DISCUSSION AND CONCLUSIONS

A discussion of the salient results obtained in Sec. IV is most easily presented by individually interpreting each figure.

Figure 3(a). The most general property of the nondiffusive process considered here is the asymmetry in $T_1^{(a)}(z,x_0)$ about $x_0=0$. In other words, a nondiffusive process beginning at the lower boundary ($x_0=-z$) is not immediately absorbed while one that departs from the upper boundary ($x_0=z$) is. As stated earlier, this is a consequence of the fact that an initial velocity away from a boundary together with a finite retention time for this velocity carries the process away from the boundary. This is different from a diffusive process wherein the velocity is infinite and its directional retention time is infinitesimal and hence the process that starts at a boundary cannot escape.

A second general property of the non-Markovian processes is that for all of them $T_1^{(a)}(z,x_0)$ exceeds the diffusive case *for all* $x_0<z$. The finite retention time of a given velocity in the former has two opposing effects: when far from a boundary, it takes the process nearer to it in a given interval than in the diffusive case (leading to an apparent shortening of T_1). On the other hand, when near a boundary it is easier for a nondiffusive process to escape from it (leading to an apparent lengthening of T_1). *A priori* it is impossible to determine which of these competing effects will win out. The figure shows that it is the second.

The third observation concerns the comparison among the nondiffusive processes. We note that $\alpha=2.5$ and $z=0.5$ lead to a coefficient $A>\lambda$ for the long-tail process so that this process admits of a greater weight of very short steps than either the rectangular or the exponential process. (Shorter steps do not make the process diffusion-like unless the velocity goes to infinity. In this discussion the velocity is held fixed at $a=1$.) Furthermore, the long-tail process allows more long intervals than the exponential, which in turn allows more than the rectangular. Intermediate-sized intervals are favored by the rectangular process. Again, it is not immediately apparent what the relative effects of these regimes are on $T_1^{(a)}(z,x_0)$. In this figure it would seem that the proportion of short steps (for a fixed velocity) may determine the ordering of T_1, with the greater weight leading to a longer mean first-passage time. We will see subsequently that the analysis is not as straightforward as this argument would imply, and that the relative ordering also depends on the barrier height z and/or on the initial state x_0.

Figure 3(b). In this graph we compare the T_1 values for two power-law indices in the long-tail distribution, $\alpha=2.5$ and 1.5. In the former the coefficient A is bigger and the tail is relatively suppressed. Correspondingly, T_1 for the former is larger than for the latter.

Figure 4(a). There are two differences between the parameter values used here and those used in Fig. 3(a): z has here been increased from 0.5 to 0.99, and α has been changed from 2.5 to 1.5. The two general features observed earlier persist, namely, that $T_1^{(a)}(z,x_0)$ is in general asymmetric about $x_0=0$, and that T_1 for a diffusive process is smaller than those of the other processes. In our discussion of Fig. 3(a) we indicated that simple arguments based on the relative weightings of long and short steps in the determination of T_1 may be misleading. Here we see the evidence for this note of caution. Thus, although the long-tail distribution with $\alpha=1.5$ admits of a greater proportion of very short intervals than does the rectangular process (cf. Fig. 2), T_1 for the latter exceeds that of the former for all x_0. Furthermore, the crossing of the rectangular and the exponential curves shows the sensitivity of the mean first-passage time to the detailed distribution $\psi(t)$ of intervals.

Figure 4(b). This graph reemphasizes the conclusion reached in Fig. 3(b). The differences between the two figures arise from the difference in z values.

Figure 5. Here we show the mean first-passage time to z or $-z$ averaged over the initial state x_0. The diffusive case again shows the shortest time to capture for each value of z. The long-tail process with $\alpha=1.5$ has the same qualitative structure as that indicated for $\alpha=2.5$, with a somewhat lower value of T_1 at each z. The exponential and rectangular results are essentially indistinguishable and are therefore drawn as a single curve. For small z this occurs because even the unaveraged mean first-passage times $T_1(z,x_0)$ are very close for these two processes [cf. Fig. 3(a)]. For large z the average over x_0 causes a cancellation between the region where the Markov process has a shorter mean first-passage time [cf. Fig. 4(a), $x_0<0$] and the region where it has a longer mean first-passage time [cf. Fig. 4(a), $x_0>0$]. A noteworthy feature of this figure is the crossing of the T_1 curves for the exponential/rectangular and the long-tail processes. As z increases for a fixed α, the value of A decreases, i.e., the proportion of short steps is reduced. Thus the process has a greater probability of remaining within the range $(-z,z)$ because it can more easily escape from the vicinity of either boundary back into the range $(-z,z)$. This effect is consistent with the discussion of Fig. 3(a).

We conclude that the mean first-passage time is a sensitive function not only of the correlation time of the fluctuations that drive the process but also of the detailed form of the temporal distribution of these fluctuations. It is worth emphasizing that this is the first method capable of dealing analytically with non-Markovian fluctuations. We have also been able to generalize the formalism to dynamical systems subject to a potential. These results will be reported in a forthcoming publication.[43]

ACKNOWLEDGMENTS

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APPENDIX

To facilitate the derivation of differential evolution equations it is convenient to introduce new variables of integration into (3.15) and (3.16). We return to the path variables $x_1=X(t_1)$ and $x_2=X(t_2)$ of Eqs. (2.9) and (2.13) and write (3.15) as

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)+\frac{1}{ab}\int_{x_0}^{z}dx_1\int_{-z}^{x_1}dx_2\,\psi_a((x_1-x_0)/a)\psi_b((x_1-x_2)/b)\times\exp\left[-s\left[\frac{(x_1-x_0)}{a}+\frac{(x_1-x_2)}{b}\right]\right]\tilde{p}^{(a)}(s;x_2)\;. \tag{A1}$$

A similar replacement can be made for (3.16).

1. Derivation of Eqs. (4.1)–(4.3) for Markov $F(t)$

When ψ_a and ψ_b are the exponential forms (2.2), Eq. (A1) becomes

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)}+\frac{\lambda_a\lambda_b}{ab}e^{(\lambda_a+s)x_0/a}\int_{x_0}^{z}dx_1\exp\left[-\left[\frac{(\lambda_a+s)}{a}+\frac{(\lambda_b+s)}{b}\right]x_1\right]\int_{-z}^{x_1}dx_2\,e^{(\lambda_b+s)x_2/b}\tilde{p}^{(a)}(s;x_2)\;. \tag{A2}$$

The derivative of (A2) with respect to x_0 is

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)=\frac{d}{dx_0}(\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})+\frac{(\lambda_a+s)}{a}[\tilde{p}^{(a)}(s;x_0)-\tilde{p}_1^{(a)}-\tilde{p}_2^{(a)}]-\frac{\lambda_a\lambda_b}{ab}e^{-(\lambda_b+s)x_0/b}\int_{-z}^{x_0}dx_2e^{(\lambda_b+s)x_2/b}\tilde{p}^{(a)}(s;x_2)\;. \tag{A3}$$

Another differentiation with respect to x_0 and some reorganization of terms yields

$$\left[\frac{d^2}{dx_0^2}+s\left[\frac{1}{b}-\frac{1}{a}\right]\frac{d}{dx_0}-\frac{s}{ab}(s+\lambda_a+\lambda_b)\right]\tilde{p}^{(a)}(s;x_0)=\left[\frac{d^2}{dx_0^2}+s\left[\frac{1}{b}-\frac{1}{a}\right]\frac{d}{dx_0}-\frac{(\lambda_a+s)(\lambda_b+s)}{ab}\right](\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})\;. \tag{A4}$$

Using the explicit forms

$$\tilde{p}_1^{(a)}(s;x_0)=e^{-(s+\lambda_a)(z-x_0)/a} \tag{A5}$$

and

$$\tilde{p}_2^{(a)}(s;x_0)=\frac{\lambda_a}{a\left[\frac{(\lambda_a+s)}{a}+\frac{(\lambda_b+s)}{b}\right]}e^{-(\lambda_b+s)(z+x_0)/b}\left\{1-\exp\left[-\left[\frac{(\lambda_a+s)}{a}+\frac{(\lambda_b+s)}{b}\right](z-x_0)\right]\right\} \tag{A6}$$

one easily shows that the right-hand side of (A4) vanishes identically.

One boundary condition is obtained by setting $z=x_0$ in Eq. (A1). The integrated term then vanishes, $\tilde{p}_1^{(a)}(s;z)\to 1$ and $\tilde{p}_2^{(a)}(s;z)\to 0$ so that

$$\tilde{p}^{(a)}(s;z)=1\;. \tag{A7}$$

The second boundary condition is obtained by setting $x_0=-z$ in Eq. (A3):

$$\left.\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)\right|_{x_0=-z}-\frac{(s+\lambda_a)}{a}\tilde{p}^{(a)}(s;-z)=-\frac{\lambda_a}{a}\;. \tag{A8}$$

2. Derivation of Eqs. (4.17)–(4.19) for rectangular $F(t)$

When ψ_a and ψ_b are the rectangular forms (4.16) with the restriction $a/\lambda_a=b/\lambda_b>z$, Eq. (A1) becomes

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)+\frac{\lambda_a\lambda_b}{4ab}e^{sx_0/a}\int_{x_0}^{z}dx_1\exp\left[-s\left[\frac{1}{a}+\frac{1}{b}\right]x_1\right]\int_{-z}^{x_1}dx_2\,e^{sx_2/b}\tilde{p}^{(a)}(s;x_2)\;. \tag{A9}$$

The x_0 derivative of (A9) is

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)=\frac{d}{dx_0}(\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})+\frac{s}{a}[\tilde{p}_2^{(a)}(s;x_0)-\tilde{p}_1^{(a)}]-\frac{\lambda_a\lambda_b}{4ab}e^{-sx_0/b}\int_{-z}^{x_0}dx_2 e^{sx_2/b}\tilde{p}^{(a)}(s;x_2)\ . \tag{A10}$$

Another derivative with respect to x_0 and some reorganization of terms yields

$$\left[\frac{d^2}{dx_0^2}+s\left[\frac{1}{b}-\frac{1}{a}\right]\frac{d}{dx_0}+\frac{1}{ab}\left[\frac{\lambda_a\lambda_b}{4}-s^2\right]\right]\tilde{p}^{(a)}(s;x_0)=\left[\frac{d^2}{dx_0^2}+s\left[\frac{1}{b}-\frac{1}{a}\right]\frac{d}{dx_0}-\frac{s^2}{ab}\right](\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})\ . \tag{A11}$$

With the explicit forms

$$\tilde{p}_1^{(a)}(s;x_0)=\frac{\lambda_a}{2a}\left[\frac{2a}{\lambda_a}-z+x_0\right]^{-(z-x_0)s/a} \tag{A12}$$

and

$$\tilde{p}_2^{(a)}(s;x_0)=\frac{\lambda_a\lambda_b}{4}e^{-s(z+x_0)/b}\left\{\exp\left[-\left[\frac{a+b}{ab}\right](z-x_0)s\right]\left[\frac{2b}{s(a+b)}\left[\frac{z}{a}-\frac{1}{\lambda_b}\right]+\frac{b^2}{(a+b)^2s^2}\right]\right.$$
$$\left.+\left[\frac{b}{s(a+b)}\left[\frac{2}{\lambda_b}-\frac{z+x_0}{a}\right]-\frac{b^2}{s^2(a+b)^2}\right]\right\}, \tag{A13}$$

the right-hand side of (A11) can be evaluated. When $a=b$ we find

$$\left[\frac{d^2}{dx_0^2}+\frac{1}{a^2}\left[\frac{\lambda^2}{4}-s^2\right]\right]\tilde{p}^{(a)}(s;x_0)=\frac{\lambda s}{a^2}e^{-(z-x_0)s/a}+\frac{\lambda^2}{4a^2}e^{(z+x_0)s/a}\ . \tag{A14}$$

One boundary condition is again obtained by setting $x_0=z$ in Eq. (A1):

$$\tilde{p}^{(a)}(s;z)=1\ . \tag{A15}$$

The second boundary condition follows from (A10) if we set $x_0=-z$:

$$\left.\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)\right|_{x_0=-z}-\frac{s}{a}\tilde{p}^{(a)}(s;-z)=\frac{\lambda}{2a}(e^{-2zs/a}-1)\ . \tag{A16}$$

3. Derivation of Eqs. (4.35)–(4.37) for long-tail $F(t)$

When $\psi(t)$ has the form (4.29), Eq. (A1) becomes

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)}+\frac{A^2}{a^2}e^{(\lambda+s)x_0/a}\int_{x_0}^{z}dx_1\, e^{-2(\lambda+s)x_1/a}\int_{-z}^{x_1}dx_2\, e^{(\lambda+s)x_2/a}\tilde{p}^{(a)}(s;x_2)\ . \tag{A17}$$

The derivative of (A17) with respect to x_0 is

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)=\frac{d}{dx_0}(\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})+\frac{(\lambda+s)}{a}[\tilde{p}^{(a)}(s;x_0)-\tilde{p}_1^{(a)}-\tilde{p}_2^{(a)}]$$
$$-\frac{A^2}{a^2}e^{-(\lambda+s)x_0/a}\int_{-z}^{x_0}dx_2\, e^{(\lambda+s)x_2/a}\tilde{p}^{(a)}(s;x_2)\ . \tag{A18}$$

Another derivation with respect to x_0 yields

$$\left[\frac{d^2}{dx_0^2}+\frac{1}{a^2}[A^2-(s+\lambda)^2]\right]\tilde{p}^{(a)}(s;x_0)=\left[\frac{d^2}{dx_0^2}-\frac{(s+\lambda)^2}{a^2}\right](\tilde{p}_1^{(a)}+\tilde{p}_2^{(a)})\ . \tag{A19}$$

Together with the explicit form

$$\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)=\frac{A}{\lambda}e^{-(\lambda+s)z/a}(e^{(\lambda+s)x_0/a}-e^{-\lambda z/a}e^{sx_0/a})$$
$$+\frac{A^2}{\lambda}\left[\frac{e^{-(2\lambda+s)z/a}}{\lambda+2s}(e^{-(\lambda+2s)z/a}e^{(\lambda+s)x_0/a}-e^{-sx_0/a})\right.$$
$$\left.-\frac{e^{-(\lambda+s)z/a}}{2(\lambda+s)}(e^{-2(\lambda+s)z/a}e^{(\lambda+s)x_0/a}-e^{-(\lambda+s)x_0/a})\right]$$
$$+\frac{B}{\alpha-1}\left[\frac{2z}{a}\right]^{1-\alpha}e^{-sz/a}\left[e^{sx_0/a}-\frac{A}{\lambda+2s}(e^{-(\lambda+2s)z/a}e^{(\lambda+s)x_0/a}-e^{-sx_0/a})\right], \quad \text{(A20)}$$

we then obtain Eq. (4.35). One boundary condition is again obtained by setting $z=x_0$ in Eq. (A1) to yield (4.36). The second boundary condition is obtained from (A18) by setting $x_0=-z$ to yield (4.37).

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