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RUBIO DE FRANCIA'S EXTRAPOLATION THEOREM FOR B_p WEIGHTS

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ABSTRACT. In this paper, we prove some of Rubio de Francia's extrapolation results for the class B_p of weights for which the Hardy operator is bounded on $L^p(w)$ restricted to decreasing functions. Applications to the boundedness of operators on $L^p_{\rm dec}(w)$ are given. We also present an extension to the B_∞ case and some connections with classical A_p theory.

1. Introduction

In 1984, J.L. Rubio de Francia [10] proved that if T is a sublinear operator that is bounded on $L^r(w)$ for every w in the Muckenhoupt class A_r ($r > 1$) with constant depending only on

$$||w||_{A_r} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_I w^{-1/(r-1)}\right)^{r-1}$$

where the supremum is taken over all cubes Q, then for every $1 < p < \infty$, T is bounded on $L^p(w)$ for every $w \in A_p$ with constant depending only on $||w||_{A_p}$. Since then, many results concerning this topic have been published (see [8], [6], [7]). From these results, it is now known that, in fact, the operator T plays no role; that is, if (f, g) are a pair of functions such that for some $1 \le p_0 < \infty$,

$$\int_{\mathbb{R}^n} f^{p_0}(x)w(x)dx \le C\int_{\mathbb{R}^n} g^{p_0}(x)w(x)dx$$

for every $w \in A_{p_0}$ with C depending on $||w||_{A_{p_0}}$, then for every $1 < p < \infty$,

$$\int_{\mathbb{R}^n} f^{p}(x)w(x)dx \le C\int_{\mathbb{R}^n} g^{p}(x)w(x)dx$$

for every $w \in A_p$ with C depending on $||w||_{A_p}$. The theory has also been generalized to the case of A_∞ weights and many interesting consequences have been derived from it.

The purpose of this paper is to develop a completely parallel theory in the setting of B_p weights. The techniques are different as usually happens with these

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two theories and things are, in some sense, clearer and more natural. We think that the results in this paper should help to clarify what is happening in the A_p context and we hope to solve that case in a forthcoming paper.

Before presenting the main results of this paper, let us just recall some important facts concerning B_p weights which will be fundamental for our purposes. First of all, let us recall that a positive and locally integrable function w on $(0,\infty)$ is called a B_p weight if the following condition holds:

$$||w||_{B_p} = \inf\left\{C>0; \int_0^r w(t)dt + r^p\int_r^\infty \frac{w(t)}{t^p}dt \le C\int_0^r w(t)dt, \forall r>0\right\} < \infty.$$

It is known ([1]) that $w \in B_p$ with $p>0$ if and only if, for every decreasing function f,

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)ds\right)^p w(t)dt \le C\int_0^\infty f^p(s)w(s)ds$$

with C depending on $||w||_{B_p}$. Observe also that $||w||_{B_p} > 1$ if w is not identically zero.

An important property that these classes of weights satisfy (see [4], Chapter 3, Section 3.3) is that, for every $p>0$ and every $w\in B_p$, there exists $\varepsilon>0$ such that $w \in B_{p-\varepsilon}$; moreover,

$$||w||_{B_{p-\varepsilon}} \le \frac{C||w||_{B_p}}{1-\varepsilon\alpha^p||w||_{B_p}}, \tag{1.1}$$

where C and $0<\alpha<1$ are universal constants and ε is such that $1-\varepsilon\alpha^p||w||_{B_p}>0$.

Since $B_p \subset B_q$ for every $0<p\le q<\infty$, we can define (similarly to A_p theory) the class B_∞ as the collection of weights belonging to some B_p; that is,

$$B_\infty = \bigcup_{p>0} B_p.$$

Let us also define

$$||w||_{B_\infty} = \inf\{||w||_{B_p}; w\in B_p\}.$$

We shall denote by C a universal constant depending possibly on p but independent of the weight w. Also C might not be the same in all instances. We write $A \lesssim B$ if there exists a universal constant C such that $A\le CB$ and $A\approx B$ if $A\lesssim B$ and $B\lesssim A$.

2. Main results

Our first result is the counterpart in this setting of the new version of Rubio de Francia's extrapolation result:

Theorem 2.1. *Let φ be an increasing function on $(0,\infty)$, let (f,g) be a pair of positive decreasing functions defined on $(0,\infty)$ and let $0<p_0<\infty$. Suppose that for every $w\in B_{p_0}$,*

$$\int_0^\infty f^{p_0}w \le \varphi(||w||_{B_{p_0}})\int_0^\infty g^{p_0}w\,.$$

Then, for every $p>0$ and $w\in B_p$,

$$\int_0^\infty f^p w \le \tilde\varphi(||w||_{B_p})\int_0^\infty g^p w\,,$$

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where

$$\tilde{\varphi}(||w||_{B_p}) = \inf_{0<\varepsilon<\frac{p_0}{p\alpha^p||w||_{B_p}}} \varphi\left(\frac{p_0}{\varepsilon}\right)^{p/p_0} \frac{C||w||_{B_p}}{1-\varepsilon\frac{p}{p_0}\alpha^p||w||_{B_p}}$$

with C *as in* (1.1).

Similarly, in the B_∞ setting:

Theorem 2.2. *Let φ be an increasing function on $(0,\infty)$, let (f,g) be a pair of positive decreasing functions defined on $(0,\infty)$ and let $0<p_0<\infty$. Suppose that for every $w\in B_\infty$,*

$$\int_0^\infty f^{p_0}w \le \varphi(||w||_{B_\infty})\int_0^\infty g^{p_0}w.$$

Then, for every $p>0$ and $w\in B_\infty$,

$$\int_0^\infty f^p w \le \varphi(1)^{p/p_0}||w||_{B_\infty}\int_0^\infty g^p w\cdot$$

In order to prove these two results, we shall use the following lemmas.

Lemma 2.3. *Let φ be an increasing function on $(0,\infty)$, let (f,g) be a pair of positive decreasing functions defined on $(0,\infty)$ and let $0<p_0<\infty$. Suppose that for every $w\in B_{p_0}$,*

$$\int_0^\infty fw \le \varphi(||w||_{B_{p_0}})\int_0^\infty gw\,.$$

Then, for every $0<\varepsilon<p_0$ and every $t>0$,

$$\int_0^t f(s)s^{p_0-1-\varepsilon}\,ds \le \varphi\left(\frac{p_0}{\varepsilon}\right)\int_0^t g(s)s^{p_0-1-\varepsilon}\,ds\cdot$$

Proof. Let $w(t)=v(t)t^{p_0-1-\varepsilon}$ with v a decreasing function and let us assume that $w\in L^1_{\text{loc}}$. Then

$$\begin{aligned}
&\int_0^r w(t)\,dt + r^{p_0}\int_r^\infty \frac{w(t)}{t^{p_0}}\,dt = \int_0^r w(t)\,dt + r^{p_0}\int_r^\infty \frac{v(t)}{t^{1+\varepsilon}}\,dt\\
&\le \int_0^r w(t)\,dt + \frac{1}{\varepsilon}v(r)r^{p_0-\varepsilon} = \int_0^r w(t)\,dt + \frac{p_0-\varepsilon}{\varepsilon}v(r)\int_0^r t^{p_0-\varepsilon-1}\,dt\\
&\le \frac{p_0}{\varepsilon}\int_0^r w(t)\,dt,
\end{aligned}$$

and hence $w\in B_{p_0}$ with constant less than or equal to p_0/ε.

In particular, taking $v(t)=\chi_{(0,s)}(t)$ and applying the hypothesis, we obtain that

$$\sup_{s>0}\frac{\int_0^s f(u)u^{p_0-1-\varepsilon}\,du}{\int_0^s g(u)u^{p_0-1-\varepsilon}\,du} \le \varphi\left(\frac{p_0}{\varepsilon}\right)<\infty$$

and the result follows. □

Let ϕ be a positive decreasing locally integrable function defined on $(0,\infty)$ and let $\Phi(x)=\int_0^x\phi(t)dt$. The generalized Hardy operator associated to ϕ is defined, for f decreasing, by

$$S_\phi f(x) = \frac{1}{\Phi(x)}\int_0^x f(t)\phi(t)dt\,.$$

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Lemma 2.4. *Let $0 < p < \infty$. Then, S_ϕ is bounded on $L^p_{\mathrm{dec}}(w)$ with constant A if and only if*

$$\int_0^r w(x)dx + \Phi(r)^p \int_r^\infty \frac{w(x)}{\Phi(x)^p}\,dx \le A^p \int_0^r w(x)dx\,, \quad \textit{for all } r > 0. \tag{2.1}$$

Proof. This result has been proved in [5] (Theorem 4.1) for the case $p > 1$. The proof also works (and is easier) for $p = 1$.

Let us now prove the case $0 < p < 1$. The necessary condition follows as in [5] by taking $f = \chi_{(0,r)}$. Conversely, let f be decreasing. Then, $f(s) \le \dfrac{1}{\Phi(s)} \displaystyle\int_0^s f(t)\phi(t)dt$ for every $s > 0$ and therefore

$$\left(\int_0^s f(t)\phi(t)dt\right)^{p-1} \le f(s)^{p-1}\Phi(s)^{p-1}.$$

Taking this into account,

$$\begin{aligned}
\int_0^\infty (S_\phi f(x))^p w(x)dx &= \int_0^\infty \left(\frac{1}{\Phi(x)}\int_0^x f(s)\phi(s)ds\right)^p w(x)dx \\
&= p\int_0^\infty \int_0^x \left(\int_0^s f(t)\phi(t)dt\right)^{p-1} f(s)\phi(s)ds\frac{w(x)}{\Phi(x)^p}dx \\
&\le p\int_0^\infty \int_0^x f(s)^p\phi(s)\Phi(s)^{p-1}ds\frac{w(x)}{\Phi(x)^p}dx\,.
\end{aligned} \tag{2.2}$$

Since f is decreasing, Corollary 2.2 in [5] gives that the chain of inequalities in (2.2) can be continued as follows:

$$\begin{aligned}
&\le p\int_0^\infty \int_0^\infty \int_0^{\lambda_{f^p}(y)} \chi_{(0,x)}(s)\phi(s)\Phi(s)^{p-1}ds\,dy\frac{w(x)}{\Phi(x)^p}dx \\
&\le p\int_0^\infty \int_0^\infty \int_0^{\min\{\lambda_{f^p}(y),x\}} \phi(s)\Phi(s)^{p-1}ds\,dy\frac{w(x)}{\Phi(x)^p}dx \\
&= \int_0^\infty \int_0^\infty \Phi(\min\{\lambda_{f^p}(y),x\})^p dy\frac{w(x)}{\Phi(x)^p}dx \\
&= \int_0^\infty \int_0^\infty \Phi(\min\{\lambda_{f^p}(y),x\})^p \frac{w(x)}{\Phi(x)^p}dx\,dy \\
&= \int_0^\infty \left(\int_0^{\lambda_{f^p}(y)} w(x)dx + \Phi(\lambda_{f^p}(y))^p \int_{\lambda_{f^p}(y)}^\infty \frac{w(x)}{\Phi(x)^p}dx\right) dy \\
&\le A^p \int_0^\infty \int_0^{\lambda_{f^p}(y)} w(x)dx\,dy = A^p\int_0^\infty f(y)^p w(y)\,dy,
\end{aligned}$$

where the last inequality is obtained from the hypothesis. □

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Proof of Theorem 2.1. Let $p > 0$, $w \in B_p$ and $0 < \varepsilon < p_0$. Using the fact that f is decreasing and Lemma 2.3, we get

$$\begin{aligned} \int_0^\infty f(t)^p w(t) dt &\le \int_0^\infty \left(\frac{p_0 - \varepsilon}{t^{p_0-\varepsilon}} \int_0^t f(s)^{p_0} s^{p_0-1-\varepsilon} ds \right)^{p/p_0} w(t)\, dt \\ &\le \varphi\left(\frac{p_0}{\varepsilon}\right)^{p/p_0} \int_0^\infty \left(\frac{p_0 - \varepsilon}{t^{p_0-\varepsilon}} \int_0^t g(s)^{p_0} s^{p_0-1-\varepsilon} ds \right)^{p/p_0} w(t)\, dt \\ &= \varphi\left(\frac{p_0}{\varepsilon}\right)^{p/p_0} \int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0}\, w(t)\, dt, \end{aligned} \tag{2.3}$$

where $\phi(t) = t^{p_0-1-\varepsilon}$. The proof will be finished once we compute A such that

$$\int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0}\, w(t)\, dt \le A \int_0^\infty g(t)^p w(t) dt,$$

and by Lemma 2.4, we only have to compute A such that

$$\int_0^r w(x) dx + r^{\frac{(p_0-\varepsilon)p}{p_0}} \int_r^\infty \frac{w(x)}{x^{\frac{(p_0-\varepsilon)p}{p_0}}} dx \le A \int_0^r w(x) dx,$$

which is equivalent to saying that $w \in B_{\frac{(p_0-\varepsilon)p}{p_0}}$ with $A = ||w||_{B_{\frac{(p_0-\varepsilon)p}{p_0}}}$.

Now, since $w \in B_p$ there exists $\tilde{\varepsilon} > 0$ so that $w \in B_{p-\tilde{\varepsilon}}$. Then, it suffices to take ε small enough so that $p - \tilde{\varepsilon} = \frac{(p_0-\varepsilon)p}{p_0}$ to get the result. Moreover, by (1.1), we have that

$$A = ||w||_{B_{\frac{(p_0-\varepsilon)p}{p_0}}} = ||w||_{B_{p-\tilde{\varepsilon}}} \le \frac{C||w||_{B_p}}{1 - \varepsilon \frac{p}{p_0} \alpha^p ||w||_{B_p}}.$$

Consequently, for every $0 < \varepsilon < \frac{p_0}{p\alpha^p ||w||_{B_p}}$,

$$\int_0^\infty f(t)^p w(t) dt \le \varphi\left(\frac{p_0}{\varepsilon}\right)^{p/p_0} \frac{C||w||_{B_p}}{1 - \varepsilon \frac{p}{p_0} \alpha^p ||w||_{B_p}} \int_0^\infty g(t)^p w(t) dt,$$

and the result follows by taking the infimum of such ε's. □

Proof of Theorem 2.2. By hypothesis we have that

$$\int_0^\infty f^{p_0} w \le \varphi(||w||_\infty) \int_0^\infty g^{p_0} w\,,$$

for every $w \in B_\infty$. Then, taking $w(t) = \chi_{(0,s)}(t) t^\beta$ with $s > 0$ and $\beta > -1$, we have that $w \in B_\infty$ and $||w||_{B_\infty} = 1$. Hence

$$\int_0^s f^{p_0}(t) t^\beta\, dt \le \varphi(1) \int_0^s g^{p_0}(t) t^\beta\, dt\,, \quad \text{for all } t > 0, \beta > -1. \tag{2.4}$$

Now let $p > 0$ and let $w \in B_\infty$ be arbitrary. Then, by definition of B_∞, there exists $q > 0$ such that $w \in B_q$. Using again that f is decreasing and inequality (2.4), we

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obtain that for every $\beta > -1$,

$$\begin{aligned}\int_0^\infty f(t)^p w(t)dt &\leq \int_0^\infty \left(\frac{1+\beta}{t^{1+\beta}}\int_0^t f(s)^{p_0} s^\beta ds\right)^{p/p_0} w(t)\,dt \\ &\leq \varphi(1)^{p/p_0}\int_0^\infty \left(\frac{1+\beta}{t^{\beta+1}}\int_0^t g(s)^{p_0} s^\beta ds\right)^{p/p_0} w(t)\,dt \\ &= \varphi(1)^{p/p_0}\int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0}\, w(t)\,dt\,,\end{aligned} \tag{2.5}$$

where $\phi(t) = t^\beta$. To finish the proof we only have to check that S_ϕ is bounded in $L^{p/p_0}_{\mathrm{dec}}(w)$ and this is equivalent to showing that $w \in B_{\frac{(1+\beta)p}{p_0}}$. Therefore, it suffices to choose $\beta > -1$ such that $\frac{(1+\beta)p}{p_0} = q$, i.e., $\beta = \frac{qp_0}{p} - 1$, to get that

$$\int_0^\infty f(t)^p w(t)dt \leq \varphi(1)^{p/p_0}||w||_{B_q}\int_0^\infty g(t)^p w(t)dt.$$

Taking the infimum of such q's we are done. □

3. Application and examples

In this section, we shall present mainly two applications which have interesting consequences. Both of them are consequences of the following observation:

Remark 3.1. It has been implicitly proved that, given $0 < p < \infty$ fixed and a pair of decreasing functions (f, g),

$$\int_0^\infty f(t)w(t)dt \leq C_w \int_0^\infty g(t)w(t)dt$$

holds for every $w \in B_p$ with constant C_w depending only on $||w||_{B_p}$ if and only if, for every $s > 0$ and every $-1 < \beta < p-1$,

$$\int_0^s f(t)t^\beta dt \lesssim C_\beta \int_0^s g(t)t^\beta dt,$$

with C_β independent of s.

Application I. The above observation is especially useful for characterizing the boundedness on $L^p_{\mathrm{dec}}(w)$ of certain operators.

Theorem 3.2. *Let T be an operator such that*

i) *for every decreasing function f, Tf is also a decreasing function whenever it is well defined;*

ii) *for every decreasing function g, a function T^*g is well defined by*

$$\int_0^\infty Tf(t)g(t)dt = \int_0^\infty f(t)T^*g(t)dt, \qquad \forall f\downarrow.$$

Let $0 < p < \infty$ be fixed. Then,

$$T : L^p_{\mathrm{dec}}(w) \longrightarrow L^p(w) \tag{3.1}$$

is bounded for every $w \in B_p$ with constant depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^s T\chi_{(0,r)}(t)t^\alpha dt \lesssim C_\alpha \min(r,s)^{\alpha+1}, \tag{3.2}$$

with C_α independent of r and s.

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Proof. If T satisfies (3.1), then taking f to be a decreasing function, we can apply Theorem 2.1 to the pair (Tf, f) to deduce that

$$T : L^1_{\text{dec}}(w) \longrightarrow L^1(w)$$

for every $w \in B_1$, and by the previous remark this is equivalent to having that, for every $s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^\infty f(t) T^*(u^\alpha \chi_{(0,s)}(u))(t)dt = \int_0^s Tf(t)t^\alpha dt \lesssim C_\alpha \int_0^s f(t)t^\alpha dt\cdot$$

Now, it is known (see [5]) that the above inequality holds for every decreasing f if and only if, for every $r > 0$,

$$\int_0^s T\chi_{(0,r)}(t)t^\alpha dt = \int_0^r T^*(u^\alpha \chi_{(0,s)}(u))(t)dt \lesssim C_\alpha \int_0^{\min(s,r)} t^\alpha dt$$
$$\approx C_\alpha \min(r,s)^{\alpha+1}$$

as we wanted to show. □

In particular, we can consider integral operators with positive kernel, which have been intensively studied in [9].

Corollary 3.3. *Let*

$$Tf(x) = \int_0^\infty f(t)k(x,t)dt$$

with k a positive kernel such that, for every decreasing function f, Tf is also a decreasing function whenever it is well defined. Then,

$$T : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$$

is bounded for every $w \in B_p$ with constant C_w depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^s \int_0^r k(x,t)x^\alpha dt\, dx \lesssim C_\alpha \min(r,s)^{\alpha+1}, \tag{3.3}$$

with C_α independent of r and s.

Similarly, in the case of two linear operators:

Corollary 3.4. *If T_1 and T_2 are two linear operators satisfying the hypothesis of Theorem* 3.2*, we have*

a)

$$\int_0^\infty (T_1 f)^p(t)w(t)dt \lesssim C_w \int_0^\infty (T_2 f)^p(t)w(t)dt \tag{3.4}$$

for every $w \in B_p$ and every decreasing function f with C_w depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^s T_1\chi_{(0,r)}(t)t^\alpha dt \lesssim C_\alpha \int_0^s T_2\chi_{(0,r)}(t)t^\alpha dt,$$

with C_α independent of r and s.

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b) *If T_j are integral operators with positive kernels k_j satisfying the hypothesis of Corollary* 3.3*, then* (3.4) *holds for every $w \in B_p$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,*

$$\int_0^s \int_0^r k_1(x,t) x^\alpha dt\, dx \lesssim C_\alpha \int_0^s \int_0^r k_2(x,t) x^\alpha dt\, dx, \tag{3.5}$$

with C_α independent of r and s.

Examples

Let us now give some examples of well known operators for which boundedness on $L^p_{\text{dec}}(w)$ is true for every $w \in B_p$ and examples in which this condition fails.

Example I. The Calderón operator.

Let $\lambda, \beta, \gamma > 0$ with $\lambda \geq \beta\gamma$ and let us consider the operator

$$Tf(x) = x^{-\lambda} \int_0^{x^\beta} t^{\gamma-1} f(t) dt.$$

Then, T is an integral operator with kernel

$$k(x,t) = x^{-\lambda} \chi_{(0,x^\beta)}(t) t^{\gamma-1}$$

and hence using Corollary 3.3 it is immediate to see the following result:

Theorem 3.5. *Let T be the Calderón operator defined above. Then, the following conditions are equivalent:*

(i) *There exists $0 < p < \infty$ such that*

$$T : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$$

is bounded for every $w \in B_p$.

(ii) *For every $0 < p < \infty$,*

$$T : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$$

is bounded for every $w \in B_p$.

(iii) *$\beta = 1$ and $\gamma = \lambda \geq 1$.*

Example II. The Riemann-Liouville fractional operator is defined by

$$R_\lambda f(x) = x^{-\lambda} \int_0^x (x-t)^{\lambda-1} f(t) dt,$$

with $0 < \lambda \leq 1$.

Theorem 3.6. *For every $0 < p < \infty$, the operator*

$$R_\lambda : L^p_{\text{dec}}(w) \longrightarrow L^p(w)$$

is bounded for every $w \in B_p$.

Proof. In this case $k(x,t) = x^{-\lambda} \chi_{(0,x)}(t)(x-t)^{\lambda-1}$. We already know that, in order to prove the result, it is enough to show that for all $-1 < \alpha < 0$ and all $r, s > 0$ we have

$$\int_0^s \int_0^r k(x,t) x^\alpha dt\, dx \lesssim C_\alpha \min(r,s)^{\alpha+1}. \tag{3.6}$$

To see this, suppose first that $s \leq r$. Then, for $x \in (0,s)$,

$$\int_0^r k(x,t) dt = \int_0^x x^{-\lambda}(x-t)^{\lambda-1} dt = \frac{1}{\lambda}.$$

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Therefore,

$$\int_0^s \int_0^r k(x,t)x^\alpha dt\,dx = \frac{1}{\lambda}\int_0^s x^\alpha dx = Cs^{\alpha+1} = C\min(r,s)^{\alpha+1}.$$

Suppose now that $r < s$. Then there are two possible cases: $s \le 2r$ and $2r < s$. In the case where $s \le 2r$ we have

$$\int_0^s \int_0^r k(x,t)x^\alpha dt\,dx \le \int_0^{2r} \int_0^{2r} k(x,t)x^\alpha dt\,dx \le C(2r)^{\alpha+1} = C\min(r,s)^{\alpha+1}.$$

If $2r < s$, then

$$\int_0^s \int_0^r k(x,t)x^\alpha dt\,dx = \int_0^{2r} \int_0^r k(x,t)x^\alpha dt\,dx + \int_{2r}^s \int_0^r k(x,t)x^\alpha dt\,dx.$$

For the first summand we proceed as in the previous case:

$$\int_0^{2r} \int_0^r k(x,t)x^\alpha dt\,dx \le \int_0^{2r} \int_0^{2r} k(x,t)x^\alpha dt\,dx \le C(2r)^{\alpha+1} = C\min(r,s)^{\alpha+1}.$$

Let us estimate the second one. By the mean value theorem applied to the function $f(u) = (x-u)^\lambda$ on the interval $[0,r]$, we have that there exists $c \in (0,r)$ such that $(x-r)^\lambda - x^\lambda = -\lambda r(x-c)^{\lambda-1}$. Then

$$\int_0^r k(x,t)dt = x^{-\lambda}\left(\frac{x^\lambda - (x-r)^\lambda}{\lambda}\right) = x^{-\lambda} r(x-c)^{\lambda-1} \le x^{-\lambda} r\frac{x^\lambda}{x-r} = \frac{r}{x-r}.$$

Therefore,

$$\int_{2r}^s \int_0^r k(x,t)x^\alpha dt\,dx \le \int_{2r}^s x^\alpha \frac{r}{x-r}dx = r\int_{2r}^s x^{\alpha-1}\frac{x}{x-r}dx.$$

Since the function $g : [2r,s] \to \mathbb{R}$ given by $g(x) = \frac{x}{x-r}$ is decreasing and $\alpha < 0$, we have that

$$\begin{aligned}\int_{2r}^s \int_0^r k(x,t)x^\alpha dt\,dx &\le 2r\left(\frac{s^\alpha - (2r)^\alpha}{\alpha}\right) = 2r\left(\frac{(2r)^\alpha - s^\alpha}{-\alpha}\right)\\ &\le C(2r)^{\alpha+1} = C\min(r,s)^{\alpha+1},\end{aligned}$$

and (3.6) is proved. □

Remark 3.7. With the same technique, we can also prove that neither the adjoint Calderón operator defined by

$$Tf(x) = x^{-\lambda}\int_{x^\beta}^1 t^{\gamma-1}f(t)dt$$

with $\lambda, \beta, \gamma > 0$ nor the Laplace operator

$$Lf(x) = \int_0^\infty e^{-xt}f(t)dt$$

satisfy the condition of boundedness on $L^p_{\mathrm{dec}}(w)$ for every $w \in B_p$.

In the first case the kernel is

$$k(x,t) = x^{-\lambda}\chi_{(x^\beta,1)}(t)t^{\gamma-1}$$

and it is enough to show that it is not true that for each $-1 < \alpha < 0$ and $r,s > 0$,

$$\int_0^s \int_0^r k(x,t)x^\alpha dt\,dx \lesssim \min(r,s)^{\alpha+1}.$$

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Let $0 < s < 1 < r$. Then

$$\int_0^r k(x,t)dt = \int_0^r x^{-\lambda}\chi_{(x^\beta,1)}(t)t^{\gamma-1}dt = x^{-\lambda}\int_{x^\beta}^1 t^{\gamma-1}dt = \frac{1}{\gamma}x^{-\lambda}(1-x^{\beta\gamma}).$$

Hence,

$$\int_0^s\int_0^r k(x,t)x^\alpha dt\,dx = \frac{1}{\gamma}\int_0^s x^{\alpha-\lambda}(1-x^{\beta\gamma})dx \geq \frac{1}{\gamma}\int_0^s x^{\alpha-\lambda}(1-s^{\beta\gamma})dx$$
$$= \frac{1-s^{\beta\gamma}}{\gamma}\int_0^s x^{\alpha-\lambda}dx = \infty\,,$$

for any α such that $-1 < \alpha < -1+\lambda$.

In the second case the kernel is $k(x,t) = e^{-xt}$. Let us take $0 < s < r$ and observe that

$$\int_0^r e^{-xt}dt = rHf(xr),$$

where H denotes the Hardy operator and $f(t) = e^{-t}$. Then, making the substitution $xr = u$, we get

$$\int_0^s x^\alpha \int_0^r k(x,t)dt\,dx = r\int_0^s x^\alpha Hf(xr)dx = \frac{1}{r^\alpha}\int_0^{sr} u^\alpha Hf(u)du$$
$$= \frac{1}{r^\alpha}\int_0^{sr} u^\alpha\frac{1-e^{-u}}{u}du.$$

If we keep $sr = 1$ and let r tend to infinity, then $\int_0^{sr} u^\alpha\frac{1-e^{-u}}{u}du = \int_0^1 u^\alpha\frac{1-e^{-u}}{u}du$ is a positive constant and, as $-1 < \alpha < 0$, $\frac{1}{r^\alpha} \to \infty$ while $\min(r,s)^{\alpha+1} = s^{\alpha+1} \to 0$.

Application II. Let $g^*(t) = \inf\{s > 0 : \lambda_g(s) \leq t\}$ be the decreasing rearrangement of g, where $\lambda_g(y) = |\{x \in \mathbb{R}^n : |g(x)| > y\}|$ is the distribution function of g with respect to Lebesgue measure, and let $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

In [3] the space $S_p(w)$ defined by

$$||f||_{S_p(w)} = \left(\int_0^\infty (f^{**}(t) - f^*(t))^p w(t)dt\right)^{1/p} < \infty$$

was studied and it was proved that it coincides with the Lorentz space $\Gamma_p(w)$ defined by

$$||f||_{\Gamma_p(w)} = \left(\int_0^\infty (f^{**}(t))^p w(t)dt\right)^{1/p} < \infty$$

if $w \in RB_p$; that is, for every $r > 0$,

$$\int_0^r w(s)ds \lesssim r^p\int_r^\infty \frac{w(s)}{s^p}ds.$$

To see this, it was proved that if $w \in RB_p$, the following inequality holds:

$$\int_0^\infty (f^{**}(t))^p w(t)dt \lesssim \int_0^\infty (f^{**}(t) - f^*(t))^p w(t)dt.$$

Now, making the change of variable $u = 1/t$ the previous inequality is the same as (3.7)

$$\int_0^\infty\left(\int_0^{\frac{1}{u}} f^*(s)ds\right)^p u^{p-2}w\Big(\frac{1}{u}\Big)du \lesssim \int_0^\infty\left(\frac{1}{u}\Big(f^{**}\Big(\frac{1}{u}\Big) - f^*\Big(\frac{1}{u}\Big)\Big)\right)^p u^{p-2}w\Big(\frac{1}{u}\Big)du.$$

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On the other hand, the following hold:
i) $w \in RB_p$ if and only if $u^{p-2} w\Big(\frac{1}{u}\Big) \in B_p$.
ii) $g(u) = \int_0^{\frac{1}{u}} f^*(s)ds$ is clearly a decreasing function.
iii) $h(u) = \frac{1}{u}\Big(f^{**}(\frac{1}{u}) - f^*(\frac{1}{u})\Big)$ is also a decreasing function (see [3]).
Therefore, inequality (3.7) can be read as

$$\int_0^\infty g(u)^p v(u) du \lesssim \int_0^\infty h(u)^p v(u) du$$

for every $v \in B_p$ with g and h being decreasing functions, and thus it is equivalent to proving that for every $s > 0$,

$$\int_0^s g(u) u^\alpha du \lesssim \int_0^s h(u) u^\alpha du$$

for every $-1 < \alpha < 0$, which can be seen with an easy computation.

4. Final comments

1) In the context of A_p weights developed in [10], [6], [7] and [3], we have a pair of positive functions (f, g) not necessarily decreasing such that, for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$, there exists a constant $C > 0$ depending only on $||w||_{A_{p_0}}$ satisfying

$$\int_0^\infty f^{p_0} w \le C \int_0^\infty g^{p_0} w\,.$$

Then, it is natural to ask whether it is true that there exists an operator T satisfying

$$f \le Tf, \qquad Tf \le Tg,$$

and

$$T : L^p(w) \longrightarrow L^p(w)$$

for every $w \in A_p$.

Observe that if this were the case, then for every $w \in A_p$,

$$\int_0^\infty f^p w \le \int_0^\infty (Tf)^p w \le \int_0^\infty (Tg)^p w \le C \int_0^\infty g^p w,$$

and we get the extrapolation result in the aforementioned papers.

Also observe that this is what happens in the B_p context since upon taking

$$Tf(t) = \left(\frac{1}{t^{p_0-\varepsilon}} \int_0^t f^{p_0}(s) s^{p_0-1-\varepsilon}\, ds\right)^{1/p_0}$$

we have that T satisfies the three conditions mentioned above for f a decreasing function.

2) In the context of the interpolation theory of Banach spaces, we also have a similar result to the ones developed in [2] (Theorems 3.8 and 5.2): Given two compatible Banach spaces $\bar{A}$ and $\bar{B}$ and a linear operator T such that, for some $0 < p < \infty$,

$$T : \bar{A}_{p,w;K} \longrightarrow \bar{B}_{p,w;K} \tag{4.1}$$

is bounded for every $w \in B_p$ with constant depending only on $||w||_{B_p}$, we have that

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for every $w \in B_q$ and every $0 < q < \infty$,

$$T : \bar{A}_{q,w;K} \longrightarrow \bar{B}_{q,w;K}$$

is bounded with constant depending only on $||w||_{B_q}$.

To see this, observe that by hypothesis,

$$\int_0^\infty \left(\frac{K(t,Tf;\bar{B})}{t}\right)^p w(t)dt \lesssim C_w \int_0^\infty \left(\frac{K(t,f;\bar{A})}{t}\right)^p w(t)dt,$$

and since $\frac{K(t,Tf;\bar{B})}{t}$ is a decreasing function, we can apply our results directly.

Moreover, we have that (4.1) holds for some p and every $w \in B_p$ (or equivalently, for every $0 < p < \infty$ and every $w \in B_p$) if and only if, for every $r > 0$ and every $-1 < \alpha < 0$,

$$\int_0^r K(t,Tf;\bar{B})t^{\alpha-1}dt \lesssim C_\alpha \int_0^r K(t,f;\bar{A})t^{\alpha-1}dt,$$

with C_α independent of $r > 0$.

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