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Fractal dimension for Gaussian colored processes

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The exact analytical expression for the Hausdorff dimension of free processes driven by Gaussian noise in n-dimensional space is obtained. The fractal dimension solely depends on the time behavior of the arbitrary correlation function of the noise, ranging from $D_X=1$ for Orstein-Uhlenbeck input noise to any real number greater than 1 for fractional Brownian motions.

I. INTRODUCTION

In recent years a great deal of interest has been devoted to the study of dynamical processes driven by Gaussian colored noise.[1] The main reason for this interest lies in the fact that colored noise is closer to physical reality than white noise. Although many properties of Gaussian colored noise are currently under intense research, there is one important feature of these processes that, to our knowledge, seems to go unnoticed, that is, the fractal geometry of their trajectories.

Fractal objects have been known as mathematical curiosities for a long time,[2] and only recently have been applied to natural phenomena.[3,4] One meets such objects in a variety of fields: Brownian motion,[4–6] fractional Brownian motion,[7] quantum mechanics,[8] disordered media, clusters and chaos,[9] random walks,[10–13] and atomic and molecular motions,[14] among many others.

In this paper we study the fractal dimension of free n-dimensional processes driven by Gaussian colored noise. We will find exact analytical expressions for the Hausdorff (fractal) dimension D_X of their trajectories. Our main conclusion is that D_X depends critically on the time behavior of the correlation function, ranging from $D_X=1$ for Orstein-Uhlenbeck input noise to any real number greater than 1 for fractional Brownian motions.

Let us consider an n-dimensional process $X(t)$ whose dynamical evolution is governed by the equation

$$\dot{\mathbf{X}}(t)=\mathbf{F}(t)\ , \tag{1}$$

where the input noise $\mathbf{F}(t)$ is Gaussian with zero mean and arbitrary (though isotropic) correlation function

$$\langle F_\mu(t)F_\nu(t')\rangle=\delta_{\mu\nu}h(t,t') \tag{2}$$

$(\mu,\nu=1,2,\ldots,n)$. Due to the linearity of Eq. (1) we see that $\mathbf{X}(t)$ is also Gaussian with $\langle\mathbf{X}(t)\rangle=\mathbf{x}_0$ $[\mathbf{x}_0\equiv\mathbf{X}(0)]$ and correlation function[15]

$$\langle X_\mu(t)X_\nu(t')\rangle=\delta_{\mu\nu}k(t,t')\ , \tag{3}$$

where

$$k(t,t')=\int_0^t d\sigma\int_0^{t'}d\sigma' h(\sigma,\sigma')\ . \tag{4}$$

To define the length of a given trajectory $X(t)$ $(0\le t\le T)$ we take a partition

$$0=t_0<t_1<\ldots<t_N=T$$

of the time interval $[0,T]$ such that $|t_j-t_{j-1}|<\delta$ $(j=1,2,\ldots,N)$ for δ arbitrary and positive. We then evaluate the length of the polygonal:

$$\sum_{j=1}^{N}|\mathbf{X}(t_j)-\mathbf{X}(t_{j-1})| \tag{5}$$

and we take the limit $\delta\to 0$. If this limit is finite the curve $\mathbf{X}(t)$ is said to be rectifiable. If the limit is infinite a possible solution to the problem of measuring the curve consists in replacing the length by the so-called *Hausdorff measure of dimension s*:[2]

$$L_s[\mathbf{X}(t)]=\lim_{\delta\to 0}\left[\inf\left\{\sum_{j=1}^{N}|\mathbf{X}(t_j)-\mathbf{X}(t_{j-1})|^s;\ |t_j-t_{j-1}|<\delta\right\}\right] \tag{6}$$

and then choosing the exponent s so that $L_s[X(t)]$ is finite and nonvanishing. This particular value of the exponent is called the *fractal (or Hausdorff) dimension* of the curve $\mathbf{X}(t)$. Note that for rectifiable curves, $s=1$. In fact, for these curves the fractal dimension and topological dimension are equal. But, in general, the fractal dimension is greater than or equal to the topological dimension.[4]

Therefore the fractal dimension of a given trajectory is the special value of the exponent s, say, $D_{\mathbf{X}}$, such that $L_{D_{\mathbf{X}}}[\mathbf{X}(t)]\neq 0$ and finite, that is,

$$L_s[\mathbf{X}(t)]=\begin{cases}0, & s>D_{\mathbf{X}}\\ \infty, & s<D_{\mathbf{X}}\end{cases} \tag{7}$$

$(0\le t\le T)$. In our case $L_s[\mathbf{X}(t)]$ and $D_{\mathbf{X}}$ are both random quantities depending functionally on a given realizations of the process; thus we will have different values of L_s and $D_{\mathbf{X}}$ for different realizations (or trajectories) of the process. Let $\langle L_s[\mathbf{X}(t)]\rangle$ be the average of the Hausdorff measure over all trajectories and let $D_{\max}$ be the least upper bound (with probability 1) of all values of $D_{\mathbf{X}}$. If $s<D_{\max}$ there will be trajectories with fractal dimension $D_{\mathbf{X}}>s$, and for these trajectories the Hausdorff measure will be infinite and therefore $\langle L_s[\mathbf{X}(t)]\rangle=\infty$. On the other hand, if $s>D_{\max}$, then necessarily $D_{\mathbf{X}}<s$ (with probability 1) and $L_s[\mathbf{X}(t)]=0$ for almost all trajectories, hence $\langle L_s[\mathbf{X}(t)]\rangle=0$.

Therefore, *the last upper bound* $D_{\max}$ for the fractal dimension of the trajectories of the process is the *special value of the exponent s,* such that

$$\langle L_s[\mathbf{X}(t)]\rangle=\begin{cases}0, & s>D_{\max}\\ \infty, & s<D_{\max}\,.\end{cases} \tag{8}$$

In order to identify the bound $D_{\max}$ our next step will be to evaluate $\langle L_s[\mathbf{X}(t)]\rangle$. From the Gaussian nature of $\mathbf{X}(t)$ it follows that the probability density function for a given trajectory to pass through the points $\mathbf{x}_j$ at times t_j $(j=0,1,\dots,N)$ is[16,17]

$$p[\mathbf{X}(t_j)=\mathbf{x}_j;\, j=0,1,\dots,N] =(2\pi)^{-n/2}(\det K)^{-n/2}\exp\left[-\tfrac{1}{2}\sum_{i,l=1}^{N}K_{il}^{-1}\mathbf{x}_i\cdot\mathbf{x}_l\right], \tag{9}$$

where K_{il}^{-1} is the inverse of the correlation matrix $K_{il}\equiv k(t_i,t_l)$ and $\mathbf{x}_i\cdot\mathbf{x}_l$ is the Euclidean scalar product. Then

$$\left\langle\sum_{j=1}^{N}\left|\mathbf{X}(t_j)-\mathbf{X}(t_{j-1})\right|^s\right\rangle=(2\pi)^{-n/2}(\det K)^{-n/2}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_N\exp\left[-\tfrac{1}{2}\sum_{i,l=1}^{N}K_{il}^{-1}\mathbf{x}_i\cdot\mathbf{x}_l\right]\sum_{j=1}^{N}|\mathbf{x}_j-\mathbf{x}_{j-1}|^s\,.$$

After some algebra we obtain[18]

$$\left\langle\sum_{j=1}^{N}\left|\mathbf{X}(t_j)-\mathbf{X}(t_{j-1})\right|^s\right\rangle =2^{s/2}\frac{\Gamma(s/2)}{\Gamma((n+s)/2)}\sum_{j=1}^{N}|\Psi(t_{j-1},t_j)|^{s/2}, \tag{10}$$

where

$$\Psi(t,t')\equiv\int_{t'}^{t}d\sigma\int_{t'}^{t}d\sigma' h(\sigma,\sigma')\,. \tag{11}$$

The function $\Psi(t,t')$ can be written in terms of the correlation function $k(t,t')$ of the process $\mathbf{X}(t)$ in the form

$$\Psi(t,t')=k(t,t)-k(t',t')-2k(t,t')\,, \tag{12}$$

and in the stationary case it reads

$$\Psi(t-t')=2[k(0)-k(t-t')]\,. \tag{13}$$

From Eqs. (6) and (10) it follows that

$$\langle L_s[\mathbf{X}(t)]\rangle=2^{s/2}\frac{\Gamma((n+s)/2)}{\Gamma(n/2)}\lim_{\delta\to 0}\left[\inf\left\{\sum_{j=1}^{N}|\Psi(t_{j-1},t_j)|^{s/2};\ |t_j-t_{j-1}|<\delta\right\}\right]. \tag{14}$$

We now suppose an equal time behavior of the function $\Psi(t,t')$ in the form

$$\Psi(t,t+\epsilon)=A(t)\epsilon^{\alpha}+O(\epsilon^{\alpha+1})\,, \tag{15}$$

where $\alpha\ge 0$ and $A(t)$ $(0\le t\le T)$ is an arbitrary function bounded by some positive constant M. Then

$$\inf\left\{\sum_{j=1}^{N}|\Psi(t_{j-1},t_j)|^{s/2};\ |t_j-t_{j-1}|\le T/N\right\} \le (MT^{\alpha})^{s/2}N^{1-s\alpha/2}\,. \tag{16}$$

The limit $\delta\to 0$ corresponds to $T/N\to 0$ with T fixed; hence

$$\langle L_s[\mathbf{X}(t)]\rangle\le(MT^{\alpha})^{s/2}\left[\lim_{N\to\infty}N^{1-s\alpha/2}\right].$$

Therefore, if $s>2/\alpha$, then $\langle L_s[\mathbf{X}(t)]\rangle=0$ and from Eq. (7) we have

$$D_{\max}\le\frac{2}{\alpha}\,. \tag{17}$$

Finally, $D_{\max}\ge D_{\mathbf{X}}$ (with probability 1), whence

$$D_X \le \frac{2}{\alpha} \tag{18}$$

for almost all trajectories of the process.

In order to find a lower bound for D_X we use a result proven in Ref. 6, p. 79. For a given trajectory the so-called s energy is the random quantity defined by[19]

$$I_s[\mathbf{X}] \equiv \int_0^T \int_0^T \frac{dt\,dt'}{|\mathbf{X}(t)-\mathbf{X}(t')|^s} .$$

As a consequence of corollary 6.6 of Ref. 6 we can affirm that if $I_s[\mathbf{X}] < \infty$ then $s < D_X$. Considering the mean value, we have that $\langle I_s[\mathbf{X}] \rangle$ finite implies $I_s[\mathbf{X}]$ finite (with probability 1). Therefore, if $\langle I_s[\mathbf{X}] \rangle < \infty$, then

$$s < D_X . \tag{19}$$

Now the average $\langle I_s[\mathbf{X}] \rangle$ can be easily evaluated as a Gaussian path integral

$$\langle I_s[\mathbf{X}] \rangle = \int_\Omega \mathcal{D}P[\mathbf{X}] \int_0^T dt \int_0^T dt' \frac{1}{|\mathbf{X}(t)-\mathbf{X}(t')|^s} , \tag{20}$$

where $\mathcal{D}P[\mathbf{X}]$ is the probability measure for the different paths on the process and Ω is the set of all of these paths. After a short manipulation we obtain [cf. Eq. (9), (Refs. 6 and 18)]

$$\langle I_s[\mathbf{X}] \rangle = 2^{-s/2} \frac{\Gamma((n-s)/2)}{\Gamma(n/2)} \int_0^T dt \int_0^T dt' |\Psi(t,t')|^{-s/2} . \tag{21}$$

Assuming an equal-time expansion for $\Psi(t,t')$ of the form given by Eq. (15) we see that the convergence of the integral (21) implies that $s < 2/\alpha$ and from Eq. (19) follows

$$D_X \ge 2/\alpha . \tag{22}$$

Combining Eqs. (18) and (22) we achieve our main result:

$$D_X = 2/\alpha . \tag{23}$$

Therefore, the fractal dimension of the trajectories of the process $\mathbf{X}(t)$ is (with probability 1) proportional to the inverse of the exponent that governs the equal-time behavior (short-time behavior for the stationary case) of the correlation function $k(t,t')$ of the process.[20] We finish this Brief Report with a few applications of Eq. (23).

A. Brownian motion

If the input noise $\mathbf{F}(t)$ is a Gaussian white noise then $\mathbf{X}(t)$ is the Brownian motion (or Wiener) process. In this case we have $h(t,t') = \delta(t-t')$ and

$$\Psi(t,t+\epsilon) = \epsilon , \tag{24}$$

whence $\alpha = 1$ and we recover the well-known result[4–6]

$$D_X = 2 . \tag{25}$$

B. Orstein-Uhlenbeck processes

We next study the fractal dimension of processes driven by Orstein-Uhlenbeck noise, which is one of the most frequent cases of colored noise appearing in the literature.[21] If the input noise $\mathbf{F}(t)$ is an Orstein-Uhlenbeck process, then

$$h(t,t') = \frac{1}{\tau} e^{-|t-t'|/\tau} , \tag{26}$$

where τ is the correlation time. The function $\Psi(t,t+\epsilon)$ is now given by

$$\Psi(t,t+\epsilon) = \frac{1}{\tau}\epsilon^2 + O(\epsilon^3) , \tag{27}$$

whence $\alpha = 2$ and the fractal dimension of the output process $X(t)$ is

$$D_X = 1 , \tag{28}$$

which equals its topological dimension.[22] Therefore, the trajectories of free processes driven by Orstein-Uhlenbeck noise present no fractal behavior.

C. Gaussian $1/f$ noise

Many physical devices exhibit fluctuations whose power spectrum is characterized by $1/f^a$ ($a > 0$). This is the so-called $1/f$ noise[23] and appears in semiconductors and metals[23,24] and also in chemical and biological systems.[24] In many situations $1/f$ noise is known to be Gaussian;[25] in this case we can evaluate the fractal dimension.

Let the input noise $\mathbf{F}(t)$ be stationary Gaussian $1/f$ noise with zero mean and power spectrum given by $1/f^a$, where $0 < a < 1$ (if $a \ge 1$, some kind of cutoff must be introduced[23–25]). The inverse Fourier transform of the power spectrum is the correlation function of the process

$$h(t,t') = \frac{c}{|t-t'|^{1-a}} ,$$

where C is a constant and $0 < a < 1$. Now

$$\Psi(t,t+\epsilon) = \frac{2c}{a(1+a)} \epsilon^{1+a} , \tag{29}$$

whence

$$D_X = \frac{2}{1+a} , \tag{30}$$

and the fractal dimension is any real number between 1 and 2, depending on the exponent of the power spectrum.

D. Fractional Brownian motion

The fractional Brownian motion process $B_\alpha(t)$ is a generalization of the ordinary Brownian motion in which the standard deviation of the increment $|\mathbf{B}_\alpha(t+T) - \mathbf{B}_\alpha(t)|$ goes as T^α with $0 < \alpha < 1$. When $\alpha = \frac{1}{2}$ the fractional Brownian motion (FBM) reduces to an ordinary Brownian motion. One important feature of FBM processes is that they show a strong interdependence between distant samples. This asymptotic dependence is the reason for their usefulness in modeling time series. Moreover, FBM is known to be self-similar and fractal.[7]

A simplified definition of FBM processes is given by the following moving average:[26]

$$B_\alpha(t)=\frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^t (t-\sigma)^{\alpha-1/2}\xi(\sigma)d\sigma \;, \tag{31}$$

where $0<\alpha<1$ and $\xi(t)$ is the n-dimensional Gaussian white noise. Assuming that $\xi(t)$ is zero centered and isotropic we easily find that $\langle B_{\alpha\mu}(t)B_{\alpha\nu}(t')\rangle=\delta_{\mu\nu}k(t,t')$, where

$$k(t,t')=\frac{1}{\Gamma^2(\alpha+\frac{1}{2})}\int_0^{\min(t,t')} d\sigma[(t-\sigma)(t'-\sigma)]^{\alpha-1/2} \;. \tag{32}$$

From the point of view of generalized functions we can write $\mathbf{B}_\alpha(t)$ to be the solution of the differential equation

$$\dot{\mathbf{B}}_\alpha(t)=\mathbf{B}_{\alpha-1}(t) \;. \tag{33}$$

Hence the function $\Psi(t,t+\epsilon)$ defined in Eq. (11) reads [cf. Eq. (12)]

$$\Psi(t,t+\epsilon)=\frac{1}{\Gamma^2(\alpha+\frac{1}{2})}\int_t^{t+\epsilon} d\sigma[(t-\sigma)(t'-\sigma)]^{\alpha-1/2}$$
$$=\frac{\epsilon^{2\alpha}}{\Gamma(1+2\alpha)} \;. \tag{34}$$

Therefore the Hausdorff dimension of the FBM is

$$D_{\mathbf{X}}=\frac{1}{\alpha} \;. \tag{35}$$

Since $0<\alpha<1$ then $D_{\mathbf{X}}$ is any real number greater than 1.

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[15] We note that neither $\mathbf{F}(t)$ nor $\mathbf{X}(t)$ is stationary.

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[17] Since Eq. (1) does not explicitly depend on $\mathbf{X}$, we may assume without loss of generality that $\mathbf{x}_0=\mathbf{X}(0)=0$.

[18] J. Llosa and J. Masoliver (unpublished).

[19] Notice that for $s=1$ $I_s[\mathbf{X}]$ is proportional to the Coulomb energy of an homogeneous charge distribution along $\mathbf{X}(t)$.

[20] Equation (2) seems to imply an absurdity when $\alpha>2$. Indeed, we should then have $D_{\mathbf{X}}<1$, which contradicts the well-known property that the Hausdorff dimension is never less than the topological dimension. Nevertheless, we have shown elsewhere (Ref. 18) that such a possibility (i.e., $\alpha>2$) is precluded by the fact that the correlation function $h(t,t')$ must be positive definite.

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[22] Equation (28) holds as far as $\epsilon^2/\tau \ll 1$. If τ goes to zero at the same rate as ϵ, then $\Psi(t,t+\epsilon)\simeq\epsilon$, which corresponds to the Brownian-motion case. We thus recover the well-known result that white noise is the limiting case of colored noise when the correlation time goes to zero.

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