Optimal generalized quantum measurements for arbitrary spin systems

A. Acín, J. I. Latorre, and P. Pascual
Departament d'Estructura i Constituents de la Matèria and IFAE, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain
(Received 15 April 1999; published 14 January 2000)

Positive-operator-valued measurements on a finite number of N identically prepared systems of arbitrary spin J are discussed. Pure states are characterized in terms of Bloch-like vectors restricted by a $SU(2J+1)$ covariant constraint. This representation allows for a simple description of the equations to be fulfilled by optimal measurements. We explicitly find the minimal positive-operator-valued measurement for the $N=2$ case, a rigorous bound for $N=3$, and set up the analysis for arbitrary N.

PACS number(s): 03.65.Bz, 03.67.$-$a

I. INTRODUCTION

A measurement on a quantum-mechanical system only provides partial information on the measured state. Even in the case where N identical copies of the system are available, the information which can be retrieved remains bounded. This fact can be quantified using the averaged fidelity based on the following general idea. Given N identical copies of a system, we may consider a two-step procedure to rate the fidelity of a measuring apparatus. First, we set up a generalized quantum-mechanical measurement [or positive-operator-valued measurement (POVM) [1,2]]. Upon performing a measurement, its outcome provides the basis for a best guess about the incoming state. The averaged fidelity quantifies how close the final guess is from the original state averaging over the latter. For any finite number N of copies of a spin J pure state system, the average fidelity is proven to be bounded by [3]

$$\overline{f}(N,J)=\frac{N+1}{N+2J+1}. \tag{1}$$

The issue at stake remains to devise the optimal and minimal measuring strategy for any quantum system.

Explicit constructions of optimal and minimal generalized quantum-mechanical measurements of spin-$\frac{1}{2}$ systems have been presented recently in Refs. [4–8]. The detailed construction is subtle and depends on whether the original system is in a pure or mixed state. The simplest case corresponds to measuring a spin-$\frac{1}{2}$ system known to be in a pure state. A generalized measurement can be constructed as a resolution of the identity made with rank-1 Hermitian operators, which are in turn built from the direct product of a given state,

$$I=\sum_{r=1}^{n} c_r^2|\Psi_r\rangle^N\, {}^N\langle\Psi_r|, \tag{2}$$

where I is then the identity in the maximal spin subspace. The important—and of possible future practical relevance—result is that the maximum averaged fidelity is attained with a finite number of operators [6]. Upon a case-by-case analysis, it is found that the minimum number, n, of such operators is a function of N and is given in the table:

N	1	2	3	4	5
n	2	4	6	10	12

The explicit form of Eq. (2) for the above cases can be found in Ref. [7].

The far more involved case of spin-$\frac{1}{2}$ mixed states has also been worked out in Ref. [8]. At variance with the pure state case, the closed expression for the maximum averaged fidelity depends on what the unbiased *a priori* distribution of density matrices is. Yet, explicit solutions for optimal measurements are found. Some remarkable properties emerge along the new construction. Let us briefly mention a few. Optimal measurements turn out to be structured using projectors on total spin eigenspaces and, within each eigenspace, on maximal spin component is some direction. This allows for a reuse of minimal and optimal results from the pure state case. Also, beyond two copies, some projectors are not of rank 1.

Explicit constructions of optimal minimal measurements are so far restricted to spin-$\frac{1}{2}$ systems, either pure or mixed. It is the purpose of this paper to extend this analysis for arbitrary spin pure states. A number of nontrivial issues must be faced at the outset. For instance, progress in the spin-$\frac{1}{2}$ case was triggered by the appropriate use of the Bloch vector labeling of density matrices associated to spinors. We shall resort to a similar representation in the case of arbitrary spin states, using representations of $SU(2J+1)$. The equivalent of a Bloch vector will be shown to obey a covariant restriction. This extra work will allow for a unified general setting of the problem of optimal measurements of arbitrary spins.

Finding explicit minimal optimal measurements remains a matter of case-by-case analysis. We shall provide explicit bounds for the minimal number of projectors, n, in POVMs. The case of $N=2$ will be fairly complete. Higher number of copies still need further ingenuity to get rigorous bounds.

II. AVERAGED FIDELITY

Consider a spin J particle which is in an unknown pure state $|\Psi\rangle$,

$$|\Psi\rangle=\begin{pmatrix} x_1+iy_1 \\ x_2+iy_2 \\ \cdots \\ x_D+iy_D \end{pmatrix}, \tag{3}$$

where $D=2J+1$ and the normalization of the state imposes $\Sigma_{i=1,\ldots,D}(x_i^2+y_i^2)=1$. Of course, we may use a different parametrization, e.g.,

$$|\Psi\rangle=\begin{pmatrix} \cos\phi \\ \sin\phi(x_2+iy_2) \\ \cdots \\ \sin\phi(x_D+iy_D) \end{pmatrix}, \tag{4}$$

with $0\leqslant\phi\leqslant\pi/2$ and $\Sigma_{i=2,\ldots,D}(x_i^2+y_i^2)=1$. Using this second parametrization and following Ref. [9] it is possible to prove that the volume element in the space of these states is

$$dV_D=4(\sin\phi)^{2D-3}\cos\phi\, d\phi\, dS_{2D-3}, \tag{5}$$

where dS_{2D-3} corresponds to the standard volume element on S_{2D-3}. The total volume is

$$V_D=\frac{4\pi^{D-1}}{(D-1)!}. \tag{6}$$

Given N identical copies of the arbitrary spin state, we have

$$|\Psi\rangle^N\equiv|\Psi\rangle\otimes|\Psi\rangle\otimes\cdots^N\cdots\otimes|\Psi\rangle. \tag{7}$$

A measurement on this enlarged system will bring richer information on $|\Psi\rangle$ than N separate measures on its respective copies [10].

Setting a generalized quantum measurement consists in providing a resolution of the identity of the type

$$\sum_{i=1}^{n} c_r^2|\Psi_r\rangle^{N\ N}\langle\Psi_r|+P_N=I, \tag{8}$$

where P_N is the projector on the space different from the one spanned from states of the form given in Eq. (7). We already have all the necessary elements to define and compute the averaged fidelity. Upon measuring $|\Psi\rangle^N$ with the above POVM, a given outcome labeled by r will result with probability $|{}^N\langle\Psi|\Psi_r\rangle^N|^2$. The natural guess for the initial pure state is, then, $|\Psi_r\rangle$ (this is only the best strategy if the initial state is known to be pure; the best guess for a mixed state is not the same state as the outcome of the POVM [8]). The overlap of this guess with the original state is just $|\langle\Psi|\Psi_r\rangle|^2$. The averaged or mean fidelity is defined as the product of the probability for r being triggered times the overlap between the ensuing guess and the original state, averaged over all possible initial unknown states,

$$\overline{f}(N,J)\equiv\frac{1}{V_{2J+1}}\sum_{r=1}^{n} c_r^2\int_0^{\pi/2}d\phi(\sin\phi)^{4J-1}\cos\phi \times\int dS_{4J-1}|{}^N\langle\Psi|\Psi_r\rangle^N|^2|\langle\Psi|\Psi_r\rangle|^2. \tag{9}$$

To evaluate the above expression, it is convenient to use the freedom to choose the integration variables to set each individual $|\Psi_r\rangle$ as a spinor with only a nonvanishing first component. Then,

$$\overline{f}(N,J)=\frac{1}{V_{2J+1}}\sum_{r=1}^{n} c_r^2\int_0^{\pi/2}d\phi(\sin\phi)^{4J-1} \times(\cos\phi)^{2N+3}S_{4J-1}. \tag{10}$$

We finally get

$$\overline{f}(N,J)=\frac{(2J)!(N+1)!}{(2J+N+1)!}\sum_{r=1}^{n} c_r^2. \tag{11}$$

This sum is easily calculated. It is just the dimension of the space spanned by the totally symmetric tensor of order N whose indices can take $2J+1$ values,

$$\sum_{r=1}^{n} c_r^2=\frac{(2J+N)!}{N!(2J)!}. \tag{12}$$

Thus,

$$\overline{f}(N,J)=\frac{N+1}{N+2J+1}, \tag{13}$$

which corresponds to Eq. (1) and was obtained in Ref. [3] using different techniques.

III. GENERALIZED BLOCH FORM OF ARBITRARY SPIN PURE STATES

It is sometimes useful to represent the state of a spin-$\frac{1}{2}$ system using the Bloch representation,

$$\rho=\frac{1}{2}I+\frac{1}{2}\vec{b}\cdot\vec{\sigma}, \tag{14}$$

where $\vec{b}$ is a vector existing within the unit sphere. Pure states correspond to the surface of the sphere, that is, $\vec{b}^2=1$. A similar but more complicated construction is possible for arbitrary spin particles.

Consider a pure state of a spin J particle. One may represent it using, e.g., Eq. (3). Alternatively we may construct its associated density matrix and write

$$\rho=\frac{1}{2J+1}I+\sqrt{\frac{J}{2J+1}}n_a\lambda_a,\quad a=1,\ldots,4J(J+1), \tag{15}$$

where λ_a are the generators of the SU$(2J+1)$ normalized by

$$\mathrm{Tr}(\lambda_a\lambda_b)=2\delta_{ab}, \tag{16}$$

and $\hat{n}$ is the normalized vector that plays the role of a generalized Bloch vector. The coefficients in Eq. (15) are chosen in such a way that $\mathrm{Tr}\,\rho=\mathrm{Tr}\,\rho^2=1$.

A simple counting of degrees of freedom shows that a spin J pure state is described by $4J$ real parameters whereas the generalized Bloch vector carries $4J(J+1)-1$. A mismatch appears for $J>\frac{1}{2}$, which implies that severe constraints must limit the subspace of valid vectors $\hat{n}$. Indeed, pure states must verify $\rho=\rho^2$, which translates into

$$d_{abc}n_an_b=\frac{2J-1}{\sqrt{J(2J+1)}}n_c \tag{17}$$

when Eq. (15) is used and where d_{abc} are the completely symmetric symbols associated to $\mathrm{SU}(2J+1)$, defined through the anticommutator of the generators of the group [11],

$$\{\lambda_a,\lambda_b\}=\frac{4}{2J+1}\delta_{ab}I+2\,d_{abc}\lambda_c, \tag{18}$$

which verify

$$d_{abb}=0,\quad d_{abc}d_{dbc}=\frac{(2J-1)(2J+3)}{2J+1}\delta_{ad}. \tag{19}$$

Some useful properties of the vectors $\hat{n}$ follow from the above general covariant constraint (17),

$$\begin{aligned} d_{abc}n_an_bn_c&=\frac{2J-1}{\sqrt{J(2J+1)}},\\ d_{abe}d_{cde}n_an_bn_cn_d&=\frac{(2J-1)^2}{J(2J+1)}, \end{aligned} \tag{20}$$

where it is clear that for spin $J=\frac{1}{2}$ the simple structure of SU(2) causes the d symbols to vanish and the right-hand side to be identically zero.

We can also deduce the useful constraint which follows from the positivity of the square of the scalar product of two arbitrary spin J pure states, which reads

$$|\langle\Psi|\Psi'\rangle|^2=\mathrm{Tr}(\rho\rho')=\frac{1}{2J+1}(1+2J\hat{n}\cdot\hat{n}')\geqslant 0. \tag{21}$$

Generalized Bloch vectors are thus constrained to have scalars products bounded by

$$\hat{n}\cdot\hat{n}'\geqslant-\frac{1}{2J}. \tag{22}$$

Two pure states are orthogonal then when the scalar product of their generalized Bloch vectors satisfies the equality in Eq. (22).

Let us illustrate the construction of a Bloch vector for the $J=1$ example. In this case, the density matrix representing the system can be connected to the standard spinorlike representation. For instance, taking $J=1$ it is easy to see that the generalized Bloch vector corresponds to Eq. (3) if

$$\begin{aligned} &n_1=\sqrt{3}(x_1x_2+y_1y_2),\quad n_2=\sqrt{3}(x_1y_2-x_2y_1),\\ &n_4=\sqrt{3}(x_1x_3+y_1y_3),\quad n_5=\sqrt{3}(x_1y_3-x_3y_1),\\ &n_6=\sqrt{3}(x_2x_3+y_2y_3),\quad n_7=\sqrt{3}(x_2y_3-x_3y_2),\\ &n_3=\frac{\sqrt{3}}{2}[x_1^2+y_1^2-(x_2^2+y_2^2)],\quad n_8=\frac{1}{2}[1-3(x_3^2+y_3^2)], \end{aligned} \tag{23}$$

and λ_a are taken in the Gell-Mann representation of SU(3) [11]. Note that symmetric and antisymmetric combinations of the spinor components build the raising and lowering generators, whereas the Casimir combinations correspond to diagonal ones. Generalization of this construction for arbitrary spin J based on the $\mathrm{SU}(2J+1)$ group is straightforward.

The advantage of using a generalized Bloch representation for arbitrary spin pure states will become apparent shortly, when all our equations will be manifestly $\mathrm{SU}(2J+1)$ covariant and real. This is equivalent to note that the difference between working with spinors, which exist in the fundamental representation of the group, or with Bloch vectors, which exist in the adjoint representation, is that the second is real.

IV. OPTIMAL MEASUREMENTS FOR A SINGLE COPY OF A SYSTEM

Let us go back to the construction of a generalized quantum measurement of arbitrary spin systems. We basically need to solve for the minimal set of $|\Psi_r\rangle$ states such that Eq. (8) is fulfilled. We have found it convenient to project out the P_N piece using

$$\sum_{r=1}^{n}c_r^2|{}^N\langle\Psi|\Psi_r\rangle^N|^2=1,\ \ \forall\ |\Psi\rangle. \tag{24}$$

This equation can also be written in the Bloch representation as

$$\sum_{r=1}^{n}c_r^2\frac{1}{(2J+1)^N}\left(1+2J\sum_a n_an_a(r)\right)^N=1, \tag{25}$$

where every $\hat{n}(r)$ corresponds to a pure state in the POVM and $\hat{n}$ to the original pure state.

It is clear that the simplest situation we may face corresponds to having a single copy of the unknown state. The optimal and minimal measurement for such a case is, of course, known to correspond to a von Neumann measurement. We shall, however, proceed in a more general way and set the *modus operandi* for the more elaborate cases as devised in Ref. [7].

Equation (24) with $N=1$ can be demonstrated (with a little effort) to be equivalent to

$$\sum_{r=1}^{n} c_r^2[x_j(r)x_k(r)+y_j(r)y_k(r)]=\delta_{jk},$$

$$\sum_{r=1}^{n} c_r^2[x_j(r)y_k(r)-x_k(r)y_j(r)]=0, \quad j,k=1,\ldots,2J+1. \tag{26}$$

Using the insight given by Eq. (25) and the result of Eq. (12), this set of $(2J+1)^2$ independent equations can be rewritten in terms of the Bloch vector as

$$\sum_{r=1}^{n} c_r^2=2J+1,$$

$$\sum_{r=1}^{n} c_r^2 n_a(r)=0, \tag{27}$$

where it is important to remember the constraints limiting $\hat{n}(r)$. For instance, scalar products between any pair $\hat{n}(r)\cdot\hat{n}(s)\geqslant-1/(2J)$, thus

$$\sum_{r\neq s} c_r^2\left(\frac{1}{2J}+\hat{n}(r)\cdot\hat{n}(s)\right)\geqslant 0. \tag{28}$$

Using the set of equations (27), the above inequality can be transformed into

$$1-c_s^2\geqslant 0, \quad \forall\ s=1,\ldots,n. \tag{29}$$

Summing over all s, we get

$$n\geqslant 2J+1. \tag{30}$$

This bound is indeed saturated by a von Neumann measurement, that is,

$$n_{\min}=2J+1,$$

$$c_s^2=1\ \ \forall\ \ s, \quad \hat{n}(r)\cdot\hat{n}(s)=-\frac{1}{2J},\ \ \forall\ r\neq s. \tag{31}$$

The explicit standard construction for $J=1$ is recovered as the solution to this $N=1$ POVM,

$$|\Psi_1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |\Psi_2\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |\Psi_3\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{32}$$

Or, alternatively,

$$\hat{n}(1)=\left(0,0,\frac{\sqrt{3}}{2},0,0,0,0,\frac{1}{2}\right),$$

$$\hat{n}(2)=\left(0,0,-\frac{\sqrt{3}}{2},0,0,0,0,\frac{1}{2}\right), \tag{33}$$

$$\hat{n}(3)=(0,0,0,0,0,0,0,-1).$$

We are now in a position to appreciate the advantage of resorting to a Bloch-like parametrization. It is easier to deal with Eq. (27) than with Eq. (26). The use of $\hat{n}(r)$ introduces a simple covariant, yet constrained, formulation. Some extra subtleties will play a relevant role in the more complicated cases.

V. OPTIMAL MEASUREMENTS FOR THE $N=2$ CASE

Let us face the case where $N=2$ identical copies of the system are at our disposal. Following the same reasoning as before, we start by writing Eq. (24) in terms of the basic spinor representation. This leads to

$$\sum_{r=1}^{n} c_r^2[x_i(r)x_j(r)+y_i(r)y_j(r)][x_k(r)x_l(r)+y_k(r)y_l(r)]$$

$$=\frac{1}{4}(2\,\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

$$\sum_{r=1}^{n} c_r^2[x_i(r)y_j(r)-x_j(r)y_i(r)][x_k(r)y_l(r)-x_l(r)y_k(r)]$$

$$=\frac{1}{4}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),$$

$$\sum_{r=1}^{n} c_r^2[x_i(r)x_j(r)+y_i(r)y_j(r)][x_k(r)y_l(r)-x_l(r)y_k(r)]$$

$$=0. \tag{34}$$

The system is now quadratic in the basic structures appearing linearly in the $N=1$ case. Using the Bloch vector representation, these $(2J+1)^2(2J^2+2J+1)$ equations can be recast into

$$\sum_{r=1}^{n} c_r^2=(2J+1)(J+1)\equiv B,$$

$$\sum_{r=1}^{n} c_r^2 n_a(r)=0, \tag{35}$$

$$\sum_{r=1}^{n} c_r^2 n_a(r)n_b(r)=B\,\frac{1}{4J(J+1)}\delta_{ab}.$$

A general pattern is emerging. Higher N optimal measurements demand a finer grained resolution of the identity. The Bloch vectors are required to satisfy isotropy conditions in $SU(2J+1)$ group space. The determination of the factor $1/[4J(J+1)]$ has been done using the fact that $\hat{n}$ is a normalized vector and Eq. (12). It is easy to verify that the set of equations (35) provides a solution for Eq. (25).

From the above basic set of equations, it is easy to get

$$\sum_{r\neq s}^{n} c_r^2 = B - c_s^2,$$

$$\sum_{r\neq s}^{n} c_r^2 \hat{n}(r)\cdot\hat{n}(s) = -c_s^2, \qquad (36)$$

$$\sum_{r\neq s}^{n} c_r^2[\hat{n}(r)\cdot\hat{n}(s)]^2 = B\frac{1}{4J(J+1)} - c_s^2.$$

Then we may argue that

$$\sum_{r\neq s} c_r^2[b+\hat{n}(r)\cdot\hat{n}(s)]^2 \geqslant 0, \qquad (37)$$

which is extremized by $b = c_s^2/(B-c_s^2)$ leading to

$$n \geqslant (2J+1)^2, \quad c_s^2 \leqslant \frac{J+1}{2J+1}, \ \forall \ s. \qquad (38)$$

For $J=\frac{1}{2}$ this bound agrees with the known solution of the tetrahedron (see the Introduction and Ref. [7]) and generalizes it in the following sense. The solution $n=(2J+1)^2$ also forces all scalar products to be $\hat{n}(r)\cdot\hat{n}(s) = -1/[4J(J+1)]$. This corresponds to a hypertetrahedron in $(2J+1)^2-1$ dimensions, exactly those of the adjoint representation of SU$(2J+1)$. Let us just write the explicit solution for $J=1$,

$$\hat{n}(1) = \left(\frac{1}{2},\frac{\sqrt{3}}{2},0,0,0,0,0,0\right),$$

$$\hat{n}(2) = \left(\frac{1}{2},-\frac{\sqrt{3}}{4},\frac{3}{4},0,0,0,0,0\right),$$

$$\hat{n}(3) = \left(\frac{1}{2},-\frac{\sqrt{3}}{4},-\frac{3}{4},0,0,0,0,0\right),$$

$$\hat{n}(4) = \left(-\frac{1}{4},0,0,0,\frac{\sqrt{6}}{4},\frac{\sqrt{3}}{4},-\frac{\sqrt{6}}{4},0\right),$$

$$\hat{n}(5) = \left(-\frac{1}{4},0,0,\frac{3\sqrt{2}}{8},-\frac{\sqrt{6}}{8},\frac{\sqrt{3}}{4},\frac{\sqrt{6}}{8},-\frac{3\sqrt{2}}{8}\right), \qquad (39)$$

$$\hat{n}(6) = \left(-\frac{1}{4},0,0,-\frac{3\sqrt{2}}{8},-\frac{\sqrt{6}}{8},\frac{\sqrt{3}}{4},\frac{\sqrt{6}}{8},\frac{3\sqrt{2}}{8}\right),$$

$$\hat{n}(7) = \left(-\frac{1}{4},0,0,0,\frac{\sqrt{6}}{4},-\frac{\sqrt{3}}{4},\frac{\sqrt{6}}{4},0\right),$$

$$\hat{n}(8) = \left(-\frac{1}{4},0,0,-\frac{3\sqrt{2}}{8},-\frac{\sqrt{6}}{8},-\frac{\sqrt{3}}{4},-\frac{\sqrt{6}}{8},-\frac{3\sqrt{2}}{8}\right),$$

$$\hat{n}(9) = \left(-\frac{1}{4},0,0,\frac{3\sqrt{2}}{8},-\frac{\sqrt{6}}{8},-\frac{\sqrt{3}}{4},-\frac{\sqrt{6}}{8},\frac{3\sqrt{2}}{8}\right).$$

There is still the need to perform the nonobvious step of finding out whether this solution does correspond to a set of spin-1 states. For completeness we give this final form of the solution, that is, the explicit states $|\Psi_1\rangle$ through $|\Psi_9\rangle$ which form the POVM,

$$|\Psi_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |\Psi_2\rangle = \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\\ 0\end{pmatrix}, \quad |\Psi_3\rangle = \begin{pmatrix}\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\\ 0\end{pmatrix},$$

$$|\Psi_4\rangle = \begin{pmatrix}\frac{1}{2}\\ i\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \quad |\Psi_5\rangle = \begin{pmatrix}\frac{1}{2}\\ i\frac{1}{2}\\ -\frac{1}{2\sqrt{2}}+i\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix},$$

$$|\Psi_6\rangle = \begin{pmatrix}\frac{1}{2}\\ i\frac{1}{2}\\ -\frac{1}{2\sqrt{2}}-i\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix}, \qquad (40)$$

$$|\Psi_7\rangle = \begin{pmatrix}\frac{1}{2}\\ -i\frac{1}{2}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \quad |\Psi_8\rangle = \begin{pmatrix}\frac{1}{2}\\ -i\frac{1}{2}\\ -\frac{1}{2\sqrt{2}}+i\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix},$$

$$|\Psi_9\rangle = \begin{pmatrix}\frac{1}{2}\\ -i\frac{1}{2}\\ -\frac{1}{2\sqrt{2}}-i\frac{\sqrt{3}}{2\sqrt{2}}\end{pmatrix}.$$

Note that all the spinors have scalar products with modulus equal to $\frac{1}{2}$.

VI. OPTIMAL MEASUREMENTS FOR THE $N=3$ CASE

The systematics of our approach are already set. It is, however, in the case of three copies where a major difference between spin $\frac{1}{2}$ and higher spin systems appears. Following an analogous reasoning to that in the preceding sections, we get

$$\sum_{r=1}^{n} c_r^2 = \frac{(2J+3)!}{3!(2J)!} \equiv C,$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) = 0,$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) n_b(r) = C \frac{1}{4J(J+1)} \delta_{ab}, \tag{41}$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) n_b(r) n_c(r) = C \frac{1}{4J(J+1)(2J+3)} \times \left(\frac{2J+1}{J}\right)^{1/2} d_{abc}.$$

We have used Eqs. (12), (19), and (20) for determining the factor $1/[4J(J+1)(2J+3)][(2J+1)/J]^{1/2}$. Again it is easy to prove that Eqs. (41) verify Eq. (25).

For the first time the right-hand side of one of the equations displays a tensor structure based on the d symbol. Such a term would vanish for $J=\frac{1}{2}$ due to the simpler structure of SU(2), but is expected for higher spins [note that the conditions (20) are zero for spin $\frac{1}{2}$].

A bound on the number of projectors appearing in a optimal POVM can be obtained following the by now standard procedure of investigating manifestly positive combinations. In this case, starting from

$$\sum_{r\neq s} \left(\frac{1}{2J} + \hat{n}(r)\cdot\hat{n}(s)\right)[b + \hat{n}(r)\cdot\hat{n}(s)]^2 \geqslant 0, \tag{42}$$

one gets

$$n \geqslant (J+1)(2J+1)^2 \tag{43}$$

and $c_s^2 \leqslant (2J+3)/[3(2J+1)]$. That is, $n \geqslant 6$ for spin $\frac{1}{2}$ (which agrees with the known result in Ref. [7]), $n \geqslant 18$ for spin 1, $n \geqslant 40$ for spin $\frac{3}{2}$, etc. Saturating this bound is impossible for certain cases as implied by the following simple argument. If the bound were to be saturated, then Eq. (42) would become a restricting condition for all scalar products. Indeed, $\hat{n}(r)\cdot\hat{n}(s)$ is either $-1/(2J)$ or else $(2J-1)/[2J(2J+3)]$ for any pair $r\neq s$. If we fix any s and assume that the minimal solution carries p scalar products of the first type and q of the second, it follows that Eq. (41) imposes $p=\frac{1}{2}J(2J+1)^2$ and $q=\frac{1}{2}J(2J+3)^2$. For any J half-integer or even this causes no problem but for odd integer values of the spin this leads to noninteger pairs, which is absurd. Thus, in such a case, the bound cannot be saturated.

VII. CONCLUSIONS

We have presented explicit solutions for minimal optimal POVMs acting on arbitrary spin J systems for the case when two copies are available. For $N=3$ we have provided a rigorous bound. The key idea to simplify the analysis consists in using Bloch representation for pure arbitrary spin states. These vectors do not span a naive $(2J+1)^2-1$ sphere, but rather an intricate subspace defined through covariant restrictions. The power of such covariance makes the set of equations simple,

$$\sum_{r=1}^{n} c_r^2 = \frac{(2J+N)!}{N!(2J)!},$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) = 0,$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) n_b(r) = \frac{(2J+N)!}{N!(2J)!} \frac{1}{4J(J+1)} \delta_{ab}, \tag{44}$$

$$\sum_{r=1}^{n} c_r^2 n_a(r) n_b(r) n_c(r) = \frac{(2J+N)!}{N!(2J)!} \frac{1}{4J(J+1)(2J+3)} \times \left(\frac{2J+1}{J}\right)^{1/2} d_{abc},$$

$$\ldots .$$

In order to analyze a given case with N copies of the spin J particle, it is necessary to retain

$$\frac{[4J(J+1)+N]!}{N![4J(J+1)]!} \tag{45}$$

equations in the system, that is, as many rows in Eq. (44) as $N+1$.

Our results confirm the expected increase of needed projectors to build a POVM as the spin of the system increases. The instances analyzed, that is, $N=1,2,3$, seem to point at a dependence of the type

$$n_{\min} \sim J^N. \tag{46}$$

ACKNOWLEDGMENTS

We are grateful to R. Tarrach and G. Vidal for continuously sharing their insight with us. Financial support from CICYT, Contract No. AEN95-0590, and from CIRIT, Contract No. 1996GR00066 is acknowledged. A.A. acknowledges a grant from MEC.

[1] M. A. Neumark, C. R. Acad. Sci. URSS **41**, 359 (1943).
[2] A. Peres, Found. Phys. **20**, 1441 (1990).
[3] D. Bruß and C. Macchiavello, e-print quant-ph/9812016.
[4] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[5] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[6] R. Derka, V. Buzek, and A. K. Ekert, Phys. Rev. Lett. **80**, 1571 (1998); e-print quant-ph/9707028.
[7] J. I. Latorre, P. Pascual, and R. Tarrach, Phys. Rev. Lett. **81**, 1351 (1998); e-print quant-ph/9803066.
[8] G. Vidal, J. I. Latorre, P. Pascual, and R. Tarrach, Phys. Rev. A **60**, 126 (1999); e-print quant-ph/9812068.
[9] R. Schack, G. M. D'Ariano, and C. M. Caves, Phys. Rev. E **50**, 972 (1994).
[10] A. Peres and W. K. Wootters, Phys. Rev. Lett. **66**, 1119 (1991).
[11] P. Pascual and R. Tarrach, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin, 1984).