# Hadronic vacuum polarization and the Lamb shift

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(Received 17 December 1998)

Recent improvements in the determination of the running of the fine-structure constant also allow an update of the hadronic vacuum-polarization contribution to the Lamb shift. We find a shift of $-3.40(7)$ kHz to the $1S$ level of hydrogen. We also comment on the contribution of this effect to the determination by elastic electron scattering of the rms radii of nuclei. [S1050-2947(99)06305-2]

PACS number(s): 31.30.Jv

Hadronic vacuum polarization (VP) [1–4] contributes to several quantities that have elicited much recent interest: $g-2$ of the muon, the effective fine-structure constant at the energy scale of the $Z$-boson mass, the hyperfine splitting in muonium and positronium [5], and the energy levels of the hydrogen atom [6].

The shaded ellipse in Fig. 1 represents the creation and subsequent annihilation of arbitrary hadronic states by virtual photons on the left (with polarization $\mu$) and right (with polarization $\nu$). For (squared) momentum transfers $q^2$, comparable to the masses of various components in the shaded ellipse, the screening of charge that defines the vacuum polarization changes with $q^2$ and leads to an effective fine-structure constant that depends on $q$: $\alpha(q^2)$. If the entire unit in Fig. 1 is inserted as a vertex correction on a lepton, it will also affect $g-2$ of that lepton via an integral over $q^2$. Finally, at very small values of $q^2$, inserting the unit between an electron and a nucleus will lead to a shift in hydrogenic energy levels.

The $S$ matrix for the process sketched in Fig. 1 (with one VP insertion) has the form (in a metric where $p^2=m^2$)

$$S^{\mu\nu}=\left(\frac{-ie_0 g^{\mu\alpha}}{q^2}\right)[i\pi^{\alpha\beta}(q^2)]\left(\frac{-ie_0 g^{\beta\nu}}{q^2}\right), \tag{1}$$

and for the process without polarization (just a single-photon propagator) is $(-ie_0^2 g^{\mu\nu}/q^2)$. The strength is determined by the bare electric charge $e_0$ located at the two ends of Fig. 1. The polarization structure function $\pi^{\alpha\beta}(q^2)$ must be gauge invariant, which requires

$$\pi^{\alpha\beta}(q^2)=-(g^{\alpha\beta}q^2-q^\alpha q^\beta)\pi(q^2). \tag{2}$$

Coupling the $q^\alpha q^\beta$ term to any conserved current (e.g., an electron or nucleus) leads to vanishing results, and simplifies the tensor structure of $S^{\mu\nu}(\sim g^{\mu\nu})$. The sequence of $0,1,2,\ldots$ insertions of $\pi^{\alpha\beta}$ into a photon propagator (Fig. 1 shows one insertion) generates the geometric series

$$S^{\mu\nu}=\frac{-ig^{\mu\nu}e_0^2}{q^2}(1-\pi+\pi^2\cdots)=\frac{-ig^{\mu\nu}}{q^2}\frac{e_0^2}{1+\pi(q^2)}. \tag{3}$$

Clearly, at $q^2=0$ we expect $e_0^2/(1+\pi(0))$ to be $e^2$, the renormalized charge (squared). Since $\pi$ itself is proportional to $e_0^2$, we form $\Delta\pi(q^2)=\pi(q^2)-\pi(0)$, rearrange, and find, to first order in $e^2$,

$$S^{\mu\nu}=\frac{-ig^{\mu\nu}}{q^2}e^2(q^2), \tag{4}$$

where

$$e^2(q^2)=\frac{e^2}{1+\Delta\pi(q^2)} \tag{5}$$

is the (squared) effective charge at the scale $q^2$. Much recent work has been devoted to the numerical determination [1–4] of $e^2(q^2)/4\pi\equiv\alpha(q^2)$.

Our purpose is to treat vacuum polarization in hydrogenic ions with nuclear charge $Ze$. The momentum-space Coulomb potential generated by Eq. (5) is given by

$$V(q^2)=\frac{Ze^2(q^2)}{q^2}=\frac{-Ze^2(q^2)}{\mathbf{q}^2}\cong\frac{-Ze^2}{\mathbf{q}^2}-Ze^2\pi'(0), \tag{6}$$

where $q^2=-\mathbf{q}^2$ for this term in Coulomb gauge, and $\Delta\pi(q^2)\cong q^2\pi'(0)+\cdots$. The first term in Eq. (6) on the right is the usual Coulomb potential and the second is the vacuum polarization potential, which we write in configuration space as

$$V_{\mathrm{VP}}(\mathbf{r})=-(4\pi Z\alpha)\pi'(0)\delta^3(\mathbf{r}). \tag{7}$$

Thus for the $n$th $S$ state, the energy shift is given by

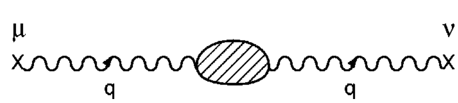


FIG. 1. Vacuum polarization insertion into a virtual photon propagator.

 

$$E_{\mathrm{VP}} \cong -(4\pi Z\alpha)\pi'(0)|\phi_n(0)|^2 = \frac{-4(Z\alpha)^4\pi'(0)\mu^3}{n^3}, \tag{8}$$

where $\mu$ is the hydrogenic reduced mass. The intrinsic nature of vacuum polarization is attraction, so we expect $\pi'(0) > 0$.

By slicing Fig. 1 through the polarization insertion, we can reexpress $\pi(q^2)$ as a dispersion relation, with an imaginary part proportional to $\sigma_h(q^2)$, the cross section for producing all hadron states in $e^+e^-$ collisions. A single subtraction then produces $\Delta\pi(q^2)$ in the form

$$\Delta\pi(q^2) = \frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{dt\,\mathrm{Im}[\Delta\pi(t)]}{t(t-q^2-i\epsilon)}, \tag{9}$$

where $\mathrm{Im}(\Delta\pi(t)) = t\sigma_h(t)/(4\pi\alpha(t))^2$. Utilizing all available $e^+e^-$ collision data (and some theory) permits an accurate interpolation of $\sigma_h(t)$, and $\Delta\pi(q^2)$ can be constructed numerically [1–4].

We require only $\pi'(0)$ for the hadronic VP (henceforth subscripted with $h$), which is given by the parameter $\ell_1$ in Sec. 1.5, and the error from Fig. 7, of Ref. [1] (note that our $\Delta\pi = -\Delta\alpha$ of Ref. [1]):

$$\pi_h'(0) = 9.3055(\pm 2.2\%)\times 10^{-3}\ \ \mathrm{GeV}^{-2}. \tag{10}$$

Equation (8) also applies to muon-pair vacuum polarization, for which $\pi_\mu'(0) = \alpha/(15\pi m_\mu^2)$, where $m_\mu$ is the muon mass. We therefore obtain

$$\pi_h'(0) = 0.671(15)\pi_\mu'(0) \equiv \delta_h \pi_\mu'(0), \tag{11}$$

and thus

$$E_{\mathrm{VP}}^{\mathrm{had}} = 0.671(15) E_{\mathrm{VP}}^{\mu}, \tag{12}$$

where, for $S$ states, Eq. (8) gives

$$E_{\mathrm{VP}}^{\mu} = \frac{-4\alpha(Z\alpha)^4\mu^3}{15\pi n^3 m_\mu^2}, \tag{13}$$

and a numerical value of $-5.07$ kHz for the $1S$ state of hydrogen. The much heavier $\tau$ lepton analogously contributes $-0.02$ kHz.

Previous values obtained for $\delta_h$ are displayed together with our value in Table I. Additional values were calculated in Refs. [10] and [6]. The estimate of Ref. [6] used only the $\rho$-meson contribution [5], which is known to give the largest fractional contribution to $\pi_h'(0)$, and results for $g-2$ of the muon are consistent with that fraction ($\sim 60\%$) [1]. All tabulated values are consistent with the more accurate Eq. (12). We also note that a noninteracting pion pair generates only $\sim 10\%$ of the total hadronic contribution.

TABLE I. Values of $\delta_h$.

| | Ref. [7] | Ref. [8] | Ref. [9] | This work |
|---|---|---|---|---|
| $\delta_h$ | 0.68 | 0.719(54) | 0.659(26) | 0.671(15) |

Finally, we repeat a caveat from Ref. [11]. In elastic-electron-scattering determinations of nuclear form factors (and hence their radii), the radiative corrections procedure [12,13] that is used to analyze the data corrects for $e^+e^-$ vacuum polarization, sometimes for the muon one, but typically not for the hadronic one. If one type of vacuum polarization is omitted, Eq. (5) then demonstrates that the effective measured form factor expressed in terms of $F_0$ (the true form factor) is

$$F_{\mathrm{eff}}(q^2) = \frac{F_0(q^2)}{1+\Delta\pi(q^2)}, \tag{14}$$

and hence the effective radius is

$$\langle r^2\rangle_{\mathrm{eff}}^{1/2} = [\langle r^2\rangle_0 - 6\pi'(0)]^{1/2}, \tag{15}$$

where $6\pi_h'(0) = 0.0022\ \ \mathrm{fm}^2$, for example. Although this is a very tiny effect, comparing a measured radius [$\langle r^2\rangle_{\mathrm{eff}}^{1/2}$ in Eq. (15)] with one determined from optical measurements *corrected for hadronic* VP [i.e., $\langle r^2\rangle_0^{1/2}$ from $F_0(q^2)$] would be inconsistent.

In summary, we have updated the calculation of the hadronic vacuum polarization correction in hydrogen using a recently obtained and more accurate value [1] for $\pi_h'(0)$. This leads to a shift of $-3.40(7)$ kHz in the $1S$ level of hydrogen. We also noted that elastic electron scattering from nuclei is not corrected for hadronic VP.

The work of J.L.F. was performed under the auspices of the U.S. Department of Energy. D.W.L.S. is grateful to the NSERC (Canada) for continued support under Research Grant No. SAPIN-3198. The work of J.M. was supported by DGES under Grant No. PB97-0915 (Spain). One of us (J.L.F.) would like to thank P. Mohr of NIST for a stimulating series of conversations, and L. Maximon of The George Washington University for information about radiative corrections.

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