First-passage times for a marginal state driven by colored noise

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The dynamical process through a marginal state (saddle point) driven by colored noise is studied. For small correlation time of the noise, the mean first-passage time and its variance are calculated using standard methods. When the correlation time of the noise is finite or large, an alternative approach, based on simple physical arguments, is proposed. It will allow us to study also the passage times of an unstable state. The theoretical predictions are tested satisfactorily by the use of computer simulations.

I. INTRODUCTION

The behavior of physical systems under the influence of stochastic forces or noises has been a subject of study for some time. Recently, the study of dynamical aspects of this subject received special attention.[1] One of the studied dynamical aspects is the influence of the noise in the relaxation process of a system from an initial state to the final steady state. Different quantities can be used to describe this dynamics. One can look at the evolution of the probability density of the dynamical variable of interest, or its statistical moments and correlations, or at the stochastic trajectory itself.

The first-passage time statistics associated with the trajectory[2] and the nonlinear relaxation time[3] for the moments have been the most common tools used in the study of relaxation dynamics. In this paper we will focus on the mean first-passage time (MFPT) and its variance.

Although all relaxation processes can be studied by the use of the MFPT technique, only a few of them have particular importance. They are the processes in which the presence of noise creates different dynamics than the pure deterministic motion. We are referring to the relaxation processes from an initial unstable state,[4-8] through a marginal state (saddle point),[9-12] or a barrier between metastable states.[2,13] These three types of processes occur only with fluctuations, and the dynamics would be very different if fluctuations were not present.

The MFPT of these processes have been obtained for the case of Gaussian white noise, which was possible because the stochastic theory of Markovian processes is well established.[2] When the noise is Gaussian, but not white, the stochastic process is not Markovian and only approximate theoretical schemes can be used.[5-8,13] Nevertheless the dominant mechanisms for the decay of unstable states[5-8] and for barrier-crossing processes[13] are now well understood even for colored noise, i.e., nonwhite. The characteristic time scales of the relaxation through a marginal point are also known but only for the

case of white noise.[9-12] The aim of this paper is to study this last process when the noise is colored. In particular, we obtain the MFPT and its variance. The accuracy of theoretical results are tested by computer simulations. The dominant mechanism of the dynamics is also explained. Our approximate procedure will be also useful in the study of some aspects of the relaxation of an unstable state not considered in previous works.[5-8]

The sections of this paper have been organized as follows. In Sec. II we introduce the explicit models we are interested in, the relevant dynamical parameters, and the mathematical techniques we will use. In Sec. III we obtain the MFPT and its variance for the relaxation process through a marginal state driven by Gaussian colored noise with small correlation time. A qualitative argument is presented in Sec. IV, which allows us to study the MFPT for unstable and marginal states when the colored noise has a large correlation time. In Sec. V we present some conclusions.

II. MODELS AND TECHNIQUES

A typical example of crossing through a saddle-point bifurcation appears in optical bistability. In this case the output intensity q of the laser obeys the following dynamical equation:[14]

$$\dot{q} = y - q - \frac{2cq}{1+q^2} + \xi(t) = -\phi'(q) + \xi(t) , \qquad (2.1)$$

where y is the input intensity and $\xi(t)$ is the noise. The potential $\phi(q)$ associated with this model has an inflection point at $q_m = [c - 1 - (c^2 - 4c)^{1/2}]^{1/2}$ if $c \geq 4$. The process of switching on corresponds to changing the value of the control parameter y from zero to a value above the threshold $y_m = q_m + 2cq_m / (1 + q_m^2)$. For this value, the inflection point has a horizontal slope, and corresponds to a marginal or saddle-node state. The system starts in $q = 0$, passes through the marginal point q_m due to the

presence of noise, and reaches its stable state at $q_0 = 2q_m/(q_m^2 - 1)$.

There are two different regimes in this evolution. Outside the marginal zone the dynamics is essentially deterministic, and the noise plays a secondary role. However, near the marginal point the potential is very flat, and then the dynamics is dominated by the noise. This implies that the almost entire action of the noise over the systems will take place there. Then, in order to study the influence of the noise, we will only need few terms of the potential expansion around this marginal point:

$$\dot{q} = -\phi'(q) + \xi(t)$$
$$= \beta + a(q - q_m)^n + O((q - q_m)^{n+1}) + \xi(t) \qquad (2.2)$$

which governs the dynamics in its vicinity. The point $q = q_m$ is the marginal point of the system when the control parameter β is equal to zero. For the model described by Eq. (2.1), $\beta = y - y_m$, the potential is a cubic power in the variable $x \equiv q - q_m$ and, therefore, $n = 2$. Different systems will have the same expansion (2.2) and then will be affected by the noise in the same way. The difference between them can be evaluated from the analysis of their deterministic equations, without considering the presence of noise, as it is explained in Ref. 12. Thus, we only need to use the simplest model which is given, for $n = 2$, by the potential

$$\phi(x) = -\frac{a}{3}x^3 - \beta x, \quad a > 0 . \qquad (2.3)$$

Models defined by Eqs. (2.2) and (2.3) have already been analyzed for $\xi(t)$ being white noise.[9-12] Our aim here is to consider a Gaussian colored noise, the Ornstein-Uhlenbeck noise, which has a correlation function

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{\tau}\exp{-\frac{|t - t'|}{\tau}} , \qquad (2.4)$$

where D is the intensity and τ the correlation time of the noise. Now the process $x(t)$ is non-Markovian and no precise theoretical scheme for solving this problem exists. The best known approximation to treat this non-Markovian problem starts from the deduction of an effective Fokker-Planck equation for the probability density $P(x, t)$ valid up to order τ and where transient terms are neglected:[7,15,16]

$$\frac{\partial}{\partial t}P(x, t) = \frac{\partial}{\partial x}\phi'(x)P(x, t)$$
$$+ \frac{\partial^2}{\partial x^2}D(1 - \tau\phi''(x))P(x, t) . \qquad (2.5)$$

This is equivalent to consider the case of a Gaussian white noise but with a renormalization of the diffusion coefficient D, which becomes a variable-dependent quantity. In this approximation, the MFPT, $\langle T \rangle$, for the system starting from x_0 and reaching x_F, and its variance $\Delta T^2 = \langle T^2 \rangle - \langle T \rangle^2$, are given, using general formulas of white noise[10-12] by

$$\langle T \rangle = \int_{x_0}^{x_F}dx_1\frac{1}{D(x_1)P_s(x_1)}\int_{-\infty}^{x_1}dx_2 P_s(x_2) , \qquad (2.6)$$

$$(\Delta T)^2 = \int_{x_0}^{x_F}dx_1\frac{1}{D(x_1)P_s(x_1)}\int_{-\infty}^{x_1}dx_2\frac{1}{D(x_2)P_s(x_2)}\int_{-\infty}^{x_2}dx_3 P_s(x_3)\int_{-\infty}^{x_3}dx_4 P_s(x_4) , \qquad (2.7)$$

where $P_s(x)$ and $D(x)$ are defined by

$$P_s(x) = \frac{1}{D(x)}\exp\left[-\int^x dy\frac{\phi'(y)}{D(y)}\right] ,$$
$$D(x) = D(1 - \tau\phi''(x)) . \qquad (2.8)$$

This quasi-Markovian approximation makes sense if the initial x_0 and final x_F values of x are far away from the marginal point, and then the non-Markovian transient effects are not important. Note also that the system defined in (2.3) has no steady state, and the function $P_s(x)$ has, in general, no physical meaning. If it would have a steady state (for instance, putting a reflecting barrier to prevent particles to go out to infinity), $P_s(x)$ would be the stationary probability density in this approximation.

Two important quantities can be constructed with the parameters a and D that will be useful to obtain some additional physical information (we will turn to that point in Sec. III). The first one is $(D/a)^{1/(n+1)}$; it has dimensions of the variable x and denotes the size of the region near the marginal point in which the dynamics is dom-

inated by the noise. The other quantity is $(a^2D^{n-1})^{-1/(n+1)}$. It has dimensions of time and is proportional to T_0, the MFPT from $-\infty$ to $+\infty$ for the system defined in Eq. (2.2), with $\beta = 0$ in the white-noise limit.[10] In the particular case of Eq. (2.3), that is $n = 2$, the value of T_0 is[11,12]

$$T_0 = \langle T(\beta = 0, \tau = 0) \rangle = [\Gamma(\tfrac{1}{3})]^2(3Da^2)^{-1/3} . \qquad (2.9)$$

where Γ denotes the gamma function.

For $\beta > 0$, there exists another relevant time scale. It is the deterministic time, T_{det}, that the system uses to go from $x = 0$ to $x = \infty$, driven by the only action of the potential $\phi(x)$. This time scale is proportional to $(a\beta^{n-1})^{-1/n}$, which is obtained from the definition

$$T_{\text{det}} = \int_0^\infty dx\frac{1}{-\phi'(x)} . \qquad (2.10)$$

Therefore Eq. (2.2) can be made dimensionless by the change

$$x' = \left[\frac{D}{a}\right]^{-1/(n+1)}(q-q_m), \quad t'=(a^2D^{n-1})^{1/(n+1)}t$$

$$(2.11)$$

with the result

$$\frac{dx'}{dt'}=x'^n+k+\eta(t') , \tag{2.12}$$

where $\eta(t')$ is a colored noise of unit intensity and scaled correlation time τ', and τ' and k are given as

$$\tau'=(a^2D^{n-1})^{1/(n+1)}\tau, \quad k=(aD^n)^{-1/(n+1)}\beta . \tag{2.13}$$

The parameter k is proportional to $(T_0/T_{\text{det}})^{n/(n-1)}$. It gives information on how far the system is from pure marginality by comparing two different characteristic times. These two parameters k and τ' are the only relevant ones in this problem, and any quantity should depend only on them through the changes given by Eqs. (2.11).

III. MFPT FOR SMALL CORRELATION TIME OF THE NOISE

We now apply Eqs. (2.5) and (2.6) to the system defined by Eq. (2.3). Making an expansion in powers of τ', we obtain the following expression for the MFPT corresponding to the evolution from $x_0 = -\infty$ to $x = \infty$:

$$(a^2D)^{1/3}\langle T\rangle = T_0+T_1\tau'+O(\tau'^2) , \tag{3.1}$$

where

$$T_0=\pi^{1/2}\int_0^\infty dx\, x^{-1/2}\exp-(\tfrac{1}{12}x^3+kx) , \tag{3.2}$$

$$T_1=\pi^{1/2}\int_0^\infty dx\, x^{1/2}\exp-(\tfrac{1}{12}x^3+kx) . \tag{3.3}$$

In a similar way, the standard deviation is, up to first order in τ'

$$(a^2D)^{1/3}\Delta T=\Delta T_0+\frac{1}{2}\frac{A}{\Delta T_0}\tau'+O(\tau'^2) , \tag{3.4}$$

where

$$(\Delta T_0)^2=4\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{x_1}dx_2\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\exp-[\tfrac{1}{3}(x_1^3+x_2^3-x_3^3-x_4^3)+k(x_1+x_2-x_3-x_4)] , \tag{3.5}$$

$$A=2\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{x_1}dx_2\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\{\exp-[\tfrac{1}{3}(x_1^3+x_2^3-x_3^3-x_4^3)+k(x_1+x_2-x_3-x_4)]\}$$

$$\times[(x_1^4+x_2^4-x_3^4-x_4^4)+2k(x_1^2+x_2^2-x_3^2-x_4^2)-4(x_3+x_4)] . \tag{3.6}$$

In Figs. 1, 2(a), and 2(b) these results are compared with the ones obtained from numerical simulation methods[16] of the system defined by Eq. (2.3). The dependence on the parameter k, Fig. 1, is fairly good, mainly in the small-τ' regime, where the expansions (3.1) and (3.4) would become more appropriate.

In the pure marginal case ($k=0$) these integrals can be evaluated

$$T_0=3^{-1/3}[\Gamma(\tfrac{1}{3})]^2 , \tag{3.7}$$

$$T_1=2\pi3^{-1/2} , \tag{3.8}$$

$$\Delta T_0=3^{-5/6}[\Gamma(\tfrac{1}{3})]^2 , \tag{3.9}$$

$$A=19.151. \ldots . \tag{3.10}$$

In this case, $n=2$, the dependence of T and ΔT on the correlation time of the noise has been also obtained from digital simulations of both of the systems defined through Eqs. (2.1) and (2.3) (with $y=y_m$ and $\beta=0$ respectively, and for different intensities of the noise), and the results can be seen in Figs. 2(a) and 2(b). They confirm the dependence of the dynamical quantities scaled by Eq. (2.3) on the scaled correlation time τ'; this dependence being the same for different physical systems. This is a
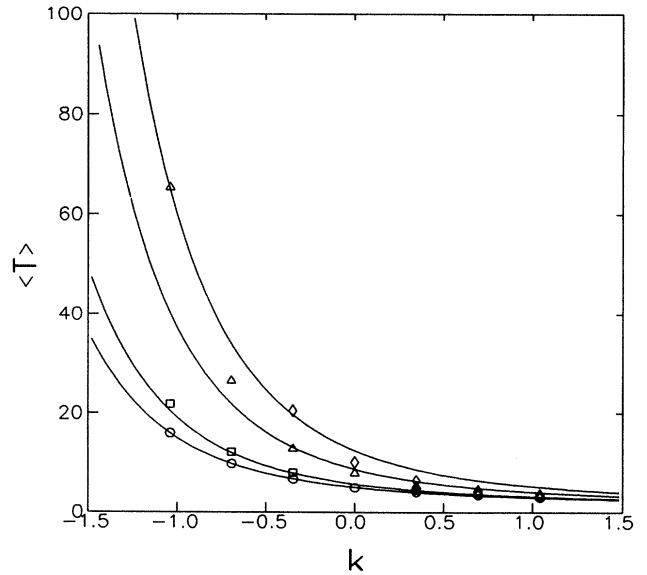

FIG. 1. $\langle T\rangle$ vs k. Symbols are simulation results of the system defined by Eq. (2.3) with $a=3$ and $D=1$ (circles, $\tau=0.01$; squares, $\tau=0.1$; triangles, $\tau=0.5$; lozenges, $\tau=1$). Solid lines, analytic results from Eqs. (3.1)–(3.3).

clear manifestation of the universal behavior near the marginal point. On the other hand, the MFPT and its variance are correctly given by Eqs. (3.1) and (3.4) with (3.7)–(3.10) in the small-τ region for both of the systems simulated. Moreover, a physical argument can be invoked to extend these results to finite values of τ, which is presented in the following section.
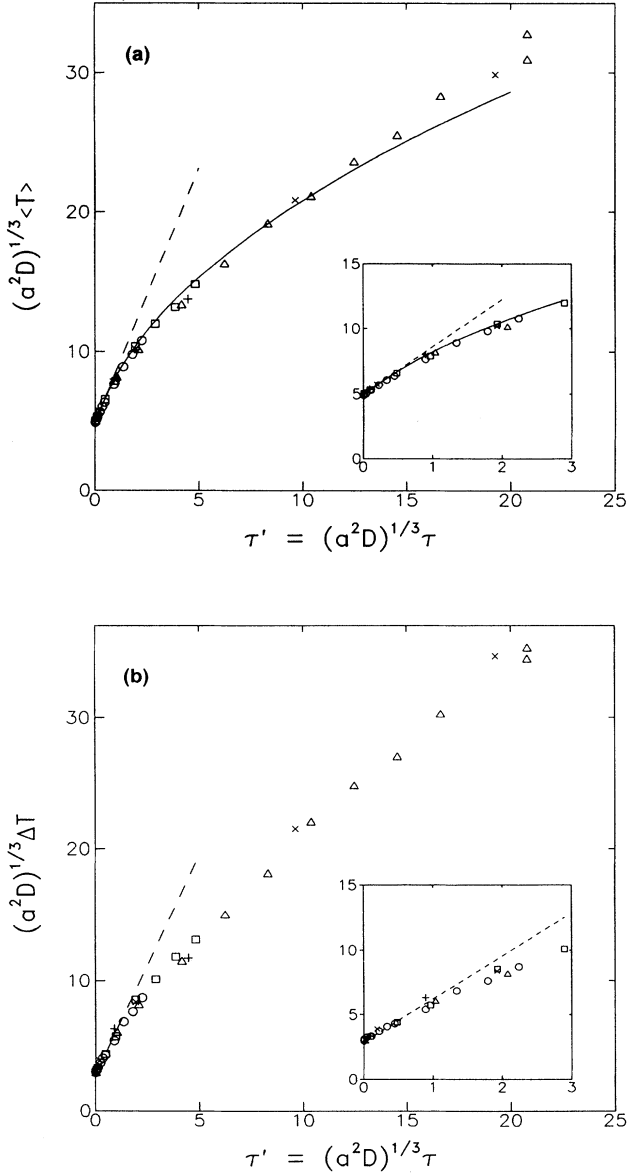

FIG. 2. (a) Scaled MFPT vs τ'. Dashed lines are the theoretical results of Eqs. (3.1), (3.7), and (3.8); solid lines are the numerical solution of Eq. (4.16). (b) Scaled standard deviation vs τ'. Dashed lines are the theoretical results of Eq. (3.4), (3.9), and (3.10). Symbols are simulation results of the system defined by Eq. (2.1) with $c=20$ ($+D=0.01$ and $\times D=0.1$) and the system defined by Eq. (2.3) with $a=3$ and $\beta=0$ (circles, $D=0.01$; squares, $D=0.1$; and triangles, $D=1$).

IV. MFPT FOR LARGE CORRELATION TIME OF THE NOISE

Let us start first from the standard linear model for the decay of an unstable system[5–8]

$$\dot{x}(t)=x(t)+\xi(t) , \qquad (4.1)$$

where the variable x is placed at $x=0$ in $t=0$, and $\xi(t)$ is the Ornstein-Uhlenbeck noise (2.4). For such a system, the evolution of the second moment of x is calculated exactly for all values of D and τ[7]

$$\langle x^2 \rangle_t = \frac{D}{1+\tau}e^{2t} - \frac{D}{1-\tau} + 2D\frac{\tau}{1-\tau^2}e^{-(1-\tau)t/\tau} . \qquad (4.2)$$

The mean time T that the system takes to leave a region of size R should be related to the time needed by the distribution of x to reach a width proportional to R

$$\langle x^2 \rangle_t = \alpha R^2 . \qquad (4.3)$$

The proportionality constant α is obtained by imposing that Eq. (4.2) has as a solution, in the double limit $\tau=0$ and $D < <R^2$, the MFPT corresponding to the weak white-noise case, which can be obtained from Eq. (2.6). This MFPT is

$$T(\tau=0)=\frac{1}{2}\ln\left[\frac{R^2}{2D}\right]+\frac{\gamma}{2}+\ln 2+O(D/R^2) . \qquad (4.4)$$

In this way, the value

$$\alpha=2e^{\gamma} \qquad (4.5)$$

is found where γ denotes the Euler constant. Then, Eq. (4.2) transforms into

$$\frac{1}{1+\tau}e^{2T}-\frac{1}{1-\tau}+2\frac{\tau}{1-\tau^2}e^{-(1-\tau)T/\tau}=\frac{2R^2}{D}e^{\gamma} . \qquad (4.6)$$

In the limit $D \ll R^2$, one can get as a byproduct the known result obtained from the quasideterministic theory (QDT) (Refs. 6 and 7).

$$T_{QDT}=\frac{1}{2}\ln\frac{R^2}{2D}+\frac{\gamma}{2}+\ln 2+\frac{1}{2}\ln(1+\tau) . \qquad (4.7)$$

Note that the substitution of T_{QDT} into the left-hand side of Eq. (4.6) gives three terms of orders 1, 0, and $(\tau-1)/2\tau$ in R^2/D, respectively, and thus the first one is dominant in that limit. In Fig. 3 both results are compared. They are equivalent for weak noise $(D/R^2 \ll 1)$, but the numerical solution of Eq. (4.6) also describes the nonweak noise situation.

For the marginal case it is impossible to find a closed equation for the evolution of $\langle x^2 \rangle$. Instead, we can relate this problem to the evolution of a system with a constant potential, that is to say, free diffusion. Let us consider the system defined by Eq. (2.2) with $\beta=0$

$$\dot{x}=ax^n+\xi(t) , \qquad (4.8)$$

where $n>1$. The dynamics outside of the marginal zone is essentially deterministic, but the noise dominates in the mentioned region of size $L \propto (D/a)^{1/(n+1)}$. Let us as-
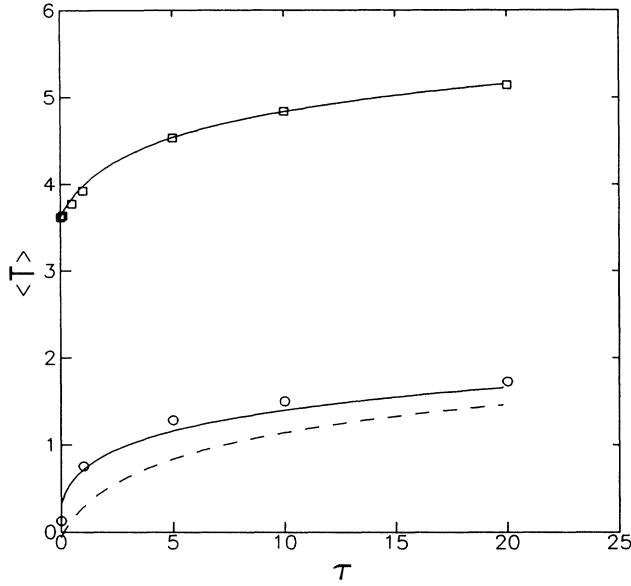
FIG. 3. MFPT vs τ for the unstable system (4.1) with $D=1$. Symbols are simulation results (squares, $R=20$; circles, $R=0.5$). The dashed line represents Eq. (4.7) (QDT theory). Solid lines are the numerical solution of Eq. (4.6). The differences between both results are indistinguishable for $R=20$.

sume that the evolution inside this region is free

$$\dot{x}=\xi(t) , \quad |x|\leq L . \tag{4.9}$$

In this case, the width of the distribution is given by

$$\langle x^2\rangle_t=2D(t-\tau(1-e^{-t/\tau})) . \tag{4.10}$$

A similar argument that the one applied to the unstable system holds here. The MFPT for Eq. (4.8) is assumed to be proportional to time t that $\langle x^2\rangle$ becomes L^2 in the free system (4.9). This implies

$$\langle x^2\rangle_t \propto \left[\frac{D}{a}\right]^{2/(n+1)} , \quad t=\mu T , \tag{4.11}$$

where μ denotes the appropriate proportionality constant. Equaling Eqs. (4.10) and (4.11) we obtain

$$\mu T-\tau(1-e^{-\mu T/\tau})\propto(a^2D^{n-1})^{-1/(n+1)} , \tag{4.12}$$

where we notice that the last expression is proportional to the white-noise time T_0. Then we introduce a new constant ν

$$\mu T-\tau(1-e^{-\mu T/\tau})=\nu T_0 . \tag{4.13}$$

Now both constants μ and ν can be determined by the same procedure that was applied to determine the constant α of Eq. (4.3). Thus, we impose that for small τ the solution of the transcendental Eq. (4.13) is of the form

$$T=T_0+\tau T_1+O(\tau^2) . \tag{4.14}$$

Introducing this expression into Eq. (4.13) and developing it up to first order on τ, we obtain

$$\mu=\nu=\frac{1}{T_1} . \tag{4.15}$$

Then Eq. (4.13) yields

$$T-T_0=\tau T_1\left[1-\exp\left[-\frac{T}{\tau T_1}\right]\right] . \tag{4.16}$$

This expression gives the passage time for the system (4.8), provided the constants T_0,T_1 are known. These two quantities can be calculated by the standard methods of Sec. III.

A problem arises in this argument if n is even, which corresponds to an asymmetric potential. In that case, a system that exits from the marginal zone by its left side crossing $x=-L$ is forced by the deterministic potential to turn back, and it really does not exit until it reaches $x=+L$. This effect becomes important when the correlation time of the noise goes to infinity. In this limit, the noise is almost constant, and the system has to wait a time of order of τ to let the noise reach a positive value, which is necessary to cross the marginal point $x=0$. This linear dependence in the τ contribution to the MFPT hides the time given by Eq. (4.16), which is asymptotically proportional to $\tau^{1/2}$ in this limit.

For $n=2$, T_0 and T_1 are given by Eqs. (3.7) and (3.8). In Fig. 2(a) the prediction of Eq. (4.16) is compared with the simulation results of the both systems (2.1) and (2.3), presenting a perfect agreement up to values of τ' as large as 10. Above that value, a linear dependence on τ' is observed, as we have commented.

V. CONCLUSIONS

We have presented an approximate theoretical scheme to calculate the dominant contributions of the colored noise on the first moments of the passage time for a relaxation process through a marginal point. This has been possible due to the fact that Markovian formulas can be used to obtain the non-Markovian effects coming from a time-independent Fokker-Planck approximation [Eq. (2.5)]. These effects are dominant in this system. Other non-Markovian effects come from the transient dynamics at the boundaries, but they are now irrelevant because they are masked by the deterministic motion.

The simple idea of substituting the real process by free diffusion in a finite interval is also profitable for large values of τ. The computer simulations have confirmed the suitability of the approximations in a wide range of the parameters. Hence we conclude that the very simple approximations presented in this paper have retained the essential physics of the dynamical process through a saddle or marginal point. They could be profitable in the study of a real system like optical bistability in lasers.[14]

ACKNOWLEDGMENTS

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