Multiplicative noise effects on relaxations from marginal states

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Relaxational processes in bistable potentials close to marginal conditions are studied under the combined effect of additive and multiplicative fluctuations. Characteristic time scales associated with the first-passage-time-distribution are analytically obtained. Multiplicative noise introduces large effects on the characteristic decay times, which is particularly significant when relaxations are mediated by fluctuations, i.e., below marginality and for small noise intensity. The relevance of our approach with respect to realistic chemical bistable systems experimentally operated under external noise influences is mentioned.

I. INTRODUCTION

Bistability is a common feature of many different physical systems in nature. Well-known examples range from optical devices [1] to chemical reactors [2]. The common level of description appropriate to such situations is based on one-variable models with multiple steady-state solutions for a well-defined range of values of the corresponding control parameter. At the limiting points of such a bistable regime, the unstable and locally stable branches of steady-state solutions coalesce into a marginally stable solution. A complementary representation, particularly useful to the discussion of dynamical processes in bistable situations such as those we propose here, is based on a potential whose extrema are the previously identified steady-state solutions. Marginality, and correspondingly the state of marginal stability, is then associated with the existence of an inflection stationary point of such a potential.

By referring to this potential picture, it is easy to understand that relaxational dynamics sufficiently close to the marginal conditions will be singularly sensitive to noise effects. Consider, for example, typical relaxational experiments crossing over a barrier between metastable states (below marginality) or passing through a saddle point (strict marginality). In both cases, these processes are not possible under pure deterministic conditions. Even, slightly above marginality, when fluctuations are not strictly necessary to make possible the relaxational process, they will considerably influence its natural time scale.

Relaxational dynamics under the influence of fluctuations is commonly formulated in terms of Langevin equations for the single relevant variable of interest [3]. Noise forces may enter into this equation in two different ways, either additively, as a constant term, or multiplicatively in a state-dependent way. It is reasonable to assume that due to their different nature either one will contribute to the relaxational process close to marginality in a different, perhaps even opposite, way. The situation with pure additive noise has been already addressed in the literature [4], but to our knowledge no attempt has been made yet to incorporate multiplicative noise effects in relaxational processes close to marginality. Since it is well known that multiplicative noise effectively results in a modification of the deterministic potential governing the decay dynamics, we expect to find distinctive and significant effects arising from multiplicative noise contributions, particularly important in situations of fluctuations activated relaxations. This constitutes the main concern of this paper.

Although formulated in this way this problem seems to have a purely formal character, the discussion reported here has a complementary motivation that comes from realistic chemical experiments. Actually, there are well-known chemical system [5], displaying bistability characteristics when operated in open well-mixed reactors, whose dynamical behavior close to marginal and critical conditions has been experimentally studied [6]. The control parameter is usually taken as the input flow rate of reactants K_0. When formulating their corresponding dynamical equations appropriate to relaxation experiments close to marginality it turns out that they can be reduced to a one-variable potential version with

$$V(x)=-\beta(x-cx^2)-ax^3+bx^4 \; . \tag{1.1}$$

In Eq. (1.1) β is a convenient dimensionless parameter directly associated with the flow rate. According to our discussion above, β is also a measure of the distance to strict marginality (which appears for $\beta=0$) and, additionally, and this is a particular specificity of this model, it introduces a quadratic term in the deterministic potential (1.1). If we now realize that even under the most optimized operating conditions it is practically impossible, as recognized by the experimentalists themselves [6], to completely suppress external disturbances affecting the control parameter K_0, we immediately recognize that the experiments mentioned above should be genuine candidates for studying the combined effects of additive and multiplicative fluctuations in relaxational dynamics close to marginality.

Once our motivation has been clearly stated, let us briefly comment on the techniques we will use to describe such relaxational processes. In principle several methods are available depending on the quantities used to describe the dynamics. In our approach here, and since we are

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primarily interested in the modifications introduced by multiplicative noise on the typical time scales of the relaxation process, we will restrict ourselves to the analysis of relaxation times evaluated in terms of the first-passage-time distribution (FPTD). An alternative approach, based on the transient dynamics of the statistical moments, would be appropriate to describe transient bimodality phenomena. Actually, both procedures have been separately and recently applied to relaxations close to marginal conditions, although considering purely additive noise contributions [4,7]. Additional related approaches existing in the literature are those of Ramirez-Piscina *et al.* [8] dealing with additive colored Gaussian noise and those of Zhu, Yu, and Roy [9] who consider, as we do here, the combined effects of additive and multiplicative noises but in the different context of decay processes from initially unstable states.

Since our primary aim concerns the discussion of generic effects that should be valid irrespective of any particular system, we will formulate the problem in terms of the simplest model appropriate to the deterministic potential (1.1). Thus, our starting Langevin equation will be written as

$$\dot{x}=\beta(1-2cx)+3ax^2-4bx^3+(1-2dx)\xi(t)\ , \qquad (1.2)$$

where $\xi(t)$ is a Gaussian white noise prescribed of zero mean and arbitrary intensity ϵ. The parameters a and b are positive valued parameters, whereas c will be assumed positive or zero with restricted values ($c<2b/3a$) in order to satisfy bistability requirements. The multiplicative noise term enters into that equation through a coupling linear function of the relevant variable $x(t)$. The parameter d may be understood as a ratio between additive and multiplicative noise intensities. In the discussion that follows, d will be taken when necessary as a variable positive parameter bounded from above in order to assure a positive definite diffusion throughout the whole relaxational dynamics. The case with $c=d\neq 0$ would correspond to the chemical example in Ref. [6]. By imposing $c=d=0$ one recovers the generic model of Colet, San Miguel, Casademunt, and Sancho in Ref. [4] dealing with additive fluctuations.

Two main effects will turn out to be particularly significant in understanding the modifications, both qualitative and quantitative, introduced by multiplicative noise in relaxational dynamics close to marginality. First, and most important, is the transformation of the deterministic potential (1.1) into an effective potential that governs the relaxational dynamics. The second, although not totally independent effect, concerns the consideration of a nonconstant diffusion coefficient. Both, in the same or opposite way, contribute to the diversity of behavior shown by the moments of the FPTD when varying the relaxation conditions (β and ϵ) and the parameter d.

In order to make our presentation as complete as possible, analytical approximated results together with exact numerical ones will be presented. They will be obtained within the appropriate theoretical framework in Sec. II and discussed, respectively, in Secs. III and IV for the first moment, the mean first-passage time (MFPT), and the first-passage-time variance (FPTV), of the first-passage-time distribution.

II. THEORETICAL FRAMEWORK

A. General equations

The Langevin equation (1.2) can be recasted in a more generic way according to

$$\dot{x}=f(x)+g(x)\xi(t)\ , \qquad (2.1)$$

where

$$f(x)=-\frac{dV(x)}{dx}=\beta(1-2cx)+3ax^2-4bx^3\ ,$$
$$g(x)=1-2dx\ . \qquad (2.2)$$

Given an initial value x_0, the time needed for the process to reach for the first time a prescribed scape value x_F is subjected to a distribution probability, known as the first-passage-time distribution. Its first moment, the MFPT denoted by T_1 and FPTV given by $(\Delta T)^2=T_2-T_1^2$, where T_2 is the second moment of the FPTD, are explicitly given by the standard theory of stochastic processes [10,11]:

$$T_1=\int_{x_0}^{x_F}dx_1\int_{-\infty}^{x_1}dx_2\frac{\exp[-U(x_1)+U(x_2)]}{D(x_2)}\ , \qquad (2.3)$$

$$(\Delta T)^2=4\int_{x_0}^{x_F}dx_1\int_{-\infty}^{x_1}dx_2\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\frac{\exp[-U(x_1)-U(x_2)+U(x_3)+U(x_4)]}{D(x_3)D(x_4)}\ , \qquad (2.4)$$

$$U(x)=\int^{x}d\eta\frac{F(\eta)}{D(\eta)}\ , \qquad (2.5)$$

where a reflecting barrier is placed at $x=-\infty$ and x_F is considered as an absorbing one. In the expressions above, $F(x)$ and $D(x)$ are, respectively, the drift and the diffusion terms associated with the stochastic process (2.1),

$$F(x)=f(x)+\epsilon g(x)\frac{dg(x)}{dx}$$
$$=\beta-2d\epsilon-2(\beta c-2d^2\epsilon)x+3ax^2-4bx^3\ ,$$
$$D(x)=\epsilon g^2(x)=\epsilon(1-2dx)^2\ . \qquad (2.6)$$

B. Approximate methods

The remainder of this section will be devoted to the consideration of several approximations to the exact expressions given above. Three different approximated methods will be presented in this subsection. The first one is based on a systematic perturbative expansion in the parameter d. A much better representation of noise effects is expected when they are fully incorporated through the effective potential $U(x)$ in (2.5). This is what we propose in the other two procedures, the difference between them resting in the consideration either of a constant diffusion $D(x)$ in (2.3) and (2.4) [method (ii)] or its first-order approximation in d [method (iii)]. Finally, all of these three methods have in common the use of a cubic truncation of the effective potential. Actually, since we are primarily interested in the first stages of the evolution during which $x(t)$ leaves a state close to marginality, saturation terms are not essential. In addition, when this cubic truncation is used the time the system spends outside the region close to marginality is almost negligible, especially for relaxations mediated by fluctuations, i.e., for low values of β and ϵ [4]. Consequently, in what follows, all the approximated results will correspond to the evolution from $x_0=-\infty$ to $x_F=\infty$, under the consideration of sufficiently small values of β and ϵ.

1. Systematic small multiplicative noise expansion

The simplest way to incorporate the effect of multiplicative noise is considering d as a small perturbative parameter. Using a systematic expansion in that parameter for the cubic approximation to the effective potential $U(x)$ as well as for the diffusion term $D(x)$ we obtain the following expression to first order for the rescaled MFPT:

$$(a^2\epsilon)^{1/3}T_1=T_1^{(0)}+dT_1^{(1)} , \tag{2.7}$$

where

$$T_1^{(0)}=\Phi(k)=\left[\frac{\pi}{3}\right]^{1/2}\int_0^{\infty}dx\, x^{-1/2}\exp(-kx-\tfrac{1}{4}x^3) \tag{2.8}$$

and

$$\begin{aligned}T_1^{(1)}&=\frac{4\beta c}{3a}\Phi^{(1)}(k,k')\\&=\frac{4\beta c}{3a}\left[\frac{\pi}{3}\right]^{1/2}\int_0^{\infty}dx(2x^{-1/2}-k'x^{1/2}+\tfrac{1}{2}x^{5/2})\\&\qquad\times\exp(-kx-\tfrac{1}{4}x^3) ,\end{aligned} \tag{2.9}$$

with

$$\begin{aligned}k&=\beta\left[1-\frac{\beta c^2}{3a}\right](a\epsilon^2)^{-1/3} ,\\k'&=\beta\left[1-\frac{2\beta c^2}{3a}\right](a\epsilon^2)^{-1/3} .\end{aligned} \tag{2.10}$$

Notice that the function $\Phi(k)$ is nothing but the universal function defined for the MFPT of the generic cubic marginal model with additive noise [4]. In a similar way, the variance is, up to first order in d,

$$(a^2\epsilon)^{2/3}(\Delta T)^2=(\Delta T)^{2^{(0)}}+d(a^2\epsilon)^{1/3}(\Delta T)^{2^{(1)}} , \tag{2.11}$$

where

$$(\Delta T)^{2^{(0)}}=\tilde{\Phi}(k)=4\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{x_1}dx_2\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\exp[-k(x_1+x_2-x_3-x_4)-x_1^3-x_2^3+x_3^3+x_4^3] \tag{2.12}$$

and

$$\begin{aligned}(\Delta T)^{2^{(1)}}&=\frac{1}{a}\tilde{\Phi}(k,k',k'',k''')\\&=\frac{8}{a}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{x_1}dx_2\int_{-\infty}^{x_2}dx_3\int_{-\infty}^{x_3}dx_4\{x_1+x_2+x_3+x_4-k''(x_1^2+x_2^2-x_3^2-x_4^2)\\&\qquad+k'''[2(1+x_1^3+x_2^3-x_3^3-x_4^3)-k'(x_1+_2-x_3-x_4)]\}\\&\qquad\times\exp[-k(x_1+x_2-x_3-x_4)-x_1^3-x_2^3+x_3^3+x_4^3] ,\end{aligned} \tag{2.13}$$

with

$$\begin{aligned}k''&=\beta\left[1-\frac{4\beta c^2}{3a}\right](a\epsilon^2)^{-1/3} ,\\k'''&=\frac{2\beta c}{3}(a^2\epsilon)^{-1/3} .\end{aligned} \tag{2.14}$$

In that case, the function $\tilde{\Phi}(k)$ is the universal function defined for the FPTV of the generic cubic marginal model with additive noise reported in Ref. [4].

2. Cubic approximation to the effective potential with constant diffusion

The cubic approximation to the effective potential $U(x)$ reads

$$U(x)\approx\frac{1}{\epsilon}(U_0+U_1x+U_2x^2+U_3x^3) \tag{2.15}$$

with

$$\begin{aligned}U_1&=\beta-2d\epsilon ,\\U_2&=\beta(2d-c)-2d^2\epsilon ,\\U_3&=a+\tfrac{4}{3}[\beta d(3d-2c)-2d^3\epsilon] ,\end{aligned} \tag{2.16}$$

whereas for the diffusion term we approximate

$$D(x) \approx \epsilon D_0 = \epsilon \left[1 - \frac{2\beta c d}{3a} \right]^2 . \tag{2.17}$$

Notice that the diffusion has been taken at a specific point of the stochastic trajectories. The value we have chosen is the inflection point of the cubic approximation to the deterministic potential (1.1).

The final expressions for the mean and variance of the first-passage-time distribution are given by

$$(U_3^2 \epsilon)^{1/3} T_1 = \frac{1}{D_0} \Phi(k) \tag{2.18}$$

and

$$(U_3^2 \epsilon)^{2/3} (\Delta T)^2 = \frac{1}{D_0^2} \tilde{\Phi}(k) , \tag{2.19}$$

where now the parameter k of the functions $\Phi(k)$ and $\tilde{\Phi}(k)$ is given by

$$k = \left[U_1 - \frac{U_2^2}{3U_3} \right] (U_3 \epsilon^2)^{-1/3} . \tag{2.20}$$

3. *Cubic approximation to the effective potential with first-order diffusion*

For the third procedure presented here, we will use the same expressions as those of the preceding subsection with respect to the effective potential, i.e., Eqs. (2.15) and (2.16). For the diffusion term we will use the following approximation:

$$\frac{1}{D(x)} \approx \frac{(1+4dx)}{\epsilon} . \tag{2.21}$$

At this level of approximation, the expressions we obtain for the mean and variance of the first-passage-time distribution read

$$(U_3^2 \epsilon)^{1/3} T_1 = (T_1)_0 + (T_1)_1 , \tag{2.22}$$

$$(U_3^2 \epsilon)^{2/3} (\Delta T)^2 = (\Delta T)_0^2 + (\Delta T)_1^2 , \tag{2.23}$$

where

$$(T_1)_0 = \Phi(k) , \tag{2.24}$$

$$(T_1)_1 = -\frac{4dU_2}{3U_3} \Phi_1(k,k') = -\frac{4dU_2}{3U_3} \left[\frac{\pi}{3} \right]^{1/2} \int_0^\infty dx \, (x^{-1/2} + k' x^{1/2}) \exp(-kx - \tfrac{1}{4} x^3) , \tag{2.25}$$

$$(\Delta T)_0^2 = \tilde{\Phi}(k) , \tag{2.26}$$

$$\begin{aligned} (\Delta T)_1^2 &= \left[\frac{4dU_2}{3U_3} \right]^2 \tilde{\Phi}_1(k,k') \\ &= \left[\frac{4dU_2}{3U_3} \right]^2 4 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \int_{-\infty}^{x_2} dx_3 \int_{-\infty}^{x_3} dx_4 [1 - k'(x_3 + x_4)] \\ &\qquad \times \exp[-k(x_1 + x_2 - x_3 - x_4) - x_1^3 - x_2^3 + x_3^3 + x_4^3] \end{aligned} \tag{2.27}$$

k being defined as in (2.20), and additionally

$$k' = \frac{3(U_3^2 \epsilon)^{1/3}}{2U_2} . \tag{2.28}$$

Notice that $(T_1)_0$ and $(\Delta T)_0^2$ would correspond respectively to the results (2.18) and (2.19) if those expressions were referred to a constant diffusion taken at the origin $x=0$.

III. RESULTS FOR THE MEAN FIRST-PASSAGE TIME

A. Approximate results

1. *Systematic small multiplicative noise expansion*

The result for the MFPT in Eq. (2.7) deserves some comments when comparing it with the corresponding result for the genuine cubic marginal model with pure additive noise [4]. The zeroth-order contribution already shows the effect of the quadratic modification in the potential (1.1) introduced through the parameter c. However, since this term actually appears multiplied by β it should not essentially modify the marginal characteristics of the dynamics expressed by (1.2) with respect to the standard case [4]. This is exactly what happens and the well-known scaling relation of the MFPT with $\epsilon^{-1/3}$ valid for the cubic marginality [4,11,12] is exactly recovered. The whole effect of the quadratic term is completely contained in a convenient redefinition of the parameter k [see Eq. (2.10)], measuring the distance to marginality: $k \propto \beta$. Since the universal function $\Phi(k)$ is a monotonically decreasing function of k [4], we can immediately conclude that the MFPT will increase due to the quadratic term in $V(x)$ [remember that in the standard case $k = \beta(a\epsilon^2)^{-1/3}$], this increment being zero at strict marginality ($\beta=0$).

Additional conclusions can be raised when taking into account the first correction originated in the multiplicative noise. Note that another independent measure of the distance to marginality is introduced through the parameter k', which only coincides with k in the generic cubic marginal case $c=0$. The last important point we want to mention is that the first-order correction, being proportional to βc, changes sign at strict marginality due to the positive character of the function $\Phi^{(1)}(k,k')$, and in addition it would be zero for the standard case $c=0$. This result admits an intuitive and approximate interpretation in terms of the cubic form of the deterministic potential used in this context. For $c=0$, its antisymmetric nature would exactly compensate the positive and negative contributions to the diffusion term (2.6), respectively, for negative and positive values of the dynamic variable $x(t)$, introduced by the multiplicative linear noise $\xi(t)$. On the other hand, for $c\neq 0$ and positive as assumed here, it happens that below marginality (when $\beta<0$) a negative modification of the potential due to the quadratic term for $x<0$ (the minimum sinks) may be compensated by an enhanced diffusion and analogously, a smaller diffusion for $x>0$ would be balanced by a positive modification of the potential (the maximum is lowered). The final result would then appear as a negative contribution: the MFPT decreases. On the contrary, this favorable balance cannot be utilized to favor relaxation conditions above marginality (when $\beta>0$) when both a steeper potential appears together with an enhanced diffusion for $x<0$, whereas both effects turn out to be negative to favor passage time for $x>0$ as they correspond to a flatter potential with a smaller diffusion, resulting in this case in a positive contribution to the MFPT.

2. *Cubic approximation to the effective potential with constant diffusion*

Under this level of approximation the MFPT is formally expressed, analogously to what happens in the standard cubic marginal situation, in terms of the universal function $\Phi(k)$ whose redefined variable k incorporates in two different ways the most significant multiplicative noise effects [see Eq. (2.20)]. The first and most important one appears through U_1 [defined in Eq. (2.16)], and can be directly associated to the negative contribution with respect to the pure additive case which was already identified in the constant term of the drift $F(x)$ in Eq. (2.6). It is then easy to understand that the correction $-2d\epsilon$, resulting in a modification itself of the strict marginal conditions, either raises the barrier height below marginality or flattens the potential above it. On the other hand, the contribution originated in the nonconstant diffusion term appears explicitly in the effective potential through U_2 in Eq. (2.16).

3. *Cubic approximation to the effective potential with first-order diffusion*

The most general comment we want to make in relation to the result (2.22) concerns the generality of the $\epsilon^{-1/3}$ behavior. Actually, (2.22), analogously to what was observed within the systematic small multiplicative noise expansion (2.7), permits us to appreciate the robustness of the $\epsilon^{-1/3}$ behavior characteristic of the MFPT at any order of the expansion in d. In addition, note the antisymmetric character of the first-order correction introduced in the explicit consideration of a nonconstant diffusion term as expressed in Eq. (2.21). It implications will be discussed in Sec. III C when comparing approximated and exact results for the MFPT.

B. Exact results

Exact results corresponding to the numerical evaluation of the Eq. (2.3) for a particular set of parameters ($a=1$, $b=2$, $c=1$, $x_0=-0.25$, $x_F=0.25$), are shown in Figs. 1–3. Figures 1(a) and 1(b) correspond to the results obtained with pure additive fluctuations ($d=0$), and respectively reproduce the variations of the MFPT with the intensity of the noise ϵ at different β and with respect to the marginality parameter β at different ϵ. Figure 1(a) clearly demonstrates the different nature of the relaxational dynamics. Above marginality ($\beta>0$) the relaxations are basically deterministic; their corresponding time

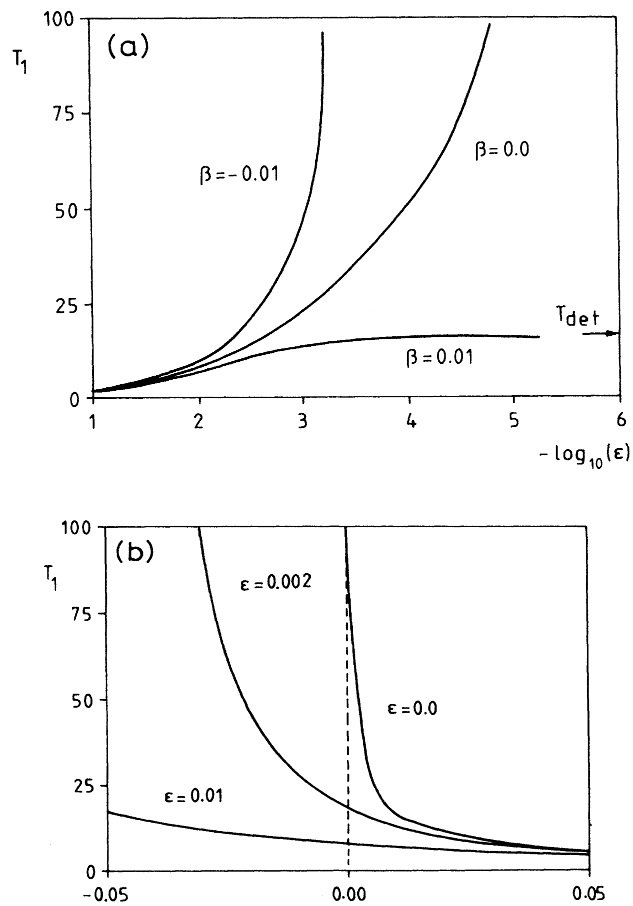

FIG. 1. Exact results for the MFPT corresponding to the stochastic dynamics given in (1.2) with pure additive noise ($d=0$) for the set of parameters given in the text. (a) T_1 vs ϵ at different β; (b) T_1 vs β at different ϵ.

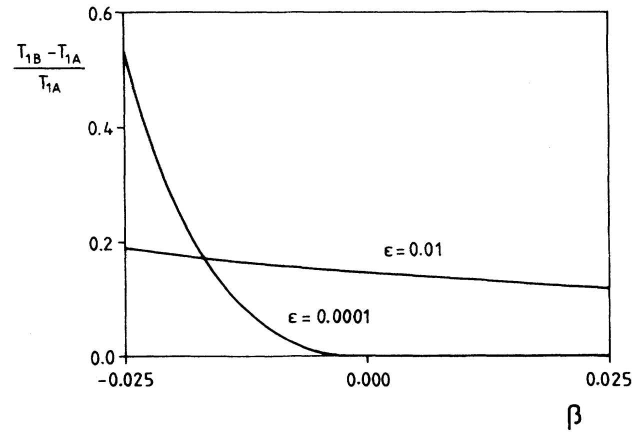

FIG. 2. Relative differences for the exact MFPT between the pure additive noise ($d=0$, noted T_{1A}) and both additive and multiplicative fluctuations ($d=1$, noted T_{1B}), plotted vs β at different ϵ. Values of the parameters as in Fig. 1.

scale softly increases with ϵ^{-1} until saturation is achieved at the corresponding deterministic times T_{det}. On the other hand, below marginality ($\beta<0$) and especially for small values of ϵ, the relaxation process proceeds via activation by fluctuations and the corresponding values of the MFPT diverges as $\epsilon\rightarrow 0$. A complementary representation of this different behavior is shown in Fig. 1(b). In particular, the well-known critical slowing down phenomenon appearing as one approaches marginality from above ($\beta\rightarrow 0^+$) is clearly manifested for deterministic relaxations.

Figures 2 and 3 show the most noticeable effects introduced by multiplicative fluctuations. Relative differences between the pure additive situation (noted A) and the additive plus multiplicative situation (noted B), the relative intensity between both fluctuations fixed arbitrarily at $d=1$ are shown in Fig. 2 for different values of β and ϵ. The first important conclusion is that multiplicative fluctuations, for large values of the relative intensity d, always slow down the relaxation dynamics, this effect being particularly significant below marginality and for small values of ϵ. We interpret this behavior as a clear signature of what, as stated previously, is the most important effect introduced by multiplicative noise on relaxational dynamics close to marginality: the modification of the potential, which is flatter above marginality and with a larger barrier below it. In particular, this effect should be crucially enhanced under conditions of relaxation mediated by fluctuations ($\beta<0$) and especially when the barrier crossing introduces larger and larger rate limiting steps on the relaxational dynamics, i.e., as $\epsilon\rightarrow 0$.

Quantitative effects introduced by multiplicative noise are better analyzed when taking d as a variable parameter. This is depicted in Fig. 3 for different values of β and ϵ. Again the different nature of the relaxational process, either deterministic for β positive or activated by fluctuations for negative values of β and especially small values of ϵ, manifests itself in the different behavior of the MFPT as the intensity of the multiplicative fluctuations increases. When the relaxation is essentially deterministic, the MFPT increases monotonously with d, this variation being unappreciable for very small values of ϵ. On the other hand, when the relaxation is definitively mediated by fluctuations the behavior of the MFPT with d is not monotonous, showing a minimum at an intermediate value of d that depends on β and ϵ [see Fig. 3(c)]. If the general trend of increasing MFPT as d increases is easily interpreted in terms of the modification of the relaxational potential, directly associated with the drift modifications appearing in Eq. (2.6), the small negative slope of T_1 with d as d increases from zero in Fig. 3(c) should be explained in terms of the nonconstant effective diffusion effect which may play a significant role when the extremely small values of d and ϵ result in a negligible modification of the potential barrier.

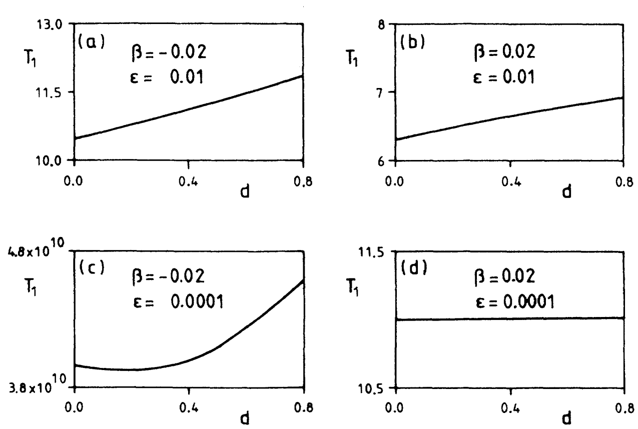

FIG. 3. Different behavior of the exact MFPT in varying the multiplicative noise intensity parameter d, at different values of β and ϵ. Values of the parameters as in Fig. 1. [Notice the largely different scale used in case (c).]

C. Comparison between approximate and exact results

Explicit numerical results corresponding to the approximate procedures proposed in Sec. II B to compute the MFPT were obtained for the same relaxation conditions and range of values of the relative noise intensity d as in Fig. 3. In general, the relative differences with respect to the exact results depend on d, with maximum discrepancies of the order of 30% within the considered range of values of d. However, this comparison is far more significant when fluctuations are expected to more largely influence the behavior of the relaxation dynamics, i.e., below marginality ($\beta<0$). So, we essentially restrict ourselves to these situations, and explicitly to the pair of noise intensities $\epsilon=10^{-2}$ and 10^{-4} considered above [cases (a) and (c) in Fig. 3]. In Fig. 4 we plot a convenient representation of the effect of multiplicative fluctuations as described by the different procedures, either approximate or exact, discussed in Sec. II. The differences between T_1 and $T_1(d=0)$, relative to the exact measure of the MFPT for $d=0$ [noted

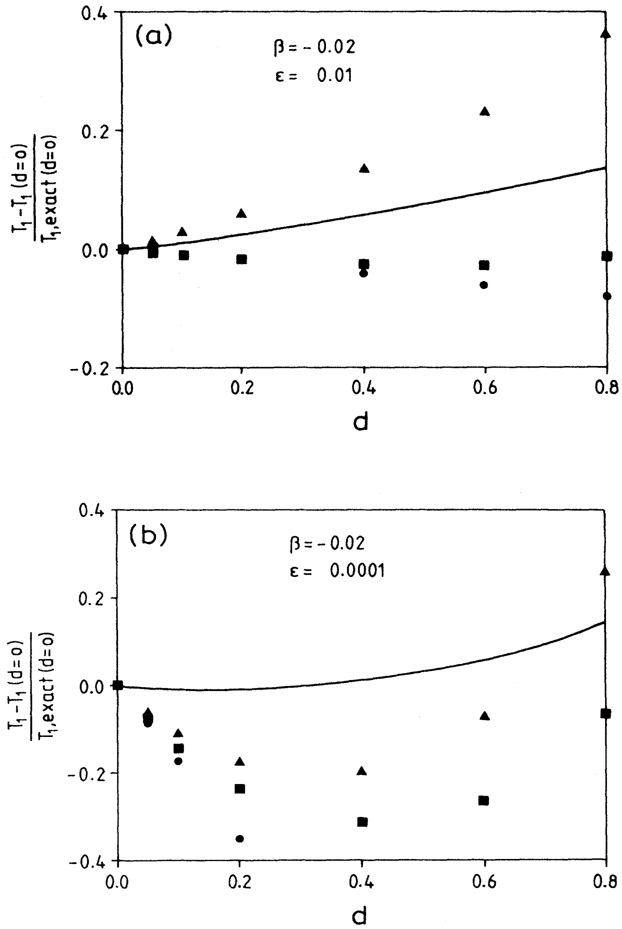

FIG. 4. Relative differences between T_1 and $T_1(d=0)$ plotted vs d, at different values of β and ϵ, for the exact and approximated methods [solid line, exact results; ●, method (i); ▲, method (ii); ■, method (iii)]. Values of the parameters as in Fig. 1.

$T_{1,\text{exact}}(d=0)$], are plotted versus d. In both cases 4(a) and 4(b) the systematic small multiplicative noise expansion [method (i)] leads to negative corrections considerably larger for small values of ϵ. Its linear behavior (as commented above in Sec. III A) can be traced down directly to Eq. (2.7). With respect to the procedure based on a cubic truncation to the effective potential with constant diffusion [method (ii)] it is worth noting that the MFPT follows exactly the same behavior as was commented upon in Sec. III B for the exact results, both in cases of small and large values of ϵ. This can be easily understood if we remember that either the effect on the potential barrier as well as on the nonconstant diffusion are simultaneously retained through the intrinsic parameter k of the scaling function $\Phi(k)$. Finally, when taking additionally a first-order explicit correction to the nonconstant diffusion [method (iii] what we obtain are determinations of the MFPT that again show analogous trends of behavior as in the preceding and exact procedures, but with systematic lower values. This is probably due to an

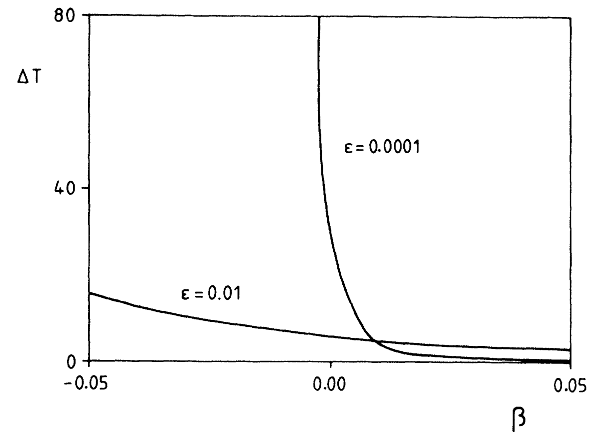

FIG. 5. Exact results for the square root of the FPTV corresponding to the stochastic dynamics given in (1.2) with pure additive noise ($d=0$) plotted vs β at different ϵ. Values of the parameters as in Fig. 1.

overestimation of the role of the diffusion in favoring the crossing over the potential barrier, which directly originates in the antisymmetric nature of the contribution there introduced as $4dx/\epsilon$, which respectively results in larger (lower) values of $D(x)$ for negative (positive) values of the relaxational variable x.

IV. RESULTS FOR THE FIRST-PASSAGE-TIME VARIANCE

The discussion of the results for the FPTV corresponding to the approximate procedures quoted in Sec. II B would be based on the same arguments employed above for the MFPT. Consequently, we will concentrate in this section on the analysis of the exact results for the FPTV. Actually, we will refer to the square root of the FPTV, noted ΔT.

Exact results corresponding to the numerical evalua-

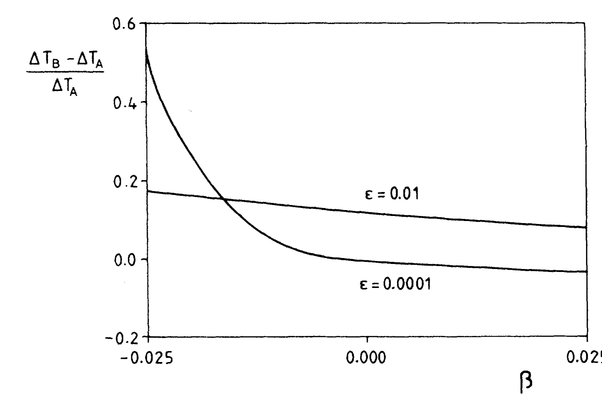

FIG. 6. Relative differences for the exact squre root of the FPTV between the pure additive noise ($d=0$, noted ΔT_A) and both additive and multiplicative fluctuations ($d=1$, noted ΔT_B), plotted vs β at different ϵ. Values of the parameters as in Fig. 1.

tion of the Eq. (2.4) for the same particular set of parameters adopted for the MFPT are shown in Figs. 5–7. Figure 5 again corresponds to the results obtained with pure additive fluctuations ($d=0$). Once more it clearly shows the different nature of the relaxations above and below marginality. Note in particular the different behavior of ΔT with ϵ in going from positive to negative values of β. Figures 6 and 7 reproduce the results arising in multiplicative noise terms. Relative differences for ΔT between the pure additive situation (noted A) and the additive plus multiplicative situation with $d=1$ (noted B), are shown in Fig. 6. Note, and this is to our understanding a significant effect introduced by multiplicative fluctuations, that both positive and negative values are found, this behavior being clearly different with respect to what was found for the MFPT at such large values of d. Larger values of ΔT originating in multiplicative fluctuations are typical of situations where the modification of the potential is determinant for the relaxational dynamics, i.e., for larger values of ϵ and especially for negative values of β even at small ϵ ($\epsilon=10^{-4}$). For such small values of ϵ, however, negative contributions to ΔT above marginality should be interpreted as due to the nonconstant effective diffusion. Note in this respect that according to (2.6), $D(x)$ is smaller than in the pure additive case for positive values of x that correspond to the more lasting portion of the relaxational dynamics above marginality due to the asymmetric nature of the potential (2.5).

Finally, Fig. 7 contains results for different values of d. A monotonically positive variation of ΔT with d appears to be the generic behavior, with the only exceptions applying to those cases for which the nonconstant diffusion is the predominant effect, i.e., for small values of ϵ and relaxations above marginality or below marginality for d very small.

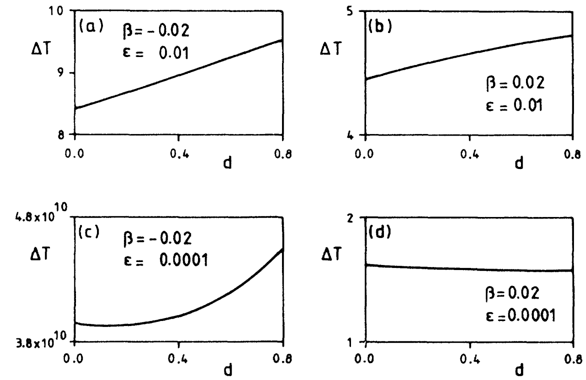

FIG. 7. Different behavior of the exact square root of the FPTV in varying the multiplicative noise intensity parameter d, at different values of β and ϵ. Values of the parameters as in Fig. 1. [Notice the largely different scale used in case (c).]

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