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SOME MODULI SPACES FOR RANK 2 STABLE REFLEXIVE SHEAVES ON $\mathbf{P}^3$

ROSA M. MIRÓ - ROIG

ABSTRACT. In [**Ma**], Maruyama proved that the set $M(c_1, c_2, c_3)$ of isomorphism classes of rank 2 stable reflexive sheaves on $\mathbf{P}^3$ with Chern classes (c_1, c_2, c_3) has a natural structure as an algebraic scheme. Until now, there are no general results about these schemes concerning dimension, irreducibility, rationality, etc. and only in a few cases a precise description of them is known.

This paper is devoted to the following cases: (i) $M(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ with $c_2 \geqslant 4$, $1 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$; and (ii) $M(-1, c_2, c_2^2 - 2(r-1)c_2)$ with $c_2 \geqslant 8$, $2 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$.

1. Introduction and preliminaries.

1.1. *Introduction*. We say that a coherent sheaf on $\mathbf{P}^n$ is reflexive if the natural map $E \to E^{\vee\vee}$ is an isomorphism. Here $E^{\vee}$ means, as usual, $\mathcal{H}om(E, \mathcal{O}_{\mathbf{P}^n})$. The systematic study of stable reflexive sheaves was begun by R. Hartshorne in [**H1**], with emphasis on the case of rank 2 stable reflexive sheaves on $\mathbf{P}^3$, which correspond to quite general curves and, moreover, they appear as soon as one makes a reduction step, even starting with vector bundles.

If a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ has Chern classes $c_1, c_2, c_3 \in \mathbf{Z}$, it is known that c_3 is the number of non-locally-free points of E, counted with suitable multiplicities. Now if E is normalized, i.e. $c_1 = 0$ or -1, then

(i) $c_2 > 0$ [**H1**, Proposition 3.3, 9.7].

(ii) $c_1 c_2 \equiv c_3 \pmod 2$ [**H1**, Corollary 2.4].

(iii)

$$0 \leqslant c_3 \leqslant \begin{cases} c_2^2 & \text{if } c_1 = -1, \\ c_2^2 - c_2 + 2 & \text{if } c_1 = 0 \end{cases} \qquad [\mathbf{H1}, \text{Theorem } 8.2].$$

(iv) if $c_1 = -1$, then [**M2**, Theorem A]

$$c_3 \notin \bigcup_{r=1}^{b(-1,c_2)} \left(c_2^2 - 2rc_2 + 2r(r+1),\ c_2^2 - 2(r-1)c_2\right),$$

where $b(-1, c_2) = (-1 + \sqrt{4c_2 - 7})/2$, and if $c_1 = 0$, then

$$c_3 \notin \bigcup_{r=2}^{b(0,c_2)} \left(c_2^2 - (2r-1)c_2 + 2r^2,\ c_2^2 - (2r-3)c_2\right),$$

where $b(0, c_2) = \sqrt{c_2 - 2}$.

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(v) Any given c_2 and c_3 satisfying (i), (ii), (iii), and (iv) actually occur [**M2**, Theorem B].

In this paper we investigate some moduli spaces of stable rank 2 reflexive sheaves on $\mathbf{P}^3$. Maruyama has shown that there is a coarse moduli scheme for stable torsion free sheaves with given Chern classes [**Ma**], but only in a few cases a precise description of the moduli $M^s_{\mathbf{P}^3}(c_1, c_2, c_3)$ is known. In [**H1**], Hartshorne studied $M^s_{\mathbf{P}^3}(-1, c_2, c_2^2)$; in [**Ch1**], Chang studied $M^s_{\mathbf{P}^3}(0, c_2, c_2^2 - c_2 + 2)$, $M^s_{\mathbf{P}^3}(0, c_2, c_2^2, -c_2)$, and $M^s_{\mathbf{P}^3}(0, c_2, c_2^2 - 3c_2 + 8)$; case $M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2c_2 + 4)$ was studied in [**M1**]. Finally Chang studied the moduli spaces of rank 2 stable reflexive sheaves with $c_2 \leqslant 3$ in [**Ch2**].

This paper is devoted to the following cases:

In §2 we study the moduli spaces $M = {}^2M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ with $c_2 > 4$, $1 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$. The main features of these spaces are that they are irreducible nonsingular and rational of dimension $c_2^2 + c_2 + 6$ if $r = 1$ and $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$ if $r \geqslant 2$. This is seen by proving that M is irreducible and nonsingular, and constructing an irreducible, nonsingular, rational family which turns out to include any rank 2 stable reflexive sheaf with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ in an open subset of M. As an application we construct a family of projectively normal curves in $\mathbf{P}^3$ (see Propositions 2.11 and 2.12, and Corollary 2.13).

In §3 we study the moduli spaces $M = {}^2M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2(r-1)c_2)$ with $c_2 \geqslant 8$ and $2 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$. We can only prove that they are irreducible and compute their dimension but at this time we do not know if the moduli spaces are reduced, rational, or nonsingular. Again this is seen by constructing an irreducible family of sheaves which turns out to contain any rank 2 stable reflexive sheaf with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$ in an open set of the moduli space.

Throughout the paper we work only with rank 2 stable reflexive sheaves with $c_1 = -1$, similar results are true for rank 2 stable reflexive sheaves with $c_1 = 0$.

As a general reference for reflexive sheaves see [**H1**].

I would like to express my thanks to S. Xambó-Descamps for many inspiring conversations and for being very generous in his advice and time, and to R. Hartshorne, who read an earlier version of this paper and suggested a number of improvements while pointing out to me several deficiencies, in particular that my former version of Proposition 2.5 was incomplete.

1.2. *Notations and preliminaries*. Throughout this paper we work over an algebraically closed field k of characteristic 0.

For a coherent sheaf E on $\mathbf{P}^3$ we will often write H^iE (resp. h^iE) for $H^i(\mathbf{P}^n, E)$ (resp. $\dim_k H^i(\mathbf{P}^n, E)$). The dual of E is written $\check{E} = \mathcal{H}om(E, 0)$.

Given two subschemes $Y_1, Y_2 \subset \mathbf{P}^3$ defined by the sheaves of ideals I_1 and I_2, respectively, we denote by $Y_1 \cup Y_2$ the subscheme of $\mathbf{P}^3$ defined by $I_1 \cap I_2$, and by $Y_1 \cap Y_2$ the subscheme of $\mathbf{P}^3$ defined by $I_1 + I_2$.

DEFINITION 1. We will say that two curves Y_1 and Y_2 of $\mathbf{P}^3$ meet in r points if $Y_1 \cap Y_2$ is a 0-dimensional scheme and length $\mathcal{O}_{Y_1 \cap Y_2} = r$.

PROPOSITION 2. *Let $Y_1, Y_2 \subset \mathbf{P}^3$ be two Cohen-Macaulay curves meeting in r points and let I_1, I_2 be the corresponding sheaves of ideals. Let $Y = Y_1 \cup Y_2$ and $Z = Y_1 \cap Y_2$. Then*

(i) $p_a(Y) = p_a(Y_1) + p_a(Y_2) - 1 + r$.

(ii) *There is an exact sequence* $0 \to \omega_{Y_1} \oplus \omega_{Y_2} \to \omega_Y \to \omega_Z \to 0$.

PROOF. See [**M2**, Proposition 4, §1].

LEMMA 3. *For each $c_2 \geqslant 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2 + 2d)$, $0 \leqslant d \leqslant r(r-1)$. Then E has an unstable plane H of order $c_2 - (r-1)$.*

PROOF. If $c_3 = c_2^2 - 2(r-1)c_2 + 2d$ with $0 \leqslant d \leqslant r(r-1)$ and $1 \leqslant r \leqslant b(-1, c_2)$, then $c_3 > \frac{1}{2}c_2^2 + c_2$, thus using [**H2**, §5.2] we conclude that E has an unstable plane H of order $c_2 - q$ for some $0 \leqslant q \leqslant \frac{1}{2}(c_2 - 3)$, and $c_2^2 - 2qc_2 \leqslant c_3 = c_2^2 - 2(r-1)c_2 + 2d \leqslant c_2^2 - 2qc_2 + 2q(q+1)$, which gives us $q = r - 1$ and so E has an unstable plane of order $c_2 - (r-1)$.

2. The moduli space $M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. In this section we classify rank 2 stable reflexive sheaves on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$, where $c_2 > 4$ and $1 \leqslant r \leqslant b(-1, c_2)$, and we show that their moduli spaces are irreducible, nonsingular, and rational.

Construction 2.1. For $c_1 = -1$ and each $c_2 \geqslant 4$, $1 \leqslant r \leqslant b(-1, c_2)$, construct E as an extension

$$(\xi) \qquad 0 \to \mathcal{O} \to E(1) \to I_Y(1) \to 0,$$

where $Y = Y_1 \cup Y_2$ is the union of a plane curve Y_1 of degree $c_2 - r$ and a plane curve Y_2 of degree r meeting Y_1 in r points.

By construction

$$c_1E = -1, \quad c_2E = c_2,$$

$$\begin{aligned} c_3E &= 2p_a - 2 + 3d = 2[p_a(Y_1) + p_a(Y_2) - 1 + r] - 2 + 3c_2 \\ &= c_2^2 - 2rc_2 + 2r(r+1), \end{aligned}$$

and E is stable.

We will call $\mathscr{F}$ the family of sheaves constructed in (2.1).

THEOREM 2.2. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, the moduli space $M = M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ is irreducible, nonsingular, and rational of dimension $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$ if $r \geqslant 2$ and $c_2^2 + c_2 + 6$ if $r = 1$.*

REMARK. The dimension is $> 8c_2 - 5$, so we have an oversized moduli space.

PROOF. The proof of the theorem consists of several steps;

1*st step*. To compute

$$\dim T_{[E]}M\big(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1)\big) = \dim \operatorname{Ext}^1(E, E)$$

$$= \begin{cases} c_2^2 + c_2 + 6 & \text{if } r = 1, \\ c_2^2 + (3 - 2r)c_2 + 2r^2 + 5 & \text{if } r \geqslant 2. \end{cases}$$

This is done in Proposition 2.3.

2nd step. To see that construction (2.1) gives an irreducible, nonsingular, rational family of dimension $c_2^2 + c_2 + 6$ if $r = 1$ and $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$ if $r \geqslant 2$. This is carried out in Proposition 2.6.

3rd step. To prove that $M = M(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ is irreducible of dimension $c_2^2 + c_2 + 6$ if $r = 1$ and $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$ if $r \geqslant 2$. This is done in Proposition 2.9.

Since M is irreducible and nonsingular, and for each rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ the dimension of $\operatorname{Ext}^1(E, E)$ is the same as the dimension of the family $\mathscr{F}$, we can conclude that $\mathscr{F}$ is an open subset of $M_{\text{red}}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ and the moduli space has the properties stated.

PROPOSITION 2.3. *For each $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Then*

$$\dim \operatorname{Ext}^1(E, E) = \begin{cases} c_2^2 + c_2 + 6, & r = 1, \\ c_2^2 + (3 - 2r)c_2 + 2r^2 + 5, & r \geqslant 2. \end{cases}$$

PROOF. By Lemma 3 of §1 there is an unstable plane H for E of order $c_2 - r$. We perform a reduction step which gives an exact sequence

$$0 \to E' \to E \to I_{ZH}(r - c_2) \to 0,$$

where Z is a 0-dimensional subscheme of H. Let $s = \operatorname{length} \mathscr{O}_Z$. The Chern classes c_i' of E' are given by $c_1' = -2$, $c_2' = r + 1$, $c_3' = r^2 + r + 2s$. The Chern classes c_i'' of the normalized sheaf $E'(1)$ are given by $c_1'' = 0$, $c_2'' = r$, $c_3'' = r^2 + r + 2s$. Since $H^0 E' = 0$, we see that $E'(1)$, and hence E', is semistable. By **[H1**, §8.2], $r^2 + r + 2s = c_3' \leqslant c_2''^2 + c_2'' = r^2 + r$, so we conclude that $s = 0$ and hence the exact sequence of the reduction step is

$$0 \to E' \to E \to \mathscr{O}_H(r - c_2) \to 0. \tag{1}$$

By **[Ok**, §2.3] $E'(1)$ admits the resolution

$$0 \to \mathscr{O}(-r-1) \to \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-r) \to E'(1) \to 0. \tag{2}$$

Thus $h^1 E'(k) = 0$, hence $h^1 E(k) = 0\ \forall k$ and $h^0 E(k) = h^0 E'(k)\ \forall k < c_2 - r$.

Applying $\operatorname{Hom}(\ , E)$ to the exact sequence (1) we get

$$\begin{aligned} 0 &\to \operatorname{Hom}(E, E) \to \operatorname{Hom}(E', E) \to \operatorname{Ext}^1(\mathscr{O}_H(r - c_2), E) \\ &\to \operatorname{Ext}^1(E, E) \to \operatorname{Ext}^1(E', E) \to \operatorname{Ext}^2(\mathscr{O}_H(r - c_2), E). \end{aligned}$$

Using the sequence (2) twisted by -1, we compute $\dim \operatorname{Ext}^1(E', E) - \dim \operatorname{Hom}(E', E)$ and we get

$$\begin{aligned} \dim \operatorname{Ext}^1(E', E) - \dim \operatorname{Hom}(E', E) &= h^0 E(r+2) - h^0 E(r+1) - h^0 E(1) - h^0 E(2) \\ &= h^0 E'(r+2) - h^0 E'(r+1) - h^0 E'(1) - h^0 E'(2) \\ &= \begin{cases} 4, & r = 1, \\ (r+2)^2 - 4, & r \geqslant 2. \end{cases} \end{aligned}$$

Next we compute $\dim \operatorname{Ext}^1(\mathcal{O}_H(r-c_2), E)$ and $\dim \operatorname{Ext}^2(\mathcal{O}_H(r-c_2), E)$. In order to do this we use the sequence

$$0 \to \mathcal{O}(r-c_2-1) \to \mathcal{O}(r-c_2) \to \mathcal{O}_H(r-c_2) \to 0$$

which gives the sequences

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathcal{O}(r-c_2), E) &\to \operatorname{Hom}(\mathcal{O}(r-c_2-1), E) \\ &\to \operatorname{Ext}^1(\mathcal{O}_H(r-c_2), E) \to \operatorname{Ext}^1(\mathcal{O}(r-c_2), E) \end{aligned}$$

and

$$\operatorname{Ext}^1(\mathcal{O}(r-c_2-1), E) \to \operatorname{Ext}^2(\mathcal{O}_H(r-c_2), E) \to \operatorname{Ext}^2(\mathcal{O}(r-c_2), E).$$

The first term in the second sequence is $H^1E(c_2-r+1)$ which vanishes, and the last term is $H^2E(c_2-r)$ which also vanishes (by [**H1**, §7.1] $h^2E(c_2-r) = h^1(\mathbf{P}^1, \oplus_i \mathcal{O}(k_i+c_2-r+1) = 0)$, so $\operatorname{Ext}^2(\mathcal{O}_H(r-c_2), E) = 0$.

The last term in the first sequence is $H^1E(c_2-r)$ which vanishes. Consequently, we get

$$\begin{aligned} \dim \operatorname{Ext}^1(\mathcal{O}_H(r-c_2), E) &= \chi E(c_2-r+1) - \chi E(c_2-r) \\ &= c_2^2 + (3-2r)c_2 + 4 + r^2 - 4r, \end{aligned}$$

as one sees using Riemann-Roch. (In fact, $\chi E(l) - \chi E(l-1) = (l+1)^2 - c_2$.)

Also $\dim \operatorname{Hom}(E,E) = 1$, since E is simple [**H1**, §3.4]. Putting these together gives

$$\dim \operatorname{Ext}^1(E,E) = \begin{cases} c_2^2 + c_2 + 6, & r = 1, \\ c_2^2 + (3-2r)c_2 + 2r^2 + 5, & r \geqslant 2. \end{cases}$$

Remark 2.3.1. In Lemma 3, §1, we prove that every rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$, $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1+\sqrt{4c_2-7})/2$, has an unstable plane H of order $c_2 - r$. Now we claim that this plane is unique.

In fact, let H_1 be any plane different from H. The exact cohomology sequence

$$H^2E'(c_2-r-3) \to H^2E'_{H_1}(c_2-r-3) \to H^3E'(c_2-r-4)$$

tells us that $H^2E_{H_1}(c_2-r-3)$ vanishes because the first and last terms vanish, as is seen using the exact sequence (2) of Proposition 2.3. If we now restrict the exact sequence (1) of Proposition 2.3 to H_1 and take cohomology we obtain $H^2E_{H_1}(c_2-r-3) = 0$. Thus H_1 is not an unstable plane of order $c_2 - r$ for E.

Remark 2.3.2. If E is a rank 2 semistable reflexive sheaf on $\mathbf{P}^3$ with $c_1 = 0$ or -1, and $E(l)$ corresponds to curve Y, let H be a plane containing a subscheme of Y of degree $d \geqslant l$. Then H is an unstable plane of order $d - c_1 - l$ [**Ch1**, §3.0].

Definition 2.4. For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Let $0 \neq s \in H^0E(1)$ and let $Y = (s)_0$. We will say that Y is the curve associated to E. We observe that the definition is not ambiguous because $h^0E(1) = 1$ (see the proof of Proposition 2.3).

Proposition 2.5. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Let Y be its associated curve. Then Y is one of the following curves*:

(i) *the union of two plane curves, of degree $c_2 - r$ and r, meeting at r points, or*

(ii) *a plane curve C of degree $c_2 - r$ with a multiplicity-two structure on a degree r subcurve Y', related by an exact sequence of the form*

$$0 \to I_Y \to I_C \to \mathcal{O}_{Y'}(-1) \to 0,$$

or

(iii) *the union of two plane curves, C of degree $c_2 - r$ and Y' of degree r, with a multiplicity-two structure along a common subline, related by an exact sequence*

$$0 \to I_Y \to I_C \to \mathcal{O}_{Y'}(-1) \to 0.$$

Proof. The curve Y associated to E is by definition the zero set of a nonzero section $0 \neq s \in H^0E(1)$, so $\deg Y = c_2E(1) = c_2$ and

$$p_a = \frac{c_2^2 - 2rc_2 + 2r(r+1) + 2 - 3c_2}{2}.$$

Now the result follows from [**S**, §7.10] replacing a and d there by $c_2 - r$ and c_2, respectively.

Proposition 2.6. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$*, Construction 2.1 *gives a family $\mathcal{F}$ which is irreducible, nonsingular, and rational of dimension*

$$\begin{cases} c_2^2 + c_2 + 6 & \text{if } r = 1, \\ c_2^2 + (3 - 2r)c_2 + 2r^2 + 5 & \text{if } r \geqslant 2. \end{cases}$$

Proof. The case $r = 1$ is studied in [**M1**]. So assume $r \geqslant 2$. Since for each $E \in \mathcal{F}$, $H^0E(1) = 1$, the isomorphism classes of the sheaves E in $\mathcal{F}$ are in one-to-one correspondence with the pairs (Y, ξ), where $Y = Y_1 \cup Y_2$ is the union of a plane curve Y_1 of degree $c_2 - r$ with a plane curve Y_2 of degree r meeting Y_1 at r points and $\xi \in H^0\omega_Y(3)$. So we want to see that such pairs are parametrized by an irreducible, nonsingular, rational variety of dimension $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$.

In order to see this, let V be the irreducible, nonsingular, and quasi-projective variety which parametrizes curves Y that are the union of a plane curve Y_1 of degree $c_2 - r$ with a plane curve Y_2 of degree r meeting Y_1 at r points. We consider the correspondence $\mathcal{Y} = \{(y, x) \in V \times \mathbf{P}^3 | x \in Y\} \subset V \times \mathbf{P}^3$, where we denote by y the point of V which corresponds to a curve Y. Let $q \colon \mathcal{Y} \to V$ and $p \colon \mathcal{Y} \to \mathbf{P}^3$ be the restrictions to $\mathcal{Y}$ of the projection maps. Thus,

$$\mathcal{Y}_y := q^{-1}(y) \simeq Y.$$

Set $\omega_{\mathcal{Y}}(3) := p^*\omega_{\mathbf{P}^3}(3)$. By the semicontinuity theorem $q_*\omega_{\mathcal{Y}}(3)$ is locally free of rank

$$\frac{c_2^2 + (3 - 2r)c_2}{2} + r^2 + r.$$

In fact, for all $y \in V$

$$h^0\left(q^{-1}(y), \omega_{\mathcal{Y}}(3)_{|q^{-1}(y)}\right) = h^0\left(Y, \omega_Y(3)\right) = \frac{c_2^2 + (3 - 2r)c_2}{2} + r^2 + r$$

$(0 \to \omega_{Y_1} \oplus \omega_{Y_2} \to \omega_Y \to \omega_{Y_1 \cap Y_2} \to 0)$. Hence the set of pairs (Y, ξ) is parametrized by $\mathbf{P}(q_*(\omega_{\mathcal{Y}}(3)))$ which is irreducible, nonsingular, and rational.

Now we compute its dimension. The choice of the plane H' containing Y_2 is three parameters, the choice of Y_2 in H' is $\binom{r+2}{2} - 1$ parameters, the choice of the plane H containing Y is three parameters, the choice of Y_1 is $\binom{c_2 - r + 2}{2} - (r+1)$ parameters, and the choice of s in $\mathbf{P}(H^0\omega_Y(3))$ is $(c_2^2 + (3 - 2r)c_2)/2 + r^2 + r - 1$. So our family depends on $c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$ parameters.

Lemma 2.7. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Then E admits the following locally free resolution*:

$$\begin{aligned} 0 &\to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \\ &\to \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2) \to E \to 0. \end{aligned}$$

Proof. We know that E has a unique unstable plane H of order $c_2 - r$ which gives us the reduction step sequence

$$0 \to E' \to E \to \mathcal{O}_H(r - c_2) \to 0 \quad \text{(cf. Proposition 2.3)}.$$

This sequence together with the sequences

$$0 \to \mathcal{O}(r-1-c_2) \to \mathcal{O}(r-c_2) \to \mathcal{O}_H(r-c_2) \to 0$$

and

$$0 \to \mathcal{O}(-r-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-r-1) \to E' \to 0$$

[**Ok**, §2.3] give the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-r-2) \arrow[r] \arrow[d] & \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \arrow[r] \arrow[d] & \mathcal{O}(r-1-c_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-r-1) \arrow[r] \arrow[d] & \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2) \arrow[r] \arrow[d] & \mathcal{O}(r-c_2) \arrow[r] \arrow[d, "g'"] \arrow[dl, "g"] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathcal{O}_H(r-c_2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Notice that g' lifts to a map $g\colon \mathcal{O}(r - c_2) \to E$ because

$$\operatorname{Ext}^1\left(\mathcal{O}(r-c_2), E'\right) = 0.$$

So we obtain the locally free resolution of E

$$\begin{aligned} 0 &\to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \\ &\to \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2) \to E \to 0. \end{aligned}$$

COROLLARY 2.8. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Then*:

(i) *$E(l)$ is generated by global sections for all $l \geqslant c_2 - r + 1$.*

(ii) *$H^1E(l) = 0$ for all $l \in \mathbf{Z}$.*

PROPOSITION 2.9. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, the moduli space $M = M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ is irreducible of dimension $c_2^2 + c_2 + 6$ if $r = 1$ and $c_2^2 + (3-2r)c_2 + 2r^2 + 5$ if $r \geqslant 2$.*

PROOF. The case $r = 1$ is studied in [**M1**].

So assume $r \geqslant 2$. By Lemma 2.7 any rank 2 stable reflexive sheaf E on $\mathbf{P}^3$ admits a locally free resolution of the following kind:

$$\begin{aligned} 0 &\to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \\ &\to \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2) \to E \to 0. \end{aligned}$$

On the other hand, the family of isomorphism classes of rank 2 stable reflexive sheaves on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$ is an open set of the family N of isomorphism classes of rank 2 coherent sheaves on $\mathbf{P}^3$ that are the cokernel of a monomorphism

$$0 \to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \to \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2).$$

So first let us compute $\dim N$. To do this, let

$$\begin{aligned} G_1 &= \operatorname{Aut}(\mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2)), \\ G_2 &= \operatorname{Aut}(\mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2)), \end{aligned}$$

and

$$\begin{aligned} S = \operatorname{Hom}(&\mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2), \\ &\mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r-c_2)). \end{aligned}$$

We consider the action

$$(G_1 \times G_2) \times S \to S, \qquad ((\varphi_1, \varphi_2), f) \mapsto \varphi_2 f \varphi_1.$$

Then $\dim N = \dim S - \dim G_1 - \dim G_2 + \dim I_f$, where

$$I_f = \{(\varphi_1, \varphi_2) \in G_1 \times G_2 / \varphi_2 f \varphi_1 = f\}$$

for any $f \in S$. But

$$\dim S = 8 + \binom{r+4}{3} + \binom{r+3}{3} + \binom{c_2 - 2r + 3}{3} + \binom{c_2 - r + 3}{3} + \binom{c_2 - r + 2}{3},$$

$$\dim G_1 = 2 + \binom{c_2 - 2r + 2}{3},$$

$$\dim G_2 = 8 + \binom{r+3}{3} + \binom{r+2}{3} + \binom{c_2 - 2r + 2}{3} + \binom{c_2 - r + 2}{3} + \binom{c_2 - r + 1}{3},$$

and

$$\dim I_f = 1 + \binom{c_2 - 2r + 1}{3}.$$

This last dimension does not depend on f, as may be seen taking for f the map represented by the matrix

$$A = \begin{pmatrix} x_0 & h \\ x_1^{r+1} & x_0^r h \\ x_2^r & x_0^{r-1} h \\ 0 & x_3 \end{pmatrix},$$

where h is a form of degree $c_2 - 2r$. In fact φ_1 is represented by a matrix of the form

$$B_1 = \begin{pmatrix} b_1 & g \\ 0 & b_2 \end{pmatrix},$$

where b_1 and b_2 are constants and g is a form of degree $c_2 - 2r - 1$, and φ_2 is represented by a matrix of the form

$$B_2 = \begin{pmatrix} a_1 & 0 & 0 & q_{c_2-2r-1} \\ q_r & a_2 & q_1 & q_{c_2-r-1} \\ q_{r-1} & 0 & a_3 & q_{c_2-r-2} \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

where a_i are constants and q_j is a form of degree j, so that I_f is isomorphic to the set of pairs (B_1, B_2) satisfying the equation $B_2 A B_1 = A$. That the dimension of I_f is indeed as claimed follows by a straightforward computation.

Now one easily obtains that $\dim N = c_2^2 + (3 - 2r)c_2 + 2r^2 + 5$. The theorem follows because, as we remarked above, M is an open subset of N.

THEOREM 2.10. *For each $c_2 > 4$, $1 \leqslant r \leqslant b(-1, c_2)$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^n$ $(n \geqslant 3)$ with $c_1 = -1$, $c_3 = c_2^2 - 2rc_2 + 2r(r+1)$. Then*

(i) *$dhE \leqslant 1$.*

(ii) *E admits the locally free resolution*

$$\begin{aligned} 0 &\to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \\ &\to \mathcal{O}(-r-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(r - c_2) \to E \to 0. \end{aligned}$$

(iii)

$$c_t(E) = \frac{(1-(r+1)t)(1-t)(1-2t)(1+(r-c_2)t)}{(1-(r+2)t)(1+(r-1-c_2)t)}.$$

Furthermore $M^s_{\mathbf{P}^n}(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1), \ldots)$ is irreducible and nonsingular of dimension

$$\begin{aligned} n - 4 &+ \binom{r+n+1}{n} + \binom{c_2 - 2r + n}{n} + \binom{c_2 - r + n}{n} + \binom{c_2 - 2r + n - 2}{n} \\ &- 2\binom{c_2 - 2r + n - 1}{n} - \binom{r + n - 1}{n} - \binom{c_2 - r + n - 2}{n} \quad \textit{if } r \geqslant 2, \\ n - 5 &+ \binom{r+n+1}{n} + \binom{c_2 - 2r + n}{n} + \binom{c_2 - r + n}{n} + \binom{c_2 - 2r + n - 2}{n} \\ &- 2\binom{c_2 - 2r + n - 1}{n} - \binom{r + n - 1}{n} - \binom{c_2 - r + n - 2}{n} \quad \textit{if } r = 1. \end{aligned}$$

PROOF. (i) Using the exact sequence $0 \to E(k-1) \to E(k) \to E_{\mathbf{P}^{n-1}}(k) \to 0$ and induction on n we get $h^1E(k) = h^2E(k) = \cdots = h^{n-2}E(k) = 0$ for all k, so by [**Ok**, §1.3] we can conclude $hdE \leqslant 1$.

(ii) Again by induction on n we get

$$h^{n-1}E(c_2 - r - n) = 1 \quad \text{and} \quad h^{n-1}E(c_2 - r - n - 1) = n.$$

Therefore there exists a linear form $f \in H^0\mathcal{O}_{\mathbf{P}^n}(1)$ such that the map

$$H^{n-1}E(c_2 - r - n - 1) \xrightarrow{\cdot f} H^{n-1}E(c_2 - r - n)$$

induced by f is zero, and thus E has an unstable hyperplane of order $c_2 - r$, which gives us the reduction step sequence

$$0 \to E' \to E \to I_{ZH}(r - c_2) \to 0. \tag{1}$$

The same argument as for Proposition 2.3 shows that $Z = \varnothing$. Thus the exact sequence (1), together with the sequences $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{\mathbf{P}^{n-1}} \to 0$, $0 \to \mathcal{O}(-r-1) \to \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(-r) \to E'(1) \to 0$ [**Ok**, §2.3] give the locally free resolution stated in (ii).

(iii) It is an immediate consequence of (ii).

The rest of the proof is done as in Proposition 2.9, so we will omit it.

Now, using all these results and the correspondence between rank 2 reflexive sheaves on $\mathbf{P}^3$ and curves in $\mathbf{P}^3$ (see [**H1**, §4]), we will construct a family of nonsingular curves which are projectively normal and for such curves we will also give a locally free resolution of their sheaves of ideals.

PROPOSITION 2.11. *For all $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$ and $l \geqslant c_2 + 1 - r$, there exists a family F of irreducible nonsingular curves Y of degree $d = l^2 + c_2 - l$ and genus $g = (c_2^2 - 2rc_2)/2 + \frac{1}{2}(l^2 + c_2 - l)(2l - 5) + r(r+1) + 1$, which are projectively normal, not contained in a surface of degree $\leqslant l - 1$, and whose sheaf of ideals admits a locally free resolution of the following kind*:

$$\begin{aligned} 0 &\to \mathcal{O}(-l-r-1) \oplus \mathcal{O}(-l+r-c_2) \oplus \mathcal{O}(-2l+1) \\ &\to \mathcal{O}(-l-r) \oplus \mathcal{O}(-l+r-c_2+1) \oplus \mathcal{O}(-l) \oplus \mathcal{O}(-l-1) \to I_Y \to 0. \end{aligned}$$

Furthermore,

$$\dim F = \begin{cases} 2c_2^2 + 2c_2 + 6 + \dfrac{2c_2^3 - 8c_2}{2} & \text{if } r = 1 \text{ and } l = c_2 + 1 - r = c_2, \\ c_2^2 + 7 + l(l + 2 - c_2) + \dfrac{c_2^2 - 3c_2}{2} & \\ \qquad + \dfrac{2l^3 + 3l^2 + l}{6} & \text{if } r = 1 \text{ and } l > c_2, \\ c_2^2 + (3 - 3r - l)c_2 + 3r^2 + r + l^2 + 2l + \dfrac{c_2^2 - 3c_2}{2} & \\ \qquad + \dfrac{2l^3 + 3l^2 + l}{6} + 3 & \text{if } r > 1 \text{ and } l = c_2 + 1 - r, \\ c_2^2 + (3 - 3r - l)c_2 + 3r^2 + r + l^2 + 2l + \dfrac{c_2^2 - 3c_2}{2} & \\ \qquad + \dfrac{2l^3 + 3l^2 + l}{2} + 4 & \text{if } r > 1 \text{ and } l > c_2 + 1 - r. \end{cases}$$

PROOF. Let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r+1))$. Recall that $E(l)$ is generated by global sections for $l \geq c_2 + 1 - r$ and $H^1E(l) = 0$ for all l (Corollary 2.8). Under these conditions the zero set Y of a general section of $E(l)$ is nonsingular. Moreover from

$$0 \to \mathcal{O} \to E(l) \to I_Y(2l-1) \to 0$$

we see that $H^1(I_Y) = H^1E(-l+1) = 0$, hence Y is connected. Thus Y is a nonsingular, irreducible curve of degree $c_2 - l + l^2$ and genus

$$\left(c_2^2 - 2rc_2\right)/2 + r(r+1) + 1 + \tfrac{1}{2}\left(l^2 + c_2 - l\right)(2l-5).$$

Since $H^1I_Y(m) \simeq H^1E(m-l+1) = 0$ for all $m \in \mathbf{Z}$, Y is projectively normal. As E is stable, Y is not contained in a surface of degree $\leq l-1$.

Now we will give a locally free resolution of I_Y. To do this, we consider the exact sequences

$$0 \to \mathcal{O} \to E(l) \to I_Y(2l-1) \to 0,$$

$$0 \to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \oplus \mathcal{O} \to \mathcal{O} \to 0,$$

and the locally free resolution of E (cf. 2.7)

$$\begin{aligned} 0 &\to \mathcal{O}(-r-2) \oplus \mathcal{O}(r-1-c_2) \\ &\to \mathcal{O}(-r-1) \oplus \mathcal{O}(r-c_2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to E \to 0. \end{aligned}$$

These sequences give the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(l-r-2) \oplus \mathcal{O}(l+r-1-c_2) & \to & H & \to & \mathcal{O} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(l-r-2) \oplus \mathcal{O}(l+r-1-c_2) & \to & G & \to & E(l) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I_Y(2l-1) & = & I_Y(2l-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $H = \mathcal{O}(l-r-2) \oplus \mathcal{O}(l+r-1-c_2) \oplus \mathcal{O}$ and $G = \mathcal{O}(l-r-1) \oplus \mathcal{O}(l+r-c_2) \oplus \mathcal{O}(l-1) \oplus \mathcal{O}(l-2)$; from which one obtains the claimed locally free resolution of I_Y.

It remains to compute the dimension of F. The choice of the sheaf E is $c_2^2 + c_2 + 6$ parameters if $r = 1$ and $c_2^2 + (3-2r)c_2 + 2r^2 + 5$ parameters if $r \geq 2$, the choice of $s \in H^0E(l)$ is

$$l^2 + r^2 + 2l - c_2 l - rc_2 + r + \frac{c_2^2 - 3c_2}{2} + \frac{2l^3 + 3l^2 + l}{6}$$

parameters and

$$\begin{aligned} \dim H^0\omega_Y(5-2l) &= \dim H^1\mathcal{O}_Y(2l-5) \\ &= \dim H^2 I_Y(2l-5) = \begin{cases} 2 & \text{if } l = c_2 + 1 - r, \\ 1 & \text{if } l > c_2 + 1 - r. \end{cases} \end{aligned}$$

Combining all these shows that dim F is

$$2c_2^2 + 2c_2 + 6 + \frac{2c_2^3 - 8c_2}{6} \quad \text{if } r = 1,\ l = c_2 + 1 - r = c_2,$$

$$c_2^2 + 7 + l^2 + 2l - c_2 l + \frac{c_2^2 - 3c_2}{2} + \frac{2l^3 + 3l^2 + l}{6} \quad \text{if } r = 1,\ l > c_2,$$

$$c_2^2 + (3 - 3r - l)c_2 + 3 + 3r^2 + r + l^2 + 2l + \frac{c_2^2 - 3c_2}{2} + \frac{2l^3 + 3l^3 + l}{6}$$
$$\text{if } r > 1,\ l = c_2 + 1 - r,$$

$$c_2^2 + (3 - 3r - l)c_2 + 4 + 3r^2 + r + l^2 + 2l + \frac{c_2^2 - 3c_2}{2} + \frac{2l^3 + 3l^2 + l}{6}$$
$$\text{if } r > 1,\ l > c_2 + 1 - r.$$

We also have the following converse:

PROPOSITION 2.12. *For all $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, and $l \geqslant c_2 + 1 - r$, if Y is a nonsingular curve of degree $d = l^2 + c_2 - l$ and genus $g = (c_2^2 - 2rc_2)/2 + \frac{1}{2}(l^2 + c_2 - l)(2l - 5) + r(r + 1) + 1$ which is not contained in a surface of degree $\leqslant l - 1$ and such that $H^0\omega_Y(-2l + 5) \neq 0$, then Y is projectively normal and I_Y admits the locally free resolution*

$$0 \to \mathcal{O}(-l - r - 1) \oplus \mathcal{O}(-l + r - c_2) \oplus \mathcal{O}(-2l + 1)$$
$$\to \mathcal{O}(-l - r) \oplus \mathcal{O}(-l + r - c_2 + 1) \oplus \mathcal{O}(-l) \oplus \mathcal{O}(-l - 1) \to I_Y \to 0.$$

PROOF. Let $0 \neq \xi \in H^0\omega_Y(-2l + 5)$, ξ determines an extension

$$0 \to \mathcal{O} \to E(l) \to I_Y(2l - 1) \to 0,$$

where E is a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2rc_2 + 2r(r + 1))$. By Corollary 2.8 $H^1E(k) = 0$ for all integer k, so $H^1I_Y(k) = 0$ for all integer k and hence Y is projectively normal. The same argument as for Proposition 2.11 shows that I_Y has the announced locally free resolution.

COROLLARY 2.13. *Each irreducible nonsingular curve C in $\mathbf{P}^3$ of degree $d = 25$ and genus $g = 73$ not contained in any surface of degree $\leqslant 4$ is projectively normal and its sheaf of ideals admits a locally free resolution of the following kind*:

$$0 \to \mathcal{O}(-7) \oplus \mathcal{O}(-9) \oplus \mathcal{O}(-9)$$
$$\to \mathcal{O}(-6) \oplus \mathcal{O}(-8) \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6) \to I_C \to 0.$$

PROOF. To prove this statement it is enough to show that, for such curves, $H^0\omega_C(-5) \neq 0$ and to apply Proposition 2.12 to the case $c_2 = 5$, $r = 1$, and $l = 5$.

As $g = 73$, C is contained in at most in one quintic surface. (If $h^0I_C(5) \geqslant 2$, the corresponding quintic surfaces must be irreducible, so C is contained in the intersection of two quintic irreducible surfaces; then, by reason of its degree, C is the complete intersection of those two quintic surfaces, so $g(C) = 76$, which is a contradiction.) Thus the exact sequence

$$0 \to I_C(5) \to \mathcal{O}(5) \to \mathcal{O}_C(5) \to 0$$

gives us $h^0\mathcal{O}_C(5) \geqslant h^0\mathcal{O}(5) - h^0I_C(5) \geqslant 55$, and using Riemann-Roch we get

$$h^0\omega_C(-5) = h^1\mathcal{O}_C(5) = h^0\mathcal{O}_C(5) - 1 + 73 - 5 \times 25 \geqslant 2 \neq 0.$$

For convenience of the reader, we give some tables:

$c=5,\quad r=1$

$l \geqslant 5$

l	5	6	7	8
d	25	35	47	61
g	73	133	222	346
$\dim F$	101	146	205	281

$c=6,\quad r=1$

$l \geqslant 6$

l	6	7	8
d	36	48	62
g	141	231	356
$\dim F$	154	213	288

$c=7,\quad r=1$

$l \geqslant 7$

l	7	8
d	49	63
g	241	267
$\dim F$	223	298

$c=8,\quad r=1$

$l \geqslant 8$

l	8
d	64
g	379
$\dim F$	310

$c=8,\quad r=2$

$l \geqslant 7$

l	7	8
d	50	64
g	248	375
$\dim F$	224	298

3. The moduli space $M = M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2(r-1)c_2)$. We begin this section studying rank 2 reflexive sheaves on $\mathbf{P}^3$ properly semistable with Chern classes $(0, \alpha, \alpha^2 + \alpha)$.

PROPOSITION 3.1. *There exists an irreducible, nonsingular and rational variety* $V = V^{ss}_{\mathbf{P}^3}(0, \alpha, \alpha^2 + \alpha)$, *whose closed points are in one-to-one correspondence with the isomorphism classes of properly semistable reflexive sheaves with Chern classes* $(0, \alpha, \alpha^2 + \alpha)$. *Furthermore*

$$\dim V = \begin{cases} 6, & \alpha = 1, \\ \alpha^2 + 4 + 2, & \alpha \geqslant 2. \end{cases}$$

PROOF. Let E be a rank 2 reflexive sheaf on $\mathbf{P}^3$, properly semistable with Chern classes $(0, \alpha, \alpha^2 + \alpha)$. E is properly semistable, so we have

$$\xi\colon 0 \to \mathcal{O} \to E \to I_Y \to 0, \qquad \xi \in H^0\omega_Y(4),$$

where Y is a curve of degree α and arithmetic genus

$$p_a(Y) = \frac{c_3E + 2 - 4d}{2} = \binom{\alpha - 1}{2}.$$

Thus Y is a plane curve of degree α. Conversely each plane curve of degree α and each $\xi \in H^0\omega_Y(4)$ determine a rank 2 reflexive sheaf properly semistable with Chern classes $(0, \alpha, \alpha^2 + \alpha)$.

Since $H^0E = 1$, the isomorphism classes of the sheaves E are in a one-to-one correspondence with the pair (Y, ξ), where Y is a plane curve of degree α and $\xi \in H^0\omega_Y(4)/k^* \simeq H^0\mathcal{O}_Y(\alpha + 1)/k^*$. So we want to see that such pairs are parametrized by an irreducible, nonsingular, rational variety of dimension 6 if $\alpha = 1$, and $\alpha^2 + 4\alpha + 2$ if $\alpha \geqslant 2$.

In order to see this, let $E \to \mathbf{P}^{3*}$ be the tautological 3-bundle. Then the projective bundle $H = \mathbf{P}(S^\alpha(E^*)) \to \mathbf{P}^{3*}$ parametrizes plane curves of degree $\alpha \geqslant 2$ (if $\alpha = 1$, set $H = \mathrm{Gr}(3, 1)$). We consider the correspondence $\mathcal{Y} = \{(y, x) \in H \times \mathbf{P}^3 | x \in Y\} \subset H \times \mathbf{P}^3$, where we denote by y the point of H which corresponds to a curve Y. Let $\pi\colon \mathcal{Y} \to H$ and $p\colon \mathcal{Y} \to \mathbf{P}^3$ be the restrictions to $\mathcal{Y}$ of the projection maps. Thus

$$\mathcal{Y}_y = \pi^{-1}(y) \simeq Y \times_H \operatorname{Spec} k(y) \simeq Y.$$

Let $\mathcal{O}_{\mathcal{Y}}(\alpha + 1) = p^*\mathcal{O}_{\mathbf{P}^3}(\alpha + 1)$. By the semicontinuity theorem $\pi_*\mathcal{O}_{\mathcal{Y}}(\alpha + 1)$ is locally free of rank $(\alpha^2 + 5\alpha)/2$ and $(\pi^*\mathcal{O}_{\mathcal{Y}}(\alpha + 1))(y) \simeq H^0(Y, \mathcal{O}_Y(\alpha + 1))$. In fact, for all $y \in H$,

$$h^0\left(\pi^{-1}(y), \mathcal{O}_{\mathcal{Y}}(\alpha + 1)|_{\pi^{-1}(y)}\right) = h^0(Y, \mathcal{O}_Y(\alpha + 1)) = (\alpha^2 + 5\alpha)/2.$$

Hence the set of pairs (Y, ξ) is parametrized by $\mathbf{P}(\pi_*\mathcal{O}_{\mathcal{Y}}(\alpha + 1))$, which has the required properties.

A property P of reflexive sheaves with Chern classes c_1, c_2, c_3 is called general if every component of $M = M^s_{\mathbf{P}^3}(c_1, c_2, c_3)$ contains a nonempty open set corresponding to sheaves which verify P. To prove the irreducibility of some moduli space $M(c_1, c_2, c_3)$ it is sufficient to exhibit a general property such that the set of E in $M(c_1, c_2, c_3)$ satisfying P is an irreducible family.

REMARK (i). Let E be a rank 2 reflexive sheaf on $\mathbf{P}^3$, properly semistable with Chern classes $(0, \alpha, \alpha^2 + \alpha)$ associated to the pair (Y, ξ), where Y is a plane curve of degree α and $\xi \in H^0\omega_Y(4)/k^* = H^0\mathcal{O}_Y(\alpha + 1)/k^*$. Then:

(a) The plane H containing Y is an unstable plane for E of order α (cf. §2, Remark 2.3.1).

(b) ξ generates $\omega_Y(4)$ except at the points where E is not free (cf. [**H1**, §4.1]).

(c) For all $\xi \in H^0\omega_Y(4)$ there is an $\eta \in H^0\mathcal{O}_{\mathbf{P}^3}(\alpha + 1)$ such that $\eta_{|Y} = \xi$.

REMARK (ii). Given a general plane curve Y of degree α and a general $\xi \in H^0\omega_Y(4)$, the rank 2 semistable reflexive sheaf E with Chern classes $(0, \alpha, \alpha^2 + \alpha)$ associated to (Y, ξ) verifies that $\mathrm{Ext}^1(E, \mathcal{O})$ is a sheaf of length $\alpha^2 + \alpha$ supported by $\alpha^2 + \alpha$

different points (Th. Bertini) obtained cutting the hypersurface of $\mathbf{P}^3$ of degree $\alpha + 1$, $\eta = 0$, with the plane curve of degree α, Y.

PROPOSITION 3.2. *Let E be a general rank 2 reflexive sheaf on $\mathbf{P}^3$ properly semistable with Chern classes $(0, \alpha, \alpha^2 + \alpha)$. Let H be its unique unstable plane of order α. Then $E_H = \mathcal{O}_H(-\alpha) \oplus I_{ZH}(\alpha)$, where $\mathcal{O}_Z \simeq \operatorname{Ext}^1(E, \mathcal{O})$.*

PROOF. There exists a plane curve Y of degree α and a $\xi \in H^0\omega_Y(4)/k^*$ such that E is given by the extension $\xi: 0 \to \mathcal{O} \to E \to I_Y \to 0$.

Restricting this sequence to H, we obtain the exact sequence $E_H \to \mathcal{O}_H(-\alpha) \to 0$.

Let F be the kernel of this map. Then F is a rank 1 torsion free sheaf, so that it must be of the form $I_{ZH}(l)$ for some $l \in \mathbf{Z}$, where Z is a 0-dimensional subscheme of H. Computing Chern classes it turns out that $l = \alpha$ and length $\mathcal{O}_Z = \alpha^2 + \alpha$, and we have the exact sequence

$$0 \to I_{ZH}(\alpha) \to E_H \to \mathcal{O}_H(-\alpha) \to 0. \tag{1}$$

Applying $\operatorname{Ext}^1(-, \mathcal{O}_H)$ we get that $Z = \operatorname{sing}(E)$. Since $\operatorname{Ext}^1(\mathcal{O}_H(-\alpha), I_{ZH}(\alpha)) = H^1(I_{ZH}(2\alpha)) = 0$ (see Remark (ii) above) the sequence (1) splits, $E_H = \mathcal{O}_H(-\alpha) \oplus I_{ZH}(\alpha)$.

Now we will study rank 2 stable reflexive sheaves on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$ with $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$. We will prove the irreducibility of their moduli spaces.

Construction 3.3. For $c_1 = -1$ and each $c_2 > 4$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, construct E as an extension

$$\xi: 0 \to \mathcal{O} \to E(r) \to I_Y(2r - 1) \to 0, \qquad \xi \in H^0\omega_Y(5 - 2r),$$

where $Y = Y_1 \cup Y_2$ is the disjoint union of a plane curve Y_1 of degree c_2 and a curve Y_2 of degree $r(r-1)$ which is the complete intersection of a surface of degree r with a surface or degree $r - 1$.

$$d = \deg Y = c_2 + r(r-1),$$

$$c_1 E = -1, \quad c_2 E = c_2, \quad c_3 E = 2p_a - 2 + d(5 - 2r) = c_2^2 - 2(r-1)c_2,$$

and E is stable by construction

$$\begin{aligned}\big(\xi \in H^0\omega_Y(5 - 2r)\big) &= H^0\omega_{Y_1}(5 - 2r) \oplus H^0\omega_{Y_2}(5 - 2r) \\ &= H^0\mathcal{O}_{Y_1}(c_2 + 2 - 2r) \oplus H^0\mathcal{O}_{Y_2}\big).\end{aligned}$$

The moduli space $M^s_{\mathbf{P}^3}(-1, c_2, c_2^2)$ was studied by R. Hartshorne [**H1**, §9.2]. We will study the cases $r \geqslant 2$.

THEOREM 3.4. *For any $c_2 \geqslant 8$, $2 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, the moduli space of stable rank 2 reflexive sheaves on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$ is irreducible of dimension $c_2^2 - 2(r-2)c_2 + c_2^2 + 4 + 3r^2 - 5r$.*

REMARK 3.4.0. The dimension of the family is $> 8c_2 - 5$ so it is an oversized family.

The proof of Theorem 3.4 follows after several propositions.

PROPOSITION 3.5. *For each $c_2 \geqslant 8$, $2 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, Construction 3.3 gives an irreducible family F of stable reflexive sheaves with $c_1 = -1$, $c_3 = c_2^2 - 2(r-1)c_2$, which has dimension $c_2^2 - 2(r-2)c_2 + c_2 + 4 + 3r^2 - 5r$.*

PROOF. The sheaf E is determined by the choices of Y_1, Y_2, and $\xi \in H^0\omega_Y(5-2r)$. These are all irreducible choices, so the family F is irreducible. Now we compute its dimension: the choice of the plane H containing Y_1 is three parameters, the choice of Y_1 in H is $\binom{c_2+2}{2}-1$ parameters. The choice of Y_2 is $(2r^3+9r^2+13r+6)/6-6$ parameters. The choice of $\xi \in H^0\omega_Y(5-2r)$ is

$$\dim \mathbf{P}\big(H^0\omega_Y(5-2r)\big) = \frac{c_2^2+7c_2-4rc_2+12-14r+4r^2}{2}$$

parameters. Then we must subtract

$$\dim \mathbf{P}\big(H^0E(r)\big) = h^0I_Y(2r-1) = h^0I_{Y_2}(2r-2) = \frac{2r^3+3r^2+r}{6}.$$

Combining all these show that the family F of sheaves E given in Construction 3.3 depends on $c_2^2-2(r-2)c_2+c_2+4+3r^2-5r$ parameters.

LEMMA 3.6. *Let E be a rank 2 semistable reflexive sheaf on $\mathbf{P}^3$ with $c_1=0$.*
(i) *If E is stable, then* $\dim \operatorname{End} E = 1$ (*E is simple*).
(ii) *If E is properly semistable, then*:
–*If* $c_2=0$, $E \simeq \mathcal{O}\oplus\mathcal{O}$, *so that* $\dim \operatorname{End}(E)=4$.
–*If* $c_2>0$, $\dim \operatorname{End} E = 2$.

PROOF. For (i) see [**H1**, §3.4.1]. For (ii) see [**Ch2**, §2.0].

PROPOSITION 3.7 (REDUCTION STEP). *Let E be a rank 2 semistable reflexive sheaf on $\mathbf{P}^3$ with $c_1=0,-1$, and let H be an unstable plane of order r. Then*
(i) *There is an exact sequence*

$$0 \to E' \to E \to I_{ZH}(-r) \to 0,$$

where Z is a 0-dimensional subscheme of H. Let $s =$ length $\mathcal{O}_Z$.
(ii) *The Chern classes c_i' of E' are given by* $c_1'=c_1-1$, $c_2'=c_2-r-c_1$, $c_3'=c_3-c_2-c_1r-r^2+2s$.
(iii) *There is a dual exact sequence*

$$0 \to E^\vee \to E'^\vee \to I_{WH}(r+1) \to 0,$$

where W is a 0-dimensional subscheme of H such that

$$w = \text{length}\,\mathcal{O}_W = c_2+c_1r+r^2-s,$$

(iv) *There is also an exact sequence*

$$0 \to \mathcal{O}_Z \to \operatorname{Ext}^1\big(E'^\vee, \mathcal{O}\big) \to \operatorname{Ext}^1\big(E^\vee, \mathcal{O}\big) \to \omega_W \to 0,$$

which gives us $s \leqslant c_3'$, $w \leqslant c_3$, and $s+c_3=w+c_3$.

REMARK. The equality $s+c_3=w+c_3$ is equivalent to the last equality of (iii).
PROOF. For (i), (ii), and (iii) see [**H1**, §9.1]
(iv) Dualizing $0 \to E^\vee \to E'^\vee \to I_{WH}(r+1) \to 0$, we get

$$\begin{aligned} 0 &\to E'^{\vee\vee} = E' \to E^{\vee\vee} = E \to \operatorname{Ext}^1\big(I_{WH}(r+1), \mathcal{O}\big) \to \operatorname{Ext}\big(E'^\vee, \mathcal{O}\big) \\ &\to \operatorname{Ext}^1\big(E^\vee, \mathcal{O}\big) \to \operatorname{Ext}^2\big(I_{WH}(r+1), \mathcal{O}\big) \to 0, \end{aligned}$$

$$\operatorname{Ext}^1\big(I_{WH}(r+1), \mathcal{O}\big) \simeq \mathcal{O}_H(-r).$$

[In fact: The exact sequence

$$0 \to I_{WH}(r+1) \to \mathcal{O}_H(r+1) \to \mathcal{O}_W \to 0$$

gives us

$$\operatorname{Ext}^1(I_{WH}(r+1), \mathcal{O}) \simeq \operatorname{Ext}^1(\mathcal{O}_H(r+1), \mathcal{O});$$

furthermore

$$\operatorname{Ext}^1(\mathcal{O}_H(r+1), \mathcal{O}) \simeq \mathcal{O}_H(-r),$$

so we have $\operatorname{Ext}^1(I_{WH}(r+1), \mathcal{O}) \simeq \mathcal{O}_H(-r)$.] In the same way

$$\operatorname{Ext}^2(I_{WH}(r+1), \mathcal{O}) \simeq \omega_W.$$

On the other hand the exact sequence

$$0 \to E' \to E \to I_{ZH}(-r) \to 0$$

tells us that $\operatorname{coker}(E \to \mathcal{O}_H(-r)) \simeq \mathcal{O}_Z$ so we have the exact sequence

$$0 \to \mathcal{O}_Z \to \operatorname{Ext}^1(E'^{\vee}, \mathcal{O}) \to \operatorname{Ext}^1(E^{\vee}, \mathcal{O}) \to \omega_W \to 0$$

which gives us $s = \operatorname{length} \mathcal{O}_Z \leqslant c_3'$, $w \leqslant c_3$, and $s + c_3 = c_3 + w$.

Let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$, $c_2 \geqslant 8$, $2 \leqslant r \leqslant (-1 + \sqrt{4c_2 - 7})/2$. By Lemma 3, §1, E has an unstable plane H of order $c_2 - (r-1)$. A reduction step for H gives us an exact sequence

$$0 \to E' \to E \to I_{ZH}((r-1) - c_2) \to 0,$$

where Z is a 0-dimensional subscheme of H. Let $s = \operatorname{length} \mathcal{O}_Z$, the Chern classes of E' are given by $c_1' = -2$, $c_2' = r$, $c_3' = -r^2 + r + 2s$. The Chern classes c_i'' of the normalized sheaf $E'(1)$ are given by $c_1'' = 0$, $c_2'' = r - 1$, $c_3'' = r - r^2 + 2s$. Since $H^0E' = 0$ we see that $E'(1)$, and hence E', is semistable. By **[H1**, §8.2] $0 \leqslant r - r^2 + 2s = c_3'' \leqslant c_2^2 + c_2 = r^2 - r$ and by Proposition 3.7 $s \leqslant c_3' = r - r^2 + 2s$. This implies that the only possible value of s is $s = r^2 - r$, so the exact sequence of the reduction step is

$$0 \to E' \to E \to I_{ZH}((r-1) - c_2) \to 0,$$

where $E'(1)$ is semistable with Chern classes $(0, r-1, r^2 - r)$, so by **[Ok**, §2.3] it admits the locally free resolution

$$0 \to \mathcal{O}(-r) \to \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-r+1) \to E'(1) \to 0.$$

REMARK. We observe in this case that not only $\mathcal{O}_Z \hookrightarrow \operatorname{Ext}^1(E'^{\vee}, \mathcal{O})$ but also $\mathcal{O}_Z \simeq \operatorname{Ext}^1(E'^{\vee}, \mathcal{O})$.

PROPOSITION 3.8. *For each $c_2 \geqslant 8$, $\mathbf{N} \ni r \leqslant (-1 + \sqrt{4c_2 - 7})/2$, let E be a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$. Then E determines*:

(a) *A plane H in $\mathbf{P}^3$.*

(b) *A semistable reflexive sheaf $E'(1) = F$ with Chern classes $(0, r-1, r^2 - r)$ whose singularity points are contained in H.*

(c) *A morphism* $\psi \in \operatorname{Hom}(E'_H(2), \mathcal{O}_H(c_2 - r + 2))$ *determined up to scalar.*

Conversely, the following data determine a rank 2 *stable reflexive sheaf on* $\mathbf{P}^3$ *with Chern classes* $(-1, c_2, c_2^2 - 2(r-1)c_2)$:

(a) *A plane* H *in* $\mathbf{P}^3$.

(b) *A general semistable reflexive sheaf on* $\mathbf{P}^3$, *with Chern classes* $(0, r-1, r^2 - r)$ *whose singularity points are contained in* H.

(c) *A general morphism* $\psi \in \operatorname{Hom}(E'_H(2), \mathcal{O}_H(c_2 - r + 2))$ *determined up to scalar* (*see the proof below for the sense given to "general" in this case*).

PROOF. We know (Lemma 3, §1) E has an unstable plane H of order $c_2 - (r-1)$ which gives us the reduction step sequence

$$(*) \qquad 0 \to E' \to E \to I_{ZH}\big((r-1) - c_2\big) \to 0,$$

where Z is a 0-dimensional subscheme of H and $r^2 - r = \text{length } \mathcal{O}_Z = c'_3$. Dualizing $(*)$ we find

$$0 \to E(1) \to E'(2) \to I_{WH}(c_2 - r + 2) \to 0,$$

where W is a 0-dimensional subscheme of H, such that length $\mathcal{O}_W = c_2^2 - 2(r-1)c_2$. The surjective map on the right factors through $E'_H(2)$

$$\begin{array}{ccccccccc} 0 & \to & E(1) & \to & E'(2) & \to & I_{WH}(c_2 - r + 2) & \to & 0 \\ & & & & \searrow & & \nearrow \psi & & \\ & & & & & E'_H(2) & & & \end{array}$$

giving a morphism $\psi \in \operatorname{Hom}(E'_H(2), \mathcal{O}_H(c_2 - r + 2))$.

Conversely for general E', $E'_H(2) = I_{ZH}(r) \oplus \mathcal{O}_H(2 - r)$ (cf. Proposition 3.2) and so to give a morphism $\psi \in \operatorname{Hom}(E'_H(2), \mathcal{O}_H(c_2 - r + 2))$ is equivalent to give 2 forms on H of degree $c_2 - 2r + 2$ and c_2. Assume the last form is not a multiple of the first, in which case we will say that ψ is general. Then the image of ψ is of the form $I_{WH}(c_2 - r + 2)$, where W is a 0-dimensional subscheme of H such that length $\mathcal{O}_W = c_2(c_2 - 2r + 2)$.

Let $E(1)$ be the kernel of the composition

$$E'(2) \to E'_H(2) \to I_{WH}(c_2 + 2 - r) \to 0.$$

Then E is a rank 2 stable reflexive sheaf on $\mathbf{P}^3$ with Chern classes $(-1, c_2, c_2^2 - 2(r-1)c_2)$.

PROOF OF THEOREM 3.4. To prove the irreducibility of some moduli space $M(c_1, c_2, c_3)$, it is sufficient to show that the set of general E in $M(c_1, c_2, c_3)$ forms an irreducible family.

In our case a general E is determined by the data in Proposition 3.8, i.e., by the choices of E', H, and ψ up to scalar. These are all irreducible choices, so the family is irreducible. If $r \geqslant 3$, the choice of E' is $(r-1)^2 + 4(r-1) + 2 = r^2 + 2r - 1$ parameters (cf. Proposition 3.1, §3). H does not contribute to the parameter count if $r \geqslant 3$, because it is determined by the $r^2 - r$ singularity points of E'. To give an epimorphism from $E'_H(2)$ to $\mathcal{O}_H(c_2 - r + 2)$ is $c_2^2 - 2(r-2)c_2 + c_2 + 2r^2 - 7r + 7$ parameters. We must subtract $\dim \operatorname{End}(E') = 2$. (Automorphism of E' acts on ψ

and gives the same kernel E.) If $r = 2$, the choice of E' is 6 parameters, the choice of H is 1 parameter and the choice of ψ is $c_2^2 + c_2 + 1$. We must subtract $\dim \mathrm{End}(E) = 2$.

Our family is irreducible and depends on $c_2^2 - 2(r-2)c_2 + c_2 + 3r^2 - 5r + 4$. So $M^s_{\mathbf{P}^3}(-1, c_2, c_2^2 - 2(r-1)c_2)$ is irreducible of dimension $c_2^2 - 2(r-2)c_2 + c_2 + 3r^2 - 5r + 4$.

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Departamento de Algebra, Universidad de Barcelona, Gran Via 585, 08007-Barcelona, Spain