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Transient and preparation colored-noise effects: The nonlinear relaxation-time approach

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We develop a singular perturbation approach to the problem of the calculation of a characteristic time (the nonlinear relaxation time) for non-Markovian processes driven by Gaussian colored noise with small correlation time. Transient and initial preparation effects are discussed and explicit results for prototype situations are obtained. New effects on the relaxation of unstable states are predicted. The approach is compared with previous techniques.

The problem of the characteristic time scales of stochastic dynamic systems has received great attention in recent years. In many cases the stochastic process under study is assumed to be driven by a white noise. This allows us to use the well established and powerful theory of Markovian processes. Nevertheless, non-Markovian processes are receiving increased attention due to their application to real systems, and also due to the challenging theoretical difficulties.

Many recent theoretical efforts and simulations, both analogical and numerical, have concentrated on the study of the characteristic time of model systems defined by stochastic differential equations of the form

$$\dot{x} = f(x) + g(x)\eta(t) , \tag{1}$$

where $f(x)$ is the deterministic force, $\eta(t)$ is a Gaussian colored noise [an Ornstein-Uhlenbeck process with zero mean and correlation $\langle \eta(t)\eta(t')\rangle = (D/\tau)\exp(|t-t'|/\tau)$] and $g(x)$ is a coupling function. The parameters D and τ are, respectively, the intensity and the correlation time of the noise.

The mean first-passage time (MFPT) was one of the first definitions of characteristic times. In the context of colored noise it was studied by means of digital[1] and analogical[2] simulations and theoretical approximations were implemented to explain the observed data.[1,2] Since then an extraordinary and fruitful controversy took place[3] with important but not definitive theoretical contributions.[4–6] The controversy came from the different ways to address the difficulties of the mathematical formulation of a non-Markovian MFPT problem, such as the derivation and the resolution of the equation obeyed by the MFPT and the treatment of the appropriate boundary conditions.

However, some of these problems seem often to be related to subtleties of the concrete mathematical definitions of the characteristic times, which have, presumably, little physical relevance. One way to avoid these problems could then be to look for other definitions of characteristic time scales. A possible point of view addresses the calculation of the smallest nonvanishing eigenvalue of a Fokker-Planck operator, whose inverse is interpreted as the characteristic time scale.[7] Although this approach is useful for metastable states, it is not appropriate for other situations.

Another characteristic time definition is the so-called linear relaxation time (LRT).[8,9] This approach was fruitfully applied to colored-noise problems but it is restricted to steady-state dynamics.[8]

It is the purpose of this paper to present a new approach which, in our opinion, circumvents some specific difficulties of the MFPT and succeeds in clarifying the different effects of colored noise in the transient evolution. The idea is to define the analog of the LRT for the transient evolution of a general statistical average $\langle \Phi(t)\rangle$ where Φ is any function of the state variables, which relaxes from an initial value $\langle \Phi\rangle_i$ to its steady-state value $\langle \Phi\rangle_{st}$. A characteristic time of this process can then be defined as

$$T_\Phi = \int_0^\infty \frac{\langle \Phi(t)\rangle - \langle \Phi\rangle_{st}}{\langle \Phi\rangle_i - \langle \Phi\rangle_{st}} dt , \tag{2}$$

which is the so-called nonlinear relaxation time[9] (NLRT) of the quantity Φ. This definition has been proposed to be a useful alternative to MFPT techniques[10] in the context of Gaussian white noise. Nevertheless, the specific advantages for colored noise problems are particularly relevant. Essentially they are twofold. First, it avoids the difficulties of the boundary conditions,[4] and second it permits a neat treatment of the different transient effects, and especially those associated with the preparation of distributed initial conditions. The effects of the initial coupling of the system variable and the noise, to which we will pay special attention throughout this paper, are not usually taken into account, despite their playing a crucial role in the understanding of the transient evolution driven by colored noise.

With the definition (2) the NLRT is exactly solvable for Gaussian white-noise problems.[10] For Gaussian colored noise, it will be solved only for small τ. However, instead of starting from an effective Fokker-Planck equation based on a τ-expansion,[11] we will set up the problem in its two-variable Markovian formulation. This is the only way to preserve the effects due to an initial coupling of the system variable x and the noise variable η we want to account for. In fact the standard effective Fokker-Planck descriptions, despite containing transient effects in its time-dependent effective diffusion function, always assume a statistical independence of those variables at $t=0$, that is, a factorization of the respective probability densities. Nevertheless, from a physical point of view it would be worth considering the case of coupled initial states such as those occurring in a system that is prepared as the steady state of another model (in general defined by $\tilde{f},\tilde{g}$), since then it is well known that the two variables are always coupled. A typical case to have in mind would be the study of the transient relaxation of a given system after an instantaneous change of a control parameter.

Therefore, we will address the augmented two-variable Markovian formulation of Eq. (1) defined by

$$\dot{x} = f(x) + (1/\epsilon) g(x)\mu ,$$
$$\dot{\mu} = -(1/\epsilon^2)\mu + (\sqrt{D}/\epsilon)\xi(t) , \tag{3}$$

with $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')$. The noise variable μ is scaled in such a way that $\langle \mu(t)\mu(t')\rangle = D\exp(|t-t'|/\epsilon^2)$ so that the relation between Eqs. (1) and (3) is given by $\mu=\epsilon\eta$ and $\epsilon^2=\tau$. The Fokker-Planck operator associated to Eq. (3) is

$$L(x,\mu) = -\frac{\partial}{\partial x} f(x) + \frac{1}{\epsilon^2}\frac{\partial}{\partial \mu}\mu - \frac{1}{\epsilon}\frac{\partial}{\partial x} g(x)\mu + \frac{D}{\epsilon^2}\frac{\partial^2}{\partial \mu^2} . \tag{4}$$

Commuting the average and the time integral in the definition (2) one can show that the NLRT associated with a function of the system variable $\Phi(x)$ is given by

$$T_\Phi = \frac{1}{\langle \Phi \rangle_i - \langle \Phi \rangle_{\rm st}} \int d\mu \int dx \Phi(x) R(x,\mu) , \tag{5}$$

where the quantity $R(x,\mu)$ obeys the equation

$$L(x,\mu) R(x,\mu) = P_{\rm st}(x,\mu) - P_i(x,\mu) . \tag{6}$$

If we knew $P_{\rm st}(x,\mu)$ and we could obtain an explicit expression for $R(x,\mu)$ from Eq. (6) we would have reduced the problem to quadrature inserting it into Eq. (5). In our case, despite Eq. (6) not being exactly solvable, standard approximate techniques can be applied. In particular a singular perturbation approach for small correlation time $\tau=\epsilon^2$ of the noise is suitable for our purposes. The method inserts the ansatz

$$R(x,\mu) = R_0(x,\mu) + \epsilon R_1(x,\mu) + \epsilon^2 R_2(x,\mu) + \cdots , \tag{7a}$$

$$P_{\rm st}(x,\mu) = P_0(x) P_{\rm st}(\mu) + \epsilon P_1(x,\mu) + \epsilon^2 P_2(x,\mu) + \cdots , \tag{7b}$$

$$P_i(x,\mu) = P_0^i(x) P_{\rm st}(\mu) + \epsilon P_4^i(x,\mu) + \epsilon^2 P_2^i(x,\mu) + \cdots \tag{7c}$$

into Eq. (5) with Eq. (4). Now collecting the different orders in ϵ one gets a set of equations which can be solved recurrently to any order. The procedure is quite cumbersome and parallels step by step that described in Ref. 12 for the calculation of the steady-state probability density of the same problem (1). The main differences come from the nonhomogeneous term of Eq. (6) and from the supplementary condition

$$\int dx R_k(x,\mu) = 0, \quad k=0,1,2,\ldots . \tag{8}$$

Here we shall consider the case in which the initial coupling is of the form of the steady state of any arbitrary preparation model of the type (1) with $\tilde{f}(x)$, $\tilde{g}(x)$, D, and τ. We have performed the algebra in this case, up to order $\epsilon^2=\tau$ (the order ϵ does not contribute). After some manipulations using the explicit form of the colored-noise steady-state probability distribution [the solution of the homogeneous part of Eq. (6) to the same order], the final result can be arranged in a compact form as

$$T = T_0(\tau) + \tau(1-T_1) + O(\tau^2) , \tag{9}$$

where we have separated the solution in two types of contributions. The first term includes the contributions which coincide with the expansion to order τ of the solution of the problem in a quasi-Markovian approximation, that is, that given by the effective Fokker-Planck operator

$$L(x) = -\frac{\partial}{\partial x} f(x) + D\frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} H(x) , \tag{10a}$$

where $H(x)$ defines any possible diffusion function whose first-order form agrees with

$$H(x) = g(x)\{1 + \tau g(x)[f(x)/g(x)]'\} + O(\tau^2) . \tag{10b}$$

If we denote the stationary solution of Eq. (10) by $P_\tau(x)$ and the corresponding to the preparation model by $\tilde{P}_\tau(x)$, the exact solution[10] of the NLRT corresponding to Eq. (10a) reads

$$T_0(\tau) = \frac{1}{\langle \tilde{\Phi}\rangle_\tau - \langle \Phi \rangle_\tau} \int_a^b \frac{F(x) I(x)}{D g(x) H(x) P_\tau(x)} dx , \tag{11}$$

where

$$F(x) = \int_a^x [\Phi(x') - \langle \Phi \rangle_\tau] P_\tau(x') dx' , \tag{12}$$

$$I(x) = \int_a^x [\tilde{P}_\tau(x') - P_\tau(x')] dx' . \tag{13}$$

The averages are taken with the corresponding steady-state probability densities and (a,b) is the domain of definition of the process $x(t)$ of Eq. (1).

The second term of the right-hand side (rhs) of Eq. (9) includes what we could call the purely non-Markovian contributions, not included in any quasi-Markovian approximation. In that sense, Eq. (9) can be interpreted as supplying the purely non-Markovian corrections to any quasi-Markovian approximation [exact up to the order one considers in Eq. (10b)], instead of giving the first-order corrections to the white-noise case.

The non-Markovian contributions of the second term of the rhs of Eq. (9) can also be separated in two parts. First there is a systematic positive amount of τ completely general for any model, any initial condition, and any relaxing function $\Phi(x)$. This is a typical non-Markovian effect analogous to that found in Ref. 8 for the LRT, which accounts for the expected slowing down of the dynamics driven by colored noise. Finally we have the coefficient T_1, which depends on the preparation of the system and reads

$$T_1 = \frac{1}{\langle \tilde{\Phi}\rangle_0 - \langle \Phi \rangle_0} \int_a^b \frac{F_0(x)}{D g(x)} \left[\frac{f(x)}{g(x)} - \frac{\tilde{f}(x)}{\tilde{g}(x)} \right] \frac{\tilde{P}_0(x)}{P_0(x)} , \tag{14}$$

where the zero subscript indicates Gaussian white noise. This coefficient contains additional transient information, particularly about the initial coupling of the variables x and μ since it contains the dependence on $\tilde{f}$ and $\tilde{g}$. This term, which could not have been obtained from any standard one-variable effective Fokker-Planck description, provides the distinction between the coupled and the decoupled initial conditions. In fact, for a decoupled initial condition T_1 would read

$$T_1 = \frac{1}{\langle\tilde{\Phi}\rangle_0 - \langle\Phi\rangle_0} \int_a^b \frac{F_0(x) P_i(x)}{D g^2(x) P_0(x)} f(x) dx , \quad (15)$$

[the term of $T_0(\tau)$ would be the same identifying $P_\tau(x)$ with $P_i(x)$]. The most remarkable particular case of Eq. (15) would be $P_i(x) = \delta(x - x_0)$. This describes the physical situation in which the system is "switched-on" instantaneously at $t=0$ being located at x_0, without any previous influence of the noise. If x_0 is a deterministically stationary point [$f(x_0)=0$] T_1 will vanish.

Coming back to the coupled case, an explicit evaluation of Eq. (14) can be performed with great generality for arbitrary f, g, D, and $\tilde{f}$, with the only restriction of $\tilde{g} = g$ and yields the simple result

$$T_1 = 1 . \quad (16)$$

This is a quite general result which applies to most of the interesting situations. First of all, it is remarkable that the sign is positive, so that this term, according to the definition of T_1 in Eq. (9), gives a decrease of the relaxation time. This goes in the opposite direction than usually expected for a colored-noise effect. Furthermore Eq. (16) implies that only the term $T_0(\tau)$ survives in Eq. (9), that is, the problem has been reduced to the effective steady-state Fokker-Planck description (10) (with a time-independent diffusion function).

The most remarkable point is that the dynamics of the system for almost any steady-state-like preparation has been reduced, having included all kinds of transient effects (to first order in τ), to an effective Gaussian white-noise dynamics, where the τ dependence enters only parametrically into the equations, so that the usual machinery of Markovian processes is applicable.

We can now check the relative importance of the different colored-noise effects and, in particular, the practical relevance of the initial coupling in different prototype situations. For instance, in the study of barrier-crossing problems, the term $T_0(\tau)$ will always be dominant, since the corrections to the diffusion will be magnified by the exponentially large time scale, typically given by the Arrhenius law.[8] The additive corrections of order τ can be neglected so that any effective Fokker-Planck approach like Eq. (10) is justified for these types of activation processes. In fact the NLRT characterization of a bistable model like $f(x) = ax - x^3$, $g(x) = 1$ with $\Phi(x) = x^n$ (n odd) and, for instance, with a δ-peaked condition at the minimum $-a^{1/2}$ ($T_1 = 0$), in the limit of small intensity D of the noise, leads to

$$T = \frac{\pi}{a\sqrt{2}} (1 + \tfrac{3}{2} a\tau + \cdots) \exp\left[\frac{a^2}{4D}\right] . \quad (17)$$

The neglected contribution $+\tau$ of Eq. (9) corresponds to the neglect of the transient terms in the effective Fokker-Planck equation. The same correction of $\frac{3}{2} a\tau$ has also been encountered for MFPT.[1,3] However, it is to be noticed that a contribution like the $\tau^{1/2}$ obtained in Ref. 4 does not appear. In that case it arose from the correct treatment of the boundary conditions in the colored-noise MFPT problem for the same activation process and can be seen as a consequence of the mathematical definition of the MFPT. The prefactor is the same as that of the MFPT with end point at the bottom of the other well, which is also the inverse of the first nonvanishing eigenvalue in the same limit.

On the other hand, the discussion about transient and preparation effects will be relevant for instance in the study of the decay of unstable states, which are very sensitive to initial conditions and whose relaxation time scale is much shorter. All studies up to now are based explicitly or implicitly on the hypothesis of statistical independence of the system variable and the noise in the initial state. For instance, Suzuki's scaling theory has been generalized recently to colored noise[13] with that assumption. In that form this theory cannot be applied to the typical switching experiments like those of Ref. 14. In fact, the relaxation of a "quenched" Ginzburg-Landau model defined by $f(x) = ax - x^3$, $g(x) = 1$ for $t > 0$ the system being prepared at $t=0$ as the steady state of $\tilde{f}(x) = -a_0 x - x^3$, $\tilde{g}(x) = 1$ ($a, a_0 > 0$) under the same noise source, is a prototype case of an initial coupling of the class we are dealing with. In these conditions, the relaxation time of the second moment [$\Phi(x) = x^2$] turns to be appreciably smaller than that corresponding to the decoupled case [$T_1 = a/(a + a_0) + O(D) < 1$]. To our knowledge, such a destabilizing effect of colored noise has never been reported before. Nevertheless, it has a clear intuitive explanation taking into account that in such an initial state the values of x are correlated with values of the noise of the same sign, that is, the noise force tends mostly to reinforce the deterministic one, whereas in the decoupled case both forces are independent at the initial stages. Despite being calculated for the NLRT, this effect is claimed to be general and should also be found for other characteristic time definitions, like MFPT, since it is related to a physical mechanism of the preparation procedure.

Following with the quenched unstable state, the remaining colored-noise effects which still coexist on the decay time with respect to the same problem with Gaussian white noise ($\tau = 0$) are still twofold. On one hand the width of the initial distribution will be smaller [to first order in τ, $\langle x_0^2(\tau)\rangle \approx D(1 - a_0\tau)/a_0$]. This will tend to slow down the decay. On the other hand, according to Eq. (10b) the effective diffusion will be greater at the initial stages [in the linear approximation and to first order in τ, $D_{\text{eff}} \approx D(1 + a\tau)$] and this will tend to accelerate the decay. A possible way to estimate the result of those competing effects, given that an exact analytic integration of Eq. (11) in our case is not possible, is to use the arguments of the scaling theory to the underlying effective Gaussian white-noise problem (for a discussion of the NLRT characterization of the decay of an unstable state with white noise see Ref. 10). This yields in the limit of small noise

$$T = T_0(\tau) \approx \begin{cases} -(1/2a)\ln[\langle x_0^2(\tau)\rangle + D_{\rm eff}(\tau)/a] + \text{const}, \\ T(\tau=0) + O(\tau^2), \end{cases} \tag{18}$$

where $T(\tau=0)$ is the Gaussian white-noise solution of the same problem.[10] It turns out that the two competing effects are exactly canceled to order τ so that the net correction with respect to the Gaussian white-noise case is of order τ^2. Therefore, the decay time is always smaller than the corresponding to the decoupled case of Ref. 13, which has always a positive first-order correction. These results call for new numerical and analogical experiments.

Finally, as a general conclusion, we advocate that the NLRT approach can be taken into account as a useful tool in the problem of the characteristic time scales of relaxation processes. In the current atmosphere of controversy surrounding mean first-passage time calculations, the NLRT approach not only could avoid some of those difficult issues, but could also provide a novel insight into the transient evolution of colored-noise driven systems.

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