Comment on "Canonical formalism for Lagrangians with nonlocality of finite extent"

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(Received 11 January 2002; revised manuscript received 10 July 2002; published 15 January 2003)

The paper by Woodward [Phys. Rev. A **62**, 052105 (2000)] claimed to have proved that Lagrangian theories with a nonlocality of finite extent are necessarily unstable. In this Comment we propose that this conclusion is false.

DOI: 10.1103/PhysRevA.67.016101 PACS number(s): 03.65.Ta, 11.10.Lm, 45.20.Jj

I. INTRODUCTION

In Ref. [1] a canonical formalism for nonlocal Lagrangians with a nonlocality of finite extent is established. It is compared with the Ostrogradski formalism [2] for local Lagrangians that depend on a finite number of derivatives of coordinates. One of its central conclusions is that Lagrangian systems with a nonlocality of finite extent have no "... possible phenomenological role [$\cdots$]. They have inherited the full Ostrogradskian instability ... ."

The aim of the present Comment is (i) to point out some defects in Ref. [1] (Sec. II) concerning the application of the variational principle that underlies the derivation of the nonlocal Eq. (2) from a Lagrangian; (ii) to stress the importance of the functional space where the variational problem is developed; this is also the functional space where the solutions must be searched (Sec. III); and (iii) to illustrate by two simple counterexamples (Sec. IV) that (a) Lagrangian systems containing derivatives of a higher order than the first are not necessarily unstable and (b) nonlocality of finite extent does not inevitably lead to instability.

II. THE NONLOCAL ACTION PRINCIPLE

Although the canonical formalism set up in Ref. [1] is derived on a general ground, it is basically illustrated by the simple nonlocal Lagrangian system

$$L[q](t)=\frac{1}{2}m\dot{q}^2\left(t+\frac{\Delta}{2}\right)-\frac{1}{2}m\omega^2q(t)q(t+\Delta) \qquad (1)$$

and the equation of motion for this Lagrangian is written as

$$\int_0^\Delta\frac{\delta L[q](t-r)}{\delta q(t)}\;dr=-m\left\{\ddot{q}(t)+\frac{1}{2}\omega^2q(t+\Delta)+\frac{1}{2}\omega^2q(t-\Delta)\right\}=0. \qquad (2)$$

It must be noticed that the latter equation as it reads does not properly correspond to the standard action principle of mechanics. Indeed, the latter states that [3] "The motion of an arbitrary mechanical system occurs in such a way that the action integral S becomes stationary for arbitrary possible variations of the configuration of the system, provided that the initial and final configurations of the system are prescribed."

On the other hand, the action integral whose variation would be the left-hand side of Eq. (2) is

$$S([q],t)=\int_0^\Delta drL[q](t-r)=\int_{t-\Delta}^t d\tau\;\;L[q](\tau) \qquad (3)$$

and Eq. (2) is equivalent to

$$\frac{\delta S([q],t)}{\delta q(t)}=0, \qquad (4)$$

where t is the same both in the numerator and the denominator. However, the Euler-Lagrange equation that follows from the action principle $\delta S=0$ is

$$\frac{\delta S([q],t)}{\delta q(t')}=0,\quad \forall\;\; t',$$

which is much more restrictive than Eq. (4).

Moreover, an equation like

$$-m\left\{\ddot{q}(t)+\frac{1}{2}\omega^2q(t+\Delta)+\frac{1}{2}\omega^2q(t-\Delta)\right\}=0, \qquad (5)$$

valid for $-\infty<t<\infty$, cannot be derived from an action integral like Eq. (3), extending over a finite interval. Indeed, the variation of the action (3) is

$$\delta S([q],t)=\left[m\dot{q}\left(\tau+\frac{\Delta}{2}\right)\;\delta q\left(\tau+\frac{\Delta}{2}\right)\right]_{t-\Delta}^{t}-m\int_{t-\Delta}^{t}d\tau\left[\ddot{q}\left(\tau+\frac{\Delta}{2}\right)\;\delta q\left(\tau+\frac{\Delta}{2}\right)+\frac{\omega^2}{2}q(\tau+\Delta)\,\delta q(\tau)+\frac{\omega^2}{2}q(\tau)\,\delta q(\tau+\Delta)\right]. \qquad (6)$$

The extremal condition $\delta S=0$ then leads to the boundary conditions $\delta q[t+(\Delta/2)]=\delta q[t-(\Delta/2)]=0$ and to the equations of motion:

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$$\begin{aligned}
&(a)\quad t-\Delta<\tau<t-\frac{\Delta}{2},\quad q(\tau+\Delta)=0,\\
&(b)\quad t-\frac{\Delta}{2}<\tau<t,\quad \ddot{q}(\tau)+\omega^2/2q(\tau+\Delta)=0,\\
&(c)\quad t<\tau<t+\frac{\Delta}{2},\quad \ddot{q}(\tau)+\omega^2/2q(\tau-\Delta)=0,\\
&(d)\quad t+\frac{\Delta}{2}<\tau<t+\Delta,\quad q(\tau-\Delta)=0,
\end{aligned} \tag{7}$$

which has the only solution $q(\tau)=0$ for $t-\Delta<\tau<t+\Delta$, as it ineluctably follows from sequentially exploiting (d), (b), (a), and (c).

Furthermore, if we alternatively try with an action extended over a larger interval,

$$S=\int_0^T d\tau\, L[q](\tau),$$

the Euler-Lagrange equations are

$$\begin{aligned}
&(i)\quad q(\tau+\Delta)=0,\\
&(ii)\quad \ddot{q}(\tau)+\frac{\omega^2}{2}q(\tau+\Delta)=0,\\
&(iii)\quad \ddot{q}(\tau)+\frac{\omega^2}{2}[q(\tau+\Delta)+q(\tau-\Delta)]=0,\\
&(iv)\quad \ddot{q}(\tau)+\frac{\omega^2}{2}q(\tau-\Delta)=0,\\
&(v)\quad q(\tau-\Delta)=0,
\end{aligned} \tag{8}$$

where the domains (i) to (v), respectively, correspond to $0<\tau<\Delta/2$; $\Delta/2<\tau<\Delta$; $\Delta<\tau<T$; $T<\tau<T+\Delta/2$, and $T+\Delta/2<\tau<T+\Delta$. Equation (8) only looks like Eq. (5) in the interval $\Delta<\tau<T$.

Conditions (i) and (v) in Eq. (8) then yield

$$q(\tau)=0,\quad \Delta<\tau<3\,\frac{\Delta}{2},\quad \text{or}\quad T-\frac{\Delta}{2}<\tau<T$$

that act as constraints on the possible solutions of (ii), (iii), and (iv) in Eq. (8). As a consequence, Eqs. (8) can be reduced to an ordinary differential equation, whose order depends on the number of times that the elementary length Δ fits into $[0,T]$.

We have thus illustrated the important role played by the integration bounds in the nonlocal action (3) as far as the Euler-Lagrange equations are concerned. The integration bounds in the action and the problems associated with them are commonly overlooked in theoretical physics literature because, in standard local cases no trouble is usually entailed by proceeding in this manner. Nonlocal cases are, however, a new ground where nothing can be taken for granted.

For a local action, the bounds of the integral also determine the functional Banach space where the variational calculus is meaningful [4], e.g., the space $\mathcal{C}^2([a,b])$ for an action integral extending over $[a,b]$. This is also the space where the solutions to the Euler-Lagrange equations have to be sought.

A way to derive Eq. (2), for t extending from $-\infty$ to ∞, from an action principle could consist in taking the integral over the whole $\mathbb{R}$:

$$S=\int_{-\infty}^{\infty} d\tau\, L[q](\tau), \tag{9}$$

but then two additional difficulties arise: on the one hand, the action S does not converge anymore for all $q\in\mathcal{C}^2(\mathbb{R})$ and, on the other, $\mathcal{C}^2(\mathbb{R})$ is not a Banach space. (The variational calculus should be then approached in terms of Fréchet spaces [5,6].) To my knowledge, it remains an open problem to establish the appropriate mathematical framework where a nonlocal equation like Eq. (5) can be derived from an action integral like Eq. (9). This results in a lack of preciseness in the definition of the functional space where the nonlocal equation has to be solved.

On possible nonstandard statements of the action principle

It could be thought[1] that the derivation of Eq. (2) from the action integral (1) does rather rest on a nonstandard version of the action principle than on the standard statement quoted right before Eq. (3), namely, "the action $S=\int_{t_1}^{t_2} ds L[q](s)$ is stationary with respect to variations $\delta q(s)$ that vanish for $s\leqslant t_1+\Delta t$ and $t_2-\Delta t\leqslant s$." Notice that it is similar to the standard statement of the action principle, where the usual "boundary point conditions" —initial and final conditions— are replaced by a sort of "boundary layer condition."

The latter nonstandard action principle leads both to the nonlocal equation

$$\ddot{q}(t)+\frac{\omega^2}{2}[q(t+\Delta t)+q(t-\Delta t)]=0\quad \text{for}\quad t_1+\Delta t<t<t_2-\Delta t \tag{10}$$

and to the "boundary layer conditions"

$$q(t)=q_1(t)\quad \text{for}\quad t\leqslant t_1+\Delta t\quad \text{and}\quad q(t)=q_2(t)\quad \text{for}\quad t_2-\Delta t\leqslant t, \tag{11}$$

where $q_1(t)$ and $q_2(t)$ are given functions defined in their respective half-lines.

Equations (10) and conditions (11) have to be solved together and, in most cases, given any set of layer data, $\{q_1(t),q_2(s);\ t\leqslant t_1+\Delta t, s\leqslant t_2-\Delta t\}$, Eq. (10) has no solution. Indeed, as is pointed out in Eq. (3) in Ref. [1], Eq. (10) can also be written as

[1]Suggested by R. P. Woodard (private communication).

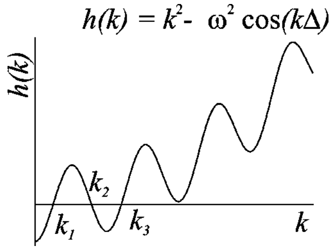

FIG. 1. $h(k)=0$ has a finite number of real roots, and the sign of the derivative $h'(k_l)$ at each root is alternating.

$$q(\tau)=-\frac{2}{\omega^2}\ddot{q}(\tau-\Delta t)-q(\tau-2\Delta t) \quad \text{for} \quad t_1+2\Delta t<\tau<t_2, \tag{12}$$

which, after n iterations permits to obtain $q(\tau)$ in terms of $q(\tau-n\Delta t)$ and $q(\tau-[n+1]\Delta t)$, whenever $\tau,\tau-(n-1)\Delta t\in[t_1+2\Delta t,t_2]$.

Now, taking n the least integer such that $(n+1)\Delta t\geqslant t_2-t_1$, $q(\tau)=q_2(\tau)$, for $t_2-\Delta t<\tau<t_2$ can be expressed in terms of $q(\tau-n\Delta t)$ and $q(\tau-[n+1]\Delta t)$, which only depend on the "lower layer datum" $q_1(t)$ because $\tau-n\Delta t\leqslant t_1+\Delta t$. This leads to a consistence condition between the upper and lower layer data, whence arbitrarily chosen data could be incompatible.

III. THE STABILITY PROBLEM

Leaving aside the difficulties just mentioned, suppose that, for some physical reasons whatsoever, we are only interested on the solutions of Eq. (5) in the Banach space,

$$\mathcal{B}=\{q\in\mathcal{C}^2(\mathbb{R});\ |q(t)|,\ |\dot{q}(t)| \ \text{and}\ |\ddot{q}(t)| \ \text{are bounded}\}.$$

The general solution of Eq. (8) is thus

$$q(t)=\sum_l (A_l e^{ik_l t}+A_l^* e^{-ik_l t}), \tag{13}$$

where $\pm k_l$ are the real solutions[2] of

$$h(k)\equiv k^2-\omega^2\cos(k\Delta)=0 \tag{14}$$

and A_l^* is the complex conjugate of A_l, to ensure that $q(t)\in\mathbb{R}$.

Notice that the number of real roots of Eq. (14) is finite. A look at Fig. 1 is enough to get convinced that they can be indexed so that

$$k_j<k_i \quad \text{if} \quad j<i \quad \text{and} \quad l=1,2,\ldots,N.$$

N can be either odd and then all roots are simple, or even, in which case $\pm k_N$ are both double. It should also be remarked that the greater is Δ, the denser is the wiggling in the graphics (Fig. 1). Therefore, N increases with Δ.

The space of solutions of Eq. (5) in $\mathcal{B}$ can be hence coordinated by $2N<\infty$ real parameters, namely, the real and imaginary parts of A_l that can be put in correspondence with the initial data: $q_0,\ \dot{q}_0,\ \ldots\ q_0^{(2N-1)}$.

The solutions of Eq. (5) in $\mathcal{B}$ are stable because a small change in the initial data $\delta q_0^{(\alpha)}$ results in a small change in the complex parameters: δA_l. Indeed, from the linearity of Eq. (5) and from the general solution (13) it follows that

$$\sum_{l=1}^{N}[\delta A_l(ik_l)^\alpha+\delta A_l^*(-ik_l)^\alpha]=\delta q_0^{(\alpha)},$$

$\alpha=0,1,\ldots,2N-1$, which can be inverted to obtain δA_l as a linear function of $\delta q_0^{(\beta)}$. Therefore, there exists $K>0$ such that $|\delta A_l|\leqslant K\|\delta q_0\|$, where $\|\delta q_0\|\equiv\sup\{|\delta q_0^{(\alpha)}|;\ \alpha=0,1,\ldots,2N-1\}$. The deviation from $q(t)$ evolves with time as

$$|\delta q(t)|=\left|\sum_{l=1}^{N}\mathrm{Re}(\delta A_l e^{ik_l t})\right|\leqslant\sum_{l=1}^{N}2|\delta A_l|\leqslant 2NK\|\delta q_0\|,$$

which proves the stability[3] of the solutions of Eq. (5) in the space $\mathcal{B}$.

Notwithstanding, if we now have a look at the Hamiltonian [Eq. (48) in Ref. [1]],

$$H(t)=\frac{1}{2}m\dot{q}^2(t)+\frac{1}{2}m\omega^2 q(t)q(t+\Delta) -\frac{1}{2}m\omega^2\int_0^\Delta ds\,\dot{q}(t+s)q(t+s-\Delta),$$

on substituting the general solution (13), we obtain that

$$H(t)=2m\sum_{l=1}^{N}k_l g(k_l)A_l A_l^* \tag{15}$$

with

$$g(k)=k+\frac{\omega^2}{2}\,\Delta\,\sin(k\Delta). \tag{16}$$

As expected, $H(t)$ is an integral of motion, but it has not a definite sign. Indeed, notice that $g(k)=h'(k)/2$ alternates sign at each root k_l [see Eq. (14) and Fig. 1]. Therefore, $g(k_l)$ is positive or negative depending on whether l is even or odd, respectively [moreover, $g(k_l)=0$ if k_l is a double root].

IV. TWO SIMPLE COUNTEREXAMPLES

A. The so-called Ostrogradskian instability

In Ref. [1] it is proved that the Hamiltonian formalism for a nonlocal Lagrangian can be obtained as a limit case for $N\to\infty$ of the Ostrogradski formalism [2] for a Lagrangian

[2]A complex value of k_l would result in an exponential growth either at $+\infty$ or $-\infty$ and then $q\notin\mathcal{B}$.

[3]In the sense of Liapounov, see [7].

that depends on the derivatives of the coordinates up to order N.[4] For $N>1$, the Ostrogradski Hamiltonian is linear on all the canonical momenta but one, namely, $P_1, \ldots, P_{N-1}$, therefore it has not a definite sign. The fact that the energy is not bounded from below is then argued to conclude that the solutions of the equations of motion are ineludibly unstable. This is what is called the *Ostrogradskian instability*. It is also shown in [1] that this drawback also holds in the limit $N \to \infty$.

Actually what has been proved there is only that the energy cannot be taken as a Liapunov function [8] to conclude the stability of the equations of motion derived from an N^{th} order Lagrangian ($N>1$). However, the fact that a sufficient condition of stability is not met does not imply instability. Let us consider the following simple counterexample:

$$L(q,\dot{q},\ddot{q}) = \frac{1}{2}\ddot{q}^2 + \frac{1}{2}B\dot{q}^2 + \frac{1}{2}Cq^2, \tag{17}$$

where B and C are two parameters which we shall later tune in order to get stability.

According to Ostrogradski theory, the canonical coordinates and momenta are [in the notation of Ref. [1], Eqs. (6) and (7)]

$$Q_1 = q, \quad Q_2 = \dot{q}, \quad P_1 = B\dot{q} - q^{(iii)}, \quad P_2 = \ddot{q}$$

and the Hamiltonian is [Eq. (9) in [1]]

$$H = \frac{1}{2}P_2^2 + P_1 Q_2 - \frac{1}{2}BQ_2^2 - \frac{1}{2}CQ_1^2. \tag{18}$$

Introducing

$$\vec{X} = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} \quad \text{and} \quad \mathbb{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ C & 0 & 0 & 0 \\ 0 & B & -1 & 0 \end{pmatrix},$$

the Hamilton equations for Eq. (18) can be then written as the linear system:

$$\frac{d}{dt}\vec{X} = \mathbb{M}\vec{X},$$

the stability of whose solutions depends on the real part of the roots of the characteristic polynomial $p_{\mathbb{M}}(\lambda) = \det(\mathbb{M} - \lambda \mathbb{I}_4)$, that is,

$$\lambda = \pm\sqrt{\frac{B \pm \sqrt{B^2 - 4C}}{2}}.$$

If the parameters are tuned so that $B<0$ and $0<C<B^2/4$, then all roots are imaginary and the system is stable [8]. The latter is not an obstacle to the fact that the Hamiltonian does not have a definite sign.

It could be argued that, although the Lagrangian system (17) is stable, it is physically irrelevant unless it can be coupled to anything else. It seems that, since the energy has not a lower bound, an unending flow of energy leaving the system cannot be prevented. To show that this is not the case, we shall now see how Eq. (17) can be stablily coupled to a harmonic oscillator. Since classical electromagnetic field is actually an infinite set of harmonic oscillators, each one characterized by its polarization and its wave vector, the next example will also serve as an indication that the stable coupling of the system (17) can be extended to a Maxwell field.

Consider the second-order Lagrangian,

$$L = \frac{1}{2}(\ddot{q}^2 + B\dot{q}^2 + Cq^2) + gqx + \frac{m}{2}(\dot{x}^2 - \omega^2 x^2). \tag{19}$$

We shall see that the parameters B and C can be tuned so that the system is stable.

The equations of motion are

$$\overset{iv}{q} = B\ddot{q} - Cq - gx, \qquad \ddot{x} = -\omega^2 x + \frac{g}{m}q, \tag{20}$$

and can be written in matrix form as

$$\frac{d}{dt}\mathbb{X} = \mathbb{G}\mathbb{X}, \tag{21}$$

where

$$\mathbb{X} = \begin{pmatrix} \dddot{q} \\ \ddot{q} \\ \dot{q} \\ q \\ \dot{x} \\ x \end{pmatrix} \quad \text{and} \quad \mathbb{G} = \begin{pmatrix} 0 & B & 0 & -C & 0 & -g \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g/m & 0 & -\omega^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The ordinary differential system is stable if the characteristic roots of $\mathbb{G}$ are all imaginary and simple. This amounts to saying that

$$P(y) = (y + \omega^2)(y^2 - By + C) + \frac{g^2}{m} \tag{22}$$

has three simple negative roots. A little bit of algebra shows that the latter is achieved if, and only if, the parameters B and C are chosen so that (1) B is a real root of the cubic equation

$$\beta(B-\omega^2)^3 + \alpha\omega^2(B-\omega^2)^2 + B\omega^4 + \frac{g^2}{m} = 0,$$

(2) C is

[4] A similar result was also obtained by Ref. [9].

$$C=\beta\frac{(B-\omega^2)^3}{\omega^2}-\frac{g^2}{m\omega^2},$$

(3) and α and β are two real parameters in the nonempty region of $\mathbb{R}^2$ defined by the inequalities,

$$\max\{0,\ 9\alpha^2-2-2(1-\alpha)^{3/2}\}\leqslant 27\beta\leqslant 9\alpha^2 -2+2(1-\alpha)^{3/2}.$$

Now, assuming that the parameters meet these conditions, the system (21) is stable in spite of the fact that the Hamiltonian does not have a definite sign. Notwithstanding, a positive definite integral of motion (a Lyapunov function [10]) can be found such that it forbids the existence of runaways.

Indeed, if all characteristic roots of Eq. (21) are imaginary, the matrix $\mathbb{G}$ can be diagonalized and all its eigenvalues are imaginary, namely, $\pm i\lambda_a$, $a=1,2,3$. Therefore, a real regular matrix $\mathbb{F}$ exists such that

$$\mathbb{G}=\mathbb{F}^{-1}\mathbb{K}\mathbb{F}\quad \text{with}$$

$$\mathbb{K}=\begin{pmatrix} 0 & -\lambda_1 & 0 & 0 & 0 & 0\\ \lambda_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda_2 & 0 & 0\\ 0 & 0 & \lambda_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\lambda_3\\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \end{pmatrix}. \qquad (23)$$

Then, the quadratic form

$$f(\dddot{q},\ddot{q},\dot{q},q,\dot{x},x)=\mathbb{X}^T\mathbb{F}^T\mathbb{F}\mathbb{X} \qquad (24)$$

(a) is positive in the whole phase space and (b) is an integral of motion. Indeed, taking Eqs. (21) and (23) into account, we readily have that

$$\begin{aligned}\frac{df}{dt}&=\dot{\mathbb{X}}^T\mathbb{F}^T\mathbb{F}\mathbb{X}+\mathbb{X}^T\mathbb{F}^T\mathbb{F}\dot{\mathbb{X}}\\ &=\mathbb{X}^T[\mathbb{G}^T\mathbb{F}^T\mathbb{F}+\mathbb{F}^T\mathbb{F}\mathbb{G}]\mathbb{X}\\ &=\mathbb{X}^T\mathbb{F}^T[\mathbb{K}^T+\mathbb{K}]\mathbb{F}\mathbb{X}=0.\end{aligned}$$

Thus, any orbit of the dynamical system (21) remains on the ellipsoid,

$$f(\dddot{q},\ddot{q},\dot{q},q,\dot{x},x)=f_0=\text{const},$$

which is bounded by

$$f_0\leqslant k^2\ \ ||\mathbb{X}_0||^2,$$

where $k=||\mathbb{F}||$ is the norm of the linear map $\mathbb{F}$ and $\mathbb{X}_0=(\dddot{q}_0,\ddot{q}_0,\dot{q}_0,q_0,\dot{x}_0,x_0)$.

B. A case of finite extent nonlocality

In the next example, the boundaries of the action integral are finite. Consider the nonlocal action $S[q]=\int_0^T dt L[q](t)$, with

$$L[q](t)=\frac{1}{2}\dot{q}^2(t)-\frac{1}{2}\omega^2q^2(t)+\frac{\omega^4}{2}q(t)\int_0^T dt'\ G(t,t')q(t'), \qquad (25)$$

where, for $(t,t')\in[0,T]^2$,

$$G(t,t')=\frac{-1}{\omega\sin\omega T}[\sin\omega(T-t')\ \sin\omega t\ \ \theta(t'-t)+\sin\omega(T-t)\ \sin\omega t'\ \ \theta(t-t')] \qquad (26)$$

and is the solution of

$$\partial_t^2G(t,t')+\omega^2G(t,t')=\delta(t-t') \qquad (27)$$

for the boundary conditions: $G(0,t')=G(t,T)=0$.

The variation $\delta S=0$ with the boundary conditions $\delta q(0)=\delta q(T)=0$ leads to the equations of motion

$$\ddot{q}(t)+\omega^2q(t)-\omega^4\int_0^T dt'\ \ G(t,t')q(t')=0. \qquad (28)$$

The solutions $q(t)$ must be sought in the Banach space $\mathcal{C}^2([0,T])$.

Differentiating twice (28) and taking (27) into account, we arrive at

$$q^{(iv)}+2\omega^2\ \ddot{q}=0. \qquad (29)$$

Hence, the solutions of Eq. (28) must be among the general solution of Eq. (29),

$$q(t)=Ae^{i\alpha t}+A^*e^{-i\alpha t}+Dt+E \qquad (30)$$

with $\alpha=\omega\sqrt{2}$. The parameters A, A^*, D, and E must fulfill the following constraints:

$$D=0\quad\text{and}\quad E=A+A^*,$$

which result from substituting Eq. (30) into Eq. (28).

The general solution of Eq. (28) is therefore

$$q(t)=A(e^{i\alpha t}+1)+A^*(e^{-i\alpha t}+1). \qquad (31)$$

The phase space for our system is thus two dimensional, and every solution is determined by the initial values q_0 and $\dot{q}_0$,

$$q_0=2(A+A^*)\quad\text{and}\quad \dot{q}_0=i\alpha(A-A^*).$$

By direct inspection of Eq. (31), we see that the solutions of Eq. (28) are stable, although the latter is derived from a Lagrangian with a nonlocality of finite extent. That is, for any $\epsilon > 0$ there exists $\rho > 0$ such that $|\delta q_0| + |\delta \dot{q}_0| < \rho$ implies that $|\delta q(t)| + |\delta \dot{q}(t)| < \epsilon$, for all t, which proves the stability.

V. CONCLUSION

We have intended to stress the crucial importance of clearly precising the Banach space where the variational principle for a nonlocal Lagrangian is formulated. This degree of precision is usually obviated in theoretical physics (i.e., for local Lagrangians) without any major problem. However, such a nonrigourous way of proceeding cannot be extrapolated to systems with a new complexity. The relevance of the above-mentioned Banach space is twofold: (i) it is where the solutions of the equations of motion must be sought and (ii) it is the function space where path integrals are to be calculated in an eventual quantization of the system.

We have also analyzed the stability of the equations of motion for a Lagrangian system presenting a nonlocality of finite extent. We have shown that the choice of the Banach space where the variational principle is meaningfully formulated is crucial to decide the stability or unstability of the system. Furthermore, we have seen that a system can be stable in spite of the fact that the Hamiltonian does not have a minimum.

Finally, we have shown by a counterexample that higher order Lagrangian systems are not necessarily unstable. The fact that a sufficient condition for stability is not fulfilled does not imply instability.

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