Solutions of the telegrapher's equation in the presence of traps

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(Received 28 February 1991; revised manuscript received 7 June 1991)

Several problems in the theory of photon migration in a turbid medium suggest the utility of calculating solutions of the telegrapher's equation in the presence of traps. This paper contains two such solutions for the one-dimensional problem, the first being for a semi-infinite line terminated by a trap, and the second being for a finite line terminated by two traps. Because solutions to the telegrapher's equation represent an interpolation between wavelike and diffusive phenomena, they will exhibit discontinuities even in the presence of traps.

PACS number(s): 05.40.+j, 42.20.−y, 42.10.−s

I. INTRODUCTION

The persistent random walk, whose diffusive analog is described by a telegrapher's equation, is possibly the simplest mathematical model allowing one to incorporate a form of momentum in addition to random or diffusive motion. A persistent random walk, in its simplest setting in discrete time and on a one-dimensional lattice, is a walk in which at each step, one chooses a probability which determines whether the random walker takes a step in the same direction as the immediately preceding one or whether the direction of motion is reversed. On appropriately scaling the lattice spacing, the time, and the persistence probability, one can derive the telegrapher's equation from the discrete evolution equation of the persistent random walk. The persistent random walk differs from the ordinary random walk in that the probabilistic element used at each step is the probability of continuing to move in a given direction rather than the probability of moving in a given direction independent of the direction of the immediately preceding step. Thus the process remains a Markov process, but of second rather than first order.

Such a model was first introduced by Fürth as a model of diffusion in a number of biological and physical problems [1] and shortly thereafter by Taylor in the analysis of turbulent diffusion [2]. The solution to a variety of forms of the telegrapher's equation on a line unbounded in both directions has been given by Goldstein [3]. Kramers [4] and Wang and Uhlenbeck [5] gave early analyses of Fokker-Planck equations in the presence of absorbing boundaries, which allow for both spatial and momentum variables. A variety of models of photon migration in a turbid medium that include forward-scattering effects [6,7] can be formulated in terms of a telegrapher's equation. There is considerable literature [8] on physical processes, particularly in the field of thermophysics, leading to a mathematical formulation in terms of a telegrapher's equation. The physics behind such applications implies that the signal propagation speed is finite rather than infinite as in the case of ordinary diffusion.

The use of a similar mathematical formulation in the design of studies of the scattering and absorption of laser radiation from human tissue [9,10] suggests that it is of some interest to study properties of the telegrapher's equation, rather than the diffusion equation, in the presence of one or more absorbing boundaries. There is little literature related to the telegrapher's equation in the presence of either absorbing or reflecting boundaries. An expression has been found for the mean first-passage time of a particle, whose motion can be described by a telegrapher's equation, to escape from a finite interval [11]. In the present paper we derive the solution of the telegrapher's equation for the probability density of the displacement of a particle diffusing on a line in the presence of one and two absorbing boundaries. The solution of the second of these allows one to calculate the survival probabilities for the same systems, and thereby recover the results of Ref. [11]. A more interesting qualitative feature to our solutions is the appearance of discontinuities in the concentration profile in the presence of traps, which illustrates the fact that the telegrapher's equation can be regarded as an interpolation between the wave and diffusion equations. Our analysis is for the case of a one-dimensional process since there is no unique multidimensional extension.

II. FORMULATION OF EQUATIONS, INITIAL AND BOUNDARY CONDITIONS

In order to derive the proper boundary conditions for the telegrapher's equation in the presence of traps, it is convenient to decompose the probability density for the position of the diffusing particle into two components, depending on whether the particle is moving along the line to the right or to the left. This corresponds to treating the persistent diffusion process as a two-state diffusion process in the sense of Ref. [12]. Let $a(x,t|x_0)$ be the probability density for a particle that moves in the direction of increasing x at time t to be at x at that time having been at x_0 initially, and let $b(x,t|x_0)$ be the corresponding density when the particle moves in the direction of decreasing x. These functions satisfy the coupled set of equations

$$\frac{\partial a}{\partial t}=-c\frac{\partial a}{\partial x}+\frac{1}{2T}(b-a)\ , \tag{1a}$$

$$\frac{\partial b}{\partial t}=c\frac{\partial b}{\partial x}+\frac{1}{2T}(a-b)\ , \tag{1b}$$

where c and T are parameters having the dimensions of velocity and time, respectively. By eliminating $b(x,t|x_0)$ in terms of $a(x,t|x_0)$ in Eq. (1a) one finds that $a(x,t|x_0)$ is the solution to a telegrapher's equation which is written

$$\frac{\partial^2 a}{\partial t^2}+\frac{1}{T}\frac{\partial a}{\partial t}=c^2\frac{\partial^2 a}{\partial x^2}\ . \tag{2}$$

It is convenient, in the following analysis, to introduce dimensionless parameters τ and y through the transformations $t=T\tau$ and $y=x/(cT)$, which is equivalent to setting both the coefficient c appearing in Eqs. (1) and (2) and the parameter T equal to 1. Since in most physical applications one cannot distinguish between right- and left-moving particles, we will be interested in calculating the probability density for the position of an arbitrary particle at time τ

$$p(y,\tau|y_0)=a(y,\tau|y_0)+b(y,\tau|y_0) \tag{3}$$

in the presence of trapping sites at $y=0$ and L, where L is now a dimensionless parameter which specifies the length of the interval. For simplicity we will take the initial conditions appropriate for a symmetric process:

$$a(y,0|y_0)=b(y,0|y_0)=\delta(y-y_0)/2\ . \tag{4}$$

A useful simplifying set of transformations for the analysis is the following:

$$a=Ae^{-\tau/2},\quad b=Be^{-\tau/2}\ . \tag{5}$$

The new functions A and B are solutions to the same set of self-adjoint equations (we give only the equation satisfied by A):

$$\frac{\partial^2 A}{\partial \tau^2}=\frac{\partial^2 A}{\partial y^2}+\frac{A}{4}\ . \tag{6}$$

The initial conditions on A and B are then identical to those in Eq. (4). Initial conditions on the time derivatives are found by returning to the dimensionless version of Eq. (1). Since the functions a and b are initially equal by Eq. (4), it follows from Eq. (1) that

$$\left.\frac{\partial a}{\partial \tau}\right|_{\tau=0}=-\frac{\delta'(y-y_0)}{2},\quad \left.\frac{\partial b}{\partial \tau}\right|_{\tau=0}=\frac{\delta'(y-y_0)}{2} \tag{7}$$

or

$$\left.\frac{\partial A}{\partial \tau}\right|_{\tau=0}=\frac{\delta(y-y_0)}{4}-\frac{\delta'(y-y_0)}{2}\ , \tag{8a}$$

$$\left.\frac{\partial B}{\partial \tau}\right|_{\tau=0}=\frac{\delta(y-y_0)}{4}+\frac{\delta'(y-y_0)}{2}\ . \tag{8b}$$

We need only concentrate on the calculation of the function $A(y,\tau|y_0)$ since $B(y,\tau|y_0)$ can be found from it through the relation

$$B=2\left[\frac{\partial A}{\partial \tau}+\frac{\partial A}{\partial y}\right]\ . \tag{9}$$

We next consider the boundary conditions to be satisfied by the functions A and B. In the case of a single trap at $y=0$, only particles that move in the negative y direction will actually be trapped when they encounter the trapping site. Particles that switch directions on encountering the trap can be considered to be reflected from the trap. That is to say, if a particle in the neighborhood of $y=0$ switches from motion in the direction of decreasing y to motion in the direction of increasing y, then it will not be trapped since trapping implies that not only does the particle arrive at $y=0$, but it must also be moving in the right direction when it does so. A convenient way of thinking about this is in terms of a two-state process, one state corresponding to motion towards increasing y, and the other to motion towards decreasing values of y. These considerations imply that the function $A(y,\tau|y_0)$ must be found subject to the boundary condition

$$A(0,\tau|y_0)=0\ . \tag{10}$$

Because of the relation given in Eq. (9), we can solve the equation for A, and then calculate the function B. This two-stage process is necessary in order to find $p(y,\tau|y_0)$.

In the case of two traps, one at $y=0$ and the other at $y=L$ we have

$$A(0,\tau|y_0)=B(L,\tau|y_0)=0\ . \tag{11}$$

These complete the set of conditions to be imposed on the solution of Eq. (6), after which the probability density for the position at time τ can be expressed in terms of the functions A and B as the sum $p(y,\tau|y_0)=(A+B)\exp(-\tau/2)$.

III. SOLUTION FOR THE SINGLE TRAP

Consider first the equation for the function $A(y,\tau|y_0)$. Because of the boundary condition on A given in Eq. (10) we make a combined Fourier-sine and Laplace transform of Eq. (6) for the function A, i.e., we calculate the function

$$\hat{A}(\omega,s|y_0)=\int_0^\infty \sin(\omega y)dy\int_0^\infty A(y,\tau|y_0)\exp(-s\tau)d\tau \tag{12}$$

from Eq. (6) taking into account the initial conditions from Eqs. (4) and (8a). The boundary condition in Eq. (10) is automatically taken care of by our use of the Fourier sine transformation. An elementary calculation yields the result

$$\hat{A}(\omega,s|y_0)=\frac{(s+\frac{1}{2})\sin(\omega y_0)+\omega\cos(\omega y_0)}{2(s^2-\frac{1}{4}+\omega^2)}\ . \tag{13}$$

The Laplace transform $\overline{A}(y,s|y_0)\equiv\mathcal{L}\{A(y,\tau|y_0)\}$ is readily found by calculating the inverse Fourier transform of this expression. The result of this calculation is

$$\overline{A}(y,s|y_0)=\begin{cases}\frac{1}{4}[\beta(s)-1](e^{-\rho(s)(y_0-y)}-e^{-\rho(s)(y_0+y)}), & y<y_0\\ \frac{1}{4}\{[\beta(s)+1]e^{-\rho(s)(y-y_0)}-[\beta(s)-1]e^{-\rho(s)(y_0+y)}\}, & y>y_0\end{cases} \tag{14}$$

in which the functions $\rho(s)$ and $\beta(s)$ are defined by

$$\rho(s)=(s^2-\tfrac{1}{4})^{1/2},\quad \beta(s)=\frac{s+\frac{1}{2}}{\rho(s)} . \tag{15}$$

The analogous transform $\overline{B}(y,s|y_0)$ is calculated by taking the Laplace transform of Eq. (9) leading to the relation

$$\overline{B}(y,s|y_0)=2\left[s\overline{A}(y,s|y_0)+\frac{\partial}{\partial y}\overline{A}(y,s|y_0)\right]-\delta(y-y_0) . \tag{16}$$

To find the Laplace transform of the total probability density $\overline{P}(y,s|y_0)$, we add the two expressions $\overline{A}$ and $\overline{B}$ found from Eqs. (14) and (16). Notice that the transform of the actual probability density for the position of the diffusing particle at time t, $\overline{p}(y,s|y_0)$, is equal to $\overline{P}(y,s+\frac{1}{2}|y_0)$. The function $\overline{P}(y,s|y_0)$ is found in this way to be

$$\overline{P}(y,s|y_0)=\frac{\beta(s)}{2}[e^{-\rho(s)|y-y_0|}-\alpha(s)e^{-\rho(s)(y_0+y)}] , \tag{17}$$

in which the function $\alpha(s)$ is

$$\alpha(s)=\frac{s-\rho(s)}{s+\rho(s)}=4[s-\rho(s)]^2 . \tag{18}$$

The function of physical interest, i.e., the probability density for the position of the diffusing particle at time τ, $p(y,\tau|y_0)$, is related to the inverse transform of the function $\overline{P}$ by $p(y,\tau|y_0)\equiv\mathcal{L}^{-1}[\overline{P}(y,s|y_0)]\exp(-\tau/2)$. Notice that the Laplace transform of $p(y,\tau|y_0)$ can be expressed more simply as $\overline{P}(y,s+\frac{1}{2}|y_0)$. One can perform the inversion of Eq. (17) in terms of known functions [15], finding

$$P(y,\tau|y_0)=f_0(\tau,|y-y_0|)-f_1(\tau,y+y_0) , \tag{19}$$

where the somewhat complicated definitions of the functions f_0 and f_1 are given in the Appendix. One finds from either Eq. (17) or (19) that the survival probability $S(\tau|y_0)\equiv\int_0^\infty p(y,\tau|y_0)dy$ satisfies $\lim_{\tau\to\infty}\tau^{1/2}S(\tau|y_0)=\text{const}$, as does the survival probability for ordinary Brownian motion in the presence of a single trap. It is also possible to find the solution for the survival probability at early times, but the result is somewhat complicated and is therefore omitted. Figure 1 shows a few curves of $p(y,\tau|0.3)$ as a function of y for different values of t. There are two notable features of these curves. The first is that the value of the probability density at the trapping point, $p(0,\tau|y_0)$, differs from 0, in contrast to its behavior for ordinary diffusion in the presence of a trap, and the second is the appearance of discontinuities in the concentration profile. There are two discontinuities in the curves in Fig. 1. Those on the right-hand side of the solution are a manifestation of the property of finite velocity, and those occurring on the left-hand side of the curves correspond to the reflection of particles that reach $y=0$, but are not trapped because they are moving in the wrong direction.

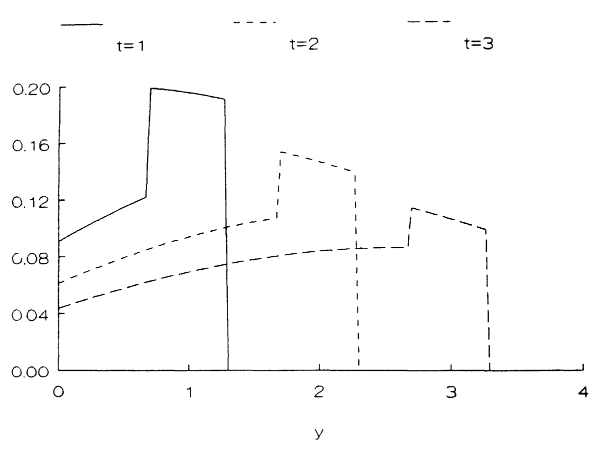

FIG. 1. Curves of $p(y,\tau|0.3)$ as a function of y in the presence of a single trapping point, as a function of y for $\tau=1$(——), 2(— — —), and 3(——). Notice that because of the finite velocity of signal transmission the curves are truncated on the right. The discontinuities on the left are the results of reflections from the trapping point, i.e., from particles which reach the trap but reverse direction when they reach that point. Note that we have omitted the δ-function contribution to the curves.

IV. SOLUTION FOR THE CASE OF TWO TRAPS

Let us now assume that there are two traps, one at $y=0$ and the second at $y=L$. In this situation it is possible to restrict our attention to just the single equation for $A(y,\tau|y_0)$ since, by the symmetry of the situation, we can express $B(y,\tau|y_0)$ in terms of this function as

$$B(y,\tau|y_0)=A(L-y,\tau|L-y_0) . \tag{20}$$

An expedient way to solve the equation for the function A is to take the Laplace transform of Eq. (6). Since the propagation velocity is equal to 1 in our system of coordinates we need only find a solution for values of $\tau>\max(y_0,L-y_0)$ because when $\tau<\min(y_0,L-y_0)$ there are no boundary effects, and when $\min(y_0,L-y_0)\le\tau\le\max(y_0,L-y_0)$ the diffusing particle is influenced by only one of the traps, thus leading us back to the single-trap result.

As before, we denote the Laplace transform of $A(y,\tau|y_0)$ by $\overline{A}(y,s|y_0)$. This function satisfies the equation

$$\frac{\partial^2 \bar{A}}{\partial y^2} - \rho^2(s)\bar{A} = -\tfrac{1}{2}[(s+\tfrac{1}{2})\delta(y-y_0) - \delta'(y-y_0)] , \tag{21}$$

where $\rho(s)$ is defined in Eq. (18). It is easy to find the Green's function for the left-hand side of this equation which, in turn, allows us to write a complete solution for the Laplace transform of the probability density of position. The solution, as in the case of the single trap, consists of two parts depending on whether y is greater or less than y_0. Specifically, we find that the function $\bar{A}(y,s|y_0)$ is

$$\begin{aligned}\bar{A}(y,s|y_0) &= \left[\frac{\alpha(1+\beta)e^{\rho(y_0-2L)} - (1-\beta)e^{-\rho y_0}}{4(1-\alpha e^{-2\rho L})}\right] \\ &\quad \times (e^{\rho y} - e^{-\rho y}), \quad y < y_0 \\ &= \left[\frac{(1+\beta)e^{\rho y_0} - (1-\beta)e^{-\rho y_0}}{4(1-\alpha e^{-2\rho L})}\right] \\ &\quad \times (e^{-\rho y} - \alpha e^{\rho(y-2L)}), \quad y > y_0\end{aligned} \tag{22}$$

where α and β are the functions defined in Eqs. (15) and (18). Equation (20) can now be used to calculate the corresponding expression for $\bar{B}(y,s|y_0)$. As before, we will be interested in calculating the function $\bar{P}(y,s|y_0)$, which is the sum $\bar{A}$ and $\bar{B}$. It is convenient to express this transform in terms of the function

$$U(\xi,s) = \frac{\beta(s)}{2[1-\alpha(s)e^{-2\rho(s)L}]}e^{-\rho(s)\xi} \tag{23}$$

in terms of which we find

$$\begin{aligned}\bar{P}(y,s|y_0) &= U(|y-y_0|,s) \\ &\quad - \sqrt{\alpha}[U(y+y_0,s) + U(2L-y-y_0,s)] \\ &\quad + \alpha U(2L-|y-y_0|,s) .\end{aligned} \tag{24}$$

The inverse of the function $U(\xi,s)$ appearing in Eq. (23) can be expanded in an infinite series of Bessel functions as shown in the Appendix. One can invert the transform in Eq. (24), writing it in a form of an infinite series reminescent of the series derived by the method of images. For this purpose let $f_n(\tau,y)$ be the combination of Bessel and other functions defined in Eq. (A3). The inverse of Eq. (24) is readily shown to be expressible as

$$\begin{aligned}P(y,\tau|y_0) = \sum_{k=0}^{\infty} \{ & f_{2k}(\tau,2kL+|y-y_0|) - f_{2k+1}(\tau,2kL+y+y_0) \\ & - f_{2k+1}[\tau,2(k+1)L-y-y_0] + f_{2k+2}[\tau,2(k+1)L-|y-y_0|]\} .\end{aligned} \tag{25}$$

Parenthetically we note that from the point of view of actual calculations it is somewhat more convenient to evaluate the inverse by a numerical inversion of the Laplace transform.

Some qualitative features are apparent from an examination of the expression for the density given in Eq. (25). For the purpose of this discussion, we define two spatial parameters ξ_1 and ξ_2 by

$$\xi_1 = \begin{cases} y+y_0, & \min(y_0,L-y_0)=y_0 \\ 2L-y-y_0, & \min(y_0,L-y_0)=L-y_0 \end{cases} \tag{26}$$

and $\xi_2 = 2L - \xi_1$, as well as three (dimensionless) time parameters

$$\tau_1 = \min(y_0,L-y_0), \quad \tau_2 = L-\tau_1, \quad \tau_3 = L+\tau_1 . \tag{27}$$

Then, for example, Eq. (25) implies that $P(y,\tau|y_0) = f_0(\tau,|y-y_0|)$ when $0<\tau<\tau_1$. Since τ is less than τ_1, this corresponds to propagation without the possibility of hitting a trap, i.e., diffusion in free space. Indeed, the solution given is just that given, for example, by Goldstein [3]. When $\tau_1<\tau<\tau_2$, the result in Eq. (25) reduces to Eq. (19) which is the solution in the case of diffusion in the presence of a single trap given in Sec. III. The effects of both boundaries are found at longer times. For example, when the times are restricted to be less than $2L$ we have

$$P(y,\tau|y_0) = \begin{cases} f_0(\tau,|y-y_0|) - f_1(t,\xi_1)f_1(\tau,\xi_2) , \\ \qquad\qquad \tau_2<\tau<\tau_3 \\ f_0(\tau,|y-t_0|) - f_1(t,\xi_1) - f_1(\tau,\xi_2) \\ \qquad + f_2(\tau,2L-|y-y_0|), \quad \tau_3<\tau<2L . \end{cases} \tag{28}$$

The discontinuities are a manifestation of the Heaviside step functions that occur in the definitions of the f_j's. When $\tau_2<\tau<\tau_3$ the discontinuity due to the term $f_0(\tau,|y-y_0|)$ has vanished, but both $f_1(\tau,\xi_1)$ and $f_2(\tau,\xi_2)$ contribute to the discontinuous behavior of the profiles, the discontinuities occurring at $y_1=\tau-y_0$ and $y_2=2L-\tau-y_0$. When $\tau=L$, $y_1=y_2$, which leads to a merging of the discontinuities, and the reversal in the concentration profile shown in Fig. 2(a). One can also observe that when $\tau<L$, one or both of the $f_1(\tau,\xi_i)$ are equal to zero, while when $\tau>L$ both of these functions differ from zero. Finally, when $\tau_3<\tau<L+\max(y_0,L-y_0)$ the rightmost discontinuity will vanish as it passes into the trap. When $\max(y_0,L-y_0)=y_0$ the discontinuity at y_1 vanishes as it passes into the trap at $y=L$ and when $\max(y_0,L-y_0)=L-y_0$ the discontinuity at y_2 vanishes at $y=0$. At sufficiently long times the discontinuities in concentration vanish and are replaced by discontinuities in the derivative only. This is illustrated by the behavior of the curve in Fig. 2(b). When $\tau>L+\max(y_0,L-y_0)$ all of the discontinuities vanish and one gets a smooth profile for the probability density of the particle position as indicated in Fig. 2(b). The discontinuities in the probability density also show up in a curve of the probability that a particle originally at y_0 remains untrapped by time τ. This is defined by

$$S(\tau|y_0)=\int_0^L P(y,\tau|y_0)dy \; . \tag{29}$$

A typical curve of $S(\tau|y_0)$ plotted as a function of τ is shown in Fig. 3. The discontinuity-free regime that appears in Fig. 2(b) corresponds to the tail of the curve plotted in Fig. 3.

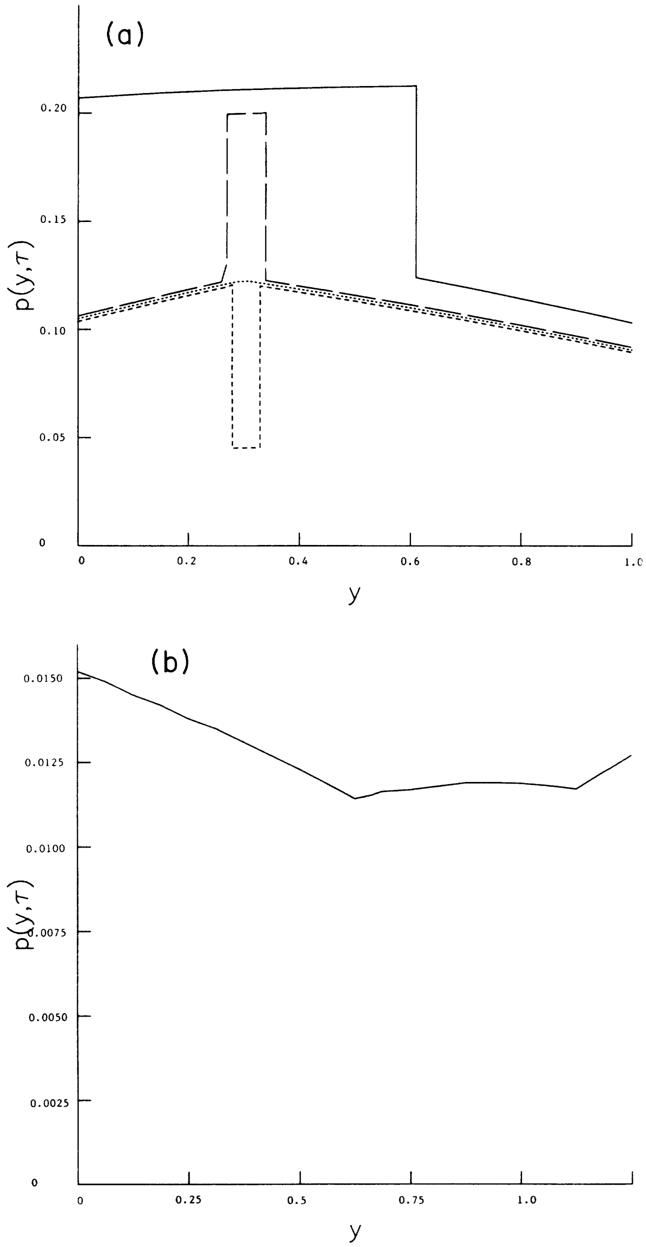

FIG. 2. (a) Curves of $p(y,\tau|0.7)$ as a function of y in the presence of trapping points at $y=0$ and 1 for $\tau=0.7$(——), 0.97(——), 1(· · · ·), and 1.03(— — —). The time $\tau=0.7$ is the earliest at which a particle can reach $y=0$ and at $\tau=1$ there is a reversal of the reflection discontinuities. (b) A curve of $p(y,\tau|0.7)$ as a function of y for $\tau=1.8$, at which time the discontinuities have disappeared. At much later times the discontinuities in derivatives also vanish. This phenomenon is a result of diffusion.

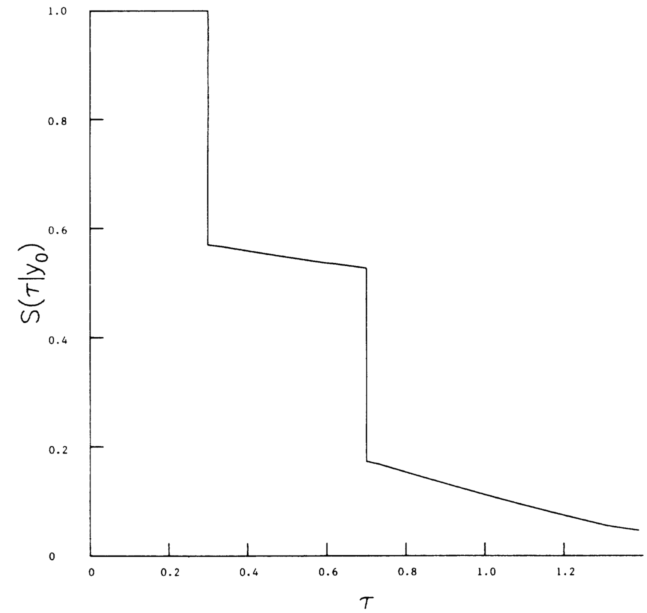

FIG. 3. Plot of $S(\tau|y_0)$ as a function of the time for $L=1$ and $y_0=0.7$. The discontinuities in this curve have the same source as those in Figs. 1 and 2, i.e., they correspond to reflected waves.

The analysis of a persistent random walk on a lattice, which is the discrete analog of the telegrapher's equation, is most conveniently carried out by means of the method of exact enumeration [15], which is basically exact within the accuracy of the computer arithmetic. Some typical results of such a calculation are shown in Fig. 2, for $L=1$ and an initial position at $y_0=0.7$. Just as in the case of the single trap, the probability density does not vanish at the boundaries. Discontinuities appear in the solution which, again, are related to the wavelike properties of the equation and correspond to reflections from the trapping point. Notice that when the discontinuities merge at $\tau=L-y_0$, they become inverted. The discontinuities in the probability density disappear for values of $\tau>L+\max(L-y_0,y_0)$. If one fixes a point in space and lets $t=r$, the central-limit theorem becomes applicable [13], and discontinuities occur only at the end points of the probability density.

It is evident from our calculations that the solution for the probability density in the presence of trapping points is not nearly as straightforward as is the case for ordinary diffusion and would be even more difficult to find in higher-dimensional analogs of the telegrapher's equation. Such a solution would, however, be useful in the analysis of a number of problems involving the migration of photons in turbid media. For example, a multidimensional generalization of the telegrapher's equation has been suggested by Ishimaru as furnishing a more accurate model for this phenomenon than the ordinary diffusion equation because it gives a crude model for forward-scattering effects [6,7]. However, we note that the present analysis is not readily extended to the solution of the multidimensional telegrapher's equation, nor is it certain that the telegrapher's equation contains the correct physics for diffusion with significant forward-scattering effects. A strong motivation for studying the effects of boundaries is the fact that a number of measurements of biomedical interest can be modeled in terms of this type of diffusion in the presence of absorbing boundaries [9,10]. The present article is merely a start in the investigation of the class of models in which there is both diffusive and wavelike motion and in which boundaries are included.

ACKNOWLEDGMENTS

The work of J. M. and J. M. P. was partially supported by the Comision Interministerial de Ciencia y Tecnologia (CICYT) under Contract No. PS87-0046, and by the Societat Catalana de Fisica (Institut Estudis Catalans).

APPENDIX: INVERSE LAPLACE TRANSFORMS OF THE FUNCTION $U(\xi,s)$ AND THE DEFINITION OF THE FUNCTIONS $f_n(\tau,x)$

To invert the transform of $U(\xi,s)$ we first expand the denominator into a power series

$$U(\xi,x)=\frac{\beta(s)}{2}\sum_{n=0}^{\infty}\alpha^n(s)\exp[-\rho(s)(\xi+2nL)] \ . \tag{A1}$$

The inverse of individual terms in this expansion can be found in terms of the Bessel functions $I_n(x)$. Defining the functions

$$g_m(\tau,x)=\left[\frac{\tau-x}{\tau+x}\right]^{m/2} I_m\left[\frac{(\tau^2-x^2)^{1/2}}{2}\right] , \tag{A2}$$

one finds [14]

$$\begin{aligned}\frac{f_n(\tau,x)}{2^n}&\equiv\mathcal{L}^{-1}\{[\beta(x)/2][s-\rho(s)]^n\exp[-\rho(s)x]\}\\ &=[g_{n-1}(\tau,x)+2g_n(\tau,x)+g_{n+1}(\tau,x)]\\ &\quad\times\frac{H(\tau-x)}{8},\quad n\geq 1\\ f_0(\tau,x)&=\frac{\delta(\tau-x)}{2}\\ &\quad+\left[g_0(\tau,x)+\frac{\tau}{(\tau^2-x^2)^{1/2}}I_1\left[\frac{(\tau^2-x^2)^{1/2}}{2}\right]\right]\\ &\quad\times\frac{H(\tau-x)}{4} \ .\end{aligned} \tag{A3}$$

These results may now be substituted into the expansion given in Eq. (A1) to find the inverse Laplace transform of the function $U(\xi,s)$, which, in turn, can be used in Eq. (24).

[1] R. Fürth, Ann. Phys. (Leipzig) **53**, 177 (1917).
[2] G. I. Taylor, Proc. London Math. Soc. **20**, 196 (1921).
[3] S. Goldstein, Q. J. Mech. Appl. Math. **4**, 129 (1950).
[4] H. Kramers, Physica **7**, 284 (1940).
[5] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).
[6] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978), Vol. 1.
[7] A. Ishimaru, Appl. Opt. **28**, 2210 (1989).
[8] D. D. Joseph and L. Preziosi, Rev. Mod. Phys. **61**, 41 (1989); **62**, 375 (1990).
[9] R. F. Bonner, S. Havlin, R. Nossal, and G. H. Weiss, J. Opt. Soc. Am. **4**, 423 (1987).
[10] B. Chance *et al.*, Proc. Natl. Acad. Sci. U.S.A. **85**, 4971 (1985).
[11] G. H. Weiss, J. Stat. Phys. **37**, 325 (1984).
[12] G. H. Weiss, J. Stat. Phys. **15**, 157 (1976).
[13] J. Masoliver, K. Lindenberg, and G. H. Weiss, Physica A **157**, 891 (1989).
[14] G. E. Roberts and H. Kaufman, *Table of Laplace Transforms* (Saunders, Philadelphia, 1968).
[15] S. Havlin and D. ben-Avraham, Adv. Phys. **36**, 695 (1987).