PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 3, Pages 691–695
S 0002-9939(03)07104-1
Article electronically published on July 16, 2003

# $2S_4 * Q_8$-EXTENSIONS IN CHARACTERISTIC 3

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(Communicated by Stephen D. Smith)

ABSTRACT. We present an explicit construction of the complete family of Galois extensions of a field $K$ of characteristic 3 with Galois group the central product $2S_4 * Q_8$ of a double cover $2S_4$ of the symmetric group $S_4$ and the quaternion group $Q_8$, containing a given $S_4$-extension of the field $K$.

Let $2S_4 * Q_8$ be the central product of a double cover $2S_4$ of the symmetric group $S_4$ and the quaternion group $Q_8$. In [2], we gave a method of construction of $2S_4 * Q_8$-extensions of a field $K$ of characteristic different from 2. In this paper we examine more closely the characteristic 3 case. Given an $S_4$-extension $L_1$ of a field $K$ containing the finite field $\mathbb{F}_9$ of 9 elements, we give explicitly all $2S_4 * Q_8$-extensions of $K$ containing $L_1$. In the case when $K = k((Y_1, Y_2, Y_3))$, the quotient field of the formal power series ring $R = k[[Y_1, Y_2, Y_3]]$ in 3 variables over an algebraically closed field $k$ of characteristic 3, the determination of all $2S_4 * Q_8$-extensions of $K$ is interesting in the framework of Abhyankar's Normal Crossings Local Conjecture (see [1]). As mentioned by Abhyankar in [1], the groups $2S_4 * Q_8$ are examples of groups $G$ such that the quotient $G/3(G)$ of $G$ by the subgroup $3(G)$ generated by all of its 3-Sylow subgroups is abelian, generated by 3 generators, but $3(G)$ does not have an abelian supplement in $G$, i.e., an abelian subgroup of $G$ generating $G$ together with $3(G)$. By a result of Harbater et al. (see [4]), this property makes them the "simplest" groups not appearing as Galois groups of a Galois extension $L$ of $K$ such that $RY_1, RY_2, RY_3$ are the only height-one primes in $R$ that are possibly ramified in $L$. We note that Harbater and Lefcourt (see [6]) have shown that every finite group can be obtained as a ramified extension of $k((Y_1, Y_2, Y_3))$. The results in this paper provide an explicit construction of $2S_4 * Q_8$-extensions in the characteristic 3 case. As a first step towards the determination of the ramification locus of the corresponding coverings, we give a formula for the discriminant.

Let us first recall the definitions and fix notation. We denote by $2S_n$ one of the two double covers of the symmetric group $S_n$ reducing to the nontrivial double cover $2A_n$ of the alternating group $A_n$ and by $Q_8$ the quaternion group, which is a double cover of the Klein group $V_4$. The group $2S_4 * Q_8$ is the central product of $2S_4$ and $Q_8$. Let $L_1|K$ be a Galois extension with Galois group the symmetric group $S_4$, for $K$ a field of characteristic different from 2. We assume that $L_1$ is given as the splitting field of a polynomial $P(X) \in K[X]$ of degree 4. We want to determine when $L_1$ is embeddable in a Galois extension of $K$ with Galois group

Received by the editors July 18, 2002 and, in revised form, November 13, 2002.
2000 *Mathematics Subject Classification.* Primary 12F12.
Partially supported by the grant BFM2000-0794-C02-01 of the Spanish Ministry of Education.

$2S_4 * Q_8$. This fact is equivalent to the existence of a Galois extension $L_2|K$ with Galois group $V_4$, disjoint from $L_1$, and such that, if $L$ is the compositum of $L_1$ and $L_2$, the Galois embedding problem

$$2S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K) \tag{1}$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension field $\widetilde{L}$ of the field $L$, which is a Galois extension of $K$ with Galois group $2S_4 * Q_8$ and such that the restriction epimorphism between the Galois groups $\mathrm{Gal}(\widetilde{L}|K) \to \mathrm{Gal}(L|K)$ agrees with the given epimorphism $2S_4 * Q_8 \to S_4 \times V_4$. If $\widetilde{L} = L(\sqrt{\gamma})$ is a solution, then the general solution is $L(\sqrt{r\gamma}), r \in K^*$. Given a Galois extension $L_1|K$ with Galois group $S_4$, in order to obtain all $2S_4 * Q_8$-extensions of $K$ containing $L_1$, we have to determine all $V_4$-extensions $L_2$ of $K$, disjoint from $L_1$, and such that the embedding problem (1) is solvable.

Let us now specify notation by writing $2^+S_n$ or $2^-S_n$ depending on whether transpositions in $S_n$ lift in the double cover to involutions or to elements of order 4. Let $E = K[X]/(P(X))$, for $P(X)$ the polynomial of degree 4 realizing $L_1$, let $Q_E$ denote the trace form of the extension $E|K$, i.e., $Q_E(x) = \mathrm{Tr}_{E|K}(x^2)$, and let $d$ be the discriminant of the polynomial $P(X)$. Let $L_2 = K(\sqrt{a}, \sqrt{b})$. We denote by w the Hasse-Witt invariant of a quadratic form and by $(\cdot,\cdot)$ a Hilbert symbol. In [2], we saw that the embedding problem $2^{\pm}S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K)$ is solvable if and only if

$$\mathrm{w}(Q_E) = (\pm 2, d).(a, b).(-1, ab) \quad \text{in } H^2(G_K, \{\pm 1\}). \tag{2}$$

From now on, we assume that $K$ is a field of characteristic 3, containing $\mathbb{F}_9$. We write $P(X) = X^4 + s_2X^2 - s_3X + s_4$. By computation of the trace form $Q_E$, we obtain

$$\mathrm{w}(Q_E) = (ds_2, (s_2^2 - s_4)s_4).$$

The equivalent condition (2) for the solvability of the embedding problem (1) then turns into

$$(ds_2, (s_2^2 - s_4)s_4) = (a, b), \tag{3}$$

that is, the equality of two Hilbert symbols.

Now, taking into account that $K$ contains $\mathbb{F}_9$, the equality of Hilbert symbols (3) holds if and only if the two quadratic forms $\langle ds_2, ms_4, dms_2s_4\rangle$ (where $m = s_2^2 - s_4$) and $\langle a, b, ab\rangle$ are $K$-equivalent (see [5, 3.2]).

*Remark* 1. We note that if $s_2 = 0$ or $s_2^2 - s_4 = 0$, the embedding problem $2S_4 \to S_4 \simeq \mathrm{Gal}(L_1|K)$ is solvable. In this case, $L_1$ is embeddable in a $2S_4 * Q_8$-extension of $K$ if and only if there exists a $V_4$-extension $L_2|K$ disjoint with $L_1$ and such that the embedding problem $Q_8 \to V_4 \simeq \mathrm{Gal}(L_2|K)$ is solvable. Any $2S_4 * Q_8$-extension of $K$ containing $L_1$ is then obtained as $L(\sqrt{\alpha\beta})$, for $L = L_1.L_2$, $L_1(\sqrt{\alpha})$ a solution to $2S_4 \to S_4 \simeq \mathrm{Gal}(L_1|K)$ and $L_2(\sqrt{\beta})$ a solution to $Q_8 \to V_4 \simeq \mathrm{Gal}(L_2|K)$, with $L_2$ a $V_4$-extension of $K$ with the conditions above. The element $\alpha$ can be computed explicitly by the method obtained in [3], the element $\beta$ is given by the formula of Witt given in [7].

**Theorem 1.** *Let $K$ be a field of characteristic* 3 *and containing* $\mathbb{F}_9$. *Let $P(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$ be a polynomial with discriminant $d$ and Galois group $S_4$, and let $L_1$ be its splitting field. Let $A \in \mathrm{GL}(3, K)$ be such that $D = A^t\langle ds_2, ms_4, dms_2s_4\rangle A$ is a diagonal matrix of the form $\langle a, b, ab\rangle$ and such that*

*$L_2 = K(\sqrt{a}, \sqrt{b})$ is a $V_4$-extension of $K$, disjoint from $L_1$. Let us choose the matrix $A$ such that $\det A = ab/(dms_2s_4)$. Let $L$ be the compositum of $L_1$ and $L_2$. Let $M$ be the matrix with entries in $L$ given by*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{ad}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{bd}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{abd}} \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

*where $(x_1, x_2, x_3, x_4)$ are the four roots of the polynomial $P(X)$ in $L_1$. Let $B \in \mathrm{GL}(4, K)$ be the matrix*

$$B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

*Let $P$ be the matrix in $\mathrm{GL}(4, K)$ given by*

$$P = \frac{1}{m}\begin{pmatrix} m & 0 & -s_2d + s_3^2s_2^4 & -s_2^3s_3d \\ 0 & dm & (s_2^2+s_4)ms_2s_3 & (s_2^2+s_4)dm \\ 0 & 0 & d - s_3^2s_2^3 & s_2^2s_3d \\ 0 & 0 & s_3s_2^2m & s_2dm \end{pmatrix}.$$

*Let $\gamma$ be the element in $L^*$ given by*

$$\gamma = \det(MPB + I).$$

*Then the general solution to the embedding problem $2^-S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K)$ is $L(\sqrt{r\gamma_1})$ for $r \in K^*$ and*

$$\gamma_1 = (1+\sqrt{a})^2(1+\sqrt{b})^2(1+\sqrt{d})^2\gamma.$$

*The general solution to the embedding problem $2^+S_4 * Q_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K)$ is $L(\sqrt{r\gamma_2})$ for $r \in K^*$ and*

$$\gamma_2 = \sqrt{d}\gamma_1.$$

*Proof.* We can check that the matrix $P$ satisfies $P^tTP = \langle 1, ds_2, ms_4, dms_2s_4\rangle$, where $T$ is the matrix of the quadratic trace form $Q_E$ in the basis $(1, x_1, x_1^2, x_1^3)$ and apply [[2], theorems 1 and 2]. In particular, the choice of the sign of the determinant of the matrix $A$ assures that the element $\gamma$ is nonzero. □

*Remark* 2. For each of the two extensions $L(\sqrt{\gamma_i})|K$, $i = 1, 2$, with $\gamma_1, \gamma_2$ given by the theorem, we consider the discriminant $\Delta_i$ of the basis $\{x_1^{i_1}x_2^{i_2}x_3^{i_3}a^{\varepsilon_1}b^{\varepsilon_2}\gamma_i^{\varepsilon_3} : 0 \le i_1 \le 3,\ 0 \le i_2 \le 2,\ i_3 = 0, 1,\ \epsilon_1, \epsilon_2, \epsilon_3 = 0, 1/2\}$, where $x_1, x_2, x_3$ are three distinct roots of the polynomial $P(X)$ realizing $S_4$. We have

$$\Delta_1 = (dab)^{96}((1-a)(1-b)(1-d))^{24}N_{L|K}(\gamma)\ ;\ \ \Delta_2 = d^{12}\Delta_1.$$

**Corollary 1.** *Let $K$ and $L_1$ be as in the theorem and assume that the extension $L_2 = K(\sqrt{a}, \sqrt{b})|K$, with $a = ds_2, b = ms_4$, has Galois group $V_4$ and is disjoint from $L_1$. Then the element $\gamma$ in the theorem corresponding to this field $L_2$ is given in terms of the coefficients $s_2, s_3, s_4$ of the polynomial $P(X)$ realising the extension $L_1|K$, of three distinct roots $x_1, x_2, x_3$ of this polynomial $P(X)$ and the square roots* *of the elements a and b, by*

$$\begin{aligned}
\gamma = {} & m^2 s_2 s_4 \sqrt{a} x_1^3 x_2^2 + 2m^2 s_2 s_4 \sqrt{a} x_1^3 x_2 x_3 + (2m s_2 d \sqrt{b} + (2s_3 s_4 m \\
& + 2s_3 m^2)\sqrt{ab}) x_1^3 x_2 + (s_4^2 m^2 \sqrt{a} + (2s_4 m^3 s_3 + d s_3 s_4 + s_4^2 s_2 s_3^3 + m^2 s_3^3 s_2 \\
& + d s_3 m + 2s_3 s_4^2 m^2 + 2m s_4 s_2 s_3^3)\sqrt{b} + (2s_4 m^2 + d + m s_2 s_3^2 \\
& + s_4 s_2 s_3^2)\sqrt{ab}) x_1^3 + (d s_2 + 2s_4 m s_3^2 + s_3^2 m^2 + s_3^2 s_4^2)\sqrt{a} x_1^2 x_2^2 x_3 + ((2s_3 m^2 \\
& + 2s_3 s_4 m)\sqrt{ab}) x_1^2 x_2^2 + (m s_2 d \sqrt{b} + (2s_3 m^2 + 2s_3 s_4 m)\sqrt{ab}) x_1^2 x_2 x_3 \\
& + ((2dm + 2s_4 m^3 + 2m^2 s_3^2 s_2 + 2s_4^2 m^2 + 2m s_4 s_2 s_3^2)\sqrt{a} + (2d s_3 m \\
& + 2d s_3 s_4)\sqrt{b}) x_1^2 x_2 + ((2s_2 s_4^2 s_3^2 + 2m s_4 s_2 s_3^2 + 2s_4^2 m^2 + 2d s_4)\sqrt{a} \\
& + (s_3 s_4^2 m^2 + s_4 m^3 s_3 + d s_3 s_4 + 2m^2 s_3^3 s_2 + 2s_4^2 s_2 s_3^3 + d s_3 m + m s_4 s_2 s_3^3)\sqrt{b} \\
& + (s_4 m^2 + m s_2 s_3^2 + s_4 s_2 s_3^2)\sqrt{ab}) x_1^2 x_3 + (2m^2 s_2 s_4 s_3 + s_4 m s_3^3 + 2s_3^3 m^2 \\
& + 2d s_2 s_3 + 2s_4^2 s_3^3)\sqrt{a} x_1^2 + (m s_2 d \sqrt{b} + (2s_3 m^2 + 2s_3 s_4 m)\sqrt{ab}) x_1 x_2^2 x_3 \\
& + ((2m s_4 s_2 s_3^2 + 2s_4^2 m^2 + 2m^2 s_3^2 s_2 + 2dm)\sqrt{a} + (2d s_3 m + s_3 s_4^2 m^2 \\
& + 2s_4^2 s_2 s_3^3 + 2m^2 s_3^3 s_2 + 2d s_3 s_4 + m s_4 s_2 s_3^3 + s_4 m^3 s_3)\sqrt{b} \\
& + (s_4 s_2 s_3^2 + m s_2 s_3^2 + s_4 m^2)\sqrt{ab}) x_1 x_2^2 \\
& + ((m^2 s_3^2 s_2 + s_4^2 m^2 + dm + m s_4 s_2 s_3^2)\sqrt{a} + (2m s_4 s_2 s_3^3 + m^2 s_3^3 s_2 \\
& + 2s_3 s_4^2 m^2 + s_4^2 s_2 s_3^3 + 2s_4 m^3 s_3)\sqrt{b} + (2d + 2s_4 m^2)\sqrt{ab}) x_1 x_2 x_3 \\
& + ((2s_4^2 s_3^3 + 2s_3^3 m^2 + 2d s_2 s_3 + s_4 m s_3^3)\sqrt{a} + (2d s_4 m + 2d m^2)\sqrt{b}) x_1 x_2 \\
& + ((s_3^3 m^2 + m^2 s_2 s_4 s_3 + 2s_4 m s_3^3 + d s_2 s_3 + s_4^2 s_3^3)\sqrt{a}) x_1 x_3 \\
& + ((2m^2 s_2 s_4^2)\sqrt{a} + (2m s_2 d s_3 + 2m^3 s_3 s_2 s_4 + 2m^2 s_2 s_4^2 s_3 + m^3 s_3^3 \\
& + s_4^3 s_3^3)\sqrt{b} + (s_4^2 s_3^2 + d s_2 + 2s_3^2 m^2 + 2m^2 s_2 s_4)\sqrt{ab}) x_1 + ((2d s_4 \\
& + 2m s_4 s_2 s_3^2 + 2s_2 s_4^2 s_3^2)\sqrt{a} + (2d s_3 m + 2d s_3 s_4)\sqrt{b} + (2m s_2 s_3^2 + d \\
& + 2s_4 s_2 s_3^2)\sqrt{ab}) x_2^2 x_3 + (2d s_2 s_3 + 2s_3^3 m^2 + 2s_4^2 s_3^3 + s_4 m s_3^3)\sqrt{a} x_2^2 \\
& + (2s_4 m s_3^3 + s_3^3 m^2 + d s_2 s_3 + s_4^2 s_3^3)\sqrt{a} x_2 x_3 + (m^2 s_2 s_4^2 \sqrt{a} + (d s_4 s_2 s_3 \\
& + m^3 s_3^3 + 2m^3 s_3 s_2 s_4 + 2m^2 s_2 s_4^2 s_3 + s_4^3 s_3^3)\sqrt{b} + (2m^2 s_2 s_4 + 2d s_2 \\
& + s_3^2 m^2 + s_4 m s_3^2)\sqrt{ab}) x_2 + ((s_3^2 s_4^3 + 2s_3^2 s_4^2 m + m^2 s_2 s_4^2 + s_4 m^2 s_3^2 \\
& + s_2 s_4 d)\sqrt{a} + (2s_4^3 s_3^3 + 2m^3 s_3^3 + m s_2 d s_3 + m^3 s_3 s_2 s_4 + m^2 s_2 s_4^2 s_3)\sqrt{b} \\
& + (m^2 s_2 s_4 + d s_2 + 2s_4 m s_3^2 + 2s_3^2 m^2)\sqrt{ab}) x_3 + m^2 s_2 s_4 d \\
& + (s_2 s_4^2 s_3^3 + d s_4 s_3 + m s_4 s_2 s_3^3)\sqrt{a} + (d s_3^2 s_4 + d s_3^2 m + m s_4 s_2 d)\sqrt{b} \\
& + (2d s_3 + m s_2 s_3^3 + s_2 s_3^3 s_4)\sqrt{ab}.
\end{aligned}$$

*Proof.* We apply the theorem with the matrix $B$ equal to the identity matrix. The computation of the corresponding determinant is simplified by applying the Cramer identities. The stated expression for the element $\gamma$ is obtained by using the symmetric functions in the four roots $x_1, x_2, x_3, x_4$ of the polynomial $P(X)$. The computations are carried out with Maple. □

**Example 1.** The polynomial $P(X) = X^4 + (Y_1 + Y_2)X^2 + Y_2X + (Y_1 + Y_3)$ has Galois group $S_4$ over the field $K = \overline{\mathbb{F}}_3((Y_1, Y_2, Y_3))$. Let $L_1$ be the splitting field of $P(X)$. We write $s_2 = Y_1 + Y_2, s_3 = -Y_2, s_4 = Y_1 + Y_3$ and consider the elements $a = ds_2, b = (s_2^2 - s_4)s_4$. To assure that the extension $L_2 = K(\sqrt{a}, \sqrt{b})|K$ is a $V_4$-extension disjoint from $L_1$ we check that the elements $a, b, ab, s_2, db, s_2b$ are not squares in $K$. To this end we use the following lemma, which is a direct consequence of the Weierstrass preparation theorem (see e.g. [8]).

**Lemma 1.** *Let $F \in R = \overline{\mathbb{F}}_3[[Y_1, Y_2, Y_3]]$ be regular and polynomial in $Y_i$ for some $i = 1, 2, 3$. If the discriminant $\Delta(F, Y_i)$ is nonzero, then $F$ is not a square in $K$.*

Let $L$ be the composition field of $L_1$ and $L_2$. By applying the corollary, we obtain extensions $L(\sqrt{\gamma_1})|K$ and $L(\sqrt{\gamma_2})|K$ with Galois groups $2^-S_4 * Q_8$ and $2^+S_4 * Q_8$, respectively, with an explicit expression for the elements $\gamma_1$ and $\gamma_2$.

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