Hydrodynamic fluctuations in fluids under external gradients

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Temperature and velocity correlation functions in a fluid subjected to conditions creating both a temperature and a velocity gradient are computed up to second order in the gradients. Temperature and velocity fluctuations are coupled due to convection and viscous heating. When the viscosity goes to infinity one gets the temperature correlation function for a solid under a temperature gradient, which contains a long-ranged contribution, quadratic in the temperature gradient. The velocity correlation function also exhibits long-range behavior. In a particular case its equilibrium term is diagonal whereas the nonequilibrium correction contains nondiagonal terms.

I. INTRODUCTION

Recently many efforts have been devoted to the study of fluctuations about nonequilibrium steady states.[1,2] It seems well established that at least in the most simple cases, as for example systems under constant external gradients, different theories come to identical results. Thus from the fluctuating-hydrodynamics,[3] kinetic-theory,[4] and master-equation approach,[5] one arrives at the result that density correlation function in a fluid under a temperature gradient consists of the equilibrium result plus a correction which includes the external gradient imposed to the system.

Until now, nonequilibrium statistical mechanics theories for fluids, have been used primarily to get expressions of the density autocorrelation function, the reason being that it can be measured by means of light-scattering experiments. Exceptions are for example[6] in which the velocity autocorrelation function of a Brownian particle in a shear flow is computed or[7] where the velocity autocorrelation function in a fluid under shear is given. In this last reference, temperature effects are eliminated by assuming that, at each point of the fluid, it is possible to extract the heat generated by the viscous heating term (homogeneous shear). In this paper this assumption is removed.

The paper is organized as follows. In Sec. II we introduce the temperature and velocity steady-state solutions. The latter corresponds to the velocity of Couette flow whereas the stationary temperature is quadratic in the shear rate due to the viscous heating term. Fluctuations are analyzed in the framework of generalized fluctuating hydrodynamics.[3] Section III is devoted to the study of temperature fluctuations. A general result is given up to quadratic order in the external gradients. When only the temperature gradient is present the static or equal-time temperature correlation function is computed. It contains a cutoff wave vector due to the perturbative expansion we used. When either the viscosity or Prandtl number are infinite one gets the static temperature correlation function in real space which introduces an algebraic correction to the local term characteristic of equilibrium. In Sec. IV we compute the velocity correlation function also to second order in gradients. For a fluid under a temperature gradient the convective term vanishes, then a nonperturbative solution of the static correlation function is found. The static velocity correlation function in real space is computed in some particular cases.

II. FLUCTUATIONS AROUND NONEQUILIBRIUM STEADY STATES

Our starting equations are the differential equations for the evolution of mass, momentum, and entropy which come from balance equations for such quantities and nonequilibrium thermodynamics linear laws. If the fluid is incompressible[8] they write

$$\nabla\cdot\mathbf{v}=0\ , \tag{2.1}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]=-\nabla p+\eta\nabla^2\mathbf{v}\ , \tag{2.2}$$

$$\rho T\left[\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right]=\lambda\nabla^2 T+2\eta(\nabla\mathbf{v})^{(S)}:(\nabla\mathbf{v})^{(S)}\ , \tag{2.3}$$

where $\mathbf{v}$ is the velocity, ρ the mass density, p the pressure, T the temperature, and s the entropy per unit mass. The transport coefficients η and λ, known as shear viscosity and thermal conductivity, respectively, come from linear laws of nonequilibrium thermodynamics. They are assumed to be constant.[9,10] The superscript (S) in $(\cdots)^{(S)}$ stands for taking the symmetric and traceless part of a tensor. Equations (2.2) and (2.3) are coupled because convection and viscous heating take place.

In general and as stated in Ref. 11 the fluid can be considered as incompressible if temperature perturbations are sufficiently small. Then variations of density are only due to variations of pressure which can be neglected if the velocity is much smaller than the velocity of sound in the fluid. On the other hand, if the thermal expansion coefficient $V^{-1}(\partial V/\partial T)_p$, V being the specific volume, vanishes the fluid can also be considered as incompressible. In fact, using

$$\delta\rho=\left[\frac{\partial\rho}{\partial p}\right]_s\delta p+\left[\frac{\partial\rho}{\partial s}\right]_p\delta s \tag{2.4}$$

and the thermodynamic relations

$$\left[\frac{\partial \rho}{\partial s}\right]_p = -\left[\frac{\partial \rho}{\partial v}\right]_s \left[\frac{\partial v}{\partial T}\right]_p \left[\frac{\partial T}{\partial s}\right]_p , \quad (2.5)$$

$$\left[\frac{\partial p}{\partial \rho}\right]_s = c^2 , \quad (2.6)$$

where c is the adiabatic sound speed, one concludes that in this case the variations of density are only due to variations of pressure which can be neglected for the same reason as in the former case. In these circumstances one also has $c_p = c_v$, where c_p and c_v are the specific heat at constant pressure and volume, respectively.

Stationary solutions of Eqs. (2.1)—(2.3) are the solutions of those equations by setting the time derivatives equal to zero. Obviously such solutions will depend on boundary conditions or in other words on external gradients. Let us consider our system bounded in the y direction and infinite in the x and z directions. Both a thermal gradient and a velocity gradient are then applied. Under such a situation boundary conditions are

$$\begin{aligned} &T(x, y = \pm L/2, z) = T_0 \pm \Delta T/2 , \\ &v_x(x, y = \pm L/2, z) = \pm u/2 , \\ &v_y = v_z = 0 , \end{aligned} \quad (2.7)$$

L being the width occupied by the fluid in the y direction and ΔT the temperature difference between both plates. The stationary solutions of (2.1)—(2.3) with (2.7) are[12]

$$T_s(\mathbf{r}) = T_0 + \tfrac{1}{4}\psi u^2 + \mathbf{r}\cdot\nabla_0 T - \psi(\mathbf{r}\cdot\nabla_0\mathbf{v})^2 , \quad (2.8)$$

$$\mathbf{v}_s(\mathbf{r}) = \mathbf{r}\cdot\nabla_0\mathbf{v} , \quad (2.9)$$

$$p_s = \text{const} , \quad (2.10)$$

where $\psi = \eta/2\lambda$ and the external gradients are

$$\nabla_0 T = \Delta T / L \hat{\mathbf{e}}_x , \quad (2.11)$$

$$\nabla_0 \mathbf{v} = \gamma \hat{\mathbf{e}}_y \hat{\mathbf{e}}_x , \quad (2.12)$$

γ being the shear rate equal to u/L and $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ the unit vectors along x and y directions, respectively. Equation (2.8) can also be written[12]

$$T_s(\mathbf{r}) = T_0 + \tfrac{1}{8}(PE)\Delta T + \Delta T y/L - \tfrac{1}{2}(PE)\Delta T y^2/L^2 , \quad (2.13)$$

where P and E are, respectively, Prandtl and Eckert numbers defined as

$$P = \nu/\alpha, \quad E = u^2/(c_p\,\Delta T) , \quad (2.14)$$

ν being the kinematic viscosity, and α the thermal diffusivity. Therefore if the product PE goes to zero the stationary solution (2.13) coincides with that of a fluid under a temperature gradient. This is essentially due to the fact that in such a case viscous heating can be neglected.

In the framework of generalized Landau-Lifshitz fluctuating hydrodynamics,[3] fluctuations evolve according to nonequilibrium thermodynamics equations in which fluctuating sources are introduced. Thus one has

$$\nabla\cdot\delta\mathbf{v} = 0 , \quad (2.15)$$

$$\rho\frac{\partial \delta\mathbf{v}}{\partial t} + \rho\mathbf{v}_s\cdot\nabla\delta\mathbf{v} + \rho\delta\mathbf{v}\cdot\nabla\mathbf{v}_s = -\nabla\delta p + \eta\nabla^2\delta\mathbf{v} + \nabla\cdot\overleftrightarrow{\sigma} , \quad (2.16)$$

$$\begin{aligned} \frac{\partial \delta T}{\partial t} &+ \mathbf{v}_s\cdot\nabla\delta T + \delta\mathbf{v}\cdot\nabla T_s \\ &= \alpha\nabla^2\delta T + 4\eta(\rho c_p)^{-1}(\nabla\mathbf{v}_s)^{(S)}:(\nabla\delta\mathbf{v})^{(S)} \\ &\quad -(\rho c_p)^{-1}\nabla\cdot\mathbf{g} , \end{aligned} \quad (2.17)$$

where $\overleftrightarrow{\sigma}$ and $\mathbf{g}$ are the fluctuating sources of momentum and internal energy, respectively, and $\delta\Lambda = \Lambda - \Lambda_s$, Λ being an unspecified hydrodynamic field. The fluctuating sources satisfy the following stochastic properties:

$$\langle \overleftrightarrow{\sigma} \rangle = \langle \mathbf{g} \rangle = 0 , \quad (2.18)$$

$$\langle \sigma_{ij}(\mathbf{r},t)\sigma_{lm}(\mathbf{r}',t') \rangle = 2k_B T_s(\mathbf{r})\eta_{ijlm}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') , \quad (2.19)$$

$$\langle g_i(\mathbf{r},t)g_j(\mathbf{r}',t') \rangle = 2k_B\lambda T_s^2(\mathbf{r})\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') , \quad (2.20)$$

$$\langle \overleftrightarrow{\sigma}\mathbf{g} \rangle = 0 , \quad (2.21)$$

where

$$\eta_{ijlm} = \eta(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) . \quad (2.22)$$

To calculate correlation functions in Fourier space, we will write Eqs. (2.8) and (2.9) in a more convenient form[3]

$$T_s(\mathbf{r}) = T_0 + \tfrac{1}{4}\psi u^2 + \delta\widetilde{T}\sin(\mathbf{q}_1\cdot\mathbf{r}) - \psi\widetilde{v}^2\sin^2(\mathbf{q}_2\cdot\mathbf{r}) , \quad (2.23)$$

$$\mathbf{v}_s(\mathbf{r}) = \widetilde{\mathbf{v}}\sin(\mathbf{q}_2\cdot\mathbf{r}) , \quad (2.24)$$

where we have defined

$$\mathbf{q}_1\delta\widetilde{T} = \nabla_0 T , \quad (2.25a)$$

$$\mathbf{q}_2\widetilde{\mathbf{v}} = \nabla_0\mathbf{v} , \quad (2.25b)$$

with $\mathbf{q}_1 = q_1\hat{\mathbf{e}}_y$ and $\mathbf{q}_2 = q_2\hat{\mathbf{e}}_y$. Obviously in the limit $\mathbf{q}_1\cdot\mathbf{r} \ll 1$ and $\mathbf{q}_2\cdot\mathbf{r} \ll 1$ both sets of equations coincide. Then calculations can be performed using Eqs. (2.23) and (2.24) instead of (2.8) and (2.9) and taking the former limits at the end.

In $(\mathbf{k},\omega)$ space, Eqs. (2.15)—(2.17) can be written, respectively, as

$$\mathbf{k}\cdot\delta\mathbf{v}(\mathbf{k},\omega) = 0 , \quad (2.26)$$

$$\delta\mathbf{v}(\mathbf{k},\omega) = G^v(\mathbf{k},\omega)(\overleftrightarrow{1} - \hat{\mathbf{k}}\hat{\mathbf{k}})\cdot\{i\mathbf{k}\cdot\overleftrightarrow{\sigma}(\mathbf{k},\omega) - \rho\widetilde{\mathbf{v}}\mathbf{q}_2\cdot\delta\mathbf{v}(\mathbf{k},\omega) + \tfrac{1}{2}\rho(\widetilde{\mathbf{v}}\cdot\mathbf{k})[\delta\mathbf{v}(\mathbf{k}+\mathbf{q}_2,\omega) - \delta\mathbf{v}(\mathbf{k}-\mathbf{q}_2,\omega)]\} , \quad (2.27)$$

$$\delta T(\mathbf{k},\omega)=G^T(\mathbf{k},\omega)\left[\frac{\tilde{\mathbf{v}}\cdot\mathbf{k}}{2}[\delta T(\mathbf{k}+\mathbf{q}_2,\omega)-\delta T(\mathbf{k}-\mathbf{q}_2,\omega)]-\delta\tilde{T}\mathbf{q}_1\cdot\delta\mathbf{v}(\mathbf{k},\omega)+i\eta(\rho c_p)^{-1}2(\mathbf{q}_2\tilde{\mathbf{v}})^{(S)}:\mathbf{k}\delta\mathbf{v}(\mathbf{k},\omega)\right.$$
$$\left.+i\psi\tilde{v}^2\mathbf{q}_2\cdot[\delta\mathbf{v}(\mathbf{k}+\mathbf{q}_2,\omega)-\delta\mathbf{v}(\mathbf{k}-\mathbf{q}_2,\omega)]-i(\rho c_p)^{-1}\mathbf{k}\cdot\mathbf{g}(\mathbf{k},\omega)\right], \tag{2.28}$$

where $\hat{\mathbf{k}}$ is the unit vector along the direct of $\mathbf{k}$ and

$$G^v(\mathbf{k},\omega)=(-i\omega\rho+\eta k^2)^{-1}, \tag{2.29}$$

$$G^T(\mathbf{k},\omega)=(-i\omega+\alpha k^2)^{-1} \tag{2.30}$$

are the equilibrium Green functions in $(\mathbf{k},\omega)$ space associated to the differential equations for the evolution of momentum and temperature, respectively. To arrive at Eq. (2.27) we have applied the operator $\nabla\times\nabla\times$ to Eq. (2.16) and we have taken into account (2.15). Moreover, the Fourier transform of an unspecified field $\Lambda(\mathbf{r},t)$ has been defined as

$$\Lambda(\mathbf{k},\omega)=\int_{-\infty}^{\infty}d\mathbf{r}\int_{-\infty}^{\infty}dt\,e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\omega t}\Lambda(\mathbf{r},t). \tag{2.31}$$

Likewise we have used the fact that $\tilde{\mathbf{v}}\cdot\mathbf{q}_2=0$. Transforming also Eqs. (2.23) and (2.24) we obtain

$$T_s(\mathbf{k})=(2\pi)^3(T_0+\tfrac{1}{4}\psi u^2)\delta(\mathbf{k})+(2\eta)^3\frac{i\delta\tilde{T}}{2}[\delta(\mathbf{k}+\mathbf{q}_1)-\delta(\mathbf{k}-\mathbf{q}_1)]+(2\pi)^3\psi\frac{\tilde{v}^2}{4}[\delta(\mathbf{k}+2\mathbf{q}_2)+\delta(\mathbf{k}-2\mathbf{q}_2)-2\delta(\mathbf{k})], \tag{2.32}$$

$$\mathbf{v}_s(\mathbf{k})=i\frac{\tilde{\mathbf{v}}}{2}(2\pi)^3[\delta(\mathbf{k}+\mathbf{q}_2)-\delta(\mathbf{k}-\mathbf{q}_2)]. \tag{2.33}$$

From Eq. (2.19) and using (2.32) one gets

$$\langle\sigma_{ij}(\mathbf{k},\omega)\sigma_{lm}(\mathbf{k}',\omega')\rangle=\Delta_{ijlm}^{(0)}+\Delta_{ijlm}^{(\delta\tilde{T})}+\Delta_{ijlm}^{(\tilde{v}^2)}, \tag{2.34}$$

where

$$\Delta_{ijlm}^{(0)}=2k_B(T_0+\tfrac{1}{4}\psi u^2)\eta_{ijlm}(2\pi)^4\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}'), \tag{2.35}$$

$$\Delta_{ijlm}^{(\delta\tilde{T})}=ik_B\delta\tilde{T}\eta_{ijlm}(2\pi)^4\delta(\omega+\omega')\times[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)], \tag{2.36}$$

$$\Delta_{ijlm}^{(\tilde{v}^2)}=\tfrac{1}{2}k_B\psi\tilde{v}^2\eta_{ijlm}(2\pi)^4\delta(\omega+\omega')\times[\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_2)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_2)-2\delta(\mathbf{k}+\mathbf{k}')]. \tag{2.37}$$

Finally, from Eq. (2.20) and by using the Fourier transform of $T_s^2(\mathbf{r})$ we obtain

$$\langle g_i(\mathbf{k},\omega)g_j(\mathbf{k},\omega)\rangle=\zeta_{ij}^{(0)}+\zeta_{ij}^{(\delta\tilde{T})}+\zeta_{ij}^{(\delta\tilde{T}^2)}+\zeta_{ij}^{(\tilde{v}^2)} \tag{2.38}$$

with

$$\zeta_{ij}^{(0)}=2k_B\lambda(T_0+\tfrac{1}{4}\psi u^2)^2\delta_{ij}(2\pi)^4\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}'), \tag{2.39}$$

$$\zeta_{ij}^{(\delta\tilde{T})}=2k_B\lambda i(T_0+\tfrac{1}{4}\psi u^2)\delta\tilde{T}\delta_{ij}(2\pi)^4\delta(\omega+\omega')\times[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)], \tag{2.40}$$

$$\zeta_{ij}^{(\delta\tilde{T}^2)}=\tfrac{1}{2}k_B\lambda\delta\tilde{T}^2\delta_{ij}(2\pi)^4\delta(\omega+\omega')\times[2\delta(\mathbf{k}+\mathbf{k}')-\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_1)], \tag{2.41}$$

$$\zeta_{ij}^{(\tilde{v}^2)}=k_B\lambda(\tfrac{1}{4}\psi u^2+T_0)\psi\tilde{v}^2\delta_{ij}(2\pi)^4\delta(\omega+\omega')\times[-2\delta(\mathbf{k}+\mathbf{k}')+\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_2)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_2)]. \tag{2.42}$$

In Eq. (2.38) we have neglected terms of order greater than two in the external gradients.

III. TEMPERATURE CORRELATION FUNCTION

In what follows we will calculate temperature correlation functions up to quadratic order in velocity and temperature gradients.

A. General case

From Eq. (2.28) one can obtain $\delta T(\mathbf{k},\omega)$ as a function of $\delta\mathbf{v}(\mathbf{k},\omega)$, $\delta\mathbf{v}(\mathbf{k}\pm\mathbf{q}_2,\omega)$, $\delta T(\mathbf{k}\pm\mathbf{q}_2,\omega)$, and $\mathbf{g}(\mathbf{k},\omega)$. Squaring and averaging one gets

$$\langle\delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle=G^T(\mathbf{k},\omega)G^T(\mathbf{k}',\omega')\times\left\{\frac{(-\mathbf{k}\mathbf{k}')}{\rho^2c_p^2}:\langle\mathbf{g}(\mathbf{k},\omega)\mathbf{g}(\mathbf{k}',\omega')\rangle-\tfrac{1}{2}(\rho c_p)^{-1}\left[\tilde{\mathbf{v}}\cdot\mathbf{k}\langle[\delta T(\mathbf{k}+\mathbf{q}_2,\omega)-\delta T(\mathbf{k}-\mathbf{q}_2,\omega)]i\mathbf{k}\cdot\mathbf{g}(\mathbf{k}',\omega')\rangle+\begin{bmatrix}\mathbf{k}\leftrightarrow\mathbf{k}'\\ \omega\leftrightarrow\omega'\end{bmatrix}\right]\right.$$
$$+\frac{\tilde{\mathbf{v}}\tilde{\mathbf{v}}:\mathbf{k}\mathbf{k}'}{4}\left\langle[\delta T(\mathbf{k}+\mathbf{q}_2,\omega)-\delta T(\mathbf{k}-\mathbf{q}_2,\omega)]\times\begin{bmatrix}\mathbf{k}\leftrightarrow\mathbf{k}'\\ \omega\leftrightarrow\omega'\end{bmatrix}\right\rangle+\delta\tilde{T}^2\mathbf{q}_1\mathbf{q}_1:\langle\delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle$$
$$\left.-\eta^2(\rho c_p)^{-2}2(q_2\tilde{\mathbf{v}})^{(S)}:\mathbf{k}\langle\delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle\mathbf{k}':2(q_2\tilde{\mathbf{v}})^{(S)}\right\}. \tag{3.1}$$

Equation (3.1) shows that temperature and velocity correlation functions are coupled to quadratic order in the gradients. In writing Eq. (3.1) we have dropped correlations between velocity and temperature. This is due to the fact that the heat flux and stress tensor are uncorrelated [see Eq. (2.21)]. For wave vectors such that $\tilde{\mathbf{v}}\cdot\mathbf{k}=0$ the second and third terms of the right-hand side of Eq. (3.1) vanish and this equation simplifies appreciably.

By using Eqs. (2.34) and (2.35) and the perturbative solution of Eqs. (2.27) and (2.28) we get from Eq. (3.1) the temperature correlation function to zeroth order in the external gradients

$$\langle \delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle^{(0)}=\frac{2k_B\lambda}{(\rho c_p)^2}(T_0+\tfrac{1}{4}\psi u^2)^2 G^T(\mathbf{k},\omega)G^T(\mathbf{k}',\omega')(-\mathbf{k}\cdot\mathbf{k}')(2\pi)^4\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') . \quad (3.2)$$

Equation (3.2) is similar to the equilibrium temperature correlation function except for the fact that there is an increase of temperature due to the viscous heating. The nonequilibrium correction up to quadratic order in gradients are also obtained from (2.27), (2.28), (2.34), (2.35), (2.38)—(2.42), and (3.1). One arrives at the linear term

$$\langle \delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle^{(\delta\tilde{T})}=\frac{2k_B\lambda}{(\rho c_p)^2}i(T_0+\tfrac{1}{4}\psi u^2)\delta\tilde{T}G^T(\mathbf{k},\omega)G^T(\mathbf{k}',\omega')(-\mathbf{k}\cdot\mathbf{k}')(2\pi)^4\delta(\omega+\omega')$$
$$\times[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)] \quad (3.3)$$

and at the nonlinear terms

$$\langle \delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle^{(\delta\tilde{T}^2)}=\frac{2k_B\lambda}{(\rho c_p)^2}(2\pi)^4\delta(\omega+\omega')G^T(\mathbf{k},\omega)G^T(\mathbf{k}',\omega')\delta\tilde{T}^2$$
$$\times\left\{[k^2/2+P(T_0+\tfrac{1}{4}\psi u^2)\rho^2 c_p G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')k^2\mathbf{q}_1\mathbf{q}_1:(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})]\delta(\mathbf{k}+\mathbf{k}')\right.$$
$$\left.+\frac{\mathbf{k}\cdot\mathbf{k}'}{4}[\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_1)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_1)]\right\} \quad (3.4)$$

and

$$\langle \delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle^{(\tilde{v}^2)}=\frac{2k_B\lambda}{(\rho c_p)^2}(2\pi)^4\delta(\omega+\omega')G^T(\mathbf{k},\omega)G^T(\mathbf{k}',\omega')\tilde{v}^2$$
$$\times\{[-k^2\psi+P\eta^2/c_p k^4 G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')(\hat{\mathbf{k}}\cdot\mathbf{q}_2)^2](T_0+\tfrac{1}{4}\psi u^2)\delta(\mathbf{k}+\mathbf{k}')$$
$$-\tfrac{1}{2}(T_0+\tfrac{1}{4}\psi u^2)\psi(\mathbf{k}\cdot\mathbf{k}')[\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_2)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_2)]\} . \quad (3.5)$$

For the sake of simplicity, all the nonequilibrium corrections have been given for wave vectors such that $\tilde{\mathbf{v}}\cdot\mathbf{k}=0$. The total temperature correlation up to quadratic order in gradients is then obtained by adding the right-hand sides of Eqs. (3.2)—(3.5).

The poles of $\langle\delta T(\mathbf{k},\omega)\delta T(\mathbf{k}',\omega')\rangle$ in frequency determine the characteristic time scale of thermal fluctuations. From Eqs. (3.2)—(3.5) we find two time scales, one associated with the thermal diffusion $\tau_T=(\alpha k^2)^{-1}$ and the other one associated with the viscous dissipation $\tau_v=(\nu k^2)^{-1}$. In the same way the poles in wave number determine the characteristic length scales $L_T\sim(\alpha/\omega)^{1/2}$ and $L_v\sim(\nu/\omega)^{1/2}$. One should notice that at equilibrium the characteristic time τ_v and length L_v are not present. This is due to the fact that in such a case thermal and velocity fluctuations are not coupled. In other words, at equilibrium the terms responsible for convection and viscous heating vanish. The characteristic length L_T and L_v represent also the penetration depth of thermal and velocity fluctuations, respectively. When we Fourier transform according to Eq. (2.31), we have tacitly assumed that, in order that our system be considered as infinite, L_T and L_v must be much smaller than the size of the system. This leads to restrictions over frequencies. However, integrals over all frequencies can be performed since the leading contributions appear in the poles of Green functions.[13]

The above results are valid in the following limits:

$$\left[\frac{c_p}{T_0'}\right]^{1/2}\frac{|\nabla_0 T|}{\min(\alpha,\nu)k^2}\ll 1, \quad \frac{|\nabla_0\mathbf{v}|}{\nu k^2}\ll 1 , \quad (3.6)$$

where $T_0'=T_0+\frac{1}{4}\psi u^2$. Notice that the above inequalities introduce a cutoff in wave vector. Therefore if any of those inequalities is not satisfied we would obtain divergences in correlation functions.

B. Fluid under a temperature gradient

Now we apply the general result of Sec. III A to the case of a fluid in which the only external perturbation is a constant thermal gradient.

We are interested in computing the equal-time correlation function. Then we will Fourier transform Eqs. (3.2)—(3.4). Due to the fact that the temperature correlation function is delta correlated in frequency, one arrives at

$$\langle \delta T(\mathbf{k},t)\delta T(\mathbf{k}',t)\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, A(\mathbf{k},\mathbf{k}',\omega)\ , \tag{3.7}$$

where $A(\mathbf{k},\mathbf{k}',\omega)$ is obtained by adding the coefficients of $\delta(\omega+\omega')$ in Eqs. (3.2)—(3.4) and changing $\omega' \leftrightarrow -\omega$. Making use of Eq. (3.7) we find the equilibrium static correlation function

$$\langle \delta T(\mathbf{k},t)\delta T(\mathbf{k}',t)\rangle^{(0)} = \frac{k_B T_0^2}{\rho c_p}(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\ . \tag{3.8}$$

The nonequilibrium linear correction, which we obtain making $q_1^2=0$ (see Appendix A), can be written

$$\langle \delta T(\mathbf{k},t)\delta T(\mathbf{k}',t)\rangle^{(\delta\widetilde{T})} = \frac{k_B}{\rho c_p} iT_0\delta\widetilde{T}(2\pi)^3[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)]\ , \tag{3.9}$$

and the nonequilibrium nonlinear correction is

$$\langle \delta T(\mathbf{k},t)\delta T(\mathbf{k}',t)\rangle^{(\delta\widetilde{T}^2)} = \frac{k_B}{\rho c_p}(2\pi)^3\left[\frac{(\delta\widetilde{T}^2)}{4}\{2\delta(\mathbf{k}+\mathbf{k}')-(1-2q_1^2/k^2)[\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_1)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_1)]\} + \frac{1}{P+1}T_0 c_p\frac{\nabla_0 T\nabla_0 T}{\alpha^2 k^4}:(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\delta(\mathbf{k}+\mathbf{k}')\right]\ . \tag{3.10}$$

Equation (3.10) contains a term proportional to $c_p\,|\nabla_0 T|^2 T_0/\alpha^2 k^4$ which leads to a divergence if we integrate over $\mathbf{k}$. To avoid this divergence and as pointed out before we must introduce the cutoff wave number $k_0^2=(c_p/T_0)^{1/2}\,|\nabla_0 T|/\alpha$. Then the static correlation is valid for $k \gg k_0$. The cutoff appears as a consequence of the coupling between the temperature and velocity fluctuations. In fact, the third term of the right-hand side of Eq. (3.10) can be rewritten in such a way that it is proportional to $P/(P+1)$; therefore if P goes to zero that term is not present.

C. Solid under a temperature gradient

Temperature correlation functions in a solid body can be computed from the fact that since for a solid ν can be considered as infinite, the Prandtl number is infinite too. Therefore in $(\mathbf{k},\omega)$ space, the temperature correlation function comes from Eqs. (3.2)—(3.4). In $(\mathbf{k},t)$ space the only difference with respect to the set of Eqs. (3.8)—(3.10) is that the third term of Eq. (3.10) cancels. Therefore we do not need to introduce any cutoff in wave vector. Fourier inversion can be made to get the temperature correlation function at different positions and equal time. One arrives at the result[14]

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle = \frac{k_B}{\rho c_v}\left[T_s^2(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \frac{|\nabla_0 T|^2}{4\pi|\mathbf{r}-\mathbf{r}'|}\right] \tag{3.11}$$

which shows that the nonequilibrium corrections are quadratic in the temperature gradient and exhibits long-range behavior. Temperature correlation function (3.11) is similar to that obtained for a finite system, using a multivariate master equation[15] or using fluctuating hydrodynamics.[2] Some details of the derivation of Eq. (3.11) are given in Appendix B.

IV. VELOCITY CORRELATION FUNCTIONS

In this section we calculate velocity correlations up to quadratic order in velocity and temperature gradients by following the procedure outlined in Sec. III.

A. General case

For the sake of simplicity we consider wave vectors such that $\widetilde{\mathbf{v}}\cdot\mathbf{k}=0$. From Eq. (2.27) one can obtain $\delta\mathbf{v}(\mathbf{k},\omega)$ as a function of $\delta\mathbf{v}(\mathbf{k}+\mathbf{q}_2,\omega)$, $\delta\mathbf{v}(\mathbf{k}-\mathbf{q}_2,\omega)$, $\delta\mathbf{v}(\mathbf{k},\omega)$, and $\vec{\sigma}(\mathbf{k},\omega)$. After neglecting terms proportional to $\widetilde{\mathbf{v}}\cdot\mathbf{k}$, if we square and average the expression of $\delta\mathbf{v}(\mathbf{k},\omega)$ one arrives at

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle = G^{v}(\mathbf{k},\omega)G^{v}(\mathbf{k}',\omega')\left\{-(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\langle\vec{\sigma}(\mathbf{k},\omega)\vec{\sigma}(\mathbf{k}',\omega')\rangle:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}'\hat{\mathbf{k}}') - i\rho\left[(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\langle\vec{\sigma}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle\cdot\mathbf{q}_2\widetilde{\mathbf{v}} + \begin{bmatrix}\mathbf{k}\leftrightarrow\mathbf{k}'\\ \omega\leftrightarrow\omega'\end{bmatrix}\right] + \rho^2\widetilde{\mathbf{v}}\widetilde{\mathbf{v}}\mathbf{q}_2\mathbf{q}_2:\langle\delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle\right\}\ . \tag{4.1}$$

By using Eqs. (2.34)—(2.37) and the perturbative solution of (2.27), Eq. (4.1) gives

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(0)} = 2k_B(T_0+\tfrac{1}{4}\psi u^2)\eta G^{v}(\mathbf{k},\omega)G^{v}(\mathbf{k}',\omega')k^2(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})(2\pi)^4\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}')\ , \tag{4.2}$$

which is similar to the equilibrium velocity correlation function with a temperature $T_0+\frac{1}{4}\psi u^2$. The nonequilibrium corrections up to linear order in gradients are

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(\tilde{v})}=-2k_B(T_0+\tfrac{1}{4}\psi u^2)\eta\rho G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')$$
$$\times[G^v(\mathbf{k}',\omega')(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\cdot\mathbf{q}_2\tilde{\mathbf{v}}+G^v(\mathbf{k},\omega)\tilde{\mathbf{v}}\mathbf{q}_2\cdot(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})]k^2(2\pi)^4\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}') \quad (4.3)$$

and

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(\delta\tilde{T})}=-ik_B\delta\tilde{T}G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\underline{\eta}:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}'\hat{\mathbf{k}}')(2\pi)^4$$
$$\times\delta(\omega+\omega')[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)]\ , \quad (4.4)$$

where $\underline{\eta}$ has been defined through Eq. (2.22).

The nonlinear corrections to the equilibrium result are

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(\delta\tilde{T}\tilde{v})}=i\rho k_B\delta\tilde{T}\,\tilde{v}G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')$$
$$\times[G^v(\mathbf{k}',\omega')(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\underline{\eta}:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}'\hat{\mathbf{k}}')\cdot\mathbf{q}_2\hat{\tilde{v}}+G^v(\mathbf{k},\omega)\hat{\tilde{v}}\mathbf{q}_2\cdot(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\underline{\eta}:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}'\hat{\mathbf{k}}')]$$
$$\times(2\pi)^4\delta(\omega+\omega')[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)] \quad (4.5)$$

and

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}';\omega')\rangle^{(\tilde{v}^2)}=G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega)\tilde{v}^2$$
$$\times\{-\tfrac{1}{2}k_B\psi(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\underline{\eta}:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}'\hat{\mathbf{k}}')[\delta(\mathbf{k}+\mathbf{k}'+2\mathbf{q}_2)+\delta(\mathbf{k}+\mathbf{k}'-2\mathbf{q}_2)-2\delta(\mathbf{k}+\mathbf{k}')]$$
$$+2\rho^2k_B\eta(T_0+\tfrac{1}{4}\psi u^2)G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')k^2\hat{\tilde{v}}\hat{\tilde{v}}[q_2^2-(\mathbf{q}_2\cdot\hat{\mathbf{k}})^2]\delta(\mathbf{k}+\mathbf{k}')\}(2\pi)^4\delta(\omega+\omega')\ , \quad (4.6)$$

where $\hat{\tilde{v}}$ is the unit vector along the direction of $\tilde{\mathbf{v}}$. The analysis of Eqs. (4.2)–(4.6) runs parallel to that made for temperature correlation function.

B. Velocity fluctuations in a fluid under a temperature gradient

In this subsection we apply our general result of Sec. IV A dropping the contribution from $\mathbf{V}_0\mathbf{v}$ or equivalently by setting $P\times E=0$. We start from Eqs. (2.15), (2.16), (2.19), and (2.23). Now Eqs. (2.16) and (2.23) are written in the form

$$\rho\frac{\partial\delta\mathbf{v}}{\partial t}=-\nabla\delta p+\eta\nabla^2\delta\mathbf{v}+\nabla\cdot\vec{\sigma}\ , \quad (4.7)$$

$$T_s(\mathbf{r})=T_0+\delta\tilde{T}\sin(\mathbf{q}_1\cdot\mathbf{r})\ . \quad (4.8)$$

From Eq. (4.7) one can see the fact that the convective current is zero; this allows us to find a nonperturbative solution for the Eq. (4.7). In $(\mathbf{k},\omega)$ space that solution is written

$$\delta\mathbf{v}(\mathbf{k},\omega)=iG^v(\mathbf{k},\omega)(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\vec{\sigma}(\mathbf{k},\omega)\ , \quad (4.9)$$

which coincides with Eq. (2.27) by setting $\tilde{v}=0$. To obtain Eq. (4.9) we have multiplied $\delta\mathbf{v}(\mathbf{k},\omega)$ by the transverse operator $(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})$ and use has been made of Eq. (2.26). From Eqs. (4.2) and (4.4) we obtain

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(0)}$$
$$=2k_BT_0\eta G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')k^2(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})$$
$$\times(2\pi)^4\delta(\omega+\omega')\delta(\mathbf{k}-\mathbf{k}') \quad (4.10)$$

and the nonequilibrium contribution

$$\langle \delta\mathbf{v}(\mathbf{k},\omega)\delta\mathbf{v}(\mathbf{k}',\omega')\rangle^{(\delta\tilde{T})}=-ik_B\delta\tilde{T}G^v(\mathbf{k},\omega)G^v(\mathbf{k}',\omega')(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\mathbf{k}:\underline{\eta}:\mathbf{k}'(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}}')(2\pi)^4\delta(\omega+\omega')$$
$$\times[\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1)-\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)]\ . \quad (4.11)$$

In the same way that we did in Sec. III B we can compute the equal-time velocity correlation function in the form

$$\langle \delta\mathbf{v}(\mathbf{k},t)\delta\mathbf{v}(\mathbf{k}',t)\rangle=\frac{1}{(2\pi)^2}\int d\omega\,B(\mathbf{k},\mathbf{k}',\omega)\ , \quad (4.12)$$

where $B(\mathbf{k},\mathbf{k}',\omega)$ is obtained by adding the coefficients of $\delta(\omega+\omega')$ in Eqs. (4.10) and (4.11), and by changing $\omega'\leftrightarrow-\omega$. Then from Eq. (4.12) we find the following result:

$$\langle \delta\mathbf{v}(\mathbf{k},t)\delta\mathbf{v}(\mathbf{k}',t)\rangle^{(0)}=\frac{k_BT_0}{\rho}(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\ , \quad (4.13)$$

$$\langle \delta\mathbf{v}(\mathbf{k},t)\delta\mathbf{v}(\mathbf{k}',t)\rangle^{(\delta\tilde{T})} = \frac{ik_B\delta\tilde{T}}{2\rho}(2\pi)^3 \left[\left[\overleftrightarrow{1} - \hat{\mathbf{k}}\hat{\mathbf{k}} + \hat{\mathbf{k}}\hat{\mathbf{k}}\frac{\hat{\mathbf{k}}\cdot\mathbf{q}_1}{k} - \mathbf{q}_1\frac{\hat{\mathbf{k}}}{k} \right] \times [\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1) - \delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}_1)] \right] . \tag{4.14}$$

To find Eq. (4.14) we have neglected terms of the order q_1^2 in developing the coefficient of the delta functions.

As we did in Sec. III C, Fourier inversion can be made to get velocity correlation function at different positions and equal time. To accomplish that we consider two particular cases.

(a) We suppose $\mathbf{k}_{\parallel}=(k_x,k_z)=0$ and transform back the k_y component of $\mathbf{k}$. Our result is (see Appendix C)

$$\langle \delta\mathbf{v}(\mathbf{k}_{\parallel}=0,y,t)\delta\mathbf{v}(\mathbf{k}'_{\parallel}=0,y',t)\rangle = \frac{k_B}{\rho}T_s(y')(2\pi)\delta(y-y') \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{4.15}$$

From Eq. (4.15) one can see that $\langle \delta v_y \delta v_y \rangle = 0$. This is due to the incompressibility of the fluid, which implies $\delta v_y(\mathbf{k}_{\parallel}=0,k_y,\omega)=0$. Notice that in this case the correlation exhibits a typical equilibrium behavior, namely, it is local in position. However, the temperature is the stationary one, then translational invariance is broken.

(b) We consider in this case $x=x'$ and $y=y'$, then $\mathbf{r}-\mathbf{r}'=(z-z')\hat{\mathbf{e}}_z$ being $\hat{\mathbf{e}}_z$ the unit vector along the z direction. One gets the following result for $z>z'$ (see Appendix D):

$$\langle \delta\mathbf{v}(x,y,z,t)\delta\mathbf{v}(x,y,z',t)\rangle = -\frac{k_B}{\rho}T_s(y)\frac{1}{4\pi|z-z'|^3}(\overleftrightarrow{1} - 3\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z) + \frac{k_B}{\rho}\frac{|\tilde{\nabla}_0 T|}{8\pi(z-z')^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \tag{4.16}$$

In this case the velocity correlation function contains a nonequilibrium contribution which is proportional to the temperature gradient instead of that quantity squared, as we got in Eq. (3.11). Moreover, such a correction introduces nondiagonal terms which also come in other problems[6] when nonequilibrium fluctuations are considered.

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APPENDIX A

We first consider the integral

$$I = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, G^T(\mathbf{k},\omega)G^T(\mathbf{k}',-\omega) \tag{A1}$$

which is the only integral in frequency that appears in the derivation of Eq. (3.9). If we consider that ω is a complex frequency, the integrand of Eq. (A1) has a simple pole $\omega=i\alpha k'^2$ in the upper half ω plane. By computing the residue of the integrand at this pole we find

$$I = \frac{1}{2\pi\alpha}\frac{1}{k^2+k'^2} . \tag{A2}$$

From Eq. (A2) the inverse Fourier transform of Eq. (3.3) is written as

$$\langle \delta T(\mathbf{k},t)\delta T(\mathbf{k}',t)\rangle^{(\delta\tilde{T})} = \frac{2k_B}{\rho c_v} iT_0\delta\tilde{T}(2\pi)^3 \times \left[\frac{k^2+\mathbf{k}\cdot\mathbf{q}_1}{2k^2+2\mathbf{k}\cdot\mathbf{q}_1+q_1^2}\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}_1) - (\mathbf{q}_1 \leftrightarrow -\mathbf{q}_1) \right] . \tag{A3}$$

If we develop the coefficients of the delta functions in Eq. (A3), we get up to order $\mathbf{q}_1$ [Eq. (3.9)].

APPENDIX B

The Fourier inversion of Eq. (3.8) gives

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle^{(0)} = \frac{k_B T_0^2}{\rho c_v}\delta(\mathbf{r}-\mathbf{r}') . \tag{B1}$$

In the same way (3.9) in real space gives

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle^{(\delta\tilde{T})} = \frac{k_B i T_0 \delta\tilde{T}}{\rho c_v}[I(\bar{q}_1) - I(-\mathbf{q}_1)] , \tag{B2}$$

where

$$I(\mathbf{q}_1) = e^{-i\mathbf{q}_1\cdot\mathbf{r}'}\delta(\mathbf{r}-\mathbf{r}') . \tag{B3}$$

By substitution of Eq. (B3) into (B2) we obtain

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle^{(\delta\tilde{T})} = \frac{2k_B T_0}{\rho c_v}\mathbf{r}'\cdot\nabla_0 T\delta(\mathbf{r}-\mathbf{r}') . \tag{B4}$$

To get Eq. (B4) we have used the fact that $\sin(\mathbf{q}_1\cdot\mathbf{r}') \simeq \mathbf{q}_1\cdot\mathbf{r}'$ and Eq. (2.25a). To make the Fourier inversion of Eq. (3.10) we take the limit $P\to\infty$. One arrives at

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle^{(\delta\tilde{T}^2)} = \frac{k_B\delta\tilde{T}^2}{4\rho c_v}\{I_1 + [I_2(\mathbf{q}_1) + I_3(\mathbf{q}_1)] + (\mathbf{q}_1 \leftrightarrow -\mathbf{q}_1)\} , \tag{B5}$$

where

$$I_1 = 2\delta(\mathbf{r}-\mathbf{r}') , \tag{B6}$$

$$I_2(\mathbf{q}_1) = -e^{-i2\mathbf{q}_1\cdot\mathbf{r}'}\delta(\mathbf{r}-\mathbf{r}') , \tag{B7}$$

$$I_3(\mathbf{q}_1) = q_1^2 e^{-i2\mathbf{q}_1\cdot\mathbf{r}'}\frac{1}{2\pi|\mathbf{r}-\mathbf{r}'|} . \tag{B8}$$

Substitution of Eqs. (B6)—(B8) into (B5) leads to

$$\langle \delta T(\mathbf{r},t)\delta T(\mathbf{r}',t)\rangle^{(\delta\tilde{T}^2)} = \frac{k_B}{\rho c_v}\left[\frac{|\nabla_0 T|^2}{4\pi|\mathbf{r}-\mathbf{r}'|}+(\mathbf{r}'\cdot\nabla_0 T)^2\delta(\mathbf{r}-\mathbf{r}')\right], \tag{B9}$$

where we have used the fact $\cos(2\mathbf{q}_1\cdot\mathbf{r}')\simeq 1-2(\mathbf{q}_1\cdot\mathbf{r}')^2$ and Eq. (2.25a). By adding Eqs. (B1), (B4), and (B9) we obtain Eq. (3.11).

APPENDIX C

The Fourier transform of (4.13) gives

$$\langle \delta\mathbf{v}(y,t)\delta\mathbf{v}(y',t)\rangle^{(0)} = \frac{k_B T_0}{\rho}2\pi\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\delta(y-y') . \tag{C1}$$

The inverse Fourier transform of Eq. (4.14) splits in the form

$$\langle \delta\mathbf{v}(y,t)\delta\mathbf{v}(y',t)\rangle^{(\delta\tilde{T})} = \frac{k_B i\delta\tilde{T}}{2\rho}\{[I_1(\mathbf{q}_1)+I_2(\mathbf{q}_1)+I_3(\mathbf{q}_1)]-(\mathbf{q}_1\leftrightarrow-\mathbf{q}_1)\} , \tag{C2}$$

where

$$I_1(\mathbf{q}_1)=2\pi e^{-i\mathbf{q}\cdot\mathbf{r}'}\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\delta(y-y') , \tag{C3}$$

$$I_2(\mathbf{q}_1)=-2\pi q_1\frac{e^{-i\mathbf{q}_1\cdot\mathbf{r}'}}{2i}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} , \tag{C4}$$

and

$$I_3(\mathbf{q}_1)=-I_2(\mathbf{q}_1) . \tag{C5}$$

By substituting Eqs. (C3)–(C5) into (C2) we obtain

$$\langle \delta\mathbf{v}(y,t)\delta\mathbf{v}(y',t)\rangle^{(\delta\tilde{T})} = \frac{k_B}{\rho}y'\,|\nabla_0 T|\,2\pi\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\delta(y-y') , \tag{C6}$$

and by adding Eqs. (C1) and (C6) one gets Eq. (4.15).

APPENDIX D

We first compute the integral

$$\overleftrightarrow{T}=\frac{1}{(2\pi)^3}\int d\mathbf{k}(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{s}} = \frac{1}{(2\pi)^3}\int d\mathbf{k}(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\tfrac{1}{2}\int_{-\infty}^{\infty}dk\,k^2\cos(ks\xi) = -\frac{1}{8\pi^2 s^2}\int d\hat{\mathbf{k}}(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\frac{\partial^2}{\partial\xi^2}\delta(s\xi) , \tag{D1}$$

where $\xi=\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_z$ and $\mathbf{s}=(z-z')\hat{\mathbf{e}}_z$. The tensor $\overleftrightarrow{T}$ splits in the form

$$\overleftrightarrow{T}=\overleftrightarrow{1}\gamma+\beta\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z . \tag{D2}$$

To compute γ and β we first contract $\overleftrightarrow{T}$ with $\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z$,

$$\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z{:}\overleftrightarrow{T}=\gamma+\beta=-\frac{1}{8\pi^2 s^2}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\xi(1-\xi^2)\frac{\partial^2}{\partial\xi^2}\delta(s\xi) = \frac{1}{2\pi s^3} . \tag{D3}$$

Now if we contract $\overleftrightarrow{T}$ with $\hat{\mathbf{e}}_x\hat{\mathbf{e}}_x$ we obtain

$$\overleftrightarrow{T}{:}\hat{\mathbf{e}}_x\hat{\mathbf{e}}_x=\gamma=-\frac{1}{8\pi^2 s^2}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\xi[1-\cos^2\varphi(1-\xi^2)] \times\frac{\partial^2}{\partial\xi^2}\delta(s\xi)=-\frac{1}{4\pi s^3} . \tag{D4}$$

From (D3) and (D4) one gets

$$\overleftrightarrow{T}=-\frac{1}{4\pi s^3}(\overleftrightarrow{1}-3\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z) . \tag{D5}$$

On the other hand, we compute the tensor

$$\overleftrightarrow{C}=\frac{q_1}{(2\pi)^3}\int d\mathbf{k}\,\hat{\mathbf{k}}_y\hat{\mathbf{k}}\hat{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\mathbf{s}}}{k} = \frac{q_1}{(2\pi)^3}\int d\hat{\mathbf{k}}\,\hat{\mathbf{k}}_y\hat{\mathbf{k}}\hat{\mathbf{k}}\tfrac{1}{2}\int_{-\infty}^{\infty}dk\,ik\sin(ks\xi) = -\frac{q_1 i}{(2\pi)^3}\int d\hat{\mathbf{k}}\,\hat{\mathbf{k}}_y\hat{\mathbf{k}}\hat{\mathbf{k}}\frac{\partial}{\partial\xi}\delta(s\xi) . \tag{D6}$$

From this last expression one can see that the only components of $\overleftrightarrow{C}$ different from zero are C_{yz} and C_{zy} with C_{yz} $(=C_{zy})$ given by

$$C_{yz}=-\frac{q_1 i}{8\pi^2 s}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\xi\sin^2\varphi(1-\xi^2)\xi\frac{\partial}{\partial\xi}\delta(s\xi) = \frac{q_1 i}{8\pi s^2} . \tag{D7}$$

Finally, we compute the tensor

$$\overleftrightarrow{D}=\frac{\mathbf{q}_1}{(2\pi)^3}\int d\mathbf{k}\,\hat{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\mathbf{s}}}{k} = \frac{\mathbf{q}_1}{(2\pi)^3}\int d\hat{\mathbf{k}}\,\hat{\mathbf{k}}\tfrac{1}{2}\int_{-\infty}^{\infty}dk\,ik\sin(ks\xi) = -\frac{i\mathbf{q}_1}{8\pi^2 s}\int d\hat{\mathbf{k}}\,\hat{\mathbf{k}}\frac{\partial}{\partial\xi}\delta(s\xi) . \tag{D8}$$

As in Eq. (D6), the only component of the tensor $\overleftrightarrow{D}$ different from zero is D_{yz}. One has

$$D_{yz}=-\frac{i q_1}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_{-1}^{1}d\xi\,\xi\frac{\partial}{\partial\xi}\delta(s\xi) = \frac{i q_1}{4\pi s^2} . \tag{D9}$$

Transforming back Eqs. (4.13) and (4.14) and employing (D1) and (D6)–(D9) one arrives at (4.16).

[1]A.-M. S. Tremblay, in *Recent Developments in Nonequilibrium Thermodynamics,* edited by J. Casas-Vázquez, D. Jou, and G. Lebon (Springer, Berlin, 1984), p. 267, and references quoted therein.

[2]J. M. Rubí, in *Recent Developments in Nonequilibrium Thermodynamics: Fluids and Related Topics,* Lecture Notes in Physics, edited by J. Casas-Vázquez, D. Jou, and J. M. Rubí, (Springer, Berlin, in press).

[3]A.-M. S. Tremblay, M. Arai, and E. D. Siggia, Phys. Rev. A **23**, 1451 (1981); G. van der Zwan, D. Bedeaux, and P. Mazur, Physica A **107**, 491 (1981).

[4]T. R. Kirkpatrick, E. G. D. Cohen, and J. R. Dorfman, Phys. Rev. A **26**, 972 (1982).

[5]H. Grabert, J. Stat. Phys. **26**, 113 (1981).

[6]M. C. Marchetti and J. W. Dufty, J. Stat. Phys. **32**, 255 (1983).

[7]E. Peacock-López and J. Keizer, Phys. Lett. **108A**, 85 (1985).

[8]S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics* (Dover, New York, 1984).

[9]As stated in Ref. 11, p. 236, the transport coefficients can be assumed to be constant if temperature variations are small. A more general hydrodynamic theory could be developed to include the effect of external gradients in transport coefficients as shown in Ref. 10.

[10]F. S. Gaeta, P. Migliardo, and F. Wanderlingh, Phys. Rev. Lett. **31**, 1181 (1973); M. Garcia-Sucre and D. Moronta, Phys. Rev. A **26**, 1713 (1982).

[11]L. D. Landau and E. M. Lifshitz, *Mécanique des Fluides* (Mir, Moscow, 1971).

[12]H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1968).

[13]D. Ronis and I. Procaccia, Phys. Rev. A **26**, 1812 (1982).

[14]A.-M. S. Tremblay, unpublished notes.

[15]G. Nicolis and M. Malek Mansour, Phys. Rev. A **29**, 2845 (1984).